Banach-Hecke Algebras and p-Adic Galois Representations

P. Schneider, J. Teitelbaum

Received: October 31, 2005
Revised: December 28, 2005

Wir lassen vom Geheimnis uns erheben
Der magischen Formelschrift, in deren Bann
Das Uferlose, Stürmende, das Leben
Zu klaren Gleichnissen gerann.
Hermann Hesse

Dedicated to John Coates

Abstract. In this paper, we take some initial steps towards illuminating the (hypothetical) p-adic local Langlands functoriality principle relating Galois representations of a p-adic field L and admissible unitary Banach space representations of $G(L)$ when G is a split reductive group over L.

2000 Mathematics Subject Classification: 11F80, 11S37, 22E50
Keywords and Phrases: Satake isomorphism, Iwahori-Hecke algebra, Banach-Hecke algebra, filtered isocrystal, crystalline Galois representation, p-adic local Langlands correspondence

In this paper, we take some initial steps towards illuminating the (hypothetical) p-adic local Langlands functoriality principle relating Galois representations of a p-adic field L and admissible unitary Banach space representations of $G(L)$ when G is a split reductive group over L. The outline of our work is derived from Breuil's remarkable insights into the nature of the correspondence between 2-dimensional crystalline Galois representations of the Galois group of $\mathbb{Q}_p$ and Banach space representations of $GL_2(\mathbb{Q}_p)$.

In the first part of the paper, we study the p-adic completion $\mathcal{B}(G,\rho)$ of the Hecke algebra $\mathcal{H}(G,\rho)$ of bi-equivariant compactly supported $\mathrm{End}(\rho)$-valued functions on a totally disconnected, locally compact group G derived from a finite dimensional continuous representation ρ of a compact open subgroup U of G. (These are the "Banach-Hecke algebras" of the title). After describing some general features of such algebras we study in particular the case where G is split reductive and $U = U_0$ is a special maximal compact or $U = U_1$ is an Iwahori subgroup of G and ρ is the restriction of a finite dimensional algebraic representation of G to U_0 or U_1.

In the smooth theory for trivial $\rho = 1_U$, by work of Bernstein, the maximal commutative subalgebra of the Iwahori-Hecke algebra is isomorphic to the group ring $K[\Lambda]$ where Λ is the cocharacter group of a maximal split torus T of G, and the spherical Hecke algebra is isomorphic by the Satake isomorphism to the ring $K[\Lambda]^W$ of Weyl group invariants. At the same time the algebra $K[\Lambda]$ may be viewed as the ring of algebraic functions on the dual maximal torus T' in the dual group G'. Together, these isomorphisms allow the identification of characters of the spherical Hecke algebra with semisimple conjugacy classes in G'. On the one hand, the Hecke character corresponds to a certain parabolically induced smooth representation; on the other, the conjugacy class in G' determines the Frobenius in an unramified Weil group representation of the field L. This is the unramified local Langlands correspondence (the Satake parametrization) in the classical case.

With these principles in mind, we show that the completed maximal commutative subalgebra of the Iwahori-Hecke algebra for ρ is isomorphic to the affinoid algebra of a certain explicitly given rational subdomain T'_ρ in the dual torus T'. The spectrum of this algebra therefore corresponds to certain points of T'. We also show that the quotient of this subdomain by the Weyl group action is isomorphic to the corresponding completion of the spherical Hecke algebra; this algebra, for most groups G, turns out to be a Tate algebra. These results may be viewed as giving a p-adic completion of the Satake isomorphism, though our situation is somewhat complicated by our reluctance to introduce a square root of q as is done routinely in the classical case. These computations take up the first four sections of the paper.

In the second part of the paper, we let $G = GL_{d+1}(L)$. We relate the subdomain of T' determined by the completion $\mathcal{B}(G,\rho)$ to isomorphism classes of a certain kind of crystalline Dieudonne module. This relationship follows Breuil's theory, which puts a 2-dimensional irreducible crystalline representation V of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ with coefficients in a field K into correspondence with a topologically irreducible admissible unitary representation of $GL_2(\mathbb{Q}_p)$ in a K-Banach space. Furthermore, this Banach space representation is a completion of a locally algebraic representation whose smooth factor comes from $D_{cris}(V)$ viewed as a Weil group representation and whose algebraic part is determined by the Hodge-Tate weights of V.

To state our relationship, let V be a $d+1$-dimensional crystalline representation of $\mathrm{Gal}(\overline{L}/L)$ in a K-vector space, where L and K are finite extensions of $\mathbb{Q}_p$. In this situation, $D_{cris}(V)$ has a K-vector space structure. Suppose further that:

i. L is embeddable into K, and fix once and for all such an embedding $L \subseteq K$;

ii. the eigenvalues of the Frobenius on $D_{cris}(V)$ lie in K;

iii. the (negatives of) the Hodge-Tate weights of $D_{cris}(V)$ are multiplicity free and are separated from one another by at least $[L : \mathbb{Q}_p]$;

iv. V is *special*, meaning that the kernel of the natural map

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \to \mathbb{C}_p \otimes_L V$$

is generated by its $\mathrm{Gal}(\overline{L}/L)$ invariants.

It follows from the Colmez-Fontaine theory that the category of such special representations is equivalent to a category of "K-isocrystals", which are K-vector spaces with a K-linear Frobenius and a filtration that is admissible in a sense very close to the usual meaning.

Given such a representation, we extract from the associated K-isocrystal its Frobenius, which we view as an element of the dual group $G'(K)$ determined up to conjugacy. The semi-simple part ζ of this element determines a point of $T'(K)$ up to the Weyl group action. From the Hodge-Tate weights, we extract a dominant cocharacter of G' and hence a highest weight ξ determining an algebraic representation $\rho = \rho_\xi$ for G. (In fact, the highest weight is a modification of the Hodge-Tate weights, but we avoid this complication in this introduction). Put together, this data yields a completion of the Iwahori-Hecke algebra, determined by the highest weight, and a character of its maximal commutative subalgebra, determined up to the Weyl group action. In other words, we obtain a simple module K_ζ for the completed spherical Hecke algebra $\mathcal{B}(G, \rho_\xi | U_0)$.

Our main result is that the existence of an admissible filtration on $D_{cris}(V)$ translates into the condition that the point of T' determined by the Frobenius lives in the subdomain T'_ρ. Conversely, we show how to reverse this procedure and, from a point of $T'_\rho(K)$ (up to Weyl action), make an isocrystal that admits an admissible filtration of Hodge-Tate type determined by ρ. See Section 5 (esp. Proposition 5.2) for the details.

It is crucial to realize that the correspondence between points of T'_ρ and isocrystals outlined above does not determine a specific filtration on the isocrystal. Except when $d = 1$ there are infinitely many choices of filtration compatible with the given data. Consequently the "correspondence" we describe is a very coarse version of a p-adic local Langlands correspondence.

To better understand this coarseness on the "representation-theoretic" side, recall that to a Galois representation V of the type described above we associate a simple module K_ζ for the completion $\mathcal{B}(G, \rho | U_0)$ of the spherical Hecke algebra. There is an easily described sup-norm on the smooth compactly induced representation $\mathrm{ind}_{U_0}^G(\rho|U_0)$; let $B_{U_0}^G(\rho|U_0)$ be the completion of this representation. We show that the completed Hecke algebra acts continuously on this space. By analogy with the Borel-Matsumoto theory constructing parabolically induced representations from compactly induced ones, and following also Breuil's approach for $GL_2(\mathbb{Q}_p)$, it is natural to consider the completed tensor product

$$B_{\xi,\zeta} := K_\zeta \widehat{\otimes}_{\mathcal{B}(G,\rho_\xi|U_0)} B^G_{U_0}(\rho_\xi|U_0) .$$

A deep theorem of Breuil-Berger ([BB]) says that, in the $GL_2(\mathbb{Q}_p)$-case, this representation in most cases is nonzero, admissible, and irreducible, and under Breuil's correspondence it is the Banach representation associated to V. In our more general situation, we do not know even that $B_{\xi,\zeta}$ is nonzero. Accepting, for the moment, that it is nonzero, we do not expect it to be admissible or irreducible, because it is associated to the entire infinite family of representations having the same Frobenius and Hodge-Tate weights as V but different admissible filtrations. We propose that $B_{\xi,\zeta}$ maps, with dense image, to each of the Banach spaces coming from this family of Galois representations. We discuss this further in Section 5.

In the last section of this paper (Section 6) we consider the shape of a p-adic local Langlands functoriality for a general L-split reductive group G over L, with Langlands dual group G' also defined over L. Here we rely on ideas from the work of Kottwitz, Rapoport-Zink, and Fontaine-Rapoport. Recall that a cocharacter ν of the dual group G' defined over K allows one to put a filtration $Fil^{\cdot}_{\rho'\circ\nu} E$ on every K-rational representation space (ρ', E) of G'. Using (a modified version of) a notion of Rapoport-Zink, we say that a pair (ν, b) consisting of an element b of $G'(K)$ and a K-rational cocharacter ν of G' is an "admissible pair" if, for any K-rational representation (ρ', E) of G', the K-isocrystal $(E, \rho'(b), Fil^{\cdot}_{\rho'\circ\nu} E)$ is admissible. Such an admissible pair defines a faithful tensor functor from the neutral Tannakian category of K-rational representations of G' to that of admissible filtered K-isocrystals. Composing this with the Fontaine functor one obtains a tensor functor to the category of "special" $\mathrm{Gal}(\overline{L}/L)$ representations of the type described earlier. The Tannakian formalism therefore constructs from an admissible pair an isomorphism class of representations of the Galois group of L in $G'(\overline{K})$.

Now suppose given an irreducible algebraic representation ρ of G. Its highest weight may be viewed as a cocharacter of G'. Under a certain technical condition, we prove in this section that there is an admissible pair (ν, b) where ν is conjugate by $G'(K)$ to a (modification of) the highest weight, and b is an element of $G'(K)$, if and only if the semisimple part of b is conjugate to an element of the affinoid domain $T'_\rho(K)$ (See Proposition 6.1). Thus in some sense this domain is functorial in the group G'.

Our work in this section relies on a technical hypothesis on G. Suppose that η is half the sum of the positive roots of G. We need $[L : \mathbb{Q}_p]\eta$ to be an integral weight of G. This happens, for example, if L has even degree over $\mathbb{Q}_p$, and in general for many groups, but not, for example, when $G = PGL_2(\mathbb{Q}_p)$. This complication has its origin in the normalization of the Langlands correspondence. Because of the square root of q issue the p-adic case seems to force the use of the "Hecke" or the "Tate" correspondence rather than the traditional unitary correspondence; but even for smooth representations this is not functorial (cf. [Del] (3.2.4-6)). It turns out that without the above integrality hypothesis one even has to introduce a square root of a specific continuous Galois character (for $L = \mathbb{Q}_p$ it is the cyclotomic character). This leads to isocrystals with a filtration indexed by half-integers. Although it seems possible to relate these to Galois representations this has not been done yet in the literature. We hope to come back to this in the future.

The authors thank Matthew Emerton for pointing out that the conditions which define our affinoid domains T'_ρ are compatible with the structure of his Jacquet functor on locally algebraic representations ([Em1] Prop. 3.4.9 and Lemma 4.4.2, [Em2] Lemma 1.6). We thank Laurent Berger, Christophe Breuil, and especially Jean-Marc Fontaine for their very helpful conversations about these results. We also want to stress that our computations in Section 4 rely in an essential way on the results of Marie-France Vigneras in [Vig]. The first author gratefully acknowledges support from UIC and CMI. During the final stages of this paper he was employed by the Clay Mathematics Institute as a Research Scholar. The second author was supported by National Science Foundation Grant DMS-0245410.

We dedicate this paper to John Coates on the occasion of his sixtieth birthday. His constant support and unrelenting enthusiasm was and is an essential source of energy and inspiration for us over all these years.

Throughout this paper K is a fixed complete extension field of $\mathbb{Q}_p$ with absolute value $|\ |$.

Added in proof: In a forthcoming joint paper by C. Breuil and P. Schneider the technical restrictions of the present paper – that $L \subseteq K$, that the crystalline Galois representations V have to be special, and that $[L : \mathbb{Q}_p]\eta$ has to be an integral weight for the split group G – will be removed. In fact, this forces a renormalization of the picture in the present paper.

1. Banach-Hecke algebras

In this section G denotes a totally disconnected and locally compact group, and $U \subseteq G$ is a fixed compact open subgroup. We let (ρ, E) be a continuous representation of U on a finite dimensional K-vector space E, and we fix a U-invariant norm $\|\ \|$ on E.

The Hecke algebra $\mathcal{H}(G,\rho)$ is the K-vector space of all compactly supported functions $\psi : G \longrightarrow \mathrm{End}_K(E)$ satisfying

$$\psi(u_1 g u_2) = \rho(u_1) \circ \psi(g) \circ \rho(u_2) \qquad \text{for any } u_1, u_2 \in U \text{ and } g \in G\ .$$

It is a unital associative K-algebra via the convolution

$$\psi_1 * \psi_2(h) := \sum_{g \in G/U} \psi_1(g) \circ \psi_2(g^{-1}h)\ .$$

Its unit element is the function

$$\psi_e(h) := \begin{cases} \rho(h) & \text{if } h \in U, \\ 0 & \text{otherwise.} \end{cases}$$

We note that any function ψ in $\mathcal{H}(G,\rho)$ necessarily is continuous. We now introduce the norm

$$\|\psi\| := \sup_{g\in G} \|\psi(g)\|$$

on $\mathcal{H}(G,\rho)$ where on the right hand side $\| \ \|$ refers to the operator norm on $\mathrm{End}_K(E)$ with respect to the original norm $\| \ \|$ on E. This norm on $\mathcal{H}(G,\rho)$ evidently is submultiplicative. By completion we therefore obtain a unital K-Banach algebra $\mathcal{B}(G,\rho)$, called in the following the Banach-Hecke algebra, with submultiplicative norm. As a Banach space $\mathcal{B}(G,\rho)$ is the space of all continuous functions $\psi : G \longrightarrow \mathrm{End}_K(E)$ vanishing at infinity and satisfying

$$\psi(u_1 g u_2) = \rho(u_1) \circ \psi(g) \circ \rho(u_2) \qquad \text{for any } u_1, u_2 \in U \text{ and } g \in G\ .$$

In the special case where $\rho = 1_U$ is the trivial representation $\mathcal{H}(G,1_U)$, resp. $\mathcal{B}(G,1_U)$, is the vector space of all K-valued finitely supported functions, resp. functions vanishing at infinity, on the double coset space $U\backslash G/U$.

A more intrinsic interpretation of these algebras can be given by introducing the compactly induced G-representation $\mathrm{ind}_U^G(\rho)$. This is the K-vector space of all compactly supported functions $f : G \longrightarrow E$ satisfying

$$f(gu) = \rho(u^{-1})(f(g)) \qquad \text{for any } u \in U \text{ and } g \in G$$

with G acting by left translations. Again we note that any function f in $\mathrm{ind}_U^G(\rho)$ is continuous. We equip $\mathrm{ind}_U^G(\rho)$ with the G-invariant norm

$$\|f\| := \sup_{g\in G} \|f(g)\|$$

and let $B_U^G(\rho)$ denote the corresponding completion. The G-action extends isometrically to the K-Banach space $B_U^G(\rho)$, which consists of all continuous functions $f : G \longrightarrow E$ vanishing at infinity and satisfying

$$f(gu) = \rho(u^{-1})(f(g)) \qquad \text{for any } u \in U \text{ and } g \in G\ .$$

Lemma 1.1: *The G-action on $B_U^G(\rho)$ is continuous.*

Proof: Since G acts isometrically it remains to show that the orbit maps

$$\begin{array}{rcl} c_f : G & \longrightarrow & B_U^G(\rho) \\ g & \longmapsto & gf\ , \end{array}$$

for any $f \in B_U^G(\rho)$, are continuous. In case $f \in \mathrm{ind}_U^G(\rho)$ the map c_f even is locally constant. In general we write $f = \lim\limits_{i\to\infty}$ as the limit of a sequence $(f_i)_{i\in\mathbb{N}}$ in $\mathrm{ind}_U^G(\rho)$. Because of

$$\|(c_f - c_{f_i})(g)\| = \|g(f - f_i)\| = \|f - f_i\|$$

the map c_f is the uniform limit of the locally constant maps c_{f_i} and hence is continuous.

One easily checks that the pairing

$$(1)\qquad \begin{array}{rcl} \mathcal{H}(G,\rho)\times \mathrm{ind}_U^G(\rho) & \longrightarrow & \mathrm{ind}_U^G(\rho) \\ (\psi, f) & \longmapsto & (\psi * f)(g) := \sum_{h\in G/U} \psi(g^{-1}h)(f(h)) \end{array}$$

makes $\mathrm{ind}_U^G(\rho)$ into a unital left $\mathcal{H}(G,\rho)$-module and that this module structure commutes with the G-action.

Lemma 1.2: *The map*

$$\begin{array}{rcl} \mathcal{H}(G,\rho) & \xrightarrow{\cong} & \mathrm{End}_G(\mathrm{ind}_U^G(\rho)) \\ \psi & \longmapsto & A_\psi(f) := \psi * f \end{array}$$

is an isomorphism of K-algebras.

Proof: For a smooth representation ρ this can be found in [Kut]. Our more general case follows by the same argument. But since we will need the notations anyway we recall the proof. The map in question certainly is a homomorphism of K-algebras. We now introduce, for any $w \in E$, the function

$$f_w(g) := \begin{cases} \rho(g^{-1})(w) & \text{if } g \in U, \\ 0 & \text{otherwise} \end{cases}$$

in $\mathrm{ind}_U^G(\rho)$. We have

$$(2)\qquad A_\psi(f_w)(g) = (\psi * f_w)(g) = \psi(g^{-1})(w) \quad \text{for any } \psi \in \mathcal{H}(G,\rho)\ .$$

This shows that the map in question is injective. To see its surjectivity we fix an operator $A_0 \in \mathrm{End}_G(\mathrm{ind}_U^G(\rho))$ and consider the function

$$\begin{array}{rcl} \psi_0 : G & \longrightarrow & \mathrm{End}_K(E) \\ g & \longmapsto & [w \mapsto A_0(f_w)(g^{-1})]\ . \end{array}$$

It clearly has compact support. Furthermore, for $u_1, u_2 \in U$, we compute

$$\begin{aligned} \psi_0(u_1 g u_2)(w) &= A_0(f_w)(u_2^{-1}g^{-1}u_1^{-1}) = \rho(u_1)[A_0(f_w)(u_2^{-1}g^{-1})] \\ &= \rho(u_1)[(u_2(A_0(f_w)))(g^{-1})] = \rho(u_1)[A_0(u_2(f_w))(g^{-1})] \\ &= \rho(u_1)[A_0(f_{\rho(u_2)(w)})(g^{-1})] = \rho(u_1)[\psi_0(\rho(u_2)(w))] \\ &= [\rho(u_1)\circ\psi_0\circ\rho(u_2)](w)\ . \end{aligned}$$

Hence $\psi_0 \in \mathcal{H}(G,\rho)$. Moreover, for any $f \in \mathrm{ind}_U^G(\rho)$ we have

$$f = \sum_{h\in G/U} h(f_{f(h)})$$

and therefore

$$\begin{aligned} A_{\psi_0}(f)(g) = (\psi_0 * f)(g) &= \sum_{h\in G/U} \psi_0(g^{-1}h)(f(h)) \\ &= \sum_{h\in G/U} A_0(f_{f(h)})(h^{-1}g) = A_0(\sum_{h\in G/U} h(f_{f(h)}))(g) \\ &= A_0(f)(g)\ . \end{aligned}$$

Hence $A_{\psi_0} = A_0$.

We evidently have $\|\psi * f\| \leq \|\psi\| \cdot \|f\|$. By continuity we therefore obtain a continuous left action of the Banach algebra $\mathcal{B}(G,\rho)$ on the Banach space $B_U^G(\rho)$ which is submultiplicative in the corresponding norms and which commutes with the G-action. This action is described by the same formula (1), and we therefore continue to denote it by $*$.

Lemma 1.3: *The map*

$$\begin{array}{ccc} \mathcal{B}(G,\rho) & \xrightarrow{\cong} & \mathrm{End}_G^{cont}(B_U^G(\rho)) \\ \psi & \longmapsto & A_\psi(f) := \psi * f \end{array}$$

is an isomorphism of K-algebras and is an isometry with respect to the operator norm on the right hand side.

Proof: (The superscript "cont" refers to the continuous endomorphisms.) By the previous discussion the map $\psi \longmapsto A_\psi$ is well defined, is a homomorphism of K-algebras, and is norm decreasing. Using the notations from the proof of Lemma 1.2 the formula (2), by continuity, holds for any $\psi \in \mathcal{B}(G,\rho)$. Using that $\|f_w\| = \|w\|$ we now compute

$$\begin{aligned} \|A_\psi\| &\geq \sup_{w\neq 0} \frac{\|\psi * f_w\|}{\|f_w\|} = \sup_{w\neq 0}\sup_{g} \frac{\|\psi(g^{-1})(w)\|}{\|w\|} = \sup_g \|\psi(g^{-1})\| \\ &= \|\psi\| \geq \|A_\psi\|\ . \end{aligned}$$

It follows that the map in the assertion is an isometry and in particular is injective. To see its surjectivity we fix an $A_0 \in \mathrm{End}_G^{cont}(B_U^G(\rho))$ and define

$$\begin{array}{rcl} \psi_0 : G & \longrightarrow & \mathrm{End}_K(E) \\ g & \longmapsto & [w \mapsto A_0(f_w)(g^{-1})]\ . \end{array}$$

Since each $A_0(f_w)$ is continuous and vanishing at infinity on G it follows that ψ_0 is continuous and vanishing at infinity. By exactly the same computations as in the proof of Lemma 1.2 one then shows that in fact $\psi_0 \in \mathcal{B}(G,\rho)$ and that $A_{\psi_0} = A_0$.

We end this section by considering the special case where (ρ, E) is the restriction to U of a continuous representation ρ of G on a finite dimensional K-vector space E. It is easy to check that then the map

$$\begin{array}{rcl} \iota_\rho : \mathcal{H}(G, 1_U) & \longrightarrow & \mathcal{H}(G,\rho) \\ \psi & \longmapsto & (\psi \cdot \rho)(g) := \psi(g)\rho(g) \end{array}$$

is an injective homomorphism of K-algebras. There are interesting situations where this map in fact is an isomorphism. We let L be a finite extension of $\mathbb{Q}_p$ contained in K, and we assume that G as well as (ρ, E) are locally L-analytic.

Lemma 1.4: *Suppose that, for the derived action of the Lie algebra $\mathfrak{g}$ of G, the $K \otimes_L \mathfrak{g}$-module E is absolutely irreducible; then the homomorphism ι_ρ is bijective.*

Proof: Using Lemma 1.2 and Frobenius reciprocity we have

$$\begin{aligned} \mathcal{H}(G,\rho) &= \mathrm{End}_G(\mathrm{ind}_U^G(\rho)) = \mathrm{Hom}_U(E, \mathrm{ind}_U^G(\rho)) \\ &= \mathrm{Hom}_U(E, \mathrm{ind}_U^G(1) \otimes_K E) \\ &= [\mathrm{ind}_U^G(1) \otimes_K E^* \otimes_K E]^U \end{aligned}$$

where the last term denotes the U-fixed vectors in the tensor product with respect to the diagonal action. This diagonal action makes the tensor product equipped with the finest locally convex topology into a locally analytic G-representation. Its U-fixed vectors certainly are contained in the vectors annihilated by the derived action of $\mathfrak{g}$. Since G acts smoothly on $\mathrm{ind}_U^G(1)$ we have

$$\begin{aligned} (\mathrm{ind}_U^G(1) \otimes_K E^* \otimes_K E)^{\mathfrak{g}=0} &= \mathrm{ind}_U^G(1) \otimes_K (E^* \otimes_K E)^{\mathfrak{g}=0} \\ &= \mathrm{ind}_U^G(1) \otimes_K \mathrm{End}_{K\otimes_L \mathfrak{g}}(E) \ . \end{aligned}$$

Our assumption on absolute irreducibility implies that $\mathrm{End}_{K\otimes_L\mathfrak{g}}(E) = K$. We therefore see that

$$\mathcal{H}(G,\rho) = [\mathrm{ind}_U^G(1) \otimes_K E^* \otimes_K E]^U = \mathrm{ind}_U^G(1)^U = \mathcal{H}(G, 1_U) \ .$$

2. Weights and affinoid algebras

For the rest of this paper $L/\mathbb{Q}_p$ is a finite extension contained in K, and G denotes the group of L-valued points of an L-split connected reductive group over L. Let $|\ |_L$ be the normalized absolute value of L, $\mathrm{val}_L : K^\times \longrightarrow \mathbb{R}$

the unique additive valuation such that $\mathrm{val}_L(L^\times) = \mathbb{Z}$, and q the number of elements in the residue class field of L. We fix a maximal L-split torus T in G and a Borel subgroup $P = TN$ of G with Levi component T and unipotent radical N. The Weyl group of G is the quotient $W = N(T)/T$ of the normalizer $N(T)$ of T in G by T. We also fix a maximal compact subgroup $U_0 \subseteq G$ which is special with respect to T (i.e., is the stabilizer of a special vertex x_0 in the apartment corresponding to T, cf. [Car]§3.5). We put $T_0 := U_0 \cap T$ and $N_0 := U_0 \cap N$. The quotient $\Lambda := T/T_0$ is a free abelian group of rank equal to the dimension of T and can naturally be identified with the cocharacter group of T. Let $\lambda : T \longrightarrow \Lambda$ denote the projection map. The conjugation action of $N(T)$ on T induces W-actions on T and Λ which we denote by $t \longmapsto {}^w t$ and $\lambda \longmapsto {}^w\lambda$, respectively. We also need the L-torus T' dual to T. Its K-valued points are given by

$$T'(K) := \mathrm{Hom}(\Lambda, K^\times) \ .$$

The group ring $K[\Lambda]$ of Λ over K naturally identifies with the ring of algebraic functions on the torus T'. We introduce the "valuation map"

$$val: \ T'(K) = \mathrm{Hom}(\Lambda, K^\times) \xrightarrow{\mathrm{val}_L \circ} \mathrm{Hom}(\Lambda, \mathbb{R}) =: V_{\mathbb{R}} \ .$$

If $X^*(T)$ denotes the algebraic character group of the torus T then, as $|\chi(T_0)| = \{1\}$, we have the embedding

$$\begin{array}{ccc} X^*(T) & \longrightarrow & \mathrm{Hom}(\Lambda, \mathbb{R}) \\ \chi & \longmapsto & \mathrm{val}_L \circ \chi \end{array}$$

which induces an isomorphism

$$X^*(T) \otimes \mathbb{R} \xrightarrow{\cong} V_{\mathbb{R}} \ .$$

We therefore may view $V_{\mathbb{R}}$ as the real vector space underlying the root datum of G with respect to T. Evidently any $\lambda \in \Lambda$ defines a linear form in the dual vector space $V_{\mathbb{R}}^*$ also denoted by λ. Let Φ denote the set of roots of T in G and let $\Phi^+ \subseteq \Phi$ be the subset of those roots which are positive with respect to P. As usual, $\check{\alpha} \in \Lambda$ denotes the coroot corresponding to the root $\alpha \in \Phi$. The subset $\Lambda^{--} \subseteq \Lambda$ of antidominant cocharacters is defined to be the image $\Lambda^{--} := \lambda(T^{--})$ of

$$T^{--} := \{t \in T : |\alpha(t)|_L \geq 1 \text{ for any } \alpha \in \Phi^+\} \ .$$

Hence

$$\Lambda^{--} = \{\lambda \in \Lambda : \mathrm{val}_L \circ \alpha(\lambda) \leq 0 \text{ for any } \alpha \in \Phi^+\} \ .$$

We finally recall that Λ^{--} carries the partial order $\leq$ given by

$$\mu \leq \lambda \quad \text{if} \quad \lambda - \mu \in \sum_{\alpha \in \Phi^+} \mathbb{R}_{\geq 0} \cdot (-\check{\alpha}) \subseteq \Lambda \otimes \mathbb{R} \ .$$

In this section we will investigate certain Banach algebra completions of the group ring $K[\Lambda]$ together with certain twisted W-actions on them. We will proceed in an axiomatic way and will give ourselves a cocycle on W with values in $T'(K)$, i.e., a map

$$\gamma : W \times \Lambda \longrightarrow K^\times$$

such that

$$(a) \qquad \gamma(w, \lambda\mu) = \gamma(w, \lambda)\gamma(w, \mu) \qquad \text{for any } w \in W \text{ and } \lambda, \mu \in \Lambda$$

and

$$(b) \qquad \gamma(vw, \lambda) = \gamma(v, {}^w\lambda)\gamma(w, \lambda) \qquad \text{for any } v, w \in W \text{ and } \lambda \in \Lambda\ .$$

Moreover we impose the positivity condition

$$(c) \qquad |\gamma(w, \lambda)| \leq 1 \qquad \text{for any } w \in W \text{ and } \lambda \in \Lambda^{--}$$

as well as the partial triviality condition

$$(d) \qquad \gamma(w, \lambda) = 1 \qquad \text{for any } w \in W \text{ and } \lambda \in \Lambda \text{ such that } {}^w\lambda = \lambda\ .$$

The twisted action of W on $K[\Lambda]$ then is defined by

$$\begin{array}{rcl} W \times K[\Lambda] & \longrightarrow & K[\Lambda] \\ (w, \sum_\lambda c_\lambda \lambda) & \longmapsto & w \cdot (\sum_\lambda c_\lambda \lambda) := \sum_\lambda \gamma(w, \lambda) c_\lambda {}^w\lambda\ . \end{array}$$

By (a), each $w \in W$ acts as an algebra automorphism, and the cocycle condition (b) guarantees the associativity of this W-action. The invariants with respect to this action will be denoted by $K[\Lambda]^{W,\gamma}$. Since Λ^{--} is a fundamental domain for the W-action on Λ it follows that $K[\Lambda]^{W,\gamma}$ has the K-basis $\{\sigma_\lambda\}_{\lambda \in \Lambda^{--}}$ defined by

$$\sigma_\lambda := \sum_{w \in W/W(\lambda)} w \cdot \lambda = \sum_{w \in W/W(\lambda)} \gamma(w, \lambda) {}^w\lambda$$

where $W(\lambda) \subseteq W$ denotes the stabilizer of λ and where the sums are well defined because of (d). Next, again using (d), we define the map

$$\begin{array}{rcl} \gamma^{dom} : \Lambda & \longrightarrow & K^\times \\ \lambda & \longmapsto & \gamma(w, \lambda) \quad \text{if } {}^w\lambda \in \Lambda^{--}\ , \end{array}$$

and we equip $K[\Lambda]$ with the norm

$$\| \sum_\lambda c_\lambda \lambda \|_\gamma := \sup_{\lambda \in \Lambda} |\gamma^{dom}(\lambda) c_\lambda|\ .$$

The cocycle condition (b) implies the identity

$$(1) \qquad \gamma^{dom}({}^w\lambda)\gamma(w, \lambda) = \gamma^{dom}(\lambda)$$

from which one deduces that the twisted W-action on $K[\Lambda]$ is isometric in the norm $\| \ \|_\gamma$ and hence extends by continuity to a W-action on the completion $K\langle\Lambda;\gamma\rangle$ of $K[\Lambda]$ with respect to $\| \ \|_\gamma$. Again we denote the corresponding W-invariants by $K\langle\Lambda;\gamma\rangle^{W,\gamma}$. One easily checks that $\{\sigma_\lambda\}_{\lambda\in\Lambda^{--}}$ is an orthonormal basis of the Banach space $(K\langle\Lambda;\gamma\rangle^{W,\gamma}, \| \ \|_\gamma)$.

Lemma 2.1: *i.* $|\gamma^{dom}(\lambda)| \geq 1$ *for any* $\lambda \in \Lambda$*;*

ii. $|\gamma^{dom}(\lambda\mu)| \leq |\gamma^{dom}(\lambda)||\gamma^{dom}(\mu)|$ *for any* $\lambda, \mu \in \Lambda$.

Proof: i. If ${}^w\lambda \in \Lambda^{--}$ then $\gamma^{dom}(\lambda) = \gamma(w,\lambda) = \gamma(w^{-1}, {}^w\lambda)^{-1}$. The claim therefore is a consequence of the positivity condition (c). ii. If ${}^w(\lambda\mu) \in \Lambda^{--}$ then, using (1), we have

$$\gamma^{dom}(\lambda\mu) = \gamma^{dom}({}^w\lambda)^{-1}\gamma^{dom}({}^w\mu)^{-1}\gamma^{dom}(\lambda)\gamma^{dom}(\mu) \ .$$

Hence the claim follows from the first assertion.

It is immediate from Lemma 2.1.i that the norm $\| \ \|_\gamma$ is submultiplicative. Hence $K\langle\Lambda;\gamma\rangle$ is a K-Banach algebra containing $K[\Lambda]$ as a dense subalgebra. Moreover, since the twisted W-action on $K\langle\Lambda;\gamma\rangle$ is by algebra automorphisms, $K\langle\Lambda;\gamma\rangle^{W,\gamma}$ is a Banach subalgebra of $K\langle\Lambda;\gamma\rangle$.

In order to compute the Banach algebra $K\langle\Lambda;\gamma\rangle$ we introduce the subset

$$T'_\gamma(K) := \{\zeta \in T'(K) : |\zeta(\lambda)| \leq |\gamma^{dom}(\lambda)| \text{ for any } \lambda \in \Lambda\}$$

of $T'(K)$. We obviously have

$$T'_\gamma(K) = val^{-1}(V^\gamma_{\mathbb{R}})$$

with

$$V^\gamma_{\mathbb{R}} := \{z \in V_{\mathbb{R}} : \lambda(z) \geq \mathrm{val}_L(\gamma^{dom}(\lambda)) \text{ for any } \lambda \in \Lambda\} \ .$$

By (a), our cocycle γ defines the finitely many points

$$z_w := -val(\gamma(w^{-1}, .)) \qquad \text{for } w \in W$$

in $V_{\mathbb{R}}$. The cocycle condition (b) implies that

$$z_{vw} = {}^v z_w + z_v \qquad \text{for any } v, w \in W \tag{2}$$

and the positivity condition (c) that

$$\lambda(z_w) \leq 0 \qquad \text{for any } w \in W \text{ and } \lambda \in \Lambda^{--} \ . \tag{3}$$

Remark 2.2: $\{z \in V_{\mathbb{R}} : \lambda(z) \leq 0 \text{ for any } \lambda \in \Lambda^{--}\} = \sum_{\alpha\in\Phi^+} \mathbb{R}_{\geq 0} \cdot \mathrm{val}_L \circ\alpha$.

Proof: This reduces to the claim that the (closed) convex hull of Λ^{--} in $V_{\mathbb{R}}^*$ is equal to the antidominant cone

$$(V_{\mathbb{R}}^*)^{--} = \{z^* \in V_{\mathbb{R}}^* : z^*(z) \leq 0 \text{ for any } z \in \sum_{\alpha\in\Phi^+} \mathbb{R}_{\geq 0}\cdot \mathrm{val}_L \circ\alpha\} \ .$$

Let $Z \subseteq G$ denote the connected component of the center of G. Then G/Z is semisimple and the sequence

$$0 \longrightarrow Z/Z_0 \longrightarrow T/T_0 \longrightarrow (T/Z)/(T/Z)_0 \longrightarrow 0$$

is exact. Hence the fundamental antidominant coweights for the semisimple group G/Z can be lifted to elements $\omega_1,\ldots,\omega_d \in V_{\mathbb{R}}^*$ in such a way that, for some $m \in \mathbb{N}$, we have $m\omega_1,\ldots,m\omega_d \in \Lambda^{--}$. It follows that

$$(V_{\mathbb{R}}^*)^{--} = (Z/Z_0)\otimes\mathbb{R} + \sum_{i=1}^{d} \mathbb{R}_{\geq 0}\cdot\omega_i$$

and

$$\Lambda^{--} \supseteq Z/Z_0 + m\cdot\sum_{i=1}^{d}\mathbb{Z}_{\geq 0}\cdot\omega_i \ .$$

We therefore obtain from (3) that

$$z_w \in \sum_{\alpha\in\Phi^+}\mathbb{R}_{\geq 0}\cdot\mathrm{val}_L\circ\alpha \qquad \text{for any } w\in W \ . \tag{4}$$

In terms of these points z_w the set $V_{\mathbb{R}}^{\gamma}$ is given as

$$\begin{aligned}&\{z\in V_{\mathbb{R}} : \lambda(z)\geq\lambda(-z_{w^{-1}}) \text{ for any } \lambda\in\Lambda,\ w\in W \text{ such that } {}^{w}\lambda\in\Lambda^{--}\}\\ &=\{z\in V_{\mathbb{R}} : {}^{w^{-1}}\lambda(z)\geq {}^{w^{-1}}\lambda(-z_{w^{-1}}) \text{ for any } w\in W \text{ and } \lambda\in\Lambda^{--}\}\\ &=\{z\in V_{\mathbb{R}} : \lambda({}^{w}z)\geq\lambda(z_w) \text{ for any } w\in W \text{ and } \lambda\in\Lambda^{--}\}\end{aligned}$$

where the last identity uses (2). Obviously $V_{\mathbb{R}}^{\gamma}$ is a convex subset of $V_{\mathbb{R}}$. Using the partial order $\leq$ on $V_{\mathbb{R}}$ defined by Φ^+ (cf. [B-GAL] Chap. VI §1.6) we obtain from Remark 2.2 that

$$V_{\mathbb{R}}^{\gamma} = \{z\in V_{\mathbb{R}} : {}^{w}z \leq z_w \text{ for any } w\in W\} \ .$$

Lemma 2.3: *$V_{\mathbb{R}}^{\gamma}$ is the convex hull in $V_{\mathbb{R}}$ of the finitely many points $-z_w$ for $w\in W$.*

Proof: From (2) and (4) we deduce that

$${}^{w}z_v + z_w = z_{wv} \geq 0 \quad \text{and hence} \quad {}^{w}(-z_v)\leq z_w$$

for any $v, w \in W$. It follows that all $-z_v$ and therefore their convex hull is contained in $V_{\mathbb{R}}^{\gamma}$. For the reverse inclusion suppose that there is a point $z \in V_{\mathbb{R}}^{\gamma}$ which does not lie in the convex hull of the $-z_w$. We then find a linear form $\ell \in V_{\mathbb{R}}^{*}$ such that $\ell(z) < \ell(-z_w)$ for any $w \in W$. Choose $v \in W$ such that $\ell_0 := {}^{v}\ell$ is antidominant. It follows that ${}^{v^{-1}}\ell_0(z) < {}^{v^{-1}}\ell_0(-z_w)$ and hence, using (2), that

$$\ell_0({}^{v}z) < \ell_0(-{}^{v}z_w) = \ell_0(z_v) - \ell_0(z_{vw})$$

for any $w \in W$. For $w := v^{-1}$ we in particular obtain

$$\ell_0({}^{v}z) < \ell_0(z_v) \ .$$

On the other hand, since $z \in V_{\mathbb{R}}^{\gamma}$, we have

$$\lambda({}^{v}z) \geq \lambda(z_v)$$

for any $\lambda \in \Lambda^{--}$ and hence for any λ in the convex hull of Λ^{--}. But as we have seen in the proof of Remark 2.2 the antidominant ℓ_0 belongs to this convex hull which leads to a contradiction.

Proposition 2.4: *i. $T'_{\gamma}(K)$ is the set of K-valued points of an open K-affinoid subdomain T'_{γ} in the torus T';*

ii. the Banach algebra $K\langle\Lambda;\gamma\rangle$ is naturally isomorphic to the ring of analytic functions on the affinoid domain T'_{γ};

iii. the affinoid domain T'_{γ} is the preimage, under the map "val", of the convex hull of the finitely many points $-z_w \in V_{\mathbb{R}}$ for $w \in W$;

iv. $K\langle\Lambda;\gamma\rangle^{W,\gamma}$ is an affinoid K-algebra.

Proof: It follows from Gordan's lemma ([KKMS] p. 7) that the monoid Λ^{--} is finitely generated. Choose a finite set of generators F^{--}, and let

$$F := \{{}^{w}\lambda : \lambda \in F^{--}\} \ .$$

Using the fact that, by construction, the function γ^{dom} is multiplicative within Weyl chambers we see that the infinitely many inequalities implicit in the definition of $T'_{\gamma}(K)$ can in fact be replaced by finitely many:

$$T'_{\gamma}(K) = \{\zeta \in T'(K) : |\zeta(\lambda)| \leq |\gamma^{dom}(\lambda)| \text{ for any } \lambda \in F\}.$$

We therefore define T'_{γ} to be the rational subset in T' given by the finitely many inequalities $|\gamma^{dom}(\lambda)^{-1}\lambda(\zeta)| \leq 1$ for $\lambda \in F$ and obtain point i. of our assertion.

Now choose indeterminates T_λ for $\lambda \in F$ and consider the commutative diagram of algebra homomorphisms

$$\begin{tikzcd}
o_K[T_\lambda : \lambda \in F] \arrow[r] \arrow[d, "\subseteq"'] & K[\Lambda]^0 \arrow[d, "\subseteq"] \\
K[T_\lambda : \lambda \in F] \arrow[r] \arrow[d, "\subseteq"'] & K[\Lambda] \arrow[d, "\subseteq"] \\
K\langle T_\lambda : \lambda \in F\rangle \arrow[r] & K\langle \Lambda; \gamma\rangle
\end{tikzcd}$$

where the horizontal arrows send T_λ to $\gamma^{dom}(\lambda)^{-1}\lambda$, where o_K is the ring of integers of K, and where $K[\Lambda]^0$ denotes the unit ball with respect to $\| \ \|_\gamma$ in $K[\Lambda]$. Again, the multiplicativity of γ^{dom} within Weyl chambers shows that all three horizontal maps are surjective. The lower arrow gives a presentation of $K\langle\Lambda;\gamma\rangle$ as an affinoid algebra. The middle arrow realizes the dual torus T' as a closed algebraic subvariety

$$\begin{aligned} \iota : T' &\longrightarrow \mathbb{A}^f \\ \zeta &\longmapsto (\zeta(\lambda))_{\lambda\in F} \end{aligned}$$

in the affine space $\mathbb{A}^f$ where f denotes the cardinality of the set F. The surjectivity of the upper arrow shows that the norm $\| \ \|_\gamma$ on $K[\Lambda]$ is the quotient norm of the usual Gauss norm on the polynomial ring $K[T_\lambda : \lambda \in F]$. Hence the kernel of the lower arrow is the norm completion of the kernel I of the middle arrow. Since any ideal in the Tate algebra $K\langle T_\lambda : \lambda \in F\rangle$ is closed we obtain

$$K\langle\Lambda;\gamma\rangle = K\langle T_\lambda : \lambda \in F\rangle / IK\langle T_\lambda : \lambda \in F\rangle \ .$$

This means that the affinoid variety $Sp(K\langle\Lambda;\gamma\rangle)$ is the preimage under ι of the affinoid unit polydisk in $\mathbb{A}^f$. In particular, $Sp(K\langle\Lambda;\gamma\rangle)$ is an open subdomain in T' which is reduced and coincides with the rational subset T'_γ (cf. [FvP] Prop. 4.6.1(4)). This establishes point ii. of the assertion. The point iii. is Lemma 2.3. For point iv., as the invariants in an affinoid algebra with respect to a finite group action, $K\langle\Lambda;\gamma\rangle^{W,\gamma}$ is again affinoid (cf. [BGR] 6.3.3 Prop. 3).

Suppose that the group G is semisimple and adjoint. Then the structure of the affinoid algebra $K\langle\Lambda;\gamma\rangle^{W,\gamma}$ is rather simple. The reason is that for such a group the set Λ^{--} is the free commutative monoid over the fundamental antidominant cocharacters $\lambda_1,\ldots,\lambda_d$. As usual we let $K\langle X_1,\ldots,X_d\rangle$ denote the Tate algebra in d variables over K. Obviously we have a unique continuous algebra homomorphism

$$K\langle X_1,\ldots,X_d\rangle \longrightarrow K\langle\Lambda;\gamma\rangle^{W,\gamma}$$

sending the variable X_i to σ_{λ_i}.

We also need a general lemma about orthogonal bases in normed vector spaces. Let $(Y, \| \, \|)$ be a normed K-vector space and suppose that Y has an orthogonal basis of the form $\{x_\ell\}_{\ell \in I}$. Recall that the latter means that

$$\| \sum_\ell c_\ell x_\ell \| = \sup_\ell |c_\ell| \cdot \|x_\ell\|$$

for any vector $\sum_\ell c_\ell x_\ell \in Y$. We suppose moreover that there is given a partial order $\leq$ on the index set I such that:

– Any nonempty subset of I has a minimal element;

– for any $k \in I$ the set $\{\ell \in I : \ell \leq k\}$ is finite.

(Note that the partial order $\leq$ on Λ^{--} has these properties.)

Lemma 2.5: *Suppose that $\|x_\ell\| \leq \|x_k\|$ whenever $\ell \leq k$; furthermore, let elements $c_{\ell k} \in K$ be given, for any $\ell \leq k$ in I, such that $|c_{\ell k}| \leq 1$; then the vectors*

$$y_k := x_k + \sum_{\ell < k} c_{\ell k} x_\ell$$

form another orthogonal basis of Y, and $\|y_k\| = \|x_k\|$.

Proof: We have

$$\|y_k\| = \max(\|x_k\|, \max_{\ell < k} |c_{\ell k}| \cdot \|x_\ell\|) = \|x_k\|$$

as an immediate consequence of our assumptions. We also have

$$x_k = y_k + \sum_{\ell < k} b_{\ell k} y_\ell$$

where $(b_{\ell k})$ is the matrix inverse to $(c_{\ell k})$ (over the ring of integers in K; cf. [B-GAL] Chap. VI §3.4 Lemma 4). Let now $x = \sum_k c_k x_k$ be an arbitrary vector in Y. We obtain

$$x = \sum_k c_k x_k = \sum_k c_k (\sum_{\ell \leq k} b_{\ell k} y_\ell) = \sum_\ell (\sum_{\ell \leq k} c_k b_{\ell k}) y_\ell \ .$$

Clearly $\|x\| \leq \sup_\ell |\sum_{\ell \leq k} c_k b_{\ell k}| \cdot \|y_\ell\|$. On the other hand we compute

$$\begin{aligned} \sup_\ell |\textstyle\sum_{\ell \leq k} c_k b_{\ell k}| \cdot \|y_\ell\| &\leq \sup_\ell \sup_{\ell \leq k} |c_k| \cdot \|y_\ell\| = \sup_\ell \sup_{\ell \leq k} |c_k| \cdot \|x_\ell\| \\ &\leq \sup_k |c_k| \cdot \|x_k\| = \|x\| \ . \end{aligned}$$

Proposition 2.6: *If the group G is semisimple and adjoint then the above map is an isometric isomorphism $K\langle X_1, \ldots, X_d\rangle \xrightarrow{\cong} K\langle \Lambda; \gamma\rangle^{W,\gamma}$.*

Proof: We write a given $\lambda \in \Lambda^{--}$ as $\lambda = \lambda_1^{m_1} \ldots \lambda_d^{m_d}$ and put

$$\widetilde{\sigma}_\lambda := \sigma_{\lambda_1}^{m_1} \cdot \ldots \cdot \sigma_{\lambda_d}^{m_d} \ .$$

It suffices to show that these $\{\widetilde{\sigma}_\lambda\}_{\lambda \in \Lambda^{--}}$ form another orthonormal basis of $K\langle \Lambda; \gamma \rangle^{W,\gamma}$. One checks that the arguments in [B-GAL] Chap. VI §§3.2 and 3.4 work, over the ring of integers in K, equally well for our twisted W-action and show that we have

$$\widetilde{\sigma}_\lambda = \sigma_\lambda + \sum_{\mu < \lambda} c_{\mu\lambda} \sigma_\mu$$

with $|c_{\mu\lambda}| \leq 1$. So we may apply Lemma 2.5.

We finish this section with a discussion of those examples of a cocycle γ which will be relevant later on.

Example 1: We fix a prime element π_L of L. Let $\xi \in X^*(T)$ be a dominant integral weight and put

$$\gamma(w, \lambda(t)) := \pi_L^{\mathrm{val}_L(\xi({}^w t)) - \mathrm{val}_L(\xi(t))} \qquad \text{for } t \in T \ .$$

This map γ obviously has the properties (a),(b), and (d). For $t \in T^{--}$ we have $\lambda({}^w t) \leq \lambda(t)$ by [B-GAL] Chap. VI §1.6 Prop. 18; since ξ is dominant we obtain

$$\mathrm{val}_L \circ \xi\left(\frac{t}{{}^w t}\right) \leq 0 \ .$$

This means that $|\gamma(w, \lambda)| \leq 1$ for $\lambda \in \Lambda^{--}$ which is condition (c). We leave it as an exercise to the reader to check that the resulting Banach algebra $K\langle \Lambda; \gamma \rangle$ together with the twisted W-action, up to isomorphism, is independent of the choice of the prime element π_L.

Example 2: A particular case of a dominant integral weight is the determinant of the adjoint action of T on the Lie algebra $\mathrm{Lie}(N)$ of the unipotent radical N

$$\Delta(t) := \det(\mathrm{ad}(t); \mathrm{Lie}(N)) \ .$$

Its absolute value satisfies

$$\delta(t) = |\Delta(t)|_L^{-1}$$

where $\delta : P \longrightarrow \mathbb{Q}^\times \subseteq K^\times$ is the modulus character of the Borel subgroup P. We let K_q/K denote the splitting field of the polynomial $X^2 - q$ and we fix a root $q^{1/2} \in K_q^\times$. Then the square root $\delta^{1/2} : \Lambda \longrightarrow K_q^\times$ of the character δ is well defined. For a completely analogous reason as in the first example the cocycle

$$\gamma(w, \lambda) := \frac{\delta^{1/2}({}^w \lambda)}{\delta^{1/2}(\lambda)}$$

has the properties $(a)-(d)$. Moreover, using the root space decomposition of $\mathrm{Lie}(N)$ one easily shows that

$$\gamma(w,\lambda(t)) = \prod_{\alpha\in\Phi^+\setminus {}^{w^{-1}}\Phi^+} |\alpha(t)|_L \ .$$

Hence the values of this cocycle γ are integral powers of q and therefore lie in K.

Example 3: Obviously the properties $(a)-(d)$ are preserved by the product of two cocycles. For any dominant integral weight $\xi \in X^*(T)$ therefore the cocycle

$$\gamma_\xi(w,\lambda(t)) := \frac{\delta^{1/2}({}^w\lambda)}{\delta^{1/2}(\lambda)} \cdot \pi_L^{\mathrm{val}_L(\xi({}^w t))-\mathrm{val}_L(\xi(t))}$$

is K-valued and satisfies $(a)-(d)$. We write

$$V_{\mathbb{R}}^\xi := V_{\mathbb{R}}^{\gamma_\xi} \qquad \text{and} \qquad T'_\xi := T'_{\gamma_\xi} \ .$$

Let $\eta \in V_{\mathbb{R}}$ denote half the sum of the positive roots in Φ^+ and put

$$\eta_L := [L:\mathbb{Q}_p]\cdot\eta \ .$$

Let

$$\xi_L := \mathrm{val}_L \circ \xi \in V_{\mathbb{R}} \ .$$

For the points $z_w \in V_{\mathbb{R}}$ corresponding to the cocycle γ_ξ we then have

$$z_w = (\eta_L+\xi_L) - {}^w(\eta_L+\xi_L) \ .$$

In particular, $V_{\mathbb{R}}^\xi$ is the convex hull of the points ${}^w(\eta_L+\xi_L)-(\eta_L+\xi_L)$ for $w\in W$. Note that, since γ_ξ has values in $L^\times$, the affinoid variety T'_ξ is naturally defined over L. Given any point $z\in V_{\mathbb{R}}$, we will write z^{dom} for the unique dominant point in the W-orbit of z.

Lemma 2.7: $V_{\mathbb{R}}^\xi = \{z\in V_{\mathbb{R}} : (z+\eta_L+\xi_L)^{dom} \le \eta_L+\xi_L\}$.

Proof: Using the formula before Lemma 2.3 we have

$$\begin{aligned} V_{\mathbb{R}}^\xi &= \{z\in V_{\mathbb{R}} : {}^w z \le (\eta_L+\xi_L) - {}^w(\eta_L+\xi_L) \text{ for any } w\in W\} \\ &= \{z\in V_{\mathbb{R}} : {}^w(z+\eta_L+\xi_L) \le \eta_L+\xi_L \text{ for any } w\in W\} \ . \end{aligned}$$

It remains to recall ([B-GAL] Chap. VI §1.6 Prop. 18) that for any $z\in V_{\mathbb{R}}$ and any $w\in W$ one has ${}^w z \le z^{dom}$.

The γ_ξ in Example 3 are the cocycles which will appear in our further investigation of specific Banach-Hecke algebras. In the following we explicitly compute the affinoid domain T'_ξ in case of the group $G := GL_{d+1}(L)$. (In case $\xi = 1$ compare also [Vig] Chap. 3.) We let $P \subseteq G$ be the lower triangular Borel subgroup and $T \subseteq P$ be the torus of diagonal matrices. We take $U_0 := GL_{d+1}(o_L)$ where o_L is the ring of integers of L. If $\pi_L \in o_L$ denotes a prime element then

$$\Lambda^{--} = \{ \begin{pmatrix} \pi_L^{m_1} & & 0 \\ & \ddots & \\ 0 & & \pi_L^{m_{d+1}} \end{pmatrix} T_0 : m_1 \geq \ldots \geq m_{d+1} \} .$$

For $1 \leq i \leq d+1$ define the diagonal matrix

$$t_i := \begin{pmatrix} \pi_L & & & & & 0 \\ & \ddots & & & & \\ & & \pi_L & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix} \quad \text{with } i \text{ diagonal entries equal to } \pi_L .$$

As a monoid Λ^{--} is generated by the elements $\lambda_1, \ldots, \lambda_{d+1}, \lambda_{d+1}^{-1}$ where $\lambda_i := \lambda(t_i)$. For any nonempty subset $I = \{i_1, \ldots, i_s\} \subseteq \{1, \ldots, d+1\}$ let $\lambda_I \in \Lambda$ be the cocharacter corresponding to the diagonal matrix having π_L at the places $i_1, \ldots, i_s$ and 1 elsewhere. Moreover let, as usual, $|I| := s$ be the cardinality of I and put $ht(I) := i_1 + \ldots + i_s$. These λ_I together with $\lambda_{\{1,\ldots,d+1\}}^{-1}$ form the W-orbit of the above monoid generators. From the proof of Prop. 2.4 we therefore know that T'_ξ as a rational subdomain of T' is described by the conditions

$$|\zeta(\lambda_I)| \leq |\gamma_\xi^{dom}(\lambda_I)|$$

for any I and

$$|\zeta(\lambda_{\{1,\ldots,d+1\}})| = |\gamma_\xi^{dom}(\lambda_{\{1,\ldots,d+1\}})| .$$

One checks that

$$|\gamma_1^{dom}(\lambda_I)| = |q|^{|I|(|I|+1)/2 - ht(I)} .$$

If the dominant integral weight $\xi \in X^*(T)$ is given by

$$\begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_{d+1} \end{pmatrix} \longmapsto \prod_{i=1}^{d+1} g_i^{a_i}$$

with $(a_1, \ldots, a_{d+1}) \in \mathbb{Z}^{d+1}$ then

$$|\gamma_\xi^{dom}(\lambda_I)| = |q|^{|I|(|I|+1)/2 - ht(I)} |\pi_L|^{\sum_{j=1}^{|I|} a_j - \sum_{i \in I} a_i} .$$

We now use the coordinates

$$\begin{array}{rcl} T'(K) & \longrightarrow & (K^\times)^{d+1} \\ \zeta & \longmapsto & (\zeta_1, \ldots, \zeta_{d+1}) \text{ with } \zeta_i := q^{i-1}\pi_L^{a_i}\zeta(\lambda_{\{i\}}) \end{array}$$

on the dual torus. In these coordinates T'_ξ is the rational subdomain of all $(\zeta_1, \ldots, \zeta_{d+1}) \in (K^\times)^{d+1}$ such that

$$\prod_{i\in I} |\zeta_i| \leq |q|^{|I|(|I|-1)/2}|\pi_L|^{\sum_{i=1}^{|I|} a_i}$$

for any proper nonempty subset $I \subseteq \{1, \ldots, d+1\}$ and

$$\prod_{i=1}^{d+1} |\zeta_i| = |q|^{d(d+1)/2}|\pi_L|^{\sum_{i=1}^{d+1} a_i} \ .$$

The advantage of these variables is the following. As usual we identify the Weyl group W with the symmetric group on the set $\{1, \ldots, d+1\}$. One checks that

$$\gamma_\xi(w, \lambda_{\{i\}}) = q^{w(i)-i}\pi_L^{a_{w(i)}-a_i}$$

for any $w \in W$ and $1 \leq i \leq d+1$. This implies that the twisted W-action on the affinoid algebra $K\langle\Lambda; \gamma_\xi\rangle$ is induced by the permutation action on the coordinates $\zeta_1, \ldots, \zeta_{d+1}$ of the affinoid domain T'_ξ. In fact, the above identity means that the cocycle γ_ξ can be written as the coboundary of an element in $T'(K)$. This is more generally possible for any group G whose derived group is simply connected (cf. [Gro] §8). We do not pursue this point of view systematically, though, since it is not compatible with general Langlands functoriality. But the problem of "splitting" the cocycle and the difficulty of reconciling the normalization of the Satake isomorphism will reappear as a technical complication in our attempt, in section 6, to treat Langlands functoriality.

3. The p-adic Satake isomorphism

Keeping the notations and assumptions introduced in the previous section we now consider a locally L-analytic representation (ρ, E) of G of the form

$$E = K_\chi \otimes_L E_L$$

where

– K_χ is a one dimensional representation of G given by a locally L-analytic character $\chi : G \longrightarrow K^\times$, and

– E_L is an L-rational irreducible representation ρ_L of G of highest weight ξ.

Let

$$E_L = \oplus_{\beta \in X^*(T)} E_{L,\beta}$$

be the decomposition into weight spaces for T. According to [BT] II.4.6.22 and Prop. II.4.6.28(ii) the reductive group G has a smooth connected affine model $\mathcal{G}$ over the ring of integers o_L in L such that $\mathcal{G}(o_L) = U_0$. We fix once and for all a U_0-invariant o_L-lattice M in E_L ([Jan] I.10.4) and let $\| \ \|$ be the corresponding U_0-invariant norm on E. The following fact is well-known.

Lemma 3.1: *We have* $M = \bigoplus_{\beta \in X^*(T)} M_\beta$ *with* $M_\beta := M \cap E_{L,\beta}$.

Proof: For the convenience of the reader we sketch the argument. Fix a weight $\beta \in X^*(T)$. It suffices to construct an element Π_β in the algebra of distributions $\mathrm{Dist}(\mathcal{G})$ which acts as a projector

$$\Pi_\beta : E_L \longrightarrow E_{L,\beta} \ .$$

Let B be the finite set of weights $\neq \beta$ which occur in E_L. Also we need the Lie algebra elements

$$H_i := (d\mu_i)(1) \in \mathrm{Lie}(\mathcal{G})$$

where $\mu_1, \ldots, \mu_r$ is a basis of the cocharacter group of T. We have

$$\underline{\gamma} := (d\gamma(H_1), \ldots, d\gamma(H_r)) \in \mathbb{Z}^r \quad \text{for any } \gamma \in X^*(T) \ .$$

According to [Hum] Lemma 27.1 we therefore find a polynomial $\Pi \in \mathbb{Q}[y_1,\ldots,y_r]$ such that $\Pi(\mathbb{Z}^r) \subseteq \mathbb{Z}$, $\Pi(\underline{\beta}) = 1$, and $\Pi(\underline{\gamma}) = 0$ for any $\gamma \in B$. Moreover [Hum] Lemma 26.1 says that the polynomial Π is a $\mathbb{Z}$-linear combination of polynomials of the form

$$\binom{y_1}{b_1} \cdot \ldots \cdot \binom{y_r}{b_r} \quad \text{with integers } b_1, \ldots, b_r \geq 0 \ .$$

Then [Jan] II.1.12 implies that

$$\Pi_\beta := \Pi(H_1, \ldots, H_r)$$

lies in $\mathrm{Dist}(\mathcal{G})$. By construction Π_β induces a projector from E_L onto $E_{L,\beta}$.

It follows that, for any $t \in T$, the operator norm of $\rho_L(t)$ on E_L is equal to

$$\|\rho_L(t)\| = \max\{|\beta(t)| : \beta \in X^*(T) \text{ such that } E_{L,\beta} \neq 0\}.$$

Lemma 3.2: *For any* $t \in T$ *we have* $\|\rho(t)\| = |\chi(t)| \cdot |\xi({}^w t)|$ *with* $w \in W$ *such that* ${}^w t \in T^{--}$.

Proof: Consider first the case $t \in T^{--}$ with $w = 1$. For any weight β occurring in E_L one has $\xi = \alpha\beta$ where α is an appropriate product of simple roots. But by definition of T^{--} we have $|\alpha(t)|_L \geq 1$ for any simple root α. For general $t \in T$ and $w \in W$ as in the assertion we then obtain

$$\begin{aligned} |\xi({}^w t)| &= \max\{|\beta({}^w t)| : E_{L,\beta} \neq 0\} \\ &= \max\{|\beta(t)| : E_{L,\beta} \neq 0\} \\ &= \|\rho_L(t)\| \ . \end{aligned}$$

Here the second identity is a consequence of the fact that the set of weights of E_L is W-invariant.

Collecting this information we first of all see that Lemma 1.4 applies and gives, for any open subgroup $U \subseteq U_0$, the isomorphism

$$\mathcal{H}(G, 1_U) \cong \mathcal{H}(G, \rho|U) \ .$$

But the norm $\| \ \|$ on $\mathcal{H}(G, \rho|U)$ corresponds under this isomorphism to the norm $\| \ \|_{\chi,\xi}$ on $\mathcal{H}(G, 1_U)$ defined by

$$\|\psi\|_{\chi,\xi} := \sup_{g\in G} |\psi(g)\chi(g)| \cdot \|\rho_L(g)\| \ .$$

If $|\chi|= 1$ (e.g., if the group G is semisimple) then the character χ does not affect the norm $\| \ \|_\xi := \| \ \|_{\chi,\xi}$. In general χ can be written as a product $\chi = \chi_1\chi_{un}$ of two characters where $|\chi_1| = 1$ and $\chi_{un}|U_0 = 1$. Then

$$\begin{array}{ccc} (\mathcal{H}(G, 1_U), \| \ \|_\xi) & \xrightarrow{\cong} & (\mathcal{H}(G, 1_U), \| \ \|_{\chi,\xi}) \\ \psi & \longmapsto & \psi \cdot \chi_{un}^{-1} \end{array}$$

is an isometric isomorphism. We therefore have the following fact.

Lemma 3.3: *The map*

$$\begin{array}{ccc} \| \ \|_\xi\text{-}completion\ of\ \mathcal{H}(G, 1_U) & \xrightarrow{\cong} & \mathcal{B}(G, \rho|U) \\ \psi & \longmapsto & \psi \cdot \chi_{un}^{-1}\rho \end{array}$$

is an isometric isomorphism of Banach algebras.

In this section we want to compute these Banach-Hecke algebras in the case $U = U_0$. By the Cartan decomposition G is the disjoint union of the double cosets U_0tU_0 with t running over T^{--}/T_0. Let therefore $\psi_{\lambda(t)} \in \mathcal{H}(G, 1_{U_0})$ denote the characteristic function of the double coset U_0tU_0. Then $\{\psi_\lambda\}_{\lambda\in\Lambda^{--}}$ is a K-basis of $\mathcal{H}(G, 1_{U_0})$. According to Lemma 3.2 the norm $\| \ \|_\xi$ on $\mathcal{H}(G, 1_{U_0})$ is given by

$$\|\psi\|_\xi := \sup_{t\in T^{--}} |\psi(t)\xi(t)| \ .$$

The $\{\psi_\lambda\}_{\lambda\in\Lambda^{--}}$ form a $\| \, \|_\xi$-orthogonal basis of $\mathcal{H}(G, 1_{U_0})$ and hence of its $\| \, \|_\xi$-completion.

The Satake isomorphism computes the Hecke algebra $\mathcal{H}(G, 1_{U_0})$. For our purposes it is important to consider the renormalized version of the Satake map given by

$$\begin{array}{rcl} S_\xi : \mathcal{H}(G, 1_{U_0}) & \longrightarrow & K[\Lambda] \\ \psi & \longmapsto & \sum_{t\in T/T_0} \pi_L^{\mathrm{val}_L(\xi(t))} (\sum_{n\in N/N_0} \psi(tn))\lambda(t) \ . \end{array}$$

On the other hand we again let K_q/K be the splitting field of the polynomial $X^2 - q$ and we temporarily fix a root $q^{1/2} \in K_q$. Satake's theorem says (cf. [Car]§4.2) that the map

$$\begin{array}{rcl} S^{norm} : \mathcal{H}(G, 1_{U_0}) \otimes_K K_q & \longrightarrow & K_q[\Lambda] \\ \psi & \longmapsto & \sum_{t\in T/T_0} \delta^{-1/2}(t) (\sum_{n\in N/N_0} \psi(tn))\lambda(t) \end{array}$$

induces an isomorphism of K_q-algebras

$$\mathcal{H}(G, 1_{U_0}) \otimes_K K_q \xrightarrow{\cong} K_q[\Lambda]^W \ .$$

Here the W-invariants on the group ring $K_q[\Lambda]$ are formed with respect to the W-action induced by the conjugation action of $N(T)$ on T. Since $\pi_L^{\mathrm{val}_L \circ \xi} \delta^{1/2}$ defines a character of Λ it is clear that S_ξ is a homomorphism of algebras as well and a simple Galois descent argument shows that S_ξ induces an isomorphism of K-algebras

$$\mathcal{H}(G, 1_{U_0}) \xrightarrow{\cong} K[\Lambda]^{W,\gamma_\xi}$$

where γ_ξ is the cocycle from Example 3 in section 2. The left hand side has the $\| \, \|_\xi$-orthogonal basis $\{\psi_\lambda\}_{\lambda\in\Lambda^{--}}$ with

$$\|\psi_{\lambda(t)}\|_\xi = |\xi(t)| \ .$$

The right hand side has the $\| \, \|_{\gamma_\xi}$-orthonormal basis $\{\sigma_\lambda\}_{\lambda\in\Lambda^{--}}$ where

$$\sigma_\lambda = \sum_{w\in W/W(\lambda)} \gamma_\xi(w, \lambda)^w\lambda$$

(cf. section 2). Since the maps

$$\begin{array}{rcl} N/N_0 & \xrightarrow{\simeq} & NtU_0/U_0 \\ nN_0 & \longmapsto & tnU_0 \end{array}$$

are bijections we have

$$\sum_{n\in N/N_0} \psi_{\lambda(s)}(tn) = |(NtU_0 \cap U_0sU_0)/U_0| =: c(\lambda(t), \lambda(s)) \qquad \text{for any } s, t \in T \ .$$

It follows that

$$\begin{aligned} S_\xi(\psi_\mu) &= \sum_{t \in T/T_0} \pi_L^{\mathrm{val}_L(\xi(t))} c(\lambda(t),\mu)\lambda(t) \\ &= \sum_{\lambda \in \Lambda^{--}} \pi_L^{\mathrm{val}_L \circ \xi(\lambda)} c(\lambda,\mu)\sigma_\lambda \qquad \text{for any } \mu \in \Lambda^{--} . \end{aligned}$$

and

$$\pi_L^{\mathrm{val}_L \circ \xi({}^w\lambda)} c({}^w\lambda,\mu) = \gamma_\xi(w,\lambda)\pi_L^{\mathrm{val}_L \circ \xi(\lambda)} c(\lambda,\mu)$$

for any $\lambda \in \Lambda^{--}$, $\mu \in \Lambda$, and $w \in W$.

The reason for the validity of Satake's theorem lies in the following properties of the coefficients $c(\lambda,\mu)$.

Lemma 3.4: *For $\lambda, \mu \in \Lambda^{--}$ we have:*

i. $c(\mu,\mu) = 1$*;*

ii. $c(\lambda,\mu) = 0$ *unless* $\lambda \leq \mu$.

Proof: [BT] Prop. I.4.4.4.

Proposition 3.5: *The map S_ξ extends by continuity to an isometric isomorphism of K-Banach algebras*

$$\| \; \|_\xi\text{-completion of } \mathcal{H}(G, 1_{U_0}) \xrightarrow{\cong} K\langle \Lambda; \gamma_\xi\rangle^{W,\gamma_\xi} .$$

Proof: Define

$$\widetilde{\psi}_\lambda := \pi_L^{-\mathrm{val}_L \circ \xi(\lambda)} \psi_\lambda$$

for $\lambda \in \Lambda^{--}$. The left, resp. right, hand side has the $\| \; \|_\xi$-orthonormal, resp. $\| \; \|_{\gamma_\xi}$-orthonormal, basis $\{\widetilde{\psi}_\lambda\}_{\lambda \in \Lambda^{--}}$, resp. $\{\sigma_\lambda\}_{\lambda \in \Lambda^{--}}$. We want to apply Lemma 2.5 to the normed vector space $(K[\Lambda]^{W,\gamma_\xi}, \| \; \|_{\gamma_\xi})$, its orthonormal basis $\{\sigma_\lambda\}$, and the elements

$$S_\xi(\widetilde{\psi}_\mu) = \sigma_\mu + \sum_{\lambda < \mu} \pi_L^{\mathrm{val}_L \circ \xi(\lambda) - \mathrm{val}_L \circ \xi(\mu)} c(\lambda,\mu)\sigma_\lambda$$

(cf. Lemma 3.4). The coefficients $c(\lambda,\mu)$ are integers and therefore satisfy $|c(\lambda,\mu)| \leq 1$. Moreover, $\lambda < \mu$ implies, since ξ is dominant, that $\mathrm{val}_L \circ \xi(\mu) \leq \mathrm{val}_L \circ \xi(\lambda)$. Hence the assumptions of Lemma 2.5 indeed are satisfied and we obtain that $\{S_\xi(\widetilde{\psi}_\lambda)\}$ is another orthonormal basis for $(K[\Lambda]^{W,\gamma_\xi}, \| \; \|_{\gamma_\xi})$.

Corollary 3.6: *The Banach algebras $\mathcal{B}(G, \rho|U_0)$ and $K\langle \Lambda; \gamma_\xi\rangle^{W,\gamma_\xi}$ are isometrically isomorphic.*

If $\xi = 1$ then, in view of Lemma 2.7, the reader should note the striking analogy between the above result and the computation in [Mac] Thm. (4.7.1) of the spectrum of the algebra of integrable complex valued functions on $U_0\backslash G/U_0$. The methods of proof are totally different, though. In fact, in our case the spherical function on $U_0\backslash G/U_0$ corresponding to a point in T_1' in general is not bounded.

Suppose that the group G is semisimple and adjoint. We fix elements $t_1, \ldots, t_d \in T^{--}$ such that $\lambda_i := \lambda(t_i)$ are the fundamental antidominant cocharacters. In Prop. 2.6 we have seen that then $K\langle\Lambda;\gamma_\xi\rangle^{W,\gamma_\xi}$ is a Tate algebra in the variables $\sigma_{\lambda_1} \ldots, \sigma_{\lambda_d}$. Hence $\mathcal{B}(G,\rho|U_0)$ is a Tate algebra as well. But it seems complicated to compute explicitly the variables corresponding to the σ_{λ_i}. Instead we may repeat our previous reasoning in a modified way.

Proposition 3.7: *Suppose that G is semisimple and adjoint; then $\mathcal{B}(G,\rho|U_0)$ is a Tate algebra over K in the variables $\frac{\psi_{\lambda_1}\cdot\rho}{\xi(t_1)}, \ldots, \frac{\psi_{\lambda_d}\cdot\rho}{\xi(t_d)}$.*

Proof: By Lemma 3.3 and Prop. 3.5 it suffices to show that $K\langle\Lambda;\gamma_\xi\rangle^{W,\gamma_\xi}$ is a Tate algebra in the variables $\xi(t_i)^{-1}S_\xi(\psi_{\lambda_i})$. We write a given $\lambda \in \Lambda^{--}$ as $\lambda = \lambda_1^{m_1} \ldots \lambda_d^{m_d}$ and put

$$\widetilde{\sigma}_\lambda := S_\xi(\widetilde{\psi}_{\lambda_1})^{m_1} \cdot \ldots \cdot S_\xi(\widetilde{\psi}_{\lambda_d})^{m_d} = S_\xi(\widetilde{\psi}_{\lambda_1}^{m_1} * \ldots * \widetilde{\psi}_{\lambda_d}^{m_d})$$

using notation from the proof of Prop. 3.5. Similarly as in the proof of Prop. 2.7 one checks that the arguments in [B-GAL] Chap. VI §§3.2 and 3.4 work, over the ring of integers in K, equally well for our twisted W-action (note that, in the language of loc. cit. and due to Lemma 3.4, the unique maximal term in $S_\xi(\widetilde{\psi}_{\lambda_i})$ is λ_i) and show that we have

$$\widetilde{\sigma}_\lambda = \sigma_\lambda + \sum_{\mu<\lambda} c_{\mu\lambda}\sigma_\mu$$

with $|c_{\mu\lambda}| \leq 1$. So we may apply again Lemma 2.5 and obtain that $\{\widetilde{\sigma}_\lambda\}$ is another orthonormal basis for $K\langle\Lambda;\gamma_\xi\rangle^{W,\gamma_\xi}$. It remains to note that $\xi(t_i)$ and $\pi_L^{\mathrm{val}_L(\xi(t))}$ only differ by a unit.

Example: Consider the group $G := GL_{d+1}(L)$. Cor. 3.6 applies to G but Prop. 3.7 does not. Nevertheless, with the same notations as at the end of section 2 a simple modification of the argument gives

$$\mathcal{B}(G,\rho|U_0) = K\Big\langle \frac{\psi_{\lambda_1}\cdot\chi_{un}^{-1}\rho}{\xi(t_1)}, \ldots, \frac{\psi_{\lambda_d}\cdot\chi_{un}^{-1}\rho}{\xi(t_d)}, \Big(\frac{\psi_{\lambda_{d+1}}\cdot\chi_{un}^{-1}\rho}{\xi(t_{d+1})}\Big)^{\pm 1}\Big\rangle \ .$$

Moreover in this case the λ_i are minimal with respect to the partial order $\leq$ so that we do have

$$\xi(t_i)^{-1}S_\xi(\psi_{\lambda_i}) = \sigma_{\lambda_i} \ .$$

Hence the above representation of $\mathcal{B}(G,\rho|U_0)$ as an affinoid algebra corresponds to the representation

$$K\langle\Lambda;\gamma_\xi\rangle^{W,\gamma_\xi} = K\langle\sigma_{\lambda_1}\dots,\sigma_{\lambda_d},\sigma_{\lambda_{d+1}}^{\pm1}\rangle\ .$$

On affinoid domains this corresponds to a map

$$T'_\xi \longrightarrow \{(\omega_1,\dots,\omega_{d+1})\in K^{d+1} : |\omega_1|,\dots,|\omega_d|\le 1, |\omega_{d+1}|=1\}$$

which, using our choice of coordinates on T' from section 2, is given by

$$(\zeta_1,\dots,\zeta_{d+1}) \longmapsto (\dots,q^{-\frac{(i-1)i}{2}}\xi(t_i)^{-1}\Sigma_i(\zeta_1,\dots,\zeta_{d+1}),\dots)$$

where

$$\Sigma_1(\zeta_1,\dots,\zeta_{d+1}) = \zeta_1+\dots+\zeta_{d+1},\ \dots\ ,\Sigma_{d+1}(\zeta_1,\dots,\zeta_{d+1}) = \zeta_1\cdot\ldots\cdot\zeta_{d+1}$$

denote the elementary symmetric polynomials.

Let us further specialize to the case $G = GL_2(L)$. Then E_L is the k-th symmetric power, for some $k\ge 0$, of the standard representation of GL_2. The highest weight of E_L is $\xi(\begin{pmatrix} t_1 & 0\\ 0 & t_2\end{pmatrix}) = t_2^k$. We obtain

$$\mathcal{B}(G,\rho|U_0) = K\big\langle X_1,(\pi_L^{-k}X_2)^{\pm1}\big\rangle\ .$$

with the variables $X_i := \psi_{\lambda_i}\cdot\chi_{un}^{-1}\rho$. The above map between affinoid domains becomes

$$(\zeta_1,\zeta_2)\longrightarrow(\zeta_1+\zeta_2,q^{-1}\pi_L^{-k}\zeta_1\zeta_2)\ .$$

4. p-adic Iwahori-Hecke algebras

With the same assumptions and notations as in the previous section we now let $U_1\subseteq U_0$ be the Iwahori subgroup such that $U_1\cap P = U_0\cap P$. In this section we will compute the Banach-Hecke algebras $\mathcal{B}(G,\rho|U_1)$. By Lemma 3.3 this means, similarly as before, computing the $\|\ \|_\xi$-completion of $\mathcal{H}(G,1_{U_1})$.

The extended affine Weyl group $\widetilde{W}$ of G is given by

$$\widetilde{W} := N(T)/T_0\ .$$

Since the Weyl group W lifts to $U_0\cap N(T)/T_0\subseteq\widetilde{W}$ we see that $\widetilde{W}$ is the semidirect product of W and Λ. The Bruhat-Tits decomposition says that G is the disjoint union of the double cosets U_1xU_1 with x running over $\widetilde{W}$. Therefore, if we let $\tau_x\in\mathcal{H}(G,1_{U_1})$ denote the characteristic function of the double coset U_1xU_1, then $\{\tau_x\}_{x\in\widetilde{W}}$ is a K-basis of $\mathcal{H}(G,1_{U_1})$. The τ_x are known to be invertible in the algebra $\mathcal{H}(G,1_{U_1})$. As a consequence of Lemma 3.2 the $\| \ \|_\xi$-norm is given by

$$\|\psi\|_\xi = \sup_{v,w\in W} \sup_{t\in T^{--}} |\psi(v\lambda({}^w t))\xi(t)| \ .$$

In particular, $\{\tau_x\}_{x\in\widetilde{W}}$ is an $\| \ \|_\xi$-orthogonal basis of $\mathcal{H}(G,1_{U_1})$ such that

$$\|\tau_x\|_\xi = |\xi({}^w t)| \qquad \text{if } v,w\in W \text{ and } t\in T \text{ such that } x = v\lambda(t) \text{ and } {}^w t \in T^{--} \ .$$

We let $\mathcal{C}$ be the unique Weyl chamber corresponding to P in the apartment corresponding to T with vertex x_0 (cf. [Car]§3.5). The Iwahori subgroup U_1 fixes pointwise the unique chamber $C \subseteq \mathcal{C}$ with vertex x_0. The reflections at the walls of $\mathcal{C}$ generate the Weyl group W. Let $s_0,\ldots,s_e \in \widetilde{W}$ be the reflections at all the walls of C and let W_{aff} denote the subgroup of $\widetilde{W}$ generated by $s_0,\ldots,s_e$. This affine Weyl group W_{aff} with the generating set $\{s_0,\ldots,s_e\}$ is a Coxeter group. In particular we have the corresponding length function $\ell : W_{aff} \longrightarrow \mathbb{N}\cup\{0\}$ and the corresponding Bruhat order $\leq$ on W_{aff}. If $\Omega \subseteq \widetilde{W}$ is the subgroup which fixes the chamber C then $\widetilde{W}$ also is the semidirect product of Ω and W_{aff}. We extend the length function ℓ to $\widetilde{W}$ by $\ell(\omega w) := \ell(w)$ for $\omega \in \Omega$ and $w \in W_{aff}$. The Bruhat order is extended to $\widetilde{W}$ by the rule $\omega w \leq \omega' w'$, for $w,w' \in W_{aff}$ and $\omega,\omega' \in \Omega$, if and only if $\omega = \omega'$ and $w \leq w'$. One of the basic relations established by Iwahori-Matsumoto is:

(1) For any $x,y \in \widetilde{W}$ such that $\ell(xy) = \ell(x) + \ell(y)$ we have $\tau_{xy} = \tau_x * \tau_y$.

It easily implies that, for any $\lambda \in \Lambda$, the element

$$\Theta(\lambda) := \tau_{\lambda_1} * \tau_{\lambda_2}^{-1} \in \mathcal{H}(G,1_{U_1})$$

where $\lambda = \lambda_1\lambda_2^{-1}$ with $\lambda_i \in \Lambda^{--}$ is independent of the choice of λ_1 and λ_2. Moreover Bernstein has shown that the map

$$\begin{array}{rcl} \Theta : K[\Lambda] & \longrightarrow & \mathcal{H}(G,1_{U_1}) \\ \lambda & \longmapsto & \Theta(\lambda) \end{array}$$

is an embedding of K-algebras.

Comment: It is more traditional (cf. [HKP] §1) to consider the embedding of K_q-algebras (with K_q/K and $q^{1/2} \in K_q$ be as before)

$$\begin{array}{rcl} \Theta^{norm} : K_q[\Lambda] & \longrightarrow & \mathcal{H}(G,1_{U_1}) \otimes_K K_q \\ \lambda & \longmapsto & \delta^{-1/2}(\lambda)\tau_{\lambda_1} * \tau_{\lambda_2}^{-1} \end{array}$$

where $\lambda = \lambda_1\lambda_2^{-1}$ with dominant λ_i. The modified map $\Theta^+ := \delta^{1/2} \cdot \Theta^{norm}$ already is defined over K. On $K[\Lambda]$ we have the involution ι_λ defined by $\iota_\Lambda(\lambda) := \lambda^{-1}$, and on $\mathcal{H}(G,1_{U_1})$ there is the anti-involution ι defined by $\iota(\psi)(g) := \psi(g^{-1})$. We then have

$$\Theta = \iota \circ \Theta^+ \circ \iota_\Lambda \ .$$

In the following we consider the renormalized embedding of K-algebras

$$\begin{array}{llcl}\Theta_\xi : & K[\Lambda] & \longrightarrow & \mathcal{H}(G, 1_{U_1}) \\ & \lambda & \longmapsto & \pi_L^{-\operatorname{val}_L \circ \xi(\lambda)} \Theta(\lambda) \ .\end{array}$$

In order to compute the norm induced, via Θ_ξ, by $\| \ \|_\xi$ on $K[\Lambda]$ we introduce the elements

$$\theta_x := q^{(\ell(x)-\ell(w)-\ell(\lambda_1)+\ell(\lambda_2))/2} \tau_w * \tau_{\lambda_1} * \tau_{\lambda_2}^{-1} \ .$$

for any $x \in \widetilde{W}$ written as $x = w\lambda_1\lambda_2^{-1}$ with $w \in W$ and $\lambda_i \in \Lambda^{--}$. Since $\ell(w) + \ell(\lambda_1) = \ell(w\lambda_1)$ (cf. [Vig] App.) we obtain from (1) that

$$\theta_x = q^{(\ell(w\lambda_1\lambda_2^{-1})-\ell(w\lambda_1)+\ell(\lambda_2))/2} \tau_{w\lambda_1} * \tau_{\lambda_2}^{-1} \ .$$

On the other hand [Vig] Lemma 1.2 (compare also [Hai] Prop. 5.4) says that, for any $x, y \in \widetilde{W}$, the number

$$(\ell(xy^{-1}) - \ell(x) + \ell(y))/2$$

is an integer between 0 and $\ell(y)$ and that

$$\tau_x * \tau_y^{-1} = q^{-(\ell(xy^{-1})-\ell(x)+\ell(y))/2}(\tau_{xy^{-1}} + Q_{x,y})$$

where $Q_{x,y}$ is a linear combination with integer coefficients of τ_z with $z < xy^{-1}$. It follows that for any $x \in \widetilde{W}$ we have

$$\theta_x = \tau_x + Q_x \tag{2}$$

where Q_x is a linear combination with integer coefficients of τ_z with $z < x$.

Lemma 4.1: *Consider two elements $x = w'\lambda$ and $y = v'\mu$ in $\widetilde{W}$ where $w', v' \in W$ and $\lambda, \mu \in \Lambda$; let $w, v \in W$ such that ${}^w\lambda, {}^v\mu \in \Lambda^{--}$; if $x \leq y$ then we have:*

i. ${}^v\mu - {}^w\lambda \in \sum_{\alpha\in\Phi^+} \mathbb{N}_0 \cdot (-\check{\alpha})$;

ii. $\|\tau_x\|_\xi \leq \|\tau_y\|_\xi$.

Proof: i. Let $w_0 \in W$ denote the longest element. We will make use of the identity

$$\{x' \in \widetilde{W} : x' \leq w_0({}^{w_0 v}\mu)\} = \bigcup_{\lambda'} W\lambda' W$$

where λ' ranges over all elements in Λ^{--} such that ${}^v\mu - \lambda' \in \sum_{\alpha\in\Phi^+} \mathbb{N}_0 \cdot (-\check{\alpha})$ (see [Ka2] (4.6) or [HKP] 7.8). Since $y \in W({}^v\mu)W$ this identity implies first that $x \leq y \leq ({}^{w_0 v}\mu)w_0$ and then that $x \in W\lambda' W$ for some $\lambda' \in \Lambda^{--}$ such that ${}^v\mu - \lambda' \in \sum_{\alpha\in\Phi^+} \mathbb{N}_0 \cdot (-\check{\alpha})$. Obviously we must have $\lambda' = {}^v\lambda$. ii. Let $\lambda = \lambda(t_1)$

and $\mu = \lambda(t_2)$. We have $\|\tau_x\|_\xi = |\xi({}^w t_1)|$ and $\|\tau_y\|_\xi = |\xi({}^v t_2)|$. Since highest weights are dominant we obtain from i. that $|\xi({}^v t_2({}^w t_1)^{-1})| \geq 1$.

It follows from Lemma 4.1.ii and formula (2) that Lemma 2.5 is applicable showing that $\{\theta_x\}_{x\in\widetilde{W}}$ is another $\|\ \|_\xi$-orthogonal basis of $\mathcal{H}(G, 1_{U_1})$ with

$$\|\theta_x\|_\xi = \|\tau_x\|_\xi \ .$$

For any $\lambda(t) = \lambda = \lambda_1\lambda_2^{-1} \in \Lambda$ with $\lambda_i \in \Lambda^{--}$ we have

$$\theta_\lambda = q^{(\ell(\lambda)-\ell(\lambda_1)+\ell(\lambda_2))/2}\pi_L^{\mathrm{val}_L(\xi(t))}\Theta_\xi(\lambda)$$

and

$$\|\theta_\lambda\|_\xi = \|\tau_\lambda\|_\xi = |\xi({}^w t)|$$

where $w \in W$ such that ${}^w t \in T^{--}$. In particular $\{\theta_\lambda\}_{\lambda\in\Lambda}$ is a $\|\ \|_\xi$-orthogonal basis of $\mathrm{im}(\Theta_\xi)$.

Lemma 4.2: *With the above notations we have*

$$q^{-(\ell(\lambda)-\ell(\lambda_1)+\ell(\lambda_2))/2} = \frac{\delta^{1/2}({}^w\lambda)}{\delta^{1/2}(\lambda)} \ .$$

Proof: Write $t = t_1t_2^{-1}$ with $\lambda(t_i) = \lambda_i$. According to the explicit formula for the length ℓ in [Vig] App. we have

$$q^{\ell(\lambda)} = \prod_{\alpha\in\Phi^+, |\alpha(t)|_L\geq 1} |\alpha(t)|_L \cdot \prod_{\alpha\in\Phi^+, |\alpha(t)|_L\leq 1} |\alpha(t)|_L^{-1}$$

and

$$q^{\ell(\lambda_i)} = \prod_{\alpha\in\Phi^+} |\alpha(t_i)|_L \ .$$

It follows that

$$q^{-(\ell(\lambda)-\ell(\lambda_1)+\ell(\lambda_2))/2} = \prod_{\alpha\in\Phi^+, |\alpha(t)|_L\leq 1} |\alpha(t)|_L$$

Since ${}^w t \in T^{--}$ we have $|{}^{w^{-1}}\alpha(t)|_L \geq 1$ for any $\alpha \in \Phi^+$. Hence $\{\alpha \in \Phi^+ : |\alpha(t)|_L < 1\} \subseteq \Phi^+ \setminus {}^{w^{-1}}\Phi^+$. By the last formula in Example 2 of section 2 the above right hand side therefore is equal to $\frac{\delta^{1/2}({}^w\lambda)}{\delta^{1/2}(\lambda)}$.

It readily follows that

$$\|\Theta_\xi(\lambda)\|_\xi = |\gamma_\xi^{dom}(\lambda)| \qquad \text{for any } \lambda \in \Lambda \ .$$

In other words
$$\Theta_\xi : (K[\Lambda], \| \ \|_{\gamma_\xi}) \longrightarrow (\mathcal{H}(G, 1_{U_1}), \| \ \|_\xi)$$
is an isometric embedding. Combining all this with Lemma 3.3 we obtain the following result.

Proposition 4.3: *i. The map*
$$\begin{array}{rcl} K\langle\Lambda;\gamma_\xi\rangle & \longrightarrow & \mathcal{B}(G,\rho|U_1) \\ \lambda & \longmapsto & \Theta_\xi(\lambda)\cdot\chi_{un}^{-1}\rho \end{array}$$
is an isometric embedding of Banach algebras;

ii. the map
$$\begin{array}{rcl} \mathcal{H}(U_0,1_{U_1})\otimes_K K\langle\Lambda;\gamma_\xi\rangle & \xrightarrow{\cong} & \mathcal{B}(G,\rho|U_1) \\ \tau_w\otimes\lambda & \longmapsto & (\tau_w * \Theta_\xi(\lambda))\cdot\chi_{un}^{-1}\rho \end{array}$$
is a K-linear isomorphism.

Remarks: 1) A related computation in the case $\xi = 1$ is contained in [Vig] Thm. 4(suite).

2) It is worth observing that the "twisted" W-action on $K\langle\Lambda;\gamma_\xi\rangle$ corresponds under the isomorphism Θ_ξ to the W-action on $\mathrm{im}(\Theta_\xi)$ given by
$$(w,\theta_\lambda) \longmapsto \theta_{^w\lambda} \ .$$

The results of this section and of the previous section are compatible in the following sense.

Proposition 4.4: *The diagram*
$$\begin{tikzcd} K\langle\Lambda;\gamma_\xi\rangle \arrow[r, "\Theta_\xi(.)\cdot\chi_{un}^{-1}\rho"] & \mathcal{B}(G,\rho|U_1) \arrow[d, "(\psi_{\lambda(1)}\cdot\chi_{un}^{-1}\rho)*."] \\ K\langle\Lambda;\gamma_\xi\rangle^{W,\gamma_\xi} \arrow[u, "\subseteq"] \arrow[r, "S_\xi^{-1}(.)\cdot\chi_{un}^{-1}\rho"] & \mathcal{B}(G,\rho|U_0) \end{tikzcd}$$
is commutative. Moreover, the image of $K\langle\Lambda;\gamma_\xi\rangle^{W,\gamma_\xi}$ under the map $\Theta_\xi(.)\cdot\chi_{un}^{-1}\rho$ lies in the center of $\mathcal{B}(G,\rho|U_1)$.

Proof: We recall that the upper, resp. lower, horizontal arrow is an isometric unital monomorphism by Prop. 4.3.i, resp. by Lemma 3.3 and Prop. 3.5. The right perpendicular arrow is a continuous linear map respecting the unit elements. It suffices to treat the case of the trivial representation $\rho = 1$. By continuity we therefore are reduced to establishing the commutativity of the diagram

$$\begin{tikzcd} K[\Lambda] \arrow[r, "\Theta"] & \mathcal{H}(G, 1_{U_1}) \arrow[d, "\psi_{\lambda(1)} *\cdot"] \\ K[\Lambda]^{W,\gamma_\xi} \arrow[u, "\subseteq"] \arrow[r, "S_1^{-1}"] & \mathcal{H}(G, 1_{U_0}) \end{tikzcd}$$

as well as the inclusion

$$\Theta(K[\Lambda]^{W,\gamma_\xi}) \subseteq center\ of\ \mathcal{H}(G, 1_{U_1}) \ .$$

It is known (cf. [HKP] Lemma 2.3.1, section 4.6, and Lemma 3.1.1) that:

– $\Theta^{norm}(K_q[\Lambda]^W) = center\ of\ \mathcal{H}(G, 1_{U_1}) \otimes_K K_q$;

– $\psi_{\lambda(1)} * \Theta^{norm} \circ S^{norm} = id$ on $\mathcal{H}(G, 1_{U_0}) \otimes_K K_q$;

– $\Theta^{norm} = \iota \circ \Theta^{norm} \circ \iota_\Lambda$ on $K_q[\Lambda]^W$.

The first identity implies the asserted inclusion. We further deduce that

$$\begin{aligned} \Theta \circ S_1 &= \iota \circ (\delta^{1/2} \cdot \Theta^{norm}) \circ \iota_\Lambda \circ (\delta^{1/2} \cdot S^{norm}) \\ &= \iota \circ \Theta^{norm} \circ \iota_\Lambda \circ S^{norm} \\ &= \Theta^{norm} \circ S^{norm} \qquad\qquad \text{on } \mathcal{H}(G, 1_{U_0}) \otimes_K K_q \end{aligned}$$

and hence that

$$\psi_{\lambda(1)} * (\Theta \circ S_1) = id \qquad\qquad \text{on } \mathcal{H}(G, 1_{U_0}) \ .$$

5. Crystalline Galois representations

We go back to the example of the group $G := GL_{d+1}(L)$ which we have discussed already at the end of section 3. But we now want to exploit Lemma 2.7. As before we fix a dominant integral weight $\xi \in X^*(T)$ that is given by

$$\begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_{d+1} \end{pmatrix} \longmapsto \prod_{i=1}^{d+1} g_i^{a_i}$$

with $(a_1, \ldots, a_{d+1}) \in \mathbb{Z}^{d+1}$. Note that the dominance means that

$$a_1 \leq \ldots \leq a_{d+1} \ .$$

Equally as before we use the coordinates

$$\begin{array}{ccl} T'(K) & \longrightarrow & (K^\times)^{d+1} \\ \zeta & \longmapsto & (\zeta_1, \ldots, \zeta_{d+1}) \text{ with } \zeta_i := q^{i-1}\pi_L^{a_i}\zeta(\lambda_{\{i\}}) \end{array}$$

on the dual torus. Some times we view ζ as the diagonal matrix in $GL_{d+1}(K)$ with diagonal entries $(\zeta_1, \ldots, \zeta_{d+1})$. On the other hand, on the root space we use the coordinates

$$\begin{array}{rcl} V_{\mathbb{R}} = \mathrm{Hom}(\Lambda, \mathbb{R}) & \longrightarrow & \mathbb{R}^{d+1} \\ z & \longmapsto & (z_1, \ldots, z_{d+1}) \text{ with } z_i := z(\lambda_{\{i\}}) \ . \end{array}$$

In these coordinates we have:

1) The points η_L and ξ_L from Example 3 in section 2 correspond to

$$\frac{[L:\mathbb{Q}_p]}{2}(-d, -(d-2), \ldots, d-2, d) \quad \text{and} \quad (a_1, \ldots, a_{d+1}) \ ,$$

respectively.

2) The map $val : T'(K) \longrightarrow V_{\mathbb{R}}$ corresponds to the map

$$\begin{array}{rcl} (K^\times)^{d+1} & \longrightarrow & \mathbb{R}^{d+1} \\ (\zeta_1, \ldots, \zeta_{d+1}) & \longmapsto & (\mathrm{val}_L(\zeta_1), \ldots, \mathrm{val}_L(\zeta_{d+1})) - \xi_L - \widetilde{\eta}_L \ . \end{array}$$

where

$$\widetilde{\eta}_L := [L:\mathbb{Q}_p](0, 1, \ldots, d) = \eta_L + \frac{[L:\mathbb{Q}_p]}{2}(d, \ldots, d) \ .$$

3) On $\mathbb{R}^{d+1}$ the partial order defined by Φ^+ is given by

$$(z_1, \ldots, z_{d+1}) \leq (z'_1, \ldots, z'_{d+1})$$

if and only if

$$z_{d+1} \leq z'_{d+1} \ , \ z_d + z_{d+1} \leq z'_d + z'_{d+1} \ , \ \ldots, \ z_2 + \ldots + z_{d+1} \leq z'_2 + \ldots + z'_{d+1}$$

and

$$z_1 + \ldots + z_{d+1} = z'_1 + \ldots + z'_{d+1} \ .$$

4) The map $z \longmapsto z^{dom}$ corresponds in $\mathbb{R}^{d+1}$ to the map which rearranges the coordinates in increasing order and which we will also denote by $(.)^{dom}$.

It now is a straightforward computation to show that Lemma 2.7 amounts to

$$T'_\xi = \{\zeta \in T' : (\mathrm{val}_L(\zeta_1), \ldots, \mathrm{val}_L(\zeta_{d+1}))^{dom} \leq \xi_L + \widetilde{\eta}_L\}.$$

Even more explicitly, T'_ξ is the domain of all $(\zeta_1, \ldots, \zeta_{d+1}) \in (K^\times)^{d+1}$ such that

$$(\mathrm{val}_L(\zeta_1), \ldots, \mathrm{val}_L(\zeta_{d+1}))^{dom} \leq (a_1, a_2 + [L:\mathbb{Q}_p], \ldots, a_{d+1} + d[L:\mathbb{Q}_p]) \ .$$

For any increasing sequence $\underline{r} = (r_1 \leq \ldots \leq r_{d+1})$ of real numbers we denote by $\mathcal{P}(\underline{r})$ the convex polygon in the plane through the points

$$(0,0),(1,r_1),(2,r_1+r_2),\ldots,(d+1,r_1+\ldots+r_{d+1}) .$$

We then may reformulate the above description of T'_ξ as follows.

Lemma 5.1: *T'_ξ is the subdomain of all $\zeta \in T'$ such that $\mathcal{P}(val(\zeta)^{dom})$ lies above $\mathcal{P}(\xi_L + \widetilde{\eta}_L)$ and both polygons have the same endpoint.*

We recall that a filtered K-isocrystal is a triple $\underline{D} = (D, \varphi, Fil^{\cdot} D)$ consisting of a finite dimensional K-vector space D, a K-linear automorphism φ of D – the "Frobenius" – , and an exhaustive and separated decreasing filtration $Fil^{\cdot} D$ on D by K-subspaces. In the following we fix the dimension of D to be equal to $d+1$ and, in fact, the vector space D to be the $d+1$-dimensional standard vector space $D = K^{d+1}$. We then may think of φ as being an element in the group $G'(K) := GL_{d+1}(K)$. The (filtration) type $type(\underline{D}) \in \mathbb{Z}^{d+1}$ is the sequence $(b_1,\ldots,b_{d+1})$, written in increasing order, of the break points b of the filtration $Fil^{\cdot} D$ each repeated $dim_K\, gr^b D$ many times. We put

$$t_H(\underline{D}) := \sum_{b\in\mathbb{Z}} b \cdot dim_K\, gr^b\, D .$$

Then $(d+1, t_H(\underline{D}))$ is the endpoint of the polygon $\mathcal{P}(type(\underline{D}))$. On the other hand we define the Frobenius type $s(\underline{D})$ of $\underline{D}$ to be the conjugacy class of the semisimple part of φ in $G'(K)$. We put

$$t_N^L(\underline{D}) := \mathrm{val}_L(\det_K(\varphi)) .$$

The filtered K-isocrystal $\underline{D}$ is called weakly L-admissible if $t_H(\underline{D}) = t_N^L(\underline{D})$ and $t_H(\underline{D}') \leq t_N^L(\underline{D}')$ for any filtered K-isocrystal $\underline{D}'$ corresponding to a φ-invariant K-subspace $D' \subseteq D$ with the induced filtration.

Proposition 5.2: *Let $\zeta \in T'(K)$ and let ξ be a dominant integral weight of G; then $\zeta \in T'_\xi(K)$ if and only if there is a weakly L-admissible filtered K-isocrystal $\underline{D}$ such that $type(\underline{D}) = \xi_L + \widetilde{\eta}_L$ and $\zeta \in s(\underline{D})$.*

Proof: Let us first suppose that there exists a filtered K-isocrystal $\underline{D}$ with the asserted properties. Then $\mathcal{P}(type(\underline{D})) = \mathcal{P}(\xi_L + \widetilde{\eta}_L)$ is the Hodge polygon of $\underline{D}$ and $\mathcal{P}(val(\zeta)^{dom})$ is its Newton polygon (relative to val_L). By [Fon] Prop. 4.3.3 (the additional assumptions imposed there on the field K are irrelevant at this point) the weak admissibility of $\underline{D}$ implies that its Newton polygon lies above its Hodge polygon with both having the same endpoint. Lemma 5.1 therefore implies that $\zeta \in T'_\xi(K)$.

We now assume vice versa that $\zeta \in T'_\xi(K)$. We let φ_{ss} be the semisimple automorphism of the standard vector space D given by the diagonal matrix with diagonal entries $(\zeta_1, \ldots, \zeta_{d+1})$. Let

$$D = D_1 + \ldots + D_m$$

be the decomposition of D into the eigenspaces of φ_{ss}. We now choose the Frobenius φ on D in such a way that φ_{ss} is the semisimple part of φ and that any D_j is φ-indecomposable. In this situation D has only finitely many φ-invariant subspaces D' and each of them is of the form

$$D' = D'_1 + \ldots + D'_m$$

with D'_j one of the finitely many φ-invariant subspaces of D_j. By construction the Newton polygon of (D, φ) is equal to $\mathcal{P}(val(\zeta)^{dom})$. To begin with consider any filtration $Fil^{\cdot} D$ of type $\xi_L + \widetilde{\eta}_L$ on D and put $\underline{D} := (D, \varphi, Fil^{\cdot} D)$. The corresponding Hodge polygon then is $\mathcal{P}(\xi_L + \widetilde{\eta}_L)$. By Lemma 5.1 the first polygon lies above the second and both have the same endpoint. The latter already says that

$$t_H(\underline{D}) = t_N^L(\underline{D}) \ .$$

It remains to be seen that we can choose the filtration $Fil^{\cdot} D$ in such a way that $t_H(\underline{D}') \leq t_N^L(\underline{D}')$ holds true for any of the above finitely many φ-invariant subspaces $D' \subseteq D$. The inequality between the two polygons which we have does imply that

$$a_1 + (a_2 + [L : \mathbb{Q}_p]) + \ldots + (a_{dim D'} + (dim\, D' - 1)[L : \mathbb{Q}_p]) \leq t_N^L(\underline{D}') \ .$$

Hence it suffices to find the filtration in such a way that we have

$$t_H(\underline{D}') \leq a_1 + (a_2 + [L : \mathbb{Q}_p]) + \ldots + (a_{dim D'} + (dim\, D' - 1)[L : \mathbb{Q}_p])$$

for any D'. But it is clear that for any filtration (of type $\xi_L + \widetilde{\eta}_L$) in general position we actually have

$$t_H(\underline{D}') = a_1 + (a_2 + [L : \mathbb{Q}_p]) + \ldots + (a_{dim D'} + (dim\, D' - 1)[L : \mathbb{Q}_p])$$

for the finitely many D'.

In order to connect this to Galois representations we have to begin with a different kind of filtered isocrystal (cf. [BM] §3.1). First of all we now suppose that K is a finite extension of $\mathbb{Q}_p$ (as always containing L). Then a filtered isocrystal over L with coefficients in K is a triple $\underline{M} = (M, \phi, Fil^{\cdot} M_L)$ consisting of a free $L_0 \otimes_{\mathbb{Q}_p} K$-module M of finite rank, a σ-linear automorphism ϕ of M – the "Frobenius" – , and an exhaustive and separated decreasing filtration $Fil^{\cdot} M_L$ on $M_L := L \otimes_{L_0} M$ by $L \otimes_{\mathbb{Q}_p} K$-submodules. Here L_0 denotes the maximal unramified subextension of L and σ its Frobenius automorphism. By abuse of notation we also write σ for the automorphism $\sigma \otimes id$ of $L_0 \otimes_{\mathbb{Q}_p} K$. We put

$$t_H(\underline{M}) := \sum_{b \in \mathbf{Z}} b \cdot dim_L\, gr^b\, M_L = [K:L] \cdot \sum_{b \in \mathbf{Z}} b \cdot dim_K\, gr^b\, M_L \ .$$

The equality is a consequence of the fact that for any finitely generated $L \otimes_{\mathbb{Q}_p} K$-module M' the identity

$$dim_L M' = [K:L] \cdot dim_K M'$$

holds true. By semisimplicity this needs to be verified only for a simple module which must be isomorphic to a field into which L and K both can be embedded and in which case this identity is obvious.

The number $t_N(\underline{M})$ is defined as $\mathrm{val}_{\mathbb{Q}_p}(\phi(x)/x)$ where x is an arbitrary nonzero element in the maximal exterior power of M as an L_0-vector space. But we have

$$\begin{aligned} t_N(\underline{M}) &= \mathrm{val}_{\mathbb{Q}_p}(\phi(x)/x) \\ &= \frac{1}{[L_0:\mathbb{Q}_p]} \cdot \mathrm{val}_{\mathbb{Q}_p}(\det_{L_0}(\phi^{[L_0:\mathbb{Q}_p]})) \\ &= \frac{1}{[L_0:\mathbb{Q}_p]} \cdot \mathrm{val}_{\mathbb{Q}_p}(\mathrm{Norm}_{K/L_0}(\det_K(\phi^{[L_0:\mathbb{Q}_p]}))) \\ &= \mathrm{val}_{\mathbb{Q}_p}(\mathrm{Norm}_{K/L_0}(\det_K(\phi))) \\ &= [K:L_0] \cdot \mathrm{val}_{\mathbb{Q}_p}(\det_K(\phi)) \\ &= [K:L] \cdot \mathrm{val}_L(\det_K(\phi)) \ . \end{aligned}$$

The filtered isocrystal $\underline{M}$ over L with coefficients in K is called weakly admissible (cf. [BM] Prop. 3.1.1.5) if $t_H(\underline{M}) = t_N(\underline{M})$ and $t_H(\underline{M}') \leq t_N(\underline{M}')$ for any subobject $\underline{M}'$ of $\underline{M}$ corresponding to a ϕ-invariant $L_0 \otimes_{\mathbb{Q}_p} K$-submodule $M' \subseteq M$ with the induced filtration on $L \otimes_{L_0} M'$.

By the main result of [CF] there is a natural equivalence of categories $V \longmapsto D_{cris}(V)$ between the category of K-linear crystalline representations of the absolute Galois group $\mathrm{Gal}(\overline{L}/L)$ of the field L and the category of weakly admissible filtered isocrystals over L with coefficients in K. It has the property that

$$dim_K\, V = rank_{L_0 \otimes_{\mathbb{Q}_p} K}\, D_{cris}(V) \ .$$

To avoid confusion we recall that a K-linear Galois representation is called crystalline if it is crystalline as a $\mathbb{Q}_p$-linear representation. We also recall that the jump indices of the filtration on $D_{cris}(V)_L$ are called the Hodge-Tate coweights of the crystalline Galois representation V (they are the negatives of the Hodge-Tate weights). Moreover, we will say that V is K-split if all eigenvalues of the Frobenius on $D_{cris}(V)$ are contained in K. This is a small technical condition which always can be achieved by extending the coefficient field K. More important is the following additional condition. We let $\mathbb{C}_p$ denote the completion of the algebraic closure $\overline{L}$. We may view V as an L-vector space through the inclusion $L \subseteq K$.

Definition: *A K-linear crystalline representation V of $\mathrm{Gal}(\overline{L}/L)$ is called special if the kernel of the natural map $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \twoheadrightarrow \mathbb{C}_p \otimes_L V$ is generated, as a $\mathbb{C}_p$-vector space, by its $\mathrm{Gal}(\overline{L}/L)$-invariants (for the diagonal action).*

On the full subcategory of special crystalline Galois representations we have a simplified form of the above equivalence of categories. This is well known (see [FR] Remark 0.3). But since we have not found any details in the literature we include them here for the convenience of the reader. We will speak of a K-isocrystal and an isocrystal over L with coefficients in K, respectively, if no filtration is prescribed. Suppose that (M, ϕ) is an isocrystal over L with coefficients in K. We then have the L_0-isotypic decomposition

$$M = \oplus_{\tau \in \Delta} M_\tau$$

where $\Delta := \mathrm{Gal}(L_0/\mathbb{Q}_p)$ and where M_τ is the K-subspace of M on which L_0 acts via the embedding $\tau : L_0 \hookrightarrow K$. One has

$$\phi(M_\tau) = M_{\tau\sigma^{-1}}$$

so that ϕ^f with $f := |\Delta|$ is an $L_0 \otimes_{\mathbb{Q}_p} K$-linear automorphism of M which respects the above decomposition. We see that $(M_1, \phi^f|M_1)$ is a K-isocrystal with $dim_K\, M_1 = rank_{L_0 \otimes_{\mathbb{Q}_p} K} M$.

Lemma 5.3: *The functor*

$$\begin{array}{ccc} \textit{category of isocrystals over } L & \xrightarrow{\sim} & \textit{category of } K\textit{-isocrystals} \\ \textit{with coefficients in } K & & \\ (M, \phi) & \longmapsto & (M_1, \phi^f|M_1) \end{array}$$

is an equivalence of categories.

Proof: Let $\mathcal{I}$ denote the functor in question. To define a functor $\mathcal{J}$ in the opposite direction let (D, φ) be a K-isocrystal. We put $M := L_0 \otimes_{\mathbb{Q}_p} D$ and $\phi := (\sigma \otimes 1) \circ \phi'$ with

$$\phi'|M_\tau := \begin{cases} \varphi & \text{if } \tau = 1, \\ id & \text{otherwise.} \end{cases}$$

Here we have used the K-linear composed isomorphism

$$D \longrightarrow L_0 \otimes_{\mathbb{Q}_p} D = M \overset{pr}{\twoheadrightarrow} M_1$$

to transport φ from D to M_1. At the same time it provides a natural isomorphism $id \simeq \mathcal{I} \circ \mathcal{J}$. The opposite natural isomorphism $id \simeq \mathcal{J} \circ \mathcal{I}$ is given by the composed maps

$$M_{\sigma^i} \xrightarrow{\phi^i} M_1 \xrightarrow{\cong} (L_0 \otimes_{\mathbb{Q}_p} M_1)_1 \xrightarrow{\sigma^{-i} \otimes \phi^{-f}} (L_0 \otimes_{\mathbb{Q}_p} M_1)_{\sigma^i}$$

for $0 \leq i \leq f-1$.

Suppose now that M_L carries a filtration $Fil^{\cdot} M_L$ making $\underline{M} := (M, \phi, Fil^{\cdot} M_L)$ into a filtered isocrystal over L with coefficients in K. Let

$$M_L = \oplus_\beta M_{L,\beta}$$

where β runs over the $\mathrm{Gal}(\overline{K}/K)$-orbits in $\mathrm{Hom}_{\mathbb{Q}_p}(L, \overline{K})$ be the L-isotypic decomposition of the $L \otimes_{\mathbb{Q}_p} K$-module M_L. The filtration on M_L induces a filtration $Fil^{\cdot} M_{L,\beta}$ on each $M_{L,\beta}$ and by the naturality of the decomposition we have

$$Fil^{\cdot} M_L = \oplus_\beta Fil^{\cdot} M_{L,\beta} \ .$$

Moreover, let β_0 denote the orbit of the inclusion map $L \subseteq K$. Then M_{L,β_0} is the K-subspace of M_L on which L acts through the inclusion $L \subseteq K$. The composite map

$$M_1 \xrightarrow{\subseteq} M \longrightarrow L \otimes_{L_0} M = M_L \xrightarrow{pr} M_{L,\beta_0}$$

is a K-linear isomorphism which we may use to transport the filtration $Fil^{\cdot} M_{L,\beta_0}$ to a filtration $Fil^{\cdot} M_1$ on M_1. In this way we obtain the filtered K-isocrystal $\underline{D} := (M_1, \phi^f | M_1, Fil^{\cdot} M_1)$. Obviously the full original filtration $Fil^{\cdot} M_L$ can be recovered from $Fil^{\cdot} M_1$ if and only if it satisfies

$$(*) \qquad gr^0 M_{L,\beta} = M_{L,\beta} \qquad \text{for any } \beta \neq \beta_0 \ .$$

Let us suppose that the condition $(*)$ is satisfied. Since gr^0, by definition, does not contribute to the number $t_H(.)$ we obviously have

$$t_H(\underline{M}) = [K : L] \cdot t_H(\underline{D}) \ .$$

On the other hand, using a normal basis of L_0 over $\mathbb{Q}_p$ as well as the inverse functor in the proof of Lemma 5.3, we compute

$$\begin{aligned} t_N(\underline{M}) &= [K : L] \cdot \mathrm{val}_L(\det_K(\phi)) \\ &= [K : L] \cdot \mathrm{val}_L(\det_K((\sigma \otimes 1) \circ (\phi^f | M_1 \oplus id_{M_\sigma} \oplus \ldots \oplus id_{M_{\sigma^{f-1}}}))) \\ &= [K : L] \cdot \mathrm{val}_L(\det_K(\phi^f | M_1)) \\ &= [K : L] \cdot t_N^L(\underline{D}) \ . \end{aligned}$$

With $\underline{M}$ any of its subobjects also satisfies the condition $(*)$. Moreover, by Lemma 5.3, the subobjects of $\underline{M}$ are in one to one correspondence with the subobjects of $\underline{D}$). It follows that $\underline{M}$ is weakly admissible if and only if $\underline{D}$ is weakly L-admissible. Hence we have the induced equivalence of categories

$$\begin{array}{c} \text{category of weakly admissible} \\ \text{filtered isocrystals over } L \text{ with} \\ \text{coefficients in } K \text{ satisfying } (*) \end{array} \xrightarrow{\sim} \begin{array}{c} \text{category of weakly} \\ L\text{-admissible filtered} \\ K\text{-isocrystals.} \end{array}$$

Suppose now that $\underline{M} = D_{cris}(V)$ of some K-linear crystalline representation of $\mathrm{Gal}(\overline{L}/L)$. By the general theory of crystalline Galois representations we have the comparison isomorphism

$$ker(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \longrightarrow\!\!\!\!\!\rightarrow \mathbb{C}_p \otimes_L V) \cong \underset{i \in \mathbf{Z}}{\oplus} \big(\mathbb{C}_p(-i) \otimes_L (\underset{\beta \neq \beta_0}{\oplus} gr^i M_{L,\beta})\big) \ .$$

It is Galois equivariant with $\mathrm{Gal}(\overline{L}/L)$ acting diagonally on the left and through the first factors on the right. For the Galois invariants we therefore obtain the formula

$$ker(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \longrightarrow\!\!\!\!\!\rightarrow \mathbb{C}_p \otimes_L V)^{\mathrm{Gal}(\overline{L}/L)} \cong \underset{\beta \neq \beta_0}{\oplus} gr^0 M_{L,\beta} \ .$$

It follows that the isocrystal $D_{cris}(V)$ satisfies the condition $(*)$ if and only if the crystalline Galois representation V is special. Altogether we obtain that the functor $V \longmapsto D_{cris}(V)_1$ induces an equivalence of categories

$$\begin{array}{c} \text{category of } K\text{-linear special} \\ \text{crystalline representations} \\ \text{of } \mathrm{Gal}(\overline{L}/L) \end{array} \xrightarrow{\sim} \begin{array}{c} \text{category of weakly} \\ L\text{-admissible filtered} \\ K\text{-isocrystals.} \end{array}$$

It satisfies

$$dim_K V = dim_K D_{cris}(V)_1 \ .$$

Finally suppose that V is a

$$(+) \qquad \begin{array}{c} (d+1)\text{-dimensional } K\text{-linear } K\text{-split special crystalline} \\ \text{representation of } \mathrm{Gal}(\overline{L}/L) \text{ all of whose Hodge-Tate coweights have} \\ \text{multiplicity one and increase at least by } [L:\mathbb{Q}_p] \text{ in each step.} \end{array}$$

Precisely in this situation there is a dominant integral $\xi = (a_1, \ldots, a_{d+1})$ such that the Hodge-Tate coweights of V are $\xi_L + \widetilde{\eta}_L$. By Prop. 5.2 we find an up to permutation unique point $\zeta \in T'_\xi(K)$ such that $\zeta \in s(D_{cris}(V))$. This means we have constructed a surjection

$$\text{set of isomorphism classes of } V\text{'s with } (+) \longrightarrow\!\!\!\!\!\rightarrow \dot{\bigcup_{\xi \ dominant}} W \backslash T'_\xi(K) \ .$$

Let us again fix a dominant $\xi = (a_1, \ldots, a_{d+1})$ and let ρ_ξ denote the irreducible rational representation of G of highest weight ξ. By Prop. 2.4 and Cor. 3.6 we have an identification

$$W\backslash T'_\xi(K) \subseteq (W\backslash T'_\xi)(K) \simeq Sp(\mathcal{B}(G, \rho_\xi | U_0))(K)$$

where $Sp(\mathcal{B}(G, \rho|U_0))(K)$ the space of K-rational points of the affinoid variety $\mathcal{B}(G, \rho_\xi|U_0)$, i.e., the space of K-valued characters of the Banach-Hecke algebra $\mathcal{B}(G, \rho_\xi|U_0)$. Our map therefore becomes a map

$$\begin{array}{c} \text{set of isomorphism classes of} \\ (d+1)\text{-dimensional } K\text{-linear } K\text{-split} \\ \text{special crystalline representations of} \\ \mathrm{Gal}(\overline{L}/L) \text{ with Hodge-Tate coweights} \\ (a_1, a_2 + [L:\mathbb{Q}_p], \ldots, a_{d+1} + d[L:\mathbb{Q}_p]) \end{array} \longrightarrow Sp(\mathcal{B}(G, \rho_\xi|U_0))(K)$$

which we write as $V \longmapsto \zeta(V)$. We point out that in this form our map is canonical in the sense that it does not depend on the choice of the prime element π_L: This choice entered into our normalization of the Satake map S_ξ and into the coordinates on T' which we used; it is easy to check that the two cancel each other out. We also note that in the limit with respect to K this map is surjective.

We finish this section with a speculation in which way the map which we have constructed above might be an approximation of a true p-adic local Langlands correspondence. We view a point $\zeta \in Sp(\mathcal{B}(G, \rho_\xi|U_0))(K)$ as a character $\zeta : \mathcal{B}(G, \rho_\xi|U_0) \longrightarrow K$. Correspondingly we let K_ζ denote the one dimensional K-vector space on which $\mathcal{B}(G, \rho_\xi|U_0)$ acts through the character ζ. We may "specialize" the "universal" Banach $\mathcal{B}(G, \rho_\xi|U_0)$-module $B^G_{U_0}(\rho_\xi|U_0)$ from section 1 to ζ by forming the completed tensor product

$$B_{\xi,\zeta} := K_\zeta \mathbin{\widehat{\otimes}}_{\mathcal{B}(G,\rho_\xi|U_0)} B^G_{U_0}(\rho_\xi|U_0) \ .$$

By construction the K-Banach space $B_{\xi,\zeta}$ still carries a continuous and isometric (for the quotient norm) action of G. A future p-adic local Langlands correspondence should provide us with a distinguished correspondence (being essentially bijective) between the fiber of our map in ζ (i.e., all V of the kind under consideration such that $\zeta(V) = \zeta$) and the isomorphism classes of all topologically irreducible "quotient" representations of $B_{\xi,\zeta}$. Unfortunately it is not even clear that the Banach spaces $B_{\xi,\zeta}$ are nonzero.

In order to describe the existing evidence for this picture we first have to recall how the characters of the Hecke algebra $\mathcal{H}(G, 1_{U_0})$ can be visualized representation theoretically. Any element $\zeta \in T'(K)$ can be viewed as a character $\zeta : T \to \Lambda \to K^\times$, and correspondingly we may form the unramified principal series representation

$$\mathrm{Ind}^G_P(\zeta)^\infty := \begin{array}{c} \text{space of all locally constant functions } F : G \longrightarrow K \text{ such that} \\ F(gtn) = \zeta(t)^{-1} F(g) \text{ for any } g \in G, t \in T, n \in N \end{array}$$

of G. The latter is a smooth G-representation of finite length. By the Iwasawa decomposition $G = U_0 P$ the subspace of U_0-invariant elements in $\mathrm{Ind}_P^G(\zeta)^\infty$ is one dimensional so that the action of $\mathcal{H}(G, 1_{U_0})$ on it is given by a character ω_ζ. On the other hand ζ defines in an obvious way a character of the algebra $K[\Lambda]$ which we also denote by ζ. Using the Satake isomorphism from section 3 one then has (cf. [Ka1] Lemma 2.4(i))

$$\omega_\zeta = \zeta \circ S_1 = (\zeta \cdot \pi_L^{-\mathrm{val}_L(\xi(.))}) \circ S_\xi \ .$$

By [Ka1] Thm. 2.7 the "specialization" in ω_ζ

$$H_{1,\zeta} := K_{\omega_\zeta} \otimes_{\mathcal{H}(G,1_{U_0})} \mathrm{ind}_{U_0}^G(1_{U_0}) \ .$$

of the "universal" $\mathcal{H}(G, 1_{U_0})$-module $\mathrm{ind}_{U_0}^G(1_{U_0})$ from section 1 is an admissible smooth G-representation. Since it also is visibly finitely generated it is, in fact, of finite length. Since $\mathrm{ind}_{U_0}^G(1_{U_0})$ as a G-representation is generated by its U_0-fixed vectors the same must hold true for any of its quotient representations, in particular for any quotient of $H_{1,\zeta}$. But the subspace of U_0-invariant vectors in $H_{1,\zeta}$ is one dimensional. It follows that $H_{1,\zeta}$ possesses a single irreducible quotient representation $V_{1,\zeta}$ – the so called spherical representation for ζ. One has the G-equivariant map

$$\begin{array}{rcl} H_{1,\zeta} & \longrightarrow & \mathrm{Ind}_P^G(\zeta)^\infty \\ 1 \otimes f & \longmapsto & f * 1_\zeta := \sum_{g \in G/U_0} f(g) g(1_\zeta) \end{array}$$

where $1_\zeta \in \mathrm{Ind}_P^G(\zeta)^\infty$ denotes the unique U_0-invariant function with value one in $1 \in G$. Hence V_ζ can also be viewed as the, up to isomorphism, unique irreducible constituent of $\mathrm{Ind}_P^G(\zeta)^\infty$ with a nonzero U_0-fixed vector.

Bringing in again the dominant integral weight ξ we have the K-linear isomorphism

$$\begin{array}{rcl} \mathrm{ind}_{U_0}^G(1_{U_0}) \otimes_K \rho_\xi & \xrightarrow{\cong} & \mathrm{ind}_{U_0}^G(\rho_\xi | U_0) \\ f \otimes x & \longmapsto & f_x(g) := f(g) g^{-1} x \ . \end{array}$$

It is G-equivariant if, on the left hand side, we let G act diagonally. On the left, resp. right, hand side we also have the action of the Hecke algebra $\mathcal{H}(G, 1_{U_0})$ through the first factor, resp. the action of the Hecke algebra $\mathcal{H}(G, \rho_\xi | U_0)$. Relative to the isomorphism ι_{ρ_ξ} between these two algebras discussed in section 1 the above map is equivariant for these Hecke algebra actions as well. (Warning: But this map does not respect our norms on both sides.) By abuse of notation we will use the same symbol to denote characters of these two Hecke algebras which correspond to each other under the isomorphism ι_{ρ_ξ}. We obtain an induced G-equivariant isomorphism

$$H_{1,\zeta} \otimes_K \rho_\xi \xrightarrow{\cong} H_{\xi,\zeta} := K_{\omega_\zeta} \otimes_{\mathcal{H}(G,\rho_\xi|U_0)} \mathrm{ind}_{U_0}^G(\rho_\xi | U_0)$$

between "specializations". Since with $V_{1,\zeta}$ also

$$V_{\xi,\zeta} := V_{1,\zeta} \otimes_K \rho_\xi$$

is irreducible as a G-representation ([ST1] Prop. 3.4) we see that $V_{\xi,\zeta}$ is the unique irreducible quotient of $H_{\xi,\zeta}$ and is also the, up to isomorphism, unique irreducible constituent of $\mathrm{Ind}_P^G(\zeta)^\infty \otimes_K \rho_\xi$ which as a U_0-representation contains $\rho_\xi|U_0$.

Assuming once more that $\zeta \in T'_\xi(K)$ we, of course, have that

$$B_{\xi,\zeta} = \text{Hausdorff completion of } H_{\xi,\zeta}$$

with respect to the quotient seminorm from $\mathrm{ind}_{U_0}^G(\rho_\xi|U_0)$. We remark that the unit ball in $\mathrm{ind}_{U_0}^G(\rho_\xi|U_0)$ and a fortiori its image in $H_{\xi,\zeta}$ are finitely generated over the group ring $o_K[G]$. Hence in order to prove that the quotient topology on $H_{\xi,\zeta}$ is Hausdorff, i.e., that the canonical map $H_{\xi,\zeta} \longrightarrow B_{\xi,\zeta}$ is injective it suffices to exhibit some bounded open G-invariant o_K-submodule in $H_{\xi,\zeta}$.

Example 1: Let $G = GL_2(\mathbb{Q}_p)$, $\xi = (a_1, a_2)$ a dominant weight, and $\zeta = (\zeta_1, \zeta_2) \in (K^\times)^2$. By the discussion at the end of section 2 the defining conditions for the affinoid domain T'_ξ are

$$|\zeta_i| \le |p|^{a_1} \quad \text{for } i = 1, 2 \quad \text{and} \quad |\zeta_1\zeta_2| = |p|^{a_1+a_2+1} .$$

The complete list of the weakly $\mathbb{Q}_p$-admissible filtered K-isocrystals with a Frobenius φ whose semisimple part is given by ζ is well known (cf. [BB] end of section 3.1): Up to conjugation we may assume that $|\zeta_1| \ge |\zeta_2|$.
Case 1: $|\zeta_1| = |p|^{a_1}$ and $|\zeta_2| = |p|^{a_2+1}$; then φ is semisimple, and there are (up to isomorphism) exactly two weakly $\mathbb{Q}_p$-admissible filtrations; one corresponds to a decomposable and the other to a reducible but indecomposable Galois representation.
Case 2: $\zeta_1 \ne \zeta_2$ with $|\zeta_i| < |p|^{a_1}$ for $i = 1, 2$; then φ is semisimple, and there is (up to isomorphism) exactly one weakly $\mathbb{Q}_p$-admissible filtration; it corresponds to an irreducible Galois representation.
Case 3: $\zeta_1 = \zeta_2$ with $|\zeta_i| < |p|^{a_1}$; then φ is not semisimple, and there is (up to isomorphism) exactly one weakly $\mathbb{Q}_p$-admissible filtration; it corresponds to an irreducible Galois representation.
In particular, the fiber of our above surjection consists of two elements in case 1 and of one element in cases 2 and 3.

On the other hand for $|\zeta_1| \ge |\zeta_2|$ the map $H_{\xi,\zeta} \xrightarrow{\cong} \mathrm{Ind}_P^G(\zeta)^\infty \otimes_K \rho_\xi$ always is an isomorphism. It therefore follows from [BB] Thm. 4.3.1 that our $B_{\xi,\zeta}$ coincides in Case 2 with the representation denoted by $\Pi(V)$ in loc. cit. Moreover, still in Case 2, by [BB] Cor.s 5.4.1/2/3 the representation of G in the Banach space $B_{\xi,\zeta}$ is topologically irreducible (in particular nonzero) and admissible in the sense of [ST2] §3. In Case 3 the same assertions are shown in [Bre] Thm. 1.3.3 under the restriction that $a_2 - a_1 < 2p - 1$ and $a_1 + a_2 \neq -3$ if $p \neq 2$, resp. $a_2 - a_1 < 2$ and $a_1 + a_2 \neq -1$ if $p = 2$.

We mention that in contrast to $B_{\xi,\zeta}$ the representation $\mathrm{Ind}_P^G(\zeta)^\infty \otimes_K \rho_\xi$ (or equivalently $\mathrm{Ind}_P^G(\zeta)^\infty$) is irreducible if and only if $\zeta_2 \neq p\zeta_1$. Hence reducibility can only occur for $a_1 = a_2$ in Case 1 and for $a_1 < a_2$ in Case 2.

It was Breuil's fundamental idea that the two dimensional crystalline Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ with distinct Hodge-Tate weights should correspond to the Banach representations $B_{\xi,\zeta}$ of $GL_2(\mathbb{Q}_p)$. Our general speculation therefore should be seen as an attempt to extend his picture. But we warn the reader that the case of GL_2 is atypical insofar as in general, given a pair (ξ, ζ), there will be infinitely many possibilities for a weakly admissible filtration.

Example 2. The unit ball $\mathrm{ind}_{U_0}^G(1_{U_0})^0$ in the normed space $\mathrm{ind}_{U_0}^G(1_{U_0})$ is a module for the unit ball $\mathcal{H}(G, 1_{U_0})^0$ in the Hecke algebra $\mathcal{H}(G, 1_{U_0})$ (for the sup-norm in both cases). For the two groups $G = GL_2(L)$ and $G = GL_3(L)$ it is known that $\mathrm{ind}_{U_0}^G(1_{U_0})^0$ is free as an $\mathcal{H}(G, 1_{U_0})^0$-module. For $G = GL_2(L)$ this is a rather elementary computation on the tree and for $G = GL_3(L)$ it is the main result in [BO] Thm. 3.2.4 (see also the paragraph after Thm. 1.5; we point out that the arguments in this paper actually prove freeness and not only flatness). Let $\{b_j\}_{j\in\mathbb{N}}$ be a basis. Then $\{1 \otimes b_j\}_j$ is a basis of $H_{1,\zeta}$ as a K-vector space, and $\sum_j o_K \cdot (1 \otimes b_j)$ is open in $H_{1,\zeta}$ for the quotient topology provided $\zeta \in T_1'(K)$. Hence the quotient topology on $H_{1,\zeta}$ is Hausdorff which means that the natural map $H_{1,\zeta} \longrightarrow B_{1,\zeta}$ is injective. In particular, $B_{1,\zeta}$ is nonzero.

Example 3: Let $G = GL_{d+1}(L)$ be general but assume that $\zeta \in \mathrm{Hom}(\Lambda, o_K^\times) \subseteq T'(K)$. Then, for any element $F \in \mathrm{Ind}_P^G(\zeta)^\infty$ the function $|F|(g) := |F(g)|$ is right P-invariant. Since G/P is compact we therefore may equip $\mathrm{Ind}_P^G(\zeta)^\infty$ with the G-invariant norm

$$\|F\| := \sup_{g\in G} |F|(g) \ .$$

Moreover, our above map

$$\mathrm{ind}_{U_0}^G(1_{U_0}) \longrightarrow H_{1,\zeta} \longrightarrow \mathrm{Ind}_P^G(\zeta)^\infty$$

then is continuous. Assuming in addition that $\zeta \in T_\xi'(K)$ we obtain by completion a G-equivariant continuous K-linear map

$$B_{1,\zeta} \longrightarrow \mathrm{Ind}_P^G(\zeta)^c \ .$$

The completion $\mathrm{Ind}_P^G(\zeta)^c$ of $\mathrm{Ind}_P^G(\zeta)^\infty$ is explicitly given by

$$\begin{aligned} \mathrm{Ind}_P^G(\zeta)^c := \quad & \text{space of all continuous functions } F : G \longrightarrow K \text{ such that} \\ & F(gtn) = \zeta(t)^{-1} F(g) \text{ for any } g \in G, t \in T, n \in N \end{aligned}$$

It is easy to show that $\mathrm{Ind}_P^G(\zeta)^c$ as a representation of G in a K-Banach space is admissible.

Conjecture: *If ζ is regular then the representation of G in the Banach space $\mathrm{Ind}_P^G(\zeta)^c$ is topologically irreducible.*

Suppose therefore that ζ is regular, i.e., is not fixed by any $1 \neq w \in W$ for the conjugation action of W on T'). It is then well known that:
– The smooth G-representation $\mathrm{Ind}_P^G(\zeta)^\infty$ is irreducible (for example by the Bernstein-Zelevinsky classification).
– The above map $H_{1,\zeta} \xrightarrow{\cong} \mathrm{Ind}_P^G(\zeta)^\infty$ is an isomorphism ([Ka1] Thm. 3.2 and Remark 3.3 or [Dat] Lemma 3.1).
The latter in particular implies that the quotient topology on $H_{1,\zeta}$ is Hausdorff and that the map $B_{1,\zeta} \longrightarrow \mathrm{Ind}_P^G(\zeta)^c$ has dense image. In this context we also mention, without proof, the following result.

Proposition 5.4: *For any two $\zeta, \zeta' \in \mathrm{Hom}(\Lambda, o_K^\times)$ the vector space of all G-equivariant continuous linear maps from $\mathrm{Ind}_P^G(\zeta)^c$ to $\mathrm{Ind}_P^G(\zeta')^c$ is zero if $\zeta \neq \zeta'$ and is $K \cdot id$ if $\zeta = \zeta'$.*

For $G = GL_2(\mathbb{Q}_p)$ the above conjecture follows from a combination of [ST1] §4 and [ST3] Thm. 7.1. If $\zeta = \zeta_{(1)} \otimes \ldots \otimes \zeta_{(d+1)}$ with unramified characters $\zeta_{(i)} : L^\times / o_L^\times \longrightarrow o_K^\times$ such that $\zeta_{(i)} \not\equiv \zeta_{(i+1)} \bmod \pi_K$ for any $1 \leq i \leq d$ then the above conjecture is a consequence of an irreducibility result in characteristic p in the thesis of R. Ollivier.

6. Weakly admissible pairs and functoriality

In the traditional Langlands program the irreducible smooth representations of a general group G over L are put into correspondence with continuous homomorphisms from the Galois group $\mathrm{Gal}(\overline{L}/L)$ (or rather the Weil-Deligne group of L) into the Langlands dual group G' of G. In order to do something in this spirit in our setting it is useful to slightly change our point of view which we motivate by looking once again at the GL_{d+1}-case. We started from a dominant weight $\xi \in X^*(T)$ and an element $\zeta \in T'(K)$ in the dual torus. Viewing ζ, by our particular choice of coordinates, as a diagonal matrix ζ_c in $G'(K) = GL_{d+1}(K)$ we considered the K-isocrystals (K^{d+1}, φ) such that ζ_c lies in the conjugacy class of the semisimple part of φ. The weight ξ was used to prescribe the type of the filtration which would make these isocrystals into filtered isocrystals. Our basic result then was that among all these filtered K-isocrystals there is at least one weakly L-admissible one if and only if $\zeta \in T'_\xi(K)$. Now we observe that ξ actually can be used to define a model filtration on K^{d+1}. Quite generally, for any K-rational cocharacter $\nu : \mathbb{G}_m \longrightarrow G'$ we decompose K^{d+1} into weight spaces

$$K^{d+1} = \oplus_{i \in \mathbb{Z}} (K^{d+1})_i$$

with respect to ν and put

$$Fil^i_\nu K^{d+1} := \oplus_{j\geq i}(K^{d+1})_j \ .$$

Because of $X^*(T) = X_*(T') \subseteq X_*(G')$ this in particular applies to $\xi\widetilde{\eta}_L$. Of course, the filtration $Fil^{\cdot}_{\xi\tilde{\eta}_L} K^{d+1}$ has no reason to be weakly L-admissible. But any other filtration of the same type as $Fil^{\cdot}_{\xi\tilde{\eta}_L} K^{d+1}$ is of the form $gFil^{\cdot}_{\xi\tilde{\eta}_L} K^{d+1} = Fil^{\cdot}_{g(\xi\tilde{\eta}_L)} K^{d+1}$ for some $g \in G'(K)$. Hence we may express our basic result also by saying that, given the pair (ξ,ζ), there is a pair $(\nu,\varphi) \in X_*(G')(K) \times G'(K)$ such that
– ν lies in the $G'(K)$-orbit of $\xi\widetilde{\eta}_L$,
– the semisimple part of φ is conjugate to ζ_c in $G'(K)$, and
– the filtered K-isocrystal $(K^{d+1},\varphi,Fil^{\cdot}_\nu K^{d+1})$ is weakly L-admissible
if and only if $\zeta \in T'_\xi(K)$.

Let now G be again a general L-split reductive group. We denote by G' its Langlands dual group which we consider to be defined over L as well (cf. [Bor]). In particular, T' is a maximal L-split torus in G'. We view our dominant $\xi \in X^*(T) = X_*(T') \subseteq X_*(G')$, as above, as a K-rational cocharacter $\xi : \mathbb{G}_m \longrightarrow G'$ and $\zeta \in T'(K) \subseteq G'(K)$. For a general pair $(\nu,b) \in X_*(G')(K) \times G'(K)$ we introduce some constructions and terminology which is borrowed from [RZ] Chap. 1. Let $REP_K(G')$ denote the category of K-rational representations of G' and let FIC_K denote the category of filtered K-isocrystals. Both are additive tensor categories. The pair (ν,b) gives rise to the tensor functor

$$\begin{array}{rcl} I_{(\nu,b)} : REP_K(G') & \longrightarrow & FIC_K \\ (\rho,E) & \longmapsto & (E,\rho(b),Fil^{\cdot}_{\rho\circ\nu} E) \ . \end{array}$$

Definition: *The pair (ν,b) is called weakly L-admissible if the filtered K-isocrystal $I_{(\nu,b)}(\rho,E)$, for any (ρ,E) in $REP_K(G')$, is weakly L-admissible.*

Suppose that (ν,b) is weakly L-admissible. Then $I_{(\nu,b)}$ can be viewed as a functor

$$I_{(\nu,b)} : REP_K(G') \longrightarrow FIC_K^{L-adm}$$

into the full subcategory FIC_K^{L-adm} of weakly L-admissible filtered K-isocrystals which, in fact, is a Tannakian category (the shortest argument for this probably is to observe that for a Galois representation the property of being special crystalline is preserved by tensor products and to use the Colmez-Fontaine equivalence of categories). Moreover, letting $Rep_K^{con}(\mathrm{Gal}(\overline{L}/L))$ denote the category of finite dimensional K-linear continuous representations of $\mathrm{Gal}(\overline{L}/L)$ we know from the last section that the inverse of the functor $D_{cris}(.)_1$ induces a tensor functor between neutral Tannakian categories

$$FIC_K^{L-adm} \longrightarrow Rep_K^{con}(\mathrm{Gal}(\overline{L}/L)) \ .$$

By composing these two functors we therefore obtain a faithful tensor functor

$$\Gamma_{(\nu,b)} : REP_K(G') \longrightarrow Rep_K^{con}(\mathrm{Gal}(\overline{L}/L))$$

which possibly is no longer compatible with the obvious fiber functors. This is measured by a G'-torsor over K ([DM] Thm. 3.2). By Steinberg's theorem ([Ste] Thm. 1.9) that $H^1(K^{nr}, G') = 0$ over the maximal unramified extension K^{nr} of K this torsor is trivial over K^{nr}. It follows then from the general formalism of neutral Tannakian categories ([DM] Cor. 2.9, Prop. 1.13) that the functor $\Gamma_{(\nu,b)}$ gives rise to a K^{nr}-homomorphism in the opposite direction between the affine group schemes of the two categories which is unique up to conjugation in the target group. For $REP_K(G')$ this affine group scheme of course is G' ([DM] Prop. 2.8). For $Rep_K^{con}(\mathrm{Gal}(\overline{L}/L))$ we at least have that the K-rational points of this affine group scheme naturally contain the Galois group $\mathrm{Gal}(\overline{L}/L)$. Hence by restriction we obtain a continuous homomorphism of groups

$$\gamma_{\nu,b} : \mathrm{Gal}(\overline{L}/L) \longrightarrow G'(K^{nr})$$

which is determined by the functor $\Gamma_{(\nu,b)}$ up to conjugation in $G'(K^{nr})$. So we see that any weakly L-admissible pair (ν, b) determines an isomorphism class of "Galois parameters" $\gamma_{\nu,b}$. We remark that if the derived group of G' is simply connected Kneser ([Kne]) showed that $H^1(K, G') = 0$ so that in this case the Galois parameter $\gamma_{\nu,b}$ already has values in $G'(K)$. Following [RZ] p. 14 and [Win] one probably can establish an explicit formula for the cohomology class in $H^1(K, G')$ of the torsor in question.

We indicated already earlier that Langlands functoriality (for smooth representations) requires to work with the normalized Satake isomorphism S^{norm}. This forces us to assume in this section that our coefficient field K contains a square root of q and to pick one once and for all. As a consequence we also have a preferred square root $\delta^{1/2} \in T'(K)$ of $\delta \in T'(K)$. Being able to work with the normalized Satake map we do not have to consider the twisted W-action on $K[\Lambda]$. But, of course, we still have a norm in the picture which depends on ξ and which is the following. We consider the automorphism of K-algebras

$$\begin{array}{rrcl} a_\xi : & K[\Lambda] & \longrightarrow & K[\Lambda] \\ & \lambda = \lambda(t) & \longmapsto & \delta^{1/2}(\lambda)\pi_L^{\mathrm{val}_L(\xi(t))}\lambda \end{array}$$

which intertwines the conjugation action by W on the source with the twisted action on the target. Pulling back along a_ξ the norm $\| \ \|_{\gamma_\xi}$ gives the norm

$$\| \sum_{\lambda\in\Lambda} c_\lambda \lambda \|_\xi^{norm} := \sup_{\lambda=\lambda(t)} |\delta^{1/2}({}^w\lambda)\pi_L^{\mathrm{val}_L(\xi({}^w t))} c_\lambda|$$

on $K[\Lambda]$ with $w \in W$ for each λ being chosen in such a way that ${}^w\lambda \in \Lambda^{--}$. Let $K\langle\Lambda;\xi\rangle$ denote the corresponding Banach algebra completion of $K[\Lambda]$. It follows from Prop. 2.4 that $K\langle\Lambda;\xi\rangle$ is the affinoid algebra of the affinoid subdomain $T'_{\xi,norm}$ obtained by pulling back T'_ξ along a_ξ. Since a_ξ induces on T' the map $\zeta \longmapsto \delta^{1/2}\pi_L^{\mathrm{val}_L \circ \xi}\zeta$ we deduce from Lemma 2.7 that

$$T'_{\xi,norm}(K) = val^{-1}(V_{\mathbb{R}}^{\xi,norm})$$

with

$$V_{\mathbb{R}}^{\xi,norm} := \{z \in V_{\mathbb{R}} : z^{dom} \leq \eta_L + \xi_L\} \ .$$

We have the commutative diagram

$$\begin{tikzcd}
& \mathcal{B}(G,\rho_\xi|U_0) \arrow[d] & \\
& \|\ \|_\xi\text{-completion of } \mathcal{H}(G,1_{U_0}) \arrow[dl, "S^{norm}"'] \arrow[dr, "S_\xi"] & \\
K\langle\Lambda;\xi\rangle^W \arrow[rr, "a_\xi"] & & K\langle\Lambda;\gamma_\xi\rangle^{W,\gamma_\xi}
\end{tikzcd}$$

in which, as a consequence of Lemma 3.3 and Prop. 3.5, all maps are isomorphisms of Banach algebras. In this section we use the left hand sequence of arrows to identify $\mathcal{B}(G,\rho_\xi|U_0)$ with the algebra of analytic functions on the affinoid space $W\backslash T'_{\xi,norm}$. In particular, this identifies $(W\backslash T'_{\xi,norm})(K)$ with the set of K-valued (continuous) characters of the Banach-Hecke algebra $\mathcal{B}(G,\rho_\xi|U_0)$.

Remark: *Using that $\delta(\lambda_{\{i\}}) = q^{-d+2(i-1)}$ the statement of Prop. 5.2 for the group $G = GL_{d+1}(L)$ becomes: $\zeta \in T'_{\xi,norm}(K)$ if and only if there is a weakly L-admissible filtered K-isocrystal $\underline{D}$ such that type$(\underline{D}) = \xi_L + \widetilde{\eta}_L$ and the semisimple part of its Frobenius is given by the diagonal matrix with entries $q^{d/2}\zeta(\lambda_{\{i\}})$.*

We note that in the case where η_L happens to be integral (i.e., if $d[L:\mathbb{Q}_p]$ is even) we can go one step further, can remove completely normalizations accidental to the group $GL_{d+1}(L)$, and can restate the above remark equivalently as follows. We have $\zeta \in T'_{\xi,norm}(K)$ if and only if there is a weakly L-admissible filtered K-isocrystal $\underline{D}$ such that *type*$(\underline{D}) = \xi_L + \eta_L$ and the semisimple part of its Frobenius is given by the diagonal matrix with entries $\zeta(\lambda_{\{i\}})$. Passing now to a general G this unfortunately forces us at present to work under the technical hypothesis that $\eta_L \in X^*(T) = X_*(T')$. This, for example, is the case if $[L:\mathbb{Q}_p]$ is even or if the group G is semisimple and simply connected. To emphasize that η_L then will be considered primarily as a rational cocharacter of T' we will use multiplicative notation and write $\xi\eta_L$ for the product of the rational cocharacters ξ and η_L. In this setting and for general G the analog of Prop. 5.2 is the following.

Proposition 6.1: *Suppose that η_L is integral, let $\xi \in X^*(T)$ be dominant, and let $\zeta \in T'(K)$; then there exists a weakly L-admissible pair (ν, b) (and hence a Galois parameter $\gamma_{\nu,b}$) such that ν lies in the $G'(K)$-orbit of $\xi\eta_L$ and b has semisimple part ζ if and only if $\zeta \in T'_{\xi,norm}(K)$.*

Proof: First let (ν, b) be a weakly L-admissible pair as in the assertion. Further let $\rho : G' \longrightarrow GL(E)$ be any K-rational representation. We then have the weakly L-admissible filtered K-isocrystal $(E, \rho(b), Fil^{\cdot}_{\rho\circ\nu} E)$. Furthermore $\rho\circ\nu$ is conjugate to $\rho \circ (\xi\eta_L)$ in $GL(E)(K)$ and $\rho(\zeta)$ is the semisimple part of $\rho(b)$. We fix a K-rational Borel subgroup $P_E \subseteq GL(E)$ and a maximal K-split torus $T_E \subseteq P_E$ such that $\rho(\zeta) \in T_E(K)$. There is a unique K-rational cocharacter $(\rho \circ \nu)^{dom} : \mathbb{G}_m \longrightarrow T_E$ which is dominant with respect to P_E and which is conjugate to $\rho\circ\nu$ in $GL(E)(K)$. Then $(\rho\circ\nu)^{dom} = (\rho\circ(\xi\eta_L))^{dom}$ corresponds to the type of the filtration $Fil^{\cdot}_{\rho\circ\nu} E$ in the sense of section 5. As in the first part of the proof of Prop. 5.2 we know from [Fon] Prop. 4.3.3 that the weak L-admissibility of our filtered isocrystal implies that the Newton polygon $\mathcal{P}((\rho(val(\zeta)))^{dom})$ lies above the Hodge polygon $\mathcal{P}((\rho \circ (\xi\eta_L))^{dom})$ with both having the same endpoint. But, as discussed before Lemma 5.1, this means that

$$(\rho(val(\zeta)))^{dom} \leq (\rho \circ (\xi\eta_L))^{dom} \ .$$

According to [FR] Lemma 2.1 the latter implies that

$$val(\zeta)^{dom} \leq (\xi\eta_L)^{dom} = \xi\eta_L \quad \text{, i.e., that} \quad \zeta \in T'_{\xi,norm}(K) \ .$$

For the reverse implication we first recall that, given any pair (ν, b) and any K-rational representation $\rho : G' \longrightarrow GL(E)$, the associated filtered K-isocrystal $(E, \rho(b), Fil^{\cdot}_{\rho\circ\nu} E)$ carries the canonical HN-filtration by subobjects (cf. [RZ] Prop. 1.4). The latter is stabilized by a unique parabolic subgroup $P^{\rho}_{(\nu,b)} \subseteq GL(E)$. We obviously have

$$\rho(b) \in P^{\rho}_{(\nu,b)}(K) \ .$$

The HN-filtrations, being functorial, equip our functor $I_{(\nu,b)}$ in fact with the structure of an exact $\otimes$-filtration in the sense of [Saa] IV.2.1.1. The exactness is trivial since the category $REP_K(G')$ is semisimple. The compatibility with the tensor product is a theorem of Faltings and Totaro (independently). It then follows from [Saa] Prop. IV.2.2.5 and Thm. IV.2.4 that

$$P_{(\nu,b)} := \bigcap_{\rho} \rho^{-1}(P^{\rho}_{(\nu,b)})$$

is a K-rational parabolic subgroup of G'. Since [Saa] only considers filtrations indexed by integers this requires the following additional observation. The category $REP_K(G')$ has a generator ([Saa] II.4.3.2) and is semisimple. From this one deduces that the jump indices in the HN-filtrations on all the values of our functor can be written with a common denominator. Hence all these HN-filtrations can be reindexed simultaneously in such a way that they become integral, and [Saa] applies. We emphasize that, denoting by $\mathbb{D}$ the protorus with character group $\mathbb{Q}$, one actually has a (not unique) K-rational homomorphism $\iota_{(\nu,b)} : \mathbb{D} \longrightarrow G'$ whose weight spaces define the HN-filtration on the functor $I_{(\nu,b)}$. Its centralizer in G' is a Levi subgroup of $P_{(\nu,b)}$.

Note that we have

$$b \in P_{(\nu,b)}(K) \ .$$

After these preliminaries we make our choice of the element b.

Lemma 6.2: *There is a regular element $b \in G'(K)$ with semisimple part ζ.*

Proof: Let $M' \subseteq G'$ denote the connected component of the centralizer of ζ in G'. We have:
– M' is connected reductive ([Ste] 2.7.a);
– M' is K-split of the same rank as G' (since $T' \subseteq M'$);
– $\zeta \in T'(K) \subseteq M'(K)$; in fact, ζ lies in the center of M'.
The regular unipotent conjugacy class in M', by its unicity ([Ste] Thm. 3.3), is defined over K. Since M' is K-split it therefore contains a point $u \in M'(K)$ ([Kot] Thm. 4.2). We put $b := \zeta u \in G'(K)$. The centralizer of b in G' contains with finite index the centralizer of u in M'. Hence b is regular in G' with semisimple part ζ.

We now fix $b \in G'(K)$ to be regular with semisimple part ζ.

Lemma 6.3: *There are only finitely many K-rational parabolic subgroups $Q \subseteq G'$ such that $b \in Q(K)$.*

Proof: Obviously it suffices to prove the corresponding statement over the algebraic closure $\overline{K}$ of K. By [Ste] Thm. 1.1 there are only finitely many Borel subgroups $Q_0 \subseteq G'$ such that $b \in Q_0(\overline{K})$. Let $Q \subseteq G'$ be any parabolic subgroup with $b \in Q(\overline{K})$. It suffices to find a Borel subgroup $Q_0 \subseteq Q$ such that $b \in Q_0(\overline{K})$. Consider the Levi quotient $\overline{M}$ of Q and the image $\overline{b} \in \overline{M}(\overline{K})$ of b. Then $\overline{b}$ is contained in some Borel subgroup $\overline{Q}_0 \subseteq \overline{M}$ (cf. [Hu1] Thm. 22.2) and we can take for Q_0 the preimage of $\overline{Q}_0$ in Q.

It follows that with ν varying over the $G'(K)$-orbit $\Xi \subseteq X_*(G')$ of $\xi\eta_L$ the family of parabolic subgroups $P_{(\nu,b)}$ actually is finite. Let $P_1, \ldots, P_m$ denote these finitely many parabolic subgroups and write

$$\Xi = \Xi_1 \cup \ldots \cup \Xi_m \qquad \text{with } \Xi_i := \{\nu \in \Xi : P_{(\nu,b)} = P_i\} \ .$$

We want to show that $\nu \in \Xi$ can be chosen in such a way that $P_{(\nu,b)} = G'$. Because then the homomorphism $\iota_{(\nu,b)} : \mathbb{D} \longrightarrow G'$ factorizes through the center of G'. Since by Schur's lemma the center of G' acts through scalars on any irreducible K-rational representation ρ of G' it follows that the HN-filtration on the filtered isocrystal $(E, \rho(b), Fil^{\cdot}_{\rho\circ\nu} E)$ for irreducible ρ has only one step. On the other hand, our assumption that $\zeta \in T'_{\xi,norm}(K)$ together with [FR] Lemma 2.1 imply that this filtered isocrystal, for any ρ, has HN-slope zero. Hence it is weakly L-admissible, first for irreducible ρ and then by passing to direct sums also for arbitrary ρ. This proves that the pair (ν, b) is weakly L-admissible.

We argue by contradiction and assume that all $P_1, \ldots, P_m \neq G'$ are proper parabolic subgroups. By [FR] Lemma 2.2.i we then find, for any $1 \leq i \leq m$, an irreducible K-rational representation $\rho_i : G' \longrightarrow GL(E_i)$ and a K-line $\ell_i \subseteq E_i$ such that

$$P_i = \text{stabilizer in } G' \text{ of } \ell_i$$

(in particular, $\ell_i \neq E_i$). We claim that $P^{\rho_i}_{(\nu,b)}$, for each $\nu \in \Xi_i$, stabilizes the line ℓ_i. To see this we have to recall the actual construction of ρ_i in loc. cit. Fix a maximal K-split torus T_i in a Levi subgroup M_i of P_i and fix a Borel subgroup $T_i \subseteq B_i \subseteq P_i$. By conjugation we may assume that all the homomorphisms $\iota_{(\nu,b)}$, for $\nu \in \Xi_i$, factorize through the center of M_i. Recall that M_i then is equal to the centralizer of $\iota_{(\nu,b)}$ in G'. Hence we may view these $\iota_{(\nu,b)}$ as elements in $X_*(T_i) \otimes \mathbb{Q}$ which lie in the interior of the facet defined by P_i (the latter follows from [Saa] Prop. IV.2.2.5.1)). Pick on the other hand a B_i-dominant character $\lambda_i \in X^*(T_i)$ which lies in the interior of the facet corresponding to P_i and let ρ_i be the rational representation of highest weight λ_i. Then, according to [FR], the highest weight space $\ell_i \subseteq E_i$ has the required property that P_i is its stabilizer in G'. Let $\lambda \in X^*(T_i)$ be any weight in E_i different from λ_i. Then $\lambda_i - \lambda$ is a nonzero linear combination with nonnegative integral coefficients of B_i-simple roots.

Claim: $(\lambda_i - \lambda)(\iota_{(\nu,b)}) > 0$

Proof: Let $\{\alpha_j : j \in \Delta\} \subseteq X^*(T_i)$ be the set of B_i-simple roots and let $J \subseteq \Delta$ denote the subset corresponding to P_i. The highest weight λ_i then satisfies

$$\lambda_i(\check{\alpha}_j) \begin{cases} = 0 & \text{if } j \in J, \\ > 0 & \text{if } j \notin J \end{cases}$$

where the $\check{\alpha}_j \in X_*(T_i)$ denote the simple coroots. On the other hand the connected center of M_i is equal to $(\bigcap_{j \in J} ker(\alpha_j))^\circ$, and we have

$$\alpha_j(\iota_{(\nu,b)}) \begin{cases} = 0 & \text{if } j \in J, \\ > 0 & \text{if } j \notin J. \end{cases}$$

We may write

$$\lambda_i - \lambda = \sum_{j \in \Delta} c_j \alpha_j \quad \text{with } c_j \in \mathbb{Z}_{\geq 0} \ .$$

Hence

$$(+) \qquad (\lambda_i - \lambda)(\iota_{(\nu,b)}) = \sum_{j \notin J} c_j \alpha_j(\iota_{(\nu,b)}) \geq 0$$

and we have to show that c_j, for at least one $j \notin J$, is nonzero. Let $\lambda' \in X^*(T_i)$ denote the unique dominant element in the orbit of λ under the Weyl group of T_i. Then λ' also is a weight occurring in E_i and we have

$$\lambda_i - \lambda' = \sum_{j \in \Delta} d_j \alpha_j \quad \text{and} \quad \lambda' - \lambda = \sum_{j \in \Delta} e_j \alpha_j \quad \text{with } d_j, e_j \in \mathbb{Z}_{\geq 0} \ .$$

In particular, $d_j + e_j = c_j$. Suppose first that $\lambda_i \neq \lambda'$. Then it suffices to find a $j \notin J$ such that $d_j > 0$. By the proof of [Hum] 13.4 Lemma B we obtain λ' from λ_i by successively subtracting simple roots while remaining inside the weights occurring in E_i in each step. But because of $\lambda_i(\check{\alpha}_j) = 0$ if $j \in J$ we know ([Hum] 21.3) that $\lambda_i - \alpha_j$ cannot be a weight occurring in E_i for any $j \in J$. This means of course that we have to have $d_j > 0$ for some $j \notin J$. Now assume that $\lambda_i = \lambda'$ so that $\lambda = {}^\sigma\lambda_i$ for some σ in the Weyl group of T_i. According to the proof of [Hum] 10.3 Lemma B we obtain λ from λ_i in the following way: Let σ_j be the reflection in the Weyl group corresponding to the simple root α_j. Write $\sigma = \sigma_{j_1} \ldots \sigma_{j_t}$ in reduced form. Then

$$\lambda_i - \lambda = \sum_{1 \leq s \leq t} \sigma_{j_{s+1}} \ldots \sigma_{j_t}(\lambda_i)(\check{\alpha}_{j_s})\alpha_{j_s}$$

with all coefficients being nonnegative integers. Since the σ_j for $j \in J$ fix λ_i we may assume that $j_t \notin J$. Then the last term in the above sum is $\lambda_i(\check{\alpha}_{j_t})\alpha_{j_t}$ whose coefficient is positive.

This claim means that ℓ_i is a full weight space of $\rho_i \circ \iota_{(\nu,b)}$. But it follows from $(+)$ also that the weight of $\mathbb{D}$ on ℓ_i is maximal with respect to the natural order on the character group $\mathbb{Q}$ of $\mathbb{D}$ among all weights of $\mathbb{D}$ occurring in E_i. Hence ℓ_i must be the bottom step in the HN-filtration of the filtered K-isocrystal $\underline{E}_{i,\nu} := (E_i, \rho_i(b), Fil^{\cdot}_{\rho_i \circ \nu} E_i)$ for each $\nu \in \Xi_i$. As such it carries the structure of a subobject $\underline{\ell}_{i,\nu} \subseteq \underline{E}_{i,\nu}$. As noted already, due to $\zeta \in T'_{\xi,norm}$, the HN-slope of $\underline{E}_{i,\nu}$ is zero. By the fundamental property of the HN-filtration (cf. [RZ] Prop. 1.4) the HN-slope of $\underline{\ell}_{i,\nu}$ then must be strictly positive which means that

$$t_H(\underline{\ell}_{i,\nu}) > t_N^L(\underline{\ell}_{i,\nu}) \ .$$

Suppose that we find an $1 \leq i \leq m$ and a $\nu \in \Xi_i$ such that ℓ_i is transversal to the filtration $Fil^{\cdot}_{\rho_i \circ \nu} E_i$. Let $(a_1, \ldots, a_r)$, resp. $(z_1, \ldots, z_r)$, denote the filtration type (in the sense of section 5), resp. the slopes written in increasing order, of the corresponding $\underline{E}_{i,\nu}$. The transversality means that $t_H(\underline{\ell}_{i,\nu}) = a_1$. On the other hand, since ℓ_i is a line we must have $t_N^L(\underline{\ell}_{i,\nu}) = z_j \geq z_1$. But because of

$\zeta \in T'_{\xi,norm}(K)$, once more [FR] Lemma 2.1, and Lemma 5.1 we have $z_1 \geq a_1$ which leads to the contradictory inequality

$$t_H(\underline{\ell}_{i,\nu}) \leq t_N^L(\underline{\ell}_{i,\nu}) \ .$$

It finally remains to justify our choice of ν. Since the filtration $Fil^{\cdot}_{\rho_i\circ\nu} E_i$ is well defined for any $\nu \in \Xi$ (and not only $\nu \in \Xi_i$) it suffices to establish the existence of some $\nu \in \Xi$ such that

$$\ell_i \text{ is transversal to } Fil^{\cdot}_{\rho_i\circ\nu} E_i \text{ for any } 1 \leq i \leq m \ .$$

Let $F_i \subset E_i$ denote the top step of the filtration $Fil^{\cdot}_{\rho_i\circ\xi\eta_L} E_i$. We have to find an element $g \in G'(K)$ such that

$$\rho_i(g)(\ell_i) \not\subseteq F_i \qquad \text{for any } 1 \leq i \leq m \ .$$

For each individual i the set $U_i := \{g \in G' : \rho_i(g)(\ell_i) \not\subseteq F_i\}$ is Zariski open in G'. Since ρ_i is irreducible the set U_i is nonempty. The intersection $U := U_1 \cap \ldots \cap U_m$ therefore still is a nonempty Zariski open subset of G'. But $G'(K)$ is Zariski dense in G' (cf. [Hu1] §34.4). Hence U must contain a K-rational point $g \in U(K)$. Then the cocharacter $\nu := g^{-1}(\xi\eta_L)$ has the properties which we needed.

We summarize that, under the integrality assumption on η_L, any K-valued character of one of our Banach-Hecke algebras $\mathcal{B}(G, \rho_\xi|U_0)$ naturally gives rise to a nonempty set of Galois parameters $\mathrm{Gal}(\overline{L}/L) \longrightarrow G'(\overline{K})$. The need to pass to the algebraic closure $\overline{K}$ comes from two different sources: First the element $\zeta \in T'_{\xi,norm}$ giving rise to a K-valued character of $\mathcal{B}(G, \rho_\xi|U_0)$ in general is defined only over a finite extension of K; secondly, to make Steinberg's theorem applicable we had to pass to the maximal unramified extension. In the spirit of our general speculation at the end of the last section we view this as an approximation to a general p-adic Langlands functoriality principle.

Without the integrality assumption on η_L one can proceed at least half way as follows. Let us fix, more generally, any natural number $r \geq 1$. We introduce the category of r-filtered K-isocrystals $FIC_{K,r}$ whose objects are triples $\underline{D} = (D, \varphi, Fil^{\cdot} D)$ as before only that the filtration $Fil^{\cdot} D$ is allowed to be indexed by $r^{-1}\mathbb{Z}$ (in particular, $FIC_K = FIC_{K,1}$). The invariants $t_H(\underline{D})$ and $t_N^L(\underline{D})$ as well as the notion of weak L-admissibility are defined literally in the same way leading to the full subcategory $FIC_{K,r}^{L-adm}$ of $FIC_{K,r}$.

Proposition 6.4: *$FIC_{K,r}^{L-adm}$ is a K-linear neutral Tannakian category.*

Proof: This follows by standard arguments from [Tot].

The tensor functor

$$\begin{array}{rcl} I_{(\nu,b)} : REP_K(G') & \longrightarrow & FIC_{K,r} \\ (\rho, E) & \longmapsto & (E, \rho(b), Fil^{\cdot}_{\rho\circ\nu} E) \end{array}$$

makes sense for any pair $(\nu, b) \in (X_*(G') \otimes r^{-1}\mathbb{Z})(K) \times G'(K)$ as does the notion of weak L-admissibility of such a pair. With these generalizations Prop. 6.1 continues to hold in complete generality (involving 2-filtered K-isocrystals) with literally the same proof. What is missing at present is the connection between the categories $FIC_{K,r}^{L-adm}$ and $Rep_K^{con}(\mathrm{Gal}(\overline{L}/L))$. This might involve a certain extension of the Galois group $\mathrm{Gal}(\overline{L}/L)$. We hope to come back to this problem in the future.

References

[BO] Bellaiche J., Otwinowska A.: Platitude du module universel pour GL_3 en caractéristique non banale. Bull. SMF 131, 507-525 (2003)

[BB] Berger L., Breuil C.: Représentations cristallines irréductibles de $GL_2(\mathbb{Q}_p)$. Preprint 2005

[Bor] Borel A.: Automorphic L-functions. In Automorphic Forms, Representations and L-Functions. Proc. Symp. Pure Math. 33 (2), pp. 27-61. American Math. Soc. 1979

[BGR] Bosch S., Güntzer U., Remmert R.: Non-Archimedean Analysis. Berlin-Heidelberg-New York: Springer 1984

[B-GAL] Bourbaki, N.: Groupes et algèbres de Lie, Chap. 4-6. Paris: Masson 1981

[Bre] Breuil C.: Invariant $\mathcal{L}$ et série spéciale p-adique. Ann. Sci. ENS 37, 559-610 (2004)

[BM] Breuil C., Mézard A.: Multiplicités modulaires et représentations de $GL_2(\mathbb{Z}_p)$ et de $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ en $\ell = p$. Duke Math. J. 115, 205-310 (2002)

[BT] Bruhat F., Tits J.: Groupes réductifs sur un corps local, I, II. Publ. Math. IHES 41, 5-252 (1972), and 60, 5-184 (1984)

[Car] Cartier P.: Representations of $\mathfrak{p}$-adic groups: a survey. In Automorphic Forms, Representations and L-Functions. Proc. Symp. Pure Math. 33 (1), pp. 111-155. American Math. Soc. 1979

[CF] Colmez P., Fontaine J.-M.: Construction des représentations semi-stable. Invent. math. 140, 1-43 (2000)

[Dat] Dat J.-F.: Caractères à valeurs dans le centre de Bernstein. J. reine angew. Math. 508, 61-83 (1999)

[Del] Deligne P.: Formes modulaires et representations de $GL(2)$. In Modular Functions of One Variable II (Eds. Deligne, Kuyk). Lecture Notes in Math. 349, pp. 55-105. Berlin-Heidelberg-New York: Springer 1973

[DM] Deligne P., Milne J. S.: Tannakian categories. In Hodge Cycles, Motives, and Shimura Varieties (Eds. Deligne, Milne, Ogus, Shih). Lecture Notes in Math. 900, pp. 101-228. Berlin-Heidelberg-New York: Springer 1982

[Em1] Emerton M.: Jacquet modules of locally analytic representations of p-adic reductive groups I: Definitions and first properties. To appear in Ann. Sci. ENS

[Em2] Emerton M.: p-adic L-functions and unitary completions of representations of p-adic reductive groups. Duke Math. J. 130, 353-392 (2005)

[Fon] Fontaine J.-M.: Modules galoisiens, modules filtrés et anneaux Barsotti-Tate. Astérisque 65, 3-80 (1979)

[FR] Fontaine J.-M., Rapoport M.: Existence de filtrations admissibles sur des isocristaux. Bull. SMF 133, 73-86 (2005)

[FvP] Fresnel J., van der Put M.: Rigid Analytic Geometry and Its Applications. Boston-Basel-Berlin: Birkhäuser 2004

[Hai] Haines T.: The combinatorics of Bernstein functions. Trans. AMS 353, 1251-1278 (2001)

[HKP] Haines T., Kottwitz R., Prasad A.: Iwahori-Hecke algebras. Preprint

[Gro] Gross B.H.: On the Satake isomorphism. In Galois Representations in Arithmetic Algebraic Geometry (Eds. Scholl, Taylor), London Math. Soc. Lect. Notes 254, pp. 223-237. Cambridge Univ. Press 1998

[Hum] Humphreys J.E.: Introduction to Lie Algebras and Representation Theory. Berlin-Heidelberg-New York: Springer 1972

[Hu1] Humphreys J.E.: Linear Algebraic Groups. Berlin-Heidelberg-New York: Springer 1987

[Hu2] Humphreys J.E.: Reflection groups and Coxeter Groups. Cambridge Univ. Press 1990

[Jan] Jantzen J.C.: Representations of Algebraic Groups. Orlando: Academic Press 1987

[Ka1] Kato S.: On Eigenspaces of the Hecke Algebra with Respect to a Good Maximal Compact Subgroup of a p-Adic Reductive Group. Math. Ann. 257, 1-7 (1981)

[Ka2] Kato S.: Spherical Functions and a q-Analogue of Kostant's Weight Multiplicity Formula. Invent. math. 66, 461-468 (1982)

[KKMS] Kempf G., Knudsen F., Mumford D., Saint-Donat B.: Toroidal Embeddings I. Lecture Notes in Math. 339. Berlin-Heidelberg-New York: Springer 1973

[Kne] Kneser M.: Galois-Kohomologie halbeinfacher algebraischer Gruppen über $\mathfrak{p}$-adischen Körpern. I. Math. Z. 88, 40-47 (1965)

[Kot] Kottwitz R.: Rational conjugacy classes in reductive groups. Duke Math. J. 49, 785-806 (1982)

[Kut] Kutzko P.: Mackey's Theorem for non-unitary representations. Proc. AMS 64, 173-175 (1977)

[Mac] Macdonald I.G.: Spherical Functions on a Group of p-Adic Type. Ramanujan Institute Publ. 2 (1971)

[RZ] Rapoport M., Zink T.: Period Spaces for p-divisible Groups. Annals Math. Studies 141. Princeton Univ. Press 1996

[Saa] Saavedra Rivano N.: Catégories Tannakiennes. Lecture Notes in Math. 265. Berlin-Heidelberg-New York: Springer 1972

[ST1] Schneider P., Teitelbaum J.: $U(\mathfrak{g})$-finite locally analytic representations. Representation Theory 5, 111-128 (2001)

[ST2] Schneider P., Teitelbaum J.: Banach space representations and Iwasawa theory. Israel J. Math. 127, 359-380 (2002)

[ST3] Schneider P., Teitelbaum J.: Algebras of p-adic distributions and admissible representations. Invent. math. 153, 145-196 (2003)

[Ste] Steinberg R.: Regular elements of semi-simple algebraic groups. Publ. IHES 25, 49-80 (1965)

[Tot] Totaro B.: Tensor products in p-adic Hodge theory. Duke Math. J. 83, 79-104 (1996)

[Vig] Vigneras M.-F.: Algèbres de Hecke affines génériques. Preprint 2004

[Win] Wintenberger J.-P.: Propriétés du groupe Tannakien des structures de Hodge p-adiques et torseur entre cohomologies cristalline et étale. Ann. Inst. Fourier 47, 1289-1334 (1997)

Peter Schneider
Mathematisches Institut
Westfälische Wilhelms-Universität
Münster
Einsteinstr. 62
D-48149 Münster, Germany
pschnei@math.uni-muenster.de
http://www.uni-muenster.de/math/u/schneider

Jeremy Teitelbaum
Department of Mathematics, Statistics and Computer Science (M/C 249)
University of Illinois at Chicago
851 S. Morgan St.
Chicago, IL 60607, USA
jeremy@uic.edu
http://raphael.math.uic.edu/~jeremy