Universal Norms of p-Units in Some Non-Commutative Galois Extensions

dedicated to Professor John Coates on the occasion of his 60th birthday

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1 Introduction.

Fix a prime number p. Let F be a finite extension of $\mathbb{Q}$ and let F_∞ be an algebraic extension of F. We will consider the $\mathbb{Z}_p$-submodule $U(F_\infty/F)$ of $O_F[1/p]^\times \otimes \mathbb{Z}_p$ defined by

$$U(F_\infty/F) = \mathrm{Image}(\varprojlim_L (O_L[1/p]^\times \otimes \mathbb{Z}_p) \to O_F[1/p]^\times \otimes \mathbb{Z}_p),$$

where L ranges over all finite extensions of F contained in F_∞ and where the inverse limit is taken with respect to the norm maps.
In the case F_∞ is the cyclotomic $\mathbb{Z}_p$-extension of F, the understanding of $U(F_\infty/F)$ is related to profound aspects in Iwasawa theory studied by Coates and other people, as we will shortly recall in §3. Concerning bigger Galois extensions F_∞/F, the following result is (essentially) contained in Corollary 3.23 of Coates and Sujatha [4] (see §3 of this paper).

Assume F_∞/F is a Galois extension and $\mathrm{Gal}(F_\infty/F)$ *is a commutative p-adic Lie group. Assume also that there is only one place of F lying over p. Then $U(F_\infty/F)$ is of finite index in $O_F[1/p]^\times \otimes \mathbb{Z}_p$.*

We ask what happens in the case of non-commutative Lie extensions.
The purpose of this paper is to prove the following theorem, which was conjectured by Coates.

Theorem 1.1. *Let $a_1, \cdots, a_r \in F$, and let*

$$F_n = F(\zeta_{p^n}, a_1^{1/p^n}, \cdots, a_r^{1/p^n}), \quad F_\infty = \cup_{n \geq 1} F_n,$$

where ζ_{p^n} denotes a primitive p^n-th root of 1. Let F^{cyc} be the cyclotomic $\mathbb{Z}_p$-extension of F. Then:

(1) The quotient group $U(F^{\mathrm{cyc}}/F)/U(F_\infty/F)$ is finite.

(2) If there is only one place of F lying over p, then $U(F_\infty/F)$ is of finite index in $O_F[1/p]^\times \otimes \mathbb{Z}_p$.

An interesting point in the proof is that we use the finiteness of the higher K-groups $K_{2n}(O_F)$ for $n \geq 1$, for this result on the muliplicative group K_1.
The author does not have any result on $\varprojlim_L O_F[1/S]^\times$ without $\otimes\mathbb{Z}_p$.
The plan of this paper is as follows. In §2, we review basic facts. In §3, we review some known results in the case F_∞/F is an abelian extension. In §4 and §5, we prove Theorem 1.1 (we will prove a slightly stronger result Theorem 5.1).
The author expresses his hearty thanks to John Coates for suggesting this subject and for advice, and to Ramdorai Sujatha for advice and the hospitality in Tata Institute where a part of this work was done.

2 Basic facts.

We prepare basic facts related to $U(F_\infty/F)$. Most materials appear in Coates and Sujatha [4]. We principally follow their notation.

2.1. Let p be a prime number, and let F be a finite extension of $\mathbb{Q}$. In the case $p = 2$, we assume F is totally imaginary, for simplicity.
Let F_∞ be a Galois extension of F such that the Galois group $G = \mathrm{Gal}(F_\infty/F)$ is a p-adic Lie group and such that only finitely many finite places of F ramify in F_∞.
Let $\mathbb{Z}_p[[G]]$ be the completed group ring of G, that is, the inverse limit of the group rings $\mathbb{Z}_p[G/U]$ where U ranges over all open subgroups of G.

2.2. We define $\mathbb{Z}_p[[G]]$-modules

$$\mathcal{Z}^i(F_\infty) \quad \text{and} \quad \mathcal{Z}^i_S(F_\infty) \quad (i \geq 0)$$

where S is a finite set of finite places of F contaning all places of F lying over p. Let

$$\mathcal{Z}^i_S(F_\infty) = \varprojlim_L H^i(O_L[1/S], \mathbb{Z}_p(1))$$

where L ranges over all finite extensions of F contained in F_∞, $O_L[1/S]$ denotes the subring of L consisting of all elements which are integral at any finite place of L not lying over S, and H^i is the étale cohomology. In the case S is the set of all places of F lying over p, we denote $\mathcal{Z}^i_S(F_\infty)$ simply by $\mathcal{Z}^i(F_\infty)$.

Since

$$(1) \quad H^1(O_L[1/S], \mathbb{Z}_p(1)) \simeq O_L[1/S]^\times \otimes \mathbb{Z}_p$$

by Kummer theory,

$$(2) \quad \mathcal{Z}^1_S(F_\infty) \simeq \varprojlim_L (O_L[1/S]^\times \otimes \mathbb{Z}_p).$$

Note that $H^i(O_L[1/S], \mathbb{Z}_p(1))$ are finitely generated $\mathbb{Z}_p$-modules and $\mathcal{Z}^i(F_\infty)$ are finitely generated $\mathbb{Z}_p[[G]]$-modules. These modules are zero if $i \geq 3$ for the reason of cohomological dimension (here in the case $p = 2$, we use our assumption F is totally imaginary).

2.3. Let $U_S(F_\infty/F)$ be the image of $\varprojlim_L (O_L[1/S]^\times \otimes \mathbb{Z}_p)$ in $O_F[1/S]^\times \otimes \mathbb{Z}_p$. Here L ranges over all finite extensions of F contained in F_∞.
The main points of the preparation in this section are the isomorphisms (1b) and (2b) below.

(1) Assume S contains all finite places of F which ramify in F_∞. Then there are canonical isomorphisms

$$(1a) \qquad H_0(G, \mathcal{Z}^2_S(F_\infty)) \simeq H^2(O_F[1/S], \mathbb{Z}_p(1)),$$

$$(1b) \quad H_1(G, \mathcal{Z}^2_S(F_\infty)) \simeq (O_F[1/S]^\times \otimes \mathbb{Z}_p)/U_S(F_\infty/F).$$

(2) Assume F_∞ contains the cyclotomic $\mathbb{Z}_p$-extension F^{cyc}. Then we have canonical isomorphisms

$$(2a) \qquad H_0(G, \mathcal{Z}^2(F_\infty/F)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq H^2(O_F[1/p], \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

$$(2b) \quad H_1(G, \mathcal{Z}^2(F_\infty)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq (O_F[1/p]^\times \otimes \mathbb{Z}_p)/U(F_\infty/F) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Here $H_m(G, ?) = \mathrm{Tor}_m^{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, ?)$ denotes the G-homology. Note that $H_m(G, M)$ are finitely generated $\mathbb{Z}_p$-modules for any finitely generated $\mathbb{Z}_p[[G]]$-module M.

(1a) and (1b) follow from the spectral sequence

$$E_2^{i,j} = H_{-i}(G, \mathcal{Z}^j_S(F_\infty)) \Rightarrow E^i_\infty = H^i(O_F[1/S], \mathbb{Z}_p(1)),$$

the isomorphisms 2.2 (1) (2), and the fact $\mathcal{Z}^j_S(F_\infty) = 0$ for $j \geq 3$. The above spectral sequence is given in [9] Proposition 8.4.8.3 in the case G is commutative. In general, we have the above spectral sequence by [6] 1.6.5 (3).
The proofs of (2a) and (2b) are given in 2.6 later.

2.4. By Kummer theory and by the well known structure theorem of the Brauer group of a global field, we have an exact sequence

$$(1) \quad 0 \to \mathrm{Pic}(O_F[1/S])\{p\} \to H^2(O_F[1/S], \mathbb{Z}_p(1)) \to \oplus_{v \in S} \mathbb{Z}_p \xrightarrow{\mathrm{sum}} \mathbb{Z}_p \to 0,$$

where $\{p\}$ denotes the p-primary part. Let

$$Y_S(F_\infty) = \varprojlim_L \mathrm{Pic}(O_L[1/S])\{p\},$$

where L ranges over all finite extensions of F contained in F_∞. In the case S is the set of all places of F lying over p, we denote $Y_S(F_\infty)$ simply by $Y(F_\infty)$. Then the exact sequences (1) with F replaced by L give an exact sequence of $\mathbb{Z}_p[[G]]$-modules

$$(2) \quad 0 \to Y_S(F_\infty) \to \mathcal{Z}^2_S(F_\infty) \to \oplus_{v\in S}\mathbb{Z}_p[[G]] \otimes_{\mathbb{Z}_p[[G_v]]} \mathbb{Z}_p \to \mathbb{Z}_p \to 0$$

where for each $v \in S$, $G_v \subset G$ is the decomposition group of a place of F_∞ lying over v.

If S contains all finite place of F which ramify in F_∞, the composite homomorphism

$$(3) \quad (O_F[1/S]^\times \otimes \mathbb{Z}_p)/U(F_\infty/F) \simeq H_1(G, \mathcal{Z}^2_S(F_\infty))$$
$$\to \oplus_{v\in S} H_1(G, \mathbb{Z}_p[[G]] \otimes_{\mathbb{Z}_p[[G_v]]} \mathbb{Z}_p) = \oplus_{v\in S} H_1(G_v, \mathbb{Z}_p)$$

induced by (1b) and (2) coincides with the homomorphism induced by the reciprocity maps

$$F_v^\times \to G_v^{\mathrm{ab}}(p) \simeq H_1(G_v, \mathbb{Z}_p)$$

of local class field theory, where G_v^{ab} denotes the abelian quotient of G_v and (p) means the pro-p part.

2.5. Assume $F_\infty \supset F^{\mathrm{cyc}}$. Then we have isomorphisms

$$\mathcal{Z}^1(F_\infty) \overset{\simeq}{\to} \mathcal{Z}^1_S(F_\infty), \quad Y(F_\infty) \overset{\simeq}{\to} Y_S(F_\infty).$$

The first isomorphism shows $U(F_\infty/F) = U_S(F_\infty/F)$.

In fact, for each finite extension L of F contained in F_∞, we have an exact sequence

$$0 \to O_L[1/p]^\times \otimes \mathbb{Z}_p \to O_L[1/S]^\times \otimes \mathbb{Z}_p \to$$
$$\to \oplus_w \mathbb{Z}_p \to \mathrm{Pic}(O_L[1/p])\{p\} \to \mathrm{Pic}(O_L[1/S])\{p\} \to 0$$

where w ranges over all places of L lying over S but not lying over p. If L' is a finite extension of F such that $L \subset L' \subset F_\infty$, and if w' is a place of L' lying over w, the transition map from $\mathbb{Z}_p$ at w' to $\mathbb{Z}_p$ at w is the multiplication by the degree of the residue extension of w'/w. Since the residue extension of v in F^{cyc}/F for v not lying over p is a $\mathbb{Z}_p$-extension, this shows that the inverse limit of $\oplus_w \mathbb{Z}_p$ for varying L is zero. Hence we have the above isomorphisms.

2.6. We prove (2a) (2b) of 2.3. Take S containing all finite places of F which ramify in F_∞. Let T be the set of all elements of S which do not lie over p.

By 2.4 (2) and by $Y(F_\infty) \xrightarrow{\simeq} Y_S(F_\infty)$ in 2.5, we have an exact sequence of $\mathbb{Z}_p[[G]]$-modules

$$0 \to \mathcal{Z}^2(F_\infty) \to \mathcal{Z}_S^2(F_\infty) \to \oplus_{v\in T}\mathbb{Z}_p[[G]] \otimes_{\mathbb{Z}_p[[G_v]]} \mathbb{Z}_p \to 0.$$

This gives a long exact sequence

$$\begin{aligned} \cdots \to H_m(G, \mathcal{Z}^2(F_\infty)) \to H_m(G, \mathcal{Z}_S^2(F_\infty)) \to \\ \to \oplus_{v\in T} H_m(G_v, \mathbb{Z}_p) \to H_{m-1}(G, \mathcal{Z}^2(F_\infty)) \to \cdots . \end{aligned}$$

Let $G^{\mathrm{cyc}} = \mathrm{Gal}(F^{\mathrm{cyc}}/F)$ and for $v \in T$, let G_v^{cyc} be the image of G_v in G^{cyc}. Then v is unramified in F^{cyc}/F, and we have a canonical isomorphism $G_v^{\mathrm{cyc}} \simeq \mathbb{Z}_p$ which sends the Frobenius of v in G_v^{cyc} to $1 \in \mathbb{Z}_p$. Let H_v $(v \in T)$ be the kernel of $G_v \to G_v^{\mathrm{cyc}}$. Since G is a p-adic Lie group and since the characteristic of the residue field of v is different from p, H_v is of dimension ≤ 1 as a p-adic Lie group. Furthermore, if H_v is infinite, for an element σ_v of G_v whose image in G_v^{cyc} is the Frobenius of v, the inner automorphism on H_v by σ_v is of infinite order as is seen from the usual description of the tame quotient of the absolute Galois group of F_v. These prove

(1) For $v \in T$, the kernel and the cokernel of the canonical map $H_m(G_v, \mathbb{Z}_p) \to H_m(G_v^{\mathrm{cyc}}, \mathbb{Z}_p)$ are finite for any m.

Since the composition $O_F[1/S]^\times \to H_1(G, \mathcal{Z}_S^2(F_\infty)) \to H_1(G_v^{\mathrm{cyc}}, \mathbb{Z}_p) = G_v^{\mathrm{cyc}} \simeq \mathbb{Z}_p$ for $v \in T$ coincides with the v-adic valuation $O_F[1/S]^\times \to \mathbb{Z}$, (1) shows that the cokernel of $H_1(G, \mathcal{Z}_S^2(F_\infty)) \to \oplus_{v\in T} H_1(G_v, \mathbb{Z}_p)$ is finite. Hence by the above long exact sequence, we have the following commutative diagram with exact rows in which the kernel of the first arrow of each row is finite.

$$\begin{array}{ccccccc} H_0(G, \mathcal{Z}^2(F_\infty)) & \to & H_0(G, \mathcal{Z}_S^2(F_\infty)) & \to & \oplus_{v\in T}\mathbb{Z}_p & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H^2(O_F[1/p], \mathbb{Z}_p(1)) & \to & H^2(O_F[1/S], \mathbb{Z}_p(1)) & \to & \oplus_{v\in T}\mathbb{Z}_p & \to & 0 \end{array}$$

By this diagram and by 2.3 (1a), we have 2.3 (2a).
We next prove 2.3 (2b). By the above (1), $H_2(G_v, \mathbb{Z}_p)$ is finite for $v \in T$. By this and by the case $m = 1$ of the above (1), we see that the complex $0 \to H_1(G, \mathcal{Z}^2(F_\infty)) \to H_1(G, \mathcal{Z}_S^2(F_\infty)) \to \oplus_{v\in T} H_1(G_v^{\mathrm{cyc}}, \mathbb{Z}_p)$ has finite homology groups. By 2.3 (1b) and by $U(F_\infty/F) = U_S(F_\infty/F)$ (2.5), the kernel of the last arrow of this complex is isomorphic to $(O_F[1/p]^\times \otimes \mathbb{Z}_p)/U(F_\infty/F)$. This proves 2.3 (2b).

3 ABELIAN EXTENSIONS (REVIEW).

In this section, we review the proof of the following result of Coates and Sujatha ([4] Cor. 3.23), and then recall some known facts on $U(F^{\mathrm{cyc}}/F)$.

PROPOSITION 3.1. *Assume F_∞/F is Galois and* $\mathrm{Gal}(F_\infty/F)$ *is a commutative p-adic Lie group. Assume further that there is only one place of F lying over p. Then:*

(1) $U(F_\infty/F)$ is of finite index in $O_F[1/p]^\times \otimes \mathbb{Z}_p$.

(2) $H_m(G, Y(F_\infty))$ and $H_m(G, \mathcal{Z}^2(F_\infty))$ are finite for any m.

In fact, this result was written in [4] in the situation $\mathrm{Gal}(F_\infty/F) \simeq \mathbb{Z}_p^2$. This was because this result appeared in [4] in the study of the arithmetic of a $\mathbb{Z}_p^2$-extension generated by p-power division points of an elliptic curve with complex multiplication. We just check here that the method of their proof works in this generality.

Proof. We may (and do) assume $F_\infty \supset F^{\mathrm{cyc}}$. In the case $p = 2$, to apply our preparation in §2, we assume F is totally imaginary without a loss of generality (we may replace F by a finite extension of F having only one place lying over p for the proof of 3.1).
(1) follows from the finiteness of $H_1(G, \mathcal{Z}^2(F_\infty))$ in (2) by 2.3 (2b). We prove (2).
We have $H_0(G, \mathcal{Z}^2(F_\infty)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq H^2(O_F[1/p], \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ by 2.3 (2a), and $H^2(O_F[1/p], \mathbb{Z}_p(1))$ is finite by the exact sequence 2.4 (1) and by the assumption that there is only one place of F lying oer p. Hence $H_0(G, \mathcal{Z}^2(F_\infty)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = 0$. This shows that $H_m(G, \mathcal{Z}^2(F_\infty)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = 0$ for any m (Serre [11]). (Here the assumption G is commutative is essential. See 5.6.) This proves $H_m(G, \mathcal{Z}^2(F_\infty))$ is finite for any m.
Let v be the unique place of F lying over p. Then by class field theory, the decomposition group G_v of v in G is of finite index in G. By the exact sequence

$$H_2(G_v, \mathbb{Z}_p) \to H_2(G, \mathbb{Z}_p) \to H_1(G, \mathcal{Z}^2(F_\infty)/Y(F_\infty)) \to H_1(G_v, \mathbb{Z}_p) \to H_1(G, \mathbb{Z}_p)$$

obtained from 2.4 (2), this shows that $H_1(G, \mathcal{Z}^2(F_\infty)/Y(F_\infty))$ and hence the kernel of $H_0(G, Y(F_\infty)) \to H_0(G, \mathcal{Z}^2(F_\infty))$ are finite. Hence $H_0(G, Y(F_\infty))$ is finite, and by Serre [11], $H_m(G, Y(F_\infty))$ is finite for any m. □

3.2. In the rest of this section, we recall some known facts about $U(F^{\mathrm{cyc}}/F)$. Let $G^{\mathrm{cyc}} = \mathrm{Gal}(F^{\mathrm{cyc}}/F)$. For a place v of F lying over p, let $G_v^{\mathrm{cyc}} \subset G^{\mathrm{cyc}}$ be the decomposition group of v (so $G_v^{\mathrm{cyc}} \simeq \mathbb{Z}_p$). Let $(\oplus_{v|p} G_v^{\mathrm{cyc}})^0$ be the kernel of the canoncial map $\oplus_{v|p} G_v^{\mathrm{cyc}} \to G^{\mathrm{cyc}}$.
Let

$$\alpha_F \;:\; (O_F[1/p]^\times \otimes \mathbb{Z}_p)/U(F^{\mathrm{cyc}}/F) \to (\oplus_{v|p} G_v^{\mathrm{cyc}})^0$$

be the homomorphism induced by the reciprocity maps of local fields F_v, which appeared in 2.4 (3).
It is known that the following conditions (1) - (3) are equivalent.

(1) $\mathrm{Ker}\,(\alpha_F)$ is finite. (That is, $U(F^{\mathrm{cyc}}/F)$ is of finite index in the kernel of $O_F[1/p]^\times \otimes \mathbb{Z}_p \to (\oplus_{v|p} G_v^{\mathrm{cyc}})^0$.)

(2) $\mathrm{Coker}\,(\alpha_F)$ is finite.

(3) $H_0(G^{\mathrm{cyc}}, Y(F^{\mathrm{cyc}}))$ is finite.

The equivalence of (1)-(3) is proved as follows. Though this is not at all an essential point, in the case $p = 2$, to apply our preparation in §2, we assume F is totally imaginary without a loss of generality (we can replace F by a finite extension of F for the proof of the equivalence). Let σ be a topological generator of G^{cyc}. Then $H_0(G^{\text{cyc}}, \mathcal{Z}^2(F^{\text{cyc}}))$ is isomorphic to the cokernel of $\sigma - 1 : \mathcal{Z}^2(F^{\text{cyc}}) \to \mathcal{Z}^2(F^{\text{cyc}})$ and $H_1(G^{\text{cyc}}, \mathcal{Z}^2(F^{\text{cyc}}))$ is isomorphic to the kernel of it. Since $\mathcal{Z}^2(F^{\text{cyc}})$ is a torsion $\mathbb{Z}_p[[G^{\text{cyc}}]]$-module, this shows that the $\mathbb{Z}_p$-rank of $H_1(G^{\text{cyc}}, \mathcal{Z}^2(F^{\text{cyc}})) \simeq (O_F[1/p] \otimes \mathbb{Z}_p)/U(F^{\text{cyc}}/F)$ is equal to the $\mathbb{Z}_p$-rank of $H_0(G^{\text{cyc}}, \mathcal{Z}^2(F^{\text{cyc}})) \simeq H^2(O_F[1/p], \mathbb{Z}_p(1))$ which is equal to the $\mathbb{Z}_p$-rank of $(\oplus_{v|p} G_v^{\text{cyc}})^0$ by 2.4 (1). Hence (1) and (2) are equivalent. The exact sequence 2.4 (2) (take $F_\infty = F^{\text{cyc}}$ and S to be the set of all places of F lying over p) shows that $\operatorname{Coker}(\alpha_F)$ is isomorphic to the kernel of $H_0(G^{\text{cyc}}, Y(F^{\text{cyc}})) \to H_0(G^{\text{cyc}}, \mathcal{Z}^2(F^{\text{cyc}})) = H^2(O_F[1/p], \mathbb{Z}_p(1))$. The image of the last map is $\operatorname{Pic}(O_F[1/p])\{p\}$ by 2.4 (1) (2), and hence is finite. Hence $\operatorname{Coker}(\alpha_F)$ is finite if and only if $H_0(G^{\text{cyc}}, Y(F^{\text{cyc}}))$ is finite.

3.3. Greenberg [7] proved that $H_0(G^{\text{cyc}}, Y(F^{\text{cyc}}))$ is finite if F is an abelian extension of $\mathbb{Q}$ (hence all (1) - (3) in 3.2 are satisfied in this case).

3.4. In the case F is totally real, by Coates [2] Theorem 1.13, $H_0(G^{\text{cyc}}, Y(F^{\text{cyc}}))$ is finite if Leopoldt conjecture for F is true.

3.5. Let F be a CM field. Let F^+ be the real part of F, and let $H_0(G^{\text{cyc}}, Y(F^{\text{cyc}}))^{\pm} \subset H_0(G^{\text{cyc}}, Y(F^{\text{cyc}}))$ be the $\pm$-part with respect to the action of the complex conjugation in $\operatorname{Gal}(F/F^+)$. Then by the above result of Coates, $H_0(G^{\text{cyc}}, Y(F^{\text{cyc}}))^+$ is finite if Leopoldt conjecture for F^+ is true. On the other hand, Conjecture 2.2 in Coates and Lichtenbaum [3] says that $H_0(G^{\text{cyc}}, Y(F^{\text{cyc}}))^-$ is finite. In [8], Gross conjectured that the kernel and the cokernel of the (-)-part α_F of α_F is finite (this finiteness is also a consequence of Conjecture 2.2 of [3]), and formulated a conjecture which relates α_F^- to the leading terms of the Taylor expansions at $s = 0$ of p-adic Artin L-functions.
Thus known conjectures support that the equivalent conditions (1) - (3) in 3.2 are satisfied by any CM field F.
A natural question arises: Are (1) - (3) in 3.2 true for any number field F?

4 A RESULT ON TOR MODULES.

The purpose of this section is to prove Proposition 4.2 below.

4.1. For a compact p-adic Lie group G, for a $\mathbb{Z}_p[[G]]$-module T, and for a continuous homomorphism $G \to \mathbb{Z}_p^\times$, let $T(\chi)$ be the $\mathbb{Z}_p[[G]]$-module whose underlying abelian group is that of T and on which $\mathbb{Z}_p[[G]]$ acts by $\mathbb{Z}_p[[G]] \to \mathbb{Z}_p[[G]] \to \operatorname{End}(T)$, where the first arrow is the automorphism $\sigma \mapsto \chi(\sigma)\sigma$ ($\sigma \in G$) of the topological ring $\mathbb{Z}_p[[G]]$ and the second arrow is the original action of $\mathbb{Z}_p[[G]]$ on T. We call $T(\chi)$ the twist of T by χ.

Proposition 4.2. *Let G be a compact p-adic Lie group, let H be a closed normal subgroup of G, and assume that we are given a finite family of closed normal subgroups H_i ($0 \le i \le r$) of G such that $\{1\} = H_0 \subset H_1 \subset \cdots \subset H_r = H$, $H_i/H_{i-1} \simeq \mathbb{Z}_p$ for $1 \le i \le r$ and such that the the action of G on H_i/H_{i-1} by inner automorphisms is given by a homomorphism $\chi_i : G/H \to \mathbb{Z}_p^\times$.*
Let M be a finitely generated $\mathbb{Z}_p[[G]]$-module, and let M' be a subquotient of the $\mathbb{Z}_p[[G]]$-module M. Let $m \ge 0$. Then there is a finite family $(S_i)_{1 \le i \le k}$ of $\mathbb{Z}_p[[G/H]]$-submodules of $H_m(H, M')$ satisfying the following (i) *and* (ii).

(i) $0 = S_0 \subset S_1 \subset \cdots \subset S_k = H_m(H, M')$.

(ii) *For each i ($1 \le i \le k$), there are a subquotient T of the $\mathbb{Z}_p[[G/H]]$-module $H_0(H, M)$ and a family $(s(j))_{1 \le j \le r}$ of non-negative integers $s(j)$ such that $\sharp\{j | s(j) > 0\} \ge m$ and such that S_i/S_{i-1} is isomorphic to the twist $T(\prod_{1 \le j \le k} \chi_j^{s(j)})$ of T.*

Note

$$H_m(H, M) = \mathrm{Tor}_m^{\mathbb{Z}_p[[H]]}(\mathbb{Z}_p, M) = \mathrm{Tor}_m^{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p[[G/H]], M)$$

for $\mathbb{Z}_p[[G]]$-modules M.
A key point in the proof of Proposition 3.1 was that for commutative rings, Tor_m vanishes if Tor_0 vanishes. This is not true for non-commutative rings. In the next section, we will use the above relation of Tor_0 and Tor_m in a non-commutative situation for the proof of Theorem 1.1.

4.3. We denote this proposition with fixed r by (A_r). Let (B_r) be the case $M = M'$ of (A_r).

Since (B_r) is a special case of (A_r), (B_r) follows from (A_r).
In 4.4, we show that conversely, (A_r) follows from (B_r). In 4.5, we prove (B_1). In 4.6, for $r \ge 1$, we prove (B_r) assuming (A_{r-1}) and (B_1). These give a proof of Prop.4.2.

4.4. We can deduce (A_r) from (B_r) as follows. Let M'' be the quotient of the $\mathbb{Z}_p[[G]]$-module M such that M' is a $\mathbb{Z}_p[[G]]$-submodule of M''. We have an exact sequence of $\mathbb{Z}_p[[G/H]]$-modules

$$H_{m+1}(H, M''/M') \to H_m(H, M') \to H_m(H, M'').$$

Then (A_r) for the pair (M, M') is obtained from (B_r) applied to M''/M' and to M'' since $H_0(H, M''/M')$ and $H_0(H, M'')$ are quotients of the $\mathbb{Z}_p[[G/H]]$-module $H_0(H, M)$.

4.5. We prove (B_1). Assume $r = 1$. Let $\chi = \chi_1$.
Note that $H \simeq \mathbb{Z}_p$. Let α be a topological generator of H, and let $N = \alpha - 1 \in \mathbb{Z}_p[[G]]$. Let $I = \mathrm{Ker}\,(\mathbb{Z}_p[[G]] \to \mathbb{Z}_p[[G/H]]) = \mathbb{Z}_p[[G]]N = N\mathbb{Z}_p[[G]]$.
We have

(1) For $\sigma \in G$, $\sigma N \sigma^{-1}$ is expressed as a power series in N with coefficients in $\mathbb{Z}_p$ which is congruent to $\chi(\sigma)N \mod N^2$. In particular, $\sigma N \sigma^{-1} \equiv \chi(\sigma)N \mod I^2$.

In fact, $\sigma N \sigma^{-1} = \alpha^{\chi(\sigma)} - 1 = (1+N)^{\chi(\sigma)} - 1 = \chi(\sigma)N + \sum_{n\geq 2} c_i N^i$ for some $c_i \in \mathbb{Z}_p$.
Concerning $H_m(H, M)$ ($m \geq 0$), we have:

(2) $N(M)$ is a $\mathbb{Z}_p[[G]]$-submodule of M, I kills $M/N(M)$, and there is an isomorphism of $\mathbb{Z}_p[[G/H]]$-modules

$$H_0(H, M) \simeq M/N(M).$$

(3) $\mathrm{Ker}\,(N : M \to M)$ is a $\mathbb{Z}_p[[G]]$-submodule of M, I kills $\mathrm{Ker}\,(N : M \to M)$, and there is an isomorphism of $\mathbb{Z}_p[[G/H]]$-modules

$$H_1(H, M) \simeq \mathrm{Ker}\,(N : M \to M)(\chi).$$

(4) $H_m(H, M) = 0$ fo $m \geq 2$.

We prove (2)–(4). We have a projective resolution

$$0 \to I \to \mathbb{Z}_p[[G]] \to \mathbb{Z}_p[[G/H]] \to 0$$

of the right $\mathbb{Z}_p[[G]]$-module $Z_p[[G/H]]$. Since $H_m(H, ?) = \mathrm{Tor}_m^{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p[[G/H]], ?)$, $H_0(H, M)$ (resp. $H_1(H, M)$) is isomorphic to the cokernel (resp. kernel) of $I \otimes_{\mathbb{Z}_p[[G]]} M \to M$, and $H_m(H, M) = 0$ for all $m \geq 2$. This proves (2) and (4). Furthermore,

$$H_1(H, M) \simeq \mathrm{Ker}\,(I \otimes_{\mathbb{Z}_p[[G]]} M \to M) \simeq I \otimes_{\mathbb{Z}_p[[G]]} \mathrm{Ker}\,(N : M \to M)$$
$$\simeq I/I^2 \otimes_{\mathbb{Z}_p[[G/H]]} \mathrm{Ker}\,(N : M \to M).$$

Consider the bijection

$$\mathrm{Ker}\,(N : M \to M) \to I/I^2 \otimes_{\mathbb{Z}_p[[G/H]]} \mathrm{Ker}\,(N : M \to M)\ ;\ x \mapsto N \otimes x.$$

By the above (1), for $\sigma \in G$, we have $\sigma N \otimes x = \chi(\sigma) N \sigma \otimes x = \chi(\sigma) N \otimes \sigma x$ in $I/I^2 \otimes_{\mathbb{Z}_p[[G/H]]} \mathrm{Ker}\,(N : M \to M)$. Hence

$$I/I^2 \otimes_{\mathbb{Z}_p[[G/H]]} \mathrm{Ker}\,(N : M \to M) \simeq \mathrm{Ker}\,(N : M \to M)(\chi)$$

as $\mathbb{Z}_p[[G/H]]$-modules. This proves (3).
Let

$$V_n = \mathrm{Ker}\,(N^n : M \to M) \quad (n \geq 0), \quad V = \cup_n V_n.$$

Then, since $\mathbb{Z}_p[[G]]N^n = N^n \mathbb{Z}_p[[G]]$, V_n is a $\mathbb{Z}_p[[G]]$-submodule of M. Since $\mathbb{Z}_p[[G]]$ is Noetherian and M is a finitely generated $\mathbb{Z}_p[[G]]$-module, $V = V_n$ for some n. That is, N is nilpotent on V. Since $\mathrm{Ker}\,(N : M/V \to M/V) = 0$, we have $H_1(H, M/V) = 0$ by (3). Hence

(5) $H_1(H, V) = H_1(H, M)$,

(6) $H_0(H, V) \to H_0(H, M)$ is injective.

Consider the monodromy filtration $(W_i)_i$ on the abelian group V given by the nilpotent endomorphism N in the sense of Deligne [5] 1.6. It is an increasing filtration characterized by the properties $N(W_i) \subset W_{i-2}$ for all i, and $N^i : \mathrm{gr}_i^W \overset{\simeq}{\to} \mathrm{gr}_{-i}^W$ for all $i \geq 0$.

(7) W_i are $\mathbb{Z}_p[[G]]$-submodules of V.

In fact, for $\sigma \in G$, the filtration $(\sigma W_i)_i$ also has the characterizing property of $(W_i)_i$ by (1).
Now we define an increasing filtration $(W'_i)_i$ of the $\mathbb{Z}_p[[G/H]]$-module $H_0(H, V)$ and an increasing filtration $(W''_i)_i$ on the $\mathbb{Z}_p[[G/H]]$-module $H_1(H, V) = H_1(H, M)$ as follows. By identifying $H_0(H, V)$ with $\mathrm{Coker}\,(N : V \to V)$, let $W'_i = W_i(\mathrm{Coker}\,(N : V \to V))$ (i.e. the image of W_i in $\mathrm{Coker}\,(N : V \to V)$). By identifying $H_1(H, V)$ with $\mathrm{Ker}\,(N : V \to V)(\chi)$, let $W''_i = W_i(\mathrm{Ker}\,(N : V \to V))(\chi)$ (i.e. $(W_i \cap \mathrm{Ker}\,(N : V \to V))(\chi)$). Then $W''_0 = H_1(H, M)$, and $W''_i = 0$ if i is sufficiently small. We prove:

(8) For any $i \geq 0$,
$$\mathrm{gr}_{-i}^{W''} \simeq \mathrm{gr}_i^{W'}(\chi^{i+1})$$
as $\mathbb{Z}_p[[G/H]]$-modules.

By the injectivity of $H_0(H, V) \to H_0(H, M)$ (6), this proves (B_1).

We prove (8). By (1), we have

(9) The map $N : \mathrm{gr}_i^W \to \mathrm{gr}_{i-2}^W$ satisfies $\sigma N \sigma^{-1} = \chi(\sigma) N$ for $\sigma \in G$.

Let $P_i \subset \mathrm{gr}_i^W$ $(i \leq 0)$ be the primitive part $\mathrm{Ker}\,(N : \mathrm{gr}_i^W \to \mathrm{gr}_{i-2}^W)$ ([5] 1.6.3). Then for $i \geq 0$, the canonical map $\mathrm{gr}_{-i}^W(\mathrm{Ker}\,(N : V \to V)) \to P_{-i}$ is an isomorphism of $\mathbb{Z}_p[[G/H]]$-modules ([5] 1.6.6). Furthermore, we have a bijection $P_{-i} \overset{\simeq}{\to} \mathrm{gr}_i^W(\mathrm{Coker}\,(N : V \to V))$ as the composition

$$P_{-i} \to \mathrm{gr}_{-i}^W \overset{N^i}{\leftarrow} \mathrm{gr}_i^W \to \mathrm{gr}_i^W(\mathrm{Coker}\,(N : V \to V))$$

([5] 1.6.4, 1.6.6, and the dual statement of 1.6.6 for $\mathrm{Coker}\,(N)$). By (9), this gives an isomorphism of $\mathbb{Z}_p[[G/H]]$-modules $P_{-i} \simeq \mathrm{gr}_i^W(\mathrm{Coker}\,(N : V \to V))(\chi^i)$. Hence we have (8).

4.6. Let $r \geq 1$. We prove (B_r) assuming (A_{r-1}) and (B_1). Let $J = H_1$. By the spectral sequence

$$E_2^{-i,-j} = H_i(H/J, H_j(J, M)) \Rightarrow E_\infty^{-m} = H_m(H, M)$$

in which $H_j(J,M) = 0$ for $j \geq 2$, we have an exact sequence of $\mathbb{Z}_p[[G/H]]$-modules

$$(1) \quad H_{m-1}(H/J, H_1(J,M)) \to H_m(H,M) \to H_m(H/J, H_0(J,M)).$$

We consider $H_{m-1}(H/J, H_1(J,M))$ first. By (B_1) applied to the triple (G,J,M), $H_1(J,M)$ is a successive extension of twists of subquotients of $H_0(J,M)$ by χ_1^i $(i \geq 1)$. By (A_{r-1}) applied the triple $(G/J, H/J, H_0(J,M))$, $H_{m-1}(H/J, ?)$ of these subquotients of $H_0(J,M)$ are successive extensions of twists of subquotients of $H_0(H/J, H_0(J,M)) = H_0(H,M)$ by $\prod_{2 \leq j \leq r} \chi_j^{s(j)}$ such that $s(j) \geq 0$ for all j and such that $\sharp(\{j \,|\, s(j) > 0\} \geq m-1$. Hence $H_{m-1}(H/J, H_1(J,M))$ is a successive extension of twists of subquotients of $H_0(H,M)$ by $\prod_{1 \leq j \leq r} \chi_j^{s(j)}$ such that $s(j) \geq 0$ for all j and such that $\sharp(\{j \,|\, s(j) > 0\} \geq m$.
We consider $H_m(H/J, H_0(J,M))$ next. By (B_{r-1}) (which is assumed since we assume (A_{r-1})) applied to the triple $(G/J, H/J, H_0(J,M))$, $H_m(H/J, H_0(J,M))$ is a successive extension of twists of subquotients of $H_0(H/J, H_0(J,M)) = H_0(H,M)$ by $\prod_{2 \leq j \leq r} \chi_i^{s(j)}$ such that $s(j) \geq 0$ for all j and such that $\sharp(\{j \,|\, s(j) > 0\} \geq m$.
By these properties of $H_{m-1}(H/J, H_1(J,M))$ and $H_m(H/J, H_0(J,M))$, the exact sequence (1) proves (B_r) (assuming (A_{r-1}) and (B_1)).

5 Some non-commutative Galois extensions.

Theorem 1.1 in Introduction is contained in Corollary 5.2 of the following Theorem 5.1, for the extension F_∞/F in Theorem 1.1 satisfies the assumption of Theorem 5.1 with $n(i) = 1$ for all i.

Theorem 5.1. *Assume that F_∞ is a Galois extension of F, $F_\infty \supset \cup_n F(\zeta_{p^n})$, and that there is a finite family of closed normal subgroups H_i $(1 \leq i \leq r)$ of $G = \mathrm{Gal}(F_\infty/F)$ satisfying the following condition. Let F^{cyc} be the cyclotomic $\mathbb{Z}_p$-extension of F and let H be the kernel of $G \to G^{\mathrm{cyc}} = \mathrm{Gal}(F^{\mathrm{cyc}}/F)$. Then $\{1\} = H_0 \subset H_1 \subset \cdots \subset H_r$, H_r is an open subgroup of H, and for $1 \leq i \leq r$, $H_i/H_{i-1} \simeq \mathbb{Z}_p$ and the action of G on it by inner automorphism is the $n(i)$-th power of the cyclotomic character $G \to \mathbb{Z}_p^\times$ for some positive integer $n(i) > 0$. Let S be any finite set of finite places of F containing all places lying over p. Then the kernel and the cokernel of the canonical maps*

$$H_m(G, \mathcal{Z}_S^2(F_\infty)) \to H_m(G^{\mathrm{cyc}}, \mathcal{Z}_S^2(F^{\mathrm{cyc}})),$$
$$H_m(G, Y(F_\infty)) \to H_m(G^{\mathrm{cyc}}, Y(F^{cyc}))$$

are finite for any m.
In particular (since $H_m(G^{\mathrm{cyc}}, ?) = 0$ for $m \geq 2$), $H_m(G, \mathcal{Z}_S^2(F_\infty))$ and $H_m(G, Y(F_\infty))$ are finite for any $m \geq 2$.

Corollary 5.2. *Let the assumption be as in Theorem 5.1. Then:*

(1) The quotient group $U(F^{cyc}/F)/U(F_\infty/F)$ *is finite.*
(2) If there is only one place of F *lying over* p*, then* $U(F_\infty/F)$ *is of finite index in* $O_F[1/p]^\times \otimes \mathbb{Z}_p$*, and* $H_m(G, Y(F_\infty))$ *and* $H_m(G, \mathcal{Z}^2(F_\infty))$ *are finite for any* m*.*

(3) If F *is an abelian extension over* $\mathbb{Q}$*, then* $H_m(G, Y(F_\infty))$ *is finite for any* m*.*

In fact, by 2.3 (2b), (1) of Corollary 5.2 follows from the finiteness of the kernel and the cokernel of $H_1(G, \mathcal{Z}^2(F_\infty)) \to H_1(G^{\rm cyc}, \mathcal{Z}^2(F^{\rm cyc}))$ which is a special case of Theorem 5.1. (2) follows from (1) and the case $F_\infty = F^{\rm cyc}$ of Proposition 3.1. (3) follows from (1) and the result of Greenberg introduced in 3.3.

Corollary 5.3. *Let the assumption be as in Theorem 5.1. Then* $H_m(G, \mathcal{Z}^1(F_\infty))$ *for* $m \geq 1$ *and the kernel of the canonical map* $H_0(G, \mathcal{Z}^1(F_\infty)) \to O_F[1/p]^\times \otimes \mathbb{Z}_p$ *are finite.*

In fact, for S containing all finite places which ramify in F_∞, since $\mathcal{Z}^1(F_\infty) \stackrel{\simeq}{\to} \mathcal{Z}^1_S(F_\infty)$ (2.5), the spectral sequence in 2.3 shows that $H_m(G, \mathcal{Z}^1(F_\infty))$ for $m \geq 1$ is isomorphic to $H_{m+2}(G, \mathcal{Z}^2_S(F_\infty))$, and the kernel of $H_0(G, \mathcal{Z}^1(F_\infty)) \to O_F[1/p]^\times \otimes \mathbb{Z}_p$ is isomorphic to $H_2(G, \mathcal{Z}^2_S(F_\infty))$. Hence this corollary follows from the finiteness of $H_m(G, \mathcal{Z}^2_S(F_\infty))$ for $m \geq 2$ in Theorem 5.1.

5.4. We prove Theorem 5.1. First in this 5.4, we show that the kernel and the cokernel of $H_m(G, \mathcal{Z}^2_S(F_\infty)) \to H_m(G^{\rm cyc}, \mathcal{Z}^2_S(F^{\rm cyc}))$ are finite for any m assuming that S contains all finite places of F which ramify in F_∞,.
We may replace F by a finite extension of F. Hence we may assume that $H_r = H$, $\cup_{n\geq 1} F(\zeta_{p^n}) = F^{\rm cyc}$, and that in the case $p = 2$, F is totally imaginary. Let $\mathfrak{p}$ be the augmentation ideal of $\mathbb{Z}_p[[G^{\rm cyc}]]$. It is a prime ideal of $\mathbb{Z}_p[[G^{\rm cyc}]]$. By the spectral sequence $E_2^{-i,-j} = H_i(G^{\rm cyc}, H_j(H, ?)) \Rightarrow E_\infty^{-m} = H_m(G, ?)$, it is sufficient to prove that $H_i(G^{\rm cyc}, H_m(H, \mathcal{Z}^2_S(F_\infty)))$ is finite for any i and for any $m \geq 1$. For a finitely generated $\mathbb{Z}_p[[G^{\rm cyc}]]$-module M, $H_i(G^{\rm cyc}, M)$ is isomorphic to $M/\mathfrak{p}M$ if $i = 0$, to the part of M annihilated by $\mathfrak{p}$ if $i = 1$, and is zero if $i \geq 2$. Applying this taking $M = H_m(H, \mathcal{Z}^2_S(F_\infty))$, we see that it is sufficient to prove

$$(1) \quad H_m(H, \mathcal{Z}^2_S(F_\infty))_{\mathfrak{p}} = 0 \quad \text{for any} \quad m \geq 1,$$

where $(?)_{\mathfrak{p}}$ denotes the localization at the prime ideal $\mathfrak{p}$.
We apply Proposition 4.2 to the case $M = M' = \mathcal{Z}^2_S(F_\infty)$. By this proposition, to prove (1), it is sufficient to show that for any subquotient T of the $\mathbb{Z}_p[[G^{\rm cyc}]]$-module $H_0(H, M) = \mathcal{Z}^2_S(F^{\rm cyc})$ and for any integer $k \geq 1$, we have $T(k)_{\mathfrak{p}} = 0$. Here $T(k)$ is the k-th Tate twist. It is sufficient to prove that $H_0(G^{\rm cyc}, T(k))$ is finite. Since $\mathcal{Z}^2_S(F^{\rm cyc})$ is a finitely generated torsion $\mathbb{Z}_p[[G^{\rm cyc}]]$-module, the $\mathbb{Z}_p[[G^{\rm cyc}]]$-module T is a successive extension of $\mathbb{Z}_p[[G^{\rm cyc}]]$-modules which are

either finite or isomorphic to $\mathbb{Z}_p[[G^{\rm cyc}]]/\mathfrak{q}$ for some prime ideal $\mathfrak{q}$ of $\mathbb{Z}_p[[G^{\rm cyc}]]$ of height one. We may assume $T \simeq \mathbb{Z}_p[[G^{\rm cyc}]]/\mathfrak{q}$. Then there is a $\mathbb{Z}_p[[G^{\rm cyc}]]$-homomorphism $\mathcal{Z}^2_S(F^{\rm cyc}) \to T$ with finite cokernel. Hence it is sufficient to prove that $H_0(G^{\rm cyc}, \mathcal{Z}^2(F^{\rm cyc})(k)))$ is finite for any $k \geq 1$. But

$$H_0(G^{\rm cyc}, \mathcal{Z}^2(F^{\rm cyc})(k))) \simeq H^2(O_F[1/S], \mathbb{Z}_p(k+1)).$$

The last group is finite by Soulé [12]. In fact, by Quillen [10] and Borel [1], $K_{2k}(O_F[1/S])$ is finite, and by Soulé [12], we have a surjective Chern class map from $K_{2k}(O_F[1/S])$ to $H^2(O_F[1/S], \mathbb{Z}_p(k+1))$.

5.5. We complete the proof of Theorem 5.1. Let S be a finite set of finite places of F which contains all places of F lying over p. Take a finite set S' of finite places of F such that $S \subset S'$ and such that S' contains all finite places of F which ramify in F_∞.

By comparing the exact sequence 2.4 (2) for F_∞/F and that for $F^{\rm cyc}/F$, we see that the finiteness of the kernel and the cokernel of $H_m(G, \mathcal{Z}^2_S(F_\infty)) \to H_m(G^{\rm cyc}, \mathcal{Z}^2_S(F^{\rm cyc}))$ for all m and that of $H_m(G, Y(F_\infty)) \to H_m(G^{\rm cyc}, Y(F^{\rm cyc}))$ for all m are consequences of the following (1) - (3).

(1) The kernel and the cokernel of $H_m(G, \mathcal{Z}^2_{S'}(F_\infty)) \to H_m(G^{\rm cyc}, \mathcal{Z}^2_{S'}(F^{\rm cyc}))$ are finite for all m.

(2) The kernel and the cokernel of $H_m(G, \mathbb{Z}_p) \to H_m(G^{\rm cyc}, \mathbb{Z}_p)$ are finite for all m.

(3) The kernel and the cokernel of $H_m(G_v, \mathbb{Z}_p) \to H_m(G_v^{\rm cyc}, \mathbb{Z}_p)$ are finite for all m and for all finite places v of F. Here $G_v \subset G$ denotes a decomposition group of a place of F_∞ lying over v, and $G_v^{\rm cyc}$ denotes the image of G_v in $G^{\rm cyc}$.

We proved (1) already in 5.4. (2) and (3) follow from the case $M = M' = \mathbb{Z}_p$ of Proposition 4.2.

REMARK 5.6. There is an example of a p-adic Lie extension F_∞/F for which there is only one place of F lying over p but $U(F_\infty/F)$ is not of finite index in $O_F[1/p]^\times \otimes \mathbb{Z}_p$. For example, let $F = \mathbb{Q}$, let E be an elliptic curve over F with good ordinary reduction at p, and let F_∞ be the field generated over F by p^n-division points of E for all n. Then $U(F_\infty/F) = \{1\}$ and is not of finite index in $O_F[1/p]^\times \otimes \mathbb{Z}_p = \mathbb{Z}[1/p]^\times \otimes \mathbb{Z}_p \simeq \mathbb{Z}_p$. In fact $U(F_\infty/F)$ must be killed by the reciprocity map of local class field theory of $\mathbb{Q}_p$ into $G_p^{\rm ab}(p) \simeq \mathbb{Z}_p^2$, where $G_p \subset G = {\rm Gal}(F_\infty/F)$ denotes the decomposition group at p, and $G_p^{\rm ab}(p)$ denotes the pro-p part of the abelian quotient of G_p. The image of $p \in \mathbb{Z}[1/p]^\times$ in $G_p^{\rm ab}(p)$ is of infinite order. This proves $U(F_\infty/F) = \{1\}$. In this case, $H_0(G, \mathcal{Z}^2(F_\infty))$ is finite, but $H_1(G, \mathcal{Z}^2(F_\infty))$ is not finite.

REMARK 5.7. There is an example of a p-adic Lie extension F_∞/F for which $G = {\rm Gal}(F_\infty/F) \simeq \mathbb{Z}_p^2$ and $H_0(G, Y(F_\infty/F))$ is not finite. Let K be an imaginary quadratic field in which p splits, let K_∞ be the unique Galois extension of K such that $\mathrm{Gal}(K_\infty/K) \simeq \mathbb{Z}_p^2$, let F be a finite extension of K in which p splits completely, and let $F_\infty = FK_\infty$. Then the $\mathbb{Z}_p$-rank of $H_1(G, Y(F_\infty))$ is $\geq [F:K]-1$ as is shown below. Hence it is not zero if $F \neq K$. In fact, from the exact sequence 2.4 (2) with S the set of all places of F lying over p, we can obtain

$$\begin{aligned}&\mathrm{rank}_{\mathbb{Z}_p} H_1(G, Y(F_\infty)) \geq \\ &\quad \geq (\sum_{v\in S} \mathrm{rank}_{\mathbb{Z}_p} H_1(G_v, \mathbb{Z}_p)) - \mathrm{rank}_{\mathbb{Z}_p} H_1(G, \mathbb{Z}_p) - \mathrm{rank}_{\mathbb{Z}}\, O_F[1/p]^\times.\end{aligned}$$

But $\mathrm{rank}_{\mathbb{Z}_p} H_1(G_v, \mathbb{Z}_p) = 2$ for any $v \in S$, $\mathrm{rank}_{\mathbb{Z}_p} H_1(G, \mathbb{Z}_p) = 2$, $\mathrm{rank}_{\mathbb{Z}} O_F[1/p]^\times = 3[F:K]-1$ by Dirichlet's unit theorem, and hence the right hand side of the above inequality is $2[F:\mathbb{Q}]-2-(3[F:K]-1) = [F:K]-1$.

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