Anticyclotomic Main Conjectures

To John Coates

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Abstract. In this paper, we prove many cases of the anticyclotomic main conjecture for general CM fields with p-ordinary CM type.

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Contents

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1. Introduction

Iwasawa's theory for elliptic curves with complex multiplication was initiated by J. Coates in the 1970s in a series of papers (for example, [CW] and [CW1]), and it is now well developed (by the effort of a handful of number-theorists) into a solid theory for abelian varieties of CM type (or one may call it Iwasawa's theory for CM fields). In this paper, we prove many cases of the anticyclotomic main conjecture for general CM fields with p-ordinary CM type.

Let M be a CM field with maximal real subfield F. The field F is totally real, and M is a totally imaginary quadratic extension of F (inside a fixed algebraic closure $\overline{F}$ of F). We fix a prime $p > 3$ *unramified* in $M/\mathbb{Q}$. We assume to have a p–ordinary CM type Σ of M. Thus, fixing an embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, the embeddings $i_p \circ \sigma$ for $\sigma \in \Sigma$ induce exactly a half Σ_p of the p–adic places of M. We identify Σ_p with a subset of prime factors of p in M. For the generator c of $\mathrm{Gal}(M/F)$, the disjoint union $\Sigma_p \sqcup \Sigma_p^c$ gives the total set of prime factors of p in M. For a multi-index $e = \sum_{\mathfrak{P}|p} e(\mathfrak{P})\mathfrak{P} \in \mathbb{Z}[\Sigma_p \sqcup \Sigma_p^c]$, we write $\mathfrak{P}^e$ for $\prod_{\mathfrak{P}|p} \mathfrak{P}^{e(\mathfrak{P})}$. We choose a complete discrete valuation ring W inside $\overline{\mathbb{Q}}_p$ finite flat and unramified over $\mathbb{Z}_p$. A Hecke character $\psi : M^\times \backslash M_{\mathbb{A}}^\times \to \mathbb{C}^\times$ is called *anticyclotomic* if $\psi(x^c) = \psi(x)^{-1}$. We call ψ has *split* conductor if the conductor of ψ is divisible only by primes split in M/F. We fix a continuous anticyclotomic character $\psi : \mathrm{Gal}(\overline{F}/M) \to W^\times$ of finite order. It is an easy consequence of class field theory(see (7.18) and [HMI] Lemma 5.31) that we can always find another Hecke character $\varphi : M_{\mathbb{A}}^\times / M^\times M_\infty^\times \to \mathbb{C}^\times$ such that $\psi(x) = \varphi^-(x) = \varphi^{-1}(x)\varphi(x^c)$. Regarding φ and ψ as Galois characters, this is equivalent to $\psi(\sigma) = \varphi^{-1}(\sigma)\varphi(c\sigma c^{-1})$ for any complex conjugation c in $\mathrm{Gal}(\overline{F}/F)$. We assume the following four conditions:

(1) *The character ψ has order prime to p with exact conductor $\mathfrak{c}\mathfrak{P}^e$ for $\mathfrak{c}$ prime to p.*
(2) *The conductor $\mathfrak{c}$ is a product of primes split in M/F.*
(3) *The local character $\psi_{\mathfrak{P}}$ is non-trivial for all $\mathfrak{P} \in \Sigma_p$.*
(4) *The restriction ψ^* of ψ to $\mathrm{Gal}(\overline{F}/M[\sqrt{p^*}])$ for $p^* = (-1)^{(p-1)/2}p$ is non-trivial.*

We study arithmetic of the unique $\mathbb{Z}_p^{[F:\mathbb{Q}]}$–extension M_∞^- of M (unramified outside p and ∞) on which $c\sigma c^{-1} = \sigma^{-1}$ for all $\sigma \in \Gamma_M^- = \mathrm{Gal}(M_\infty^-/M)$. The extension M_∞^-/M is called the anticyclotomic tower over M. Let $M(\psi)/M$ be the class field with ψ inducing the isomorphism $\mathrm{Gal}(M(\psi)/M) \cong \mathrm{Im}(\psi)$. Let $L_\infty/M_\infty^- M(\psi)$ be the maximal p–abelian extension unramified outside Σ_p. Each $\gamma \in \mathrm{Gal}(L_\infty/M)$ acts on the normal subgroup $X = \mathrm{Gal}(L_\infty/M_\infty^- M(\psi))$ continuously by conjugation, and by the commutativity of X, this action factors through $\mathrm{Gal}(M(\psi)M_\infty^-/M)$. We have a canonical splitting $\mathrm{Gal}(M(\psi)M_\infty^-/M) = \Gamma_M^- \times G_{tor}(\psi)$ for the maximal torsion subgroup $G_{tor}(\psi) \cong \mathrm{Im}(\psi)$. Since ψ is of order prime to p, it factors through the maximal torsion subgroup $G_{tor}(\psi)$. Then we look into the Γ_M^-–module: $X[\psi] = X \otimes_{\mathbb{Z}_p[G_{tor}(\psi)],\psi} W$.

As is well known, $X[\psi]$ is a $W[[\Gamma_M^-]]$–module of finite type, and it is a torsion module by a result of Fujiwara (cf. [H00] Corollary 5.4 and [HMI] Theorem 5.33) generalizing the fundamental work of Wiles [W] and Taylor-Wiles [TW]. Thus we can think of the characteristic element $\mathcal{F}^-(\psi) \in W[[\Gamma_M^-]]$ of the module $X[\psi]$. As we have seen in [HT1] and [HT2], we have the anticyclotomic p–adic Hecke L–function $L_p^-(\psi) \in \overline{W}[[\Gamma_M^-]]$ (constructed by Katz), where $\overline{W}$ is the completed p–adic integer ring of the maximal unramified extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$. We regard $W \subset \overline{W}$. Then we prove

Theorem. *We have the identity: $\mathcal{F}^-(\psi) = L_p^-(\psi)$ up to a unit in $\overline{W}[[\Gamma_M^-]]$.*

The condition $p > 3$ is necessary because at one point we need to choose a prime ideal $\mathfrak{q}$ of F with $N_{F/\mathbb{Q}}(\mathfrak{q}) \not\equiv \pm 1 \mod p$. By implementing our idea more carefully, we might be able to include the prime $p = 3$, but there is no hope (without a new idea) of including $p = 2$. The condition (1) is probably inessential, and it could be avoided by using the nearly ordinary Galois deformation with fixed p–power order nearly ordinary characters instead of the minimal one we used, although some of our argument has to be done more carefully to incorporate p–power order characters. In such a generalization, we probably need to assume (2-4) replacing ψ by the Teichmüller lift of $\psi \mod \mathfrak{m}_W$ for the maximal ideal $\mathfrak{m}_W$ of W. The condition (2) is imposed to guarantee the local representation at the prime $\mathfrak{l}$ given by $\mathrm{Ind}_{M_\mathfrak{l}}^{F_\mathfrak{l}} \varphi_\mathfrak{l}$ is reducible; otherwise, we possibly need to work with quaternionic modular forms coming from a quaternion algebra ramifying at an inert or ramified prime $\mathfrak{l}|\mathfrak{c}$, adding further technicality, though we hope that the obstacle is surmountable. The condition (3) is a rigidity condition for nearly ordinary Galois deformation of $\mathrm{Ind}_M^F \varphi$, assuring the existence of the "universal" (not "versal") deformation ring. To remove this, we need to somehow invent a reasonable requirement to rigidify the deformation problem. The condition (4) is a technical assumption in order to form a Taylor-Wiles system to identify the deformation ring with an appropriate Hecke algebra (see [TW], [Fu] and [HMI] Sections 3.2–3).

The type of the assertion (in the theorem) is called the anticyclotomic main conjecture for CM fields. The main conjecture for imaginary quadratic fields (including the cyclotomic $\mathbb{Z}_p$–extension) and its anticyclotomic version for imaginary quadratic fields have been proved by K. Rubin [R] and [R1] refining Kolyvagin's method of Euler systems, and basically at the same time, the anticyclotomic conjecture was treated by J. Tilouine (and B. Mazur) [Ti] and [MT] (for imaginary quadratic cases) by a method similar to the one exploited here combined with the class number formula of the ring class fields. A partial result towards the general conjecture was studied in [HT1], [HT2] and [H05d].

The present idea of the proof is a refinement of those exploited in [HT1], [HT2] and [H05d] Theorem 5.1, where we have proven $L_p^-(\psi)|\mathcal{F}^-(\psi)$ in $\overline{W}[[\Gamma_M^-]]$. One of the main ingredients of the proof is the congruence power series $H(\psi)\in W[[\Gamma_M^-]]$ of the CM–component of the universal nearly ordinary Hecke algebra $\mathbf{h}$ for $GL(2)_{/F}$. In the joint works with Tilouine, we took $\mathbf{h}$ of (outside p) level $N_{M/F}(\mathfrak{C})d(M/F)$ for the conductor $\mathfrak{C}$ of φ and the relative discriminant $d(M/F)$ of M/F. In this paper, as in [H05d] Section 2.10, we take the Hecke algebra of level $\mathfrak{N}(\psi)$ which is a product of $\mathfrak{c}\cap F$ and $d(M/F)$ (introducing a new type of Neben character determined by φ with $\psi=\varphi^-$). Fujiwara formulated his results in [Fu] using such level groups. Another important ingredient is the divisibility proven in [H05d] Corollary 5.5:

$$\text{(L)}\qquad (h(M)/h(F))L_p^-(\psi^-)\big|H(\psi) \text{ in } \overline{W}[[\Gamma_M^-]].$$

Here $h(M)$ (resp. $h(F)$) is the class number of M (resp. F). On the other hand, Fujiwara's result already quoted implies (see [Fu], [HT2], [H00] and [HMI] Sections 3.2–3 and 5.3):

$$\text{(F)}\qquad H(\psi)=(h(M)/h(F))\mathcal{F}^-(\psi^-) \text{ up to units in } \overline{W}[[\Gamma_M^-]].$$

Thus we need to prove:

$$\text{(R)}\qquad H(\psi)(\kappa)\big|(h(M)/h(F))L_p^-(\psi^-)(\kappa) \text{ in } \overline{W}$$

for a (single) weight κ specialization, where $\Phi(\kappa)$ is the value of a power series $\Phi\in\overline{W}[[\Gamma_M^-]]$ at $\kappa\in\mathrm{Spec}(W[[\Gamma_M^-]])(W)$. By (L) and Nakayama's lemma, the reverse divisibility (R) (specialized at κ) implies the theorem. In the (finite dimensional) space $S_\kappa^{n.ord}(\mathfrak{N}(\psi)p^\infty,\varepsilon_\lambda;W)$ of nearly p–ordinary cusp forms of weight κ with coefficients in W and with suitable Neben character ε_λ, we have a CM Hecke eigenform $f(\lambda)$ of a Hecke character λ of weight κ (regarded as a Galois character) such that λ^- factors through $\mathrm{Gal}(M(\psi)M_\infty^-/M)$ and $\lambda^-|_{G_{tor}(\psi)}=\psi$. We write $\mathfrak{N}(\lambda)$ $(\mathfrak{N}(\lambda)|\mathfrak{N}(\psi)p^\infty)$ for the level of $f(\lambda)$. This form studied in [H91] is of minimal level (possibly of level smaller than that of the primitive form). Since the CM local ring $\mathcal{R}$ of $\mathbf{h}$ is a Gorenstein ring (see [Fu], [H00] Corollary 5.3 (3) and [HMI] Proposition 1.53 and Theorem 3.59), the number $H(\psi)(\kappa)$ is the maximal denominator of the numbers $\frac{(f(\lambda),f)}{(f(\lambda),f(\lambda))}$ in W as f running through all elements of $S_\kappa(\mathfrak{N}(\lambda),\varepsilon_\lambda;W)$ (see again [H00] Corollary 5.3 (1) and [H86] Proposition 3.9), where $(\cdot,\cdot)$ is the Petersson inner product of level $\mathfrak{N}(\lambda)$. As seen in [HT1] Theorem 7.1 and [H05d]

Proposition 5.6, we have $\pi^{\kappa_1-\kappa_2+\Sigma}(f(\lambda), f(\lambda)) = c_1(h(M)/h(F))L(1,\lambda^-)$ for an innocuous constant $c_1 \in W$ (for the constant c_1, see (7.17)). The quotient $\frac{\pi^{2(\kappa_1-\kappa_2)}W_p(\lambda^-)(f(\lambda),f(\lambda))}{\Omega^{2(\kappa_1-\kappa_2)}}$ is then the value $(h(M)/h(F))L_p^-(\psi^-)(\kappa) \in \overline{W}$ (up to units in W). Here $W_p(\lambda^-)$ is the local Gauss sum of λ^- at p, Ω is the Néron period of the abelian variety of CM type Σ (defined over $\overline{\mathbb{Q}} \cap W$), and the exponent $\kappa_1 - \kappa_2$ is determined by the weight κ. Since $H(\psi)(\kappa)$ is the maximal denominator of $\frac{(f(\lambda),f)}{(f(\lambda),f(\lambda))}$, what we need to show (to prove (R)) is the W–integrality of $\frac{\pi^{2(\kappa_1-\kappa_2)}W_p(\lambda^-)(f(\lambda),f)}{\Omega^{2(\kappa_1-\kappa_2)}}$ for all $f \in S_\kappa(\mathfrak{N}(\lambda), \varepsilon_\lambda; W)$. This we will show by a detailed analysis of the residue formulas of generalized Eisenstein series, which we call Shimura series, on orthogonal groups of signature $(n, 2)$. The series have been introduced in [Sh1] and [Sh2], and we take those associated with a theta series of M and the determinant (quadratic form) of $M_2(F)$. The validity of the q–expansion principle is very important to show the W–integrality, because we write the Petersson inner product as a value of a modular form (with integral q–expansion) at a CM point of (the product of two copies of) the Hilbert modular variety. This modular form is obtained as the residue of a Shimura series. However in the split case, the orthogonal similitude group of signature $(2, 2)$ over F is isogenous to the product $GL(2) \times GL(2)_{/F}$; so, basically we are dealing with Hilbert modular forms, and the q–expansion principle is known by a work of Ribet (see [PAF] Theorem 4.21).

Another important point is to write down every W–integral Hilbert cusp form as a W–integral linear combination of theta series of the definite quaternion algebra unramified at every finite (henselian) place. Such a problem over $\mathbb{Q}$ was first studied by Eichler (his basis problem) and then generalized to the Hilbert modular case by Shimizu and Jacquet-Langlands in different manners. We scrutinize the integrality of the Jacquet-Langlands-Shimizu correspondence (mainly using duality between Hecke algebras and their spaces of cusp forms; see [H05b]). At the last step of finalizing the W–integral correspondence, we again need a result of Fujiwara: Freeness theorem in [Fu] of quaternionic cohomology groups as Hecke modules, which is valid again under the assumptions (1-4) for cusp forms with complex multiplication (see [HMI] Corollary 3.42). The everywhere unramified definite quaternion algebra exists only when the degree $[F : \mathbb{Q}]$ is even; so, we will at the end reduce, by a base-change argument, the case of odd degree to the case of even degree.

The identity: $(h(M)/h(F))L_p^-(\psi^-) = H(\psi)$ resulted from our proof of the theorem is the one (implicitly) conjectured at the end of [H86] (after Theorem 7.2) in the elliptic modular case. A similar conjecture made there for Eisenstein congruences has now also been proven by [O] under some mild assumptions.

2. Siegel's theta series for $GL(2) \times GL(2)$

Since the Shimura series has an integral presentation as a Rankin-Selberg convolution of Siegel's theta series and a Hilbert modular form, we recall here the definition and some properties of the theta series we need later.

2.1. Symmetric Domain of $O(n,2)$. We describe the symmetric domain associated to an orthogonal group of signature $(n,2)$, following [Sh1] Section 2. Let V be a $n+2$–dimensional space over $\mathbb{R}$. We consider a symmetric bilinear form $S : V \times V \to \mathbb{R}$ of signature $(n,2)$ with $n > 0$. We define an orthogonal similitude group G by

$$G(\mathbb{R}) = \left\{\alpha \in \mathrm{End}_{\mathbb{R}}(V) \middle| S(\alpha x, \alpha y) = \nu(\alpha) S(x,y) \;\text{ with } \nu(\alpha) \in \mathbb{R}^\times\right\}. \tag{2.1}$$

We would like to make explicit the symmetric hermitian domain $G(\mathbb{R})^+/\mathbb{R}^\times C$ for a maximal compact subgroup $C \subset G(\mathbb{R})^+$ for the identity connected component $G(\mathbb{R})^+$ of $G(\mathbb{R})$. We start with the following complex submanifold of $V_{\mathbb{C}} = V \otimes \mathbb{C}$:

$$\mathcal{Y}(S) = \left\{v \in V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} \middle| S[v] = S(v,v) = 0,\ S(v,\overline{v}) < 0\right\}.$$

Since S is indefinite over $\mathbb{C}$, the space $\mathcal{Y}(S)$ is always non-empty. Obviously $g \in G(\mathbb{R})$ with $\nu(g) > 0$ acts on $\mathcal{Y}(S)$ by $v \mapsto gv$.

Take $v \in \mathcal{Y}(S)$, and write W for the subspace spanned over $\mathbb{R}$ by $v + \overline{v}$ and $iv - i\overline{v}$ for $i = \sqrt{-1}$. Then we have

$$\begin{aligned}
&S(v+\overline{v}, v+\overline{v}) = 2S(v,\overline{v}) < 0\\
&S(iv - i\overline{v}, iv - i\overline{v}) = 2S(v,\overline{v}) < 0\\
&S(v+\overline{v}, iv - i\overline{v}) = -i \cdot S(v,\overline{v}) + i \cdot S(\overline{v}, v) = 0.
\end{aligned}$$

This shows that $S|_W$ is negative definite. Let $W^\perp = \left\{w \in V \middle| S(w,W) = 0\right\}$. Then we have an orthogonal decomposition: $V = W \oplus W^\perp$ and $S|_{W^\perp}$ is positive definite. We then define a positive definite bilinear form

$$P_v(x,y) = -S(x_W, y_W) + S(x_{W^\perp}, y_{W^\perp})$$

for the orthogonal projections x_W to W and $x_{W^\perp}$ to $W^\perp$ of x. The bilinear form P_v is called the *positive majorant* of S indexed by $v \in \mathcal{Y}(S)$. If $g \in G(\mathbb{R})$ fixes $v \in \mathcal{Y}(S)$, g fixes by definition the positive definite form P_v. Thus g has to be in the compact subgroup $O(P_v)$ made up of orthogonal matrices preserving P_v. Thus $G(\mathbb{R})^+/O(P_v) \hookrightarrow \mathcal{Y}(S)$. If we have two $v, w \in \mathcal{Y}(S)$, then by Sylvester's theorem, we find $g \in G(\mathbb{R})^+$ such that $gv = w$, and hence $G(\mathbb{R})^+/O(P_v) \cong \mathcal{Y}(S)$.

Writing $P_v[x] = P_v(x,x)$ for $x = cv + \overline{cv} + z$ with $c \in \mathbb{C}$ and $z \in W^\perp$, we see

(2.2)
$$\begin{aligned} P_v[x] - S[x] =& P_v(cv + \overline{cv} + z, cv + \overline{cv} + z) - S(cv + \overline{cv} + z, cv + \overline{cv} + z) \\ =& -2c^2 S[v] - 2\overline{c}^2 S[\overline{v}] - 4|c|^2 S(v,\overline{v}) + S[z] - S[z] \\ =& 4|c|^2 S(v,\overline{v}) = -4S(v,\overline{v})^{-1}|S(x,v)|^2 \geq 0. \end{aligned}$$

We now make explicit the domain $\mathcal{Y}(S)$ as a hermitian bounded matrix domain.

Proposition 2.1. *We have a $\mathbb{C}$–linear isomorphism $A : V_{\mathbb{C}} \cong \mathbb{C}^{n+2}$ such that*
$$S(x,y) = {}^t(Ax) \cdot RAy, \;\; S(\overline{x},y) = {}^t\overline{(Ax)} \cdot QAy,$$
where R and Q are real symmetric matrices given by
$$R = \begin{pmatrix} 1_n & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \;\; Q = \begin{pmatrix} 1_n & 0 \\ 0 & -1_2 \end{pmatrix}.$$

Proof. Choose a base $v_1, \ldots, v_{n+2}$ of V over $\mathbb{R}$, identify V with $\mathbb{R}^{n+2}$ by sending $\sum_{i=1}^{n+2} x_i v_i \mapsto {}^t(x_1, \ldots, x_{n+2}) \in \mathbb{R}^{n+2}$ and use the same symbol S for the symmetric matrix $(S(v_i,v_j))_{i,j}$. Then $S(x,y) = {}^tx \cdot Sy$ for $x, y \in V = \mathbb{R}^{n+2}$. By a theorem of Sylvester, S is equivalent (in $GL_{n+2}(\mathbb{R})$) to Q; so, we find an invertible matrix $X \in GL_{n+2}(\mathbb{R})$ with ${}^tX \cdot SX = Q$.

Choose $B = \mathrm{diag}[1_n, \sqrt{2}^{-1} \left(\begin{smallmatrix} 1 & 1 \\ -i & i \end{smallmatrix}\right)]$. Then by computation ${}^t\overline{B} \cdot QB = Q$ and ${}^tBQB = R$. Then $x \mapsto Ax$ for $A = (XB)^{-1} = B^{-1}X^{-1}$ does the desired job. □

By our choice of A, the map $\alpha \mapsto A\alpha A^{-1}$ gives an isomorphism of Lie groups:

(2.3)
$$\begin{aligned} &\iota : G(\mathbb{R}) \cong G(Q,R) \\ &= \left\{\alpha \in GL_{n+2}(\mathbb{C}) \middle| {}^t\alpha \cdot R\alpha = \nu(\alpha)R, \; {}^t\overline{\alpha} \cdot q\alpha = \nu(\alpha)Q \;\text{ with } \alpha \in \mathbb{R}^\times\right\}, \end{aligned}$$

and the map: $v \mapsto Av$ gives an isomorphism of complex manifolds:

(2.4)
$$j : \mathcal{Y}(S) \cong \mathcal{Y}(Q,R) = \left\{u \in \mathbb{C}^{n+2} \middle| {}^tu \cdot Ru = 0, \; {}^t\overline{u} \cdot Qu < 0\right\}.$$

These two maps are equivariant:
$$\iota(\alpha)j(v) = j(\alpha v).$$

We are going to show that $\mathcal{Y}(Q,R)$ has two connected components. Write $u = {}^t(u_1, \ldots, u_{n+2}) \in \mathcal{Y}(Q,R)$. Then we have
$$\left(\sum_{i=1}^n u_i^2\right) - 2u_{n+1}u_{n+2} = {}^tu \cdot Ru = 0,$$
$$\sum_{i=1}^n |u_i|^2 < |u_{n+1}|^2 + |u_{n+2}|^2 \Leftrightarrow {}^t\overline{u} \cdot Qu < 0.$$

Assume $|u_{n+1}| = |n_{n+2}|$ towards contradiction. Then we see

$$\sum_{j=1}^{n} |u_j|^2 \geq |\sum_{j=1}^{n} u_j^2| = 2|u_{n+1}u_{n+2}| = |u_{n+1}|^2 + |u_{n+2}|^2,$$

a contradiction; hence we have either $|u_{n+1}| > |u_{n+2}|$ or $|u_{n+1}| < |u_{n+2}|$. These two cases split the domain $\mathcal{Y}(Q,R)$ into two pieces of connected components. To see each component is connected, we may assume that $|u_{n+2}| > |u_{n+1}|$ by interchanging indices if necessary; so, $u_{n+2} \neq 0$. Put $z_j = U_j/u_{n+2}$ for $j \leq n$, and define a column vector $z = {}^t(z_1, z_2, \ldots, z_n)$. Then $w = u_{n+1}/u_{n+2} = {}^tz \cdot z/2$, and defining

$$\mathfrak{Z} = \mathfrak{Z}_n = \left\{ z \in \mathbb{C}^n \middle| {}^tz \cdot \overline{z} < 1 + \frac{1}{4}|{}^tz \cdot z|^2 < 2 \right\}, \tag{2.5}$$

$\mathbb{C}^\times \times \mathfrak{Z}$ is isomorphic to the connected component of $\mathcal{Y}(Q,R)$ given by $|u_{n+2}| > |u_{n+1}|$ via $(\lambda, z) \mapsto \lambda \mathcal{P}(z)$, where

$$\mathcal{P}(z) = {}^t(z, ({}^tz \cdot z)/2, 1). \tag{2.6}$$

From this expression, it is plain that $\mathcal{Y}(Q,R)$ has two connected components. We define the action of $\alpha \in G(\mathbb{R})$ on $\mathfrak{Z}$ and a factor of automorphy $\mu(\alpha; z)$ ($z \in \mathfrak{Z}$) by

$$\iota(\alpha)\mathcal{P}(z) = \mathcal{P}(\alpha(z))\mu(\alpha; z). \tag{2.7}$$

We look into spherical functions on $V_{\mathbb{C}}$. Choose a base $v_1, \ldots, v_d$ of V over $\mathbb{R}$. By means of this base, we identify V with $\mathbb{R}^d$ ($d = n+2$); so, $v \mapsto (x_1, \ldots, x_d)$ if $v = \sum_j x_j v_j$. We take the dual base v_j^* so that $S(v_i^*, v_j) = \delta_{ij}$ for the Kronecker symbol δ_{ij} and define a second-degree homogeneous differential operator Δ by

$$\Delta = \sum_{i,j} S(v_i^*, v_j^*) \frac{\partial^2}{\partial x_i \partial x_j}.$$

A polynomial function $\eta : V \to \mathbb{C}$ is called a spherical function if $\Delta\eta = 0$. Writing $S = (S(v_i, v_j))$, we see that this definition does not depend on the choice of the base v_j, because $\Delta = {}^t\partial S^{-1}\partial$ for $\partial = {}^t(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d})$. Since $\partial({}^twSx) = Sw$ for a constant vector $w = (w_1, \ldots, w_d)$, we find that, for $k \geq 2$

$$\Delta({}^twSx)^k = k{}^t\partial(S^{-1}Sw)({}^twSx)^{k-1} = k(k-1)({}^twSw)({}^twSx)^{k-2}.$$

Thus the polynomial function $x \mapsto S(w,x)^k$ for $k \geq 2$ is spherical if and only if $S[w] := S(w,w) = 0$. All homogeneous spherical functions of degree $k \geq 2$ are linear combination of $S(w,x)^k$ for a finite set of spherical vectors w with $S[w] = 0$. In particular, for $v \in \mathcal{Y}(S)$, the function $x \mapsto S(v,x)^k$ is a spherical function.

Note here that for $v \in \mathcal{Y}(S)$, $S[v] = 0$ and $S(v,x) = -P_v(v,x)$, because $P(v,x) = P(v, x_W) + P(v, x_{W^\perp}) = -S(v, x_W) = -S(v,x)$. Define $\partial_v = \widetilde{v} \cdot \partial$, where $\widetilde{v} = (\lambda_1, \ldots, \lambda_d)$ when $v = \sum_j \lambda_j v_j$. Then we have, by computation,

$$\partial_v S[x] = 2S(v,x), \ \partial_v P_v[x] = 2P_v(v,x) = -S(v,x). \tag{2.8}$$

We define a Schwartz function Ψ on V for each $\tau = \xi + i\eta \in \mathfrak{H}$ and $v \in \mathcal{Y}(S)$ by

$$\Psi(\tau; v; w) = \mathbf{e}(\frac{1}{2}(S[w]\xi + iP_v[w]\eta)) = \exp(\pi i(S[w]\xi + iP_v[w]\eta)).$$

We see by computation using (2.8)

$$(\partial_v^k \Psi)(\tau; v; w) = (2\pi i)^k (\overline{\tau} S(v, w))^k \Psi(\tau; v; w). \tag{2.9}$$

2.2. $SL(2) \times SL(2)$ as an orthogonal group. We realize the product as an orthogonal group of signature $(2, 2)$, and hence this group gives a special case of the orthogonal groups treated in the previous subsection.

Let $V = M_2(\mathbb{R})$, and consider the symmetric bilinear form $S : V \times V \to \mathbb{R}$ given by $S(x, y) = \mathrm{Tr}(xy^\iota)$, where $yy^\iota = y^\iota y = \det(y)$ for 2×2 matrices y. We let $(a, b) \in GL_2(\mathbb{R}) \times GL_2(\mathbb{R})$ act on V by $x \mapsto axb^\iota$. Then

$$\begin{aligned} S(axb^\iota, ayb^\iota) = \mathrm{Tr}(axb^\iota by^\iota a^\iota) &= \det(b)\mathrm{Tr}(axy^\iota a^\iota) \\ &= \det(b)\mathrm{Tr}(xy^\iota a^\iota a) = \det(a)\det(b)S(x, y). \end{aligned}$$

Thus we have an isomorphism

$$(GL_2(\mathbb{R}) \times GL_2(\mathbb{R})) / \{\pm(1, 1)\} \hookrightarrow G(\mathbb{R})$$

with $\nu(a, b) = \det(a)\det(b)$. Since the symmetric space of $G(\mathbb{R})$ has dimension 2 over $\mathbb{C}$, the above isomorphism has to be onto on the identity connected component. Since $G(\mathbb{R})$ has four connected components (because $\mathcal{Y}(S)$ has two), the above morphism has to be a surjective isomorphism because $GL_2(\mathbb{R}) \times GL_2(\mathbb{R})$ has four connected components:

$$(GL_2(\mathbb{R}) \times GL_2(\mathbb{R})) / \{\pm(1, 1)\} \cong G(\mathbb{R}). \tag{2.10}$$

Since the symmetric domain of $GL_2(\mathbb{R}) \times GL_2(\mathbb{R})$ is isomorphic to $\mathfrak{H} \times \mathfrak{H}$ for the upper half complex plane $\mathfrak{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$, we find that $\mathfrak{Z} \cong \mathfrak{H} \times \mathfrak{H}$.

We are going to make this isomorphism: $\mathfrak{Z} \cong \mathfrak{H} \times \mathfrak{H}$ more explicit. We study $\mathcal{Y} = \mathcal{Y}(S)$ more closely. Since $V_\mathbb{C} = M_2(\mathbb{C})$, writing $v = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{C})$, we have from the definition:

$$\mathcal{Y} = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{C}) \middle| ad = bc,\ a\overline{d} - b\overline{c} + d\overline{a} - c\overline{b} < 0 \right\}.$$

Pick $v = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{Y}$, and suppose that $c = 0$. Then by the defining equation of $\mathcal{Y}$, $ad = 0 \Rightarrow 0 = a\overline{d} + d\overline{a} < 0$, which is a contradiction. Thus $c \neq 0$; so, we define for v as above, $z = \frac{a}{c}$ and $w = -\frac{d}{c}$. Then $-zw = \frac{b}{c}$, and hence (see [Sh2] II (4.6))

$$v = cp(z, w) \quad \text{with} \quad p(z, w) = \left(\begin{smallmatrix} z & -wz \\ 1 & -w \end{smallmatrix}\right) = -{}^t(z, 1)(w, 1)\varepsilon, \tag{2.11}$$

where $\varepsilon = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Again by the equation defining $\mathcal{Y}$,

$$S(p(z, w), \overline{p(z, w)}) = (w - \overline{w})(z - \overline{z}) = -z\overline{w} + zw - \overline{z}w + \overline{zw} < 0. \tag{2.12}$$

From this, it is clear that $\mathcal{Y} \cong \mathbb{C}^\times \times \left(\mathfrak{H}^2 \sqcup \overline{\mathfrak{H}}^2\right)$. By this isomorphism, for $\alpha \in G(\mathbb{R})$, we can define its action $\alpha(z,w) \in \left(\mathfrak{H}^2 \sqcup \overline{\mathfrak{H}}^2\right)$ and a factor $\mu(\alpha; z, w) \in \mathbb{C}^\times$ of automorphy by

$$\alpha \cdot p(z,w) = p(\alpha(z,w))\mu(\alpha; z, w).$$

By a direct computation, writing $j(v,z) = cz+d$ for $v = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $v(z) = \frac{az+b}{cz+d}$, we have, for $(\alpha, \beta) \in GL_2(\mathbb{R}) \times GL_2(\mathbb{R})$,

$$\alpha p(z,w)\beta^\iota = p(\alpha(z), \beta(w))j(\alpha, z)j(\beta, w). \tag{2.13}$$

Thus

$$(\alpha,\beta)(z,w) = (\alpha(z), \beta(w)) \quad \text{and} \quad \mu((\alpha,\beta);(z,w)) = j(\alpha,z)j(\beta,w).$$

We define a spherical function

$$v \mapsto [v; z, w]^k = S(v, p(z,w))^k \tag{2.14}$$

for a positive integer $k > 0$. This function is spherical because $S[p(z,w)] = 2\det p(z,w) = 0$, and we have

$$[\alpha v \beta^\iota; z, w] = j(\alpha^\iota, z)j(\beta^\iota, w)[v; \alpha^{-1}(z), \beta^{-1}(w)]. \tag{2.15}$$

2.3. Growth of theta series. Let F be a totally real field with integer ring O and B be a quaternion algebra over F. The algebra B can be $M_2(F)$. Let $x \mapsto x^\iota$ be the main involution of B; so, $xx^\iota = N(x)$ and $x + x^\iota = \mathrm{Tr}(x)$ for the reduced norm $N : B \to F$ and the reduced trace $\mathrm{Tr} : B \to F$. We consider the symmetric bilinear form $S : B \times B \to F$ given by $S(x,y) = \mathrm{Tr}(xy^\iota)$.

Writing I for the set of all archimedean places of F, we split $I = I_B \sqcup I^B$ so that $B \otimes_{F,\sigma} \mathbb{R} \cong M_2(\mathbb{R}) \Leftrightarrow \sigma \in I_B$. Thus for $\sigma \in I^B$,

$$B \otimes_{F,\sigma} \mathbb{R} \cong \mathbb{H} = \left\{ \left(\begin{smallmatrix} a & b \\ -\overline{b} & \overline{a} \end{smallmatrix}\right) \middle| a, b \in \mathbb{C} \right\}.$$

We identify $B_\sigma = B \otimes_{F,\sigma} \mathbb{R}$ with $M_2(\mathbb{R})$ or $\mathbb{H}$ for each $\sigma \in I$. Thus $G(\mathbb{Q}) = (B^\times \times B^\times)/\{\pm(1,1)\}$ is the orthogonal group of (B,S). Since S at $\sigma \in I^B$ is positive definite, $G(\mathbb{R}) \cong (GL_2(\mathbb{R}) \times GL_2(\mathbb{R}))^{I_B} \times (\mathbb{H}^\times \times \mathbb{H}^\times)^{I^B}/\{\pm(1,1)\}$. For each $b \in B_\infty = B \otimes_{\mathbb{Q}} \mathbb{R}$, writing $b = (b_\sigma)$ for σ–component $b_\sigma \in B_\sigma$, we define

$$[b; z, w]^k = \prod_{\sigma \in I_B} [b_\sigma; z_\sigma, w_\sigma]^{k_\sigma} \quad (k = \sum_{\sigma \in I_B} k_\sigma \sigma \in \mathbb{Z}[I_B]), \tag{2.16}$$

where $[b_\sigma; z_\sigma, w_\sigma]$ is as in (2.14) defined for $B_\sigma = M_2(\mathbb{R})$. For $\sigma \in I^B$, we pick a homogeneous spherical polynomial $\varphi_\sigma : B_\sigma \to \mathbb{C}$ of degree κ_σ, and put $\varphi = \prod_{\sigma \in I^B} \varphi_\sigma$ and $\kappa = \sum_\sigma \kappa_\sigma \in \mathbb{Z}[I^B]$. We define an additive character $\mathbf{e}_F : F_{\mathbb{C}} = F \otimes_{\mathbb{Q}} \mathbb{C} \to \mathbb{C}^\times$ by $\mathbf{e}_F(z) = \exp(2\pi i \sum_\sigma z_\sigma)$ $(z = (z_\sigma)_{\sigma \in I})$ identifying $F_{\mathbb{C}}$ with $\mathbb{C}^I$ as $\mathbb{C}$–algebras. Writing $\mathrm{Tr} : F_{\mathbb{C}} \to \mathbb{C}$ for the trace map, we have $\mathbf{e}_F(z) = \mathbf{e}(\mathrm{Tr}(z))$.

We consider Siegel's theta series defined for $0 \le k \in \mathbb{Z}[I_B]$ and a Schwartz-Bruhat function $\phi : B_{\mathbb{A}^{(\infty)}} \to \mathbb{C}$:

$$\eta^{-I}\theta_k(\tau; z, w; v, \phi\varphi) = \sum_{\ell \in B} [\ell; z, w]^k (\phi\varphi)(\ell) \mathbf{e}_F(\frac{1}{2}(\xi S[\ell] + i\eta P_{p(z,w)}[\ell])) \tag{2.17}$$

$$= \sum_{\ell \in B} [\ell; z, w]^k (\phi\varphi)(\ell) \mathbf{e}(\frac{1}{2}\mathrm{Tr}(S[\ell]\tau)) \mathbf{e}\left(\frac{i}{2} \sum_{\sigma \in I_B} \frac{\eta_\sigma \left|[\ell_\sigma; z_\sigma, w_\sigma]\right|^2}{\mathrm{Im}(z_\sigma)\,\mathrm{Im}(w_\sigma)}\right),$$

where $\tau = \xi + i\eta \in \mathfrak{H}^I$, $\eta^I(\tau) = \prod_\sigma \eta_\sigma$ and the last equality follows from (2.12). Since the majorant $P_{p(z,w)}$ is positive definite, the theta series is rapidly decreasing with respect to τ towards the cusp ∞, as long as $\varphi(0)[0; z, w]^k = 0$ (in other words, as long as $k + \kappa > 0$). Since the infinity type $k + \kappa$ does not change under the transformation $\tau \mapsto \alpha(\tau)$ for $\alpha \in SL_2(F)$, the theta series is rapidly decreasing towards any given cusp if $k + \kappa > 0$. Otherwise it is slowly increasing (see below Proposition 2.3).

2.4. Partial Fourier transform. We are going to compute in the following subsection the Fourier expansion of the theta series (introduced in the earlier subsections) with respect to (z, w) when $B = M_2(F)$. This is non-trivial, because θ is defined by its Fourier expansion with respect to the variable τ. A key idea is to compute the partial Fourier transform of each term of the theta series and to resort to the Poisson summation formula. In this subsection, we describe the computation of the partial Fourier transform.

The Schwartz function on $B_\infty = B \otimes_{\mathbb{Q}} \mathbb{R} = M_2(F_\infty)$ which gives rise to the theta series $\theta_0(\tau; z, w; \phi)$ is given by

$$u \mapsto \Psi_0(u) = \eta^I \mathbf{e}_F(\det(u)\tau + \frac{\eta}{2yt}|[u; z, w]|^2)$$

for $\tau = \xi + i\eta$, $z = x + yi$ and $w = r + ti$ with $\xi, x, r \in F_\infty$ and $\eta, y, t \in F_{\infty+}^\times$. Here $F_{\infty+}^\times$ is the identity connected component of $F_\infty^\times$. We define

$$\Psi_k(u) = \prod_\sigma \Psi_{k_\sigma,\sigma}(u_\sigma) \quad (0 \le k = \sum_\sigma k_\sigma \sigma \in \mathbb{Z}[I]) \text{ and} \tag{2.18}$$

$$\Psi_{k_\sigma,\sigma}(u_\sigma) = \eta_\sigma^{k_\sigma+1} [u_\sigma; z_\sigma, w_\sigma]^{k_\sigma} \mathbf{e}(\det[u_\sigma]\tau_\sigma + i\frac{\eta_\sigma}{2y_\sigma t_\sigma}|[u_\sigma; z_\sigma, w_\sigma]|^2).$$

We write the variable $u = \left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right)$ for two row vectors u_j and write individually $u_1 = (a, b)$ and $u_2 = (c, d)$. The partial Fourier transform ϕ^* of ϕ is given by

$$\phi^*\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \int_{F_\infty^2} \phi\left(\begin{smallmatrix} a' & b' \\ c & d \end{smallmatrix}\right) \mathbf{e}_F(ab' - ba') da' db', \tag{2.19}$$

where $ab' - ba' = \frac{1}{2}S\left[\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right)\right]$ and $da' = \otimes_\sigma da'_\sigma$ for the Lebesgue measure da'_σ on the σ–component $\mathbb{R}$ of F_∞. By applying complex conjugation, we have

$$\overline{\phi^*}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\overline{\phi}\right)^*\left(\begin{smallmatrix} -a & -b \\ c & d \end{smallmatrix}\right). \tag{2.20}$$

We compute first the partial Fourier transform to the action of $\mathcal{U}(F_\infty)\times GL_2(F_\infty)$, where $\mathcal{U}(X)$ is made up of upper unipotent matrices with right shoulder entry in X. We first deal with $(1,\beta)$ with $\beta\in GL_2(F_\infty)$:

$$
\begin{aligned}
(\phi\circ(1,\beta))^*\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) &= \int_{F^2_\infty}\phi\left(\left(\begin{smallmatrix} a' & b'\\ c & d\end{smallmatrix}\right)\beta^\iota\right)\mathbf{e}_F(-(a',b')\varepsilon^t(a,b))da'db'\\
&\overset{(a',b')\beta^\iota\mapsto(a',b')}{=}|N(\det(\beta)|^{-1}\int_{F^2_\infty}\phi\left(\begin{smallmatrix} a' & b'\\ c & d\end{smallmatrix}\right)\mathbf{e}_F(-(a',b')\beta^{-\iota}\varepsilon^t(a,b))da'db'\\
&=|N(\det(\beta)|^{-1}\int_{F^2_\infty}\phi\left(\begin{smallmatrix} a' & b'\\ c & d\end{smallmatrix}\right)\mathbf{e}_F(-(a',b')\beta^{-\iota}\varepsilon^t\beta^{-1t}(a,b))da'db'\\
&=|N(\det(\beta)|^{-1}\phi^*\circ\left(\left(\begin{smallmatrix} 1 & 0\\ 0 & \det(\beta)\end{smallmatrix}\right),\beta^{-1}\right)\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right).
\end{aligned}
$$

We now compute $(\phi\circ(\alpha,1))^*$ for $\alpha\in\mathcal{U}(F_\infty)$:

$$
\begin{aligned}
(\phi\circ(\left(\begin{smallmatrix} 1 & x\\ 0 & 1\end{smallmatrix}\right),1))^*\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) &= \int_{F^2_\infty}\phi\left(\begin{smallmatrix} a'+xc & b'+xd\\ c & d\end{smallmatrix}\right)\mathbf{e}_F(ab'-ba')da'db'\\
&\overset{(a'+xc,b'+xd)\mapsto(a',b')}{=}\int_{F^2_\infty}\phi\left(\begin{smallmatrix} a' & b'\\ c & d\end{smallmatrix}\right)\mathbf{e}_F(ab'-ba')da'db'\mathbf{e}_F(-x(ad-bc))\\
&= \mathbf{e}_F(-x(ad-bc))\phi^*\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right).
\end{aligned}
$$

Summarizing the above computation, we get for $(\left(\begin{smallmatrix} 1 & x\\ 0 & 1\end{smallmatrix}\right),1)\in\mathcal{U}(F_\infty)\times SL_2(F_\infty)$

$$
(\phi\circ(\left(\begin{smallmatrix} 1 & x\\ 0 & 1\end{smallmatrix}\right),\beta))^*(u)=\mathbf{e}_F(-x\det(u))\phi^*\circ(1,\beta^{-1})(u). \tag{2.21}
$$

By (2.15), for $(\alpha,\beta)\in SL_2(F_\infty)\times SL_2(F_\infty)$, we have

$$
\begin{aligned}
[\alpha u\beta^\iota;z,w]=S(\alpha u\beta^\iota;p(z,w))&=S(u;\alpha^{-1}p(z,w)\beta^{-\iota})\\
&=[u,\alpha^{-1}(z),\beta^{-1}(w)]j(\alpha^{-1},z)j(\beta^{-1},z).
\end{aligned}
$$

To compute the partial Fourier transform of Ψ_k, we may therefore assume that $r=x=0$. Then the computation for Ψ^*_0 is reduced to, writing $u'=\left(\begin{smallmatrix} a' & b'\\ c & d\end{smallmatrix}\right)$ (and omitting the subscript σ),

$$
\begin{aligned}
&\int_{F^2_\sigma}\Psi_{0,\sigma}(u')\mathbf{e}(ab'-ba')da'db'= \\
&\int_{\mathbb{R}^2}\eta\mathbf{e}\left(\xi\det u'+\frac{i\eta}{2}(\frac{ta'^2}{y}+\frac{b'^2}{yt}+\frac{yd^2}{t}+ytc^2)\right)\mathbf{e}(ab'-ba')da'db'.
\end{aligned}\tag{2.22}
$$

We then invoke the following formula:

$$
\int_{-\infty}^{\infty}\exp(-\pi za'^2)\mathbf{e}(a'b)da'=z^{-1/2}\exp(-\frac{\pi b^2}{z}),
$$

where $z \in \mathbb{C} - \mathbb{R}_-$ ($\mathbb{R}_-$: the negative real line) and $z^{-1/2}$ is the branch of the square root which is positive real if z is positive real. Then (2.22) is equal to

$$\begin{aligned}(2.23)\quad & y_\sigma \exp(-\pi\eta^{-1}\left(\frac{y_\sigma}{t_\sigma}(d\xi_\sigma - b)^2 + y_\sigma t_\sigma(c\xi_\sigma - a)^2\right)\mathbf{e}\left(\frac{i\eta_\sigma}{2}(\frac{y_\sigma}{t_\sigma}d^2 + y_\sigma t_\sigma c^2)\right) \\ &= y_\sigma \exp\left(-\pi\frac{y_\sigma}{\eta_\sigma}(\frac{1}{t_\sigma}|d\tau_\sigma - b|^2 + t|c\tau_\sigma - a|^2)\right).\end{aligned}$$

By computation, we have

$$(2.24)\qquad t|\tau c - a|^2 + t^{-1}|\tau d - b|^2 = t^{-1}|[u;\tau,it]|^2 + 2\eta\det(u).$$

Thus we get

$$\Phi_0(u) = \Psi_0^*(u) = \prod_\sigma \Psi_{0,\sigma}^*(u_\sigma),$$

$$\begin{aligned}(2.25)\quad \Phi_{0,\sigma}(u) = \Psi_{0,\sigma}^*\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) &= y_\sigma \exp\left(-\pi\frac{y_\sigma}{\eta_\sigma}(\frac{1}{t_\sigma}|d\tau_\sigma - b|^2 + t_\sigma|c\tau_\sigma - a|^2)\right) \\ &= y_\sigma \exp\left(-2\pi y_\sigma\det(u) - \pi\frac{y_\sigma}{\eta_\sigma t_\sigma}|[u;\tau_\sigma, it_\sigma]|^2\right).\end{aligned}$$

In order to compute the partial Fourier transform of Ψ_k, we consider the following differential operator

$$(2.26)\qquad \partial_\sigma = S\left(p(\tau_\sigma, w_\sigma), {}^t\left(\begin{smallmatrix} \frac{\partial}{\partial a} & \frac{\partial}{\partial b} \\ \frac{\partial}{\partial c} & \frac{\partial}{\partial d}\end{smallmatrix}\right)^\iota\right) = \tau_\sigma\frac{\partial}{\partial a} - w_\sigma\tau_\sigma\frac{\partial}{\partial b} + \frac{\partial}{\partial c} - w_\sigma\frac{\partial}{\partial d}.$$

Since we have, for $u = \left(\begin{smallmatrix} a' & b' \\ c & d\end{smallmatrix}\right)$,

$$\begin{aligned}\tau_\sigma\frac{\partial}{\partial a}\mathbf{e}(ab' - ba') &= 2\pi i\tau_\sigma b'\mathbf{e}(ab' - ba') \\ -w_\sigma\tau_\sigma\frac{\partial}{\partial b}\mathbf{e}(ab' - ba') &= 2\pi i w_\sigma\tau_\sigma a'\mathbf{e}(ab' - ba') \\ \frac{\partial}{\partial c}\Psi_{0,\sigma}(u) &= (-2\pi i b'\tau_\sigma - \pi\frac{\eta_\sigma}{y_\sigma t_\sigma}(w_\sigma z_\sigma\overline{[u;z_\sigma,w_\sigma]} + \overline{w}_\sigma\overline{z}_\sigma[u;z_\sigma,w_\sigma]))\Psi_{0,\sigma}(u) \\ -w_\sigma\frac{\partial}{\partial d}\Psi_{0,\sigma}(u) & \\ = -(2\pi i a'\tau_\sigma w_\sigma &- \pi\frac{\eta_\sigma}{y_\sigma t_\sigma}(w_\sigma z_\sigma\overline{[u;z_\sigma,w_\sigma]} + w_\sigma\overline{z}_\sigma[u;z_\sigma,w_\sigma]))\Psi_{0,\sigma}(u).\end{aligned}$$

Taking the fact that $\overline{w}_\sigma - w_\sigma = 2it_\sigma$, $z_\sigma = iy_\sigma$ and

$$\partial_\sigma([u;z_\sigma,w_\sigma]) = \partial_\sigma(S_\sigma(u,p(z_\sigma,w_\sigma)) = S_\sigma(p(\tau_\sigma,w_\sigma),p(z_\sigma,w_\sigma)) = 0$$

into account, we have

$$(2.27)\qquad \partial_\sigma(\Psi_{j,\sigma}(u)\mathbf{e}(ab' - ba')) = 2\pi\Psi_{j+1,\sigma}(u)\mathbf{e}(ab' - ba')$$

for all integers $j \geq 0$.

To complete the computation, we need to compute $\partial_\sigma \Phi_{j,\sigma}(u)$. We have, noting that we are restricting ourselves to $w_\sigma = it_\sigma$:

$$\begin{aligned}
\tau_\sigma \frac{\partial}{\partial a}\Phi_{0,\sigma}(u) &= \pi \frac{y_\sigma t_\sigma}{\eta_\sigma}(\tau_\sigma(c\overline{\tau}_\sigma - a) + \tau_\sigma(c\tau_\sigma - a))\Phi_{0,\sigma}(u) \\
-it_\sigma\tau_\sigma \frac{\partial}{\partial b}\Phi_{0,\sigma}(u) &= -\pi i \frac{y_\sigma}{\eta_\sigma}(\tau_\sigma(d\overline{\tau}_\sigma - b) + \tau_\sigma(d\tau_\sigma - b))\Phi_{0,\sigma}(u) \\
\frac{\partial}{\partial c}\Phi_{0,\sigma}(u) &= -\pi \frac{y_\sigma t_\sigma}{\eta_\sigma}(\tau_\sigma(c\overline{\tau}_\sigma - a) + \overline{\tau}_\sigma(c\tau_\sigma - a))\Phi_{0,\sigma}(u) \\
-it_\sigma \frac{\partial}{\partial d}\Phi_{0,\sigma}(u) &= \pi i \frac{y_\sigma}{\eta_\sigma}(\tau_\sigma(d\overline{\tau}_\sigma - b) + \overline{\tau}_\sigma(d\tau_\sigma - b))\Phi_{0,\sigma}(u).
\end{aligned}$$

From this we get, taking the fact:

$$it_\sigma(c\tau_\sigma - a) + d\tau_\sigma - b = [u; \tau, it_\sigma]$$

into account, we have

$$\partial_\sigma \Phi_{0,\sigma}(u) = 2\pi y_\sigma [u; \tau_\sigma, it_\sigma]\Phi_{0,\sigma}(u).$$

Since $\partial_\sigma([u;\tau,w]) = 0$, we again obtain, when $z = iy$ and $w = it$,

$$\partial_\sigma(\Phi_{j,\sigma})(u) = 2\pi\Phi_{j+1}(u), \tag{2.28}$$

where $\Phi_{j,\sigma}(u) = y_\sigma^{j+1}[u;\tau,w]^j\Phi_{0,\sigma}(u)$. By (2.27) and (2.28) combined, we get, at this moment for $z = iy$ and $w = it$,

$$(\Psi_k)^*(u) = \Phi_k(u), \tag{2.29}$$

where $\Phi_k(u) = \prod_\sigma \Phi_{k_\sigma,\sigma}(u_\sigma)$ and $\Psi_k(u) = \prod_\sigma \Psi_{k_\sigma,\sigma}(u_\sigma)$.

We are going to compute the partial Fourier transform for general (z, w) and show that (2.29) is valid in general under a suitable description of Φ for general (z, w): To do this, we write

$$\Psi_{j,\sigma}^{z_\sigma,w_\sigma,\tau_\sigma}(u) = \eta_\sigma^{j+1}[u; z_\sigma, w_\sigma]^j \mathbf{e}\left(\det(u)\tau_\sigma + i\frac{\eta_\sigma}{2y_\sigma t_\sigma}\left|[u; z_\sigma, w_\sigma]\right|^2\right).$$

Since $[u, \alpha(z_\sigma), \beta(w_\sigma)]j(\alpha, z)j(\beta, w) = [\alpha^{-1}u\beta^{-\iota}; z_\sigma, w_\sigma]$ by (2.13) and (2.14) combined, we have

$$\Psi_{j,\sigma}^{z_\sigma,w_\sigma,\tau_\sigma} = \Psi_{j,\sigma}^{iy_\sigma,it_\sigma,\tau_\sigma} \circ \left(\left(\begin{smallmatrix} 1 & -x_\sigma \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & -r_\sigma \\ 0 & 1 \end{smallmatrix}\right)\right).$$

Then by (2.21),

$$\left(\phi \circ \left(\left(\begin{smallmatrix} 1 & -x_\sigma \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & -r_\sigma \\ 0 & 1 \end{smallmatrix}\right)\right)\right)^*(u) = \mathbf{e}(x_\sigma \det(u))\phi^* \circ \left(1, \left(\begin{smallmatrix} 1 & r_\sigma \\ 0 & 1 \end{smallmatrix}\right)\right)$$

and applying this to $\Psi_{j,\sigma}^{z_\sigma,w_\sigma,\tau_\sigma}$, we get from (2.29)

$$\left(\Psi_{j,\sigma}^{z_\sigma,w_\sigma,\tau_\sigma}\right)^*(u) = \mathbf{e}(x_\sigma \det(u))\Phi_{j,\sigma}^{iy_\sigma,it_\sigma,\tau_\sigma}\left(u\left(\begin{smallmatrix} 1 & -r_\sigma \\ 0 & 1 \end{smallmatrix}\right)\right),$$

where

$$\Phi_{j,\sigma}^{iy_\sigma,it_\sigma,\tau_\sigma}(u) = [u_\sigma; \tau_\sigma, it_\sigma]^j y_\sigma^{j+1}\exp\left(-2\pi y_\sigma \det(u) - \pi\frac{y_\sigma}{\eta_\sigma t_\sigma}|[u; \tau_\sigma, it_\sigma]|^2\right).$$

Define
(2.30)

$$\Phi_k(u) = \Phi_k^{z,w,\tau}(u) = \prod_\sigma \Phi_{k_\sigma,\sigma}^{z_\sigma,w_\sigma,\tau_\sigma}(u) \quad \text{for}$$

$$\Phi_{j,\sigma}(u) = \Phi_{j,\sigma}^{z_\sigma,w_\sigma,\tau_\sigma}(u) = y_\sigma^{j+1}[u;\tau_\sigma,w_\sigma]^j \mathbf{e}\left(\det(u)z_\sigma + \frac{iy_\sigma}{2\eta_\sigma t_\sigma}|[u;\tau_\sigma,w_\sigma]|^2\right).$$

Using this definition, (2.29) is valid for general $(z,w,\tau) \in \mathfrak{H}^I \times \mathfrak{H}^I \times \mathfrak{H}^I$. In other words, we have the reciprocal formula:

$$\Phi_k^{z,w,\tau} = \Psi_k^{\tau,w,z} \quad \text{and} \quad \left(\Psi_k^{z,w,\tau}\right)^*(u) = \Psi_k^{\tau,w,z}(u). \tag{2.31}$$

By (2.20) (and (2.15)), we also have

$$\left(\overline{\Psi_k^{z,w,\tau}}\right)^*(u) \tag{2.32}$$

$$= \prod_\sigma \left(y_\sigma^{k_\sigma+1}[u_\sigma;-\overline{\tau}_\sigma,\overline{w}_\sigma]^{k_\sigma} \mathbf{e}\left(\det(u_\sigma)\overline{\tau}_\sigma + \frac{y_\sigma}{2\eta_\sigma t_\sigma}|[u_\sigma;-\overline{\tau}_\sigma,\overline{w}_\sigma]|^2\right)\right).$$

2.5. Fourier expansion of theta series. Write $V = M_2(F)$. We choose on $F_{\mathbb{A}^{(\infty)}} = F \otimes_{\mathbb{Q}} \mathbb{A}^{(\infty)}$ the standard additive Haar measure da so that

$$\int_{\widehat{O}} da = 1 \quad \text{for} \quad \widehat{O} = O \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \quad (\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p).$$

At infinity, we choose the Lebesgue measure $\otimes_\sigma da_\sigma$ on $F_\infty = \prod_{\sigma\in I}\mathbb{R}$. Then we take the tensor product measure $du = da \otimes db \otimes dc \otimes dd$ for $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in V_{\mathbb{A}}$.

Let $\phi : V_{\mathbb{A}} = M_2(F_{\mathbb{A}}) \to \mathbb{C}$ be a Schwartz-Bruhat function, and assume that $\phi = \prod_v \phi_v$ for $\phi_v : V \otimes \mathbb{Q}_v \to \mathbb{C}$. We define the partial Fourier transform of ϕ for $\phi : V_{\mathbb{A}} \to \mathbb{C}$ by the same formula as in (2.19):

$$\phi^*\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \int_{F_{\mathbb{A}}^2} \phi\left(\begin{smallmatrix} a' & b' \\ c & d \end{smallmatrix}\right) \mathbf{e}_{\mathbb{A}}(ab'-ba')da'db', \tag{2.33}$$

where $\mathbf{e}_{\mathbb{A}} : F_{\mathbb{A}}/F \to \mathbb{C}^\times$ is the additive character with $\mathbf{e}_{\mathbb{A}}(x_\infty) = \mathbf{e}_F(x_\infty)$ for $x_\infty \in F_\infty$. We further assume that $\phi_\infty = \Psi_k^{z,w,\tau}$ studied in the previous subsection. Then we define

$$\Theta(\phi) = \sum_{\ell\in V}\phi(\ell). \tag{2.34}$$

Writing $\phi^{(\infty)}$ for the finite part of ϕ and regarding it as a function on $V \subset V_{\mathbb{A}^{(\infty)}}$, we find

$$\Theta(\phi) = \eta^k \theta_k(\tau;z,w;\phi^{(\infty)}).$$

Since $\int_{F_{\mathbb{A}}/F} da = \sqrt{|D|}$ for the discriminant D of F, the measure $|D|^{-1}da'db'$ has volume 1 for the quotient $F_{\mathbb{A}}^2/F^2$. Thus $|D|^{-1}\phi^*$ gives the partial Fourier transform with respect to volume 1 measure $|D|^{-1}da'db'$. The Poisson summation formula (with respect to the discrete subgroup $F^2 \subset F_{\mathbb{A}}^2$) is valid for the volume 1 measure (cf. [LFE] Section 8.4), we have the following result:

Proposition 2.2. *We have* $\Theta(\phi) = |D|^{-1}\Theta(\phi^*)$. *In terms of* θ_k, *we have*

$$\eta^k\theta_k(\tau; z, w; \phi^{(\infty)}) = |D|^{-1}y^k\theta_k(z; \tau, w; \phi^{*(\infty)}).$$

We could say that the right-hand-side of this formula gives the Fourier expansion of the theta series in terms of the variable z.

Proposition 2.3. *Let*

$$\Gamma^\tau(\phi^*) = \{\gamma \in SL_2(F) | \phi^{*(\infty)}(\gamma u) = \chi_\tau(\gamma)\phi^{*(\infty)}(u)\}$$
$$\Gamma^{z,w}(\phi) = \{(\gamma, \delta) \in SL_2(F)^2 | \phi^{(\infty)}(\gamma u \delta^{-1}) = \chi_{z,w}(\gamma, \delta)\phi^{(\infty)}(u)\}.$$

for characters $\chi_\tau : \Gamma^\tau(\phi^*) \to \mathbb{C}^\times$ *and* $\chi_{z,w} : \Gamma^{z,w}(\phi) \to \mathbb{C}^\times$ *Suppose that* $\phi_\infty = \Psi_k^{z,w,\tau}$. *Then for* $(\alpha, \beta, \gamma) \in \Gamma^\tau(\phi^*) \times \Gamma^{z,w}(\phi)$, *we have*

$$\Theta(\phi)(\alpha(\tau); \beta(z), \gamma(w))$$
$$= \Theta(\phi)(\tau; z, w)\chi_\tau(\alpha)^{-1}\chi_{z,w}(\beta, \gamma)^{-1}j(\alpha, \tau)^{-k}j(\beta, z)^{-k}j(\gamma, w)^{-k}.$$

More generally, for general $\alpha \in SL_2(F)$, *we have*

$$\Theta(\phi)(\alpha(\tau); z, w)j(\alpha, \tau)^k = |D|^{-1}\Theta(\phi^* \circ \alpha) = \Theta(\Phi),$$

where $\phi^* \circ \alpha(u) = \phi^*(\alpha u)$ *and* $\Phi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = (\phi^* \circ \alpha)^*\left(\begin{smallmatrix} -a & -b \\ c & d \end{smallmatrix}\right)$. *Similarly, for* $(\beta, \gamma) \in SL_2(F)$, *we have*

$$\Theta(\phi)(\tau; z, w)j(\beta, z)^k j(\gamma, w)^k = \Theta(\phi \circ (\beta, \gamma)),$$

where $\phi \circ (\beta, \gamma)(u) = \phi(\beta u \gamma^{-1})$.

Proof. Since the argument is similar, we prove the formula in details for the action on τ. Write $\Gamma = \Gamma^\tau(\phi^*)$. We use the expression $\Theta(\phi) = |D|^{-1}\Theta(\phi^*)$. By (2.15), we have

$$\frac{|[\gamma^{-1}\ell; \tau_\sigma, w_\sigma]|^2}{\eta(\tau_s)} = \frac{|[\ell; \gamma(\tau_\sigma), w_\sigma]|^2}{\eta(\gamma(\tau_s))}, \quad [\gamma^{-1}\ell; \tau, w]^k = [\ell; \gamma(\tau), w]^k j(\gamma, \tau)^k.$$

Then, up to $y^{k+I}\mathbf{e}_F(\det(\ell)z)$ (independent of τ), $\Theta(\phi^*)$ is the sum of the following terms over $\ell \in \Gamma\backslash M_2(F)$ and $\gamma \in \Gamma$:

$$\chi_\tau(\gamma)\phi^*(\ell)Y_\ell(\gamma(\tau))j(\gamma, \tau)^k,$$

where $Y_\ell(\tau) = [\ell; \tau, w]^k \exp(-\pi \sum_\sigma \frac{y_\sigma}{t_\sigma} \frac{|[\ell; \tau_\sigma, w_\sigma]|^2}{\eta_\sigma(\tau_\sigma)})$. Thus we need to prove the automorphic property with respect to τ for

$$f(\tau) = \sum_{\gamma \in \Gamma/\Gamma_\ell} \chi_\tau(\gamma)Y_\ell(\gamma(\tau))j(\gamma, \tau)^k,$$

where $\Gamma_\ell \subset \Gamma$ is the stabilizer of ℓ. We see

$$f(\alpha(\tau)) = \sum_{\gamma \in \Gamma/\Gamma_\ell} \chi_\tau(\gamma)Y_\ell(\gamma\alpha(\tau))j(\gamma, \alpha(\tau))^k$$
$$= \sum_{\gamma \in \Gamma/\Gamma_\ell} \chi_\tau(\gamma)Y_\ell(\gamma\alpha(\tau))j(\gamma\alpha, \tau)^k j(\alpha, \tau)^{-k} \overset{\gamma\alpha \mapsto \gamma}{=} \chi_\tau(\alpha)^{-1}f(\tau)j(\alpha, \tau)^{-k}.$$

This shows the first assertion for τ. As for the assertion with respect to (z, w), we argue similarly looking into the terms of $\Theta(\phi)$.

For the action of general α, the argument is similar for $\Theta(\phi^*)$. To return to $\Theta(\phi)$, we need to use the Fourier inversion formula $(\phi^*)^*\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \phi\left(\begin{smallmatrix} -a & -b \\ c & d \end{smallmatrix}\right)$. We leave the details to the attentive readers. $\square$

3. q–Expansion of Shimura series

The Shimura series for $GL(2) \times GL(2)$ is defined for $0 < k \in \mathbb{Z}[I]$ and $0 \le m \in \mathbb{Z}[I]$ in [Sh2] II (4.11) by

$$(3.1)\quad H(z,w;s) = H_{k,m}(z,w;s;\phi^{(\infty)},f)$$
$$= [U] \sum_{0 \neq \alpha \in M_2(F)/U} \phi^{(\infty)}(\alpha) a(-\det(\alpha), f) |\det(\alpha)|^m [\alpha; z, w]^{-k} \left|[\alpha; z, w]\right|^{-2sI}$$

for $(z, w) \in \mathfrak{H}^I \times \mathfrak{H}^I$. When $m = 0$, we simply write H_k for $H_{k,0}$. The positivity of k means that $k \ge 0$ and $k_\sigma > 0$ for at least one $\sigma \in I$. Here f is a Hilbert modular form given by the Fourier expansion: $\sum_{\xi \in F} a(\xi, f) \mathbf{e}_F(\xi\tau)$ for $\tau \in \mathfrak{H}^I$ of weight ℓ ($\mathbf{e}_F(\xi\tau) = \exp(2\pi i \sum_\sigma \xi^\sigma \tau_\sigma)$) with $a(\xi, f) = 0$ if $\xi^\sigma < 0$ for some $\sigma \in I$, U is a subgroup of finite index of the group $O_+^\times$ of all totally positive units for which each term of the above sum is invariant, $[U] = [O_+^\times : U]^{-1}$ and $\phi^{(\infty)} : M_2(F_{\mathbb{A}^{(\infty)}}) \to \mathbb{C}$ is a locally constant compactly supported function (a Schwartz-Bruhat function). To have invariance of the terms under the unit group U, we need to assume

$$(3.2)\quad k - \ell - 2m = [k - \ell - 2m] I \quad (I = \sum_{\sigma \in I} \sigma) \text{ for an integer } [k - \ell - 2m].$$

The series (3.1) converges absolutely and locally uniformly with respect to all variables s, z, w if

$$(3.3)\qquad \mathrm{Re}(s) > n + 2 + 2\theta(f) - [k - \ell - 2m]$$

as was shown in [Sh2] I Proposition 5.1 and Theorem 5.2, where $\theta(f) = -1$ when f is a constant, and otherwise, $\theta(f) = \theta \ge -\frac{1}{2}$ with $|a(\xi, f)\xi^{-\ell/2}| = O(|N(\xi)|^\theta)$ for the norm map $N = N_{F/\mathbb{Q}}$. This series is a generalization of Eisenstein series, because if we take $f = 1$ (so $\ell = 0$ and $m = 0$), the series gives an Eisenstein series for $GL(2) \times GL(2)$ over F.

We are going to compute the Fourier expansion of the Shimura series. We summarize here how we proceed. We have already computed the Fourier expansion of $\Theta(\phi)(\tau; z, w)$ with respect to z, and it is equal to $|D|^{-1}\Theta(\phi^*)(z; \tau, w)$ for the partial Fourier transform ϕ^* of ϕ. By the integral expression of the series given in [Sh2] I Section 7, the series (actually its complex conjugate) is the Rankin-Selberg convolution product of $\Theta(\phi)$ and f with respect to the variable

τ. Since integration with respect to τ preserves Fourier expansion of $\Theta(\phi)$ with respect to z, what we need to compute is

$$\int_{\Gamma\backslash\mathfrak{H}^I} \Theta(\phi^*)(z;\tau,w)f(\tau)E(\tau;0)d\mu(\tau)$$

for the invariant measure $d\mu(\tau)$ for a suitable holomorphic Eisenstein series $E(\tau;0)$. This has been actually done, though without referring the result as the Fourier expansion of the series $H_k(z,w;0)$, in [Sh2] II Proposition 5.1 (replacing $f(w)$ and variable w there by $E(\tau;0)f(\tau)$ and τ). We recall the integral expression in Subsection 3.1 and the computation of Proposition 5.1 in [Sh2] II in Subsection 3.2. We shall do this to formulate our result in a manner optimal for our later use.

3.1. INTEGRAL EXPRESSION. Let Γ be a congruence subgroup of $SL_2(F)$ which leaves $\theta_k(\tau;z,w;\phi^{(\infty)})$ and f fixed; thus, $\Gamma\subset\Gamma^\tau(\phi^*)$. The stabilizer Γ_∞ of the infinity cusp has the following canonical exact sequence:

$$\begin{array}{ccccccc} 0\to & \mathfrak{a} & \longrightarrow & \Gamma_\infty & \longrightarrow & U & \to 1 \\ & a & \mapsto & \left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right) & & & \\ & & & \left(\begin{smallmatrix}\epsilon & a\\ 0 & \epsilon^{-1}\end{smallmatrix}\right) & \mapsto & \epsilon & \end{array} \tag{3.4}$$

for a fractional ideal $\mathfrak{a}$ and a subgroup $U\subset O^\times$ of finite index. By shrinking Γ a little, we may assume that $U\subset O_+^\times$. We recall the integral expression of the Shimura series involving Siegel's theta series given in [Sh2] I (7.2) and II (6.5b):

$$[U]N(\mathfrak{a})^{-1}\sqrt{|D|}^{-1}\int_{F_{\infty+}^\times/U^2}\left(\int_{F_\infty/\mathfrak{a}}\Theta(\phi)d^m f(\tau)d\xi\right)\eta^{(s-1)I}d^\times\eta, \tag{3.5}$$

where $d^m=\prod_\sigma\left(\frac{1}{2\pi i}\frac{\partial}{\partial\tau_\sigma}\right)^{m_\sigma}$, $\phi(u)=\phi^{(\infty)}(u^{(\infty)})\Psi_k^{z,w,\tau}(u_\infty)$ and $d^\times\eta$ is the multiplicative Haar measure given by $\otimes_\sigma(\eta_\sigma^{-1}d\eta_\sigma)$. We first compute the inner integral: if $\mathrm{Re}(s)\gg 0$,

$$N(\mathfrak{a})^{-1}\sqrt{|D|}^{-1}\int_{F_\infty/\mathfrak{a}}\Theta(\phi)d^m f(\tau)d\xi =$$
$$\sum_{\alpha\in V,\beta\in F}\phi^{(\infty)}(\alpha)a(\beta,f)|\beta|^m[\alpha;z,w]^k\exp(-\pi(2\beta+P_{z,w}(\alpha))\eta)\eta^{k+I}\delta_{\det(\alpha),-\beta},$$

because for $C=N(\mathfrak{a})^{-1}\sqrt{|D|}^{-1}$

$$C\int_{F_\infty/\mathfrak{a}}\mathbf{e}_F((\det(\alpha)+\beta)\xi)d\xi=\delta_{\det(\alpha),-\beta}=\begin{cases}1 & \text{if } \det(\alpha)=-\beta,\\ 0 & \text{otherwise.}\end{cases}$$

To compute the outer integral, when $\det(\alpha)=-\beta$, we note from (2.2) that $P_{z,w}[\alpha]=S[\alpha]+\frac{|[\alpha;z,w]|^2}{yt}$ for $S[\alpha]=2\det(\alpha)$ and that

$$\exp(-\pi(2\beta+P_{z,w}(\alpha))\eta)=\exp(\pi(2\det(\alpha)-P_{z,w}(\alpha))\eta)=\exp(-\pi\frac{|[\alpha;z,w]|^2}{yt}\eta).$$

Here we have integrated term wise (with respect to) the summation of Θ and the Fourier expansion of f, which can be justified by the locally uniform and absolute convergence of the Fourier expansions of Θ and f as long as the resulting series is absolutely convergent (Lebesgue's term wise integration theorem). The convergence of the series is guaranteed by (3.3) if $\mathrm{Re}(s)$ is large. Again spreading the integral $\int_{F^\times_{\infty+}/U^2}\sum_{\epsilon\in U}\Phi(\varepsilon^2\eta)d^\times\eta$ to the whole $F^\times_{\infty+}$ for $\Phi(\eta)=\exp(-\pi\frac{|[\alpha;z,w]|^2\eta}{yt})$, we see that (as long as the latter integral is absolutely convergent) the integral (3.5) is equal to

(3.6)
$$[U]\sum_{\alpha\in V/U}\phi^{(\infty)}(\alpha)a(-\det(\alpha),f)|\det(\alpha)|^m[\alpha;z,w]^k \times\int_{F^\times_{\infty+}}\exp(-\pi\mathrm{Tr}(\frac{|[\alpha;z,w]|^2}{yt}\eta))\eta^{k+sI}d^\times\eta.$$

We know

$$\begin{aligned}&[\alpha;z,w]^k\int_{F^\times_{\infty+}}\exp(-\pi\frac{|[\alpha;z,w]|^2}{yt}\eta)\eta^{k+sI}d^\times\eta\\ &\quad=2^{1-[F:\mathbb{Q}]}\pi^{-k-sI}\Gamma_F(k+sI)y^{k+sI}t^{k+sI}[\alpha;z,w]^k|[\alpha;z,w]|^{-2s-2k}\\ &\qquad=2^{1-[F:\mathbb{Q}]}\pi^{-k-sI}\Gamma_F(k+sI)y^{k+sI}t^{k+sI}[\alpha;\overline{z},\overline{w}]^{-k}|[\alpha;z,w]|^{-2s},\end{aligned}$$

where $\Gamma_F(k)=\prod_\sigma\Gamma(k_\sigma)$, and as for the factor $2^{1-[F:\mathbb{Q}]}$, see [LFE] page 271. Thus we conclude

$$\begin{aligned}&2^{1-[F:\mathbb{Q}]}\pi^{-k-sI}\Gamma_F(k+sI)y^{k+sI}t^{k+sI}\overline{H_{k,m}(z,w;\overline{s};\overline{\phi}^{(\infty)},f_c)}\\ &\quad=[U]N(\mathfrak{a})^{-1}\sqrt{|D|}^{-1}\int_{F^\times_{\infty+}/U^2}\int_{F_\infty/\mathfrak{a}}\Theta(\phi)d^mf(\tau)\eta^{(s-1)I}d\xi d^\times\eta,\end{aligned}$$

where $f_c(z)=\overline{f(-\overline{z})}$. In other words, by taking complex conjugation, we have, for $\widetilde{\phi}=\overline{\phi}^{(\infty)}\phi_\infty$,

$$\begin{aligned}(3.7)\quad&2^{1-[F:\mathbb{Q}]}\pi^{-k-sI}\Gamma_F(k+sI)y^{k+sI}t^{k+sI}H_{k,m}(z,w;s;\phi^{(\infty)},f)\\ &\quad=[U]N(\mathfrak{a})^{-1}\sqrt{|D|}^{-1}\int_{F^\times_{\infty+}/U^2}\int_{F_\infty/\mathfrak{a}}\overline{\Theta(\widetilde{\phi})}\overline{d^mf_c(\tau)}\eta^{(s-1)I}d\xi d^\times\eta.\end{aligned}$$

The above formula (3.7) is only valid for s satisfying (3.3). However, by Rankin-Selberg convolution, we can analytically continue the function H to a meromorphic function on the whole s–plane (see [Sh2] I Section 7). We recall the process. We first assume that $m=0$. Since $\Gamma_\infty\backslash\mathfrak{H}^I\cong(F^\times_{\infty+}/U^2)\times(F_\infty/\mathfrak{a})$, we can rewrite the above integral as

$$\begin{aligned}
&2^{1-[F:\mathbb{Q}]}\pi^{-k-sI}\Gamma_F(k+sI)y^{k+sI}t^{k+sI}H_k(z,w;s;\phi^{(\infty)},f)\\
&=[U]N(\mathfrak{a})^{-1}\sqrt{|D|}^{-1}\int_{\Gamma_\infty\backslash\mathfrak{H}^I}\overline{\Theta(\widetilde{\phi})}\overline{f_c(\tau)}\eta^{sI}d\mu(\tau)\\
&=N(\mathfrak{a})^{-1}\sqrt{|D|}^{-1}\int_{\Gamma\backslash\mathfrak{H}^I}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\left(\overline{\Theta(\widetilde{\phi})}\overline{f_c(\tau)}\eta^{sI}\right)\circ\gamma d\mu(\tau) \qquad (3.8)\\
&=N(\mathfrak{a})^{-1}\sqrt{|D|}^{-1}\int_{\Gamma\backslash\mathfrak{H}^I}\overline{\Theta(\widetilde{\phi})}\overline{f_c(\tau)E(\tau;\overline{s})}d\mu(\tau),
\end{aligned}$$

where $d\mu(\tau)$ is the invariant measure $\eta^{-2I}d\xi d\eta$ on $\mathfrak{H}^I$, and by Proposition 2.3,

$$E(\tau;s)=E_{k-\ell}(\tau;s)=\eta^{sI}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}j(\gamma,\tau)^{\ell-k}|j(\gamma,\tau)|^{-2sI}. \qquad (3.9)$$

In general, if $m\neq 0$, we use the formula (see [Sh2] I (1.16a)):

$$d^m=\sum_{0\le j\le m}\binom{m}{j}\frac{\Gamma_F(\ell+m)}{\Gamma_F(\ell+j)}(4\pi\eta)^{j-m}\delta_\tau^j(\ell)$$

for $\delta_\tau^\sigma(j)=\frac{1}{2\pi i}\left(\frac{j}{\tau_\sigma-\overline{\tau}_\sigma}+\frac{\partial}{\partial\tau_\sigma}\right)$ and

$$\delta_\tau^j(\ell)=\prod_\sigma\left(\delta_\tau^\sigma(\ell_\sigma+2j_\sigma-2)\cdots\delta_\tau^\sigma(\ell_\sigma+2)\delta_\tau^\sigma(\ell_\sigma)\right).$$

The binomial coefficients $\binom{m}{j}$ is the product of individual ones $\binom{m_\sigma}{j_\sigma}$ over $\sigma\in I$. Since $\delta_\tau^j(\ell)$ preserves automorphy (but not holomorphy), we can write $d^m f_c$ as a linear combination of $\delta_\ell^u f_c$, which is an automorphic form of weight $\ell+2u$ on the same Γ, and therefore the above computation still works.

The integral (3.8) (in general for $m\ge 0$) is convergent for all $s\in\mathbb{C}$ except for s giving rise to a singularity of the Eisenstein series, because $\Theta(\phi)$ for $k>0$ does not have constant term at any cusp; so, it is rapidly decreasing. Thus the integral of (3.8) converges absolutely for any slowly increasing automorphic form $f(\tau)$ as long as $E(\tau;s)$ is finite. This is the proof of the analytic continuation given in [Sh1] Section 13. This proof works well even when $k=0$ for cusp forms f.

3.2. Computation of q–expansion. We assume that $m=0$. We are going to compute the Fourier expansion of $\int_{\Gamma\backslash\mathfrak{H}^I}\Theta(\phi)g(\tau)d\mu(\tau)$ for an eigenform $g(\tau)$ of Laplacian Δ_σ: $\Delta_\sigma g=(s_\sigma^2-\frac{1}{4})g$ ($s_\sigma\in\mathbb{C}$) for all $\sigma\in I$, where $\Delta_\sigma=\eta_\sigma^2\left(\frac{\partial^2}{\partial\xi_\sigma^2}+\frac{\partial^2}{\partial\eta_\sigma^2}\right)$. We assume that $(\Theta(\phi)g)(\gamma(\tau))=(\Theta(\phi)g)(\tau)$ for all $\gamma\in\Gamma$. By (2.2), $\Theta(\phi)=|D|^{-1}\Theta((\phi)^*)$ is the sum of the following terms:

$$\prod_\sigma y_\sigma^{k_\sigma+1}[\alpha^\sigma;\tau_\sigma,w_\sigma]^{k_\sigma}\mathbf{e}\left(\det(\alpha^\sigma)z_\sigma+\frac{iy_\sigma}{2\eta_\sigma t_\sigma}|[\alpha^\sigma;\tau_\sigma,w_\sigma]|^2\right).$$

By (2.15), we have, for $\gamma \in \Gamma$,

$$\log(Y_\sigma(\tau_\sigma)) = \frac{-\pi y_\sigma}{\eta_\sigma t_\sigma}|[(\gamma\alpha)^\sigma; \tau_\sigma, w_\sigma]|^2$$
$$= \frac{-\pi y_\sigma}{\eta_\sigma t_\sigma}|[\alpha^\sigma; \gamma^{-1}(\tau_\sigma), w_\sigma]|^2 |j(\gamma^{-1}, z)|^2$$
$$= \frac{-\pi y_\sigma}{\mathrm{Im}(\gamma^{-1}(\tau_\sigma))t_\sigma}|[\alpha^\sigma; \gamma^{-1}(\tau_\sigma), w_\sigma]|^2.$$

This shows

(3.10)
$$y^{-(k+I)}\int_{\Gamma\backslash\mathfrak{H}^I}\Theta(\phi)g(\tau)d\mu(\tau)$$
$$= \sum_{\alpha\in\Gamma\backslash M_2(F)}\mathbf{e}_F(\det(\alpha)z)\phi^{(\infty)}(\alpha)\int_{\Gamma\backslash\mathfrak{H}^I}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}[\alpha;\gamma(\tau),w]^k g(\gamma(\tau))Y(\gamma(\tau))d\mu$$
$$= \sum_{\alpha\in\Gamma\backslash M_2(F)}\mathbf{e}_F(\det(\alpha)z)\phi^{(\infty)}(\alpha)\int_{\Gamma_\alpha\backslash\mathfrak{H}^I}[\alpha;\tau,w]^k Y(\tau)g(\tau)d\mu,$$

where $\Gamma_\alpha = \{\gamma \in \Gamma | \gamma\alpha = \alpha\}$ and $Y(z) = \prod_\sigma \mathbf{e}(Y(z_\sigma))$. If $\det(\alpha) \neq 0$, then $\Gamma_\alpha = \{1\}$.

We first compute the general term: $\int_{\mathfrak{H}^I}[\alpha;\tau,w]^k Y(\tau)g(\tau)d\mu(\tau)$. For that, we recall [Sh2] Lemma 5.2 and the discussion after the lemma:

Lemma 3.1. *Let $\alpha \in GL_2(F)$. Let $P(\tau, w) = \exp(-\sum_\sigma \frac{u_\sigma}{\eta_\sigma t_\sigma}|[\alpha, \overline{\tau}_\sigma, w_\sigma]|^2)$ for $\tau, w \in \mathfrak{H}^I$ with $0 < u_\sigma \in \mathbb{R}$. Assume that the integral $\int_{\mathfrak{H}^I} P(\tau, w)g(\tau)d\mu(\tau)$ is convergent. If $\Delta_\sigma g = (s_\sigma^2 - \frac{1}{4})g$ and $\det(\alpha)$ is totally positive, we have*

$$\text{(3.11)}\quad \int_{\mathfrak{H}^I} P(\tau,w)g(\tau)d\mu(\tau)$$
$$= \pi^{[F:\mathbb{Q}]/2}(\det(\alpha)u^{-1})^{I/2}\exp(-2\sum_\sigma \det(\alpha^\sigma)u_\sigma)K(\det(\alpha)u, s)g|_k\alpha(w)$$

for the modified Bessel function:

$$K(u,s) = \prod_\sigma \int_0^\infty \exp(-u_\sigma(x_\sigma + x_\sigma^{-1}))x_\sigma^{s_\sigma - 1}dx_\sigma,$$

where $g|_k\alpha(w) = \det(\alpha)^{k-I}g(\alpha(w))j(\alpha,z)^{-k}$. If $\det(\alpha)$ is not totally positive and g is holomorphic, the integral (3.11) *vanishes, as long as it converges.*

By the above lemma, taking $g = f$ (so, $g = f$ is holomorphic), only non-trivial case is when $\det(\alpha)$ is totally negative, and noting the fact that $K(u, \frac{1}{2}) =$

$\pi^{[F:\mathbb{Q}]/2}u^{-I/2}\exp(-2\sum_\sigma u_\sigma)$, we have

$$\int_{\mathfrak{H}^I}\exp(-\pi\sum_\sigma\frac{y_\sigma}{\eta_\sigma(\tau_\sigma)t_\sigma}|[\alpha^\sigma;\tau_\sigma,w_\sigma]|^2)[\alpha;\tau,w]^k f(\tau)d\mu(\tau)$$
$$\overset{\tau\mapsto\alpha(\overline{\tau})}{=}\int_{\mathfrak{H}^I}\exp(-\pi\sum_\sigma\frac{y_\sigma}{|\eta_\sigma(\alpha(\tau_\sigma))|t_\sigma}|[\alpha^\sigma;\alpha(\overline{\tau}_\sigma),w_\sigma]|^2)[\alpha;\alpha(\overline{\tau}),w]^k f(\tau)d\mu(\tau)$$
$$\overset{(2.15)}{=}\int_{\mathfrak{H}^I}\exp(-\pi\sum_\sigma\frac{|\det(\alpha^\sigma)|y_\sigma}{\eta_\sigma(\tau_\sigma)t_\sigma}|[1;\overline{\tau}_\sigma,w_\sigma]|^2)$$
$$\times[1;\overline{\tau},w]^k\det(\alpha)^k j(\alpha,\overline{\tau})^{-k}f(\alpha(\overline{\tau}))d\mu(\tau)$$
$$\overset{s_\sigma=1/2}{=}(-1)^{[F:\mathbb{Q}]}(-2i)^k t^k y^{-I}\exp(-4\pi\sum_\sigma|\det(\alpha^\sigma)|y_\sigma)f|_k\alpha(\overline{w}).$$

If $\alpha\neq 0$ and $\det(\alpha)=0$, then Γ_α is equal to $\Gamma\cap\beta\mathcal{U}(F)\beta^{-1}$ for $\beta\in GL_2(F)$. By a variable change, we may assume that $\alpha=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$. Then $\Gamma_\alpha=\Gamma\cap\mathcal{U}(F)$, and we have an isomorphism: $\mathfrak{a}\cong\Gamma_\alpha$ by $\mathfrak{a}\ni a\mapsto\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right)$, where $\mathfrak{a}$ is a fractional ideal of F. In this case, $[\alpha;\tau,w]=-w$. We then have

$$(3.12)\quad\int_{\Gamma_\alpha\backslash\mathfrak{H}^I}[\alpha;\tau,w]^k Y(\tau)f(\tau)d\mu(\tau)$$
$$=\int_{F^\times_{\infty+}}[\alpha;\tau,w]^k Y(\tau)\int_{F_\infty/\mathfrak{a}}f(\xi+i\eta)d\xi\eta^{-2I}d\eta$$
$$=N(\mathfrak{a})\sqrt{|D|}a(0,f)\int_{F^\times_{\infty+}}(-w)^k\exp(-\pi\sum_\sigma\frac{y_\sigma}{\eta_\sigma t_\sigma}|w|^2)\eta^{-2I}d\eta$$
$$\overset{\eta\mapsto\eta^{-1}}{=}N(\mathfrak{a})\sqrt{|D|}a(0,f)(-w)^k\int_{F^\times_{\infty+}}\exp(-\pi\sum_\sigma\frac{\eta_\sigma y_\sigma}{t_\sigma}|w|^2)d\eta$$
$$=\pi^{-1}N(\mathfrak{a})\sqrt{|D|}a(0,f)(-w)^k\frac{t_\sigma}{y_\sigma}|w|^{-2I},$$

where $f(\tau)=\sum_{\delta\in F}a(\delta,f)\mathbf{e}_F(\delta\tau)$.

Thus we obtain the following version of [Sh2] II Proposition 5.1 for $B=M_2(F)$:

THEOREM 3.2. *Suppose that f is a holomorphic cusp form of weight $k>0$. Let Γ be a congruence subgroup of $SL_2(F)$ fixing $f(\tau)\Theta(\phi)(\tau)$. Then we have*

$$(-1)^{[F:\mathbb{Q}]}|D|\int_{\Gamma\backslash\mathfrak{H}^I}\Theta(\phi)(\tau;z,w)f(\tau)d\mu(\tau)$$
$$=(-2i)^k t^k y^k\sum_{\alpha\in\Gamma\backslash M_2(F);\det(\alpha)\ll 0}\phi^{*(\infty)}(\alpha)\mathbf{e}_F(\det(\alpha)\overline{z})f|_k\alpha(\overline{w}),$$

where $f|\alpha(\overline{w})=\det(\alpha)^{k-I}f(\alpha(\overline{w}))j(\alpha,\overline{w})^{-k}$ for $\alpha\in M_2(F)$ with totally negative determinant.

Taking complex conjugate of the above expansion and replacing the pair (ϕ,f) in the above theorem by $(\widetilde{\phi}=\overline{\phi}^{(\infty)}\phi_\infty,f_cE(w;0))$, we get

Corollary 3.3. *We have, if f is a holomorphic cusp form of weight ℓ with $k-\ell=[k-\ell]I$ for an integer $[k-\ell]>0$,*

$$\begin{aligned}H_k(z,w;0;\phi^{(\infty)},f) &= 2^{[F:\mathbb{Q}]-1}[U]|D|^{-3/2}N(\mathfrak{a})^{-1}\frac{(2\pi i)^k}{\Gamma_F(k)}\\ &\times \sum_{\alpha\in\Gamma\backslash M_2(F);\det(\alpha)\gg 0}\phi^{*(\infty)}(\epsilon\alpha)\mathbf{e}_F(\det(\alpha)z)(fE_{k-\ell}(w;0))|_k\alpha(w),\end{aligned}$$

where $\epsilon=\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$.

We can apply the above theorem to the following integral:

$$\int_{\Gamma\backslash\mathfrak{H}^I}\Theta(\widetilde{\phi})(s-1)E_{k-\ell}(\tau;s)\overline{f}_c(\tau)d\mu(\tau)$$

at $s=1$ when $k=\ell$, because $E(\tau,s)$ has a simple pole at $s=1$ whose residue is a constant $c_\Gamma\neq 0$ (independent of τ). We then have

Corollary 3.4. *We have, if f is a holomorphic cusp form of weight k,*

$$\begin{aligned}\mathrm{Res}_{s=1}H_k(z,w;s;\phi^{(\infty)},f) &= c_\Gamma(-i)^{[F:\mathbb{Q}]}2^{-1}[U]|D|^{-3/2}N(\mathfrak{a})^{-1}\frac{(2\pi i)^{k+I}}{\Gamma_F(k+I)}\\ &\times y^{-I}t^{-I}\sum_{\alpha\in\Gamma\backslash M_2(F);\det(\alpha)\gg 0}\phi^{*(\infty)}(\epsilon\alpha)\mathbf{e}_F(\det(\alpha)z)f|_k\alpha(w),\end{aligned}$$

where $c_\Gamma=\mathrm{Res}_{s=1}E(w;s)$.

For the exact value of the constant $c_\Gamma\neq 0$, see [H99] (RES3) page 173.

4. Evaluation at CM points

We follow [Sh2] I Sections 5 and 8 to write down the evaluation of the Shimura series at some special CM points in terms of Rankin-Selberg L–functions.

4.1. CM points. We fix the "identity" embedding $(\sigma_0:F\to\overline{\mathbb{Q}})\in I$. Let (z_0,w_0) be a point in $\mathfrak{H}^I$ such that $M=F[z_{0,\sigma_0}]$ and $L=F[w_{0,\sigma_0}]$ are totally imaginary quadratic extensions of F (so, CM fields). Let $Y=M\otimes_F L$, and we embed Y into $M_2(F)\otimes_F M_2(F)=M_4(F)$ by $(a,b)\mapsto\rho_M(a)\otimes\rho_L(b)$ with

$$\left(\begin{smallmatrix}z_0a\\a\end{smallmatrix}\right)=\rho_M(a)\left(\begin{smallmatrix}z_0\\1\end{smallmatrix}\right)\quad\text{and}\quad\left(\begin{smallmatrix}w_0b\\b\end{smallmatrix}\right)=\rho_L(b)\left(\begin{smallmatrix}w_0\\1\end{smallmatrix}\right).$$

We see easily that $\rho_M(a)^\iota=\rho_M(a^c)$ and $\rho_L(b)^\iota=\rho_L(b^c)$ for complex conjugation c. We regard $V=M_2(F)$ as a Y–module for the multiplicative semi-group Y via $\rho_M\otimes\rho_L$; in other words, $(a,b)v=\rho_M(a)v\rho_L^\iota(b)$.

We have four distinct Y–eigenvectors $p(z_{0,\sigma},w_{0,\sigma})$, $p(z_{0,\sigma},\overline{w}_{0,\sigma})$, $p(\overline{z}_{0,\sigma},w_{0,\sigma})$ and $p(\overline{z}_{0,\sigma},\overline{w}_{0,\sigma})$ in $M_2(\mathbb{C})=V\otimes_{F,\sigma}\mathbb{C}$, whose eigenvalues of (a,b) are $(a^{\widetilde{\sigma}}b^{\widetilde{\sigma}c})$, $(a^{\widetilde{\sigma}}b^{\widetilde{\sigma}})$, $(a^{\widetilde{\sigma}c}b^{\widetilde{\sigma}c})$ and $(a^{\widetilde{\sigma}c}b^{\widetilde{\sigma}})$, respectively, for an extension $\widetilde{\sigma}$ of σ to the composite LM. Since $V\otimes_{F,\sigma}\mathbb{C}$ is free of rank 1 over $Y_\sigma=Y\otimes_{F,\sigma}\mathbb{C}$, $V=M_2(F)$ is free of rank 1 over Y (because $\mathbb{C}$ is faithfully flat over F). Thus we find $v\in V$ such that $V=Yv$. Then $S_Y:(y,y')\mapsto S(yv,y'v)$ gives a non-degenerate symmetric F–bilinear form on Y with $S_Y(yy', y'') = S_Y(y', y^c y'')$, and we can write $S_Y(x, y) = \mathrm{Tr}_{Y/F}(\delta x y^c)$ for $\delta \in Y^\times$ with $\delta^c = \delta$.

Suppose now that $L = M$. Then $Y \cong M \oplus Y_0$ with $Y_0 \cong M$, the first projection to M is given by $a \otimes b \mapsto ab^c$ and the second to Y_0 is given by $a \otimes b \mapsto ab$. Since c is an automorphism of M, $p(z, w)$ and $p(\overline{z}_{0,\sigma}, \overline{w}_{0,\sigma})$ belongs to $Mv \otimes_{F,\sigma} \mathbb{C} \subset Y_\sigma v = V_\sigma$. The vectors $p(z_{0,\sigma}, w_{0,\sigma})$ and $p(\overline{z}_{0,\sigma}, \overline{w}_{0,\sigma})$ are orthogonal to $Y_0 \otimes_{F,\sigma} \mathbb{C}$. In other words,

$$Y_0 = \left\{ y \in Y \middle| \widetilde{\sigma}(y) = c\widetilde{\sigma}(y) = 0 \text{ for all } \sigma \in I \right\}.$$

Thus $\Sigma = \sum_{\sigma \in I} \widetilde{\sigma}$ gives rise to a CM type of M (with $\Sigma \sqcup \Sigma c$ giving all complex embeddings of M). This shows: writing $V \ni \alpha = av \oplus bv$ with $a \in M$ and $b \in Y_0$

$$\begin{aligned}
[\alpha^\sigma; z_{0,\sigma}, w_{0,\sigma}] &= S(\alpha^\sigma, p(z_{0,\sigma}, w_{0,w_0})) = S(av_\sigma, p(z_{0,\sigma}, w_{0,w_0})) \\
&= S(v_\sigma, a^c p(z_{0,\sigma}, w_{0,w_0})) = a^{\widetilde{\sigma}c}[v_\sigma; z_{0,\sigma}, w_{0,\sigma}], \qquad (4.1)\\
[\alpha; z_0, w_0]^{-k} \left|[\alpha; z_0, w_0]\right|^{-2sI} &= C^{-k\Sigma}|C^\Sigma|^{-2s} a^{-ck\Sigma} N(a)^{-s},
\end{aligned}$$

where $C = [v_\sigma; z_{0,\sigma}, w_{0,\sigma}]$ and $N(a)$ is the absolute norm of $a \in M$. Here we have written $k\Sigma = \sum_{\sigma \in I} k_\sigma \widetilde{\sigma}$ and $ck\Sigma = \sum_{\sigma \in I} k_\sigma \widetilde{\sigma} c$.

Since $p(z_{0,\sigma}, w_{0,\sigma})$ and $p(\overline{z}_{0,\sigma}, \overline{w}_{0,\sigma})$ span (by the definition of $\mathcal{Y}(S)$ in Subsection 2.1) a scalar extension to $\mathbb{C}$ of a subspace on which S_σ is negative definite, S is totally positive definite on $W = Y_0 v$, because every vector in W is orthogonal to $p(z_{0,\sigma}, w_{0,\sigma})$ and $p(\overline{z}_{0,\sigma}, \overline{w}_{0,\sigma})$. We write S_W for the restriction of S to W. By this fact, writing $\delta = -\delta_M \oplus \delta_0$ for $\delta_M \in M$ and $\delta_0 \in Y_0$, then δ_M is a totally positive element of F; so, we may assume that $\delta_M = \frac{1}{2}$ by changing v if necessary. Similarly, we may choose $\delta_0 = \frac{1}{2}$.

4.2. Special values of Shimura series. As we have explained already, we choose v as in previous subsection so that

$$S_Y((a, b), (a', b')) = \frac{1}{2}\mathrm{Tr}_{M/F}(-a{a'}^c + b{b'}^c). \qquad (4.2)$$

We see, supposing

$$\phi^{(\infty)}(u) = \phi_M \otimes \phi_0 \qquad (4.3)$$

for functions $\phi_M : Mv \to \mathbb{C}$ and $\phi_0 : Y_0 v \to \mathbb{C}$,

$$\begin{aligned}
&C^{k\Sigma}|C^\Sigma|^{2s} H_k(z_0, w_0; s; \phi^{(\infty)}, f) \\
&\qquad = [U] \sum_{\alpha \in M/U} \phi_M(\alpha v) \sum_{\beta \in Y_0/U} \phi_0(\beta v) a(\alpha\alpha^c - \beta\beta^c, f) \alpha^{-ck\Sigma} N(\alpha)^{-s},
\end{aligned}$$

where C is as in (4.1). We now define $\theta(\phi_0) = \sum_{\beta \in Y_0} \phi_0(\beta v) \mathbf{e}_F(\beta\beta^c z)$. Then for $f'(z) = \theta(\phi_0) f(z) = \sum_{\xi \in F} a(\xi, f') \mathbf{e}_F(\xi z)$, we have

$$a(\xi, f') = \sum_{\beta \in Y_0/U} a(\xi - \beta\beta^c, f) \phi_0(\beta v),$$

which is a finite sum because $\{x \in Y_0 \otimes_F \mathbb{R} | x^\sigma x^{\sigma c} < \xi^\sigma \ \forall \sigma\}$ is a compact set. Thus we have, under (4.2) and (4.3)

$$\begin{aligned} & C^{k\Sigma}|C^\Sigma|^{2s} H_k(z_0, w_0; s; \phi^{(\infty)}, f) \\ (4.4) \qquad & \qquad = [U] \sum_{\alpha \in M/U} \phi_M(\alpha v) a(\alpha\alpha^c, f') \alpha^{-ck\Sigma} N(\alpha)^{-s}. \end{aligned}$$

In general, $\phi^{(\infty)}|\det^m|$ is a constant linear combination of the functions satisfying (4.3); so, $H(z_0, w_0; s)$ is a linear combination of the series of the above type. The series (4.4) is the Rankin convolution of f' and the theta series $\theta(\phi_{k,M})$ of the norm form of M for $\phi_{k,M}(\alpha) = \alpha^{k\Sigma}\phi_M(\alpha v)$ (see (4.9)).

4.3. An explicit formula of Petersson inner product. For a given theta series $\theta_M(\phi)$ of weight $k + I$ of a CM field M/F, we are going to write down the inner product $\langle \theta_M(\phi), f'_c \rangle$ for a special value of a modular form on $GL(2) \times GL(2)$, taking $f' = f\theta_M(\phi')$ for another theta series $\theta_M(\phi')$ of weight I of M. Here $f'_c(z) = \overline{f'(-\overline{z})}$; so, f'_c is a holomorphic modular form whose Fourier coefficients (at the infinity) are the complex conjugate of those of f'. The modular form is given by, up to an explicit constant,

$$\mathrm{Res}_{s=1} H_k(z, w; s; \phi' \otimes \phi^{(\infty)}, f).$$

We will later in Section 7 deduce from this the integrality of $\frac{\pi^{2k+2I}\langle g, \theta(\phi)\rangle}{\Omega^{2(k+I)}}$ for the period Ω of the Néron differential of the abelian variety of CM-type sitting at the evaluation point (z_0, w_0).

Let f and g be Hilbert modular forms on $\Gamma \subset SL_2(F)$ with Fourier expansion $f = \sum_{\xi \in F} a(\xi, f)\mathbf{e}_F(\xi\tau)$ and $g = \sum_{\xi \in F} a(\xi, g)\mathbf{e}_F(\xi\tau)$ for $z \in \mathfrak{H}^I$. We take the ideal $\mathfrak{a} \subset F$ and the unit group $U \subset O_+^\times$ as in (3.4). Let ℓ and κ be the weights of f and g respectively. We suppose that one of f and g is a cusp form so that $\overline{f}g$ is rapidly decreasing.

We let $\epsilon \in U$ act on $\mathfrak{H}^I$ by $\tau \mapsto \epsilon^2\tau$. Then $f(\epsilon^2\tau) = \epsilon^{-\ell} f(\tau)$ and $g(\epsilon^2\tau) = \epsilon^{-\kappa} g(\tau)$. Then the function $\overline{f}g(\tau)\eta^{(\ell+\kappa)/2}$ is U–invariant. We then consider

(4.5)
$$D(s; f, g) = [U^2]N(\mathfrak{a})^{-1}\sqrt{|D|}^{-1} \int_{F_\infty/\mathfrak{a}} \int_{F^\times_{\infty+}/U^2} \overline{f}(\tau)g(\tau)\eta^{sI+(\ell+\kappa)/2} d\xi d^\times\eta.$$

We now assume that

$$(4.6) \qquad \ell \equiv \kappa \mod 2\mathbb{Z}[I] + \mathbb{Z}I.$$

Thus we find $m \in \mathbb{Z}[I]$ such that $\ell - \kappa - 2m \in \mathbb{Z}I$. Replacing Γ by

$$\left\{\gamma \in \Gamma \middle| (\overline{f}g\eta^{(\ell+\kappa)/2} \circ \gamma)(\tau) = (\overline{f}g\eta^{(\ell+\kappa)/2})(\tau) j(\gamma, \overline{\tau})^\ell j(\gamma, \tau)^\kappa |j(\gamma, \tau)^{-\ell-\kappa}| \right\}$$

if necessary, we have

$$(4.7)\quad [U^2]^{-1}N(\mathfrak{a})\sqrt{|D|}D(s;f,g) = \int_{\Gamma\backslash\mathfrak{H}^I} \overline{f}(\tau)g(\tau)\eta^{\ell-m}E_{[\ell-\kappa-2m]I,m}(\tau;s+1-\frac{[\ell-\kappa-2m]}{2})d\mu(\tau),$$

where

$$E_{nI,m}(\tau;s) = \eta^{sI}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\left(\frac{j(\gamma,\overline{\tau})}{j(\gamma,\tau)}\right)^m j(\gamma,\tau)^{-nI}|j(\gamma,\tau)|^{-2sI}.$$

When $m=0$, we write simply E_{nI} for $E_{nI,0}$ ($n\in\mathbb{Z}$). Since $E_{nI,m}(\tau;s)$ has meromorphic continuation on the whole s–plane as a slowly increasing function (outside its singularity), the above integral gives an analytic continuation of $D(s;f,g)$ to the whole complex s–plane. In particular if $\ell=\kappa$, the L–function $D(s;f,g)$ can have a pole at $s=0$ because in that case, we can choose $m=0$ and $E_0(\tau,s)$ has a simple pole at $s=1$ with a constant residue.

By the same calculation as in Subsection 3.1, we have

$$(4.8)\quad \begin{aligned} &2^{[F:\mathbb{Q}]-1}(4\pi)^{sI+(\ell+\kappa)/2}D(s;f,g) \\ &= [U^2]\Gamma_F(sI+(\ell+\kappa)/2)\sum_{0\ll\xi\in F^\times/U^2}\overline{a(\xi,f)\xi^{-\ell/2}}a(\xi,g)\xi^{-\kappa/2}N(\xi)^{-s}\end{aligned}$$

if $\mathrm{Re}(s)>\theta(f)+\theta(g)+1$ for $\theta(f)$ as in Section 3.

Let us recall the theta series defined below (4.4):

$$\theta(\phi_{k,M}) = \sum_{a\in M}\phi_{k,M}(a)\mathbf{e}_F(aa^c\tau)$$

for $\phi_{k,M}(a)=a^{k\Sigma}\phi_M(a)$ as in (4.4). We compute $D(s;f'_c,\theta(\phi_{k,M}))$ for a cusp form f' of weight $\ell+I$:

$$\begin{aligned} &2^{[F:\mathbb{Q}]-1}\frac{(4\pi)^{sI+(\ell+k+2I)/2}}{\Gamma_F(sI+(\ell+k+2I)/2)}D(s;f'_c,\theta(\phi_{k,M})) \\ &= [U^2]\sum_{0\ll\xi\in F^\times/U^2}a(\xi,f')\xi^{-(\ell+I)/2}a(\xi,\theta(\phi_{k,M}))\xi^{-(k+I)/2}N(\xi)^{-s} \\ &= [U^2]\sum_{\alpha\in M^\times/U}\phi_M(\alpha)a(\alpha\alpha^c,f')\alpha^{k\Sigma}(\alpha\alpha^c)^{-(k+\ell+2I)/2}N(\alpha)^{-s} \\ &= [U^2]\sum_{\alpha\in M^\times/U}\phi_M(\alpha)a(\alpha\alpha^c,f')\alpha^{k\Sigma}(\alpha^{-k\Sigma}\alpha^{-kc\Sigma})N(\alpha)^{-s-1+(k-\ell)/2} \\ &= [U^2]\sum_{\alpha\in M^\times/U}\phi_M(\alpha)a(\alpha\alpha^c,f')\alpha^{-kc\Sigma}N(\alpha)^{-s-1+(k-\ell)/2}\end{aligned}$$

From this, we get under the notation and the assumption of (4.4)

$$
\begin{aligned}
(4.9)\qquad & 2^{1-[F:\mathbb{Q}]}C^{k\Sigma}|C^{\Sigma}|^{2s}(4\pi)^{-sI-k}\Gamma_F(sI+k)H_k(z_0,w_0;s;\phi^{(\infty)},f)\\
&\qquad = [U:U^2]D(s-1+\frac{[k-\ell]}{2};f'_c,\theta(\phi_{k,M})),
\end{aligned}
$$

where ℓ is the weight of f (so, weight of f'_c is $\ell+I$). Note here that $[U:U^2]=2^{[F:\mathbb{Q}]-1}$.

Since $E_0(\tau;s)$ has a simple pole at $s=1$ with constant residue $c_\Gamma \neq 0$, if $k=\ell$ and $\phi^{(\infty)}=\phi_M\otimes\phi_0$, we have from (4.7)

$$
\begin{aligned}
(4.10)\qquad & 4^{1-[F:\mathbb{Q}]}C^{k\Sigma}|C^{\Sigma}|^{2}(4\pi)^{-k-I}\Gamma_F(k+I)\mathrm{Res}_{s=1}H_k(z_0,w_0;s;\phi^{(\infty)},f)\\
&\qquad = \mathrm{Res}_{s=1}D(s-1;f'_c,\theta(\phi_{k,M}))\\
&\qquad = [U^2]N(\mathfrak{a})^{-1}\sqrt{|D|}^{-1}c_\Gamma\langle\theta(\phi_{k,M}),f'_c\rangle_\Gamma,
\end{aligned}
$$

where

$$\langle g,f\rangle_\Gamma=\int_{\Gamma\backslash\mathfrak{H}^I} g(\tau)\overline{f(\tau)}\eta^k d\mu(\tau).$$

Let $\Psi_f(z,w)$ be the modular from on $GL(2)\times GL(2)$ given by the Fourier expansion:

$$\Psi_f(z,w)=\sum_{\alpha\in\Gamma\backslash M_2(F),\det(\alpha)\gg 0}\phi^{*(\infty)}(\epsilon\alpha)\mathbf{e}_F(\det(\alpha)z)f|_k\alpha(w)$$

as in Corollary 3.4. Then taking Γ sufficiently small and combining Corollary 3.4 and (4.10), we get the following explicit formula:

Theorem 4.1. *Let f be a Hilbert modular cusp form of weight k. Then we have*

$$\langle\theta(\phi_{k,M}),f'_c\rangle_\Gamma=2^{-k-2I}|D|^{-1}C^{k\Sigma}|C^{\Sigma}|^2 i^k\,\mathrm{Im}(z_0)^{-I}\,\mathrm{Im}(w_0)^{-I}\Psi_f(z_0,w_0)$$

under the notation of (4.4).

This type of results enabled Shimura to get a rationality result of the Petersson inner product of quaternionic cusp forms of CM type with respect to CM periods (for example, see [Sh2] II Section 3).

5. Jacquet-Langlands-Shimizu correspondence

It is a well known result of Jacquet-Langlands and Shimizu that if we choose level appropriately, the space of quaternionic automorphic forms can be embedded into the space of Hilbert modular forms keeping the Hecke operator action. We are going to recall the result, scrutinizing integrality of the correspondence.

5.1. HILBERT MODULAR FORMS AND HECKE ALGEBRAS. Let us recall the definition of the adelic Hilbert modular forms and their Hecke ring of level $\mathfrak{N}$ for an integral ideal $\mathfrak{N}$ of F (cf. [H96] Sections 2.2-4).

We first recall formal Hecke rings of double cosets. We consider the following open compact subgroup of $GL_2(F_{\mathbb{A}^{(\infty)}})$:

$$U_0(\mathfrak{N}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\widehat{O}) \middle| c \equiv 0 \mod \mathfrak{N}\widehat{O} \right\}, \tag{5.1}$$

where $\widehat{O} = O \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ and $\widehat{\mathbb{Z}} = \prod_\ell \mathbb{Z}_\ell$. Then we introduce the following semi-group

$$\Delta_0(\mathfrak{N}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(F_{\mathbb{A}^{(\infty)}}) \cap M_2(\widehat{O}) \middle| c \equiv 0 \mod \mathfrak{N}\widehat{O}, d_{\mathfrak{N}} \in O_{\mathfrak{N}}^\times \right\}, \tag{5.2}$$

where $d_{\mathfrak{N}}$ is the projection of $d \in \widehat{O}$ to $\prod_{\mathfrak{l}|\mathfrak{N}} O_{\mathfrak{l}}$ for prime ideals $\mathfrak{l}$. Writing T_0 for the maximal diagonal torus of $GL(2)_{/O}$ and putting

$$D_0 = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in T_0(F_{\mathbb{A}^{(\infty)}}) \cap M_2(\widehat{O}) \middle| d_{\mathfrak{N}} = 1 \right\}, \tag{5.3}$$

we have (e.g. [MFG] 3.1.6)

$$\Delta_0(\mathfrak{N}) = U_0(\mathfrak{N}) D_0 U_0(\mathfrak{N}). \tag{5.4}$$

Formal finite linear combinations $\sum_\delta c_\delta U_0(\mathfrak{N})\delta U_0(\mathfrak{N})$ of double cosets of $U_0(\mathfrak{N})$ in $\Delta_0(\mathfrak{N})$ form a ring $R(U_0(\mathfrak{N}), \Delta_0(\mathfrak{N}))$ under convolution product (see [IAT] Chapter 3 or [MFG] 3.1.6). The algebra is commutative and is isomorphic to the polynomial ring with variables $\{T(\mathfrak{l}), T(\mathfrak{l},\mathfrak{l})\}_{\mathfrak{l}}$, $T(\mathfrak{l})$ for primes $\mathfrak{l}$ corresponding to the double coset $U_0(\mathfrak{N}) \left(\begin{smallmatrix} \varpi_{\mathfrak{l}} & 0 \\ 0 & 1 \end{smallmatrix}\right) U_0(\mathfrak{N})$ and $T(\mathfrak{l},\mathfrak{l})$ for primes $\mathfrak{l} \nmid \mathfrak{N}$ corresponding to $U_0(\mathfrak{N})\varpi_{\mathfrak{l}} U_0(\mathfrak{N})$, where $\varpi_{\mathfrak{l}}$ is a prime element of $O_{\mathfrak{l}}$.

The double coset ring $R(U_0(\mathfrak{N}), \Delta_0(\mathfrak{N}))$ naturally acts on the space of adelic modular forms whose definition we now recall. Since $T_0(O/\mathfrak{N}')$ is canonically a quotient of $U_0(\mathfrak{N}')$, a character $\varepsilon : T_0(O/\mathfrak{N}') \to \mathbb{C}^\times$ can be considered as a character of $U_0(\mathfrak{N}')$. Writing $\varepsilon\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) = \varepsilon_1(a)\varepsilon_2(d)$, if $\widetilde{\varepsilon} = \varepsilon_1\varepsilon_2^{-1}$ factors through $O/\mathfrak{N}$ for $\mathfrak{N}|\mathfrak{N}'$, then we can extend the character ε of $U_0(\mathfrak{N}')$ to $U_0(\mathfrak{N})$ by putting $\varepsilon(u) = \varepsilon_2(\det(u))\widetilde{\varepsilon}(a)$ for $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U_0(\mathfrak{N})$. Writing $\varepsilon^- = \widetilde{\varepsilon}^{-1}$, $\varepsilon(u)$ has another expression $\varepsilon(u) = \varepsilon_1(\det(u))\varepsilon^-(d)$, because they induce the same character on $U_0(\mathfrak{N}')$ and on $U_0(\mathfrak{N}) \cap SL_2(\widehat{O})$. Hereafter we use the expression $\varepsilon(u) = \varepsilon_1(\det(u))\varepsilon^-(d)$ (although $\varepsilon(u) = \varepsilon_2(\det(u))\widetilde{\varepsilon}(a)$ is used in [Fu] and [HMI]; we note that (κ_1, κ_2) in this paper corresponds to (κ_2, κ_1) in [HMI] and [PAF]). We fix an arithmetic character $\varepsilon_+ : F_{\mathbb{A}}^\times / F^\times \to \mathbb{C}^\times$ with $\varepsilon_+|_{\widehat{O}^\times} = \varepsilon_1\varepsilon_2$ and $\varepsilon_\infty(x) = x^{-(\kappa_1+\kappa_2-I)}$. We use the symbol ε for the triple $(\varepsilon_1, \varepsilon_2, \varepsilon_+)$; thus, we may regard ε as a character of $U_0(\mathfrak{N})F_{\mathbb{A}}^\times$ by $\varepsilon(uz) = \varepsilon(u)\varepsilon_+(z)$ for $z \in F_{\mathbb{A}}^\times$ and $u \in U_0(\mathfrak{N})$. If we replace ε_+ by its p–adic avatar $\widehat{\varepsilon}_+$, we get a p–adic character $\widehat{\varepsilon}$ of $U_0(\mathfrak{N})F_{\mathbb{A}}^\times$.

We identify the group of algebraic characters $X^*(T_0)$ of T_0 with $\mathbb{Z}[I]^2$ so that $\kappa = (\kappa_1, \kappa_2) \in \mathbb{Z}[I]^2$ sends $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$ to $x^{-\kappa_1}y^{-\kappa_2} = \prod_{\sigma \in I}(\sigma(x)^{-\kappa_{1,\sigma}}\sigma(y)^{-\kappa_{2,\sigma}})$. To each $\kappa \in X^*(T_0)$, we associate a factor of automorphy:

$$J_\kappa(g,\tau) = \det(g)^{\kappa_2 - I} j(g,\tau)^{\kappa_1-\kappa_2+I} \quad \text{for } g \in GL_2(F_\infty) \text{ and } \tau \in \mathfrak{H}^I. \tag{5.5}$$

Then we define $S_\kappa(\mathfrak{N},\varepsilon;\mathbb{C})$ to be the space of functions $f : GL_2(F_\mathbb{A}) \to \mathbb{C}$ satisfying the following conditions (e.g. [H96] Section 2.2):

(S1) We have $f(\alpha x u z) = \varepsilon_+(z)\varepsilon(u) f(x) J_\kappa(u_\infty, \mathbf{i})^{-1}$ for all $\alpha \in GL_2(F)$, $z \in F_\mathbb{A}^\times$ and $u \in U_0(\mathfrak{N})C_\mathbf{i}$ for the stabilizer $C_\mathbf{i}$ in $GL_2^+(F_\infty)$ of $\mathbf{i} = (\sqrt{-1},\dots,\sqrt{-1}) \in \mathfrak{Z} = \mathfrak{H}^I$, where $GL_2^+(F_\infty)$ is the identity connected component of $GL_2(F_\infty)$;

(S2) Choosing $u \in GL_2(F_\infty)$ with $u(\mathbf{i}) = \tau$ for each $\tau \in \mathfrak{H}^I$, define $f_x(\tau) = f(xu_\infty)J_\kappa(u_\infty,\mathbf{i})$ for each $x \in GL_2(F_{\mathbb{A}^{(\infty)}})$. Then f_x is a holomorphic function on $\mathfrak{Z}$ for all x;

(S3) $f_x(\tau)$ is rapidly decreasing towards the cusp ∞.

If we replace the word: "rapidly decreasing" in (S3) by "slowly increasing", we get the definition of the space of modular forms $M_\kappa(\mathfrak{N},\varepsilon;\mathbb{C})$. It is easy to check (e.g. [MFG] 3.1.5 and [HMI] 2.3.5) that the function f_x in (S2) satisfies the classical automorphy condition:

$$f(\gamma(\tau)) = \varepsilon(x^{-1}\gamma x)^{-1} f(\tau) J_\kappa(\gamma,\tau) \quad \text{for all } \gamma \in \Gamma_{0,x}(\mathfrak{N}), \tag{5.6}$$

where $\Gamma_{0,x}(\mathfrak{N}) = xU_0(\mathfrak{N})x^{-1}GL_2^+(F_\infty) \cap GL_2(F)$, and $GL_2^+(F_\infty)$ is the subgroup of $GL_2(F_\infty)$ made up of matrices with totally positive determinant. Also by (S3), f_x is rapidly decreasing towards all cusps of Γ_x (e.g. [MFG] (3.22)). It is well known that $M_\kappa = 0$ unless $\kappa_1 + \kappa_2 = [\kappa_1+\kappa_2]I$ for $[\kappa_1+\kappa_2] \in \mathbb{Z}$. We write simply $[\kappa]$ for $[\kappa_1+\kappa_2] \in \mathbb{Z}$ if $M_\kappa \neq 0$. In [H88a] Section 2, the space S_κ is written as $S^*_{k,\widehat{w}}$ for $k = \kappa_1 - \kappa_2 + I$ and $\widehat{w} = I - \kappa_2$, and the action of Hecke operators is the same as specified in [H88a] (2.9e), which we recall now.

In order to define the Hecke operator action on the space of automorphic forms, we fix a prime element $\varpi_\mathfrak{l}$ of the $\mathfrak{l}$–adic completion $O_\mathfrak{l}$ of O for each prime ideal $\mathfrak{l}$ of F. We extend $\varepsilon^- : \widehat{O}^\times \to \mathbb{C}^\times$ to $F^\times_{\mathbb{A}^{(\infty)}} \to \mathbb{C}^\times$ just by putting $\varepsilon^-(\varpi_\mathfrak{l}^m) = 1$ for $m \in \mathbb{Z}$. This is possible because $F_\mathfrak{l}^\times = O_\mathfrak{l}^\times \times \varpi_\mathfrak{l}^\mathbb{Z}$ for $\varpi_\mathfrak{l}^\mathbb{Z} = \{\varpi_\mathfrak{l}^m | m \in \mathbb{Z}\}$. Similarly, we extend ε_2 to $F^\times_{\mathbb{A}^{(\infty)}}$. Then we define $\varepsilon(u) = \varepsilon_1(\det(u))\varepsilon^-(d)$ for $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta_0(\mathfrak{N})$. Let $\mathcal{U}$ be the unipotent algebraic subgroup of $GL(2)_{/F}$ defined by

$$\mathcal{U}(A) = \left\{ \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \middle| a \in A \right\}.$$

For each $U_0(\mathfrak{N})yU_0(\mathfrak{N}) \in R(U_0(\mathfrak{N}),\Delta_0(\mathfrak{N}))$, we can decompose

$$U_0(\mathfrak{N})yU_0(\mathfrak{N}) = \bigsqcup_{t \in T_0(F_\mathbb{A}^{(\infty)}), u \in \mathcal{U}(\widehat{O})} utU_0(\mathfrak{N})$$

for finitely many u and t (see [IAT] Chapter 3 or [MFG] 3.1.6). We define

$$f|[U_0(\mathfrak{N})yU_0(\mathfrak{N})](x) = \sum_{t,u} \varepsilon(t)^{-1} f(xut). \tag{5.7}$$

It is easy to check that this operator preserves the space $M_\kappa(\mathfrak{N},\varepsilon;\mathbb{C})$ and $S_\kappa(\mathfrak{N},\varepsilon;\mathbb{C})$ by verifying (S1-3) for $f|[U_0(\mathfrak{N})yU_0(\mathfrak{N})]$. This action for y with

$y_{\mathfrak{N}} = 1$ is independent of the choice of the extension of ε to $T_0(F_{\mathbb{A}})$. When $y_{\mathfrak{N}} \neq 1$, we may assume that $y_{\mathfrak{N}} \in D_0 \subset T_0(F_{\mathbb{A}})$, and in this case, t can be chosen so that $t_{\mathfrak{N}} = y_{\mathfrak{N}}$ (so $t_{\mathfrak{N}}$ is independent of single right cosets in the double coset). If we extend ε to $T(F_{\mathbb{A}}^{(\infty)})$ by choosing another prime element $\varpi'_{\mathfrak{l}}$ and write the extension as ε', then we have

$$\varepsilon(t_{\mathfrak{N}})[U_0(\mathfrak{N})yU_0(\mathfrak{N})] = \varepsilon'(t_{\mathfrak{N}})[U_0(\mathfrak{N})yU_0(\mathfrak{N})]',$$

where the operator on the right-hand-side is defined with respect to ε'. Thus the sole difference is the root of unity $\varepsilon(t_{\mathfrak{N}})/\varepsilon'(t_{\mathfrak{N}}) \in \mathrm{Im}(\varepsilon|_{T_0(O/\mathfrak{N})})$. Since it depends on the choice of $\varpi_{\mathfrak{l}}$, we make the choice once and for all, and write $T(\mathfrak{l})$ for $[U_0(\mathfrak{N}) \left(\begin{smallmatrix}\varpi_{\mathfrak{l}} & 0\\ 0 & 1\end{smallmatrix}\right) U_0(\mathfrak{N})]$ (if $\mathfrak{l}|\mathfrak{N}$). By linearity, these action of double cosets extends to the ring action of the double coset ring $R(U_0(\mathfrak{N}), \Delta_0(\mathfrak{N}))$.

To introduce rationality structure on the space of modular forms, we recall Fourier expansion and q–expansion of modular forms (cf. [H96] Sections 2.3–4 and [HMI] Proposition 2.26, where the order of κ_j $(j = 1, 2)$ is reversed; so, (κ_1, κ_2) here corresponds to (κ_2, κ_1) in [HMI]). We fix an embedding $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ once and for all and identify $\overline{\mathbb{Q}}$ with the subfield of all algebraic numbers in $\mathbb{C}$. We also choose a differental idele $d \in F_{\mathbb{A}}^{\times}$ with trivial prime-to–$\mathfrak{d}$ part: $d^{(\mathfrak{d})} = 1$. Thus $d\widehat{O} = \mathfrak{d}\widehat{O}$ for the absolute different $\mathfrak{d}$ of F. Each member f of $M_\kappa(\mathfrak{N}, \varepsilon; \mathbb{C})$ has Fourier expansion of the following form:

(5.8)
$$f\left(\begin{smallmatrix}y & x\\ 0 & 1\end{smallmatrix}\right) = |y|_{\mathbb{A}} \left\{ a_0(yd, f)|u|_{\mathbb{A}}^{[\kappa_2]} + \sum_{0 \ll \xi \in F} a(\xi yd, f)(\xi y_\infty)^{-\kappa_2} \mathbf{e}_F(i\xi y_\infty)\mathbf{e}_{\mathbb{A}}(\xi x) \right\}.$$

Here $y \mapsto a(y, f)$ and $a_0(y, f)$ are functions defined on $y \in F_{\mathbb{A}}^{\times}$ only depending on its finite part $y^{(\infty)}$. The function $a(y, f)$ is supported by the set $(\widehat{O} \times F_\infty) \cap F_{\mathbb{A}}^{\times}$. When $f \in S_\kappa(\mathfrak{N}, \varepsilon; \mathbb{C})$, $a_0(y, f) = 0$; so, we just ignore the constant term $a_0(y, f)$. When κ_2 is not in $\mathbb{Z}I$, we have $S_\kappa = M_\kappa$; so, we ignore the constant term if $[\kappa_2] \in \mathbb{Z}$ is not well defined. Let $F[\kappa]$ be the field fixed by $\{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F) | \kappa\sigma = \kappa\}$, which is the field of rationality of the character $\kappa \in X^*(T_0)$. Write $O[\kappa]$ for the integer ring of $F[\kappa]$. We also define $O[\kappa, \varepsilon]$ for the integer ring of the field $F[\kappa, \varepsilon]$ generated by the values of ε (on finite ideles) over $F[\kappa]$. We call an idele $y \in F_{\mathbb{A}}^{\times}$ *integral* if $y^{(\infty)} \in \widehat{O}$. Then for any $F[\kappa, \varepsilon]$–algebra A inside $\mathbb{C}$, we define

(5.9)
$$M_\kappa(\mathfrak{N}, \varepsilon; A) = \left\{ f \in M_\kappa(\mathfrak{N}, \varepsilon; \mathbb{C}) \middle| a_0(y, f), a(y, f) \in A \text{ as long as } y \text{ is integral} \right\}$$
$$S_\kappa(\mathfrak{N}, \varepsilon; A) = M_\kappa(\mathfrak{N}, \varepsilon; A) \cap S_\kappa(\mathfrak{N}, \varepsilon; \mathbb{C}).$$

Using rationality of (the canonical models of) the Hilbert modular variety (studied by Shimura and others), we can interpret $S_\kappa(\mathfrak{N}, \varepsilon; A)$ (resp. $M_\kappa(\mathfrak{N}, \varepsilon; A)$) as the space of A–rational global sections of a line bundle of the variety defined over A; so, we have, by the flat base-change theorem (e.g.

[GME] Lemma 1.10.2),

$$(5.10)\quad M_\kappa(\mathfrak{N},\varepsilon;A)\otimes_A\mathbb{C}=M_\kappa(\mathfrak{N},\varepsilon;\mathbb{C})\ \text{ and }\ S_\kappa(\mathfrak{N},\varepsilon;A)\otimes_A\mathbb{C}=S_\kappa(\mathfrak{N},\varepsilon;\mathbb{C})$$

Since Hecke operators are induced by algebraic correspondences on the product of two Hilbert modular varieties defined over A (e.g. [GME] 4.2.1 and [PAF] 4.2.5), the action of the Hecke operators $T(\mathfrak{l})$ and $T(\mathfrak{l},\mathfrak{l})$ preserves the A–rational space of modular forms (see below (5.15) for a more concrete argument showing the Hecke operator stability). We define the Hecke algebra $h_\kappa(\mathfrak{N},\varepsilon;A)\subset \mathrm{End}_A(S_\kappa(\mathfrak{N},\varepsilon;A))$ by the A–subalgebra generated by the Hecke operators $T(\mathfrak{l})$ and $T(\mathfrak{l},\mathfrak{l})$ for all prime ideals $\mathfrak{l}$ (here we agree to put $T(\mathfrak{l},\mathfrak{l})=0$ if $\mathfrak{l}|\mathfrak{N}$). In the same manner, we define $H_\kappa(\mathfrak{N},\varepsilon;A)\subset \mathrm{End}_A(M_\kappa(\mathfrak{N},\varepsilon;A))$.

5.2. q–Expansion of p–integral modular forms. We recall the rational prime p and the embedding $i_p:\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}_p$. Then for any $\overline{\mathbb{Q}}_p$–algebras A, we define

(5.11)
$$M_\kappa(\mathfrak{N},\varepsilon;A)=M_\kappa(\mathfrak{N},\varepsilon;\overline{\mathbb{Q}})\otimes_{\overline{\mathbb{Q}},i_p}A \text{ and } S_\kappa(\mathfrak{N},\varepsilon;A)=S_\kappa(\mathfrak{N},\varepsilon;\overline{\mathbb{Q}})\otimes_{\overline{\mathbb{Q}},i_p}A.$$

By linearity, $y\mapsto a(y,f)$ and $a_0(y,f)$ extend to functions on $F_{\mathbb{A}}^\times\times M_\kappa(\mathfrak{N},\varepsilon;A)$ with values in A. Let $\mathcal{N}:F_{\mathbb{A}}^\times/F^\times\to\overline{\mathbb{Q}}_p^\times$ be the p–cyclotomic character defined by $\mathcal{N}(y)=y_p^{-I}|y^{(\infty)}|_{\mathbb{A}}^{-1}$. Then we define the q–expansion coefficients (at p) of $f\in M_\kappa(\mathfrak{N},\varepsilon;A)$ by

$$(5.12)\qquad \mathbf{a}_p(y,f)=y_p^{-\kappa_2}a(y,f)\ \text{ and }\ \mathbf{a}_{0,p}(y,f)=\mathcal{N}(yd^{-1})^{[\kappa_2]}a_0(y,f).$$

Here we note that $a_0(y,f)=0$ unless $[\kappa_2]\in\mathbb{Z}$ is well defined. We now define for any p–adically complete $O[\kappa,\varepsilon]$–algebra A in $\widehat{\mathbb{Q}}_p$ (the p–adic completion of $\overline{\mathbb{Q}}_p$)

(5.13)
$$M_\kappa(\mathfrak{N},\varepsilon;A)=\left\{f\in M_\kappa(\mathfrak{N},\varepsilon;\widehat{\mathbb{Q}}_p)\middle|\mathbf{a}_{0,p}(y,f),\mathbf{a}_p(y,f)\in A\text{ for integral }y\right\}$$
$$S_\kappa(\mathfrak{N},\varepsilon;A)=M_\kappa(\mathfrak{N},\varepsilon;A)\cap S_\kappa(\mathfrak{N},\varepsilon;\widehat{\mathbb{Q}}_p).$$

These spaces have geometric meaning as the space of A–integral global sections of a line bundle of the Hilbert modular variety of level $\mathfrak{N}$ (e.g. [HT1] 1.3 and [HMI] 4.3.7).

The formal q–expansion of f has values in the space of functions on $F^\times_{\mathbb{A}^{(\infty)}}$ with values in the formal monoid algebra $A[[q^\xi]]_{\xi\in F_+}$ of the multiplicative semi-group F_+ made up of totally positive elements, which is given by

$$(5.14)\qquad f(y)=\mathcal{N}(y)^{-1}\left\{\mathbf{a}_{0,p}(yd,f)+\sum_{\xi\gg0}\mathbf{a}_p(\xi yd,f)q^\xi\right\}.$$

We choose a complete representative set $\{a_i\}_{i=1,\dots,h}$ in finite ideles for the strict idele class group $F^\times\backslash F_{\mathbb{A}}^\times/\widehat{O}^\times F_{\infty+}^\times$. Let $\mathfrak{a}_i=a_iO$. Write $t_i=\left(\begin{smallmatrix}a_id^{-1}&0\\0&1\end{smallmatrix}\right)$ and consider $f_i = f_{t_i}$ as defined in (S2). The collection $(f_i)_{i=1,\dots,h}$ determines f, because

$$GL_2(F_{\mathbb{A}}) = \bigsqcup_{i=1}^{h} GL_2(F)t_iU_0(\mathfrak{N})GL_2^+(F_\infty)$$

by the approximation theorem. Then as observed in [H88a] Section 4 (and [PAF] 4.2.10), $f(a_id^{-1})$ gives the q–expansion over A of f_i at the Tate abelian variety with a_iO–polarization $\mathrm{Tate}_{\mathfrak{a}_i^{-1},O}(q)$ as in [HT1] 1.7. Thus by the q–expansion principle ([HT1] 1.7 and [HMI] 4.2.8), the q–expansion: $y \mapsto f(y)$ determines f uniquely (for any algebra A for which the space of A–integral modular forms is well defined).

We write $T(y)$ for the Hecke operator acting on $M_\kappa(\mathfrak{N},\varepsilon;A)$ corresponding to the double coset decomposition of

$$\mathcal{T}(y) = \left\{x \in \Delta_0(\mathfrak{N}) \middle| \det(x)\widehat{O} = y\widehat{O}\right\}$$

for $y \in \widehat{O} \cap F^\times_{\mathbb{A}(\infty)}$. We renormalize $T(y)$ to create a new operator $\mathbb{T}(y)$ by $\mathbb{T}(y) = y_p^{-\kappa_2}T(y)$. Since this only affects $T(y)$ with $y_p \neq 1$, $\mathbb{T}(\mathfrak{l}) = T(\varpi_{\mathfrak{l}}) = T(\mathfrak{l})$ if $\mathfrak{l} \nmid p$. However $\mathbb{T}(\mathfrak{p}) \neq T(\mathfrak{p})$ for primes $\mathfrak{p}|p$. This renormalization is optimal to have the stability of the A–integral spaces under Hecke operators. We define $\langle \mathfrak{l} \rangle = N(\mathfrak{l})T(\mathfrak{l},\mathfrak{l})$ for $\mathfrak{l} \nmid \mathfrak{N}$. This new action also preserves the integrality as long as $[\kappa] \geq 0$ (cf. [H96] Section 2.2 and [HMI] Theorem 2.28). We have the following formula of the action of $T(\mathfrak{l})$ and $T(\mathfrak{l},\mathfrak{l})$ (e.g. [H96] Section 2.4):

$$\mathbf{a}_p(y, f|\mathbb{T}(\mathfrak{l})) = \begin{cases} \mathbf{a}_p(y\varpi_{\mathfrak{l}}, f) + \mathbf{a}_p(y\varpi_{\mathfrak{l}}^{-1}, f|\langle \mathfrak{l} \rangle) & \text{if } \mathfrak{l} \nmid \mathfrak{N} \\ \mathbf{a}_p(y\varpi_{\mathfrak{l}}, f) & \text{if } \mathfrak{l}|\mathfrak{N}. \end{cases} \tag{5.15}$$

From this, it is plain that $T(\mathfrak{l})$ preserves the space $S_\kappa(\mathfrak{N},\varepsilon;A)$ if either $p|\mathfrak{N}$ or $[\kappa] \geq 0$, because $\mathbf{a}_p(y\varpi_{\mathfrak{l}}^{-1}, f|\langle \mathfrak{l} \rangle) = \varpi_{\mathfrak{l},p}^{-2\kappa_2}N(\mathfrak{l})\varepsilon_+(\mathfrak{l})\mathbf{a}_p(y,f)$. We hereafter assume

$$\text{Either } p|\mathfrak{N} \text{ or } [\kappa] \geq 0 \text{ and } \kappa_1 - \kappa_2 \geq I. \tag{5.16}$$

We define $h_\kappa(\mathfrak{N},\varepsilon;A)$ again by the A–subalgebra of $\mathrm{End}_A(S_\kappa(\mathfrak{N},\varepsilon;A))$ generated by $\mathbb{T}(\mathfrak{l})$ and $\langle \mathfrak{l} \rangle$ over A for all primes $\mathfrak{l}$ (for a p–adically complete $O[\kappa,\varepsilon]$–algebra A).

We can think of the subgroup $U(\mathfrak{N})$ of $U_0(\mathfrak{N})$ made of matrices $u \in U_0(\mathfrak{N})$ whose reduction modulo $\mathfrak{N}$ are upper unipotent. Then for any subgroup U with $U(\mathfrak{N}) \subset U \subset U_0(\mathfrak{N})$, we can think of the space of cusp forms $S_\kappa(U,\varepsilon;\mathbb{C})$ made up of cusp forms satisfying (S1-3) for U in place of $U_0(\mathfrak{N})$. We have Hecke operators $\mathbb{T}(y)$ corresponding to $(U \cdot D_0U) \cap \mathcal{T}(y)$ acting on $S_\kappa(U,\varepsilon;A)$. Then in the same manner of $S_\kappa(\mathfrak{N},\varepsilon;A)$, we define $S_\kappa(U,\varepsilon;A)$ and the Hecke algebra $h_\kappa(U,\varepsilon;A)$ as the A–subalgebra of $\mathrm{End}_A(S_\kappa(U,\varepsilon;A))$ generated by $\mathbb{T}(y)$ and $\langle \mathfrak{l} \rangle$.

PROPOSITION 5.1. *Let A be an $O[\kappa,\varepsilon]$–algebra for which the space of cusp forms $S_\kappa(\mathfrak{N},\varepsilon;A)$ is well defined (by* (5.9) *or* (5.11) *or* (5.13)*). Write $H =$*

$h_\kappa(\mathfrak{N},\varepsilon;A)$ and $S = S_\kappa(\mathfrak{N},\varepsilon;A)$. Let V be an H–module and V' be an A–module of finite type with an A–bilinear product $\langle\ ,\ \rangle : V \times V' \to A$. Then we have:

(1) *The formal q–expansion for $v \in V$ and $w \in V'$:*

$$f(v\otimes w)(y) = \mathcal{N}(y)^{-1}\left\{\sum_{\xi\gg 0}\langle v|\mathbb{T}(\xi y d), w\rangle q^\xi\right\}$$

gives a unique element of S.

(2) *The map $v \otimes w \mapsto f(v\otimes w)$ gives an A–linear map of $V \otimes_A V'$ into S with $f((v|\mathbb{T}(y))\otimes w) = f(v\otimes w)|\mathbb{T}(y)$. If further V' is an H–module and $\langle v|h, w\rangle = \langle v, w|h\rangle$ for all $v \in V$, $w \in V'$ and $h \in H$, then the map f induces an H–linear map: $V \otimes_H V' \to S_\kappa(\mathfrak{N},\varepsilon;A)$.*

(3) *Suppose that R is an A–algebra direct summand of H, and put $V(R) = RV$ and $S(R) = RS$. If $V(R)$ is R–free of finite rank and $\mathrm{Hom}_A(V(R),A)$ is embedded into V' by the pairing $\langle\ ,\ \rangle$, then the map $f : V(R)\otimes_A V' \to S(R)$ is surjective.*

The formulation of this proposition is suggested by the expression of the theta correspondence given in [Sh2] II, Theorem 3.1.

Proof. We have an isomorphism $\iota : \mathrm{Hom}_A(H,A) \cong S$ given by $\mathbf{a}_p(y,\iota(\phi)) = \phi(\mathbb{T}(y))$ (see [H88a] Theorem 5.11, [H91] Theorem 3.1 and [H96] Section 2.6), which is an H–linear map (that is, $\iota(\phi\circ h) = \iota(\phi)|h$). Since V is an H–module, $h \mapsto \langle v|h, w\rangle$ gives an element of $\mathrm{Hom}_A(H,A)$ and hence an element in S. The element has the expression as in (1) by the above explicit form of ι. The assertion (2) is then clear from (1). As for (3), by the isomorphism $\mathrm{Hom}_A(V(R),A) \hookrightarrow V'$, each element of $\mathrm{Hom}(R,A) \cong S(R)$ is a finite A–linear combination of $h \mapsto \langle v|h,w\rangle$ for $v \in V(R)$ and $w \in V'$; so, the surjectivity follows. □

5.3. Integral correspondence. In order to create a proto-typical example of the module V in Proposition 5.1, we study here cohomology groups on quaternionic Shimura varieties. See [H94] and [H88a] for more details of such cohomology groups.

Let B be a quaternion algebra over F. We write G for the algebraic group defined over $\mathbb{Q}$ such that $G(A) = (B\otimes_{\mathbb{Q}} A)^\times$ for each $\mathbb{Q}$–algebra A. Let $d(B)^2$ be the discriminant of B. We assume that $p \nmid d(B)$ and that

$$B\otimes_{F,\sigma}\mathbb{R} \cong \begin{cases} M_2(\mathbb{R}) & \text{if } \sigma\in I_B \\ \mathbb{H} & \text{if } \sigma \in I - I_B = I^B, \end{cases} \tag{5.17}$$

where $\mathbb{H}$ is the Hamilton quaternion algebra over $\mathbb{R}$.

We fix once and for all an extension of $\sigma : F \hookrightarrow \overline{\mathbb{Q}}$ to $\sigma : \overline{F} \cong \overline{\mathbb{Q}}$ for an algebraic closure $\overline{F}/F$. We take a quadratic extension K/F inside $\overline{F}$ so that $K\otimes_{F,\sigma}\mathbb{R} \cong \mathbb{R}\times\mathbb{R}$ as F–algebras for $\sigma \in I_B$, $K\otimes_F F_{\mathfrak{p}} \cong F_{\mathfrak{p}}\times F_{\mathfrak{p}}$ for primes $\mathfrak{p}|p$

and $B\otimes_F K \cong M_2(K)$. We can always choose such a quadratic extension K as long as $p\nmid d(B)$. These condition automatically implies $K\otimes_F\mathbb{R}\cong\mathbb{C}$ for $\sigma\in I^B$.

We identify $B\otimes_F K$ with $M_2(K)$ by the above isomorphism. We fix maximal orders O_B and O_K of B and K, respectively, and we suppose that

$$O_B\otimes_O O_K\subset M_2(O_K). \tag{5.18}$$

We fix an isomorphism $O_{B,\mathfrak{l}}\cong M_2(O_{\mathfrak{l}})$ so that for the p–adic place $\mathfrak{p}|p$ induced by $i_p\circ\sigma$, this isomorphism coincides with the one: $O_B\hookrightarrow M_2(O_K)\xrightarrow{i_p\circ\sigma} M_2(O_{\mathfrak{p}})$. For an integral ideal $\mathfrak{N}_0$ of F prime to $d(B)$, putting $\mathfrak{N}=\mathfrak{N}_0 d(B)$, we define

$$U_0^B(\mathfrak{N})=\left\{x\in G(\mathbb{A})\middle|x_{\mathfrak{N}_0}=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\quad\text{with}\quad c\in\mathfrak{N}_0 O_{\mathfrak{N}_0}\right\}, \tag{5.19}$$

where $O_{\mathfrak{N}_0}=\prod_{\mathfrak{l}|\mathfrak{N}_0}O_{\mathfrak{l}}$. Similarly we define $\Delta_0^B(\mathfrak{N})\subset B\otimes_{\mathbb{Q}}\mathbb{A}^{(\infty)}$ so that it is the product of local components $\Delta_{\mathfrak{l}}$ which coincide with the local components of $\Delta_0(\mathfrak{N})$ as long as $\mathfrak{l}\nmid d(B)$ and $\Delta_{\mathfrak{l}}=O_{B,\mathfrak{l}}$ if $\mathfrak{l}|d(B)$. Again we can think of the double coset ring $R(U_0^B(\mathfrak{N}),\Delta_0^B(\mathfrak{N}))$. We have $T(\mathfrak{l})$ and $T(\mathfrak{l},\mathfrak{l})$ in $R(U_0^B(\mathfrak{N}),\Delta_0^B(\mathfrak{N}))$ for $\mathfrak{l}\nmid d(B)$, because the local component at $\mathfrak{l}$ of $\Delta_0^B(\mathfrak{N})$ is identical to that of $\Delta_0(\mathfrak{N})$. For $\mathfrak{l}|d(B)$, we take $\alpha_{\mathfrak{l}}\in O_{B,\mathfrak{l}}$ so that its reduced norm generates $\mathfrak{l}O_{\mathfrak{l}}$. Then we define $T(\mathfrak{l})=-U_0^B(\mathfrak{N})\alpha_{\mathfrak{l}}U_0^B(\mathfrak{N})$ for $\mathfrak{l}|d(B)$, and we have

$$R(U_0(\mathfrak{N}),\Delta_0(\mathfrak{N}))\cong R(U_0^B(\mathfrak{N}),\Delta_0^B(\mathfrak{N})). \tag{5.20}$$

The above isomorphism brings $T(\mathfrak{l})$ and $T(\mathfrak{l},\mathfrak{l})$ to the corresponding elements in the right-hand-side.

For a given ring A, we consider the following module $L(\kappa;A)$ over the multiplicative semi-group $M_2(A)$: Let $n=\kappa_1-\kappa_2-I\in\mathbb{Z}[I]$. We suppose that $n\geq 0$ (implying $n_\sigma\geq 0$ for all $\sigma\in I$), and we consider polynomials with coefficients in A of $(X_\sigma,Y_\sigma)_{\sigma\in I}$ homogeneous of degree n_σ for each pairs (X_σ,Y_σ). The collection of all such polynomials forms an A–free module $L(\kappa;A)$ of rank $\prod_\sigma(n_\sigma+1)$.

Suppose that A is a closed $O_K[\kappa,\varepsilon]$-algebra (via i_p) of $\widehat{\mathbb{Q}}_p$. Then $i_p(\sigma(\delta_p))$ (which we write simply $\sigma(\delta_p)$) for $\delta\in G(\mathbb{A})$ can be regarded as an element in $M_2(A)$. We let $\Delta_0^B(\mathfrak{N})$ act on $L(\kappa;A)$ as follows:

$$\delta\Phi\left(\left(\begin{smallmatrix}X_\sigma\\Y_\sigma\end{smallmatrix}\right)\right)=\varepsilon(\delta)N(\delta)^{\kappa_2}\Phi\left(\left(\sigma(\delta^\iota)\left(\begin{smallmatrix}X_\sigma\\Y_\sigma\end{smallmatrix}\right)\right)\right). \tag{5.21}$$

Here $N(\delta)$ is the reduced norm of B. We also let $z\in F_{\mathbb{A}}^\times$ act on $L(\kappa;A)$ through scalar multiplication by $\widehat{\varepsilon}_+(z)=\varepsilon_+(z)z_p^{-\kappa_1-\kappa_2+I}$ (the p–adic avatar of ε_+). We write $L(\kappa\varepsilon;A)$ for the module $L(\kappa;A)$ with this $\Delta_0^B(\mathfrak{N})F_{\mathbb{A}}^\times$–action. By the condition: $\kappa_1+\kappa_2\in\mathbb{Z}I$, if $U\subset U_0^B(\mathfrak{N})$ is sufficiently small open compact subgroup, central elements in $\Gamma_x=xUx^{-1}\cap G(\mathbb{Q})$ acts trivially on $L(\kappa\varepsilon;A)$.

We let $g\in G(\mathbb{R})$ with $N(g)\gg 0$ act on $\mathfrak{H}^{I_B}$ (by the linear fractional transformation) component-wise via $g_\sigma=\sigma(g)\in GL_2(K\otimes_{K,\sigma}\mathbb{R})=GL_2(\mathbb{R})$. We put $C_{\sigma+}$ for the stabilizer of $\sqrt{-1}$ in the identity connected component of $(B \otimes_{F,\sigma} \mathbb{R})^\times$ and define

$$C_{\infty+} = \prod_{\sigma \in I_B} C_{\sigma+} \times \prod_{\sigma \in I^B} (B \otimes_{F,\sigma} \mathbb{R})^\times.$$

Thus we have $\mathfrak{H}^{I_B} \cong G(\mathbb{R})^+/C_{\infty+}$ by $g(\mathbf{i}) \leftrightarrow g$ ($\mathbf{i} = (\sqrt{-1}, \dots, \sqrt{-1}) \in \mathfrak{H}^{I_B}$) for the identity connected component $G(\mathbb{R})^+$ of $G(\mathbb{R})$. For any open compact subgroup $U \subset U_0^B(\mathfrak{N})$, we think of the complex manifold associated to the Shimura variety:

$$Y(U) = G(\mathbb{Q})\backslash G(\mathbb{A})/F_{\mathbb{A}}^\times U \cdot C_{\infty+}.$$

We write simply $Y_0^B(\mathfrak{N})$ for $Y(U_0^B(\mathfrak{N}))$.

If U is sufficiently small so that the image $\overline{\Gamma}_{U,x}$ of $\Gamma_{U,x} = xUx^{-1}G^+(\mathbb{R}) \cap G(\mathbb{Q})$ in $G(\mathbb{R})/F_\infty^\times$ acts freely on $\mathfrak{H}^{I_B}$ for all $x \in G(\mathbb{A}^{(\infty)})$, and the action of $\Gamma_{U,x}$ on $L(\kappa\varepsilon; A)$ factors through $\overline{\Gamma}_{U,x}$. Then we can define an étale space over $Y(U)$:

$$\mathcal{L}(\kappa\varepsilon; A) = G(\mathbb{Q})\backslash \left(G(\mathbb{A}) \times L(\kappa\varepsilon; A)\right)/F_{\mathbb{A}}^\times U \cdot C_{\infty+},$$

where $\gamma(x, \Phi)uz = (\gamma xuz, u^\iota \widehat{\varepsilon}_+(z)\Phi)$ for $u \in U \cdot C_{\infty+}$, $z \in F_{\mathbb{A}}^\times$ and $\gamma \in G(\mathbb{Q})$. This étale space gives rise to the sheaf $L(\kappa\varepsilon; A)_{/Y(U)}$ of locally constant sections. We consider the sheaf cohomology group $H^q(Y(U), L(\kappa\varepsilon; A))$.

Since $Y(U) \cong \sqcup_x \overline{\Gamma}_x \backslash \mathfrak{H}^{I_B}$ for finitely many x with $x_p = 1$, we have a canonical isomorphism (cf. [H94] page 470):

$$H^q(Y(U), L(\kappa\varepsilon; A)) \cong \bigoplus_x H^q(\overline{\Gamma}_{U,x}, L(\kappa\varepsilon; A)), \tag{5.22}$$

where the right-hand-side is the direct sum of the group cohomology of the $\overline{\Gamma}_x$–module $L(\kappa\varepsilon; A)$. The kernel $E = \mathrm{Ker}(\Gamma_{U,x} \to \overline{\Gamma}_{U,x})$ is a subgroup of units $O^\times$. Since $\kappa_1 + \kappa_2 \in \mathbb{Z}I$, the action of $\epsilon \in E$ on $L(\kappa\varepsilon; A)$ is the multiplication by $\widehat{\varepsilon}_+(\epsilon)N(\epsilon)^{[\kappa]+1} = 1$. Even if $\overline{\Gamma}_{U,x}$ does not act freely on the module $L(\kappa\varepsilon; A)$, we still have $Y(U) \cong \bigsqcup_x \overline{\Gamma}_x \backslash \mathfrak{H}^{I_B}$ for finitely many x with $x_p = 1$, we can *define* the left-hand-side of (5.22) by the right hand side of (5.22).

We choose U sufficiently small as above so that $[U_0^B(\mathfrak{N}) : U]$ is prime to p (this is a condition on p). Then we have the trace map Tr (that is, the transfer map in group cohomology) and the restriction map Res:

$$\mathrm{Tr} : H^q(Y^B(U), L(\kappa\varepsilon; A)) \to H^q(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; A))$$
$$\mathrm{Res} : H^q(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; A)) \to H^q(Y^B(U), L(\kappa\varepsilon; A)).$$

Since $\mathrm{Tr} \circ \mathrm{Res}$ is the multiplication by $[U_0(\mathfrak{N}) : U]$, we have

$$H^q(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; A)) = H^q(Y^B(U), L(\kappa\varepsilon; A))/\mathrm{Ker}(\mathrm{Tr}) = \mathrm{Im}(\mathrm{Res}). \tag{5.23}$$

We can always choose a multiple $\mathfrak{N}' = \mathfrak{N}\mathfrak{q}$ (by a prime $\mathfrak{q}$) of $\mathfrak{N}$ so that $\overline{\Gamma}_{0,x}(\mathfrak{N}')$ acts freely on $\mathfrak{H}^{I_B}$.

As defined in [H88a] Section 7 and [H94] Section 4, where $L(\kappa\varepsilon; A)$ is written as $L(n, v, \varepsilon; A)$ for $v = \kappa_2$ and $n = \kappa_1 - \kappa_2 - I$, we have a natural action of the ring $R(U_0^B(\mathfrak{N}), \Delta_0^B(\mathfrak{N}))$ on the cohomology group $H^q(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; A))$. For our later use, we recall the definition of the action when $q = 0$: In this case, we may regard each cohomology class as a global section $f : B_{\mathbb{A}}^{\times} \to L(\kappa\varepsilon; A)$ with $f(\alpha x u) = u^{\iota} f(x)$ for $\alpha \in B^{\times}$ and $u \in U_0^B(\mathfrak{N}) B_{\infty}^{\times}$. Decomposing $U_0^B(\mathfrak{N}) \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) U_0^B(\mathfrak{N}) = \bigsqcup_{\varpi} \varpi U_0^B(\mathfrak{N})$, we have

$$f|\mathbb{T}(y) = y_p^{-\kappa_2} \sum_{\varpi} \varpi f(x\varpi^{-\iota}). \tag{5.24}$$

Let W be a valuation ring as in the introduction. We assume that $h_\kappa(\mathfrak{N}, \varepsilon; W)$ is well defined and $O_K[\kappa, \varepsilon]$ is embedded into W via i_p. Let V be the image of $H^q(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; W))$ in $H^q(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; W \otimes_{\mathbb{Z}} \mathbb{Q}))$. By the Eichler-Shimura isomorphism (between the space of cusp forms on $G(\mathbb{A})$ and the cohomology group; e.g. [H94] Proposition 3.1 and (10.4)) combined with the Jacquet-Langlands-Shimizu correspondence (e.g. [H88a] Theorem 2.1, Proposition 2.3 and [H81] 2.12), the above cohomology group and its compactly supported version (denoted by $H_c^q(Y_0^B(\mathfrak{N}), \cdot)$) are the module over the Hecke algebra $H_\kappa(\mathfrak{N}, \varepsilon; W \otimes_{\mathbb{Z}} \mathbb{Q})$. Since

$$H_\kappa(\mathfrak{N}, \varepsilon; W \otimes_{\mathbb{Z}} \mathbb{Q}) = h_\kappa(\mathfrak{N}, \varepsilon; W \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus E$$

as an algebra direct sum for the Eisenstein part E, for the idempotent 1_h of the cuspidal part $h_\kappa(\mathfrak{N}, \varepsilon; W \otimes_{\mathbb{Z}} \mathbb{Q})$, we can define the cuspidal cohomology groups by

$$H^q_{cusp}(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; W \otimes_{\mathbb{Z}} \mathbb{Q})) = 1_h H^q(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; W \otimes_{\mathbb{Z}} \mathbb{Q})).$$

The natural map from compactly supported cohomology group into the cohomology group without support condition actually induces an isomorphism

$$1_h H_c^q(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; W \otimes_{\mathbb{Z}} \mathbb{Q})) \cong H^q_{cusp}(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; W \otimes_{\mathbb{Z}} \mathbb{Q})).$$

We then put

$$H^q_{cusp}(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; W)) = H^q_{cusp}(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; W \otimes_{\mathbb{Z}} \mathbb{Q})) \cap \mathrm{Im}(i) \tag{5.25}$$

for the natural morphism

$$i : H^q(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; W)) \to H^q(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; W \otimes_{\mathbb{Z}} \mathbb{Q})).$$

We consider the duality pairing $[\ ,\ ]$ on $L(\kappa\varepsilon; A)$ (for $\mathbb{Q}$–algebra A) introduced in [H94] Section 5:

$$\left[\sum_{0 \le j \le n} b_j X^{n-j} Y^j, \sum_{0 \le j \le n} a_j X^{n-j} Y^j\right] = \sum_j (-1)^j \binom{n}{j}^{-1} b_{n-j} a_j \in A, \tag{5.26}$$

where $n = \kappa_1 - \kappa_2 - I$, $\binom{n}{j} = \prod_{\sigma \in I} \binom{n_\sigma}{j_\sigma}$ and for example $X^j = \prod_{\sigma \in I} X_\sigma^{j_\sigma}$. As $U_0(\mathfrak{N}) F_{\mathbb{A}}^{\times}$-modules, this pairing satisfies:

$$[uz\Phi, uz\Phi'] = \varepsilon^2(u) \widehat{\varepsilon}_+(z)^2 N_{B/F}(u_p)^{\kappa_1 + \kappa_2 - I} [\Phi, \Phi'], \tag{5.27}$$

where $N_{B/F} : B \to F$ is the reduced norm map.

Define $\kappa^* = (-\kappa_2, 1-\kappa_1)$ and $\varepsilon^* = \varepsilon^{-1}$. Thus $[\kappa^*] \leq 1 \Leftrightarrow [\kappa] \geq 0$. Then the pairing $[\ ,\ ]$ induces $U_0(\mathfrak{N})F_{\mathbb{A}}^{\times}$–equivariant pairing

$$[\ ,\ ] : L(\kappa\varepsilon; A) \times L(\kappa^*\varepsilon^*; A) \to A.$$

We now choose $q = |I_B| = \dim_{\mathbb{C}} \mathfrak{H}^{I_B}$. Then the cup product pairing induces ([H94] (5.3)) a non-degenerate pairing:

$$(\ ,\ ) : H^q_{cusp}(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; W)) \times H^q_{cusp}(Y_0^B(\mathfrak{N}), L(\kappa^*\varepsilon^*; W)) \to W \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Thus we obtain from Proposition 5.1 the following result:

Proposition 5.2. *Let $V = H^q_{cusp}(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; W))$. Let V^* be the dual W–lattice of V in $H^q_{cusp}(Y_0^B(\mathfrak{N}), L(\kappa^*\varepsilon^*; W \otimes_{\mathbb{Z}} \mathbb{Q}))$ under the Poincaré duality:*

$$(\ ,\ ) : H^q_{cusp}(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; W)) \times H^q_{cusp}(Y_0^B(\mathfrak{N}), L^*(\kappa^*\varepsilon^*; W)) \to W \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Then we have a $h_\kappa(\mathfrak{N}, \varepsilon; W)$–linear map

$$f : V \otimes_W V^* \to S_\kappa(\mathfrak{N}, \varepsilon; W)$$

defined by the q–expansion:

$$f(v \otimes w) = \mathcal{N}(y)^{-1} \sum_{0 \ll \xi} (v | \mathbb{T}(\xi y d), w) q^\xi,$$

where we regard $V \otimes_W V^*$ as an $h_\kappa(\mathfrak{N}, \varepsilon; W)$–module through the left factor V.

A similar fact for the matrix coefficients of $T(y)$ in place of $(v|T(y), w)$ has been proven in [Sh2] Theorem 3.1 by analytic means without using the Jacquet-Langlands-Shimizu correspondence.

We have $H^q_{cusp}(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; W)) = H^q(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; W))$ under the following two conditions:

- (V1) The character $\kappa\varepsilon : T_0(\widehat{O}) \to W^\times$ does not factor through the reduced norm map $N : T_0(\widehat{O}^{(d(B))}) \hookrightarrow G(\widehat{O}^{(d(B))}) \to (\widehat{O}^{(d(B))})^\times$. In particular, if $\kappa_1 \neq \kappa_2$, this condition is satisfied.
- (V2) The quaternion algebra B is a division algebra. In particular, this condition is satisfied if $|I_B| < [F : \mathbb{Q}]$.

6. Ordinary cohomology groups

We are going to prove that the morphism $f : V(R) \otimes_W V(R) \to S(R)$ in Proposition 5.1 for V in Proposition 5.2 is surjective for the nearly ordinary local ring R (associated to a mod p irreducible Galois representation), when B is unramified at every finite place and $q = |I_B| \leq 1$. A key to the proof is the R–freeness of $V(R)$ proven by Fujiwara [Fu] (see [HMI] Corollary 3.42). Another important ingredient of the proof is the self duality of $V(R)$ over W.

6.1. Freeness as Hecke modules. We recall here a special case of Fujiwara's result in [Fu] "Freeness Theorem" of the component $V(R)$ for a local ring R of the Hecke algebra $h_\kappa(\mathfrak{N},\varepsilon;W)$ (see also [HMI] Corollary 3.42). To state the result, we need to have a good description of the modular nearly ordinary Galois representation; so, we recall the description. We call a local ring R of $h_\kappa(\mathfrak{N},\varepsilon;W)$ nearly ordinary if the projection of $\mathbb{T}(p)$ to R is a unit. We hereafter always assume

(ord) *R is nearly ordinary with $\kappa_1-\kappa_2 \geq I$, that is, $\kappa_{1,\sigma}-\kappa_{2,\sigma}\geq 1$ for all σ.*

(unr) *$F/\mathbb{Q}$ is unramified at $\mathbb{Q}$.*

We write $\mathfrak{N}'$ for the product of primes $\mathfrak{l}\nmid p$ for which one of ε_1 and ε_2 ramifies; so, $\mathfrak{N}'\subset\mathfrak{N}^{(p)}$. For a W–algebra homomorphism $\lambda: h_\kappa(\mathfrak{N},\varepsilon;W)\to W$ factoring through R (such a λ is called *nearly ordinary*), we have a Galois representation $\rho=\rho_\lambda:\mathrm{Gal}(\overline{F}/F)\to GL_2(W)$ (e.g. [H96] 2.8 and [MFG] 5.6.1) such that

(G1) ρ is continuous and is absolutely irreducible over $W\otimes_{\mathbb{Z}}\mathbb{Q}$;

(G2) ρ is unramified outside $\mathfrak{N}'p$;

(G3) For primes $\mathfrak{l}$ outside $\mathfrak{N}'p$, we have
$$\det(1_2-\rho(Frob_{\mathfrak{l}})X)=1-\lambda(T(\mathfrak{l}))X+\lambda(\langle\mathfrak{l}\rangle)X^2;$$

(G4) For the decomposition group $D_{\mathfrak{p}}\subset\mathrm{Gal}(\overline{F}/F)$ at each prime $\mathfrak{p}|p$, we have an exact sequence of $D_{\mathfrak{p}}$–modules: $0\to\epsilon_{\mathfrak{p}}\to\rho|_{D_{\mathfrak{p}}}\to\delta_{\mathfrak{p}}\to 0$ with one dimensional character $\delta_{\mathfrak{p}}$ satisfying $\delta_{\mathfrak{p}}([y;F_{\mathfrak{p}}])=\lambda(\mathbb{T}(y))$ for the local Artin symbol $[y;F_{\mathfrak{p}}]$ of $y\in F_{\mathfrak{p}}^\times$.

Writing $\mathbb{F}$ for the residue field of W, the semi-simplification $\overline{\rho}=\overline{\rho}_R$ of the reduction of ρ modulo the maximal ideal $\mathfrak{m}_W$ of W is independent of λ by (G2-3) (cf. [MFG] Corollary 2.8 combined with the Chebotarev density). In particular, if $\overline{\rho}$ is irreducible, the isomorphism class of $\rho \mod \mathfrak{m}_W$ for the maximal ideal $\mathfrak{m}_W$ is unique, and always we have $(\rho \mod \mathfrak{m}_W)\cong\overline{\rho}$.

We shall recall some terminology from (formal) deformation theory of Galois representations. See [MFG] Section 2.3 for basics of formal deformation theory of representations. Let H be a subgroup of $\mathrm{Gal}(\overline{F}/F)$. We call a representation $\rho: H\to GL_2(A)$ for a local proartinian W–algebra A with residue field $\mathbb{F}$ a *deformation* over H of $\overline{\rho}$ if $\rho\equiv\overline{\rho}|_H \mod \mathfrak{m}_A$. Let $\chi=\det(\rho_\lambda)/\mathcal{N}^{[\kappa]}$ for the p–adic cyclotomic character $\mathcal{N}$. Then χ is of finite order. For any character $\varphi: D_{\mathfrak{l}}\to A^\times$, let $C(\varphi)$ denote the conductor of φ; thus, $C(\varphi)=1$ if φ is unramified, and $C(\varphi)=\mathfrak{l}^m$ if $y\mapsto\varphi([y,F_{\mathfrak{l}}])$ factors through $F_{\mathfrak{l}}^\times/(1+\mathfrak{l}^mO_{\mathfrak{l}})$ but not $F_{\mathfrak{l}}^\times/(1+\mathfrak{l}^{m-1}O_{\mathfrak{l}})$ for $m>0$. We assume the following four conditions on ρ_λ:

(H1) χ is of order prime to p.

(H2) For primes $\mathfrak{l}|\mathfrak{N}p$, write $D_{\mathfrak{l}}$ for the decomposition group at $\mathfrak{l}$. Then we have $\rho|_{D_{\mathfrak{p}}}\cong\left(\begin{smallmatrix}\epsilon_{\mathfrak{l}} & *\\ 0 & \delta_{\mathfrak{l}}\end{smallmatrix}\right)$ with $\delta([y,F_{\mathfrak{l}}])=\lambda(\mathbb{T}(y))$. This condition actually follows for $\mathfrak{l}|p$ from near ordinarity of λ as already remarked in (G4).

(H3) If a prime $\mathfrak{l}|\mathfrak{N}$ but $\mathfrak{l}\nmid p$, then the restriction of $\delta_{\mathfrak{l}}$ and $\epsilon_{\mathfrak{l}}$ to the inertia subgroup $I_{\mathfrak{l}}$ of $D_{\mathfrak{l}}$ is of order prime to p.

(H4) If $\epsilon_{\mathfrak{p}} \equiv \delta_{\mathfrak{p}}\mathcal{N} \mod \mathfrak{m}_W$ on $I_{\mathfrak{p}}$ for a prime $\mathfrak{p}|p$, the following five conditions have to be met: (i) the character $\epsilon_{\mathfrak{p}}$ is of order prime to p, (ii) $\kappa = (I, 0)$, (iii) $\rho_\lambda|_{I_{\mathfrak{p}}}$ is associated to a p–divisible group over an unramified extension of $O_{\mathfrak{p}}$, (iv) $\mathfrak{p} \nmid \mathfrak{N}$, and (v) $\epsilon_{\mathfrak{p}}\delta_{\mathfrak{p}}^{-1}(y)y^{-I} = 1$ for all $y \in O_{\mathfrak{p}}^{\times}$.

We write $\overline{\delta}_{\mathfrak{l}} = (\delta_{\mathfrak{l}} \mod \mathfrak{m}_W)$ and $\overline{\epsilon}_{\mathfrak{l}} = (\epsilon_{\mathfrak{l}} \mod \mathfrak{m}_W)$. We assume the following two local conditions on $\overline{\rho}$.

(H5) For all $\mathfrak{p}|p$, $\overline{\delta}_{\mathfrak{p}} \neq \overline{\epsilon}_{\mathfrak{p}}$.

(H6) For $\mathfrak{l}|\mathfrak{N}$ and $\mathfrak{l} \nmid p$, the $\mathfrak{l}$–primary part of $\mathfrak{N}$ coincides with $C(\overline{\epsilon}_{\mathfrak{l}}\overline{\delta}_{\mathfrak{l}}^{-1})$.

Thus $\overline{\rho}$ could ramify at a prime $\mathfrak{l} \nmid \mathfrak{N}$, and by (H3), $\mathfrak{N}'$ gives the product of primes (outside p) at which $\overline{\rho}$ ramifies. We assume the following global condition on $\overline{\rho}$:

(H7) $\overline{\rho}$ is absolutely irreducible over $\mathrm{Gal}(\overline{F}/F[\sqrt{p^*}])$ for $p^* = (-1)^{(p-1)/2}p$.

We choose a quaternion algebra $B_{/F}$ so that $d(B) = 1$ and ramified at most infinite places (that is I^B is as large as possible). This implies:

$$I_B = \{\sigma_1\} \text{ if } [F:\mathbb{Q}] \text{ is odd, and } I_B = \emptyset \text{ if } [F:\mathbb{Q}] \text{ is even.} \tag{6.1}$$

We now quote the following special case of "Freeness Theorem" in Section 0 in [Fu] (see [HMI] Corollary 3.42 for a proof of this Fujiwara's result):

Theorem 6.1. *Suppose the conditions* (6.1), (ord), (unr), (H1-7) *and* $p > 3$. *Then* $V(R)$ *for* $V = H^q(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon; W))$ $(q = |I_B|)$ *is free of rank* 2^q *over the local ring* R. *Even if we ease the condition* (H4) *to allow the case where the* $\mathfrak{p}$*–primary part of* $\mathfrak{N}$ *is equal to* $\mathfrak{p}$ *for primes* $\mathfrak{p}|p$, *the same assertion holds as long as* $[F:\mathbb{Q}]$ *is even.*

This is a special case of Fujiwara's result. In particular, we do not need to assume unramifiedness of p in F, but we use the assumption (unr) anyway in our later application; so, we have imposed it.

Proof. Here is a brief account of how to deduce the above theorem either from [HMI] Corollary 3.42 or from [Fu], because the set of the assumptions imposed in these works appears different. In [HMI] Corollary 3.42, the theorem is proven under the assumptions:

(A) $[F:\mathbb{Q}]$ is even;
(B) $\kappa = (I, 0)$;
(C) the assumptions (H1–3) and (H5–7);
(D) the milder condition than (H4) as stated in the theorem.

As can be easily seen, the conditions (A–D) implies the assumptions actually stated in Corollary 3.42 of [HMI]: the absolute irreducibility of $\overline{\rho}$ over $F[\mu_p]$ (written as $(\mathrm{ai}_F[\mu_p])$ in [HMI]) which follows from (H7), the conditions (h1–4) in [HMI] 3.2.1, (ds_Q) which is (H5) and (H6), and the conditions (Q1–6) (for $Q = \emptyset$) in [HMI] Section 3.2.1. These conditions exhaust all the assumptions of Corollary 3.42 of [HMI] except for the condition (sm1). The condition: $p > 3$ and the unramifiedness of p in $F/\mathbb{Q}$ implies $[F[\mu_p]:F] > 2$, which is the last assumption (sm1) in Corollary 3.42 of [HMI]. We only use this theorem under the four conditions (A–D); so, logically, for the proof of the main theorem of this paper, it is sufficient to quote [HMI] Corollary 3.42.

For the sake of completeness, we now reduce the theorem in the case not covered under (A–D) to [Fu] (the version of 1999). Recall that $\mathfrak{N}'$ is the product of all primes (outside p) at which $\overline{\rho}$ ramifies. We consider an open compact subgroup $U(\overline{\rho}) = \prod_{\mathfrak{l}} U_{\mathfrak{l}}(\overline{\rho}) \subset U_0(\mathfrak{N})$ and a character $\nu_{\mathfrak{l}}$ of $U_{\mathfrak{l}}(\overline{\rho})$ with values in $W^{\times}$ defined as follows:

(1) $U_{\mathfrak{l}}(\overline{\rho}) = GL_2(O_{\mathfrak{l}})$ in $B_{\mathfrak{l}}^{\times}$ if $\mathfrak{l} \nmid \mathfrak{N}p$, and $\nu_{\mathfrak{l}}$ is the trivial character;

(2) Suppose that $\mathfrak{l}|\mathfrak{N}'$. If $\overline{\epsilon}_{\mathfrak{l}} \neq \overline{\delta}_{\mathfrak{l}}$ on $I_{\mathfrak{l}}$, then $\mathfrak{l}|\mathfrak{N}$,

$$U_{\mathfrak{l}}(\overline{\rho}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U_0(\mathfrak{N})_{\mathfrak{l}} \middle| a \equiv 1 \mod \mathfrak{l}, (d \mod \mathfrak{l}) \text{ is of } p\text{–power order} \right\}$$

and $\nu_{\ell}(u) = \delta_{\mathfrak{l}}([\det(u), F_{\mathfrak{l}}])$. If $\overline{\epsilon}_{\mathfrak{l}} = \overline{\delta}_{\mathfrak{l}}$ on $I_{\mathfrak{l}}$, $U_{\mathfrak{l}}(\overline{\rho}) = GL_2(O_{\mathfrak{l}})$ (so $\mathfrak{l} \nmid \mathfrak{N}$) and $\nu_{\mathfrak{l}}(u) = \delta_{\mathfrak{l}}([\det(u), F_{\mathfrak{l}}])$.

(3) For $\mathfrak{p}|p$, define $\nu_{\mathfrak{p}}(u) = \epsilon_{\mathfrak{p}}([\det(u), F_{\mathfrak{p}}])(\det(u))^{-\kappa_2}$ for $u \in GL_2(O_{\mathfrak{p}})$, which is a finite order character and can be regarded as a character with values in $W^{\times}$. If $\overline{\epsilon}_{\mathfrak{p}} \neq \overline{\delta}_{\mathfrak{p}}\overline{\omega}$ on $I_{\mathfrak{p}}$ for $\overline{\omega} = (\mathcal{N} \mod \mathfrak{m}_W)$, then $\mathfrak{p}|\mathfrak{N}$ and

$$U_{\mathfrak{p}}(\overline{\rho}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U_0(p)_{\mathfrak{p}} \middle| a \equiv 1 \mod \mathfrak{p}, (d \bmod \mathfrak{p}) \text{ is of } p\text{–power order} \right\}.$$

If $\overline{\epsilon}_{\mathfrak{p}} = \overline{\delta}_{\mathfrak{p}}\overline{\omega}_{\mathfrak{p}}$ on $I_{\mathfrak{p}}$, then $U_{\mathfrak{p}} = GL_2(O_{\mathfrak{p}})$ (and $\mathfrak{p} \nmid \mathfrak{N}$).

Let $U = \mathrm{Ker}(\prod_{\mathfrak{l}} \nu_{\mathfrak{l}} : U(\overline{\rho}) \to W^{\times})$. Since the restriction of operators of $h_\kappa(U, \varepsilon; W)$ to $S_\kappa(\mathfrak{N}, \varepsilon; W)$ induces a surjective algebra homomorphism $\pi : h_\kappa(U, \varepsilon; W) \twoheadrightarrow h_\kappa(\mathfrak{N}, \varepsilon; W)$, we have a unique local ring R_U of $h_\kappa(U, \varepsilon; W)$ through which $\lambda \circ \pi$ factors. Then R_U is nearly ordinary. For a smaller open compact subgroup U' with $U(\mathfrak{N}\mathfrak{q}) \subset U' \subset U$ for a suitable prime $\mathfrak{q}$ outside $\mathfrak{N}'p$, it is proven in [Fu] that

(1) For all $x \in G(\mathbb{A}^{(\infty)})$, $\overline{\Gamma}_{U',x}$ is torsion-free and acts on $\mathfrak{H}^{I_B}$ freely;

(2) The action of $\Gamma_{U',x}$ on $L(\kappa\varepsilon; A)$ factors through $\overline{\Gamma}_{U',x}$ for all $x \in G(\mathbb{A}^{(p\infty)})$;

(3) The Hecke algebra $h_\kappa(U', \varepsilon; W)$ has a local ring R' with $R' \cong R$ as W–algebras;

(4) Let $V_U = H^q(Y^B(U), L)$ and $V' = H^q(Y^B(U'), L)$ for $L = L(\kappa\varepsilon; W))$ and $q = |I_B|$. Then the restriction map $\mathrm{Res} : H^q(Y_0^B(\mathfrak{N}), L) \to H^q(Y^B(U), L)$ composed with the multiplication by the idempotent of R' induces a W–linear map: $V_U(R_U) \cong V'(R')$ which is $\mathbb{T}(y)$–equivariant as long as $y_{\mathfrak{q}} = 1$;

(5) R_U is generated by $\mathbb{T}(y)$ with $y_{\mathfrak{q}} = 1$;

(6) $V'(R') \cong R'^r$ for some r.

In [Fu], U' and U are written as $K_{\mathcal{D},y}$ and $K_{\mathcal{D}}$, respectively. This is enough to conclude that $V(R_U)$ is R_U–free. On the other hand, for the Sylow p–subgroup S of $U_0(\mathfrak{N})/U$, R_U is $W[S]$–free of finite rank. Then $R \cong R_U \otimes_{W[S],\varepsilon} W$, where ε is the algebra homomorphism $W[S] \to W$ induced by the character ε of S.

This fact follows from the freeness of the Hecke algebra over the group algebra (under (unr) and $p > 2$), for example, [H02] Corollary 4.3, [H05a] Corollary 9.3 or [PAF] 4.2.11–12. In the above papers, the symbol N is used for the prime-to-p–part of the present level $\mathfrak{N}$. Similarly, V_U is $W[S]$–free of finite rank by [H89] Theorem 3.8. Thus we have

$$V(R) = V_U(R) \otimes_{W[S],\varepsilon} W \cong (R_U \otimes_{W[S],\varepsilon} W)^r \cong R^r$$

for a suitable integer r. Actually $r = 2^q \leq 2$, because $V \otimes \mathbb{Q}$ is of rank 2^q over the (rational) Hecke algebra.

As for the easing of the condition (H4) on $\mathfrak{N}$, it follows from the same argument, replacing $\mathfrak{N}$ by $\mathfrak{N} \cap \prod_{\mathfrak{p}\in P} \mathfrak{p}$, because this is the case where the deformation is unrestricted at $\mathfrak{p} \in P$, which has been dealt with in [Fu] assuming that, for example, $[F : \mathbb{Q}]$ is even (see [HMI] Section 3.2). $\square$

By the theory of p–adic analytic families of nearly ordinary cusp forms (see [H89], [H96] Section 2.7 and [HMI] 3.2.8, 3.3.4 and 4.3.9), we can ease slightly the conditions necessary to have freeness of $V(R)$ over R. We shall describe this generalization for our later use. Let $\mathbf{G} = \mathbf{G}(\mathfrak{N}') = Cl_F^+(\mathfrak{N}'p^\infty) \times (O_p \times O/\mathfrak{N}'^{(p)})^\times$, where $Cl_F^+(\mathfrak{N}'p^n)$ is the strict ray class group modulo $\mathfrak{N}'p^n$ of F, and

$$Cl_F^+(\mathfrak{N}'p^\infty) = \varprojlim_n Cl_F^+(\mathfrak{N}'p^n) = F_{\mathbb{A}}^\times / F^\times U_F(\mathfrak{N}')^{(p)} F_{\infty+}^\times$$

with $U_F(\mathfrak{N}') = \widehat{O}^\times \cap (1+\mathfrak{N}'\widehat{O})$. We have a natural homomorphism $\iota : T_0(O_p) \to \mathbf{G}$ sending (a, b) to $(a^{-1}, a^{-1}b)$. Each element $(z, y) \in \mathbf{G}$ acts on $f \in S_\kappa(U, \varepsilon; A)$ by $f|(z, y)(x) = f|\mathbb{T}(y)(xz)$ (for $U \subset U_0(\mathfrak{N}')$). Let Γ_0 be the maximal torsion-free quotient of $\mathbf{G}$ (which is independent of $\mathfrak{N}'$ up to isomorphisms), and fix a splitting $\mathbf{G} = \Gamma_0 \times \mathbf{G}_{tor}$. We consider the Iwasawa algebra $W[[\Gamma_0]]$. For an integral domain $\mathbb{I}$ finite flat over $W[[\Gamma_0]]$, we define

$$\mathcal{A}(\mathbb{I}) = \left\{ P \in \mathrm{Hom}_W(\mathbb{I}, \overline{\mathbb{Q}}_p) \middle| P \circ \iota \sim \kappa \text{ with } \kappa_1 - \kappa_2 \geq I \text{ and } [\kappa] \geq 0 \right\},$$

where $\varphi \sim \psi$ if $\varphi = \psi$ locally on $T_0(O_p)$ (in other words, $\varphi\psi^{-1}$ is of finite order). For each $P \in \mathcal{A}(\mathbb{I})$, we write $\kappa(P)$ and ε_P for the corresponding algebraic character of T_0 and the character of

$$g = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) z \in T_0(O_p \times (O/\mathfrak{N}'^{(p)}) F_{\mathbb{A}}^\times \subset GL_2(F_{\mathbb{A}})$$

given by $\mathbf{G} \ni g \mapsto P(T(ab^{-1}))P(\langle bz \rangle)\varepsilon_{tor}(g)$, where ε_{tor} is the restriction of ε to the torsion part $\mathbb{G}_{tor}$ (regarded as a character of $\mathbf{G}$). Thus we can form a triple $(\varepsilon_{P,1}, \varepsilon_{P,2}, \varepsilon_{P+})$ out of ε_P so that $\varepsilon_P(g) = \varepsilon_{P,1}(a)\varepsilon_{P,2}(b)\varepsilon_{P+}(z)$. For a given nearly ordinary Hecke eigenform $f \in S_\kappa(\mathfrak{N}, \varepsilon; W)$ with $\kappa_1 - \kappa_2 \geq I$, decomposing $\mathbf{G}$ into a product $\Gamma_0 \times \Delta$ for a finite subgroup Δ, we write $\varepsilon(P) = \varepsilon_P \varepsilon|_\Delta$. Thus for a suitable $P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ whose weight is κ, we find $\varepsilon(P) = \varepsilon$. Then there exist $\mathbb{I}_{/W[[\Gamma_0]]}$ as above and a unique family of Hecke eigenforms $\{f_P\}_{P\in\mathcal{A}(\mathbb{I})}$ containing f and satisfying the following two conditions:

(1) $f_P \in S_{\kappa(P)}(\mathfrak{N}_P, \varepsilon(P); W[\varepsilon_P])$ for the conductor $\mathfrak{N}_P$ of the character $\varepsilon(P)^-$, where $W[\varepsilon_P]$ is a subring of $\overline{\mathbb{Q}}_p$ generated over W by the values of ε_P;
(2) There exists a function $\mathbf{a} : F_{\mathbb{A}}^{\times} \to \mathbb{I}$ such that $\mathbf{a}_p(y, f_P) = \mathbf{a}(y)(P)$ for all $y \in F_{\mathbb{A}}^{\times}$ and all $P \in \mathcal{A}(\mathbb{I})$.

COROLLARY 6.2. *Let $\{f_P\}_{P\in\mathcal{A}(\mathbb{I})}$ be the family of nearly p–ordinary Hecke eigenforms as above. Write R_P be the local ring of $h_{\kappa(P)}(\mathfrak{N}_P, \varepsilon(P); W[\varepsilon_P])$ through which the algebra homomorphism λ_P of the Hecke algebra given by $f_P|\mathbb{T}(y) = \lambda_P(\mathbb{T}(y))f_P$ factors. If one member $f \in S_\kappa(\mathfrak{N}, \varepsilon; W)$ satisfies the assumptions* (H1-7)*, $V(R_P)$ is R_P–free of rank 2^q, where $V = H^q(Y_0^B(\mathfrak{N}_P), L(\kappa(P)\varepsilon(P); W[\varepsilon_P]))$ and $q = 0, 1$ by* (6.1).

Proof. We choose U' as in the proof of Theorem 6.1 and write $U_0'(\mathfrak{N}') = U' \cap U_0(\mathfrak{N}')$. We consider the limit $\mathcal{V} = \varinjlim_n H^q_{n.ord}(Y^B(U' \cap U(p^n)), L(\kappa\varepsilon; W) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$, where $H^q_{n.ord} = eH^q$ for the idempotent $e = \lim_{n\to\infty} \mathbb{T}(p)^{n!}$. The module $\mathcal{V}$ is naturally a module over $\mathbf{G}(\mathfrak{N}')$ and hence over $W[[\Gamma_0]]$. Then in the same manner in [H89] Corollary 3.5 and Theorem 3.8, we can prove that for the Pontryagin dual $\mathbf{V}'$ of $\mathcal{V}$,

$$\mathbf{V}'/P\mathbf{V}' \cong H^q_{n.ord}(Y^B(U_0'(\mathfrak{N}_P)), L(\kappa(P)\varepsilon(P); W[\varepsilon_P]))$$

as Hecke modules and that $\mathbf{V}'$ is $W[[\Gamma_0]]$–free module of finite rank. We write V_P' for the Hecke module of the right-hand-side of the above formula. Then we define $\mathbf{h}' \subset \mathrm{End}_{W[[\Gamma_0]]}(\mathbf{V}')$ by the $W[[\Gamma_0]]$–subalgebra generated by $\mathbb{T}(y)$ for all integral ideles y. As proved under (unr) and $p > 3$ in [PAF] Corollaries 4.31–32 or [H02] Corollary 4.3 (where the assumption is $p > 2$ and N denotes the prime-to–p part of the present $\mathfrak{N}'$), $\mathbf{h}'$ is $W[[\Gamma_0]]$–algebra free of finite rank, whose rank is equal to $\mathrm{rank}_{W[\varepsilon_P]}\, h_P'$ for $h_P' = h^{n.ord}_{\kappa(P)}(U_0'(\mathfrak{N}\mathfrak{p}^{e(P)}), \varepsilon(P); W[\varepsilon_P])$. Since they have the same generators $\mathbb{T}(y)$'s, $\mathbf{h}'/P\mathbf{h}'$ surjects down to h_P'. By comparing their rank over $W[\varepsilon_P]$, we find $\mathbf{h}'/P\mathbf{h}' \cong h_P'$ canonically sending $\mathbb{T}(y)$ to $\mathbb{T}(y)$. Since R' is the direct summand of $h'_{P_0} \subset h_\kappa(\mathfrak{N}, \varepsilon; W)$, by Hensel's lemma (cf. [BCM] III.4.6), $\mathbf{h}'$ has a unique local ring $\mathbf{R}' \subset \mathbf{h}'$ with $\mathbf{R}'/P_0\mathbf{R}' \cong R'$. We put $\mathbf{V}'(\mathbf{R}') = \mathbf{R}'\mathbf{V}'$, which is $W[[\Gamma_0]]$–free module of finite rank. Since $\mathbf{V}'(\mathbf{R}')/P_0\mathbf{V}'(\mathbf{R}') \cong V'(R')$, which is a free of finite rank over $R' = \mathbf{R}'/P_0\mathbf{R}'$, we choose a lift $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ in $\mathbf{V}'$ of a base of $V'(R')$ over R'. Then the $\mathbf{R}'$–linear map $\pi : \mathbf{R}'^r \to \mathbf{V}'(\mathbf{R}')$ given by $(h_1, \dots, h_r) \mapsto \sum_j h_j \mathbf{v}_j$ is surjective by Nakayama's lemma applied to $\mathbf{R}'$ and ideal P_0. By comparing the rank over $W[[\Gamma_0]]$, we find that π is an isomorphism. Thus $\mathbf{V}'(\mathbf{R}')$ is free of rank $r = 2^q$ over $\mathbf{R}'$.

We define R_P' by $R_P' = \mathbf{R}/P\mathbf{R} \subset h_P'$. Then $R_P' \cong R_P$ canonically, and $V_P'(R_P') \cong V_P(R_P)$ for $V_P = H^q_{n.ord}(U_0'(\mathfrak{N}_P), L(\kappa(P)\varepsilon(P); W[\varepsilon_P])$ in the same manner as in the proof of Theorem 6.1. This finishes the proof. □

6.2. INDUCED REPRESENTATIONS. We are going to verify the assumption of the freeness theorem: Theorem 6.1 for induced representations from CM fields.

We first recall a cusp form f on $GL_2(F_{\mathbb{A}})$ with complex multiplication by a CM field M. Let M/F be a CM field with integer ring R and choose a CM type Σ:

$$I_M = \mathrm{Hom}_{\mathrm{field}}(M, \overline{\mathbb{Q}}) = \Sigma \sqcup \Sigma c$$

for complex conjugation c. To assure the assumption (ord), we need to assume that the CM type Σ is p–ordinary, that is, the set Σ_p of p–adic places induced by $i_p \circ \sigma$ for $\sigma \in \Sigma$ is disjoint from $\Sigma_p c$ (its conjugate by the generator c of $\mathrm{Gal}(M/F)$). The existence of such an ordinary CM type implies that all prime factors of p in F split in M/F. For each $k \in \mathbb{Z}[I]$, we write $k\Sigma = \sum_{\sigma\in\Sigma} k_{\sigma|F}\sigma$.

We choose $\kappa_1 > \kappa_2$ with $\kappa_1 + \kappa_2 = [\kappa]I$ for an integer $[\kappa]$. We then choose a Hecke character λ of conductor $\mathfrak{C}\mathfrak{P}^e$ ($\mathfrak{C}$ prime to p) such that

$$\lambda((\alpha)) = \alpha^{\kappa_1\Sigma + c\kappa_2\Sigma} \quad \text{for } \alpha \in M^\times \text{ with } \alpha \equiv 1 \mod \mathfrak{C}\mathfrak{P}^e,$$

where $\mathfrak{P}^e = \prod_{\mathfrak{P}\in\Sigma_p}\left(\mathfrak{P}^{e(\mathfrak{P})}\mathfrak{P}^{ce(\mathfrak{P}^c)}\right)$ for $e = \sum_{\mathfrak{P}\in\Sigma_p\sqcup\Sigma_p c} e(\mathfrak{P})\mathfrak{P}$. We also decompose $\mathfrak{C} = \prod_{\mathfrak{L}} \mathfrak{L}^{e(\mathfrak{L})}$ for prime ideals $\mathfrak{L}$ of M. We extend λ to a p–adic idele character $\widehat{\lambda} : M_{\mathbb{A}}^\times/M^\times M_\infty^\times \to \overline{\mathbb{Q}}_p^\times$ so that $\widehat{\lambda}(a) = \lambda(aO)a_p^{-\kappa_1\Sigma - c\kappa_2\Sigma}$. By class field theory, we may regard $\widehat{\lambda}$ as a character of $\mathrm{Gal}(\overline{F}/M)$. Any character φ of $\mathrm{Gal}(\overline{F}/M)$ of the form $\widehat{\lambda}$ as above is called "of weight κ". For a prime ideal $\mathfrak{L}$ of M outside p, we write $\lambda_{\mathfrak{L}}$ for the restriction of $\widehat{\lambda}$ to $M_{\mathfrak{L}}^\times$. For $\mathfrak{P} \in \Sigma_p$, we define $\lambda_{\mathfrak{P}}(x) = \widehat{\lambda}(x)x^{\kappa_1\Sigma}$ for $x \in M_{\mathfrak{P}}^\times$ and $\lambda_{\mathfrak{P}^c}(x) = \widehat{\lambda}(x)x^{c\kappa_2\Sigma}$ for $x \in M_{\mathfrak{P}^c}^\times$. Then $\lambda_{\mathfrak{L}}$ for all prime ideals $\mathfrak{L}$ is a continuous character of $M_{\mathfrak{L}}^\times$ with values in $\overline{\mathbb{Q}}$ whose restriction to $R_{\mathfrak{L}}^\times$ is of finite order. By the condition $\kappa_1 > \kappa_2$, $\widehat{\lambda}$ cannot be of the form $\widehat{\lambda} = \phi \circ N_{M/F}$ for an idele character $\phi : F_{\mathbb{A}}^\times/F^\times F_{\infty+}^\times \to \overline{\mathbb{Q}}_p^\times$.

We define a function $F_{\mathbb{A}}^\times \ni y \mapsto \mathbf{a}_p(y, \theta(\lambda))$ supported by integral ideles by

$$\mathbf{a}_p(y,\theta(\lambda)) = \sum_{x\in M_{\mathbb{A}}^\times, xx^c = y, x_{\Sigma_p}=1} \widehat{\lambda}(x) \quad \text{if } y \text{ is integral.} \tag{6.2}$$

where x runs over elements in $M_{\mathbb{A}(\infty)}^\times/(\widehat{R}^{(p\mathfrak{C}\mathfrak{C}^c)})^\times$ satisfying the following three conditions: (i) xR is an integral ideal of M, (ii) $N_{M/F}(x) = y$ and (iii) $x_{\mathfrak{Q}} = 1$ for primes $\mathfrak{Q}$ in Σ_p and $\mathfrak{Q}|\mathfrak{C}$. The q–expansion determined by the coefficients $\mathbf{a}_p(y,\theta(\lambda))$ gives a unique element $\theta(\lambda) \in S_\kappa(\mathfrak{N}', \varepsilon'_\lambda; \overline{\mathbb{Q}})$ ([HT1] Theorem 6.1), where $\mathfrak{N}' = N_{M/F}(\mathfrak{C}\mathfrak{P}^e)d(M/F)$ for the discriminant $d(M/F)$ of M/F and ε'_λ is a suitable "Neben" character.

We decompose $\mathfrak{C} = \mathfrak{F}\mathfrak{F}^c\mathfrak{I}$ so that $\mathfrak{F}\mathfrak{F}_c$ is a product of split primes and $\mathfrak{I}$ for the product of inert or ramified primes, $\mathfrak{F} + \mathfrak{F}_c = R$ and $\mathfrak{F} \subset \mathfrak{F}_c^c$. We put $\mathfrak{f} = \mathfrak{F} \cap F$ and $\mathfrak{i} = \mathfrak{I} \cap F$. Assuming that λ^- has split conductor, we describe the Neben character ε_λ of the minimal form $f(\lambda)$ in the automorphic representation $\pi(\lambda)$ generated by $\theta(\lambda)$. The character ε_λ is possibly different from ε'_λ and is given as follows:

(1) For $\mathfrak{l}|\mathfrak{f}$, we identify $T_0(O_{\mathfrak{l}}) = O_{\mathfrak{l}}^\times \times O_{\mathfrak{l}}^\times$ with $R_{\mathfrak{L}^c}^\times \times R_{\mathfrak{L}}^\times$ with this order for the prime ideal $\mathfrak{L}|(\mathfrak{l}R \cap \mathfrak{F})$. We define $\varepsilon_{\lambda,\mathfrak{l}}$ by the restriction of $\lambda_{\mathfrak{L}^c} \times \lambda_{\mathfrak{L}}$ to $T_0(O_{\mathfrak{l}})$.
(2) For $\mathfrak{p}|p$, identify $T_0(O_{\mathfrak{p}})$ with $R_{\mathfrak{P}^c}^\times \times R_{\mathfrak{P}}^\times$ for $\mathfrak{P}|\mathfrak{p}$ in Σ_p, we define $\varepsilon_{\lambda,\mathfrak{p}}$ by the restriction of $\lambda_{\mathfrak{P}^c} \times \lambda_{\mathfrak{P}}$ to $T_0(O_{\mathfrak{p}})$.
(3) For $\mathfrak{l}|\mathfrak{i}d(M/F)$, we choose a character $\phi_{\mathfrak{l}} : F_{\mathfrak{l}}^\times \to \mathbb{C}^\times$ such that $\lambda_{\mathfrak{L}} = \phi_{\mathfrak{l}} \circ N_{M_{\mathfrak{L}}/F_{\mathfrak{l}}}$ (this is possible because λ^- has split conductor). Then we define $\varepsilon_{\lambda,1,\mathfrak{l}}(a) = \phi_{\mathfrak{l}}$ and $\varepsilon_{\lambda,2,\mathfrak{l}}(d) = \left(\frac{M_{\mathfrak{L}}/F_{\mathfrak{l}}}{d}\right)\lambda_{\mathfrak{L}}(d)$, where $\mathfrak{L}$ is the prime factor of $\mathfrak{l}$ in M and $\left(\frac{M_{\mathfrak{L}}/F_{\mathfrak{l}}}{d}\right)$ is the quadratic residue symbol for $M_{\mathfrak{L}}/F_{\mathfrak{l}}$.
(4) The central character $\varepsilon_{\lambda+}$ is given by the product of the restriction of λ to $F_{\mathbb{A}}^\times$ and the quadratic character $\left(\frac{M/F}{}\right)$ of the CM field M/F.

We now give an explicit description of $f(\lambda)$ without assuming that λ^- has split conductor. Let Ξ_{pr} be the set of prime factors $\mathfrak{l}$ of $\mathfrak{N}' = d(M/F)N_{M/F}(\mathfrak{C}\mathfrak{P}^e)$ where $\pi_{\mathfrak{l}}$ is principal. If λ^- has split conductor, Ξ_{pr} is the full set of prime factors of $\mathfrak{N}'$. Otherwise, $\mathfrak{l} \in \Xi_{pr}$ if and only if either $\mathfrak{l}|\mathfrak{f}$ or $\mathfrak{l}|\mathfrak{i}$ and

$$\lambda_{\mathfrak{L}}(x) = \phi_{\mathfrak{l}}(xx^c) \quad \text{for a character } \phi_{\mathfrak{l}} : F_{\mathfrak{l}}^\times \to \mathbb{C}^\times. \tag{6.3}$$

For $\mathfrak{l} \in \Xi_{pr}$, taking a prime $\mathfrak{L}|\mathfrak{l}$ in M, we have

$$\pi_{\mathfrak{l}}(\lambda) \cong \begin{cases} \pi(\lambda_{\mathfrak{L}^c}, \lambda_{\mathfrak{L}}) & \text{if } \mathfrak{l}|\mathfrak{f} \text{ and } \mathfrak{L}|\mathfrak{F}, \\ \pi(\phi_{\mathfrak{l}}, \left(\frac{M_{\mathfrak{L}}/F_{\mathfrak{l}}}{}\right)\phi_{\mathfrak{l}}) & \text{if } \mathfrak{l}|\mathfrak{i}. \end{cases} \tag{6.4}$$

We split $\mathfrak{N}'$ into a product $\mathfrak{N}_1\mathfrak{N}_2$ of co-prime ideals so that $\mathfrak{N}_1$ is made up of primes in Ξ_{pr}. Writing $\pi_{\mathfrak{l}}(\lambda) = \pi(\eta_{\mathfrak{l}}, \eta'_{\mathfrak{l}})$ for characters $\eta_{\mathfrak{l}}, \eta'_{\mathfrak{l}} : F_{\mathfrak{l}}^\times \to \mathbb{C}^\times$, we write $C_{\mathfrak{l}}$ for the conductor of $\eta_{\mathfrak{l}}^{-1}\eta'_{\mathfrak{l}}$. Define the minimal level of $\pi(\lambda)$ by

$$\mathfrak{N}(\lambda) = \mathfrak{N}_2 \prod_{\mathfrak{l} \in \Xi_{pr}} C_{\mathfrak{l}}.$$

We write $\Xi = \{\mathfrak{L}|\mathfrak{L} \supset \mathfrak{F}\mathfrak{P}^\Sigma, \mathfrak{L} \supset \mathfrak{N}(\lambda)\}$ for primes $\mathfrak{L}$ of M and define

$$\mathbf{a}_p(y, f(\lambda)) = \begin{cases} \sum_{xx^c=y, x_\Xi=1} \widehat{\lambda}(x)x_p^{(\kappa_1-\kappa_2)\Sigma} & \text{if } y \text{ is integral,} \\ 0 & \text{otherwise,} \end{cases} \tag{6.5}$$

where x runs over $(\widehat{R} \cap M_{\mathbb{A}^{(\infty)}}^\times/(R^{(\Xi)})^\times$ with $x_{\mathfrak{L}} = 1$ for $\mathfrak{L} \in \Xi$. The value $\widehat{\lambda}(x)$ is well defined modulo $(R^{(\Xi)})^\times$ as long as $x_\Xi = 1$ for the following reason: For primes $\mathfrak{l}|\mathfrak{N}(\lambda)$ non-split in M/F, by the condition $xx^c = y$, x is determined up to a unit u with $uu^c = 1$. Since $\lambda_{\mathfrak{L}}(u) = \phi_{\mathfrak{l}}(uu^c) = 1$, the value $\lambda_{\mathfrak{L}}(x_{\mathfrak{L}})$ is well defined. For $\mathfrak{L} \in \Xi$, by imposing $x_{\mathfrak{L}} = 1$, the condition $xx^c = y$ implies $x_{\mathfrak{L}^c} = y_{\mathfrak{l}}$; so, the value $\lambda_{\mathfrak{L}}(x_{\mathfrak{l}})$ is again well defined. As for a split prime $\mathfrak{l} \nmid \mathfrak{N}(\lambda)$ but $\mathfrak{l}|N_{M/F}(\mathfrak{C})$, we have $\lambda_{\mathfrak{L}}|_{O_{\mathfrak{l}}^\times} = \lambda_{\mathfrak{L}^c}|_{O_{\mathfrak{l}}^\times}$, so $\lambda_{\mathfrak{L}}(u_{\mathfrak{L}})\lambda_{\mathfrak{L}^c}(u_{\mathfrak{L}^c}) = 1$ because $uu^c = 1$ implies $u_{\mathfrak{L}} = u_{\mathfrak{L}^c}^{-1}$ identifying $R_{\mathfrak{L}}$ and $R_{\mathfrak{L}^c}$ with $O_{\mathfrak{l}}$. As for $\mathfrak{p}|p$

with $\mathfrak{p} \nmid \mathfrak{N}(\lambda)$, if $(uu^c) = 1$, we have

$$\widehat{\lambda}(u)u^{(\kappa_1-\kappa_2)\Sigma} = u^{-\kappa_1\Sigma - c\kappa_2\Sigma + (\kappa_1-\kappa_2)\Sigma} = (uu^c)^{-\kappa_2} = 1.$$

So again, $\widehat{\lambda}(x)x_p^{(\kappa_1-\kappa_2)\Sigma}$ is well-defined modulo such local units.

For a principal series representation $\pi(\eta', \eta)$ of $GL_2(F_{\mathfrak{l}})$, if $\eta|_{O_{\mathfrak{l}}^\times} = \eta'|_{O_{\mathfrak{l}}^\times}$, we have $\pi(\eta', \eta) \cong \eta \otimes \pi(\eta^{-1}\eta', 1)$ and $\pi(\eta^{-1}\eta', 1)$ is spherical; thus we have a unique spherical vector $v \neq 0$ in $\pi(\eta^{-1}\eta', 1)$ with $v|T(\mathfrak{l}) = (1+\eta^{-1}\eta'(\varpi_{\mathfrak{l}}))v$. The corresponding vector $v' = v \otimes \eta$ in $\pi(\eta', \eta)$ has minimal level fixed by $SL_2(O_{\mathfrak{l}})$ with $v'|T(y) = (\eta(y)+\eta'(y))v'$. If the conductor $C_{\mathfrak{l}}$ of $\eta^{-1}\eta'$ is non-trivial, again by the same argument, we find $v' \neq 0$ in $\pi_{\mathfrak{l}}(\lambda)$ such that $v'|T(y) = \eta(y)v'$ and $v'|u = \varepsilon(u)v'$ $(u \in U_0(C_{\mathfrak{l}})_{\mathfrak{l}})$, where $\varepsilon(u) = \eta(\det(u))(\eta^{-1}\eta'(a))$ for $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U_0(C_{\mathfrak{l}})_{\mathfrak{l}}$. This shows that $f(\lambda)$ is a classical modular form in $M_\kappa(\mathfrak{N}(\lambda), \varepsilon_\lambda; \overline{\mathbb{Q}})$ if λ^- has split conductor. The form $f(\lambda)$ is a common eigenform of Hecke operators $\mathbb{T}(y)$. The p–adic Galois representation ρ_λ associated to $f(\lambda)$ is the induced representation $\mathrm{Ind}_M^F \widehat{\lambda}$, regarding $\widehat{\lambda}$ as a character of $\mathrm{Gal}(\overline{F}/M)$ by class field theory. By regularity: $\kappa_1 > \kappa_2$, $\widehat{\lambda}(c\sigma c^{-1}) \neq \widehat{\lambda}(\sigma)$ for $\sigma \in \mathrm{Gal}(\overline{F}/M)$, ρ_λ is absolutely irreducible by Mackey's theorem, and $f(\lambda)$ is a cusp form.

We take the coefficient ring W to be free of finite rank over $\mathbb{Z}_p$. Assuming that λ^- has split conductor ($\Leftrightarrow \pi(\lambda)$ is principal at every finite place), we shall study when $f(\lambda)$ satisfies the conditions (H1-7) of Theorem 6.1. We take a character φ of $\mathrm{Gal}(\overline{F}/M)$ of order prime to p such that $\widehat{\lambda}\varphi^{-1} \equiv 1 \mod \mathfrak{m}_W$ and define $\psi = \varphi^-$. Suppose that λ and φ coincides on $R_{\mathfrak{L}}^\times$ if $\mathfrak{L} \nmid p$. Then the conditions (2) and (3) on ψ in the introduction are an interpretation of principality of $\pi(\lambda)$ at every finite place. To interpret the four conditions (1-4) on ψ in the introduction in terms of φ, let $G(\mathfrak{C}) = M_{\mathbb{A}}^\times / \overline{M^\times U_M(\mathfrak{C})^{(p)} M_\infty^\times}$, where

$$U_M(\mathfrak{C})^{(p)} = \left\{ x \in \widehat{R}^\times \middle| x_p = 1,\ x \equiv 1 \mod \mathfrak{C}\widehat{R} \right\}.$$

The first conditions (1) on ψ can be stated in terms of φ as follows:

(h1) *φ has order prime to p with exact conductor $\mathfrak{C}\mathfrak{P}^e$ for $\mathfrak{C}$ prime to p.*

Thus φ factors through the maximal prime-to-p quotient of $G(\mathfrak{C})$ which can be regarded canonically as a subgroup of $G(\mathfrak{C})$, because $G(\mathfrak{C})$ is almost p–profinite. The conditions (2-4) in the introduction imply the following three assertions:

(h2) *For all prime factors $\mathfrak{L}|\mathfrak{I}$, $\varphi_{\mathfrak{L}} = \phi_{\mathfrak{l}} \circ N_{M/F}$ for a character $\phi_{\mathfrak{l}} : F_{\mathfrak{l}}^\times \to W^\times$.*

(h3) *$\varphi_{\mathfrak{P}} \neq \varphi_{\mathfrak{P}^c}$ for all $\mathfrak{P} \in \Sigma_p$.*

(h4) *Over $\mathrm{Gal}(\overline{F}/M[\sqrt{p^*}])$, we have $\varphi_c \neq \varphi$, where $\varphi_c(\sigma) = \varphi(c\sigma c^{-1})$.*

We write $G_{tor}(\mathfrak{C})$ for the maximal torsion subgroup of $G(\mathfrak{C})$.

Theorem 6.3. *Assume* (6.1) *and the four conditions* (h1-4). *Let λ_k : $G(\mathfrak{C}) \to \overline{\mathbb{Q}}_p^\times$ be an arithmetic Galoischaracter of weight $k_1\Sigma + ck_2\Sigma$ ($k_j \in \mathbb{Z}[I]$) such that $k_1 > k_2$ and $\lambda_k|_{G_{tor}(\mathfrak{C})} = \varphi$. Then for the local ring R of $h_k(\mathfrak{N}(\lambda_k), \varepsilon_{\lambda_k}; W[\lambda_k])$ corresponding to $f(\lambda_k)$, the R–component $V(R)$ of*

$V = H^q(Y_0^B(\mathfrak{N}(\lambda_k)), L(k\varepsilon_{\lambda_k}; W[\lambda_k]))$ is R–free of rank 2^q, where $W[\lambda_k]$ is the complete discrete valuation ring inside $\overline{\mathbb{Q}}_p$ generated by the values of λ_k over W and $q = |I_B| \leq 1$.

Proof. We take a sufficiently large κ with $\kappa_1 > \kappa_2$ and $\kappa_1 + \kappa_2 = [\kappa]I$ for $0 \leq [\kappa] \in \mathbb{Z}$ so that $\zeta^{\kappa_1\Sigma + c\kappa_2\Sigma} = 1$ for all $\zeta \in R^\times$ and $\kappa \equiv 0 \mod (Q-1)\mathbb{Z}[I]$ for $Q = |\mathbb{F}|$. Then we can find a Hecke character λ with the following properties:

(1) *We have $\lambda((\alpha)) = \alpha^{\kappa_1\Sigma + c\kappa_2\Sigma}\varphi((\alpha))$ for all $\alpha \in M^\times$ prime to $\mathfrak{C}\mathfrak{P}^e$;*
(2) $\lambda \equiv \varphi \mod \mathfrak{m}_W$.

We are going to show for $f(\lambda)$ the assumptions (H1-7) except for (H4) of Theorem 6.1. Thus if (H4) is not applicable to $\mathrm{Ind}_M^F \widehat{\lambda}$, we get the result for $f(\lambda_k)$ by Corollary 6.2, because $f(\lambda)$ is a member of the p–adic family of modular forms determined by $f(\lambda_k)$. Otherwise, we modify the choice of λ.

We verify condition (H1-3) and (H5-7) one by one. We always have a character λ_1 of conductor 1 with $\lambda_1((\alpha)) = \alpha^{\kappa_1\Sigma + c\kappa_2\Sigma}$ for all $\alpha \in M^\times$ and $\lambda_1 \equiv 1 \mod \mathfrak{m}_W$ by our choice of κ; so, $\lambda/\lambda_1 \equiv \varphi \mod \mathfrak{m}_W$. We may assume that $\lambda/\lambda_1 = \varphi$.

- By the above choice of λ_1, we have $\det \rho_{\lambda_1} = \mathcal{N}^{[\kappa]}\left(\frac{M/F}{}\right)$ and $\det \rho_\lambda = \mathcal{N}^{[\kappa]}\widetilde{\varphi}\left(\frac{M/F}{}\right)$, where $\widetilde{\varphi}$ is the Galois character corresponding to the pull back of φ as a Hecke character of $M_{\mathbb{A}}^\times$ to $F_{\mathbb{A}}^\times$. Then χ in (H1) is given by $\widetilde{\varphi}\left(\frac{M/F}{}\right)$, which has order prime to p because $p > 2$. This shows (H1).
- By (h2), we have for $\mathfrak{l}|\mathfrak{N}(\lambda)p$,
$$\rho_\lambda|_{D_\mathfrak{l}} \cong \begin{cases} \begin{pmatrix} \widetilde{\lambda} & 0 \\ 0 & \widetilde{\lambda}_c \end{pmatrix} & \text{if } \mathfrak{l} = \mathfrak{L}\overline{\mathfrak{L}}\ (\mathfrak{L} \neq \overline{\mathfrak{L}}) \text{ in } M, \\ \begin{pmatrix} \widetilde{\lambda}_\mathfrak{l} & 0 \\ 0 & \widetilde{\lambda}_\mathfrak{l}\left(\frac{M/F}{}\right) \end{pmatrix} & \text{if } \mathfrak{l} \text{ is inert or ramified in } M/F. \end{cases}$$
We can choose $\widehat{\lambda}_c$ to corresponds to $\widehat{\lambda}_{\mathfrak{P}^c}$ for $\mathfrak{P} \in \Sigma_p$ with $\mathfrak{P}|\mathfrak{l}$ if $\mathfrak{l}|p$. Then by construction (or the definition of κ_2), we have $\delta_\mathfrak{l} = \widehat{\lambda}_{\mathfrak{P}^c}$. This shows (H2).
- Since λ_1 is of conductor 1, we find that $\widehat{\lambda}|_{I_\mathfrak{l}} = \varphi|_{I_\mathfrak{l}}$, which is of order prime to p. This shows (H3).
- Since $\widehat{\lambda} \equiv \varphi \mod \mathfrak{m}_W$, (h3) implies that $\overline{\delta}_\mathfrak{p} \neq \overline{\varepsilon}_\mathfrak{p}$; so, (H5) follows from (h3).
- The condition (H6) follows from the definition of $\mathfrak{N}(\lambda)$ and (h1), because $C(\overline{\varepsilon}_\mathfrak{l}\overline{\delta}_\mathfrak{l}^{-1})$ is equal to $C(\varepsilon_\mathfrak{l}\delta_\mathfrak{l}^{-1})$ by (H3) already verified. By our definition of $\mathfrak{N}(\lambda)$, its $\mathfrak{l}$ part coincides with $C(\varepsilon_\mathfrak{l}\delta_\mathfrak{l}^{-1})$.
- The condition (H7) follows from (h4) by Mackey's theorem.

Thus as long as $\widehat{\lambda} \not\equiv \widehat{\lambda}_c\mathcal{N} \mod \mathfrak{m}_W$ on $I_\mathfrak{p}$ for every $\mathfrak{p}|p$, we have verified the theorem.

Now assume that

$$P = \left\{\mathfrak{p}|p\middle|\widehat{\lambda} \equiv \widehat{\lambda}_c\mathcal{N} \mod \mathfrak{m}_W \text{ on } I_{\mathfrak{p}}\right\}$$

is non-empty. Let $\overline{R^\times}$ (resp. $\overline{O^\times}$) be the p–adic closure of $R^\times$ (resp. $O^\times$) in $R_p^\times$ for $R_p = R \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Since M cannot have p–th root of unity (by ordinarity of Σ and unramifiedness of p in $F/\mathbb{Q}$), $[R^\times : O^\times]$ is prime to p; so, $\overline{R^\times/O^\times}$ has order prime to p. We consider the character $x \mapsto x^\Sigma$ of $R_p^\times \to W^\times$, which has values in a complete valuation subring A of W unramified and finite over $\mathbb{Z}_p$. Let $A_p^\times$ be the maximal p–profinite subgroup of $A^\times$, which is canonically a direct factor of $A^\times$, because A is unramified over $\mathbb{Z}_p$. Let $x \mapsto \langle x\rangle$ be the projection of $x \in A^\times$ to $A_p^\times$. Thus $\langle x\rangle \equiv 1 \mod \mathfrak{m}_W$ for all $x \in A^\times$ and $\langle\zeta\rangle = 1$ for all roots of unity ζ in A. Thus $x \mapsto \langle x^\Sigma\rangle$ is a character of $R_p^\times/\overline{R^\times}$, which is a subgroup of finite index of $G(1)$. We can extend this character to a character $\widehat{\chi}$ of $G(1)$ so that $\widehat{\chi} \equiv 1 \mod \mathfrak{m}_W$ on $G(1)$. This is possible for the following reason: We first extend the character to a character $\chi' : G(1) \to W^\times$, which is always possible, replacing W by its finite extension if necessary. Then we take a Teichmüller lift ε of the reduction ($\chi' \mod \mathfrak{m}_W$). Then $\widehat{\chi} = \varepsilon^{-1}\chi'$ gives the desired extension. By our construction, $\widehat{\chi}$ is the p–adic avatar of an arithmetic Hecke character χ whose infinity type is Σ.

We now take the Teichmüller lift $\widehat{\lambda}_0$ of ($\widehat{\lambda} \mod \mathfrak{m}_W$), which is a p–adic avatar of a finite order character $\lambda_0 : G(\mathfrak{C}) \to W^\times$. Then $\lambda' = \lambda_0\chi$ is of infinite type Σ and satisfies $\lambda' \equiv \lambda \equiv \varphi \mod \mathfrak{m}_W$. For $x \in R_p$, we write $\omega(x) = \lim_{n\to\infty} x^{[R:pR]^n} \in R_p$ for $x \in R_p$. Since p is unramified in $M/\mathbb{Q}$, the Teichmüller lift of ($x^k \mod \mathfrak{m}_W$) for $k \in \mathbb{Z}[\Sigma \cup \Sigma c]$ is given by $\omega(x)^k$ (in other words, the operations k and ω commute). Thus, at the place $\mathfrak{p} \in P$, by the above process of construction, $\lambda'^-(x_{\mathfrak{p}}) = \mathcal{N}^{-1}(x_{\mathfrak{p}})$ for $x_{\mathfrak{p}} \in R_{\mathfrak{P}} \cap F_{\mathfrak{P}}^\times$ ($\mathfrak{P} \in \Sigma_p$ with $\mathfrak{P}|\mathfrak{p}$), and the level $\mathfrak{N}(\lambda')$ of $f(\lambda')$ is prime to all $\mathfrak{p} \in P$. Thus $f(\lambda')$ has weight $(I, 0)$ and its Galois representation satisfies (H4). Then the theorem follows from Corollary 6.2, since $f(\lambda_k)$ comes from the same local ring of the universal nearly ordinary Hecke algebra $\mathbf{h}$ as the local ring of the p–adic family of Hecke eigenforms determined by $f(\lambda)$ or $f(\lambda')$. □

For our later use, we shall compute the q–expansion of classical modular forms associated to $f(\lambda)$. Pick $y \in F_{\mathbb{A}}^\times$ with $y_p = y_\infty = 1$. Then by the definition of $\widehat{\lambda}$ and (6.5), we get the following formula of the complex Fourier coefficients:

$$a(\xi yd, f(\lambda)) = \sum_{xx^c=\xi yd, x_\Xi=1} \lambda(xR),$$

where $xR = F \cap x\widehat{R}$ and x runs over $(\widehat{R} \cap M_{\mathbb{A}^{(\infty)}}^\times)/R^{(\Xi)}$ for Ξ as in (6.5). This shows that for $f_{\mathrm{diag}[y,1]}$ in (S2),

$$f_{\mathrm{diag}[y,1]}(\tau) = N(\mathfrak{y})^{-1} \sum_{\mathfrak{A};\mathfrak{A}\mathfrak{A}^c\sim\mathfrak{y}\mathfrak{d}} \lambda(\mathfrak{A})\alpha^{-\kappa_2}\theta(\lambda;\mathfrak{A}),$$

where $\mathfrak{A}$ runs over a complete representative set for ideal classes of M with $\mathfrak{A}\mathfrak{A}^c = \alpha\mathfrak{y}\mathfrak{d}$ ($\mathfrak{y} = y\widehat{O} \cap F$) for a totally positive $\alpha \in F$ and

$$\theta(\lambda;\mathfrak{A}) = \sum_{\xi \in \mathfrak{A}^{-1}/\mu(M)} \lambda(\xi^{(\infty\Xi)})(\xi\xi^c)^{-\kappa_2} q^{\alpha\xi\xi^c}. \tag{6.6}$$

Here we regard λ as an idele character $\lambda : M_{\mathbb{A}}^{\times}/M^{\times}$ by putting

$$\lambda(x) = \lambda(xR)x_{\infty}^{-\kappa_1\Sigma - c\kappa_2\Sigma},$$

and ξ runs over elements in $\mathfrak{A}^{-1}$ such that $\xi\mathfrak{A}$ is outside Ξ for Ξ as in (6.5). As a locally constant function on $\widehat{\mathfrak{A}}^{-1}$, the p–component of $\phi'_1 : \xi \mapsto \lambda(\xi^{(\Xi)})$ is given by λ_p^{-1} restricted to $\mathfrak{A}_p^{-1}$ by the following reason: ϕ'_1 is the characteristic function of $\mathfrak{A}_{\mathfrak{l}}^{-1}$ for $\mathfrak{l}$ outside the conductor $C(\lambda)$, and taking $\xi \in \mathfrak{A}^{-1}$ with $\xi \equiv 1 \mod C^{(p)}(\lambda)$, we see that $\phi'_1(\xi) = \lambda(\xi^{(\Xi)}) = \lambda(\xi^{(p)}) = \lambda(\xi_p)^{-1}$.

The modular form $\theta(\lambda;\mathfrak{A})$ is of weight $\kappa\varepsilon$ on

$$\Gamma_0(\mathfrak{N}(\lambda);\mathfrak{y}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL_2(F) \middle| a, d \in O,\ b \in \mathfrak{y},\ c \in \mathfrak{N}(\lambda)\mathfrak{y}^{-1} \right\}.$$

6.3. Self-duality. Let $L^*(\kappa\varepsilon;W)$ be the dual lattice of $L(\kappa;W)$ under the pairing $[\ ,\ ]$ introduced in Subsection 5.3. Then by definition, $L^*(\kappa\varepsilon;W) \subset L(\kappa\varepsilon;W)$ and the quotient $L(\kappa\varepsilon;W)/L^*(\kappa\varepsilon;W)$ is spanned by $X^{n-j}Y^j$ for $0 < j < n$.

Since

$$U_0(\mathfrak{N}p)_p \operatorname{diag}[p,1] U_0(\mathfrak{N}p)_p = \bigsqcup_{u \mod pO_p} \left(\begin{smallmatrix} p & u \\ 0 & 1 \end{smallmatrix} \right) U_0(\mathfrak{N}p)_p,$$

the action of $\left(\begin{smallmatrix} p & u \\ 0 & 1 \end{smallmatrix} \right)$ on $L(\kappa\varepsilon;W)/L^*(\kappa\varepsilon;W)$ (even after dividing by p^{κ_2}) is nilpotent. Thus the projector $e = \lim_{n\to\infty} \mathbb{T}(p)$ kills the cohomology group:

$$H^r_*(Y, L(\kappa\varepsilon;W)/L^*(\kappa\varepsilon;W)) \quad (Y = Y_0^B(\mathfrak{N}))$$

for any $r \geq 0$, and hence by cohomology sequence, we get a canonical isomorphism for $Y = Y_0^B(\mathfrak{N})$:

$$H^r_{*,n.ord}(Y, L^*(\kappa\varepsilon;W)) \cong H^r_{*,n.ord}(Y, L(\kappa\varepsilon;W)), \tag{6.7}$$

where H^r_* is either compactly supported or usual cohomology group. We define the action of Hecke operators $\mathbb{T}(y)$ and $\langle \mathfrak{l} \rangle$ on $H^r_*(Y, L^*(\kappa^*\varepsilon^*;W))$ via the adjoint action under $[,\ ]$ of the semi-group $\Delta_0(\mathfrak{N})$. Then the operator is integral if either $p|\mathfrak{N}$ or $[\kappa] \leq 1 \Leftrightarrow [\kappa] \geq 0$. Thus in the same way, we get

$$H^r_{*,n.ord}(Y, L(\kappa^*\varepsilon^*;W)) \cong H^r_{*,n.ord}(Y, L^*(\kappa^*\varepsilon^*;W)). \tag{6.8}$$

As we have seen in [H88a] Theorem 10.1, $H^r_*(Y, L(\kappa\varepsilon;W) \otimes (\mathbb{Q}_p/\mathbb{Z}_p))$ is p–divisible if $|I_B| \leq 1$. Then by looking into the cohomology sequence attached to the short exact sequence:

$$0 \to L(\kappa\varepsilon;W) \to L(\kappa\varepsilon;W \otimes \mathbb{Q}_p) \to L(\kappa\varepsilon;W) \otimes (\mathbb{Q}_p/\mathbb{Z}_p) \to 0,$$

$H^r_*(Y,L(\kappa\varepsilon;W))$ is free of finite rank over W, and we get a perfect pairing:

$$(\ ,\ )_W : H^q_{n.ord}(Y,L(\kappa\varepsilon;W)) \times H^q_{c,n.ord}(Y,L(\kappa^*\varepsilon^*;W)) \to W$$

of W–free modules. For the moment, assume that $F \neq \mathbb{Q}$. Then $Y^B(U)$ is compact; so, $H^q_{cusp,n.ord} = H^q_{n.ord} = H^q_{c,n.ord}$, and we have the perfect duality pairing

$$(\ ,\ )_W : H^q_{cusp,n.ord}(Y,L(\kappa\varepsilon;W)) \times H^q_{cusp,n.ord}(Y,L(\kappa^*\varepsilon^*;W)) \longrightarrow W \tag{6.9}$$

As already verified in [H88b] Theorem 3.1 for $F = \mathbb{Q}$, the assertion (6.9) holds even for $F = \mathbb{Q}$; so, we do not need to assume $F \neq \mathbb{Q}$ anymore. We thus have

Corollary 6.4. *Under the assumptions and notations of* Corollary 6.2*, the map $(v,w) \mapsto f(v \otimes w)$ induces a surjective linear map: $V(R_P) \otimes_{R_P} V'(R_P) \twoheadrightarrow S(R_P)$ for all $P \in \mathcal{A}(\mathbb{I})$, where $S = S_{\kappa(P)}(\mathfrak{N},\varepsilon(P);W[\varepsilon_P])$, $V = H^q(Y,L(\kappa(P)\varepsilon(P);W[\varepsilon_P]))$, $V' = H^q(Y,L(\kappa(P)^*\varepsilon(P)^*;W[\varepsilon_P]))$. If $q = |I_B| = 0$, f is an isomorphism: $V(R_P) \otimes_{R_P} V'(R_P) \cong S(R_P)$.*

7. Proof of the theorem

We shall prove the theorem in the introduction under the assumptions (h1-4) on φ, which are equivalent to the assumptions (1-4) in the introduction once we have chosen φ with $\psi = \varphi^-$. We first recall integrality results due to Shimura [ACM] Section 32 and Katz [K] II on the values of modular forms and then prepare preliminary results on integral decomposition of quaternionic quadratic spaces. After that, we prove the theorem in the case where the degree $[F:\mathbb{Q}]$ is even. The odd degree case will be reduced to the even degree case.

7.1. Integrality of values of modular forms.

By the approximation theorem,

$$GL_2(F)\backslash GL_2(F^{(\infty)}_{\mathbb{A}})/U_0(\mathfrak{N}) \cong F^\times\backslash F^\times_{\mathbb{A}^{(\infty)}}/\det(U_0(\mathfrak{N})) \cong Cl_F \quad \text{via } y \mapsto \det(y)$$

for the class group Cl_F of F. From this, $f \in S_\kappa(\mathfrak{N},\varepsilon;W)$ is determined by the q–expansions $\{f(y)\}_y$. Writing $\mathfrak{y} = y\widehat{O} \cap F$ for the ideal corresponding to the idele y and setting $\widetilde{y} = \left(\begin{smallmatrix} y & 0 \\ 0 & 1\end{smallmatrix}\right)$, $f(y)$ is the q–expansion at the Tate AVRM $\mathrm{Tate}_{\mathfrak{y}^*,O}(q)$ (in [K] 1.1) of the classical modular form $f_{\widetilde{y}}$ (of (S2) in Subsection 5.1) of weight $k = \kappa_1 - \kappa_2 + I$ on the following congruence subgroup:

$$\Gamma_0(\mathfrak{N};\mathfrak{y}) = \left\{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL_2(F) \middle| a,d \in O,\ b \in \mathfrak{y},\ c \in \mathfrak{y}^{-1}\mathfrak{N}\right\}. \tag{7.1}$$

Here $\mathfrak{y}^* = \mathfrak{y}^{-1}\mathfrak{d}^{-1}$ for the absolute different $\mathfrak{d}$ of F.

A classical modular form with q–expansion coefficients in W on a slightly smaller Γ_1–type congruence subgroup:

$$\Gamma(\mathfrak{N};\mathfrak{y}) = \left\{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(\mathfrak{N};\mathfrak{y}) \middle| a \equiv d \equiv 1 \mod \mathfrak{N}\right\} \tag{7.2}$$

has a moduli theoretic interpretation, which we recall in the following paragraph. We write $S_k(\Gamma(\mathfrak{N};\mathfrak{y});A)$ for the space of the classical cusp forms on $\Gamma(\mathfrak{N};\mathfrak{y})$ of weight k with q–expansion coefficients in A.

Let A be a valuation ring with residual characteristic p. An abelian scheme $X_{/A'}$ over an A–algebra A' is called an abelian variety with real multiplication by O (AVRM) if it has an embedding: $O \hookrightarrow \mathrm{End}(X_{/A'})$ of algebras such that $H^0(X, \Omega_{X/O}) = (O \otimes_{\mathbb{Z}} A')\omega$ for a nowhere vanishing differential ω. Here we have used the unramifiedness of F at p (otherwise, we need to formulate this condition as $H^0(X, \Omega_{X/O}) = (\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} A')\omega$). Each Hilbert modular form $f \in S_k(\Gamma(\mathfrak{N};\mathfrak{y}); A)$ can be regarded as a function of quintuples: $(X, \lambda, i, \omega, A')$ made up of an A–algebra A', an AVRM X over A', a polarization λ whose polarization ideal is given by $\mathfrak{y}^*$, an embedding $i : \mu_{\mathfrak{N}} \hookrightarrow X$ of group schemes over A' and a differential ω as above (see, for more details of AVRM's, [K] 1.0 and [PAF] Section 4.1). Here $\mu_{\mathfrak{N}}$ is the group scheme made up of $\mathfrak{N}$–torsion points of $\mathbb{G}_m \otimes \mathfrak{d}^{-1}$, that is, $\mu_{\mathfrak{N}}(A) = \{\zeta \in \mathbb{G}_m \otimes \mathfrak{d}^{-1}(A) | \mathfrak{N}\zeta = 0\}$, regarding $\mathbb{G}_m \otimes \mathfrak{d}^{-1}(A)$ as an additive group. Every ingredient of the quintuple has to be defined over A'. As a function of $(X, \lambda, i, \omega)_{/A'}$, f satisfies the following conditions (see [HMI] 4.2.7):

(M1) $f(X', \lambda', i', \omega') = \rho(f(X, \lambda, i, \omega))$ if $\rho : A' \to C$ is an A–algebra homomorphism and $(X', \lambda', i', \omega')_{/C} \cong (X, \lambda, i, \omega) \times_{A',\rho} C$. Here "$\cong$" implies: $\phi : X \times_A C \cong X'_{/C}$ as AVRM's, ${}^t\phi \circ \lambda' \circ \phi = \lambda \times_{A'} C$, $\phi \circ i \equiv i'$ and $\phi^*\omega' = \omega$.

(M2) f vanishes at all cusps, that is, the q–expansion of f at every Tate quintuple vanishes at $q = 0$.

(M3) $f(X, \lambda, i, \alpha\omega) = \alpha^{-k} f(X, \lambda, i, \omega)$ for $\alpha \in (A' \otimes_{\mathbb{Z}} O)^\times$.

The "Neben" character $\varepsilon : U_0(\mathfrak{N}) \to \overline{\mathbb{Q}}^\times$ restricted to $U_0^1(\mathfrak{N}) = U_0(\mathfrak{N}) \cap SL_2(\widehat{O})$ factors through $U_0^1(\mathfrak{N})/U^1(\mathfrak{N})$ for $U^1(\mathfrak{N}) = U(\mathfrak{N}) \cap SL_2(\widehat{O})$ (the conductor of ε^- is $\mathfrak{N}$), because $\varepsilon(u) = \varepsilon_1(\det(u))\varepsilon^-(d)$ for $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Thus to evaluate $f \in S_\kappa(\mathfrak{N}, \varepsilon; A)$ at an AVRM X of CM type, we only need to specify $\mu_{\mathfrak{N}} \hookrightarrow X$.

Let M/F be the CM quadratic extension in the introduction. Recall the decomposition: $\mathfrak{C} = \mathfrak{F}\mathfrak{F}_c\mathfrak{I}$ of the conductor of the Hecke character λ such that $\mathfrak{F} + \mathfrak{F}_c = R$ with $\mathfrak{F} \subset \mathfrak{F}_c^c$ and $\mathfrak{I}$ is made up of primes non-split in M/F. By (h2) (for $\varphi = \lambda$), the prime factors of $\mathfrak{N}(\lambda)$ are either split or ramified over F. If $\mathfrak{l}|\mathfrak{N}(\lambda)$ and $\mathfrak{l} = \mathfrak{L}\overline{\mathfrak{L}}$ (with $\mathfrak{L} \neq \overline{\mathfrak{L}}$) in M, we may choose $\mathfrak{L}$ so that $\mathfrak{L} \supset \mathfrak{F}$. The exponent of $\mathfrak{l}$ in $\mathfrak{N}(\lambda)$ is less than or equal to that of $\mathfrak{L}$ in $\mathfrak{F}$. Thus to evaluate $f(\lambda)$ at a CM point, we need to specify the level structure for the level $d(M/F)\mathfrak{f}$ ($\mathfrak{f} = \mathfrak{F} \cap F$). Actually we later need the level structure at other primes non-split in M/F; so, we first specify level structure for split primes and then extend the definition to non-split primes. We shall do this first for an abelian variety of CM type Σ with multiplication by R. Hereafter $\mathfrak{F}$ is an integral ideal of R with $\mathfrak{F} + \mathfrak{F}^c = R$ and prime to p (because we need to be more careful for primes dividing p).

Let $\overline{W}$ be as in the introduction. Define $\mathcal{W} = i_p^{-1}(\overline{W}) \subset \overline{\mathbb{Q}}$, which is a valuation ring unramified over $\mathbb{Z}_{(p)}$ with algebraically closed residue field $\overline{\mathbb{F}}$.

We consider $X(R)_{/\mathbb{C}}$ to be the algebraization of the complex torus $\mathbb{C}^\Sigma/R^\Sigma$, where $R^\Sigma = \{(a^\sigma)_{\sigma\in\Sigma}|a \in R\}$ and $\mathbb{C}^\Sigma = R^\Sigma \otimes_{\mathbb{Z}} \mathbb{R}$. Since $X(R)$ has complex multiplication, it can be defined over $\overline{\mathbb{Q}}$ and hence over a number field (see [ACM] 12.4). By the main theorem of complex multiplication (see [ACM] 18.6), $X(R)$ and its ℓ–divisible group for any prime ℓ outside p are actually defined over an infinite extension K of $\mathbb{Q}$ unramified at p. By the criterion of good reduction by unramifiedness of ℓ–power torsion points (see [ST]), we find a model $X(R)_{/\mathcal{W}}$ of $X(R)_{/K}$.

By choosing $\delta \in M$ with $\mathrm{Im}(\sigma(\delta)) > 0$ for $\sigma \in \Sigma$, we have a polarization pairing $(x, y) \mapsto \mathrm{Tr}_{M/\mathbb{Q}}(\delta x c(y))$. This pairing identifies $R \wedge_O R$ with $\mathfrak{y}$ for a suitable choice of a fractional ideal $\mathfrak{y} \subset F$ (prime to p) and induces a $\mathfrak{y}^*$–polarization $\lambda = \lambda(R)$. Thus we have the CM-triple $(X(R), \lambda(R), i(R), \omega(R))_{/\mathcal{W}}$, choosing $\omega(R)$ so that $H^0(X(R), \Omega_{X(R)/\mathcal{W}}) = (O \otimes_{\mathbb{Z}} \mathcal{W})\omega(R)$.

Since $\overline{W}$ has algebraically closed residue field, for any integer m prime to p, we have $X(R)[m] = \{x \in X(R)(\overline{W})|mx = 0\} \cong (\mathbb{Z}/m\mathbb{Z})^{[M:\mathbb{Q}]}$ and $\mu_m \cong \mathbb{Z}/m\mathbb{Z}$ as group schemes over $\overline{W}$. Thus we define the level $\mathfrak{f}$–structure to be

$$\mu_{\mathfrak{f}} \cong O/\mathfrak{f} \cong X(R)[\mathfrak{F}] = \{x \in X(R)(\overline{W})|\mathfrak{F}x = 0\}.$$

Since the Frobenius map of $\overline{\mathbb{F}}_p$ acts by multiplication by p (times a unit) on $\Omega_{X(R)/\overline{W}}$, the p–divisible group $X(R)[\mathfrak{P}^\infty]_{/\mathcal{W}}$ for $\mathfrak{P} \in \Sigma_p$ is connected. Since the residue field of $\mathcal{W}$ is algebraically closed, we see that $X(R)[\mathfrak{P}^e] \cong \mu_{\mathfrak{P}^e}$ over $\mathcal{W}$ (for $e = (e(\mathfrak{P}))_{\mathfrak{P}\in\Sigma_p}$), which gives rise to the level $\mathfrak{p}^e$–structure we need.

Since $R \wedge R \cong \mathfrak{y}$, we can choose a base w_1 and w_2 of R so that $R = Ow_1 + \mathfrak{y}w_2$. For any integral ideal $\mathfrak{q}$ prime to p, we choose a generator $\varpi_{\mathfrak{q}}$ of $\mathfrak{q}O_{\mathfrak{q}}$. Fixing an isomorphism $O/\mathfrak{q} \cong \mathfrak{y}/\mathfrak{q}$, we embed $O/\mathfrak{q} \cong \mathfrak{y}/\mathfrak{q}/\mathfrak{y} \hookrightarrow \mathfrak{q}^{-1}R_{\mathfrak{q}}/R_{\mathfrak{q}} \cong X(R)[\mathfrak{q}]$ by sending x to $\varpi_{\mathfrak{q}}^{-1}xw_2 \in M_{\mathfrak{q}}/R_{\mathfrak{q}}$, which gives the level $\mathfrak{q}$–structure on $X(R)$. We choose the base $w = (w_1, w_2)$ so that the level $\mathfrak{p}^e\mathfrak{f}$–structure we have chosen coincides with the one for $\mathfrak{q}$ if $\mathfrak{p}^e\mathfrak{f} + \mathfrak{q}$ is non-trivial. We may always choose w so that $w_0 = w_1/w_2 \in \mathfrak{H}^I$. Therefore choosing the base (w_1, w_2) is almost equivalent to the choice of a point $w_0 \in \mathfrak{H}^I$ modulo $\Gamma(\mathfrak{N}, \mathfrak{y})$ for $\mathfrak{N} = \mathfrak{q} \cap \mathfrak{f}\mathfrak{p}^e$. We write the level structure as $i(R) : \mu_{\mathfrak{N}} \hookrightarrow X(R)[\mathfrak{N}]$.

The above definition of the quadruple $x(R) = (X(R), \Lambda(R), i(R), \omega(R))_{/\mathcal{W}}$ can be generalized to ideals of an O–order of R. Let $\mathfrak{m}$ be an integral ideal of F prime to $p\mathfrak{f}$. Let $R' = O + \mathfrak{m}R$ be the O–order of M of conductor $\mathfrak{m}$. We take a proper fractional ideal $\mathfrak{A}$ of R' prime to $p\mathfrak{f}\mathfrak{q}d(M/F)$. A fractional R'–ideal $\mathfrak{A}$ is called R'*–proper* if $\{x \in M|x\mathfrak{A} \subset \mathfrak{A}\} = R'$. The polarization pairing on R (so on M) induces the polarization $\Lambda(\mathfrak{A})$ on $\mathfrak{A}$. We identify $\mathfrak{A} \wedge \mathfrak{A}$ with a fractional ideal $\mathfrak{y}(\mathfrak{A})$ of F under this pairing. It is easy to verify $\mathfrak{y}(\mathfrak{A}) = \mathfrak{y}(R)\mathfrak{m}N_{M/F}(\mathfrak{A})$. Then we can choose a base w of $\mathfrak{A}$ so that $\mathfrak{A} = Ow_1 + \mathfrak{y}(\mathfrak{A})w_2$ and $w_0(\mathfrak{A}) = w_1/w_2 \in \mathfrak{H}^I$. This choice w gives rise to the level structure $i(\mathfrak{A}) : \mu_{\mathfrak{N}} \hookrightarrow X(\mathfrak{A})[\mathfrak{N}]$. We can always find an étale constant subgroup $C \cong O/\mathfrak{c}$ ($\mathfrak{c}$ prime to $\mathfrak{fq}d(M/F)p$) in $X(R)$ such that the étale quotient $X(\mathfrak{A}) = X(R)/C$ over $\mathcal{W}$ (e.g. [GME] 1.8.3) gives a model over $\mathcal{W}$ of $\mathbb{C}^\Sigma/\mathfrak{A}^\Sigma$. Since $\mathfrak{c}$ is prime to $p\mathfrak{f}$, the level structure $i(R)$ and the differential $\omega(R)$ induce a unique level structure and a unique differential $\omega(\mathfrak{A})$ on $X(\mathfrak{A})$. We make a choice w so that the two level structures (one coming from $i(R)$ and another from the base w) coincide at primes where the two are well defined. Thus we have a unique point $w_0(\mathfrak{A}) \in \mathfrak{H}^I/\Gamma_1(\mathfrak{N}, \mathfrak{y})$. Having w is equivalent to having the quadruple $x(\mathfrak{A}) = (X(\mathfrak{A}), \Lambda(\mathfrak{A}), i(\mathfrak{A}), \omega(\mathfrak{A}))$ over $\mathbb{C}$.

Supposing that $f \in S_k(\Gamma(\mathfrak{N}; \mathfrak{y}); \mathcal{W})$ (and regarding f as a complex modular form), we may interpret the value $f(x(\mathfrak{A}))$ in terms of evaluation at a CM point $w_0(\mathfrak{A}) \in \mathfrak{H}^I$. For each $z = (z_1, z_2)$ with $z_0 := \frac{z_1}{z_2} \in \mathfrak{H}^I$, we consider the lattice $L_z = L_z^{\mathfrak{y}} = 2\pi i(Oz_1 + \mathfrak{y}z_2) \subset F_{\mathbb{C}} = F \otimes_{\mathbb{Q}} \mathbb{C}$. We define a pairing $\langle\ ,\ \rangle : F_{\mathbb{C}} \times F_{\mathbb{C}} \to \mathbb{R}$ by $\langle 2\pi i(az_1 + bz_2), 2\pi i(cz_1 + dz_2)\rangle = ad - bc$, which induces a $\mathfrak{y}^*$–polarization $\lambda_z = \lambda_z^{\mathfrak{y}}$ on the complex torus $X_z = X_z^{\mathfrak{y}} = F_{\mathbb{C}}/L_z$. Thus we can algebraize X_z to an abelian variety $X_{z/\mathbb{C}}$. We have a canonical level $\mathfrak{N}$–structure $i_z : (\mathfrak{d}^{-1} \otimes O/\mathfrak{N}) \cong 2\pi i(\mathfrak{y}z_2 \otimes O/\mathfrak{N}) \subset X_z(\mathbb{C})$ as long as $\mathfrak{y}$ is prime to $\mathfrak{N}$. Then the analytic value of f at z is given by

$$z_2^{-k} f((z_0, 1)) = f(z) = f(x_z^{\mathfrak{y}}) \ \text{ for } x_z^{\mathfrak{y}} = (X_z, \lambda_z, i_z, du), \tag{7.3}$$

where u is the variable $(u_\sigma)_{\sigma \in I}$ with $u_\sigma \in \mathbb{C}$ identifying $F_{\mathbb{C}}$ with $\mathbb{C}^I$ as $\mathbb{C}$–algebras.

Defining the canonical period $\Omega \in F_{\mathbb{C}}^\times = (\mathbb{C}^\times)^\Sigma$ by

$$\omega(R) = \Omega du \tag{7.4}$$

and choosing $\mathfrak{y}$ so that $R = (2\pi i)^{-1} L_{z_0}^{\mathfrak{y}}$, we find $x(\mathfrak{A}) \cong x_z^{\mathfrak{y}(\mathfrak{A})}$ and

$$f(x(\mathfrak{A})) = \frac{(2\pi i)^k f(z)}{\Omega^k} \in \mathcal{W} \ \text{ up to units in } \mathcal{W}, \tag{7.5}$$

because $\omega(\mathfrak{A})/\omega(R) \in (O \otimes_{\mathbb{Z}} \mathcal{W})^\times$ (see [ACM] Section 32 and [K] II). Here writing $\Omega = (\Omega_\sigma) \in \mathbb{C}^\Sigma$, $\Omega^k = \prod_{\sigma \in \Sigma} \Omega_\sigma^{k_\sigma}$.

Since $\mathcal{W}$–integral modular forms $f(z, w)$ of weight (k, k) for the product of congruence subgroups: $\Gamma(\mathfrak{N}; \mathfrak{y}) \times \Gamma(\mathfrak{N}'; \mathfrak{y}')$ classify the pairs of test objects: $(x_z^{\mathfrak{y}}, x_w^{\mathfrak{y}'})$, the same formula is valid (by the same proof given in [K]): up to units in $\mathcal{W}$,

$$f(x(\mathfrak{A}), x(\mathfrak{B})) = \frac{(2\pi i)^{2k} f(z, w)}{\Omega^{2k}}. \tag{7.6}$$

7.2. Error terms of integral decomposition. Let B be a quaternion algebra over F. Let M/F be a CM field with integer ring R. We are going to compute error terms of O–integral decomposition of an O–lattice of B as an integral quadratic space into a direct sum of two O–lattices of M with its norm form.

We fix a maximal order O_B of B. For an embedding $i : R \hookrightarrow B$ of O–algebras, since i is an embedding of O–algebras, we have $\mathrm{Tr}(i(a)) = \mathrm{Tr}_{M/F}(a)$ for the reduced trace Tr of B and $i(a)i(a)^\iota = N_{M/F}(a) = aa^c$. This shows $i(a^c) = i(a)^\iota$ for the main involution ι of B.

Let L be an O–lattice in B. We consider the two orders:

$$O^l_L = \{x \in B | xL \subset L\} \ \text{ and } \ O^r_L = \{x \in B | Lx \subset L\}. \tag{7.7}$$

We suppose to have two embedding $l : R \hookrightarrow B$ and $r : R \hookrightarrow B$. Thus L becomes an $R_l \otimes_O R_r$–module by $(a \otimes b)\ell = l(a)\ell r(b)$, where $R_l = l^{-1}(l(R) \cap O^l_L)$ and $R_r = r^{-1}(r(R) \cap O^r_L)$. Since $K^m \otimes K^n \cong M_{m\times n}(K)$, we find that $M_m \otimes_K M_n(K) \cong M_{mn}(K)$ as K–algebras. By extending scalars to M, we find $B \otimes_F M \cong M_2(M)$, and the above argument applied to the extended algebra $M_2(M)$ shows that the embedding $l \otimes r : R_l \otimes_O R_r \hookrightarrow \mathrm{End}_O(L)$ is injective. Therefore B is a free $M \otimes_F M$–module of rank 1. When we regard B as an M–vector space, we agree to use right multiplication by $\alpha \in M$ given by $\alpha b = b \cdot r(\alpha)$. Therefore $M \otimes_F M$ is identified with $M \oplus M$ by $a \otimes b \mapsto (ab, a^c b)$ for the generator c of $\mathrm{Gal}(M/F)$. Then we define $L^1 = (1,0)L$ and $L^2 = (0,1)L$ for the idempotents $(1,0),(0,1) \in M \oplus M$. Since $L^M = L^1 \oplus L^2 \supset L$, we can define $L_j = L^j \cap L$. Then $L_M = L_1 \oplus L_2 \subset L$. Since $(1,0)B$ is the eigenspace of $M \oplus M$ killed by the right factor M, we have

$$L_2 = \left\{x \in L \middle| S(L_1, x) = 0\right\},$$

because multiplication by units in $(M \otimes_F M)^\times$ preserves the inner product $S(x,y) = \mathrm{Tr}(xy^\iota)$ up to scalar similitude. By S, we have the orthogonal projection π_1 of B to ML_1 and π_2 to ML_2. Then we may have defined $L^M = \pi_1(L) \oplus \pi_2(L)$. Indeed, π_1 (resp. π_2) is given by the multiplication by $(1,0)$ (resp. $(0,1) \in M \otimes_F M$). We want to determine primes dividing the index $[L^M : L_M]$. Here is the result:

Lemma 7.1. *Let $d(R_l/O)$ (resp. $d(R_r/O)$) be the relative discriminant of R_l/O (resp. of R_r/O). Then we have $d(R_l/O)d(R_r/O)L^M \subset L_M$.*

Proof. The process constructing L^M and L_M can be done at each localization $B_\mathfrak{p}$ for primes $\mathfrak{p}$ of O. Then $L_{i,\mathfrak{p}} = L_\mathfrak{p} \cap M_\mathfrak{p} L_i$ and $\pi_j(L_\mathfrak{p}) = \pi_j(L)_\mathfrak{p}$. If a prime $\mathfrak{p}$ of O is unramified in R_r and R_l, we have $R_{l,\mathfrak{p}} \otimes_{O_\mathfrak{p}} R_{r,\mathfrak{p}} \cong R_\mathfrak{p} \oplus R_\mathfrak{p}$, and hence $L^M_\mathfrak{p} = L_{M,\mathfrak{p}}$ by definition. More generally, by the definition of the discriminant, we have

$$d(R_l/O)d(R_r/O)(R \oplus R) \subset R_l \otimes R_r \subset M \otimes_F M.$$

This shows the desired assertion. $\square$

For a prime $\mathfrak{l}$ outside the discriminant of B/F, identifying $B_\mathfrak{l}$ with $M_2(F_\mathfrak{l})$, we define the Eichler order of level $\mathfrak{l}^m$ by

$$\widehat{O}_0(\mathfrak{l}^m)_\mathfrak{l} = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(F_\mathfrak{l}) \middle| c \in \mathfrak{l}^m O_\mathfrak{l} \right\}.$$

Here $\widehat{O}_0(\mathfrak{l}^0)_\mathfrak{l}$ is the fixed maximal order of $B_\mathfrak{l}$. We then put for ideals $\mathfrak{N}$ outside the discriminant of B/F, $\widehat{O}_0(\mathfrak{N}) = \prod_\mathfrak{l} \widehat{O}_0(\mathfrak{l}^{e(\mathfrak{l})})$, where $\mathfrak{N} = \prod_\mathfrak{l} \mathfrak{l}^{e(\mathfrak{l})}$ is the prime decomposition of $\mathfrak{N}$ (for $\mathfrak{l} \nmid \mathfrak{N}$, we agree to put $e(\mathfrak{l}) = 0$).

We identify B_p with $M_2(F_p)$ so that r and l both bring $(x, y) \in M_p = M_{\Sigma_p} \times M_{\Sigma_p c}$ onto $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$ in $M_2(F_p)$. For $\mathfrak{l}|\mathfrak{f} = \mathfrak{F} \cap O$, we take the factor $\mathfrak{L}|\mathfrak{l}$ so that $\mathfrak{L}|\mathfrak{F}$, and we identify $B_\mathfrak{l}$ with $M_2(F_\mathfrak{l})$ bringing $(x, y) \in M_\mathfrak{l} = M_\mathfrak{L} \times M_{\overline{\mathfrak{L}}}$ to $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$ in $M_2(F_\mathfrak{l})$. For $\mathfrak{l}|D(M/F)$, we embed $M_\mathfrak{l}$ by $r = l$ into the $O_\mathfrak{l}$–order of $M_2(F_\mathfrak{l})$ generated by the scalar in $O_\mathfrak{l}$ and $\Delta_0^B(\mathfrak{l})$, that is the Eichler order $\widehat{O}_0(D(M/F))_\mathfrak{l}$ of level $D(M/F)_\mathfrak{l}$.

Proposition 7.2. *Suppose the following three conditions:*

(a) $p\mathfrak{N}$ *is prime to* $\mathfrak{D} = d(R_r/O)d(R_l/O)$*;*

(b) $L_{\mathfrak{f}p} = \widehat{O}_0(\mathfrak{f}\mathfrak{p}^e)_{\mathfrak{f}p} \subset B_{\mathfrak{f}p}$ *for the conductor* $\mathfrak{p}^e = \prod_{\mathfrak{p}|p} \mathfrak{p}^{e(\mathfrak{p})}$ *of* ε_2*;*

(c) $\varepsilon_{1,p}$ *is trivial on* $O_p^\times$ *and* $\kappa = (I, 0)$.

Let $v \in L(\kappa\varepsilon; \overline{W}) = \overline{W}$ *and* $w \in L(\kappa^*\varepsilon^*; \overline{W}) = \overline{W}$. *Then* $\phi : L \to \overline{W}$ *given by* $\phi(\gamma) = [\gamma v, w]$ *is a* $\overline{W}$*–integral linear combination of functions of the form* $\phi_1 \otimes \phi_2$ *for functions* $\phi_j : L^j \to \overline{W}$ *such that*

(1) $\phi_1(x) = \phi_{1,p}(x_p)\phi_1^{(p)}(x^{(p)})$ *(resp.* $\phi_2(x) = \phi_{2,p}(x_p)\phi_2^{(p)}(x^{(p)})$*), where we embed* $x \in M$ *into* $M_p \times M^{(p)}$ *by* $x \mapsto (x_p, x^{(p)})$ *and for a* $\mathbb{Z}$*–module* $X \subset B$, $X^{(p)} = X \otimes_\mathbb{Z} \widehat{\mathbb{Z}}^{(p)}$ *with* $\widehat{\mathbb{Z}}^{(p)} = \prod_{\ell \nmid p} \mathbb{Z}_\ell$*;*

(2) $\phi_{2,p}\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right) = \varepsilon_2(d)$ *if* $\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right) \in O_p^2$ *and vanishes outside* $O_p \times O_p^\times \subset O_p^2 = L_p^2$*;*

(3) $\phi_{1,p}$ *is the characteristic function of* $L_p^1 \cong O_p \times \mathfrak{p}^e O_p$*;*

(4) $\phi_j^{(p)}$ $(j = 1, 2)$ *factors through the finite quotient* $L^j/\mathfrak{f}\mathfrak{D}L^j$ *of* $L^{j,(p)}$*;*

(5) *the function* ϕ_j *is supported on* L^j *and has values in* $\overline{W}$*.*

Proof. We regard ϕ as a function of $B_\mathbb{A}^{(\infty)} = B_p \times B_\mathbb{A}^{(p\infty)}$ supported on $\widehat{L}$ so that $\phi(b) = \phi_p(b_p)\phi^{(p)}(b^{(p)})$ for $\phi_p = \phi|_{B_p}$ and $\phi^{(p)} = \phi|_{B_\mathbb{A}^{(p\infty)}}$. We identify B_p with

$$M_2(F_p) = M_p \oplus M_p = \left(\begin{smallmatrix} R_{\Sigma_p c} & R_{\Sigma_p c} \\ R_{\Sigma_p} & R_{\Sigma_p} \end{smallmatrix}\right).$$

Then $\phi_p\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \varepsilon_1(a)\varepsilon_2(d)[v, w]$ if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \widehat{O}_0(\mathfrak{p}^e)_p$. This shows the desired assertion for ϕ_p.

As for the component outside p, we only need to prove that the characteristic function $\chi_{L^{(p)}}$ of $L^{(p)}$ is a finite $\overline{W}$–linear combination of tensor products of $\overline{W}$–integral locally constant functions. Note that any additive character $L^M/L_M \to \overline{W}^\times$ is a tensor product of $\overline{W}$–integral valued additive characters of $L^M/L_M = L^{M,(p)}/L_M^{(p)}$, because $[L^M : L]$ is a product of primes dividing the discriminant $\mathfrak{D}$ by the proposition. We then have $\chi_L = [L^M : L]^{-1} \sum_\psi \psi$, where ψ running through all additive characters of $L^{M,(p)}/L^{(p)}$. Note that $\psi = \psi_1 \otimes \psi_2$ with locally constant additive characters of $\psi_j : L^j \to \overline{W}^\times$.

Thus we may take $\phi_1(x_p, x^{(p)}) = \phi_{1,p}(x_p)\psi_1(x^{(p)})\varepsilon_2^{(p)}(x^{c(p)})$ and $\phi_2(y_p, y^{(p)}) = \phi_{2,p}(y_p)\psi_1(y^{(p)})\varepsilon_1^{(p)}(y^{(p)})$ for $(x, y) \in L^1 \oplus L^2$. Since ψ_j (resp. $\varepsilon_j^{(p)}$) factors through $L^j/\mathfrak{D}L^j$ by Lemma 7.1 (resp. $L^j/\mathfrak{f}L^j$ by definition), we conclude that $\phi_j^{(p)}$ factors through $L^j/\mathfrak{f}\mathfrak{D}L^j$. □

Let $B = M_2(F)$. We choose two fractional ideals $\mathfrak{B}_1$ and $\mathfrak{B}_2$ of M. Then we decompose $\mathfrak{A} = Oz_1 + \mathfrak{a}z_2$ and $\mathfrak{B} = Ow_1 + \mathfrak{b}w_2$ with $z_0 = z_1/z_2 \in \mathfrak{H}^I$ and $w_0 = w_1/w_2 \in \mathfrak{H}^I$. The regular representation l of R on $\mathfrak{B}_1$ given by $l(\alpha)\left(\begin{smallmatrix} z_0 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} z_0\alpha \\ \alpha \end{smallmatrix}\right)$ gives an embedding of R into

$$O_L^l = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \middle| a, d \in O,\ b \in \mathfrak{a},\ c \in \mathfrak{a}^{-1} \right\}.$$

Similarly we define an embedding $r : R \hookrightarrow O_L^r$ replacing z_0 by w_0, where

$$O_L^r = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \middle| a, d \in O,\ b \in \mathfrak{b},\ c \in \mathfrak{b}^{-1} \right\}.$$

We consider the tensor product: $\mathfrak{B}_1 \otimes_O \mathfrak{B}_2$ and $L = l(\mathfrak{B}_1)v \cdot r(\mathfrak{B}_2) \subset M_2(F)$ for a suitable $v \in M_2(F)$.

We want to determine the factors of L_M and L^M. Since L^1 is the projection of L to the first factor M of $M \otimes_F M = M \oplus M$, writing the projection to the first factor as $(a \otimes b) \mapsto a^c b$ (so the projection to the second factor is given by $(a \otimes b) \mapsto a^c b^c$), we have $L^1 \cong \mathfrak{B}_1^c \mathfrak{B}_2$ and $L^2 \cong \mathfrak{B}_1^c \mathfrak{B}_2^c$.

Since $R \otimes_O R$ can be identified with

$$\left\{ (a, b) \in R \oplus R \middle| a \equiv b \mod \mathfrak{d}(M/F) \right\}$$

inside $R \oplus R \subset M \otimes_F M$ for the relative different $\mathfrak{d}(M/F)$ for M/F, we see that $L_1 \cong \mathfrak{B}_1^c \mathfrak{B}_2 \mathfrak{d}(M/F)$ and $L_2 \cong \mathfrak{B}_1^c \mathfrak{B}_2^c \mathfrak{d}(M/F)$.

Remark 7.1. We analyze the choice of v locally at primes $\mathfrak{p}|p$ of F when $\mathfrak{B}_{j,\mathfrak{p}} = R_\mathfrak{p}$ for $j = 1, 2$. Since the prime ideal $\mathfrak{p}$ is split into $\mathfrak{P}\mathfrak{P}^c$ with $\mathfrak{P} \in \Sigma_p$ in M, by choosing the base (e_1, e_2) for $e_1 = (1, 0), e_2 = (0, 1)$ of $R_\mathfrak{p} = R_{\mathfrak{P}^c} \oplus R_\mathfrak{P}$ over $O_\mathfrak{p}$, we may assume that $l(\alpha) = r(\alpha) = \left(\begin{smallmatrix} \alpha^c & 0 \\ 0 & \alpha \end{smallmatrix}\right)$. Then we choose v to be $b = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. By computation, we have

$$l(\alpha)b \cdot r(\beta^c) = \left(\begin{smallmatrix} \alpha^c & 0 \\ 0 & \alpha \end{smallmatrix}\right) b \left(\begin{smallmatrix} \beta & 0 \\ 0 & \beta^c \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha^c\beta & \alpha^c\beta^c \\ \alpha\beta & \alpha\beta^c \end{smallmatrix}\right).$$

This shows that $l(R_\mathfrak{p})b \cdot r(R_\mathfrak{p}) = M_2(O_\mathfrak{p})$, and regarding $M_2(O_\mathfrak{p})$ as an $R_\mathfrak{p}$–module via $\alpha x = l(\alpha)x$, we find

$$M_2(O_\mathfrak{p}) = \left(\begin{smallmatrix} R_{\mathfrak{P}^c} & R_{\mathfrak{P}^c} \\ R_\mathfrak{P} & R_\mathfrak{P} \end{smallmatrix}\right).$$

Take $O_\mathfrak{p}$ basis $w = (w_1, w_2)$ and $z = (z_1, z_2)$ of $R_\mathfrak{p}$ in M so that $w \equiv z \equiv (e_1, e_2) \mod \mathfrak{p}^m$ for $m \geq e(\mathfrak{p})$ for $e(\mathfrak{p})$ as in Proposition 7.2.

We define $p(z, w) = z_2 w_2 p(z_0, w_0)$ and $[u; z, w] = S(u, p(z, w))$ (the homogeneous form of $[u; z_0, w_0]$). Then we find $[b; z, w] = (z_1 - z_2)(w_2 - w_1)$ and that $[b; z, w]$ is a $\mathfrak{p}$–adic unit.

7.3. PROOF. We first suppose that $[F:\mathbb{Q}]$ is even. Then we have a definite quaternion algebra $B_{/F}$ with $d(B/F)=1$ and $I_B=\emptyset$. We write $G_{/\mathbb{Q}}$ for the algebraic group associated to $B^\times$.

We fix a maximal order O_B and identify $\widehat{O}_B$ with $M_2(\widehat{O})$ once and for all. Thus $\widehat{O}_0(\mathfrak{N})\subset\widehat{O}_B$ is an open compact subring. We have $U_0^B(\mathfrak{N})=\widehat{O}_0(\mathfrak{N})^\times$. We fix complete representative sets $\{a_1,\dots,a_h\}$ for $G(\mathbb{Q})\backslash G(\mathbb{A})/U_0^B(\mathfrak{N})G(\mathbb{R})F_{\mathbb{A}}^\times$ with $a_{i,\mathfrak{N}p}=a_\infty=1$ and $Z\subset(F_{\mathbb{A}}^\times)^{(\mathfrak{N}p\infty)}$ for $Cl_F=F_{\mathbb{A}}^\times/F^\times\widehat{O}^\times F_\infty^\times$. We consider

$$\Delta_{ijz}(\mathfrak{N})=a_i^{-\iota}z\cdot\Delta_0^B(\mathfrak{N})a_j^\iota\cap B, O_{ijz}(\mathfrak{N})=a_i^{-\iota}z\cdot\widehat{O}_0(\mathfrak{N})a_j^\iota\cap B\ (z\in Z)\quad\text{and}\quad\Gamma_0^i(\mathfrak{N})=G^1(\mathbb{Q})\cap a_iU_0^B(\mathfrak{N})a_i^{-1}G(\mathbb{R}),\tag{7.8}$$

where $G^1(A)=\{g\in G(A)|gg^\iota=1\}$. Thus $\Delta_{ijz}(\mathfrak{N})\subset O_{ijz}(\mathfrak{N})$. Note here that $\{a_iz|z\in Z\}_{i=1,\dots,h}$ gives a complete representative set for $G(\mathbb{Q})\backslash G(\mathbb{A})/U_0^B(\mathfrak{N})G(\mathbb{R})$.

Let $\phi\in H^0(Y_0^B(\mathfrak{N}),L(\kappa\varepsilon;\overline{W}))$; so, we may regard $\phi:G(\mathbb{A})\to L(\kappa\varepsilon;\overline{W})$ with $\phi(\gamma xu)=u^\iota\phi(x)$ for $u\in U_0^B(\mathfrak{N})F_{\mathbb{A}}^\times G(\mathbb{R})F_{\mathbb{A}}^\times$ and $\gamma\in G(\mathbb{Q})$. Similarly, we choose $\phi^*\in H^0(Y_0^B(\mathfrak{N}),L(\kappa^*\varepsilon^*;\overline{W}))$. Then

$$(\phi,\phi^*)_{\overline{W}}=\sum_{i=1}^h[\phi(a_i),\phi^*(a_i)].$$

Pick $y\in F_{\mathbb{A}}^\times$ with $y_p=y_\infty=1$. Supposing ξyd is integral, we consider $\mathbb{T}(\xi yd)$ for $0\ll\xi\in F$. By (unr), we have $d_p=1$. We choose a decomposition

$$U_0^B(\mathfrak{N})\left(\begin{smallmatrix}\xi yd&0\\0&1\end{smallmatrix}\right)U_0^B(\mathfrak{N})=\bigsqcup_{\varpi}\varpi U_0^B(\mathfrak{N}).$$

Here we can choose ϖ so that $\varpi\varpi^\iota=\xi yd$, because

$$U\backslash U\left(\begin{smallmatrix}y&0\\0&1\end{smallmatrix}\right)U/U=U_0^B(\mathfrak{N})\backslash U_0^B(\mathfrak{N})\left(\begin{smallmatrix}y&0\\0&1\end{smallmatrix}\right)U_0^B(\mathfrak{N})/U_0^B(\mathfrak{N})$$

writing $U=\{u\in U_0^B(\mathfrak{N})|uu^\iota=1\}$. Thus $\varpi_p\varpi_p^\iota=\xi$. Then

$$\phi|\mathbb{T}(\xi yd)(x)=\xi^{-\kappa_2}\sum_{\varpi}\varpi_{p\mathfrak{N}}\phi(x\varpi^{-\iota}).$$

Since $a_i\varpi^{-\iota}\in\sqcup_{j,z}G(\mathbb{Q})a_jz\cdot U_0^B(\mathfrak{N})G(\mathbb{R})$, we can write $a_i\varpi^{-\iota}=\gamma_ia_ju_iz$ for $\gamma_i^{-\iota}\in\Delta_{ijz}(\mathfrak{N})$ and $u_i\in U_0^B(\mathfrak{N})G(\mathbb{R})$. Thus we have, writing $\mathfrak{a}_i=N_{B/F}(a_i)\widehat{O}\cap F$ and $\mathfrak{y}=y\widehat{O}\cap F$, $\mathfrak{a}_j\mathfrak{y}\mathfrak{d}\mathfrak{z}^2\xi=N(\gamma_i^{-1})\mathfrak{a}_i\ \mathfrak{z}=z\widehat{O}\cap F$; in other words, $\mathfrak{y}^{-1}\mathfrak{d}^{-1}\mathfrak{a}_i\mathfrak{a}_j^{-1}\mathfrak{z}^{-2}$ is generated by a totally positive element $\alpha_{ijz}\in F$ prime to $p\mathfrak{N}$. Thus we have

$$\xi=\alpha_{ijz}\gamma_i^{-1}\gamma_i^{-\iota}\ \text{ up to totally positive units.}$$

Then we see, up to totally positive units,

$$\phi|\mathbb{T}(\xi yd)(a_i)=\xi^{-\kappa_2}\sum_{\varpi}\varpi_{p\mathfrak{N}}\phi(a_i\varpi^{-\iota})=\alpha_{ijz}^{-\kappa_2}\sum_{\gamma_i}N_{B/F}(\gamma_i)^{\kappa_2}\gamma_i^{-\iota}\phi(a_j).$$

Here, extending $\varepsilon : U_0(\mathfrak{N}) \to \overline{\mathbb{Q}}^\times$ to $\varepsilon : U_0(\mathfrak{N})F_{\mathbb{A}}^\times \to \overline{\mathbb{Q}}_p^\times$ by the p–adic avatar $\widehat{\varepsilon}_+ : F_{\mathbb{A}}^\times/F^\times \to \overline{\mathbb{Q}}_p^\times$ of the central character ε_+, we have

$$N_{B/F}(\gamma_i)^{\kappa_2}\gamma_i^{-\iota}\phi(a_j) = \varepsilon(\gamma_i^{-\iota})\phi(a_j)\left(\sigma(\gamma_i^{-1})\left(\begin{smallmatrix} X_\sigma \\ Y_\sigma \end{smallmatrix}\right)\right),$$

which is p–integral if $\phi(a_j)$ is in $L(\kappa\varepsilon;\overline{W})$.

Since B is totally definite ($|I_B| \leq 1$ and $|I_B| \equiv [F:\mathbb{Q}] \mod 2 \Rightarrow I_B = \emptyset$), $\overline{\Gamma}_0^i(\mathfrak{N}) = \Gamma_0^i(\mathfrak{N})/O^\times$ is a finite group. We then put $e_i = |\overline{\Gamma}_0^i(\mathfrak{N})|$. Defining

$$\Theta_{ijz}(v,w) = \frac{1}{e_ie_j}\sum_{\gamma\in\Delta_{ijz}(\mathfrak{N})\cap Supp(\varepsilon)} N_{B/F}(\gamma)^{-\kappa_2}[\gamma v, w]q^{\alpha_{ijz}\gamma\gamma^\iota}$$

for $v \in H^0(\Gamma_0^j(\mathfrak{N}), L(\kappa\varepsilon;\overline{W}))$ and $w \in H^0(\Gamma_0^i(\mathfrak{N}), L(\kappa^*\varepsilon^*;\overline{W}))$ (and rewriting $\gamma_i^{-\iota}$ as γ), we find for $y \in F_{\mathbb{A}}^\times$ with $y_p = 1$

$$f(\phi\otimes\phi^*)(y) = N(y)^{-1}\sum_{i,j,z;\mathfrak{a}_i\mathfrak{a}_j^{-1}\mathfrak{z}^{-2}\sim\mathfrak{y}\mathfrak{d}} \alpha_{ijz}^{-\kappa_2}\Theta_{ijz}(\phi(a_j),\phi^*(a_i)), \tag{7.9}$$

where $\mathfrak{a} \sim \mathfrak{b}$ indicates that the two ideals belong to the same strict class in F. Here Θ_{ijz} is a theta series of the O–lattice $\Delta_{ijz}(\mathfrak{N})$ and is a Hilbert modular form of weight $\kappa\varepsilon$ on $\Gamma_0(\mathfrak{N};\mathfrak{y})$ for $\mathfrak{y} = F \cap y\widehat{O}$. Since the pairing: $[\ ,\ ]$ is p–integral valued on $L(\kappa\varepsilon;\overline{W}) \times L(\kappa^*\varepsilon^*;\overline{W})$ and α_{ijz} is prime to $p\mathfrak{N}$, the theta series has p–integral Fourier coefficients (except possibly for the constant term). The constant term does not show up if $\phi \in H^0_{n.ord}(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon;W)) \subset H^0_{cusp}(Y_0^B(\mathfrak{N}), L(\kappa\varepsilon;W))$. Thus restricting ϕ to the ordinary part, $f(\phi\otimes\phi^*)$ has to be cuspidal (cf. [H88a] Theorem 6.2) and hence, the constant term vanishes. We may forget about the integrality problem stemming from the denominator: e_ie_j.

We choose an ideal $\mathfrak{A}$ of M with $N_{M/F}(\mathfrak{A}) \sim \mathfrak{y}$. We choose $\alpha \gg 0$ with $\mathfrak{y}^{-1}\mathfrak{d}^{-1}N_{M/F}(\mathfrak{A}) = (\alpha)$. Then we consider the theta series defined in (6.6):

$$\theta(\lambda;\mathfrak{A}) = \sum_{\gamma\in\mathfrak{A}^{-1}} \lambda(\gamma)(\gamma\gamma^c)^{-\kappa_2}q^{\alpha\gamma\gamma^c}$$

for a Hecke character λ of conductor $\mathfrak{C}$ with $\lambda(\alpha) = \alpha^{\kappa_1\Sigma+\kappa_2\Sigma c}$ if $\alpha \equiv 1 \mod \mathfrak{C}$. Strictly speaking, we need to divide the above series by $|\mu(M)|$ (see (6.6)), but $|\mu(M)|$ is prime to p by the unramifiedness of p in $M/\mathbb{Q}$. So we forget about $|\mu(M)|$. Here we have freedom of choosing $\mathfrak{A}$ in its ideal class (by changing $\alpha \gg 0$ suitably).

We define the reversed Petersson inner product $(f,g) = \langle g,f\rangle = \overline{\langle f,g\rangle}$ to make it linear with respect to the right variable g. By the variable change $z \mapsto -\overline{z}$, we have

$$(f,g) = \langle f_c, g_c\rangle \quad \text{for } f_c(z) = \overline{f(-\overline{z})}. \tag{7.10}$$

Unless the following condition is met:

$$\kappa = (I, 0) \text{ and } (\lambda^-)^*(\mathfrak{P}^c) \equiv 1 \mod \mathfrak{m}_W \text{ for some } \mathfrak{P} \in \Sigma_p, \tag{7.11}$$

we have proven in [H05d] Proposition 5.6 the following equality up to units in $\overline{W}$:

$$L_p(\widehat{\lambda}^-) = \frac{(2\pi i)^{2(\kappa_1-\kappa_2)} W_p(\lambda^-)(f(\lambda), f(\lambda))_{\mathfrak{N}}}{\Omega^{2(\kappa_1-\kappa_2)}} \in \overline{W}, \tag{MT}$$

where $W_p(\lambda) = \prod_{\mathfrak{P}\in\Sigma_p} W(\lambda_{\mathfrak{P}})$ and

$$W(\lambda_{\mathfrak{P}}) = N(\mathfrak{P}^{-e(\mathfrak{P})})\lambda(\varpi_{\mathfrak{P}}^{-e(\mathfrak{P})}) \sum_{u\in(R/\mathfrak{P}^{e(\mathfrak{P})})^\times} \lambda_{\mathfrak{P}}(u)\mathbf{e}_M\left(\frac{u}{\varpi_{\mathfrak{P}}^{e(\mathfrak{P})}}\right)$$

if $e(\mathfrak{P}) > 0$ and $W(\lambda_{\mathfrak{P}}) = 1$ otherwise. We would like to show (choosing λ in the p–adic analytic family so that (7.11) does not hold)

$$\frac{(2\pi i)^{2(\kappa_1-\kappa_2)} W_p(\lambda^-)(\theta(\lambda;\mathfrak{A}), \Theta_{ijz}(\phi(a_j), \phi^*(a_j)))_\Gamma}{\Omega^{2(\kappa_1-\kappa_2)}} \in \overline{W} \tag{GL}$$

for $\Gamma = \Gamma_0(\mathfrak{N}(\lambda); \mathfrak{y})$ and the optimal CM period Ω defined in (7.4), as long as $\phi \in V(R)$ and $\phi^* \in V^*(R)$ for $R = R_P$ as in Corollary 6.4 for P associated to λ.

We write $O_i(\mathfrak{N})$ for $O_{iiz}(\mathfrak{N})$ with $z = 1$. We choose an embedding $i_0 : M \hookrightarrow B$. We may then realize B as

$$B = \left\{ \left(\begin{smallmatrix} a^c & b^c \\ b\eta & a \end{smallmatrix}\right) \middle| a, b \in M \right\}$$

with O_B containing $\left(\begin{smallmatrix} a^c & b^c \\ b\eta & a \end{smallmatrix}\right)$ if $a, b \in R$. We define $i_1(a) = \left(\begin{smallmatrix} a^c & 0 \\ 0 & a \end{smallmatrix}\right) \in B$. For primes $\mathfrak{l}$ split in M/F, we assume that our identification $B_{\mathfrak{l}} \cong M_2(F_{\mathfrak{l}})$ is induced by completing $\mathfrak{L}$–adically the above expression of B choosing one prime factor $\mathfrak{L}|\mathfrak{l}$ in M. Taking $a_1 = 1$, we find that $i_1(R) \subset O_1(\mathfrak{N})$ if $\mathfrak{N}$ is made of primes split in M/F. Suppose now that $\mathfrak{N}$ contains primes non-split in M/F. For a given finite set S of primes, we can conjugate the embedding i_1 by a norm 1 element $u_{\mathfrak{l}}$ ($\mathfrak{l} \in S$) so that $ui_1u^{-1}(R_S) \subset O_1(\mathfrak{N})_S$ ($O_1(\mathfrak{N})_S = O_1(\mathfrak{N}) \otimes_O O_S$ for the localization $O_S = \prod_{\mathfrak{l}\in S} O_{\mathfrak{l}}$). By the strong approximation theorem, choosing one prime $\mathfrak{q}$ of F, we can write $u = \gamma u'$ with $\gamma \in G(\mathbb{Q})$ and $u' \in U_0^B(\mathfrak{N})B_{\mathfrak{q}}^\times$. Thus changing i_1 by $\gamma i_1 \gamma^{-1}$, we may assume that for any given $\mathfrak{N}$ that $i_1(R_1) \subset O_1(\mathfrak{N})$ for an O–order $R_1 \subset R$ of $\mathfrak{q}$–power conductor. We identify $M_{\mathbb{A}}^\times$ with the image in $G(\mathbb{A})$ under i_1.

If $d(M/F) \neq 1$, we find $b_1, \ldots, b_j$ in $M_{\mathbb{A}}^\times$ so that $N_{M/F}(b_j)$ gives a complete representative set for $F^\times\backslash F_{\mathbb{A}}^\times/\widehat{O}^\times(F_{\mathbb{A}}^\times)^2$. By the reduced norm map: $N_{B/F} : G(\mathbb{A}) \to F_{\mathbb{A}+}^\times$, we have a surjection:

$$G(\mathbb{Q})\backslash G(\mathbb{A})/U_0^B(\mathfrak{N})G(\mathbb{R})F_{\mathbb{A}}^\times \twoheadrightarrow F_+^\times\backslash F_{\mathbb{A}+}^\times/\widehat{O}^\times(F_{\mathbb{A}}^\times)^2.$$

Thus we can choose $\{a_i = b_js_k\} = \{b_j\} \times \{s_k\}$ so that $N_{B/F}(s_k) = 1$. Then again by the strong approximation theorem, we can write $s_k = \gamma_k u_k$ with $u_k \in U_0^B(\mathfrak{N})B_{\mathfrak{q}}^\times$ and $\gamma_k \in G(\mathbb{Q})$. Since b_j commutes with $i_1(R_1)$, conjugation by b_j does not alter i_1. Then defining $i_j : M \to B$ by $\gamma_k i_1 \gamma_k^{-1}$ and putting R_j to be the inverse image under i_j of $i_j(M) \cap O_j(\mathfrak{N})$, we find that R_j is an O–order of M of $\mathfrak{q}$–power conductor.

Suppose now that $d(M/F) = 1$. In this case, the image of $M_{\mathbb{A}}^{\times}$ in the class group $F_+^{\times} \backslash F_{\mathbb{A}+}^{\times}/\widehat{O}^{\times} F_{\infty+}^{\times}$ under the norm map is of index two; so, we need to add one more element $b' \in G(F_{\mathfrak{q}})$ with $N_{M/F}(b)$ generating $\mathfrak{q}O_{\mathfrak{q}}$, choosing the prime $\mathfrak{q}$ to be inert in M/F. Then the representatives a_i can be chosen as $b_j s_k$ or $b_j b' s_k$ for $s_k \in SL_2(F_{\mathfrak{q}})$ and $b_j \in M_{\mathbb{A}}^{\times}$. Thus, by the same argument as above, we find again an O–order R_j of $\mathfrak{q}$–power conductor and an embedding $i_j : R_j \hookrightarrow O_j(\mathfrak{N})$. We have now proven:

Lemma 7.3. *Let the notation be as above. By choosing a prime ideal $\mathfrak{q}$ of F outside any given finite set of primes, we can embed the order $O + \mathfrak{q}^m R \subset M$ of $\mathfrak{q}$–power conductor into $O_j(\mathfrak{N})$ for all $j = 1, 2, \ldots, h$, if the conductor $\mathfrak{q}^m$ is sufficiently deep.*

We write R_j for $O_j(\mathfrak{N}) \cap R$. By the above lemma, we assume that R_j is of conductor $\mathfrak{q}^{m(j)}$. We choose later $\mathfrak{q}$ in a way optimal to our proof. We regard $L_{ijz} = O_{ijz}(\mathfrak{N})$ as $R_i \otimes_O R_j$ module by i_1: $(\alpha, \beta)b = \alpha b \beta$. Since $M \otimes_F M = M \oplus M$, writing 1_r (resp. 1_l) the idempotent of left and right factors, we split $O_{ijz}(\mathfrak{N}) \subset L_{ijz}^M = 1_r L_{ijz} \oplus 1_l L_{ijz}$. The index $[L_{ijz}^M : L_{ijz}]$ is a product of a power of $\mathfrak{q}$ and primes ramifying in M/F, which we can choose to be prime to p. Then as studied in Subsection 7.2, we can write Θ_{ijz} of level $\mathfrak{N}$ as a p–integral linear combination of $\theta(\phi_1)\theta(\phi_2)$ of theta series of $L_{ijz}^r = 1_r L_{ijz}$ and $L_{ijz}^l = 1_l L_{ijz}$, respectively. The functions ϕ_k $(k = 1, 2)$ can be chosen to be p–integral.

We now bound the level of $\theta(\phi_k)$. To make the argument simple, first assume that $i_1(R) \subset O_1(\mathfrak{N})$, $a_i = b_{i'}$ and $a_j = b_{j'}$, and we choose that $b_{i'}$ so that $b_{i',\mathfrak{l}} = 1$ for all primes $\mathfrak{l} | \mathfrak{N} p \cdot d(M/F)\mathfrak{q}$. Note that $b_{i'} z \cdot O_0(\mathfrak{N}) b_{j'}^{-1} = \mathfrak{z} \mathfrak{b}_{i'} \otimes \mathfrak{b}_{j'}^{-1}$ as $R \otimes_O R$–modules for $\mathfrak{b}_{i'} = (b_{i'} \widehat{R} \cap M)$, we find from the discussion at the end of the previous section that $L_{ijz}^1 = \mathfrak{z} \mathfrak{b}_{i'}^c \mathfrak{b}_{j'}^{-1}$ and $L_{ijz}^2 = \mathfrak{z} \mathfrak{b}_{i'}^c \mathfrak{b}_{j'}^{-c}$. Thus we find that $\mathfrak{y}\mathfrak{d} = \mathfrak{a}_i^{-1} \mathfrak{a}_j \mathfrak{z}^2 = N_{M/F}(L_{ijz}^1) = N_{M/F} L_{ijz}^2$.

As explained in the introduction, we take φ with $\psi = \varphi^-$. We may assume that the weight κ of $f(\varphi)$ is $(I, 0)$. We than take a weight κ member $f(\lambda)$ of the p–adic family (associated with φ: $\widehat{\lambda}|_{G_{tor}(\mathfrak{c})} = \varphi$) with complex multiplication by M. To avoid (7.11) ($\Leftrightarrow$ (MT)), we choose ε so that it is non-trivial at all $\mathfrak{p}|p$. Replacing φ by $\varphi\eta$ for a finite order character $\eta : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \overline{W}^{\times}$ does not alter the anticyclotomic part φ^-. By a theorem of Chevalley ([Ch]), we can choose η so that $\eta_{\mathfrak{l}} = \lambda_{\mathfrak{l}}^{-1}$ on the inertia group at $\mathfrak{l}$ for every prime $\mathfrak{l}$ in any given finite set of prime ideals. Thus we may assume

(7.12) *λ has conductor prime to $\Sigma_p c$.*

Write $\mathfrak{N} = \mathfrak{N}(\lambda)$. Under this assumption, $\kappa = \kappa^*$, $\varepsilon^* = \varepsilon^{-1}$ and $[v, w] = vw$ by identifying $L(\kappa\varepsilon; \overline{W}) = \overline{W}$ (on which $\Delta_0(\mathfrak{N})$ acts via multiplication by ε) and $L(\kappa^*\varepsilon^*; \overline{W}) = \overline{W}$. Then

$$[\gamma\phi(a_j), \phi^*(a_i)] = \varepsilon(\gamma_{\mathfrak{f}p})\phi(a_j)\phi^*(a_i).$$

Regarding the character $\varepsilon : \Delta_0^B(\mathfrak{N})_{\mathfrak{f}p} \to \overline{W}^\times$ as a function ε_{ijz} of $B \otimes_{\mathbb{Q}} \mathbb{A}^{(\infty)}$ supported on $\widehat{\Delta}_{ijz}(N) = a_i^\iota z \Delta_0^B(\mathfrak{N}) a_j^{-\iota}$ by $\varepsilon_{ijz}(x) = \varepsilon(x_{\mathfrak{f}p})$ ($\widehat{\Delta}_{ijz}(N)_{\mathfrak{f}p} = \Delta_0^B(N)_{\mathfrak{f}p}$), the function $\chi_{ijz} : \gamma \mapsto \varepsilon(\gamma)[\gamma\phi(a_j), \phi^*(a_i)]$ is the function ε_{ijz} multiplied by the p–integral constant: $\phi(a_j)\phi^*(a_i)$. Write down χ_{ijz} as a sum $\chi_{ijz} = \sum_{\phi_1,\phi_2} \phi_1 \otimes \phi_2$ for finitely many p–integral locally constant functions $\phi_1 : L^1_{ijz} \to \overline{W}$ and $\phi_2 : L^2_{ijz} \to \overline{W}$. By Proposition 7.2, $\phi_{2,p}(x_p) = \lambda_{\Sigma_p}(x_{\Sigma_p})$ on $R_{\Sigma_p c} \times R^\times_{\Sigma_p}$ and is supported by $(R_{\Sigma_p c} \times R^\times_{\Sigma_p}) \subset L^2_{ijz,p}$ (and $\phi_{1,p}$ is the characteristic function of $L^1_{ijz,p} = R_{\Sigma_p c} \times \mathfrak{p}^e R_{\Sigma_p}$).

By the proof of Proposition 7.2, we find that $\phi_k^{(p)}$ $(k = 1, 2)$ factors through $L^k_{ijz}/\mathfrak{d}(M/F)\mathfrak{f}L^k_{ijz}$. Thus $\theta(\phi_k)$ is at least automorphic with respect to the congruence subgroup $\Gamma_0(\mathfrak{N}(\lambda); \mathfrak{y}) \cap \Gamma(d(M/F)^2; \mathfrak{y})$, where

$$\Gamma(\mathfrak{N}; \mathfrak{y}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(\mathfrak{N}; \mathfrak{y}) \middle| a \equiv d \equiv 1 \mod \mathfrak{N} \right\}.$$

This follows from the fact that ϕ_k as above is a linear combination of p–integral functions χ of the lattice $(\mathfrak{z}\mathfrak{b}^c_{j'}, \mathfrak{b}^{-1}_{i'})$ modulo $(\mathfrak{z}\mathfrak{b}^c_{j'}, \mathfrak{b}^{-1}_{i'}\mathfrak{d}(M/F))$ for a sufficiently large m and the fact that $\theta(\chi) = \sum_{\xi \in M} \chi(\xi) q^{\alpha_{ijz}\xi\xi^c}$ has the level as described above.

More generally, when $a_i = b_{i'}s$ and $a_j = b_{j'}s'$ for s or s' with norm 1 in $B^\times_{\mathfrak{q}}$, R_i and R_j could have conductor a power of $\mathfrak{q}$; so, the same argument yields that $\theta(\phi_k)$ is on $\Gamma = \Gamma_0(\mathfrak{N}(\lambda); \mathfrak{y}) \cap \Gamma(d(M/F)^2\mathfrak{q}^m; \mathfrak{y})$ for a sufficiently large m.

As seen in (6.6), the $\mathfrak{y}$–component of $f(\lambda)$ is given by a p–integral finite sum $|\mu(M)|^{-1} \sum_{\mathfrak{A}} \lambda(\mathfrak{A})\theta(\lambda; \mathfrak{A})$ of theta series of the form:

$$\theta(\lambda; \mathfrak{A}) = \sum_{\xi \in \mathfrak{A}^{-1}} \lambda(\xi^{(\Xi\infty)}) q^{\alpha\xi\xi^c},$$

where $\mathfrak{A}\mathfrak{A}^c = \alpha\mathfrak{y}\mathfrak{d}$ (with $\alpha \gg 0$ in F). Here the sum $\sum_{\mathfrak{A}} \lambda(\mathfrak{A})\theta(\lambda; \mathfrak{A})$ is over ideal classes of M whose norm isequivalent to $\mathfrak{y}\mathfrak{d}$. By choosing $v \in M_2(F)$ and $(z_0, w_0) \in \mathfrak{H}^I \times \mathfrak{H}^I$ as in Section 4, we identify $M_2(F)$ with $M \oplus M$. Then we choose $\mathcal{L} = \mathfrak{A}^{-1} \oplus L^2_{ijz}$ as an O–lattice of $M_2(F)$. Since we have freedom of changing $\mathfrak{A}$ in its ideal class, we may assume that the p–adic completion $\mathcal{L}_p = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is equal to $M_2(O_p)$ in $M_2(F_p) = B_p$, because $L^2_{ijz} = O_p \oplus O_p$. Then $\mathcal{L}^1 = \mathfrak{A}^{-1}$ and $\mathcal{L}^2 = L^2_{ijz}$. We take $\phi'_1 : \mathcal{L}^1 \to \overline{W}$ so that $\theta(\phi'_1) = \theta(\lambda; \mathfrak{A})$. Then $\phi'_1(\xi) = \lambda(\xi^{(\Xi\infty)})$ and $\phi'_{1,\Sigma_p} = \lambda^{-1}_{\Sigma_p}(\xi_{\Sigma_p})$, and $\phi'_{1,\Sigma_p c}$ is the characteristic function of $R_{\Sigma_p c}$.

We choose two ideals $\mathfrak{B}_1$ and $\mathfrak{B}_2$ of M and $v \in M_2(F)$ very close p–adically to $b = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \in M_2(F_p)$ as in Remark 7.1 so that $L = l(\mathfrak{B}_1)v \cdot r(\mathfrak{B}_2) \subset \mathcal{L}$ with $\mathcal{L}/L$ killed by a power of $d(M/F)\mathfrak{q}$. Here we need to introduce another prime $\mathfrak{q}$, because $\mathcal{L}^2$ is stable only by an O–order in M of $\mathfrak{q}$–power conductor. We choose the base $\mathfrak{B}_1 = Oz_1 + \mathfrak{y}z_2$ and $\mathfrak{B}_2 = Ow_1 + \mathfrak{y}w_2$ again as in Remark 7.1. Let $z = (z_1, z_2)$ and $w = (w_1, w_2)$. Thus $z_0 = z_1/z_2$ and $w_0 = w_1/w_2$ are both in $\mathfrak{H}^I$. Thus we have from Theorem 4.1 and (7.10) that:

$$(\theta(\phi_1'), \theta(\phi_1)\theta(\phi_2))_\Gamma = \langle \theta(\phi_1')_c, \theta(\phi_1 \otimes \phi_2)_c \rangle_\Gamma = C\Psi(z, w)$$

for a constant $C \in \overline{W}^\times$ and a congruence subgroup

$$\Gamma = \Gamma_0(\mathfrak{N}(\lambda); \mathfrak{y}) \cap \Gamma(d(M/F)^2\mathfrak{q}^m; \mathfrak{y}) \quad (m \gg 0).$$

Here $\Psi(z, w)$ is the homogeneous version of the modular form:

$$\Psi(z_0, w_0) = \sum_{0 \ll \alpha \in \Gamma \backslash M_2(F)} \phi^{*(\infty)}(\epsilon\alpha)\mathbf{e}_F(\det(\alpha)z_0)\theta(\phi_1)|_1\alpha(w_0) \tag{7.13}$$

for the partial Fourier transform ϕ^* of $\phi = \overline{\phi}_1' \circ c \otimes \phi_2$, because $\theta(\phi)_c(z) = \overline{\theta(\phi)(-\overline{z})} = \theta(\overline{\phi \circ c})$. The constant C is prime to p (that is, $i_p(C) \in W^\times$) because of the following reason: Since Ψ is of weight (I, I), the homogeneous form is given by $z_2^I w_2^I \Psi(z, w) = \Psi(z_0, w_0)$. Since v is very close p–adically to b, we may assume that $v = b$. Then by Theorem 4.1 and $[b; z, w] = (z_1 - z_2)(w_2 - w_1)$, we have

$$C = z_2^I w_2^I \operatorname{Im}(z_0)^{-I} \operatorname{Im}(w_0)^{-I} [b; z_0, w_0]^I |[b; z_0, w_0]|^{2I}$$
$$= \frac{(z_1 - z_2)(w_2 - w_1)|(z_1 - z_2)(w_2 - w_1)|^2}{(z_1\overline{z}_2 - \overline{z}_1 z_2)(w_1\overline{w}_2 - \overline{w}_1 w_2)},$$

whose image under i_p is easily seen to be in $W^\times$ (by our choice of the base z and w as in Remark 7.1).

The local partial Fourier transform preserves p–integral Schwartz-Bruhat functions on $M_2(F_\mathfrak{l})$ as long as $\mathfrak{l} \nmid p$. Since $M_p = M_{\Sigma_p} \oplus M_{\Sigma_p c}$, we find

$$M_2(F_p) = M_p \oplus M_p = \begin{pmatrix} M_{\Sigma_p c} & M_{\Sigma_p c} \\ M_{\Sigma_p} & M_{\Sigma_p} \end{pmatrix}.$$

The first column is the factor M_p carrying $\overline{\phi}_{1,p}' \circ c$. The function $\overline{\phi}_{1,p}' \circ c$ is supported on R_p. Since complex conjugation interchanges a and c (see Proposition 7.2), we see from (6.6) that $(\overline{\phi}_{1,p}' \circ c)\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right) = \lambda_{\Sigma_p}(a)$ (because we have assumed that λ has conductor prime to Σ_p^c: (7.12)). Similarly, $\phi_{2,p}\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right) = \lambda_{\Sigma_p}(d)$ for $b \in R_{\Sigma_p c}$ and $d \in R_{\Sigma_p}$. Thus $\phi_p(a, b)$ equals to $\chi(a, b)\lambda_{\Sigma_p}(a)$ for the characteristic function χ of $R_p = O_p^\times \times O_p$. The partial Fourier transform is with respect to the variables "$(a^{(p)}, b^{(p)})$" keeps p–integrality by the Fourier inversion formula. Thus we may concentrate on the p–component. Define for each $\mathfrak{P} \in \Sigma$, $\Phi_\lambda(x)$ to be equal to $\lambda_\mathfrak{P}(x)$ if $x \in R_\mathfrak{P}^\times$ and 0 outside $R_\mathfrak{P}^\times$. Then the Fourier transform of Φ_λ is given by $W(\lambda_\mathfrak{P})\lambda_\mathfrak{P}(\varpi_\mathfrak{P}^{e(\mathfrak{P})})\Phi_{\overline{\lambda}}(\varpi_\mathfrak{P}^{e(\mathfrak{P})}x)$ (see [BNT]

Proposition 13 in VII.7). Thus we need to prove $W(\lambda_{\mathfrak{P}}^-)W(\lambda_{\mathfrak{P}})\lambda(\varpi_{\mathfrak{P}}^{e(\mathfrak{P})}) \in \overline{W}$. This can be done as follows: writing $e = e(\mathfrak{P})$ and $\varpi = \varpi_{\mathfrak{P}}$,

$$\begin{aligned}(7.14)\quad W(\lambda_{\mathfrak{P}}^-)\lambda(\varpi^e)W(\lambda_{\mathfrak{P}}) &= N(\mathfrak{P}^{-2e})\lambda(\varpi^e)\lambda(\varpi^{-ce})G(\lambda_{\mathfrak{P}}^{-1})G(\lambda_{\mathfrak{P}}) \\ &= \lambda_{\mathfrak{P}}(-1)\lambda(\varpi^{-ce})\lambda(\varpi^e)N(\mathfrak{P}^{-e}),\end{aligned}$$

where $G(\chi) = \sum_{u \in R/\mathfrak{P}^e} \chi(u)\mathbf{e}_M(\frac{u}{\varpi^e})$ for the conductor $\mathfrak{P}^e$ of χ. Note here that the infinity type of λ is $-\Sigma$, and hence $\lambda(\varpi^e)$ is up to unit equal to $\varpi^{e\Sigma}$ which is equal to $N(\mathfrak{P}^e)$ up to units in $\overline{W}$. This shows the desired integrality.

Since the partial Fourier transform with respect to the character $\mathbf{e}_{\mathbb{A}}(ab' - ba')$ interchanges (a,b), the support of $\overline{\phi}_p^*$ is contained in

$$\begin{pmatrix} O_p & \varpi^{-e}O_p^\times \\ O_p & O_p \end{pmatrix} = \tau^{-1}\begin{pmatrix} O_p & O_p \\ \mathfrak{p}^e O_p & O_p^\times \end{pmatrix} \subset M_2(F_p),$$

where $\tau = \left(\begin{smallmatrix} 0 & -1 \\ \varpi^e & 0 \end{smallmatrix}\right)$.

The function $\phi_{1,p}$ is the characteristic function of $R_{\Sigma_p c} \times \mathfrak{p}^e R_{\Sigma_p}^\times$. Since τ normalizes $U_0(\mathfrak{p}^e)_p$, we can choose complete representative set $\mathcal{R}$ for

$$U_0(\mathfrak{p}^e)_p \backslash \left(\begin{pmatrix} O_p & O_p \\ \mathfrak{p}^e O_p & O_p^\times \end{pmatrix} \times GL_2(F_{\mathbb{A}^{(p\infty)}})\right)$$

such that $\alpha \in \mathcal{R}$ can be written as $\tau^{-1}\beta$ with p–component β_p is upper triangular (e.g. [MFG] 3.1.6) with p–adic unit at the lower bottom corner. The Hecke operator UxU for $x \in \mathcal{R}$ preserves the p–integral structure of $S_\kappa(\Gamma; W)$ (the space of cusp forms on Γ with W–integral Fourier coefficients). This fact follows, for example, [H88a] Theorem 4.11, and actually, if $x \in GL_2(F)$ has upper triangular p–component with p–adic unit at the lower bottom corner, the action of $\theta \mapsto \theta|_1 x$ on modular forms preserves p–integrality since it is basically given by $\theta(z) \mapsto \theta(az)$ for totally positive a. Thus the action of β: $\theta(\phi_1)|_1\tau^{-1} \mapsto \theta(\phi_1)|_1\tau^{-1}\beta$ in (7.13) preserves the p–integrality (see Theorem 4.9 in [H88a]), and $\theta(\phi_1)|_1\tau^{-1}\beta$ has p–integral q–expansion with respect to the variable w if $\theta(\phi_1)|\tau^{-1}$ is p–integral. Thus we need to prove that $\theta(\phi)|_1\tau^{-1}$ has p–integral q–expansion coefficients, in order to show $\Psi(z,w)$ in (7.13) has p–integral q–expansion. Since $\theta(\phi)|\tau'$ for $\tau' = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is given by $\theta(\widehat{\phi}_1)$ for the Fourier transform $\widehat{\phi}_1$ of ϕ_1 regarding it as a function on $M_{\mathbb{A}}$. The p–integrality only depends on the p–part $\phi_{1,p}$ of ϕ_1. By computation, $\widehat{\phi}_{1,p}$ is $N(\mathfrak{p}^{-e})$ times the characteristic function of $R_{\Sigma_p c} \times \mathfrak{p}^{-e}R_{\Sigma_p}$. Taking ϖ^{-e} in O, we find that $\theta(\phi_1)|\tau^{-1}$ is equal to $\theta(\widehat{\phi}_1)|_1\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi^{-e} \end{smallmatrix}\right)(w) = \varpi^e\theta(\widehat{\phi}_1)(\varpi^e w)$ up to a p–adic unit. Since $\varpi^e N(\mathfrak{p}^{-e})$ is a p–adic unit, we get the desired integrality.

By the q–expansion principle, we conclude from (7.6)

$$(7.15)\qquad \frac{(2\pi i)^{2\Sigma}W_p(\lambda^-)(\theta(\lambda;\mathfrak{A}), \theta(\phi_1)\theta(\phi_2))_\Gamma}{\Omega^{2\Sigma}} \in \overline{W}.$$

This shows

$$[\overline{\Gamma}_0(\mathfrak{N}(\lambda);\mathfrak{y}):\overline{\Gamma}]\frac{(2\pi i)^{2\Sigma}W_p(\lambda^-)(\theta(\lambda;\mathfrak{A}),\Theta_{ijz})_{\Gamma_0(\mathfrak{N}(\lambda);\mathfrak{y})}}{\Omega^{2\Sigma}}\in\overline{W}.$$

Write $(\theta(\lambda;\mathfrak{A}),f)_\Gamma$ for the inner product $(\theta(\lambda;\mathfrak{A}),f_\mathfrak{y})$ for the $\mathfrak{y}$–component $f_\mathfrak{y}$ of $f\in S_\kappa(\mathfrak{N},\varepsilon;W)$. Since any $f\in S_\kappa(\mathfrak{N}(\lambda),\varepsilon;\overline{W})$ is a $\overline{W}$–linear combination of Θ_{ijz} by Corollary 6.4, we conclude

$$[\overline{\Gamma}_0(\mathfrak{N}(\lambda);\mathfrak{y}):\overline{\Gamma}]\frac{(2\pi i)^{2\Sigma}W_p(\lambda^-)(\theta(\lambda;\mathfrak{A}),S_\kappa(\mathfrak{N}(\lambda),\varepsilon;\overline{W}))_{\Gamma_0(\mathfrak{N}(\lambda);\mathfrak{y})}}{\Omega^{2\Sigma}}\subset\overline{W}.$$

Since $[\overline{\Gamma}_0(\mathfrak{N}(\lambda);\mathfrak{y}):\overline{\Gamma}]$ is a factor of

$$N_{F/\mathbb{Q}}(\mathfrak{N}')^2\prod_{\mathfrak{l}|\mathfrak{N}'}\left(1-\frac{1}{N_{F/\mathbb{Q}}(\mathfrak{l})^2}\right)$$

for $\mathfrak{N}'=d(M/F)^2\mathfrak{q}^m$, if $p\nmid(N(\mathfrak{l})\pm1)$ for all primes $\mathfrak{l}|d(M/F)\mathfrak{q}$, we get

$$\frac{(2\pi i)^{2\Sigma}W_p(\lambda^-)(\theta(\lambda;\mathfrak{A}),S_\kappa(\mathfrak{N}(\lambda),\varepsilon;\overline{W}))_{\Gamma_0(\mathfrak{N}(\lambda);\mathfrak{y})}}{\Omega^{2\Sigma}}\subset\overline{W}. \tag{7.16}$$

We can choose $\mathfrak{q}$ (by unramifiedness of p in $F/\mathbb{Q}$ and $p\geq5$) so that

$$p\nmid(N_{M/F}(\mathfrak{q})\pm1).$$

Thus if $p\nmid(N_{M/F}(\mathfrak{l})\pm1)$ for all primes $\mathfrak{l}|d(M/F)$, we conclude $H(\varphi)|\frac{h(M)}{h(F)}L^-(\varphi)$ as we explained in the introduction. Here $H(\varphi)$ is the congruence power series with respect to the nearly ordinary Hecke algebra $\mathbf{h}(\mathfrak{N}(\varphi),\varepsilon_\varphi;W)$ interpolating $h_\kappa^{n.ord}(\mathfrak{N}(\lambda),\varepsilon_\lambda;W)$ (for all $\kappa\varepsilon\in\mathcal{A}(\mathbb{I})$). Thus $H(\varphi)$ divides thecongruence power series H in [HT1] but could be smaller if $\mathfrak{C}\cap\mathfrak{C}^c$ contains non-trivial prime factor. In [HT1], we had an extra factor $\Delta(M/F;\mathfrak{C})$ which is equal to the product of the Euler factors of $L(s,\alpha)L(s,\varphi^{-1}\varphi_c)$ for primes outside p in $\mathfrak{C}\cap\mathfrak{C}^c$. This comes from the formula of the inner product of $\theta(\lambda)$ in [HT1] Theorem 7.1. After doing the same computation for $f(\lambda)$ of smaller level instead of $\theta(\lambda)$ and writing $k=\kappa_1-\kappa_2+I$ (see [H05d] (5.5)), we get the exact formula, if λ^- has split conductor:

$$\begin{aligned}(7.17)\quad &(f(\lambda)^u,f(\lambda)^u)_{\mathfrak{N}(\lambda)}\\ &\qquad=D\cdot N_{F/\mathbb{Q}}(\mathfrak{N}(\lambda))2^{-2k+1}\pi^{-(k+I)}\Gamma_F(k+I)L(1,Ad(f(\lambda)))\end{aligned}$$

under the terminology of [HT1] Section 7 without any error terms. Here $D=N(\mathfrak{d})$ is the discriminant of $F/\mathbb{Q}$.

Here is how to remove the condition: $p\nmid(N_{M/F}(\mathfrak{l})\pm1)$ for primes $\mathfrak{l}$ in the discriminant $d(M/F)$. The idea is to make quadratic base-change (and then descent). As a target of the base-change, we can find a totally real quadratic extension F'/F unramified at p such that $d(M'/F')$ for the composite $M'=MF'$ does not contain prime factors as above. Then for M'/F', we get the assertion. We later choose F' more carefully so that we can effectively descend back to F again. Let χ be the character $\mathrm{Gal}(F'/F)\cong\{\pm1\}$ restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/M)$. Suppose that we find a character η of $\mathrm{Gal}(\overline{\mathbb{Q}}/M)$ of conductor

$\mathfrak{C}'$ such that $\eta^{c-1} = \chi$.

We can always assume that η is of order prime to p by taking the Teichmüller lift of ($\eta \mod \mathfrak{m}_{\overline{W}}$). Let Γ_M be as in the introduction and we consider the universal character $\widetilde{\varphi} : G(\mathfrak{C}) \to W[[\Gamma_M]]$ with branch character φ. Put Ψ : $\mathrm{Gal}(\overline{F}/F) \to GL_2(\mathbb{I})$ be the induced Galois representation $\mathrm{Ind}_M^F \widetilde{\varphi}$. Then we have

$$Ad(\Psi) \cong \alpha \oplus \mathrm{Ind}_M^F(\widetilde{\psi}) \quad \text{for } \alpha = \left(\frac{M/F}{\quad}\right).$$

Thus

$$Ad(\Psi) \otimes \chi \cong \alpha\chi \oplus \mathrm{Ind}_M^F(\widetilde{\varphi\eta}^{-1}\widetilde{\varphi\eta}_c).$$

By Fujiwara's "$R = T$" theorem [Fu] (actually its $\mathbb{I}$–adic version: [HMI] Theorem 3.59), under the assumption (h1-4), the congruence power series $H(\varphi)$ gives the characteristic power series of the Selmer group

$$\mathrm{Sel}(Ad(\Psi)) = \mathrm{Hom}(Cl^-, \mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{I} \oplus \mathrm{Sel}(\psi),$$

where $Cl^- = Cl_M/Cl_F$ for the class groups Cl_M (resp. Cl_F) of M (resp. F).

We need to argue more for the character $\varphi\eta$, because $\varphi\eta$ may not satisfy the condition (h2). We choose F' so that $F'_{\mathfrak{l}} = M_{\mathfrak{l}}$ for all $\mathfrak{l}|2pd(M/F)$ and F'/F ramifies outside $2d(M/F)$ only at primes which split in M/F. This is possible for the following reason: We take an element $\delta \in O$ so that $M = F[\sqrt{\delta}]$. Then we take a high power $\mathfrak{a} = (2pd(M/F))^m$ so that any element $u \in F$ with $u \equiv 1 \mod \mathfrak{a}$ is a square in $F_{\mathfrak{l}}$ for all $\mathfrak{l}|2pd(M/F)$. Then for the infinite set $\Xi = \{\varepsilon \in O | \varepsilon \equiv \delta \mod \mathfrak{a},\ \varepsilon \gg 0\}$, we can find an infinite set of primes $\mathfrak{q} = (\varepsilon\delta^{-1})$ which splits in M/F. Then we define $F' = F[\sqrt{\varepsilon}]$. By our choice, $(\varepsilon) = \mathfrak{q}(\delta)$, and hence if a prime outside $2d(M/F)$ ramifies in F'/F, it has to be $\mathfrak{q}$, which splits in M/F.

We shall show that for the above choice of F', $\varphi\eta$ satisfies (h2). In fact, suppose that $\mathfrak{l}$ remains prime in M/F. Then if $\eta_{\mathfrak{l}}^{c-1} = \chi_{\mathfrak{l}} \neq 1$, then $\chi_{\mathfrak{l}}$ has to ramify, and hence F'/F ramifies at $\mathfrak{l}$. By our choice of F', $\mathfrak{l}$ splits in M/F, a contradiction. If $\mathfrak{l}$ ramifies in M, $\chi_{\mathfrak{l}}$ restricted in $\mathrm{Gal}(\overline{M}_{\mathfrak{l}}/M_{\mathfrak{l}})$ is trivial because $F'_{\mathfrak{l}} = M_{\mathfrak{l}}$. This shows that $\varphi_{\mathfrak{l}}\eta_{\mathfrak{l}}$ is c–invariant, and hence by local class field theory, it is a pull-back of a character of $F_{\mathfrak{l}}^{\times}$ by the norm. Thus $\varphi\eta$ satisfies (h2), and the congruence power series $H(\varphi\eta)$ still gives the exact characteristic power series of $\mathrm{Sel}(Ad(\Psi'))$, where $\Psi' = \mathrm{Ind}_M^F \widetilde{\varphi\eta}$. This is the beauty of taking level $\mathfrak{N}(\varphi)$ (not the deeper level: $N_{M/F}(\mathfrak{C})d(M/F)$ taken in [HT1] and [HT2]). Writing the congruence power series for $\widehat{\varphi} = \varphi \circ N_{M'/M}$ as $H(\widehat{\varphi})$, by the base change (cf. [H00] Proposition 2.4), we have (by $p > 2$),

$$\mathrm{Sel}(Ad(\mathrm{Ind}_{M'}^{F'} \widetilde{\varphi})) = \mathrm{Sel}(Ad(\Psi)) \oplus \mathrm{Sel}(Ad(\Psi) \otimes \chi),$$

which implies

$$H(\widehat{\varphi}) = H(\varphi)H(\varphi\eta)\frac{h(M'')}{h(M)},$$

where M'' is the third (and unique) CM quadratic extension of F inside $M' = MF'$.

If $\chi = \eta^{1-c}$ for a Hecke character η of M, $\chi\psi$ is again anti-cyclotomic. We have shown in [H05d] Corollary 5.5:

$$(h(M)/h(F))L^-(\psi)|H(\varphi) \ \text{ and } \ (h(M)/h(F))L^-(\psi\eta)|H(\varphi\eta),$$

which is enough to conclude the equality for each (by Nakayama's lemma):

$$(h(M)/h(F))L^-(\psi) = H(\varphi) \ \text{ and } \ (h(M)/h(F))L^-(\psi\eta) = H(\varphi\eta)$$

from $(h(M')/h(F'))L^-(\widehat{\psi}) = H(\widehat{\varphi})$ we have already proven.

We now prove the anticyclotomy of χ: $\chi = \eta^{c-1}$. Let $\chi : M_{\mathbb{A}}^\times/M^\times \to \{\pm 1\}$ be the quadratic idele character corresponding to M'/M. We want to have a finite order Hecke character $\eta : M_{\mathbb{A}}^\times \to \mu_{\mathfrak{N}}$ such that $\eta^{c-1} = \chi$, where $\eta^c(x) = \eta(c(x))$ for $x \in M_{\mathbb{A}}^\times$.

Let k be a number field. By class field theory, any continuous character of $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$ can be regarded as a continuous idele character: $C_k = k_{\mathbb{A}}^\times/k^\times \to \mathbb{T}$, where

$$\mathbb{T} = \left\{ z \in \mathbb{C} \middle| |z| = 1 \right\}.$$

A given continuous character of C_k is of finite order if and only if it is trivial on the identity component of the infinite part $k_\infty^\times$ of $k_{\mathbb{A}}^\times$ (cf. [MFG] Proposition 2.2). By Artin reciprocity, any continuous character of C_k trivial on the identity component of $k_\infty^\times \subset k_{\mathbb{A}}^\times$ can be viewed as a (finite order) character of $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$ canonically.

Looking at the exact sequence:

$$1 \to M^\times \to M_{\mathbb{A}}^\times \to C_M \to 1,$$

by Hilbert's theorem 90 applied to $M^\times$ and $\mathrm{Gal}(M/F) = \langle c \rangle$, we find

$$H^0(\mathrm{Gal}(M/F), C_M) = C_F,$$

and the kernel of $c-1 : x \mapsto x^{c-1}$ is given by C_F. A character $\phi : C_M \to \mathbb{T}$ is of the form $\phi = \eta^{c-1}$ if and only if ϕ is trivial on C_F. Since $\mathrm{Gal}(M'/F) \cong (\mathbb{Z}/2\mathbb{Z})^2$, we find a quadratic character α of C_F such that $\chi = \alpha \circ N_{M/F}$. This shows that $\chi(x) = \alpha(xx^c) = \alpha(x^2) = 1$ for $x \in C_F$. Thus we can write $\chi = \eta^{c-1}$ for a character $\eta : C_M \to \mathbb{T}$.

To have η factor through the Galois group of the maximal abelian extension of M, we need to show that η can be chosen so that its restriction to $M_\infty^\times$ is trivial. Since $\chi = \eta^{c-1}$ is trivial on $M_\infty^\times$, η is trivial on $(M_\infty^\times)^{c-1} = \mathrm{Ker}(N_{M/F} : M_\infty^\times \to F_\infty^\times)$. Thus $\eta|_{M_\infty^\times}$ factors through $N_{M/F} : M_\infty^\times \to F_{\infty+}^\times$. Replacing η by $\eta(\xi \circ N_{M/F})$ for a Hecke character ξ of F, we may assume that η is trivial on $M_\infty^\times$. This finishes the proof for even degree field.

We now assume that F has odd degree. The above trick of taking totally real quadratic extensions F'/F reduces the proof to the even degree case of M'/F'; so, we get the theorem. $\square$

As we have seen that $\psi = \varphi^-$ if and only if ψ is trivial on C_F. If ψ is anticyclotomic, then $\psi(x^c) = \psi(x^{-1})$ ($\Leftrightarrow \psi = 1$ on $N_{M/F}(M_{\mathbb{A}}^\times)$). Thus $\psi|_{C_F}$ is either the character of M/F or trivial. Since ψ is a Hecke character of $M_{\mathbb{A}}^\times$ of finite order, its infinity type is trivial; so, ψ has to be trivial on C_F. This shows

(7.18) If ψ is anticyclotomic, then $\psi = \varphi^-$ for a Hecke character φ of M.

We leave the reader to show that we can take φ to be of finite order (see [HMI] Lemma 5.31).

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