The 2-adic Eigencurve is Proper.

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Abstract. Coleman and Mazur ask whether the Eigencurve has any "holes". We answer their question in the negative for the 2-adic Eigencurve of tame level one.

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Dedicated to John Coates on his 60th birthday.

1 Introduction

In [7], Coleman and Mazur construct a rigid analytic space $\mathcal{E}$ (the "Eigencurve") that parameterizes overconvergent and therefore classical modular eigenforms of finite slope. The geometry of $\mathcal{E}$ is at present poorly understood, and seems quite complicated, especially over the centre of weight space. Recently, some progress has been made in understanding the geometry of $\mathcal{E}$ in certain examples (see for example [3],[4]). Many questions remain. In this paper, we address the following question raised on p5 of [7]:

> Do there exist p-adic analytic families of overconvergent eigenforms of finite slope parameterized by a punctured disc, and converging, at the puncture, to an overconvergent eigenform of infinite slope?

We answer this question in the negative for the 2-adic eigencurve of tame level 1. Another way of phrasing our result is that the map from the eigencurve to weight space satisfies the *valuative criterion of properness*, and it is in this sense that the phrase "proper" is used in the title, since the projection to weight space has infinite degree and so is not technically proper in the sense of rigid analytic geometry. One might perhaps say that this map is "functorially proper". Our approach is based on the following simple idea. One knows (for instance, from [1]) that finite slope eigenforms of integer weight may be

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analytically continued far into the supersingular regions of the moduli space. On the other hand, it turns out that eigenforms in the kernel of U do not extend as far. Now one can check that a limit of highly overconvergent eigenforms is also highly overconvergent, and this shows that given a punctured disc as above, the limiting eigenform cannot lie in the kernel of U.

The problem with this approach is that perhaps the most natural definition of "highly overconvergent" is not so easy to work with at non-integral weight. The problem stems from the fact that such forms of non-integral weight are not defined as sections of a line bundle. In fact Coleman's definition of an overconvergent form of weight κ is a formal q-expansion F for which F/E_κ is overconvergent of weight 0, where E_κ is the p-deprived weight κ Eisenstein series. One might then hope that the overconvergence of F/E_κ would be a good measure of the overconvergence of F. One difficulty is that if F is an eigenform for the Hecke operators, the form F/E_κ is unlikely to be an eigenform. This does not cause too much trouble when proving that finite slope eigenforms overconverge a long way, as one can twist the U-operator as explained in [5] and apply the usual techniques. We outline the argument in sections 2 and 3 of this paper. On the other hand we do not know how to prove general results about (the lack of) overconvergence of forms in the kernel of U in this generality. Things would be easier if we used $V(E_\kappa)$ to twist from weight κ to weight 0, but unfortunately the results we achieve using this twist are not strong enough for us to get the strict inequalities that we need.

The approach that we take in our "test case" of $N = 1$ and $p = 2$ is to control the kernel of U in weight κ by explicitly writing down the matrix of U (and of $2VU - \mathrm{Id}$) with respect to a carefully-chosen basis. To enable us to push the argument through, however, we were forced to diverge from Coleman's choice of twist. We define the overconvergence of F, not in terms of F/E_κ, but rather in terms of F/h^s for some explicit modular form h. The benefit of our choice of h is that it is nicely compatible with the explicit formulae developed in [3], and hence we may prove all our convergence results by hand in this case. Our proof that eigenforms of finite slope overconverge "as far as possible" is essentially standard. The main contribution of this paper is to analyze the overconvergence (or lack thereof) of eigenforms in the kernel of the U operator in this case.

One disadvantage of our approach is that the power series defining h^s only converges for s sufficiently small and hence our arguments only deal with forms whose weights lie in a certain disc at the centre of weight space. However, recently in [4], the 2-adic level 1 eigencurve was shown to be a disjoint union of copies of weight space near the boundary of weight space, and hence is automatically proper there.

2 DEFINITIONS

Let $\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \cdots$ denote the classical level 1 weight 12 modular form (where $q = e^{2\pi i\tau}$). Set

$$f = \Delta(2\tau)/\Delta(\tau) = q + 24q^2 + 300q^3 + 2624q^4 + \cdots,$$

a uniformizer for $X_0(2)$, and

$$h = \Delta(\tau)^2/\Delta(2\tau) = \prod_{n\geq 1}\left(\frac{1-q^n}{1+q^n}\right)^{24} = 1 - 48q + 1104q^2 - 16192q^3 + \ldots$$

a modular form of level 2 and weight 12. Note that the divisor of h is $3(0)$, where (0) denotes the zero cusp on $X_0(2)$, and hence that

$$h^{1/3} = \prod_{n\geq 1}\left(\frac{1-q^n}{1+q^n}\right)^{8}$$

is a classical modular form of weight 4 and level 2.

We briefly review the theory of overconvergent p-adic modular forms, and make it completely explicit in the setting we are interested in, namely $p=2$ and tame level 1. Let $\mathbb{C}_2$ denote the completion of an algebraic closure of $\mathbb{Q}_2$. Normalize the norm on $\mathbb{C}_2$ such that $|2| = 1/2$, and normalize the valuation $v : \mathbb{C}_2^\times \to \mathbb{Q}$ so that $v(2) = 1$. Choose a group-theoretic splitting of v sending 1 to 2, and let the resulting homomorphism $\mathbb{Q} \to \mathbb{C}_2^\times$ be denoted $t \mapsto 2^t$. Define $v(0) = +\infty$. Let $\mathcal{O}_2$ denote the elements of $\mathbb{C}_2$ with non-negative valuation.

If $r \in \mathbb{Q}$ with $0 < r < 2/3$ (note that $2/3 = p/(p+1)$ if $p = 2$) then there is a rigid space $X_0(1)_{\geq 2^{-r}}$ over $\mathbb{C}_2$ such that functions on this space are r-overconvergent 2-adic modular functions. Let $X[r]$ denote the rigid space $X_0(1)_{\geq 2^{-r}}$. By Proposition 1 of the appendix to [3], we see that $X[r]$ is simply the closed subdisc of the j-line defined by $|j| \geq 2^{-12r}$. We will also need to use (in Lemma 6.13) the rigid space $X[2/3]$, which we define as the closed subdisc of the j-line defined by $|j| \geq 2^{-8}$. The parameter q can be viewed as a rigid function defined in a neighbourhood of ∞ on $X[r]$, and hence any rigid function on $X[r]$ can be written as a power series in q; this is the q-expansion of the form in this rigid analytic setting. Moreover, it is well-known that the classical level 2 form f descends to a function on $X[r]$ (for any $r < 2/3$), with the same q-expansion as that given above.

For $0 < r < 2/3$, define $M_0[r]$ to be the space of rigid functions on $X[r]$, equipped with its supremum norm. Then $M_0[r]$ is a Banach space over $\mathbb{C}_2$ — it is the space of r-overconvergent modular forms of weight 0. An easy calculation using the remarks after Proposition 1 of the appendix to [3] shows that the set $\{1, 2^{12r}f, 2^{24r}f^2, \ldots, (2^{12r}f)^n, \ldots\}$ is an orthonormal Banach basis for $M_0[r]$, and we endow $M_0[r]$ once and for all with this basis.

We define $\mathcal{W}$ to be the open disc of centre 1 and radius 1 in the rigid affine line over $\mathbb{C}_2$. If $w \in \mathcal{W}(\mathbb{C}_2)$ then there is a unique continuous group homomorphism $\kappa : \mathbb{Z}_2^\times \to \mathbb{C}_2^\times$ such that $\kappa(-1) = 1$ and $\kappa(5) = w$; moreover this establishes a bijection between $\mathcal{W}(\mathbb{C}_2)$ and the set of even 2-adic weights, that is, continuous group homomorphisms $\kappa : \mathbb{Z}_2^\times \to \mathbb{C}_2^\times$ such that $\kappa(-1) = 1$. Note that if k is an even integer then the map $x \mapsto x^k$ is such a homomorphism, and we refer to this weight as weight k. Let $\tau : \mathbb{Z}_2^\times \to \mathbb{C}_2^\times$ denote the character with kernel equal to $1 + 4\mathbb{Z}_2$, and let $\langle\cdot\rangle$ denote the character $x \mapsto x/\tau(x)$; this character corresponds to $w = 5 \in \mathcal{W}(\mathbb{C}_2)$. If $t \in \mathbb{C}_2$ with $|t| < 2$ then we may define $5^t := \exp\bigl(t\log(5)\bigr) \in \mathcal{W}(\mathbb{C}_2)$ and we let $\langle\cdot\rangle^t$ denote the homomorphism $\mathbb{Z}_2^\times \to \mathbb{C}_2^\times$ corresponding to this point of weight space. One checks easily that the points of weight space corresponding to characters of this form are $\{w \in \mathcal{W}(\mathbb{C}_2) : |w - 1| < 1/2\}$.

We now explain the definitions of overconvergent modular forms of general weight that we shall use in this paper. Recall $h = \prod_{n\geq 1}(1 - q^n)^{24}/(1 + q^n)^{24}$. Define $h^{1/8}$ to be the formal q-expansion $\prod_{n\geq 1}((1 - q^n)^3/(1 + q^n)^3$. Now

$$(1 - q^n)/(1 + q^n) = 1 - 2q^n + 2q^{2n} - \cdots \in 1 + 2q\mathbb{Z}[[q]]$$

and hence $h^{1/8} \in 1 + 2q\mathbb{Z}[[q]]$. Write $h^{1/8} = 1 + 2qg$ with $g \in \mathbb{Z}[[q]]$. If S is a formal variable then we define $h^S \in 1 + 16qS\mathbb{Z}_2[[8S, q]]$ to be the formal binomial expansion of $(1 + 2qg)^{8S}$. If $s \in \mathbb{C}_2$ with $|s| < 8$ then we define h^s to be the specialization in $1 + 2q\mathcal{O}_2[[q]]$ of h^S at $S = s$. In fact for the main part of this paper we shall only be concerned with h^s when $|s| < 4$.

If $s \in \mathbb{C}_2$ with $|s| < 8$, then define $\mu(s) := \min\{v(s), 0\}$, so $-3 < \mu(s) \leq 0$. Define $\mathcal{X}$ to be the pairs (κ, r) (where $\kappa : \mathbb{Z}_2^\times \to \mathbb{C}_2^\times$ and $r \in \mathbb{Q}$) such that there exists $s \in \mathbb{C}_2$ with $|s| < 8$ satisfying

- $\kappa = \langle\cdot\rangle^{-12s}$, and
- $0 < r < 1/2 + \mu(s)/6$.

Note that the second inequality implies $r < 1/2$, and conversely if $|s| \leq 1$ and $0 < r < 1/2$ then $(\langle\cdot\rangle^{-12s}, r) \in \mathcal{X}$.

For $(\kappa, r) \in \mathcal{X}$, and only for these (κ, r), we define the space $M_\kappa[r]$ of r-overconvergent forms of weight κ thus. Write $\kappa = \langle\cdot\rangle^{-12s}$ and define $M_\kappa[r]$ to be the vector space of formal q-expansions $F \in \mathbb{C}_2[[q]]$ such that Fh^s is the q-expansion of an element of $M_0[r]$. We give $M_\kappa[r]$ the Banach space structure such that multiplication by h^s induces an isomorphism of Banach spaces $M_\kappa[r] \to M_0[r]$, and we endow $M_\kappa[r]$ once and for all with the orthonormal basis $\{h^{-s}, h^{-s}(2^{12r}f), h^{-s}(2^{12r}f)^2, \ldots\}$.

Remark 2.1. *We do not consider the question here as to whether, for all $(\kappa, r) \in \mathcal{X}$, the space $M_\kappa[r]$ is equal to the space of r-overconvergent modular forms of weight κ as defined by Coleman (who uses the weight κ Eisenstein series E_κ to pass from weight κ to weight 0). One could use the methods* *of proof of §5 of [4] to verify this; the issue is verifying whether $E_\kappa h^s$ is r-overconvergent and has no zeroes on $X[r]$. However, we do not need this result — we shall prove all the compactness results for the U operator that we need by explicit matrix computations, rather than invoking Coleman's results. Note however that our spaces clearly coincide with Coleman's if $\kappa = 0$, as the two definitions coincide in this case. Note also that for $r > 0$ sufficiently small (depending on $\kappa = \langle\cdot\rangle^{-12s}$ with $|s| < 8$), the definitions do coincide, because if $E_1 := 1 + 4q + 4q^2 + \cdots$ denotes the weight 1 level 4 Eisenstein series, then $h/E_1^{12} = 1 - 96q + \cdots$ is overconvergent of weight 0, has no zeroes on $X[r]$ for $r < 1/3$, and has q-expansion congruent to 1 mod 32. Hence for $r > 0$ sufficiently small, the supremum norm of $(h/E_1^{12}) - 1$ on $X[r]$ is t with $t < 1/2$ and $|s|t < 1/2$, and this is enough to ensure that the power series $(h/E_1^{12})^s$ is the q-expansion of a function on $X[r]$ with supremum norm at most 1. Hence instead of using powers of h to pass between weight κ and weight 0, we could use powers of E_1. Finally, Corollary B4.5.2 of [5] shows that if $\kappa = \langle\cdot\rangle^{-12s}$ then there exists $r > 0$ such that E_1^{-12s}/E_κ is r-overconvergent, which suffices.*

Recall that if X and Y are Banach spaces over a complete field K with orthonormal bases $\{e_0, e_1, e_2, \ldots\}$ and $\{f_0, f_1, f_2, \ldots\}$, then by the *matrix* of a continuous linear map $\alpha : X \to Y$ we mean the collection $(a_{ij})_{i,j\geq 0}$ of elements of K such that $\alpha(e_j) = \sum_{i\geq 0} a_{ij} f_i$. One checks that

- $\sup_{i,j} |a_{ij}| < \infty$, and
- for all j we have $\lim_{i\to\infty} |a_{ij}| = 0$,

and conversely that given any collection $(a_{ij})_{i,j\geq 0}$ of elements of K having these two properties, there is a unique continuous linear map $\alpha : X \to Y$ having matrix $(a_{ij})_{i,j\geq 0}$ (see Proposition 3 of [10] and the remarks following it for a proof). When we speak of "the matrix" associated to a continuous linear map between two spaces of overconvergent modular forms, we will mean the matrix associated to the map using the bases that we fixed earlier.

If R is a ring then we may define maps U, V and W on the ring $R[[q]]$ by

$$U\left(\sum a_n q^n\right) = \sum a_{2n} q^n,$$
$$V\left(\sum a_n q^n\right) = \sum a_n q^{2n},$$

and

$$W\left(\sum a_n q^n\right) = \sum (-1)^n a_n q^n.$$

Recall that $U(V(G)F) = GU(F)$ for F, G formal power series in q, and that $V : R[[q]] \to R[[q]]$ is a ring homomorphism. The operator W is not standard (or at least, our notation for it is not standard), but is also a ring homomorphism (it sends $f(q)$ to $f(-q)$) and one also checks easily that $W = 2VU - \mathrm{Id}$. We shall show later on that there are continuous linear maps between various spaces of overconvergent modular forms which correspond to U and W, and will write down explicit formulae for the matrices associated to these linear maps.

3 The U operator on overconvergent modular forms

Our goal in this section is to make precise the statement in the introduction that finite slope U-eigenforms overconverge a long way. Fix $r \in \mathbb{Q}$ with $0 < r < 1/2$. We will show that if $(\kappa, r) \in \mathcal{X}$ then the U-operator (defined on q-expansions) induces a continuous linear map $M_\kappa[r] \to M_\kappa[r]$, and we will compute the matrix of this linear map (with respect to our chosen basis of $M_\kappa[r]$). We will deduce that if $0 < \rho < r$ and F is ρ-overconvergent with $UF = \lambda F \neq 0$ then F is r-overconvergent. These results are essentially standard but we shall re-prove them, for two reasons: firstly to show that the arguments still go through with our choice of twist, and secondly to introduce a technique for computing matrices of Hecke operators in arbitrary weight that we shall use when analyzing the W operator later.
It is well-known that the U-operator induces a continuous linear map $U : M_0[r] \to M_0[r]$, and its associated matrix was computed in [3]. Now choose $m \in \mathbb{Z}_{\geq 0}$, and set $k = -12m$. One checks that $(k, r) \in \mathcal{X}$. If $\phi \in M_0[r]$ then

$$h^m U\left(h^{-m}\phi\right) = h^m U\left(\Delta(2\tau)^{-m} f^{2m}\phi\right)$$

$$= h^m \Delta(\tau)^{-m} U\left(f^{2m}\phi\right) = f^{-m} U\left(f^{2m}\phi\right).$$

A simple analysis of the q-expansion of $f^{-m}U(f^{2m}\phi)$ shows that it has no pole at the cusp of $X[r]$ and hence $f^{-m}U(f^{2m}\phi) \in M_0[r]$. We deduce that U induces a continuous map $M_k[r] \to M_k[r]$, and moreover that the matrix of this map (with respect to the basis fixed earlier) equals the matrix of the operator $U_k := f^{-m}Uf^{2m}$ acting on $M_0[r]$. We now compute this matrix.

Lemma 3.1. *For $m \in \mathbb{Z}_{\geq 0}$ and $k = -12m$ as above, and $j \in \mathbb{Z}_{\geq 0}$, we have*

$$U_k\left((2^{12r} f)^j\right) = \sum_{i=0}^{\infty} u_{ij}(m)(2^{12r} f)^i,$$

where $u_{ij}(m)$ is defined as follows: we have $u_{00}(0) = 1$, $u_{ij}(m) = 0$ if $2i - j < 0$ or $2j - i + 3m < 0$, and

$$u_{ij}(m) = \frac{3(i + j + 3m - 1)!(j + 2m)2^{8i-4j+12r(j-i)}}{2(2i - j)!(2j - i + 3m)!}$$

if $2i - j \geq 0$, $2j - i + 3m \geq 0$, and i, j, m are not all zero.

Proof. The case $m = 0$ of the lemma is Lemma 2 of [3], and the general case follows easily from the fact that $U_k = f^{-m}Uf^{2m}$. Note that in fact all the sums in question are finite, as $u_{ij}(m) = 0$ for $i > 2j + 3m$. □

Now for $i, j \in \mathbb{Z}_{\geq 0}$ define a polynomial $u_{ij}(S) \in \mathbb{C}_2[S]$ by $u_{ij}(S) = 0$ if $2i < j$, $u_{ij}(S) = 2^{12ir}$ if $2i = j$, and

$$u_{ij}(S) = \frac{3 \cdot 2^{12r(j-i)}(j + 2S)2^{8i-4j}}{2(2i - j)!} \prod_{\lambda=1}^{2i-j-1} (2j - i + \lambda + 3S)$$

if $2i > j$. One checks easily that evaluating $u_{ij}(S)$ at $S = m$ for $m \in \mathbb{Z}_{\geq 0}$ gives $u_{ij}(m)$, so there is no ambiguity in notation. Our goal now is to prove that for all $s \in \mathbb{C}_2$ such that $|s| < 8$ and $(\langle\cdot\rangle^{-12s}, r) \in \mathcal{X}$, the matrix $(u_{ij}(s))_{i,j\geq 0}$ is the matrix of the U-operator acting on $M_\kappa[r]$ for $\kappa = \langle\cdot\rangle^{-12s}$ (with respect to the basis of $M_\kappa[r]$ that we fixed earlier).

Say $s \in \mathbb{C}_2$ with $|s| < 8$, define $\kappa = \langle\cdot\rangle^{-12s}$, set $\mu = \min\{v(s), 0\}$, and say $0 < r < 1/2 + \mu/6$. Then $(\kappa, r) \in \mathcal{X}$. Note that $v(as + b) \geq \mu$ for any $a, b \in \mathbb{Z}$, and $3 + \mu - 6r > 0$.

Lemma 3.2. *(a) One has $v(u_{ij}(s)) \geq (3 + \mu - 6r)(2i - j) + 6rj$.*
(b) There is a continuous linear map $U(s) : M_0[r] \to M_0[r]$ with matrix $u_{ij}(s)$. Equivalently, there is a continuous linear map $U(s) : M_0[r] \to M_0[r]$ such that

$$U(s)\left((2^{12r}f)^j\right) = \sum_{i=0}^{\infty} u_{ij}(s)(2^{12r}f)^i.$$

Proof. (a) This is a trivial consequence of our explicit formula for $u_{ij}(s)$, the remark about $v(as + b)$ above, and the fact that $v(m!) \leq m - 1$ if $m \geq 1$ (see Lemma 6.2).

(b) Recall that $u_{ij}(s) = 0$ if $2i < j$. Hence by (a) we see that $|u_{ij}(s)| \leq 1$ for all i, j. It remains to check that for all j we have $\lim_{i\to\infty} v(u_{ij}(s)) = +\infty$ which is also clear from (a). □

Note that $U(s) = U_{-12s}$ if $s = m \in \mathbb{Z}_{\geq 0}$.

In fact the same argument gives slightly more. Choose $\epsilon \in \mathbb{Q}$ with $0 < \epsilon < \min\{r, 1/2 + \mu/6 - r\}$. Then $(\kappa, r + \epsilon) \in \mathcal{X}$.

Theorem 3.3. *The endomorphism $U(s)$ of $M_0[r]$ is the composite of a continuous map $M_0[r] \to M_0[r + \epsilon]$ and the restriction $M_0[r + \epsilon] \to M_0[r]$.*

Proof. Define $w_{ij}(s) = u_{ij}(s)/2^{12\epsilon i}$. By the previous lemma we have

$$v(w_{ij}(s)) \geq (2i - j)(3 + \mu - 6r - 6\epsilon) + 6j(r - \epsilon)$$

and $w_{ij}(s) = 0$ if $j > 2i$. In particular $v(w_{ij}(s)) \geq 0$ for all i, j, and moreover for all j we have $\lim_{i\to\infty} w_{ij}(s) = 0$. The continuous linear map $M_0[r] \to M_0[r+\epsilon]$ with matrix $(w_{ij}(s))_{i,j\geq 0}$ will hence do the job. □

As usual say $|s| < 8$, $\kappa = \langle\cdot\rangle^{-12s}$ and $(\kappa, r) \in \mathcal{X}$.

Corollary 3.4. *The map $U(s) : M_0[r] \to M_0[r]$ is compact and its characteristic power series is independent of r with $0 < r < 1/2 + \mu/6$. Furthermore if $0 < \rho < r$ then any non-zero $U(s)$-eigenform with non-zero eigenvalue on $M_0[\rho]$ extends to an element of $M_0[r]$.*

Proof. This follows via standard arguments from the theorem; see for example Proposition 4.3.2 of [7], although the argument dates back much further. □

Keep the notation: $|s| < 8$, $\kappa = \langle\cdot\rangle^{-12s}$, $\mu = \min\{v(s), 0\}$ and $0 < r < 1/2 + \mu/6$, so $(\kappa, r) \in \mathcal{X}$. We now twist $U(s)$ back to weight κ and show that the resulting compact operator is the U-operator (defined in the usual way on power series).

Proposition 3.5. *The compact endomorphism of $M_\kappa[r]$ defined by $\phi \mapsto h^{-s}U(s)(h^s\phi)$ is the U-operator, i.e., sends $\sum a_n q^n$ to $\sum a_{2n} q^n$.*

Proof. It suffices to check the proposition for $\phi = h^{-s}(2^{12r}f)^j$ for all $j \in \mathbb{Z}_{\geq 0}$, as the result then follows by linearity. If S is a formal variable then recall that we may think of h^S as an element of $1 + 16qS\mathcal{O}_2[[8S, q]])$ and in particular as an invertible element of $\mathcal{O}_2[[8S, q]]$. Write h^{-S} for its inverse. We may think of $(h^S)U(h^{-S}(2^{12r}f)^j)$ as an element of $\mathcal{O}_2[[8S, q]]$ (though not yet as an element of $M_0[r]$). Write

$$(h^S)U(h^{-S}(2^{12r}f)^j) = \sum_{i\geq 0} \tilde{u}_{ij}(S)(2^{12r}f)^i$$

with $\tilde{u}_{ij}(S) \in \mathcal{O}_2[[8S]] \otimes \mathbb{C}_2$ (this is clearly possible as $f = q + \ldots$). The proposition is just the statement that the power series $\tilde{u}_{ij}(S)$ equals the polynomial $u_{ij}(S)$. Now there exists some integer $N \gg 0$ such that both $2^N u_{ij}(S)$ and $2^N \tilde{u}_{ij}(S)$ lie in $\mathcal{O}_2[[8S]]$ (as $u_{ij}(S)$ is a polynomial). Furthermore, Lemma 3.1 shows that $u_{ij}(m) = \tilde{u}_{ij}(m)$ for all $m \in \mathbb{Z}_{\geq 0}$ and hence $2^N(u_{ij}(S) - \tilde{u}_{ij}(S))$ is an element of $\mathcal{O}_2[[8S]]$ with infinitely many zeroes in the disc $|8s| < 1$, so it is identically zero by the Weierstrass approximation theorem. □

Corollary 3.6. *If $(\kappa, r) \in \mathcal{X}$ and $\kappa = \langle\cdot\rangle^{-12s}$ then U is a compact operator on $M_\kappa[r]$ and its characteristic power series coincides with the characteristic power series of $U(s)$ on $M_0[r]$. Furthermore $F \in M_\kappa[r]$ is an eigenvector for U iff $Fh^s \in M_0[r]$ is an eigenvector for $U(s)$.*

Proof. Clear. □

The utility of these results is that they allow us to measure the overconvergence of a finite slope form F of transcendental weight by instead considering the associated form Fh^s in weight 0. This will be particularly useful to us later on in the case when F is in the kernel of U. We record explicitly what we have proved. By an overconvergent modular form of weight κ we mean an element of $\bigcup_r M_\kappa[r]$, where r runs through the $r \in \mathbb{Q}$ for which $(\kappa, r) \in \mathcal{X}$.

Corollary 3.7. *If $(\kappa, r) \in \mathcal{X}$ and f is an overconvergent modular form of weight κ which is an eigenform for U with non-zero eigenvalue, then f extends to an element of $M_\kappa[r]$.*

Proof. This follows from 3.4 and 3.5. □

In fact we will need a similar result for families of modular forms, but our methods generalize to this case. We explicitly state what we need.

COROLLARY 3.8. *Let $A \subseteq \mathcal{W}$ be an affinoid subdomain, say $0 < \rho < r < 1/2$, and assume that for all $\kappa \in A(\mathbb{C}_2)$ we have $(\kappa, r) \in \mathcal{X}$. Let $F \in \mathcal{O}(A)[[q]]$ be an analytic family of ρ-overconvergent modular forms, such that $UF = \lambda F$ for some $\lambda \in \mathcal{O}(A)^\times$. Then F is r-overconvergent.*

□

4 THE W OPERATOR ON OVERCONVERGENT MODULAR FORMS

We need to perform a similar analysis to the previous section with the operator W. Because $W = 2VU - \mathrm{Id}$ we know that W induces a continuous linear map $V : M_0[r] \to M_0[r]$ for $r < 1/3$ (for r in this range, U doubles and then V halves the radius of convergence). Our goal in this section is to show that, at least for $\kappa = \langle\cdot\rangle^{-12s}$ with $|s| < 8$, there is an operator on weight κ overconvergent modular forms which also acts on q-expansions in this manner, and to compute its matrix.

We proceed as in the previous section by firstly introducing a twist of W. If $m \in \mathbb{Z}_{\geq 0}$, if $k = -12m$ and if $\phi \in M_0[r]$ then the fact that $h(q)/h(-q) = (f(-q)/f(q))^2$ implies

$$h^m W(h^{-m}\phi) = f^{-2m} W(f^{2m}\phi)$$

and so we define the operator W_k on $M_0[r]$, $r < 1/3$, by $W_k := f^{-2m} W f^{2m} : M_0[r] \to M_0[r]$.

Set $g = Wf$, so $g(q) = f(-q) = -q + 24q^2 - 300q^3 + \ldots$. Because $g = 2VUf - f = 48Vf + 4096(Vf)^2 - f$, we see that the g can be regarded as a meromorphic function on $X_0(4)$ of degree at most 4. Similarly f may be regarded as a function on $X_0(4)$ of degree 2. Now the meromorphic function

$$(1 + 48f - 8192f^2 g)^2 - (1 + 16f)^2(1 + 64f)$$

on $X_0(4)$ has degree at most 16 but the first 1000 terms of its q-expansion can be checked to be zero on a computer, and hence this function is identically zero. We deduce the identity

$$g = \frac{1 + 48f - (1 + 16f)\sqrt{1 + 64f}}{8192f^2},$$

where the square root is the one of the form $1 + 32f + \ldots$, and one verifies using the binomial theorem that $g = \sum_{i \geq 1} c_i f^i$ with

$$\begin{aligned} c_i &:= (-1)^i 2^{4i-4} \left(\frac{(2i+2)!}{(i+1)!(i+2)!} - \frac{(2i)!}{i!(i+1)!} \right) \\ &= (-1)^i 2^{4(i-1)} \frac{3(2i)!}{(i-1)!(i+2)!} \end{aligned}$$

The other ingredient we need to compute the matrix of W_k is a combinatorial lemma.

Lemma 4.1. *If $j \geq 1$ and $i \geq j+1$ are integers then*

$$\sum_{a=j}^{i-1} \frac{3(2a+j-1)!j(2i-2a)!}{(a-j)!(a+2j)!(i-a-1)!(i-a+2)!} = \frac{(2i+j)!(j+1)}{(i-j-1)!(i+2j+2)!}.$$

Proof. Set $k = i-1-a$ and $n = i-1-j$ and then eliminate the variables i and a; the lemma then takes the form

$$\sum_{k=0}^{n} F(j,n,k) = G(j,n)$$

and, for fixed n and k, both $F(j,n,k)$ and $G(j,n)$ are rational functions of j. The lemma is now easily proved using Zeilberger's algorithm (regarding j as a free variable), which proves that the left hand side of the equation satisfies an explicit (rather cumbersome) recurrence relation of degree 1; however it is easily checked that the right hand side is a solution to this recurrence relation, and this argument reduces the proof of the lemma to the case $n = 0$, where it is easily checked by hand. □

We now compute the matrix of W_k on $M_0[r]$ for $r < 1/3$ and $k = -12m$, $m \in \mathbb{Z}_{\geq 0}$.

Lemma 4.2. *For $j \geq 0$ we have*

$$W_k\big((2^{12r}f)^j\big) = \sum_{i=0}^{\infty} \eta_{ij}(m)(2^{12r}f)^i,$$

where $\eta_{ij}(m)$ is defined as follows: we have $\eta_{ij}(m) = 0$ if $i < j$, $\eta_{ii}(m) = (-1)^i$, and for $i > j$ we define

$$\eta_{ij}(m) = \frac{(2i+j-1+6m)!3(j+2m)\cdot 2^{(4-12r)(i-j)}(-1)^i}{(i-j)!(i+2j+6m)!}.$$

Proof. We firstly deal with the case $m = 0$, by induction on j. The case $j = 0$ is easily checked as $\eta_{i0}(0) = 0$ for $i > 0$, and the case $j = 1$ follows from the fact that $c_i 2^{12r(1-i)} = \eta_{i1}(0)$ for $i \geq 1$, as is easily verified. For $j \geq 1$ we have $W(f^{j+1}) = f(-q)^{j+1} = g \cdot W(f^j) = (\sum_{t\geq 1} c_t f^t)W(f^j)$, and so to finish the $m = 0$ case it suffices to verify that for $j \geq 1$ and $i \geq j+1$ we have $\eta_{i\,j+1}(0) = 2^{12r}\sum_{a=0}^{i-1} c_{i-a}2^{-12r(i-a)}\eta_{aj}(0)$, which quickly reduces to the combinatorial lemma above.
Finally we note that because $\eta_{i+2m\,j+2m}(0) = \eta_{ij}(m)$, the general case follows easily from the case $m = 0$ and the fact that $W_k = f^{-2m}Wf^{2m}$. □

As before, we now define polynomials $\eta_{ij}(S)$ by $\eta_{ij}(S) = 0$ if $i < j$, $\eta_{ii}(S) = (-1)^i$, and

$$\eta_{ij}(S) = \frac{3(j+2S)2^{(4-12r)(i-j)}(-1)^i}{(i-j)!}\prod_{\lambda=1}^{i-j-1}(i+2j+\lambda+6S)$$

for $i > j$. We observe that $\eta_{ij}(S)$ specializes to $\eta_{ij}(m)$ when $S = m \in \mathbb{Z}_{\geq 0}$. Now if $|s| < 8$ and $\kappa = \langle\cdot\rangle^{-12s}$, and we set $\lambda = \min\{v(2s), 0\} > -2$, then we check easily that $v(\eta_{ij}(s)) \geq (3 - 12r + \lambda)(i - j) + 1$, so for $12r < 3 + \lambda$ we see that $(\eta_{ij}(s))_{i,j\geq 0}$ is the matrix of a continuous endomorphism $W(s)$ of $M_0[r]$. Moreover, arguments analogous to those of the previous section show that if furthermore $(\kappa, r) \in \mathcal{X}$ (so $M_\kappa[r]$ is defined), then the endomorphism of $M_\kappa[r]$ defined by sending ϕ to $h^{-s}W(s)(h^s\phi)$ equals the W operator as defined on q-expansions. Note that if $|s| \leq 4$ then $12r < 3 + \lambda$ implies $(\kappa, r) \in \mathcal{X}$.

5 Strategy of the proof.

We have proved in Corollary 3.7 that overconvergent modular forms f such that $Uf = \lambda f$ with $\lambda \neq 0$ overconverge "a long way". Using the W-operator introduced in the previous section we will now prove that overconvergent modular forms $f = q + \cdots$ such that $Uf = 0$ cannot overconverge as far. We introduce a definition and then record the precise statement.

Definition 5.1. *If $x \in \mathbb{C}_2$ then set $\beta = \beta(x) = \sup\{v(x - n) : n \in \mathbb{Z}_2\}$, allowing $\beta = +\infty$ if $x \in \mathbb{Z}_2$, and define $\nu = \nu(x)$ as follows: $\nu = \beta$ if $\beta \leq 0$, $\nu = \beta/2$ if $0 \leq \beta \leq 1$, and in general*

$$\nu = \sum_{k=1}^{n} 1/2^k + (\beta - n)/2^{n+1}$$

if $n \leq \beta \leq n + 1$. Finally define $\nu = 1$ if $\beta = +\infty$.

The meaning of the following purely elementary lemma will become apparent after the statement of Theorem 5.3.

Lemma 5.2. *Say $s \in \mathbb{C}_2$ with $|s| < 4$ and furthermore assume $2s \notin \mathbb{Z}_2^\times$. Then for all $s' \in \mathbb{C}_2$ with $|s - s'| \leq 1$, we have $0 < \frac{3+\nu(2s)}{12} < \frac{1}{2} + \frac{\mu(s')}{6}$.*

Proof. We have $\nu(2s) > -1$ and so certainly $\frac{3+\nu(s)}{12} > 0$. The other inequality can be verified on a case-by-case basis. We sketch the argument.
If $|s| > 2$ then $|s'| = |s| > 2$ and $\nu(2s) - 1 = v(s) = v(s') = \mu(s')$; the inequality now follows easily from the fact that $\mu(s') > -2$.
If $|s| \leq 2$ but $2s \notin \mathbb{Z}_2$ then $0 < \beta(2s) < \infty$ and $\nu(2s) < 1$; now $|s'| \leq 2$ and hence $\mu(s') \geq -1$, thus $\frac{3+\nu(2s)}{12} < \frac{1}{3} \leq \frac{1}{2} + \frac{\mu(s')}{6}$.
Finally if $2s \in \mathbb{Z}_2$ then we are assuming $2s \notin \mathbb{Z}_2^\times$ and hence $s \in \mathbb{Z}_2$ so $|s| \leq 1$ and hence $|s'| \leq 1$. Hence $\mu(s') = 0$ and we have $\frac{3+\nu(2s)}{12} = \frac{1}{3} < \frac{1}{2} + \frac{\mu(s')}{6}$. $\square$

Again say $|s| < 4$ and $2s \notin \mathbb{Z}_2^\times$. Write $\kappa = \langle\cdot\rangle^{-12s}$, and $\nu = \nu(2s)$. Let $G = q + \cdots$ be an overconvergent form of weight κ (by which we mean an element of $M_\kappa[\rho]$ for some $\rho \in \mathbb{Q}_{>0}$ sufficiently small). The theorem we prove in the next section (which is really the main contribution of this paper) is

Theorem 5.3. *If $G = q + \cdots$ satisfies $UG = 0$, then $F := h^s G \in M_0[\rho]$ does not extend to an element of $M_0[r]$ for $r = \frac{3+\nu}{12}$. Equivalently, $G \notin M_\kappa[r]$.*

Note that by Lemma 5.2 we have $(\kappa, r) \in \mathcal{X}$ so the theorem makes sense. Furthermore, by Corollary 3.7, overconvergent eigenforms of the form $q + \cdots$ in the kernel of U overconverge less than finite slope overconvergent eigenforms. Note also that if $2s \in \mathbb{Z}_2^\times$ then $\nu(2s) = 1$ and for κ, r as above we have $(\kappa, r) \notin \mathcal{X}$. We deal with this minor annoyance in the last section of this paper.

6 The Kernel of U

In this section we prove Theorem 5.3. We divide the argument up into several cases depending on the value of s. We suppose that $|s| < 4$ and $2s \notin \mathbb{Z}_2^\times$, and we set $\kappa = \langle \cdot \rangle^{-12s}$. Define $\nu = \nu(2s)$ as in the previous section, and set $r = \frac{3+\nu}{12}$. For simplicity we drop the s notation from $\eta_{ij}(s)$ and write

$$\begin{aligned}\eta_{ij} &= \frac{3(j+2s)2^{(4-12r)(i-j)}(-1)^i}{(i-j)!} \prod_{t=1}^{i-j-1} (i + 2j + t + 6s) \\ &= \frac{3(j+2s)2^{(1-\nu)(i-j)}(-1)^i}{(i-j)!} \prod_{t=1}^{i-j-1} (i + 2j + t + 6s).\end{aligned}$$

Say $G = q + \cdots$ as in Theorem 5.3 is ρ-overconvergent for some $0 < \rho < r$, so $F = h^s G \in M_0[\rho]$. If we expand F as

$$F = \sum_{j \geq 1} \tilde{a}_j (2^{12\rho} f)^j$$

then it follows that $\tilde{a}_1 \neq 0$. Recall also that $\tilde{a}_j \to 0$ as $j \to \infty$. On the other hand, $F = -W(s)F$, and so

$$\tilde{a}_i = -\sum_{j=1}^{\infty} \tilde{a}_j \tilde{\eta}_{i,j},$$

where $\tilde{\eta}_{ij}$ denotes the matrix of $W(s)$ on $M_0[\rho]$ (so $\eta_{ij} = \tilde{\eta}_{ij} 2^{12(r-\rho)(j-i)}$). We deduce from this that if we define $a_i = \tilde{a}_i 2^{12(\rho - r)i}$ then $F = \sum_{j \geq 1} a_j (2^{12r} f)^j$ and

$$a_i = -\sum_{j \geq 1} a_j \eta_{ij}.$$

Note in particular that the sum converges even if $W(s)$ does not extend to a continuous endomorphism of $M_0[r]$ or if F does not extend to an element of $M_0[r]$. In fact our goal is to show that the a_i do not tend to zero, and in particular that F does not extend to an element of $M_0[r]$.

LEMMA 6.1. *Suppose F is as above. Suppose also that there exist constants c_1 and $c_3 \in \mathbb{R}$, an infinite set I of positive integers, and for each $i \in I$ constants $N(i)$ and $c_2(i)$ tending to infinity as $i \to \infty$ and such that*

(i) $v(\eta_{i1}) \leq c_1$, *for all* $i \in I$.

(ii) $v(\eta_{ij}) \geq c_2(i)$ *for all* $i \in I$ *and* $2 \leq j \leq N(i)$.

(iii) $v(\eta_{ij}) \geq c_3$ *for all* $i \in I$ *and* $j \in \mathbb{Z}_{\geq 0}$.

Then the a_i do not tend to zero as $i \to \infty$, and hence F does not extend to a function on $M_0[r]$.

Proof. Assume $a_i \to 0$. Recall that we assume $a_1 \neq 0$. By throwing away the first few terms of I if necessary, we may then assume that for all $i \in I$ we have

(1) $c_2(i) > v(a_1) + c_1 - \min\{v(a_j) : j \geq 1\}$, and

(2) $\min\{v(a_j) : j > N(i)\} > v(a_1) + c_1 - c_3$.

We now claim that for all $i \in I$ we have $v(a_1\eta_{i1}) < v(a_j\eta_{ij})$ for all $j > 1$. The reason is that if $j \leq N(i)$ the inequality follows from equation (1) above, and if $j > N(i)$ it follows from (2). Now from the equality

$$a_i = -\sum_{j=1}^{\infty} a_j\eta_{ij}$$

we deduce that $v(a_i) = v(a_1\eta_{i1})$ is bounded for all $i \in I$, contradicting the fact that $a_i \to 0$. $\square$

The rest of this section is devoted to establishing these inequalities for suitable I and r. We start with some preliminary lemmas.

LEMMA 6.2. *1. If $m \geq 1$ then $v(m!) \leq m - 1$, with equality if and only if m is a power of 2.*

2. If $m \geq 0$ then $v(m!) \geq (m-1)/2$, with equality if and only if $m = 1, 3$.

3. If $n \geq 0$ and $0 \leq m < 2^n$ then setting $t = 2^n - m$ we have $m - v(m!) \geq n - (t/2)$.

Proof. 1 and 2 follow easily from

$$v(m!) = \lfloor m/2 \rfloor + \lfloor m/4 \rfloor + \lfloor m/8 \rfloor + \dots.$$

For 3, we have $m!(m+1)(m+2)\dots(2^n-1)(2^n) = (2^n)!$ and for $0 < d < 2^n$ we have $v(d) = v(2^n - d)$, so $v((m+1)(m+2)\dots(2^n-1)) = v((t-1)!) \geq (t-2)/2$ by 2. Finally $v((2^n)!) = 2^n - 1$ by 1. Hence $v(m!) \leq 2^n - 1 - n - (t-2)/2 = 2^n - n - (t/2)$ and so $m - v(m!) \geq 2^n - t - (2^n - n - (t/2)) = n - (t/2)$. $\square$

Lemma 6.3. *Let $m \in \mathbb{Z}$ be arbitrary and set $\beta = \beta(x)$ and $\nu = \nu(x)$ as in Definition 5.1.*

1. *If $\beta \leq 0$ then $v(x+n) = \nu$ for all $n \in \mathbb{Z}$, hence the valuation of $\prod_{t=1}^{N}(x+m+t)$ is $N\nu$.*
2. *If $0 < \beta < \infty$ and if N is a power of 2 with $N \geq 2^{\lceil \beta \rceil}$ then the valuation of*
$$\prod_{t=1}^{N}(x+m+t)$$
is exactly $N\nu$.
3. *If $0 < \beta < \infty$ and if $N \geq 0$ is an arbitrary integer then the valuation of*
$$\prod_{t=1}^{N}(x+m+t)$$
is v, where $|v - N\nu| < \beta$.
4. *If $\beta = \infty$ and if $N \geq 0$ is an arbitrary integer then the valuation of $\prod_{t=1}^{N}(x+m+t)$ is at least $v(N!)$.*

Proof. (1) is obvious and (2) is easy to check (note that $v(x+n)$ is periodic with period $2^{\lceil \beta \rceil}$). For part (3), say $n = \lfloor \beta \rfloor$. Now about half of the terms in this product are divisible by 2, about a quarter are divisible by 4, and so on. More precisely, this means that the largest possible power of 2 that can divide this product is

$$\begin{aligned}
&\lceil N/2 \rceil + \lceil N/4 \rceil + \ldots + \lceil N/2^n \rceil + (\beta - n)\lceil N/2^{n+1} \rceil \\
<&(N/2+1) + (N/4+1) + \ldots + (N/2^n+1) + (\beta - n)(N/2^{n+1}+1) \\
=&N\nu + \beta.
\end{aligned}$$

A similar argument shows that the lowest possible power of 2 dividing this product is strictly greater than $N\nu - \beta$.
For part (4), if $\beta = \infty$ then $x \in \mathbb{Z}_2$ and by a continuity argument it suffices to prove the result for x a large positive integer, where it is immediate because the binomial coefficient $\binom{x+m+N}{N}$ is an integer.

□

Now set $x = 2s$ and let $\beta = \beta(2s), \nu = \nu(2s)$. Note that if $\beta \leq 0$ then $\mu = \beta - 1$, and if $\beta \geq 1$ then $\mu = 0$.
Recall $\eta_{ij} = 0$ if $i < j$, $\eta_{ii} = (-1)^i$, and if $i > j$ we have

$$\eta_{ij} = \frac{3(j+2s)2^{(1-\nu)(i-j)}(-1)^i}{(i-j)!} \prod_{t=1}^{i-j-1}(i+2j+t+6s).$$

In particular, for $i > j$ we have

$$(*) \quad v(\eta_{i,j}) = (1-\nu)(i-j) - v((i-j)!) + v(j+2s) + v\left(\prod_{t=1}^{i-j-1}(i+2j+t+6s)\right).$$

We shall continually refer to $(*)$ in what follows.

Proposition 6.4. *Say $\beta \leq 0$ (and hence $\nu = \beta$).*

1. *If $j \geq i$ then $v(\eta_{ij}) \geq 0$, and if $j < i$ then $v(\eta_{ij}) = i - j - v((i-j)!) \geq 1$.*
2. *If $i = 2^n + 1$ then $v(\eta_{i1}) = 1$ and if $1 < j < i$ then $v(\eta_{ij}) \geq n - (j-1)/2$.*

Proof. 1 is immediate from $(*)$ and Lemma 6.3(1). Now 2 can be deduced from 1, using part 1 of Lemma 6.2 for the first part and part 3 of Lemma 6.2 for the second. □

We now prove:

Lemma 6.5. *Theorem 5.3 is true if $-1 < \beta \leq 0$ (i.e., if $2 \leq |s| < 4$).*

Equivalently, if $2 \leq |s| < 4$ and $\kappa = \langle\cdot\rangle^{-12s}$, and if $G = q + \ldots$ is a non-zero weight κ overconvergent form in $\ker(U)$, then $F = h^s G$ does not converge as far as $M_0[1/4 + \nu/12]$, where $\nu = \nu(2s)$ as above.

Proof. This will be a direct application of lemma 6.1. We set $I = \{2^n + 1 : n \in \mathbb{Z}_{>0}\}$, and if $i = 2^n + 1$ we define $c_2(i) = (n+1)/2$ and $N(i) = n$. We set $c_1 = 1$ and $c_3 = 0$. Now assumptions (i) and (ii) of Lemma 6.1 follow from Proposition 6.4(2), and (iii) follows from Proposition 6.4(1). □

Let us now consider the case when $0 < \beta < \infty$.

Proposition 6.6. *Let $0 < \beta < \infty$.*

1. *If $j < i$ then $v(\eta_{i,j}) - \big((i-j) - v((i-j)!) - \nu\big) \in [-\beta, 2\beta]$.*
2. *If $j < i$ then*
$$v(\eta_{ij}) \geq 1 - \beta - \nu.$$
If $i = 2^n + 1$ then
$$v(\eta_{i1}) \leq 2\beta - \nu + 1$$
and if $1 < j < i$ then
$$v(\eta_{ij}) \geq n - (j+1)/2 - \nu - \beta.$$

Proof. From the definition of β, the valuation of $j + 2s$ lies in $[0, \beta]$. The result then follows from $(*)$ and lemma 6.3, part 3. Part 2 follows from part 1 and Lemma 6.2, parts (1) and (3), applied to $(i-j)!$. □

LEMMA 6.7. *Theorem 5.3 is true if* $0 < \beta < \infty$, *that is, if* $|s| \leq 2$ *and* $2s \notin \mathbb{Z}_2$.

Proof. Again this is an application of lemma 6.1. Set $I = \{2^n + 1 : n \in \mathbb{Z}_{>0}\}$, $c_1 = 2\beta - \nu + 1$, $c_3 = \min\{0, 1 - \beta - \nu\}$, and if $i = 2^n + 1$ then set $N(i) = n$ and $c_2(i) = (n+1)/2 - \nu - \beta$. Conditions (i)–(iii) of Lemma 6.1 hold by Proposition 6.6(2). □

The only cases of Theorem 5.3 left to deal with are those with $\beta = +\infty$, that is, $2s \in \mathbb{Z}_2$. Because the theorem does not deal with the case $2s \in \mathbb{Z}_2^\times$ we may assume from now on that $2s \in 2\mathbb{Z}_2$, so $s \in \mathbb{Z}_2$. We next deal with the case $s \in \mathbb{Z}_2$ and $6s \notin \mathbb{N}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the positive integers. In this case, we shall again use Lemma 6.1 with i of the form $i = 2^n + 1$. However, it will turn out that only certain (although infinitely many) n will be suitable.
Since we assume $s \in \mathbb{Z}_2$ we have $\beta = +\infty$, so $\nu = 1$ and hence

$$(**) \qquad \eta_{ij} = \frac{3(j+2s)(-1)^i}{(i-j)!} \prod_{t=1}^{i-j-1} (i + 2j + t + 6s).$$

Let $u \in \mathbb{Z}_2$. Define functions $f_n(u)$ as follows:

$$f_n(u) = (2^n + u)(2^n + u + 1) \cdots (2^{n+1} - 1 + u) = \prod_{\tau=0}^{2^n - 1} (2^n + u + \tau).$$

LEMMA 6.8. *For any* $u \in \mathbb{Z}_2$ *there exist infinitely many values of* n *for which*

$$v_n(f(u)) = v((2^n)!) \text{ or } v((2^n)!) + 1.$$

Proof. For each n, define an integer $0 < u_n \leq 2^n$ by setting $u \equiv u_n \mod 2^n$. If $0 \leq \tau \leq 2^n - 1$ and $\tau \neq 2^n - u_n$, then

$$v(2^n + u + \tau) = v(u_n + \tau).$$

Since τ takes on every equivalence class modulo 2^n, It follows from the definition of f_n that

$$v(f_n(u)) = v((2^n - 1)!) + v(2^{n+1} + u - u_n).$$

If $u \not\equiv u_n \mod 2^{n+1}$ then $v(2^{n+1} + u - u_n) = v(2^n)$ and $v(f_n(u)) = v((2^n)!)$. There are infinitely many n satisfying this condition unless $u \equiv u_n \mod 2^{n+1}$ for all sufficiently large n. Yet this implies $u_n = u_{n+1}$ for all sufficiently large n, and subsequently that $u = u_n$. In this case we have $v(2^{n+1} + u - u_n) = v(2^{n+1})$, and $v(f_n(u)) = v((2^n)!) + 1$. □

COROLLARY 6.9. *There are infinitely many* n *such that if* $i = 2^n + 1$ *then* $v(\eta_{i1}) \in \{0, 1\}$.

Proof. Let $i = 2^n + 1$ and $j = 1$, and assume $n \geq 1$. By $(**)$ we have

$$\eta_{i1} = \frac{3(1+2s)(-1)}{(2^n)!} \prod_{t=1}^{2^n-1} (2^n + 3 + t + 6s).$$

Let $u = 6s + 4 \in 2\mathbb{Z}_2$ and set $\tau = t - 1$. Then

$$\eta_{i1} = \frac{(1-u)}{(2^n)!} \prod_{\tau=0}^{2^n-2} (2^n + u + \tau) = \frac{f_n(u)}{(2^n)!} \cdot \frac{1-u}{u - 1 + 2^{n+1}}$$

and the result follows from Lemma 6.8 and the fact that $u \in 2\mathbb{Z}_2$. $\square$

Let us now turn to estimating η_{ij} for general i, j.

Lemma 6.10. *If $i, j \in \mathbb{Z}_{\geq 0}$ then $v(\eta_{ij}) \geq 0$.*

Proof. By continuity, it suffices to verify the result for $6s$ a large positive even integer. It is clear if $i \leq j$ so assume $i > j$. Now because the product of N successive integers is divisible by $N!$ we see (putting one extra term into the product) that both $x_1 := \frac{i+2j+6s}{3(j+2s)}\eta_{ij}$ and $x_2 := \frac{2i+j+6s}{3(j+2s)}\eta_{ij}$ are integers. The result now follows as $\eta_{ij} = 2x_1 - x_2$. $\square$

Set $I_0 = \{i = 2^n + 1 : v(\eta_{i1}) \in \{0, 1\}\}$. Then I_0 is infinite by Corollary 6.9. We will ultimately let I be a subset of I_0. We must analyze η_{ij} for $i \in I_0$ and $1 < j$ small. Note that if $i = 2^n + 1$ and $j \geq 2$, then

$$\frac{\eta_{i,j}}{\eta_{i,1}} = 2^n \cdot \frac{(j+2s)}{(1+2s)} \cdot \prod_{t=1}^{j-2} (i - j + t) \cdot \frac{\prod_{t=1}^{j-1}(2i + t + 6s)}{\prod_{t=1}^{2j-2}(i + 2 + t + 6s)}$$

Since $6s \notin -\mathbb{N}$, $3 + 6s + t \neq 0$. Thus for any N there exists n_0 depending on N such that for all $n \geq n_0$ we have $v(i + 2 + 6s + t) = v(3 + 6s + t)$ for all $t \leq 2N - 2$. In particular, for fixed N and sufficiently large n (with $i = 2^n + 1$),

$$v(\eta_{ij}) \geq n - v\left(\prod_{t=0}^{2j-2} (3 + 6s + t)\right) + v(\eta_{i1}).$$

Lemma 6.11. *For any constants $c_2 \in \mathbb{R}$ and $N \in \mathbb{Z}_{\geq 1}$, there exists $n_1 = n_1(c_2, N)$ such that for all $n \geq n_1$ such that $i = 2^n + 1 \in I_0$, we have $v(\eta_{ij}) \geq c_2$ for $2 \leq j \leq N$.*

Proof. Set $M = v(\prod_{t=0}^{2N-2}(3+6s+t))$ and choose n_1 such that $n_1 - M \geq c_2$. $\square$

We may now prove:

Lemma 6.12. *Theorem 5.3 is true if $s \in \mathbb{Z}_2$ and $6s \notin -\mathbb{N}$.*

Proof. We apply lemma 6.1 as follows. Set $c_1 = 1$ and $c_3 = 0$. We build I as follows. As m runs through the positive integers, set $N = c_2 = m$, define n_1 as in Lemma 6.11, choose $n \geq n_1$ such that $i := 2^n + 1 \in I_0$ and such that i is not yet in I; now add i to I and define $N(i) = c_2(i) = t$. The conditions of lemma 6.1 are then satisfied. □

The final case in our proof of Theorem 5.3 is the case $6s \in -2\mathbb{N}$, which corresponds to weight $k = -12s \in 4\mathbb{N}$. We shall not use Lemma 6.1 in this case, but give a direct argument.

Because our level structure is so small it is convenient to temporarily augment it to get around representability issues. Choose some auxiliary odd integer N and consider the compact modular curve Y over $\mathbb{Q}_2$ whose cuspidal points parameterize elliptic curves with a subgroup of order 2 and a full level N structure (note that this curve is not in general connected). There is a sheaf ω on Y, and classical modular forms of weight k and level 2 are, by definition, $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$-invariant global sections of $\omega^{\otimes k}$ on Y.

For $0 < r \leq 2/3$ let $Y[r]$ denote the pre-image of $X[r]$ via the forgetful functor. Recall that there is a compact operator U on $H^0(Y[r], \omega^{\otimes k})$ for $r < 2/3$ and $k \in \mathbb{Z}$.

Lemma 6.13. *If $k \in \mathbb{Z}$ and $f \in H^0(Y[1/3], \omega^{\otimes k})$ is in the kernel of U, then $f = 0$.*

Remark 6.14. *The lemma is not special to $p = 2$; the proof shows that non-zero p-adic modular forms in the kernel of U are never $1/(p+1)$-overconvergent.*

Proof. Say $f \in H^0(Y[1/3], \omega^{\otimes k})$ is arbitrary. If E is an elliptic curve over a finite extension of $\mathbb{Q}_2$, equipped with with a subgroup C of order 2 and a full level N structure L, and such that the corresponding point $(E, C, L) \in Y$ is in $Y[1/3]$, then one can regard $f(E, C, L)$ as an element of $H^0(E, \Omega^1)^{\otimes k}$. Now define $g \in H^0(Y[2/3], \omega^{\otimes k})$ by

$$g(E, L) = \sum_{D \neq C} (\mathrm{pr})^* f(E/D, \overline{C}, \overline{L}),$$

where the sum is over the subgroups $D \neq C$ of E of order 2, pr denotes the projection $E \to E/D$, and a bar over a level structure denotes its natural push-forward. An easy calculation using Tate curves (see for example Proposition 5.1 of [1]) shows that $g = 2Uf$, and hence if $Uf = 0$ then $g = 0$. In particular if E is an elliptic curve with no canonical subgroup and we fix a full level N structure L on E, then then $(E, C, L) \in Y[2/3]$ for all C, and $g(E, C, L) = 0$ for all C implies that $\sum_{D \neq C} (\mathrm{pr})^* f(E/D, E[2]/D, \overline{L}) = 0$ for all C. Summing, one deduces that $\sum_D (\mathrm{pr})^* f(E/D, E[2]/D, \overline{L}) = 0$ and hence that $f(E/D, E[2]/D, \overline{L}) = 0$ for all D of order 2. This implies that f is identically zero on the "boundary" of $Y[1/3]$ and hence that f is identically zero. □

We deduce

LEMMA 6.15. *Theorem 5.3 is true for* $6s \in -2\mathbb{N}$.

Proof. If $G \in M_k[1/3]$ then $G = h^{k/12}F$ and, because $k = -12s \in 4\mathbb{N}$, we know that $h^{k/12}$ is a classical modular form of level 2 and hence an element of $H^0(Y[1/3], \omega^{\otimes(k/12)})$. Thus the preceding lemma applies to G and we conclude that $G = 0$. $\square$

Theorem 5.3 now follows from Lemmas 6.5, 6.7, 6.12 and 6.15.

7 THERE ARE NOT TOO MANY HOLES IN THE EIGENCURVE.

We begin with a simple rigid-analytic lemma that forms the basis to our approach. Let X be a connected affinoid variety, and let V be a non-empty admissible open affinoid subdomain of X. Let $B = \mathrm{Sp}(\mathbb{C}_2\langle T\rangle)$ denote the closed unit disc, and let $A = \mathrm{Sp}(\mathbb{C}_2\langle T, T^{-1}\rangle)$ denote its "boundary", the closed annulus with inner and outer radii both 1.

LEMMA 7.1. *If f is a function on $V \times B$ and the restriction of f to $V \times A$ extends to a function on $X \times A$, then f extends to a function on $X \times B$.*

Proof. We have an inclusion $\mathcal{O}(X) \subseteq \mathcal{O}(V)$, as X is connected, and we know $f \in \mathcal{O}(V)\langle T\rangle$ and $f \in \mathcal{O}(X)\langle T, T^{-1}\rangle$. But the intersection of these two rings is $\mathcal{O}(V)\langle T\rangle$. $\square$

Let $\mathcal{E}$ denote the 2-adic eigencurve of tame level 1, and let $\mathcal{W}$ denote 2-adic weight space. We recall that because 2 is a regular prime, $\mathcal{E}$ is a disjoint union $\mathcal{E}^{\mathrm{Eis}} \coprod \mathcal{E}^{\mathrm{cusp}}$, and the natural map from the Eisenstein component $\mathcal{E}^{\mathrm{Eis}}$ to weight space is an isomorphism. One can also check from the definition of the eigencurve in [2] that the cuspidal component $\mathcal{E}^{\mathrm{cusp}}$ of E represents the functor on rigid spaces over $\mathcal{W}$ sending a rigid space $Y \to \mathcal{W}$ to the set of normalized overconvergent finite slope cuspidal eigenforms of "weight Y", that is, formal power series $\sum a_n q^n \in \mathcal{O}(Y)[[q]]$ with $a_1 = 1$ and a_2 a unit, which are eigenforms for all the Hecke operators and, when divided by the pullback of the Eisenstein family to Y, become overconvergent functions on $Y \times X[0]$. Let B denote the closed unit disc and let $B^\times$ denote B with the origin removed. Suppose we have a map $\phi : B^\times \to \mathcal{E}$ such that the induced map $B^\times \to \mathcal{W}$ extends (necessarily uniquely) to a map $B \to \mathcal{W}$. Let $\kappa_0 \in \mathcal{W}(\mathbb{C}_2)$ denote the image $0 \in B(\mathbb{C}_2)$ under this map. The theorem we prove in this section is

THEOREM 7.2. *If $\kappa_0 \notin \{\langle\cdot\rangle^{-12s} : 2s \in \mathbb{Z}_2^\times\}$ then the map $\phi : B^\times \to \mathcal{E}$ extends to a map $B \to \mathcal{E}$.*

Proof. If the image of ϕ is contained in $\mathcal{E}^{\mathrm{Eis}}$ then the theorem is automatic, because the projection $\mathcal{E}^{\mathrm{Eis}} \to \mathcal{W}$ is an isomorphism. Hence we may assume that $\phi : B^\times \to \mathcal{E}^{\mathrm{cusp}}$. If $|\kappa_0(5) - 1| > 1/8$ then we are finished by the main theorem of [4]. Assume from now on that $|\kappa_0(5) - 1| \leq 1/8$. Then the map ϕ corresponds to a family $\sum a_n q^n$ of overconvergent eigenforms over $B^\times$. Furthermore, the supremum norm of each a_n is at most 1 (because Hecke operators on overconvergent p-adic modular forms have eigenvalues with norm at most 1) and, analogous to the analysis of isolated singularities of holomorphic functions, one checks easily that this is enough to ensure that each a_n extends to a function on B. Our task is to analyze the "limiting" power series $\sum a_n(0)q^n$.

More precisely, we now have a formal power series $\sum_{n\geq 1} a_n q^n$ in $\mathcal{O}(B)[[q]]$. To prove the theorem we must check that this formal power series is a finite slope overconvergent form of weight B. We are assuming $|\kappa_0(5)-1|\leq 1/8$ and hence $\kappa_0=\langle\cdot\rangle^{-12s}$ with $|s|<4$. Now assume also that $2s\notin\mathbb{Z}_2^\times$. Set $r=\frac{3+\nu(2s)}{12}$. After shrinking B if necessary, we may assume that for all $b\in B$ we have $\kappa_b=\langle\cdot\rangle^{-12s'}$ with $|s-s'|\leq 1$. By Lemma 5.2 we have $(\kappa_b,r)\in\mathcal{X}$ for all $b\in B$, and by Corollary 3.8 we see that on the boundary of B our function $\sum a_n q^n$ is r-overconvergent, it being a finite slope eigenform for U here. Moreover, the coefficients a_n are all bounded by 1 on all of B. Now applying Lemma 7.1 with $X=X[r]$ and V a small disc near infinity such such that q (the q-expansion parameter) is a well-defined function on V, we deduce that $\sum a_n q^n$ is r-overconvergent on all of B.

All that we need to show now is that $a_2\in\mathcal{O}(B)^\times$. It suffices to prove that $a_2(0)\neq 0$, as we know that $a_2(b)\neq 0$ for all $0\neq b\in B$. But $\sum a_n(0)q^n=q+\ldots$ is an r-overconvergent form of weight κ_0, so by Theorem 5.3 (note that this is where all the work is) we deduce $a_2(0)\neq 0$. Hence $a_2\in\mathcal{O}(B)^\times$ and $\sum a_n q^n$ is an overconvergent cuspidal finite slope eigenform of weight B, which induces the map $B\to\mathcal{E}^{\mathrm{cusp}}$ which we seek. $\square$

8 There are no holes in the eigencurve

In the previous section we showed that if there are any holes in the eigencurve, then they lie above weights of the form $\{\langle\cdot\rangle^{-12s}:2s\in\mathbb{Z}_2^\times\}$. To show that in fact there are no holes in the eigencurve, we redo our entire argument with a second, even more non-standard, twist and show that using this twist we may deduce that the only holes in the eigencurve lie above the set $\{\langle\cdot\rangle^{2-12s}:2s\in\mathbb{Z}_2^\times\}$. Because there is no $s\in\frac{1}{2}\mathbb{Z}_2^\times$ such that $\frac{12s-2}{12}\in\frac{1}{2}\mathbb{Z}_2^\times$ this finishes the argument. We sketch the details.

Let $E_2=1+24q+24q^2+96q^3+\ldots$ denote the holomorphic Eisenstein series of weight 2 and level $\Gamma_0(2)$. We define $\mathcal{X}'=\{(\kappa\langle\cdot\rangle^2,r):(\kappa,r)\in\mathcal{X}\}$. If $|s|<8$ then set $\kappa'=\langle\cdot\rangle^{2-12s}$. If r is such that $(\kappa',r)\in\mathcal{X}'$, we define $M'_{\kappa'}[r]$ to be the vector space of formal q-expansions $F\in\mathbb{C}_2[[q]]$ such that Fh^s/E_2 is the q-expansion of an element of $M_0[r]$. For $r>0$ sufficiently small this definition is easily checked to coincide with the usual definition. We shall be using this definition with r quite large and again we neglect to verify whether the two definitions coincide in the generality in which we use them. We give $M'_{\kappa'}[r]$ the Banach space structure such that multiplication by h^s/E_2 is an isometric isomorphism $M'_{\kappa'}[r]\to M_0[r]$, and endow $M'_{\kappa'}[r]$ once and for all with the orthonormal basis $\{E_2h^{-s},E_2h^{-s}(2^{12r}f),E_2h^{-s}(2^{12r}f)^2,\ldots\}$. Note that the

reason that this definition gives us more than our original definition of $M_\kappa[r]$ is that if k is an even integer with $2||k$ then $(k,1/3)\notin\mathcal{X}$ but $(k,1/2-\epsilon)\in\mathcal{X}'$, so we can "overconverge further" for such weights.
If $\theta=q(d/dq)$ is the operator on formal q-expansions, then one checks that $U\theta=2\theta U$. Moreover, it is well-known that $\theta f=fE_2$ and hence $\theta f^j=jf^jE_2$ for any $j\geq 0$. Hence our formulae for the coefficients of U acting on $M_0[r]$ will give rise to formulae for the coefficients of U acting on $M'_2[r]$, which was the starting point for the arguments in section 3. We give some of the details of how the arguments should be modified. If $m\in\mathbb{Z}_{\geq 0}$ and $k'=2-12m$ then we define a continuous operator $U'_{k'}$ on $M_0[r]$ by $U'_{k'}(\phi)=E_2^{-1}h^mU(E_2h^{-m}\phi)$. One checks that this is indeed a continuous operator by verifying that it has a basis $(u'_{ij}(m))_{i,j\geq 0}$ defined by $u'_{ij}(m)=0$ for $2i<j$ or $2j-i+3m<0$, $u'_{00}(0)=1$, and

$$u'_{ij}(m)=\frac{3(i+j+3m-1)!(i+m)2^{8i-4j+12r(j-i)}}{(2i-j)!(2j-i+3m)!}$$

otherwise. One checks that for i,j fixed there is a polynomial $u'_{ij}(S)$ interpolating $u'_{ij}(m)$ and that for $|s|<8$ with $\mu=\min\{v(s),0\}$ we have $v(u'_{ij}(s))\geq(\mu+3-6r)(2i-j)+6rj$ as before. Hence for $|s|<8$, $\kappa'=\langle\cdot\rangle^{2-12s}$ and $r\in\mathbb{Q}$ such that $(\kappa',r)\in\mathcal{X}'$, the matrix $(u'_{ij}(s))_{i,j\geq 0}$ defines a compact operator $U'(s)$ on $M_0[r]$. Furthermore we have $U'(s)(\phi)=E_2^{-1}h^sU(E_2h^{-s}\phi)$, and in particular $U:M'_{\kappa'}[r]\to M'_{\kappa'}[r]$ is well-defined and compact. Moreover, $U'(s)$ increases overconvergence and any eigenvector for $U'(s)$ on $M_0[r]$ with non-zero eigenvalue extends to $M_0[r']$ for any r' such that $0<r'<1/2+\mu(s)/6$. Finally, these arguments also work for families of modular forms and the analogue of Corollary 3.8 remains true in this setting.
Similar arguments work in section 4. One checks that $2V\theta=\theta V$ and hence $VU\theta=2V\theta U=\theta VU$. Hence θ commutes with W and one now deduces from our explicit formulae for W in weight $-12m$ that in weight $2-12m$ the matrix for W is given by $W_k=[\eta'_{ij}]$, where:

$$\eta'_{ij}=\frac{(2i+j-1+6m)!3(i+2m)\cdot 2^{(4-12r)(i-j)}(-1)^i}{(i-j)!(i+2j+6m)!}.$$

We remark that the only difference in this formula is that $(j+2m)$ has been replaced by $(i+2m)$. One finds that the arguments at the end of this section apply *mutatis mutandis* in this case.
The analogue of Theorem 5.3 is that if $|s|<4$ and $2s\notin\mathbb{Z}_2^\times$ and $\kappa'=\langle\cdot\rangle^{2-12s}$ then an overconvergent infinite slope form of weight κ' is not r-overconvergent, for $r=\frac{3+\nu(2s)}{12}$. The proof follows the same strategy, although some of the lemmas in section 6 need minor modifications; for example in Lemma 6.10 we set $x_1=\frac{i+2j+6s}{3(i+2s)}\eta'_{ij}$ and $x_2:=\frac{2i+j+6s}{3(i+2s)}\eta'_{ij}$, and the result follows as $\eta'_{ij}=2x_2-x_1$. Note that E_2 can be regarded as an element of $H^0(Y[1/3],\omega^{\otimes 2})$ so that Lemma 6.13 does not need modification.
We deduce our main theorem:

Theorem 8.1. *If $\phi : B^\times \to \mathcal{E}$ and the induced map $B^\times \to \mathcal{W}$ extends to a map $\psi : B \to \mathcal{W}$, then ϕ extends to a map $B \to \mathcal{E}$.*

Proof. If $\psi(0) \notin \{\langle\cdot\rangle^{-12s} : 2s \in \mathbb{Z}_2^\times\}$ then we use Theorem 7.2, and if it is then we use the modification explained above. $\square$

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