On the Equivariant Tamagawa Number Conjecture for Tate Motives, Part II.

Dedicated to John Coates

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Abstract. Let K be any finite abelian extension of $\mathbb{Q}$, k any subfield of K and r any integer. We complete the proof of the equivariant Tamagawa Number Conjecture for the pair $(h^0(\mathrm{Spec}(K))(r), \mathbb{Z}[\mathrm{Gal}(K/k)])$.

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1. Introduction

Let K/k be a Galois extension of number fields with group G. For each complex character χ of G denote by $L(\chi, s)$ the Artin L-function of χ and let $\hat{G}$ be the set of irreducible characters. We call

$$\zeta_{K/k}(s) = (L(\chi, s))_{\chi\in\hat{G}}$$

the equivariant Dedekind Zeta function of K/k. It is a meromorphic function with values in the center $\prod_{\chi\in\hat{G}}\mathbb{C}$ of $\mathbb{C}[G]$. The 'equivariant Tamagawa number conjecture' that is formulated in [9, Conj. 4], when specialized to the motive $M := \mathbb{Q}(r)_K := h^0(\mathrm{Spec}(K))(r)$ and the order $\mathfrak{A} := \mathbb{Z}[G]$, gives a cohomological interpretation of the leading Taylor coefficient of $\zeta_{K/k}(s)$ at any integer argument $s = r$. We recall that this conjecture is a natural refinement of the seminal 'Tamagawa number conjecture' that was first formulated by Bloch and Kato in [5] and then both extended and refined by Fontaine and Perrin-Riou [18] and Kato [27]. If $K = k$ and $r \in \{0, 1\}$ then the conjecture specializes to the analytic class number formula and is therefore already a theorem.

The most succinct formulation of the equivariant Tamagawa number conjecture asserts the vanishing of a certain element $T\Omega(M,\mathfrak{A}) = T\Omega(\mathbb{Q}(r)_K, \mathbb{Z}[G])$ in the relative algebraic K-group $K_0(\mathbb{Z}[G],\mathbb{R})$. Further, the functional equation of Artin L-functions is reflected by an equality

$$(1) \quad T\Omega(\mathbb{Q}(r)_K,\mathbb{Z}[G]) + \psi^*(T\Omega(\mathbb{Q}(1-r)_K,\mathbb{Z}[G]^{\mathrm{op}})) = T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G])$$

where ψ^* is a natural isomorphism $K_0(\mathbb{Z}[G]^{\mathrm{op}},\mathbb{R}) \cong K_0(\mathbb{Z}[G],\mathbb{R})$ and $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G])$ is an element of $K_0(\mathbb{Z}[G],\mathbb{R})$ of the form

$$(2)\ T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G]) = L^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G]) + \delta_{K/k}(r) + R\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G]).$$

Here $L^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G])$ is an 'analytic' element constructed from the archimedean Euler factors and epsilon constants associated to both $\mathbb{Q}(r)_K$ and $\mathbb{Q}(1-r)_K$, the element $\delta_{K/k}(r)$ reflects sign differences between the regulator maps used in defining $T\Omega(\mathbb{Q}(r)_K,\mathbb{Z}[G])$ and $T\Omega(\mathbb{Q}(1-r)_K,\mathbb{Z}[G]^{\mathrm{op}})$ and $R\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G])$ is an 'algebraic' element constructed from the various realisations of $\mathbb{Q}(r)_K$. (We caution the reader that the notation in (1) and (2) is slightly different from that which is used in [9] - see §3.1 for details of these changes.)
In this article we shall further specialize to the case where K is an *abelian* extension of $\mathbb{Q}$ and prove that $T\Omega(\mathbb{Q}(r)_K,\mathbb{Z}[G]) = 0$ for all integers r and all subgroups G of $\mathrm{Gal}(K/\mathbb{Q})$. In fact, taking advantage of previous work in this area, the main new result which we prove here is the following refinement of the results of Benois and Nguyen Quang Do in [1].

Theorem 1.1. *If K is any finite abelian extension of $\mathbb{Q}$, G any subgroup of $\mathrm{Gal}(K/\mathbb{Q})$ and r any strictly positive integer, then $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G]) = 0$.*

We now discuss some interesting consequences of Theorem 1.1. The first consequence we record is the promised verification of the equivariant Tamagawa number conjecture for Tate motives over absolutely abelian fields. This result therefore completes the proof of [17, Th. 5.1] and also refines the main result of Huber and Kings in [25] (for more details of the relationship between our approach and that of [25] see [11, Intro.]).

Corollary 1.2. *If K is any finite abelian extension of $\mathbb{Q}$, G any subgroup of $\mathrm{Gal}(K/\mathbb{Q})$ and r any integer, then $T\Omega(\mathbb{Q}(r)_K,\mathbb{Z}[G]) = 0$.*

Proof. If $r \leq 0$, then the vanishing of $T\Omega(\mathbb{Q}(r)_K,\mathbb{Z}[G])$ is proved modulo powers of 2 by Greither and the first named author in [11, Cor. 8.1] and the argument necessary to cover the 2-primary part is provided by the second named author in [17]. For any $r > 0$, the vanishing of $T\Omega(\mathbb{Q}(r)_K,\mathbb{Z}[G])$ then follows by combining Theorem 1.1 with the equality (1). $\square$

Corollary 1.3. *The conjecture of Bloch and Kato* [5, Conj. (5.15)] *is valid for the Riemann-Zeta function at each integer strictly bigger than* 1.

Proof. If r is any integer strictly bigger than 1, then [5, Th. (6.1)] proves the validity of [5, Conj. (5.15)] for the leading term of the Riemann Zeta function at $s = r$, modulo powers of 2 and a certain compatibility assumption [5, Conj. (6.2)] concerning the 'cyclotomic elements' of Deligne and Soulé in algebraic

K-theory. But the latter assumption was verified by Huber and Wildeshaus in [26] and Corollary 1.2 for $K = k = \mathbb{Q}$ now resolves the ambiguity at 2. □

For any finite group G the image of the homomorphism $\delta_G : K_0(\mathbb{Z}[G], \mathbb{R}) \to K_0(\mathbb{Z}[G])$ that occurs in the long exact sequence of relative K-theory is equal to the locally-free class group $\mathrm{Cl}(\mathbb{Z}[G])$. In the following result we use the elements $\Omega(K/k, 1), \Omega(K/k, 2), \Omega(K/k, 3)$ and $w(K/k)$ of $\mathrm{Cl}(\mathbb{Z}[\mathrm{Gal}(K/k)])$ that are defined by Chinburg in [13].

Corollary 1.4. *If K is any finite abelian extension of $\mathbb{Q}$ and k is any subfield of K, then one has $\Omega(K/k, 1) = \Omega(K/k, 2) = \Omega(K/k, 3) = w(K/k) = 0$. In particular, the Chinburg conjectures are all valid for K/k.*

Proof. In this first paragraph we do not assume that K is Galois over $\mathbb{Q}$ or that $G := \mathrm{Gal}(K/k)$ is abelian. We recall that from [10, (31)] one has

$$\delta_G(\psi^*(T\Omega(\mathbb{Q}(0)_K, \mathbb{Z}[G]^{\mathrm{op}}))) = \Omega(K/k, 3) - w(K/k).$$

Further, [4, Prop. 3.1] implies δ_G sends $L^{\mathrm{loc}}(\mathbb{Q}(1)_K, \mathbb{Z}[G]) + \delta_{K/k}(1)$ to $-w(K/k)$ whilst the argument used in [4, §4.3] shows that $R\Omega^{\mathrm{loc}}(\mathbb{Q}(1)_K, \mathbb{Z}[G])$ is equal to the element $R\Omega^{\mathrm{loc}}(K/k, 1)$ defined in [7, §5.1.1]. Hence, from [7, Rem. 5.5], we may deduce that

$$\delta_G(T\Omega^{\mathrm{loc}}(\mathbb{Q}(1)_K, \mathbb{Z}[G])) = -w(K/k) + \Omega(K/k, 2). \tag{3}$$

We now assume that G is abelian. Then G has no irreducible complex symplectic characters and so the very definition of $w(K/k)$ ensures that $w(K/k) = 0$. Hence by combining the above displayed equalities with Theorem 1.1 (with $r = 1$) and Corollary 1.2 (with $r = 0$) we may deduce that $\Omega(K/k, 2) = \Omega(K/k, 3) = 0$. But from [13, (3.2)] one has $\Omega(K/k, 1) = \Omega(K/k, 2) - \Omega(K/k, 3)$, and so this also implies that $\Omega(K/k, 1) = 0$. □

For finite abelian extensions $K/\mathbb{Q}$ in which 2 is unramified, an alternative proof of the equality $\Omega(K/k, 2) = 0$ in Corollary 1.4 was first obtained by Greither in [21].

Before stating our next result we recall that, ever since the seminal results of Fröhlich in [19], the study of Quaternion extensions has been very important to the development of leading term conjectures in non-commutative settings. In the following result we provide a natural refinement of the main result of Hooper, Snaith and Tran in [24] (and hence extend the main result of Snaith in [35]).

Corollary 1.5. *Let K be any Galois extension of $\mathbb{Q}$ for which $\mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to the Quaternion group of order 8 and k any subfield of K. Then one has $T\Omega^{\mathrm{loc}}(\mathbb{Q}(1)_K, \mathbb{Z}[\mathrm{Gal}(K/k)]) = 0$.*

Proof. We set $G := \mathrm{Gal}(K/\mathbb{Q})$ and let Γ denote the maximal abelian quotient of G with E the subfield of K such that $\Gamma = \mathrm{Gal}(E/\mathbb{Q})$ (so $E/\mathbb{Q}$ is biquadratic). We set $T\Omega^{\mathrm{loc}} := T\Omega^{\mathrm{loc}}(\mathbb{Q}(1)_K, \mathbb{Z}[G])$ and $T\Omega^{\mathrm{loc}}_E := T\Omega^{\mathrm{loc}}(\mathbb{Q}(1)_E, \mathbb{Z}[\Gamma])$.
Then from [9, Th. 5.1 and Prop. 4.1] we know that $T\Omega^{\mathrm{loc}}(\mathbb{Q}(1)_K, \mathbb{Z}[\mathrm{Gal}(K/k)])$ and $T\Omega^{\mathrm{loc}}_E$ are equal to the images of $T\Omega^{\mathrm{loc}}$ under the natural homomorphisms

$K_0(\mathbb{Z}[G],\mathbb{R}) \to K_0(\mathbb{Z}[\mathrm{Gal}(K/k)],\mathbb{R})$ and $K_0(\mathbb{Z}[G],\mathbb{R}) \to K_0(\mathbb{Z}[\Gamma],\mathbb{R})$ respectively. In particular, it suffices to prove that $T\Omega^{\mathrm{loc}} = 0$.
Now [4, Cor. 6.3(i)] implies that $T\Omega^{\mathrm{loc}}$ is an element of finite order in the subgroup $K_0(\mathbb{Z}[G],\mathbb{Q})$ of $K_0(\mathbb{Z}[G],\mathbb{R})$ and so [10, Lem. 4] implies that $T\Omega^{\mathrm{loc}} = 0$ if and only if both $T\Omega^{\mathrm{loc}}_E = 0$ and $\delta_G(T\Omega^{\mathrm{loc}}) = 0$. But Theorem 1.1 implies $T\Omega^{\mathrm{loc}}_E = 0$ and, since $\delta_G(T\Omega^{\mathrm{loc}}) = -w(K/\mathbb{Q}) + \Omega(K/\mathbb{Q},2)$ (by (3)), the main result of Hooper, Snaith and Tran in [24] implies that $\delta_G(T\Omega^{\mathrm{loc}}) = 0$. □

The following result provides the first generalization to wildly ramified extensions of the algebraic characterization of tame symplectic Artin root numbers that was obtained by Cassou-Noguès and Taylor in [12].

Corollary 1.6. *Let K be any Galois extension of $\mathbb{Q}$ for which $G := \mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to the Quaternion group of order 8. Then the Artin root number of the (unique) irreducible 2-dimensional complex character of G is uniquely determined by the algebraic invariant $R\Omega^{\mathrm{loc}}(\mathbb{Q}(1)_K,\mathbb{Z}[G])$ in $K_0(\mathbb{Z}[G],\mathbb{R})$.*

Proof. This is a direct consequence of combining Corollary 1.5 with a result of Breuning and the first named author [7, Th. 5.8] and the following facts: $L^{\mathrm{loc}}(\mathbb{Q}(1)_K,\mathbb{Z}[G])+\delta_{K/\mathbb{Q}}(1)$ is equal to -1 times the element $\hat{\partial}^1_G(\epsilon_{K/\mathbb{Q}}(0))$ used in [7, §5.1.1] and $R\Omega^{\mathrm{loc}}(\mathbb{Q}(1)_K,\mathbb{Z}[G])$ is equal to the element $R\Omega^{\mathrm{loc}}(K/\mathbb{Q},1)$ defined in loc. cit. □

To prove Theorem 1.1 we shall combine some classical and rather explicit computations of Hasse (concerning Gauss sums) and Leopoldt (concerning integer rings in cyclotomic fields) with a refinement of some general results proved in [9, §5] and a systematic use of the Iwasawa theory of complexes in the spirit of Kato [27, 3.1.2] and Nekovář [32] and of the generalization of the fundamental exact sequence of Coleman theory obtained by Perrin-Riou in [34].
We would like to point out that, in addition to the connections discussed above, there are also links between our approach and aspects of the work of Kato [28], Fukaya and Kato [20] and Benois and Berger [2]. In particular, the main technical result that we prove (the validity of equality (16)) is closely related to [28, Th. 4.1] and hence also to the material of [20, §3]. Indeed, Theorem 1.1 (in the case $r = 1$) provides a natural generalization of the results discussed in [20, §3.6]. However, the arguments of both loc. cit. and [28] do not cover the prime 2 and also leave open certain sign ambiguities, and much effort is required in the present article to deal with such subtleties.
Both authors were introduced to the subject of Tamagawa number conjectures by John Coates. It is therefore a particular pleasure for us to dedicate this paper to him on the occasion of his sixtieth birthday.
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2. Equivariant local Tamagawa numbers

In this article we must compute explicitly certain equivariant local Tamagawa numbers, as defined in [9]. For the reader's convenience, we therefore first quickly review the general definition of such invariants. All further details of this construction can be found in loc. cit.

2.1. We fix a motive M that is defined over $\mathbb{Q}$ (if M is defined over a general number field as in [9], then we use induction to reduce to the base field $\mathbb{Q}$) and we assume that M is endowed with an action of a finite dimensional semisimple $\mathbb{Q}$-algebra A.

We write $H_{dR}(M)$ and $H_B(M)$ for the de Rham and Betti realisations of M and for each prime number p we denote by $V_p = H_p(M)$ the p-adic realisation of M. We fix a $\mathbb{Z}$-order $\mathfrak{A}$ in A such that, for each prime p, if we set $\mathfrak{A}_p := \mathfrak{A}\otimes_{\mathbb{Z}}\mathbb{Z}_p$, then there exists a full projective Galois stable $\mathfrak{A}_p$-sublattice T_p of V_p. We also fix a finite set S of places of $\mathbb{Q}$ containing ∞ and all primes of bad reduction for M and then set $S_p := S \cup \{p\}$ and $S_{p,f} := S_p \setminus \{\infty\}$.

For any associative unital ring R we write $D^{\rm perf}(R)$ for the derived catgeory of perfect complexes of R-modules. We also let $\mathrm{Det}_R : D^{\rm perf}(R) \to V(R)$ denote the universal determinant functor to the Picard category of virtual objects of R (which is denoted by $P \mapsto [P]$ in [9]) and $\otimes_R$ the product functor in $V(R)$ (denoted by $\boxtimes$ in [9]). In particular, if R is commutative, then Det_R is naturally isomorphic to the Knudsen-Mumford functor to graded invertible R-modules. We denote by $\mathbf{1}_R$ a unit object of $V(R)$ and recall that the group $K_1(R)$ can be identified with $\mathrm{Aut}_{V(R)}(L)$ for any object L of $V(R)$ (and in particular therefore with $\pi_1(V(R)) := \mathrm{Aut}_{V(R)}(\mathbf{1}_R)$). For each automorphism $\alpha : W \to W$ of a finitely generated projective R-module W we denote by $\mathrm{Det}_R(\alpha|W)$ the element of $K_1(R)$ that is represented by α. We let $\zeta(R)$ denote the centre of R.

If X is any R-module upon which complex conjugation acts as an endomorphism of R-modules, then we write X^+ and X^- for the R-submodules of X upon which complex conjugation acts as multiplication by 1 and -1 respectively.

For any $\mathbb{Q}$-vector space W we set $W_{\mathbb{C}} = W \otimes_{\mathbb{Q}} \mathbb{C}$, $W_{\mathbb{R}} = W \otimes_{\mathbb{Q}} \mathbb{R}$ and $W_p = W \otimes_{\mathbb{Q}} \mathbb{Q}_p$ for each prime p.

2.2. The virtual object

$$\Xi^{\rm loc}(M) := \mathrm{Det}_A(H_{dR}(M)) \otimes_A \mathrm{Det}_A^{-1}(H_B(M))$$

is endowed with a canonical morphism

$$\vartheta_\infty^{\rm loc} : A_{\mathbb{R}} \otimes_A \Xi^{\rm loc}(M) \cong \mathbf{1}_{A_{\mathbb{R}}}.$$

To describe this morphism we note that the canonical period isomorphism $H_{dR}(M)_{\mathbb{C}} \cong H_B(M)_{\mathbb{C}}$ induces an isomorphism of $A_{\mathbb{R}}$-modules

$$H_{dR}(M)_{\mathbb{R}} = (H_{dR}(M)_{\mathbb{C}})^+ \cong (H_B(M)_{\mathbb{C}})^+ \qquad (4)$$

and that there is also a canonical isomorphism of $A_{\mathbb{R}}$-modules

$$\begin{aligned}(5)\quad (H_B(M)_{\mathbb{C}})^+ &= (H_B(M)^+ \otimes_{\mathbb{Q}} \mathbb{R}) \oplus (H_B(M)^- \otimes_{\mathbb{Q}} \mathbb{R}(2\pi i)^{-1}) \\ &\cong (H_B(M)^+ \otimes_{\mathbb{Q}} \mathbb{R}) \oplus (H_B(M)^- \otimes_{\mathbb{Q}} \mathbb{R}) = H_B(M)_{\mathbb{R}}\end{aligned}$$

where the central map results from identifying $\mathbb{R}(2\pi i)^{-1}$ with $\mathbb{R}$ by sending $(2\pi i)^{-1}$ to 1.
By applying $\mathrm{Det}_{A_{\mathbb{R}}}$ to the composite of (4) and (5) one obtains a morphism $(\vartheta_\infty^{\mathrm{loc}})' : A_{\mathbb{R}} \otimes_A \Xi^{\mathrm{loc}}(M) \cong \mathbf{1}_{A_{\mathbb{R}}}$ and $\vartheta_\infty^{\mathrm{loc}}$ is defined in [9, (57)] to be the composite of $(\vartheta_\infty^{\mathrm{loc}})'$ and the 'sign' elements $\epsilon_B := \mathrm{Det}_A(-1 \mid H_B(M)^+)$ and $\epsilon_{dR} := \mathrm{Det}_A(-1 \mid F^0 H_{dR}(M))$ of $\pi_1(V(A_{\mathbb{R}})) \cong K_1(A_{\mathbb{R}})$.

2.3. Following [9, (66), (67)], we set

$$\Lambda_p(S, V_p) := \left(\bigotimes_{\ell \in S_{p,f}} \mathrm{Det}_{A_p}^{-1} R\Gamma(\mathbb{Q}_\ell, V_p) \right) \otimes_{A_p} \mathrm{Det}_{A_p}^{-1}(V_p),$$

and let

$$\theta_p : A_p \otimes_A \Xi^{\mathrm{loc}}(M) \cong \Lambda_p(S, V_p)$$

denote the morphism in $V(A_p)$ obtained by taking the product of the morphisms $\theta_p^{\ell\text{-part}}$ for $\ell \in S_{p,f}$ that are discussed in the next subsection.

2.4. There exists a canonical morphism in $V(A_p)$ of the form

$$\theta_p^{p\text{-part}} : A_p \otimes_A \Xi^{\mathrm{loc}}(M) \to \mathrm{Det}_{A_p}^{-1} R\Gamma(\mathbb{Q}_p, V_p) \otimes_{A_p} \mathrm{Det}_{A_p}^{-1}(V_p).$$

This morphism results by applying Det_{A_p} to each of the following: the canonical comparison isomorphism $H_B(M)_p \cong V_p$; the (Poincaré duality) exact sequence $0 \to (H_{dR}(M^*(1))/F^0)^* \to H_{dR}(M) \to H_{dR}(M)/F^0 \to 0$; the canonical comparison isomorphisms $(H_{dR}(M)/F^0)_p \cong t_p(V_p)$ and $(H_{dR}(M^*(1))/F^0)_p^* \cong t_p(V_p^*(1))^*$; the exact triangle

$$(6)\qquad R\Gamma_f(\mathbb{Q}_p, V_p) \to R\Gamma(\mathbb{Q}_p, V_p) \to R\Gamma_f(\mathbb{Q}_p, V_p^*(1))^*[-2] \to$$

which results from [9, (18) and Lem. 12a)]; the exact triangle

$$(7)\qquad t_p(W)[-1] \to R\Gamma_f(\mathbb{Q}_p, W) \to \left(D_{\mathrm{cris}}(W) \xrightarrow{1-\varphi_v} D_{\mathrm{cris}}(W)\right) \to$$

of [9, (22)] for both $W = V_p$ and $W = V_p^*(1)$, where the first term of the last complex is placed in degree 0 and $\mathrm{Det}_{A_p}\left(D_{\mathrm{cris}}(W) \xrightarrow{1-\varphi_v} D_{\mathrm{cris}}(W)\right)$ is identified with $\mathbf{1}_{A_p}$ via the canonical morphism $\mathrm{Det}_{A_p}(D_{\mathrm{cris}}(W)) \otimes_{A_p} \mathrm{Det}_{A_p}^{-1}(D_{\mathrm{cris}}(W)) \to \mathbf{1}_{A_p}$.
For each $\ell \in S_{p,f} \setminus \{p\}$ there exists a canonical morphism in $V(A_p)$ of the form

$$\theta_p^{\ell\text{-part}} : \mathbf{1}_{A_p} \cong \mathrm{Det}_{A_p}^{-1} R\Gamma(\mathbb{Q}_\ell, V_p).$$

For more details about this morphism see Proposition 7.1.

2.5. From [9, (71), (78)] we recall that there exists a canonical object $\Lambda_p(S,T_p)$ of $V(\mathfrak{A}_p)$ and a canonical morphism in $V(A_p)$ of the form

$$\theta'_p : \Lambda_p(S,V_p) \cong A_p \otimes_{\mathfrak{A}_p} \Lambda_p(S,T_p)$$

(the definitions of $\Lambda_p(S,T_p)$ and θ'_p are to be recalled further in §7.2). We set

$$\vartheta_p^{\mathrm{loc}} := \epsilon(S,p) \circ \theta'_p \circ \theta_p : A_p \otimes_A \Xi^{\mathrm{loc}}(M) \cong A_p \otimes_{\mathfrak{A}_p} \Lambda_p(S,T_p)$$

where $\epsilon(S,p)$ is the element of $\pi_1(V(A_p))$ that corresponds to multiplication by -1 on the complex $\bigoplus_{\ell\in S_{p,f}} R\Gamma_{/f}(\mathbb{Q}_\ell, V_p)$ which is defined in [9, (18)].
If M is a direct factor of $h^n(X)(t)$ for any non-negative integer n, smooth projective variety X and integer t, then [9, Lem. 15b)] implies that the data

$$(\prod_p \Lambda_p(S,T_p), \Xi^{\mathrm{loc}}(M), \prod_p \vartheta_p^{\mathrm{loc}}; \vartheta_\infty^{\mathrm{loc}}),$$

where p runs over all prime numbers, gives rise (conjecturally in general, but unconditionally in the case of Tate motives) to a canonical element $R\Omega^{\mathrm{loc}}(M,\mathfrak{A})$ of $K_0(\mathfrak{A},\mathbb{R})$. For example, if A is commutative, then $\mathbf{1}_{A_\mathbb{R}} = (A_\mathbb{R}, 0)$ and $K_0(\mathfrak{A},\mathbb{R})$ identifies with the multiplicative group of invertible $\mathfrak{A}$-sublattices of $A_\mathbb{R}$ and, with respect to this identification, $R\Omega^{\mathrm{loc}}(M,\mathfrak{A})$ corresponds to the (conjecturally invertible) $\mathfrak{A}$-sublattice Ξ of $A_\mathbb{R}$ that is defined by the equality

$$\vartheta_\infty^{\mathrm{loc}}\left(\bigcap_p (\Xi^{\mathrm{loc}}(M) \cap (\vartheta_p^{\mathrm{loc}})^{-1}(\Lambda_p(S,T_p)))\right) = (\Xi, 0),$$

where the intersection is taken over all primes p.

2.6. We write $L_\infty({}_AM, s)$ and $\epsilon({}_AM, 0)$ for the archimedean Euler factor and epsilon constant that are defined in [9, §4.1]. Also, with $\rho \in \mathbb{Z}^{\pi_0(\mathrm{Spec}(\zeta(A_\mathbb{R})))}$ denoting the algebraic order at $s=0$ of the completed $\zeta(A_\mathbb{C})$-valued L-function $\Lambda({}_{A^{op}}M^*(1), s)$ that is defined in loc. cit., we set

$$\mathcal{E}({}_AM) := (-1)^\rho \epsilon({}_AM, 0) \frac{L_\infty^*({}_{A^{op}}M^*(1), 0)}{L_\infty^*({}_AM, 0)} \in \zeta(A_\mathbb{R})^\times.$$

Following [9, §5.1], we define

$$L^{\mathrm{loc}}(M,\mathfrak{A}) := \hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(\mathcal{E}({}_AM)) \in K_0(\mathfrak{A},\mathbb{R})$$

where $\hat{\delta}^1_{\mathfrak{A},\mathbb{R}} : \zeta(A_\mathbb{R})^\times \to K_0(\mathfrak{A},\mathbb{R})$ is the 'extended boundary homomorphism' of [9, Lem. 9] (so, if A is commutative, then $L^{\mathrm{loc}}(M,\mathfrak{A}) = \mathfrak{A}\cdot\mathcal{E}({}_AM) \subset A_\mathbb{R}$). Finally, we let

$$T\Omega^{\mathrm{loc}}(M,\mathfrak{A})' := L^{\mathrm{loc}}(M,\mathfrak{A}) + R\Omega^{\mathrm{loc}}(M,\mathfrak{A}) \in K_0(\mathfrak{A},\mathbb{R}) \tag{8}$$

denote the 'equivariant local Tamagawa number' that is defined in [9, just prior to Th. 5.1].

3. Normalizations and notation

3.1. Normalizations. In this section we fix an arbitrary Galois extension of number fields K/k, set $G := \mathrm{Gal}(K/k)$ and for each integer t write $T\Omega(\mathbb{Q}(t)_K, \mathbb{Z}[G])'$ for the element of $K_0(\mathbb{Z}[G], \mathbb{R})$ that is defined (unconditionally) by [9, Conj. 4(iii)] in the case $M = \mathbb{Q}(t)_K$ and $\mathfrak{A} = \mathbb{Z}[G]$.

Let r be a strictly positive integer. Then the computations of [10, 17] show that [9, Conj. 4(iv)] requires that the morphism $\vartheta_\infty : \mathbb{R}\otimes_{\mathbb{Q}} \Xi(\mathbb{Q}(1-r)_K) \to \mathbf{1}_{V(\mathbb{R}[G])}$ constructed in [9, §3.4] should be normalized by using -1 times the Dirichlet (resp. Beilinson if $r > 1$) regulator map, rather than the Dirichlet (resp. Beilinson) regulator map itself as used in [9]. To incorporate this observation we set

$$T\Omega(\mathbb{Q}(1-r)_K, \mathbb{Z}[G]) := T\Omega(\mathbb{Q}(1-r)_K, \mathbb{Z}[G])' + \delta_{K/k}(r) \tag{9}$$

where $\delta_{K/k}(r)$ is the image under the canonical map $K_1(\mathbb{R}[G]) \to K_0(\mathbb{Z}[G], \mathbb{R})$ of the element $\mathrm{Det}_{\mathbb{Q}[G]}(-1 \mid K_{2r-1}(\mathcal{O}_K)^* \otimes_{\mathbb{Z}} \mathbb{Q})$. To deduce the validity of (1) from the result of [9, Th. 5.3] it is thus also necessary to renormalise the definition of either $T\Omega(\mathbb{Q}(r)_K, \mathbb{Z}[G])'$ or of the element $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])'$ defined by (8). Our proof of Theorem 1.1 now shows that the correct normalization is to set

$$T\Omega(\mathbb{Q}(r)_K, \mathbb{Z}[G]) := T\Omega(\mathbb{Q}(r)_K, \mathbb{Z}[G])'$$

and

$$T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G]) := T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])' + \delta_{K/k}(r). \tag{10}$$

Note that the elements defined in (9) and (10) satisfy all of the functorial properties of $T\Omega(\mathbb{Q}(1-r)_K, \mathbb{Z}[G])'$ and $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])'$ that are proved in [9, Th. 5.1, Prop. 4.1]. Further, with these definitions, the equalities (1) and (2) are valid and it can be shown that the conjectural vanishing of $T\Omega^{\mathrm{loc}}(\mathbb{Q}(1)_K, \mathbb{Z}[G])$ is compatible with the conjectures discussed in both [4] and [7].

Thus, in the remainder of this article we always use the notation $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])$ as defined in (10).

3.2. The abelian case. Until explicitly stated otherwise, in the sequel we consider only abelian groups. Thus, following [9, §2.5], we use the graded determinant functor of [29] in place of virtual objects (for a convenient review of all relevant properties of the determinant functor see [11, §2]). However, we caution the reader that for reasons of typographical clarity we sometimes do not distinguish between a graded invertible module and the underlying invertible module.

We note that, when proving Theorem 1.1, the functorial properties of the elements $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[\mathrm{Gal}(K/k)])$ allow us to assume that $k = \mathbb{Q}$ and also that K is generated by a primitive N-th root of unity for some natural number $N \not\equiv 2 \mod 4$. Therefore, until explicitly stated otherwise, we henceforth fix the following notation:

$$K := \mathbb{Q}(e^{2\pi i/N}); \quad G := \mathrm{Gal}(K/\mathbb{Q}); \quad M := \mathbb{Q}(r)_K, \quad r \geq 1; \quad A := \mathbb{Q}[G].$$

For any natural number n we also set $\zeta_n := e^{2\pi i/n}$ and denote by σ_n the resulting complex embedding of the field $\mathbb{Q}(\zeta_n)$.

For each complex character η of G we denote by $e_\eta = \frac{1}{|G|}\sum_{g\in G}\eta(g^{-1})g$ the associated idempotent in $A_{\mathbb{C}}$. For each $\mathbb{Q}$-rational character (or equivalently, $\mathrm{Aut}(\mathbb{C})$-conjugacy class of $\mathbb{C}$-rational characters) χ of G we set $e_\chi = \sum_{\eta\in\chi} e_\eta \in A$ and denote by $\mathbb{Q}(\chi) = e_\chi A$ the field of values of χ. There is a ring decomposition $A = \prod_\chi \mathbb{Q}(\chi)$ and a corresponding decomposition $Y = \prod_\chi e_\chi Y$ for any A-module Y. We make similar conventions for $\mathbb{Q}_p$-rational characters of G.

4. An explicit analysis of $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])$

In this section we reduce the proof of Theorem 1.1 to the verification of an explicit local equality (cf. Proposition 4.4).

4.1. The archimedean component of $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])$. In this subsection we explicate the morphism $\vartheta_\infty^{\mathrm{loc}}$ defined in §2.2 and the element $\mathcal{E}({}_AM) \in A_{\mathbb{R}}^\times$ defined in §2.6.

The de Rham realization $H_{dR}(M)$ of M identifies with K, considered as a free A-module of rank one (by means of the normal basis theorem). The Betti realisation $H_B(M)$ of M identifies with the $\mathbb{Q}$-vector space Y_Σ with basis equal to the set $\Sigma := \mathrm{Hom}(K, \mathbb{C})$ of field embeddings and is therefore also a free A-module of rank one (with basis σ_N). We set $Y_\Sigma^{-1} := \mathrm{Hom}_A(Y_\Sigma, A)$. Then, by [9, Th. 5.2], we know that $(\vartheta_\infty^{\mathrm{loc}})^{-1}((\mathcal{E}({}_AM)^{-1}, 0))$ belongs to $\Xi^{\mathrm{loc}}(M) = (K \otimes_A Y_\Sigma^{-1}, 0)$ and we now describe this element explicitly.

Proposition 4.1. *We define an element $\epsilon_\infty := \sum_\chi \epsilon_{\infty,\chi} e_\chi$ of $A^\times$ by setting*

$$\epsilon_{\infty,\chi} := \begin{cases} -2 & \text{if } \chi(-1) = (-1)^r \\ -\frac{1}{2} & \text{if } \chi(-1) = -(-1)^r \text{ and } (\chi \neq 1 \text{ or } r > 1) \\ \frac{1}{2} & \text{if } \chi = 1 \text{ and } r = 1. \end{cases}$$

Then

$$(\vartheta_\infty^{\mathrm{loc}})^{-1}((\mathcal{E}({}_AM)^{-1}, 0)) = (\epsilon_\infty \beta_N \otimes \sigma_N^{-1}, 0) \in (K \otimes_A Y_\Sigma^{-1}, 0)$$

where σ_N^{-1} is the (unique) element of Y_Σ^{-1} which satisfies $\sigma_N^{-1}(\sigma_N) = 1$ and β_N is the (unique) element of $K = \prod_\chi e_\chi K$ which satisfies

$$e_\chi \beta_N := [K : \mathbb{Q}(\zeta_{f_\chi})]^{-1}(r-1)! f_\chi^{r-1} \cdot e_\chi \zeta_{f_\chi}$$

for all $\mathbb{Q}$-rational characters χ of G.

Proof. For each Dirichlet character η of G the functional equation of $L(\eta, s)$ is

$$L(\eta, s) = \frac{\tau(\eta)}{2i^\delta}\left(\frac{2\pi}{f_\eta}\right)^s \frac{1}{\Gamma(s)\cos(\frac{\pi(s-\delta)}{2})} L(\bar{\eta}, 1-s)$$

where f_η is the conductor of η and

$$\tau(\eta) = \sum_{a=1}^{f_\eta} \eta(a) e^{2\pi i a/f_\eta}; \quad \eta(-1) = (-1)^\delta,\ \delta \in \{0, 1\} \tag{11}$$

(cf. [36, Ch. 4]). Thus, by its very definition in §2.6, the η-component of the element $\mathcal{E}({}_AM)^{-1}$ of $A_{\mathbb{C}} = \prod_\eta \mathbb{C}$ is the leading Taylor coefficient at $s = r$ of the meromorphic function

$$(-1)^{\rho_\eta}\frac{2i^\delta}{\tau(\eta)}\left(\frac{f_\eta}{2\pi}\right)^s\Gamma(s)\cos(\frac{\pi(s-\delta)}{2}); \qquad \rho_\eta = \begin{cases} 1 & r=1,\eta=1\\ 0 & \text{else.}\end{cases}$$

Hence we have

$$\mathcal{E}({}_AM)_\eta^{-1} = \begin{cases} \frac{2i^\delta}{\tau(\eta)}\left(\frac{f_\eta}{2\pi}\right)^r(r-1)!(-1)^{\frac{r-\delta}{2}}, & r-\delta \text{ even}\\ (-1)^{\rho_\eta}\frac{2i^\delta}{\tau(\eta)}\left(\frac{f_\eta}{2\pi}\right)^r(r-1)!(-1)^{\frac{r-\delta+1}{2}}\frac{\pi}{2}, & r-\delta\text{ odd}\end{cases}$$

which, after collecting powers of i and using the relation $\tau(\eta)\tau(\bar\eta) = \eta(-1)f_\eta$, can be written as

$$\mathcal{E}({}_AM)_\eta^{-1} = \begin{cases} 2\,\tau(\bar\eta)(2\pi i)^{-r}f_\eta^{r-1}(r-1)!, & r-\delta\text{ even}\\ (-1)^{\rho_\eta+1}\frac{1}{2}\,\tau(\bar\eta)(2\pi i)^{-(r-1)}f_\eta^{r-1}(r-1)!, & r-\delta\text{ odd.}\end{cases}$$

Lemma 4.2. *The isomorphism $Y_{\Sigma,\mathbb{C}}^+ = (H_B(M)_{\mathbb{C}})^+ \cong H_B(M)_{\mathbb{R}} = Y_{\Sigma,\mathbb{R}}$ in (5) is given by*

$$\sum_{g\in G}\alpha_g g^{-1}\sigma_N \mapsto \sum_{g\in G/<c>}\big(\Re(\alpha_g)(1+(-1)^rc) - 2\pi\Im(\alpha_g)(1-(-1)^rc)\big)g^{-1}\sigma_N$$

where $c\in G$ is complex conjugation, G acts on Σ via $(g\sigma)(x) = \sigma(g(x))$ and $\Re(\alpha)$, resp. $\Im(\alpha)$, denotes the real, resp. imaginary, part of $\alpha\in\mathbb{C}$.

Proof. An element $x := \sum_{g\in G}\alpha_g g^{-1}\sigma_N$ of $Y_{\Sigma,\mathbb{C}}$ belongs to the subspace $Y_{\Sigma,\mathbb{C}}^+$ if and only if one has $\alpha_{gc} = (-1)^r\bar\alpha_g$ for all $g\in G$. Writing

$$\alpha_g = \Re(\alpha_g) - (2\pi i)^{-1}(2\pi)\Im(\alpha_g), \quad \bar\alpha_g = \Re(\alpha_g) + (2\pi i)^{-1}(2\pi)\Im(\alpha_g)$$

we find that

$$x = \sum_{g\in G/<c>}\big(\Re(\alpha_g)(1+(-1)^rc) - (2\pi i)^{-1}2\pi\Im(\alpha_g)(1-(-1)^rc)\big)g^{-1}\sigma_N.$$

But $\sum_{g\in G/<c>}(2\pi i)^{-1}2\pi\Im(\alpha_g)(1-(-1)^rc)g^{-1}\sigma_N \in H_B(M)^-\otimes_{\mathbb{Q}}\mathbb{R}\cdot i$ and the central map in (5) sends $(2\pi i)^{-1}$ to 1. This implies the claimed result. $\square$

The canonical comparison isomorphism $K_{\mathbb{C}} = H_{dR}(M)_{\mathbb{C}} \cong H_B(M)_{\mathbb{C}} = Y_{\Sigma,\mathbb{C}}$ which occurs in (4) sends any element β of K to

$$\sum_{g\in G}\sigma_N(g\beta)(2\pi i)^{-r}g^{-1}\sigma_N = \sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times}\sigma_N\tau_a(\beta)(2\pi i)^{-r}\tau_a^{-1}\sigma_N$$

where $\tau_a(\zeta) = \zeta^a$ for each N-th root of unity ζ. In particular, after composing this comparison isomorphism with the isomorphism of Lemma 4.2 we find that ζ_f is sent to the following element of $Y_{\Sigma,\mathbb{R}}$

$$\sum_a\big(\Re(e^{2\pi ia/f}(2\pi i)^{-r})(1+(-1)^rc) - 2\pi\Im(e^{2\pi ia/f}(2\pi i)^{-r})(1-(-1)^rc)\big)\tau_a^{-1}\sigma_N$$

where the summation runs over all elements a of $(\mathbb{Z}/N\mathbb{Z})^\times/\pm 1$. For each Dirichlet character η the η-component of this element is equal to $e_\eta\sigma_N$ multiplied by

$$\begin{aligned}
&\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times/\pm 1} (2\pi)^{-r}\Re(e^{2\pi i a/f} i^{-r})\overline{\eta(a)}\cdot 2\\
=&\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times/\pm 1} (2\pi i)^{-r}(e^{2\pi i a/f}+(-1)^r e^{-2\pi i a/f})\overline{\eta(a)}\\
=&(2\pi i)^{-r}\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times} e^{2\pi i a/f}\overline{\eta(a)}
\end{aligned}$$

if $\eta(-1)=(-1)^r$ (so $\delta-r$ is even), resp. by

$$\begin{aligned}
&-2\pi\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times/\pm 1} (2\pi)^{-r}\Im(e^{2\pi i a/f} i^{-r})\overline{\eta(a)}\cdot 2\\
=&-2\pi\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times/\pm 1} (2\pi i)^{-r}\frac{e^{2\pi i a/f}-(-1)^r e^{-2\pi i a/f}}{i}\overline{\eta(a)}\\
=&(2\pi i)^{-(r-1)}\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times} e^{2\pi i a/f}\overline{\eta(a)}
\end{aligned}$$

if $\eta(-1)=-(-1)^r$ (so $\delta-r$ is odd). Taking $f=f_\eta$ we find that the (η-part of the) morphism $(\vartheta_\infty^{\mathrm{loc}})': A_\mathbb{R}\otimes_A \Xi^{\mathrm{loc}}(M)\cong(A_\mathbb{R},0)$ defined in §2.2 sends

$$(12)\quad e_\eta\zeta_{f_\eta}\otimes_\mathbb{C} e_\eta\sigma_N^{-1}\mapsto\begin{cases}(2\pi i)^{-r}[K:\mathbb{Q}(f_\eta)]\tau(\bar\eta) & \text{if } \eta(-1)=(-1)^r\\ (2\pi i)^{-(r-1)}[K:\mathbb{Q}(f_\eta)]\tau(\bar\eta) & \text{if } \eta(-1)=-(-1)^r.\end{cases}$$

Now $\vartheta_\infty^{\mathrm{loc}}$ is defined to be the composite of $(\vartheta_\infty^{\mathrm{loc}})'$ and the sign factors ϵ_{dR} and ϵ_B that are defined at the end of §2.2. But it is easily seen that $e_{dR}=1$, that $(\epsilon_B)_\chi=-1$ for $\chi(-1)=(-1)^r$ and that $(\epsilon_B)_\chi=1$ otherwise. Thus, upon comparing (12) with the description of $\mathcal{E}({}_AM)_\eta^{-1}$ before Lemma 4.2 one verifies the statement of Proposition 4.1. □

4.2. Reduction to the p-primary component. By [9, Th. 5.2] we know that $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G])$ belongs to the subgroup $K_0(\mathbb{Z}[G],\mathbb{Q})$ of $K_0(\mathbb{Z}[G],\mathbb{R})$. Recalling the direct sum decomposition $K_0(\mathbb{Z}[G],\mathbb{Q})\cong\bigoplus_\ell K_0(\mathbb{Z}_\ell[G],\mathbb{Q}_\ell)$ over all primes ℓ from [9, (13)], we may therefore prove Theorem 1.1 by showing that, for each prime ℓ, the projection $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G])_\ell$ of $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G])$ to $K_0(\mathbb{Z}_\ell[G],\mathbb{Q}_\ell)$ vanishes. Henceforth we therefore fix a prime number p and shall analyze $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G])_p$.

We denote by

$$T_p:=\mathrm{Ind}_K^\mathbb{Q}\mathbb{Z}_p(r)\subset V_p:=\mathrm{Ind}_K^\mathbb{Q}\mathbb{Q}_p(r)=H_p(M)$$

the natural lattice in the p-adic realisation V_p of M. Then by combining the definition of $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K,\mathbb{Z}[G])$ from (10) (and (8)) together with the explicit description of Proposition 4.1 one finds that $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])_p = 0$ if and only if

$$\mathbb{Z}_p[G] \cdot \epsilon(r)\epsilon(S,p) \cdot \theta'_p \circ \theta_p((\epsilon_\infty\beta_N \otimes \sigma_N^{-1}, 0)) = \Lambda_p(S, T_p)$$

where θ_p is as defined in §2.3, $\Lambda_p(S,T_p)$, θ'_p and $\epsilon(S,p) \in A_p^\times$ are as discussed in §2.5 and we have set $\epsilon(r) := \mathrm{Det}_A(-1 \mid K_{2r-1}(\mathcal{O}_K)^* \otimes_{\mathbb{Z}} \mathbb{Q}) \in A^\times$.

Lemma 4.3. *We set*

$$\epsilon_p := \mathrm{Det}_{A_p}(2|V_p^+)\,\mathrm{Det}_{A_p}(2|V_p^-)^{-1} \in A_p^\times.$$

Then, with ϵ_∞ as defined in Proposition 4.1, there exists an element $u(r)$ of $\mathbb{Z}_p[G]^\times$ such that $\epsilon(r)\epsilon(S,p)\epsilon_\infty = u(r)\epsilon_p$.

Proof. We recall that $\epsilon(S,p)$ is a product of factors $\mathrm{Det}_{A_p}(-1|R\Gamma_{/f}(\mathbb{Q}_\ell, V_p))$. Further, the quasi-isomorphism $R\Gamma_{/f}(\mathbb{Q}_\ell, V_p) \cong R\Gamma_f(\mathbb{Q}_\ell, V_p^*(1))^*[-2]$ from [9, Lem. 12a)] implies that each such complex is quasi-isomorphic to a complex of the form $W \to W$ (indeed, this is clear if $\ell \neq p$ and is true in the case $\ell = p$ because the tangent space of the motive $\mathbb{Q}(1-r)_K$ vanishes for $r \geq 1$) and so one has $\epsilon(S,p) = 1$.
We next note that if $\epsilon(r) := \sum_\chi \epsilon(r)_\chi e_\chi$ with $\epsilon(r)_\chi \in \{\pm 1\}$, then the explicit structure of the $\mathbb{Q}[G]$-module $K_{2r-1}(\mathcal{O}_K)^* \otimes_{\mathbb{Z}} \mathbb{Q}$ (cf. [17, p. 86, p. 105]) implies that $\epsilon(r)_\chi = 1$ if either $r = 1$ and χ is trivial or if $\chi(-1) = (-1)^r$, and that $\epsilon(r)_\chi = -1$ otherwise.
Thus, after recalling the explicit definitions of ϵ_∞ and ϵ_p, it is straightforward to check that the claimed equality $\epsilon(r)\epsilon(S,p)\epsilon_\infty = u(r)\epsilon_p$ is valid with $u(r) = -(-1)^r c$ where $c \in G$ is complex conjugation. □

The element ϵ_p in Lemma 4.3 is equal to the element ϵ_{V_p} that occurs in Proposition 7.2 below (with $V_p = \mathrm{Ind}_K^{\mathbb{Q}}\mathbb{Q}_p(r)$). Hence, upon combining Lemma 4.3 with the discussion which immediately precedes it and the result of Proposition 7.2 we may deduce that $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])_p = 0$ if and only if

$$(13)\quad \mathbb{Z}_p[G]\cdot\theta_p((\beta_N \otimes \sigma_N^{-1}, 0)) = \left(\bigotimes_{\ell | Np} \mathrm{Det}^{-1}_{\mathbb{Z}_p[G]} R\Gamma(\mathbb{Q}_\ell, T_p)\right) \otimes_{\mathbb{Z}_p[G]} (T_p^{-1}, -1).$$

Here we have set $T_p^{-1} := \mathrm{Hom}_{\mathbb{Z}_p[G]}(T_p, \mathbb{Z}_p[G])$ and also used the fact that, since T_p is a free rank one $\mathbb{Z}_p[G]$-module, one has $\mathrm{Det}^{-1}_{\mathbb{Z}_p[G]}(T_p) = (T_p^{-1}, -1)$.
Now Shapiro's Lemma allows us to identify the complexes $R\Gamma(\mathbb{Q}_\ell, T_p)$ and $R\Gamma(\mathbb{Q}_\ell, V_p)$ with $R\Gamma(K_\ell, \mathbb{Z}_p(r))$ and $R\Gamma(K_\ell, \mathbb{Q}_p(r))$ respectively. Further, the complex $R\Gamma(K_p, \mathbb{Q}_p(r))$ is acyclic outside degree 1 for $r > 1$, and for $r = 1$ one has a natural exact sequence of $\mathbb{Q}_p[G]$-modules

$$(14)\qquad 0 \to \hat{\mathcal{O}}_{K_p}^\times \to \hat{K}_p^\times \cong H^1(K_p, \mathbb{Q}_p(1)) \xrightarrow{\mathrm{val}} \prod_{v|p} \mathbb{Q}_p \cong H^2(K_p, \mathbb{Q}_p(1)) \to 0$$

where the first isomorphism is induced by Kummer theory and the second by the invariant map on the Brauer group. Our notation here is that $\hat{M} :=$

$(\varprojlim_n M/p^nM) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for any abelian group M. We let

$$K_p = D_{dR}(V_p) \xrightarrow{\exp} H^1_f(K_p, \mathbb{Q}_p(r))$$

denote the exponential map of Bloch and Kato for the representation V_p of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. This map is bijective (since $r > 0$) and $H^1_f(K_p, \mathbb{Q}_p(r))$ coincides with $\hat{\mathcal{O}}^{\times}_{K_p}$ for $r = 1$ and with $H^1(K_p, \mathbb{Q}_p(r))$ for $r > 1$ (cf. [5]). Also, both source and target for the map exp are free A_p-modules of rank one. By using the sequence (14) for $r = 1$ we therefore find that for each $r \geq 1$ there exists an isomorphism of graded invertible A_p-modules of the form

$$\widetilde{\exp} : (K_p, 1) \xrightarrow{\exp} (H^1_f(K_p, \mathbb{Q}_p(r)), 1) \cong \mathrm{Det}^{-1}_{A_p} R\Gamma(K_p, \mathbb{Q}_p(r)). \tag{15}$$

For any subgroup $H \subseteq G$ we set

$$e_H := \sum_{\chi(H)=1} e_\chi = \frac{1}{|H|} \sum_{g \in H} g.$$

Also, for each prime ℓ we denote by J_ℓ and D_ℓ the inertia and decomposition groups of ℓ in G. For $x \in A_p$ we then set

$$e_\ell(x) := 1 + (x-1)e_{J_\ell} \in A_p$$

(so $x \mapsto e_\ell(x)$ is a multiplicative map that preserves the maximal $\mathbb{Z}_p$-order in A_p) and we denote by $\mathrm{Fr}_\ell \in G \subset A$ any choice of a Frobenius element.

PROPOSITION 4.4. *We define an element $e^*_p(1 - p^{r-1}\,\mathrm{Fr}_p^{-1})$ of $A_p^\times$ by setting*

$$e_\chi e^*_p(1 - p^{r-1}\,\mathrm{Fr}_p^{-1}) = \begin{cases} e_\chi e_p(1 - p^{r-1}\,\mathrm{Fr}_p^{-1}), & \text{if } r > 1 \text{ or } \chi(D_p) \neq 1 \\ |D_p/J_p|^{-1} e_\chi, & \text{otherwise.} \end{cases}$$

Then one has $T\Omega^{\mathrm{loc}}(\mathbb{Q}(r)_K, \mathbb{Z}[G])_p = 0$ if and only if

$$\mathbb{Z}_p[G] \cdot \prod_{\substack{\ell \mid N \\ \ell \neq p}} e_\ell(-\mathrm{Fr}_\ell^{-1})\, e_p(1 - \frac{\mathrm{Fr}_p}{p^r})^{-1} e^*_p(1 - p^{r-1}\,\mathrm{Fr}_p^{-1})\, \widetilde{\exp}((\beta_N, 1)) = \mathrm{Det}^{-1}_{\mathbb{Z}_p[G]} R\Gamma(K_p, \mathbb{Z}_p(r)). \tag{16}$$

Proof. It suffices to prove that (16) is equivalent to (13).
Now, by its definition in §2.3, the morphism θ_p which occurs in (13) is induced by taking the tensor product of the morphisms

$$\theta_p^{p\text{-part}} : A_p \otimes_A \Xi^{\mathrm{loc}}(M) \cong \mathrm{Det}^{-1}_{A_p} R\Gamma(K_p, \mathbb{Q}_p(r)) \otimes_{A_p} (V_p^{-1}, -1),$$

where we set $V_p^{-1} := \mathrm{Hom}_{A_p}(V_p, A_p)$, and for each prime $\ell \mid N$ with $\ell \neq p$

$$\theta_p^{\ell\text{-part}} : (A_p, 0) \cong \mathrm{Det}^{-1}_{A_p} R\Gamma(K_\ell, \mathbb{Q}_p(r)).$$

In addition, for $W = V_p$ the exact triangle (7) identifies with

$$K_p[-1] \to R\Gamma_f(\mathbb{Q}_p, V_p) \to \left(D_{\mathrm{cris}}(V_p) \xrightarrow{1 - p^{-r}\,\mathrm{Fr}_p} D_{\mathrm{cris}}(V_p) \right)$$

(with this last complex concentrated in degrees 0 and 1), and there is a canonical quasi-isomorphism

$$R\Gamma_f(\mathbb{Q}_p, V_p^*(1))^*[-2] \cong \left(D_{\mathrm{cris}}(V_p) \xrightarrow{1-p^{r-1}\,\mathrm{Fr}_p^{-1}} D_{\mathrm{cris}}(V_p)\right),$$

where the latter complex is concentrated in degrees 1 and 2. The identity map on $D_{\mathrm{cris}}(V_p)$ therefore induces isomorphisms of graded invertible A_p-modules

$$(17)\quad (K_p, 1) \cong \mathrm{Det}_{A_p}^{-1} R\Gamma_f(\mathbb{Q}_p, V_p); \quad (A_p, 0) \cong \mathrm{Det}_{A_p} R\Gamma_f(\mathbb{Q}_p, V_p^*(1))^*[-2].$$

The morphism $\theta_p^{p\text{-part}}$ is thus induced by (17) and (6) together with the (elementary) comparison isomorphism

$$\gamma : Y_{\Sigma,p} = H_B(M)_p \cong H_p(M) = V_p$$

between the Betti and p-adic realizations of M. On the other hand, the isomorphism $\widetilde{\exp}$ arises by passing to the cohomology sequence of (6) and then also using the identifications in (14) if $r = 1$. Hence, from [8, Lem. 1, Lem. 2], one has

$$(18)\qquad \theta_p^{p\text{-part}} = e_p(1 - p^{-r}\,\mathrm{Fr}_p)^{-1} e_p^*(1 - p^{r-1}\,\mathrm{Fr}_p^{-1})\widetilde{\exp} \otimes_{A_p} \gamma^{-1}.$$

Now if $\ell \neq p$, then Proposition 7.1 below implies that

$$\mathrm{Det}_{A_p}(-\sigma_\ell \ell^{-1}|(V_p)_{I_\ell})^{-1} \cdot \theta_p^{\ell\text{-part}}((\mathbb{Z}_p[G], 0)) = \mathrm{Det}_{\mathbb{Z}_p[G]}^{-1} R\Gamma(K_\ell, \mathbb{Z}_p(r)).$$

Thus, since $\gamma(\sigma_N)$ is a $\mathbb{Z}_p[G]$-basis of T_p, we find that (13) holds if and only if the element

$$\prod_{\substack{\ell | N \\ \ell \neq p}} \mathrm{Det}_{A_p}(-\sigma_\ell \ell^{-1}|(V_p)_{I_\ell})\, e_p(1 - p^{-r}\,\mathrm{Fr}_p)^{-1} e_p^*(1 - p^{r-1}\,\mathrm{Fr}_p^{-1})\,\widetilde{\exp}((\beta_N, 1))$$

is a $\mathbb{Z}_p[G]$-basis of $\mathrm{Det}_{\mathbb{Z}_p[G]}^{-1} R\Gamma(K_p, \mathbb{Z}_p(r))$. But

$$\mathrm{Det}_{A_p}(-\sigma_\ell \ell^{-1}|(V_p)_{I_\ell}) = \mathrm{Det}_{A_p}(-\mathrm{Fr}_\ell^{-1}\,\ell^{r-1}|A_p \cdot e_{J_\ell}) = e_\ell(-\mathrm{Fr}_\ell^{-1})e_\ell(\ell^{r-1})$$

and so Proposition 4.4 is implied by Lemma 4.5 below with u equal to the function which sends 0 to ℓ^{r-1} and all non-zero integers to 1. □

Lemma 4.5. *Fix a prime number $\ell \neq p$. If $u : \mathbb{Z} \to \mathbb{Z}_p[G]^\times$ is any function such that $\ell - 1$ divides $u(0) - u(1)$ in $\mathbb{Z}_p[G]$, then the element $\sum_\chi u(\mathrm{ord}_\ell(f_\chi))e_\chi$ is a unit of $\mathbb{Z}_p[G]$.*

Proof. If $\ell - 1$ divides $u(0) - u(1)$, then $\ell - 1$ divides $(u(1) - u(0))/u(1)u(0) = u(0)^{-1} - u(1)^{-1}$. It follows that the function u^{-1} also satisfies the hypothesis of the lemma and so it suffices to prove that the element $x_u := \sum_\chi u(\mathrm{ord}_\ell(f_\chi))e_\chi$ belongs to $\mathbb{Z}_p[G]$.

To this end, we let $J_\ell = J_{\ell,0} \subseteq G$ denote the inertia subgroup at ℓ and $J_{\ell,k} \subseteq J_{\ell,k-1} \subseteq \cdots \subseteq J_{\ell,1} \subseteq J_{\ell,0}$ its canonical filtration, so that a character χ satisfies $\mathrm{ord}_\ell(f_\chi) = k$ if and only if $\chi(J_{\ell,k}) = 1$ (and $\chi(J_{\ell,k-1}) \neq 1$ if $k > 0$). Then

$$x_u = \sum_{k=0}^{k=K} u(k)(e_{J_{\ell,k}} - e_{J_{\ell,k-1}}) = \sum_{k=0}^{k=K-1} (u(k) - u(k+1))e_{J_{\ell,k}} + u(K)e_{J_{\ell,K}}$$

where $K = \mathrm{ord}_\ell(N)$ and we have set $e_{J_{\ell,-1}} := 0$. For $k \geq 1$ one has $e_{J_{\ell,k}} \in \mathbb{Z}_p[G]$ since $J_{\ell,k}$ is an ℓ-group and $\ell \neq p$. If $K = 0$, then $e_{J_{\ell,0}} = e_{J_{\ell,K}} = 1$ also lies in $\mathbb{Z}_p[G]$. Otherwise the assumptions that $\ell - 1$ divides $u(0) - u(1)$ and that $\ell \neq p$ combine to imply that

$$(u(0) - u(1))e_{J_{\ell,0}} = \frac{u(0) - u(1)}{(\ell - 1)\ell^{K-1}} \sum_{g \in J_{\ell,0}} g \in \mathbb{Z}_p[G],$$

as required. □

5. Local Iwasawa theory

As preparation for our proof of (16) we now prove a result in Iwasawa theory. We write

$$N = N_0 p^\nu; \qquad \nu \geq 0,\ p \nmid N_0.$$

For any natural number n we set $G_n := \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$. We also let $\mathbb{Q}(\zeta_{Np^\infty})$ denote the union of the fields $\mathbb{Q}(\zeta_{Np^m})$ over $m \geq 0$ and set $G_{Np^\infty} := \mathrm{Gal}(\mathbb{Q}(\zeta_{Np^\infty})/\mathbb{Q})$. We then define

$$\Lambda := \mathbb{Z}_p[[G_{Np^\infty}]] = \varprojlim_n \mathbb{Z}_p[G_{Np^n}] \cong \mathbb{Z}_p[G_{N_0\tilde{p}}][[T]].$$

Here we have set $\tilde{p} := p$ for odd p and $\tilde{p} := 4$ for $p = 2$, and the isomorphism depends on a choice of topological generator of $\mathrm{Gal}(\mathbb{Q}(\zeta_{Np^\infty})/\mathbb{Q}(\zeta_{N_0\tilde{p}})) \cong \mathbb{Z}_p$. We also set

$$T_p^\infty := \varprojlim_n \mathrm{Ind}_{\mathbb{Q}(\zeta_{Np^n})}^{\mathbb{Q}} \mathbb{Z}_p(r).$$

This is a free rank one Λ-module upon which the absolute Galois group $G_\mathbb{Q} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts by the character $(\chi_{\mathrm{cyclo}})^r\tau^{-1}$ where $\chi_{\mathrm{cyclo}} : G_\mathbb{Q} \to \mathbb{Z}_p^\times$ is the cyclotomic character and $\tau : G_\mathbb{Q} \to G_{Np^\infty} \subseteq \Lambda^\times$ is the tautological character. In this section we shall describe (in Proposition 5.2) a basis of the invertible Λ-module $\mathrm{Det}_\Lambda^{-1} R\Gamma(\mathbb{Q}_p, T_p^\infty)$.

We note first that the cohomology of $R\Gamma(\mathbb{Q}_p, T_p^\infty)$ is naturally isomorphic to

$$H^i(\mathbb{Q}_p, T_p^\infty) \cong \begin{cases} (\varprojlim_n \mathbb{Q}(\zeta_{Np^n})_p^\times/p^n) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r-1) & i = 1 \\ \prod_{v|p} \mathbb{Z}_p(r-1) & i = 2 \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

where the limit is taken with respect to the norm maps (and $\mathbb{Q}(\zeta_{Np^n})_p = \mathbb{Q}(\zeta_{Np^n}) \otimes_\mathbb{Q} \mathbb{Q}_p$ is a finite product of local fields). The valuation map induces a natural short exact sequence

$$0 \to \tilde{Z} := \varprojlim_n \mathcal{O}^\times_{\mathbb{Q}(\zeta_{Np^n})_p}/p^n \to \varprojlim_n \mathbb{Q}(\zeta_{Np^n})_p^\times/p^n \xrightarrow{\mathrm{val}} \prod_{v|p} \mathbb{Z}_p \to 0 \tag{20}$$

and in addition Perrin-Riou has constructed an exact sequence [34, Prop. 4.1.3]

$$0 \to \prod_{v|p} \mathbb{Z}_p(r) \to \tilde{Z}(r-1) \xrightarrow{\theta_r^{PR}} R \to \prod_{v|p} \mathbb{Z}_p(r) \to 0 \tag{21}$$

where

$$R := \{f \in \mathbb{Z}[\zeta_{N_0}]_p[[X]] \mid \psi(f) := \sum_{\zeta^p=1} f(\zeta(1+X)-1) = 0\}$$

and $\mathbb{Z}[\zeta_{N_0}]_p$ denotes the finite étale $\mathbb{Z}_p$-algebra $\mathbb{Z}[\zeta_{N_0}] \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We remark that, whilst p is assumed to be odd in [34] the same arguments show that the sequence (21) exists and is exact also in the case $p = 2$. The $\mathbb{Z}_p$-module R carries a natural continuous G_{Np^∞}-action [34, 1.1.4], and with respect to this action all maps in (19), (20) and (21) are Λ-equivariant. In addition, if $r = 1$, then the exact sequence (21) is due to Coleman and the map θ_1^{PR} is given by

$$\theta_1^{PR}(u) = \left(1 - \frac{\phi}{p}\right) \log(f_u) \tag{22}$$

where f_u is the (unique) Coleman power series of the norm compatible system of units u with respect to $(\zeta_{p^n})_{n\geq 1}$ and one has $\phi(f_u)(X) = f_u^{\mathrm{Fr}_p}((1+X)^p - 1)$.

Lemma 5.1. *The Λ-module R is free of rank one with basis*

$$\beta_{N_0}^\infty := \xi_{N_0}(1+X); \qquad \xi_{N_0} := \sum_{N_1|d|N_0} \zeta_d$$

where $N_1 := \prod_{\ell|N_0} \ell$.

Proof. The element ξ_{N_0} is a $\mathbb{Z}_p[G_{N_0}]$-basis of $\mathbb{Z}[\zeta_{N_0}]_p$. Indeed, this observation (which is due originally to Leopoldt [30]) can be explicitly deduced from [31, Th. 2] after observing that the idempotents ε_d of loc. cit. belong to $\mathbb{Z}_p[G_{N_0}]$. On the other hand, Perrin-Riou shows in [33, Lem. 1.5] that if W is the ring of integers in any finite unramified extension of $\mathbb{Z}_p$, then $W[[X]]^{\psi=0}$ is a free rank one $W[[G_{p^\infty}]]$-module with basis $1 + X$ (her proof applies for all primes p, including $p = 2$). Since $\mathbb{Z}[\zeta_{N_0}]_p$ is a finite product of such rings W and $G_{Np^\infty} \cong G_{N_0} \times G_{p^\infty}$, the result follows. □

Proposition 5.2. *Let Q be the total ring of fractions of Λ (so Q is a finite product of fields). Using Lemma 5.1, we regard $\beta_{N_0}^\infty$ as a Q-basis of*

$$R \otimes_\Lambda Q \cong \tilde{Z}(r-1) \otimes_\Lambda Q \cong H^1(\mathbb{Q}_p, T_p^\infty) \otimes_\Lambda Q \cong (\mathrm{Det}_\Lambda^{-1} R\Gamma(\mathbb{Q}_p, T_p^\infty)) \otimes_\Lambda Q,$$

where the first isomorphism is induced by $(\theta_r^{PR} \otimes_\Lambda Q)^{-1}$, the second by (19) and the $(r-1)$-fold twist of (20) and the third by (19). Then one has

$$\Lambda \cdot \beta_{N_0}^\infty = \mathrm{Det}_\Lambda^{-1} R\Gamma(\mathbb{Q}_p, T_p^\infty) \subset (\mathrm{Det}_\Lambda^{-1} R\Gamma(\mathbb{Q}_p, T_p^\infty)) \otimes_\Lambda Q.$$

Proof. We note first that, since Λ is noetherian, Cohen-Macauley and semilocal, it is enough to prove that $\beta_{N_0}^\infty$ is a $\Lambda_\mathfrak{q}$-basis of $\mathrm{Det}_{\Lambda_\mathfrak{q}}^{-1} R\Gamma(\mathbb{Q}_p, T_p^\infty)_\mathfrak{q}$ for all height one prime ideals $\mathfrak{q}$ of Λ (see, for example, [17, Lem. 5.7]). In view of (19), (20) and (21) this claim is immediate for prime ideals $\mathfrak{q}$ which are not in the support of the (torsion) Λ-modules $\prod_{v|p} \mathbb{Z}_p(r-1)$ and $\prod_{v|p} \mathbb{Z}_p(r)$. On the other hand, since these modules are each p-torsion free, any prime $\mathfrak{q}$ which does lie in their support is *regular* in the sense that $p \notin \mathfrak{q}$ (see, for example, [17, p. 90]). In particular, in any such case $\Lambda_{\mathfrak{q}}$ is a discrete valuation ring and so it suffices to check cancellation of the Fitting ideals of the occurring torsion modules. But the Fitting ideal of $H^2(\mathbb{Q}_p, T_p^\infty)_{\mathfrak{q}}$ cancels against that of the module $(\prod_{v|p}\mathbb{Z}_p(r-1))_{\mathfrak{q}}$ which occurs in the $(r-1)$-fold twist of (20), whilst the Fitting ideals of the kernel and cokernel of θ_r^{PR} obviously cancel against each other. □

6. Descent calculations

In this section we deduce equality (16) as a consequence of Proposition 5.2 and thereby finish the proof of Theorem 1.1.

At the outset we note that the natural ring homomorphism

$$\Lambda \to \mathbb{Z}_p[G] \subseteq \mathbb{Q}_p[G] = \prod_\chi \mathbb{Q}_p(\chi) \tag{23}$$

induces an isomorphism of perfect complexes of $\mathbb{Z}_p[G]$-modules

$$R\Gamma(\mathbb{Q}_p, T_p^\infty) \otimes^{\mathbb{L}}_\Lambda \mathbb{Z}_p[G] \cong R\Gamma(\mathbb{Q}_p, T_p)$$

and hence also an isomorphism of determinants

$$\mathrm{Det}_\Lambda^{-1} R\Gamma(\mathbb{Q}_p, T_p^\infty) \otimes_\Lambda \mathbb{Z}_p[G] \cong \mathrm{Det}^{-1}_{\mathbb{Z}_p[G]} R\Gamma(\mathbb{Q}_p, T_p).$$

Taken in conjunction with Proposition 5.2, this shows that $(\beta^\infty_{N_0} \otimes_\Lambda 1, 1)$ is a $\mathbb{Z}_p[G]$-basis of the graded module $\mathrm{Det}^{-1}_{\mathbb{Z}_p[G]} R\Gamma(\mathbb{Q}_p, T_p)$. Hence, if we define an element u of $\mathbb{Q}_p[G]^\times$ by means of the equality

$$\prod_{\ell | N_0} e_\ell(-\mathrm{Fr}_\ell^{-1})\, e_p(1-\frac{\mathrm{Fr}_p}{p^r})^{-1} e_p^*(1-p^{r-1}\,\mathrm{Fr}_p^{-1})\,\widetilde{\exp}((\beta_N, 1)) \tag{24}$$
$$= (u\cdot \beta^\infty_{N_0} \otimes_\Lambda 1, 1)$$

then it is clear that the equality (16) is valid if and only if $u \in \mathbb{Z}_p[G]^\times$.

6.1. The unit u'.

To prove that the element u defined in (24) belongs to $\mathbb{Z}_p[G]^\times$ we will compare it to the unit described by the following result.

Lemma 6.1. *There exists a unit $u' \in \mathbb{Z}_p[G]^\times$ such that for any integer k with $0 \le k \le \nu$ and any $\mathbb{Q}_p$-rational character χ of G the element $e_\chi(\zeta_{p^k}\xi_{N_0}^{\mathrm{Fr}_p^{-k}})$ is equal to*

$$\begin{cases} \chi(u')\prod_{\ell|N_0, \ell\nmid f_\chi}\frac{1}{\ell-1}\prod_{\ell|N_0} e_\ell(-\mathrm{Fr}_\ell^{-1})e_\chi\zeta_{f_\chi}, & \text{if } k=\mathrm{ord}_p(f_\chi)\\ \chi(u')(-\mathrm{Fr}_p^{-1})\prod_{\ell|N,\ell\nmid f_\chi}\frac{1}{\ell-1}\prod_{\ell|N_0} e_\ell(-\mathrm{Fr}_\ell^{-1})e_\chi\zeta_{f_\chi}, & \text{if } k=1,\ \mathrm{ord}_p(f_\chi)=0\\ 0, & \text{otherwise.}\end{cases}$$

Proof. For $d \mid N_0$ and $k \ge 0$ we set $d_k := p^k d$ and

$$a(d) := (d,1) \in (\mathbb{Z}/p^\nu\mathbb{Z})^\times \times (\mathbb{Z}/N_0\mathbb{Z})^\times \cong (\mathbb{Z}/N\mathbb{Z})^\times \cong G$$

so that $\zeta_{p^k}\zeta_d^{\mathrm{Fr}_p^{-k}} = \zeta_{d_k}^{a(d)}$. Since $\xi_{N_0} = \sum_{N_1|d|N_0}\zeta_d$ Lemma 6.2 below implies

$$e_\chi(\zeta_{p^k}\xi_{N_0}^{\mathrm{Fr}_p^{-k}}) = \sum_{N_1|d|N_0, f_\chi|d_k} \frac{\phi(f_\chi)}{\phi(d_k)}\mu(\frac{d_k}{f_\chi})\chi^{-1}(\frac{d_k}{f_\chi})\chi(a(d))e_\chi\zeta_{f_\chi}. \tag{25}$$

The only non-vanishing summands in (25) are those for which the quotient d_k/f_χ is both square-free and prime to f_χ. Given the nature of the summation condition there is a unique such summand corresponding to

$$d_k/f_\chi = \begin{cases} \prod_{\ell|N_0, \ell\nmid f_\chi}\ell, & \text{if } k = \mathrm{ord}_p(f_\chi) \\ \prod_{\ell|N, \ell\nmid f_\chi}\ell, & \text{if } k = 1 \text{ and } \mathrm{ord}_p(f_\chi) = 0. \end{cases}$$

If neither of these conditions on k and $\mathrm{ord}_p(f_\chi)$ is satisfied, then $e_\chi(\zeta_{p^k}\xi_{N_0}^{\mathrm{Fr}_p^{-k}}) = 0$. By using the multiplicativity of μ, ϕ and χ and the equalities $\mu(\ell) = -1$ and $\phi(\ell) = \ell - 1$ we then compute that (25) is equal to

$$\begin{cases} \chi(a(d_\chi))\prod_{\ell|N_0, \ell\nmid f_\chi}\left(\frac{1}{\ell-1}(-\chi^{-1}(\ell))\right)e_\chi\zeta_{f_\chi}, & \text{if } k = \mathrm{ord}_p(f_\chi) \\ \chi(a(d_\chi))\prod_{\ell|N, \ell\nmid f_\chi}\left(\frac{1}{\ell-1}(-\chi^{-1}(\ell))\right)e_\chi\zeta_{f_\chi}, & \text{if } k = 1,\ \mathrm{ord}_p(f_\chi) = 0 \\ 0, & \text{otherwise} \end{cases}$$

where d_χ is the index of the unique nonvanishing summand in (25), i.e. $d_\chi = f_{\chi,0}\prod_{\ell|N_0, \ell\nmid f_\chi}\ell$ with $f_{\chi,0}$ the prime to p-part of f_χ. Now the element

$$u' := \sum_\chi \chi\big(a(d_\chi)\big)e_\chi \in \mathbb{Q}_p[G]^\times$$

belongs to $\mathbb{Z}_p[G]^\times$ by Lemma 4.5 (indeed the function $d \mapsto a(d)$ is multiplicative, $d_\chi = d(\mathrm{ord}_\ell(f_\chi))$ is a function of $\mathrm{ord}_\ell(f_\chi)$ only and satisfies $d(0) = d(1)$ as such). From here the explicit description of Lemma 6.1 follows because the definition of e_ℓ ensures that $\prod_{\ell|N_0, \ell\nmid f_\chi}\big(-\chi^{-1}(\ell)\big) = \prod_{\ell|N_0} e_\ell(-\mathrm{Fr}_\ell^{-1})e_\chi$. □

Lemma 6.2. *For any $\mathbb{Q}$-rational (resp. $\mathbb{Q}_p$-rational) character χ of $G \cong (\mathbb{Z}/N\mathbb{Z})^\times$, any $d \mid N$ and any* primitive *d-th root of unity ζ_d^a we have*

$$e_\chi\zeta_d^a = \begin{cases} 0, & \text{if } f_\chi \nmid d \\ \dfrac{\phi(f_\chi)}{\phi(d)}\mu(\dfrac{d}{f_\chi})\chi^{-1}(\dfrac{d}{f_\chi})\chi(a)e_\chi\zeta_{f_\chi}, & \text{if } f_\chi \mid d \end{cases} \tag{26}$$

in K (resp. K_p). Here $\phi(m)$ is Euler's ϕ-function, $\mu(m)$ is the Möbius function and $\chi(m) = 0$ if $(m, f_\chi) > 1$.

Proof. Recall that we view a $\mathbb{Q}$-rational character χ as the tautological homomorphism $G \to A^\times \to \mathbb{Q}(\chi)^\times$ to the field $\mathbb{Q}(\chi) := e_\chi A$ which is a direct ring factor of A. Thus, any complex embedding $j : \mathbb{Q}(\chi) \to \mathbb{C}$ induces a complex character $j\chi = \eta : G \to \mathbb{C}^\times$. We set $b := N/d$. Then under the $\mathbb{C}$-linear map $\sigma_N : K_\mathbb{C} \to \mathbb{C}$ the element

$$(j\, e_\chi)\zeta_d^a = e_\eta\zeta_N^{ab} = \frac{1}{|G|}\sum_{g\in G}\eta(g^{-1})g\,\zeta_N^{ab} = \frac{1}{\phi(N)}\sum_{x \bmod N}\bar{\eta}(x)\zeta_N^{xab} \in K_\mathbb{C}$$

is sent to the general Gaussian sum $\phi(N)^{-1}\tau(\bar\eta_N|\zeta_N^{ab})$ in the notation of Hasse [22, §20.1]. By [22, §20.2.IV] we have

$$\tau(\bar\eta_N|\zeta_N^{ab}) = \begin{cases} 0, & f_\eta\nmid d\\ \frac{\phi(N)}{\phi(d)}\mu(\frac{d}{f_\chi})\bar\eta(\frac{d}{f_\chi})\eta(a)\tau(\bar\eta), & f_\eta\mid d\end{cases}$$

where the Gaussian sum $\tau(\eta)$ attached to the character η is as defined in (11). For $d=f_\chi$ and $\zeta_d^a=\zeta_{f_\chi}$ we find $\tau(\bar\eta)=\phi(f_\chi)\sigma_N((j\,e_\chi)\zeta_{f_\chi})$. This yields the image of (26) under σ_N. Note that $K_{\mathbb{C}}\cong\prod_{g\in G}\mathbb{C}$ via $x\mapsto(\sigma_N gx)_{g\in G}$ and both sides of (26) are multiplied by $\chi(g)$ after applying g. Since $j\chi(g)=\eta(g)$ is a scalar and σ_N is $\mathbb{C}$-linear we find that (26) holds in $K_{\mathbb{C}}$, hence in K, hence also in K_p for all p. □

Given Lemma 6.1, our proof of Theorem 1.1 will be complete if we can show that $uu'\in\mathbb{Z}_p[G]^\times$. Recalling Lemma 4.5 it thus suffices to prove that for each $\mathbb{Q}_p$-rational character χ one has

$$\chi(uu') = \frac{f_{\chi,0}^{r-1}\prod_{\ell|N_0,\ell\nmid f_\chi}(\ell-1)}{[\mathbb{Q}(\zeta_{N_0}):\mathbb{Q}(\zeta_{f_{\chi,0}})]} \tag{27}$$

where $f_{\chi,0}$ denotes for the prime to p-part of f_χ. (In this regard note that the expression on the right hand side of (27) belongs to $\mathbb{Z}_p^\times$.)

We shall use explicit descent computations to prove that (27) is a consequence of the definition of u in (24). To this end, for each $\mathbb{Q}_p$-character χ of G we let $\mathfrak{q}_\chi$ denote the kernel of the homomorphism $\Lambda\to\mathbb{Q}_p(\chi)$ in (23). Then $\mathfrak{q}_\chi$ is a regular prime ideal of Λ and $\Lambda_{\mathfrak{q}_\chi}$ is a discrete valuation ring with residue field $\mathbb{Q}_p(\chi)$. To apply [17, Lem. 5.7] we need to describe a $\Lambda_{\mathfrak{q}_\chi}$-basis of $H^1(\mathbb{Q}_p,T_p^\infty)_{\mathfrak{q}_\chi}$ and for this purpose we find it convenient to split the argument into several different cases.

6.2. THE CASE $r>1$ OR $\chi(D_p)\neq 1$. In this subsection we shall prove (27) for all characters χ except those which are trivial on D_p in the case that $r=1$. In particular, the material of this section completes the proof of Theorem 1.1 in the case $r>1$.

We note first that if either $r>1$ or $\chi(D_p)\neq1$, then $\mathfrak{q}_\chi$ does not lie in the support of either $\prod_{v|p}\mathbb{Z}_p(r-1)$ or $\prod_{v|p}\mathbb{Z}_p(r)$. Hence, modulo the identifications made in Proposition 5.2, it follows from (19), (20) and (21) that $\beta_{N_0}^\infty$ is a $\Lambda_{\mathfrak{q}_\chi}$-basis of $H^1(\mathbb{Q}_p,T_p^\infty)_{\mathfrak{q}_\chi}=(\mathrm{Det}_\Lambda^{-1}R\Gamma(\mathbb{Q}_p,T_p^\infty))_{\mathfrak{q}_\chi}$ and that $\beta_{N_0}^\infty\otimes_{\Lambda_{\mathfrak{q}_\chi}}1$ is equal to the image of $\beta_{N_0}^\infty$ under the composite map

$$H^1(\mathbb{Q}_p,T_p^\infty)_{\mathfrak{q}_\chi}\to H^1(\mathbb{Q}_p,T_p^\infty)_{\mathfrak{q}_\chi}\otimes_{\Lambda_{\mathfrak{q}_\chi}}\mathbb{Q}_p(\chi)\cong H^1(K_p,\mathbb{Q}_p(r))\otimes_{A_p}\mathbb{Q}_p(\chi)$$

where the isomorphism is induced by the vanishing of $H^2(\mathbb{Q}_p,T_p^\infty)_{\mathfrak{q}_\chi}$ (cf. [17, Lem. 5.7]).

6.2.1. *The descent diagram.* By an obvious semi-local generalization of the argument of [1, §2.3.2] there exists a commutative diagram of Λ-modules

$$\begin{array}{ccc} \tilde{Z}(r-1) & \xrightarrow{\theta_r^{PR}} & R \\ \downarrow & & \downarrow{\scriptstyle \Xi_{r,\nu}} \\ H^1(K_p,\mathbb{Q}_p(r)) & \xleftarrow{(r-1)!\exp} & K_p \end{array} \tag{28}$$

where $\nu=\mathrm{ord}_p(N)$ is as defined at the beginning of §5,

$$\Xi_{r,\nu}(f)=\begin{cases}\sum_{k=1}^{\nu}p^{rk-\nu}f^{\mathrm{Fr}_p^{-k}}(\zeta_{p^k}-1)+p^{-\nu}(1-\frac{\mathrm{Fr}_p}{p^r})^{-1}f(0), & \nu\geq 1\\ \mathrm{Tr}_{K(\zeta_p)/K}(\Xi_{r,1}(f)), & \nu=0\end{cases}$$

is the map of [1, Lem. 2.2.2] and the choice of Frobenius element $\mathrm{Fr}_p\in G\cong G_{N_0}\times G_{p^\nu}$ is that which acts trivially on p-power roots of unity. (We are grateful to Laurent Berger for pointing out that the methods of [3] show that the diagram (28) commutes even in the case $p=2$.)
Now for $f=\beta_{N_0}^\infty=\xi_{N_0}(1+X)$ this formula gives

$$\Xi_{r,\nu}(\beta_{N_0}^\infty)=\begin{cases}\sum\limits_{k=1}^{\nu}p^{rk-\nu}\zeta_{p^k}\,\mathrm{Fr}_p^{-k}\,\xi_{N_0}+p^{-\nu}\big(1-\dfrac{\mathrm{Fr}_p}{p^r}\big)^{-1}\xi_{N_0}, & \nu\geq 1\\ \mathrm{Tr}_{K(\zeta_p)/K}\left(p^{r-1}\zeta_p\,\mathrm{Fr}_p^{-1}+\dfrac{1}{p}(1-\dfrac{\mathrm{Fr}_p}{p^r})^{-1}\right)\xi_{N_0}= & \\ \left(-p^{r-1}\,\mathrm{Fr}_p^{-1}+\dfrac{p-1}{p}(1-\dfrac{\mathrm{Fr}_p}{p^r})^{-1}\right)\xi_{N_0}= & \\ (1-\dfrac{\mathrm{Fr}_p}{p^r})^{-1}(1-p^{r-1}\,\mathrm{Fr}_p^{-1})\xi_{N_0}, & \nu=0.\end{cases}$$

In addition, since either $r>1$ or $\chi(D_p)\neq 1$, one has $e_\chi\exp=e_\chi\widetilde{\exp}$ and so the commutativity of (28) implies that the e_χ-projection of the defining equality (24) is equivalent to an equality in $e_\chi K_p$ of the form

$$\begin{aligned}\prod_{\ell\mid N_0}e_\ell(-\mathrm{Fr}_\ell^{-1})\,e_p(1-\frac{\mathrm{Fr}_p}{p^r})^{-1}e_p(1-p^{r-1}\,\mathrm{Fr}_p^{-1})e_\chi\beta_N&\\ =\chi(u)(r-1)!\left(\sum_{k=1}^{\nu}p^{rk-\nu}e_\chi(\zeta_{p^k}\xi_{N_0}^{\mathrm{Fr}_p^{-k}})+p^{-\nu}\left(1-\frac{\mathrm{Fr}_p}{p^r}\right)^{-1}e_\chi\xi_{N_0}\right)&\end{aligned}\tag{29}$$

if $\nu\geq 1$, resp.

$$\prod_{\ell\mid N_0}e_\ell(-\mathrm{Fr}_\ell^{-1})e_\chi\beta_N=\chi(u)(r-1)!e_\chi\xi_{N_0}\tag{30}$$

if $\nu=0$.

6.2.2. *The case* $\mathrm{ord}_p(f_\chi) > 0$. In this case $\nu > 0$ and $e_p(x)e_\chi = e_\chi$ for all $x \in A_p^\times$ and so we may leave out all factors of the form $e_p(-)$ on the left hand side of (29). In addition, Lemma 6.1 implies that the only non-vanishing term in the summation on the right hand side of (29) is that corresponding to $k = \mathrm{ord}_p(f_\chi)$ and moreover that (29) is equivalent to an equality

$$e_\chi\beta_N = \chi(uu')(r-1)!p^{rk-\nu}\prod_{\ell|N_0,\ell\nmid f_\chi}\frac{1}{\ell-1}e_\chi\zeta_{f_\chi}.$$

Now, since $k = \mathrm{ord}_p(f_\chi)$ and $\nu = \mathrm{ord}_p(N)$, we have

$$p^{rk-\nu}\prod_{\ell|N_0,\ell\nmid f_\chi}\frac{1}{\ell-1} = \frac{[\mathbb{Q}(\zeta_{N_0}):\mathbb{Q}(\zeta_{f_{\chi,0}})]}{f_{\chi,0}^{r-1}\prod_{\ell|N_0,\ell\nmid f_\chi}(\ell-1)}\frac{f_\chi^{r-1}}{[K:\mathbb{Q}(\zeta_{f_\chi})]}.$$

To deduce the required equality (27) from the last two displayed formulas one need only substitute the expression for $e_\chi\beta_N$ given in Proposition 4.1.

6.2.3. *The case* $\mathrm{ord}_p(f_\chi) = 0$ *and* $\nu > 0$. In this case Lemma 6.1 shows that the only non-zero terms in the summation on the right hand side of (29) are those which correspond to $k = 0$ and $k = 1$. Moreover, one has $e_p(x)e_\chi = xe_\chi$ for $x \in A_p^\times$. By Lemma 6.1, equation (29) is thus equivalent to

$$(31)\quad \left(1-\frac{\mathrm{Fr}_p}{p^r}\right)^{-1}(1-p^{r-1}\,\mathrm{Fr}_p^{-1})e_\chi\beta_N =$$
$$\chi(uu')(r-1)!\prod_{\ell|N_0,\ell\nmid f_\chi}\frac{1}{\ell-1}\left(\frac{p^{r-\nu}}{p-1}(-\mathrm{Fr}_p^{-1}) + p^{-\nu}\left(1-\frac{\mathrm{Fr}_p}{p^r}\right)^{-1}\right)e_\chi\zeta_{f_\chi}.$$

But

$$\begin{aligned}
&\frac{p^{r-\nu}}{p-1}(-\mathrm{Fr}_p^{-1}) + p^{-\nu}\left(1-\frac{\mathrm{Fr}_p}{p^r}\right)^{-1}\\
=&\frac{p^{-\nu+1}}{p-1}\left(1-\frac{\mathrm{Fr}_p}{p^r}\right)^{-1}\left(p^{r-1}(-\mathrm{Fr}_p^{-1})\left(1-\frac{\mathrm{Fr}_p}{p^r}\right)+\frac{p-1}{p}\right)\\
=&\frac{1}{\phi(p^\nu)}\left(1-\frac{\mathrm{Fr}_p}{p^r}\right)^{-1}(1-p^{r-1}\,\mathrm{Fr}_p^{-1})
\end{aligned}$$

and so (31) implies that

$$e_\chi\beta_N = \chi(uu')(r-1)!\frac{1}{\phi(p^\nu)}\prod_{\ell|N_0,\ell\nmid f_\chi}\frac{1}{\ell-1}e_\chi\zeta_{f_\chi}.$$

The required equality (27) follows from this in conjunction with the equality

$$\frac{1}{\phi(p^\nu)}\prod_{\ell|N_0,\ell\nmid f_\chi}\frac{1}{\ell-1} = \frac{[\mathbb{Q}(\zeta_{N_0}):\mathbb{Q}(\zeta_{f_{\chi,0}})]}{f_{\chi,0}^{r-1}\prod_{\ell|N_0,\ell\nmid f_\chi}(\ell-1)}\frac{f_\chi^{r-1}}{[K:\mathbb{Q}(\zeta_{f_\chi})]}$$

and the expression for $e_\chi\beta_N$ given in Proposition 4.1.

6.2.4. *The case* $\nu = \mathrm{ord}_p(N) = 0$. In this case (27) results directly upon substituting the formulas of Proposition 4.1 and Lemma 6.1 (with $k = 0$) into (30).

6.3. The case $r = 1$ and $\chi(D_p) = 1$. In this case $\mathfrak{q}_\chi$ lies in the support of $\prod_{v|p} \mathbb{Z}_p(r-1)$ (but not of $\prod_{v|p} \mathbb{Z}_p(r)$) and $\beta^\infty_{N_0}$ is not a $\Lambda_{\mathfrak{q}_\chi}$-basis of $H^1(\mathbb{Q}_p, T_p^\infty)_{\mathfrak{q}_\chi}$. We fix a generator γ of $\mathbb{Z}_p \cong \mathrm{Gal}(\mathbb{Q}(\zeta_{Np^\infty})/K(\zeta_p)) \subseteq G_{Np^\infty}$ and then define a uniformizer of $\Lambda_{\mathfrak{q}_\chi}$ by setting

$$\varpi := 1 - \gamma.$$

The p-adic places of the fields $K = \mathbb{Q}(\zeta_{N_0 p^\nu})$ and $\mathbb{Q}(\zeta_{N_0 p^\infty})$ are in natural bijection. We fix one such place v_0 and set

$$\begin{aligned} \eta^\infty := (1 - \zeta_{p^n})_{n \geq 1} \in \quad & \varprojlim_n (\mathbb{Q}(\zeta_{N_0 p^n})^\times_{v_0})/p^n \\ & \subseteq \varprojlim_n \prod_{v|p} (\mathbb{Q}(\zeta_{N_0 p^n})^\times_v)/p^n = H^1(\mathbb{Q}_p, T_p^\infty). \end{aligned}$$

Then the image $\bar{\eta}^\infty$ of η^∞ in

$$H^1(\mathbb{Q}_p, T_p^\infty)_{\mathfrak{q}_\chi}/\varpi \subseteq H^1(K_p, \mathbb{Q}_p(1)) \otimes_{A_p} \mathbb{Q}_p(\chi) =: H^1(K_p, \mathbb{Q}_p(1))_\chi$$

coincides with that of $p \in \mathbb{Q}(\zeta_{N_0})^\times_{v_0}$ and so is non-zero. In particular therefore, η^∞ is a $\Lambda_{\mathfrak{q}_\chi}$-basis of $H^1(\mathbb{Q}_p, T_p^\infty)_{\mathfrak{q}_\chi}$. Now, by [17, Lem. 5.7] there is an exact sequence

$$0 \to H^1(\mathbb{Q}_p, T_p^\infty)_{\mathfrak{q}_\chi}/\varpi \to H^1(K_p, \mathbb{Q}_p(1))_\chi \xrightarrow{\beta} H^2(K_p, \mathbb{Q}_p(1))_\chi \to 0$$

where β is the χ-projection of the composite homomorphism

$$H^1(K_p, \mathbb{Q}_p(1)) \cong \hat{K}_p^\times \to \prod_{v|p} \mathbb{Q}_p \cong H^2(K_p, \mathbb{Q}_p(1)); \quad u_v \to \frac{\mathrm{Tr}_{K_v/\mathbb{Q}_p}(\log_p(u_v))}{\log_p(\chi_{\mathrm{cyclo}}(\gamma))}$$

(see [17, Lem. 5.8] and its proof). This exact sequence induces an isomorphism

$$\phi_\varpi : \mathrm{Det}^{-1}_{\mathbb{Q}_p(\chi)} R\Gamma(\mathbb{Q}_p, V_p)_\chi \cong H^1(\mathbb{Q}_p, T_p^\infty)_{\mathfrak{q}_\chi}/\varpi$$

and [17, Lem. 5.7] implies that, modulo the identifications made in Proposition 5.2, one has

$$\bar{\lambda} \cdot \beta^\infty_{N_0} \otimes_{\Lambda_{\mathfrak{q}_\chi}} 1 = \phi_\varpi^{-1}(\bar{\eta}^\infty) \tag{32}$$

where the elements $\lambda \in \Lambda^\times_{\mathfrak{q}_\chi}$ and $e \in \mathbb{Z}$ are defined by the equality

$$(\theta_1^{PR})^{-1}_{\mathfrak{q}_\chi}(\lambda \cdot \beta^\infty_{N_0}) = \varpi^e \eta^\infty \in H^1(\mathbb{Q}_p, T_p^\infty)_{\mathfrak{q}_\chi} \tag{33}$$

and $\bar{\lambda}$ denotes the image of λ in $\Lambda_{\mathfrak{q}_\chi}/\varpi$. This description of $\bar{\eta}^\infty$ implies that

$$\mathrm{val}(\bar{\eta}^\infty) = \beta(\exp(e_\chi b)),$$

where 'val' is the normalized valuation map which occurs in (14) and

$$b := |D_p|^{-1} \log_p(\chi_{\mathrm{cyclo}}(\gamma)) \in \mathbb{Q}_p \subseteq K_{v_0} \subseteq \prod_{v|p} K_v = K_p.$$

LEMMA 6.3. *The element λ that is defined in (33) belongs to Λ and in $\mathbb{Q}(\zeta_{N_0})_p \cong \prod_{v|p} \mathbb{Q}(\zeta_{N_0})_v$ one has*

$$b = -|D_p|^{-1}\left(1 - \frac{1}{p}\right)^{-1} \lambda \cdot \xi_{N_0}.$$

This formula uniquely determines the image $\bar{\lambda}$ of λ in $\mathbb{Q}_p(\chi)$.

Proof. Since $\beta^{\infty}_{N_0}$ is a basis of the free rank one Λ-module R (by Lemma 5.1) we have

$$\lambda \cdot \beta^{\infty}_{N_0} = \theta_1^{PR}((\eta^{\infty})^{\varpi}) \tag{34}$$

for some element λ of Λ, which then also satisfies the condition (33).
The map θ_1^{PR} is described explicitly by (22). Further, with respect to the system $(\zeta_{p^n})_{n\geq 1}$, the Coleman power series that is associated to the norm compatible system of units $(\eta^{\infty})^{\varpi} = (\eta^{\infty})^{(1-\gamma)}$ is equal to

$$f(X) := \frac{X}{(1+X)^{\chi_{\mathrm{cyclo}}(\gamma)} - 1} \equiv \chi_{\mathrm{cyclo}}(\gamma)^{-1} \mod (X).$$

Thus, by computing constant terms in the power series identity (34) we obtain equalities

$$\begin{aligned}
\lambda \cdot \xi_{N_0} &= (1 - \frac{\phi}{p}) \log(f(X))\Big|_{X=0} \\
&= (1 - \frac{1}{p}) \log_p(\chi_{\mathrm{cyclo}}(\gamma)^{-1}) \\
&= -(1 - \frac{1}{p})|D_p| b
\end{aligned}$$

as required to finish the proof of the first sentence of the lemma. On the other hand, the second sentence of the lemma is clear because ξ_{N_0} is a $\mathbb{Q}_p[G_{N_0}]$-basis of $\mathbb{Q}(\zeta_{N_0})_p$ and $\mathbb{Q}_p(\chi) = \Lambda_{\mathfrak{q}_\chi}/\varpi$ is a quotient of $\mathbb{Q}_p[G_{N_0}]$. $\square$

With $\widetilde{\exp}$ denoting the map in (15), the last lemma implies that

$$\begin{aligned}
\phi_{\varpi}^{-1}(\bar{\eta}^{\infty}) &= \bar{\eta}^{\infty} \wedge \exp(e_\chi b) \otimes \beta(\exp(e_\chi b))^{-1} \\
&= -\exp(e_\chi b) \wedge \bar{\eta}^{\infty} \otimes \mathrm{val}(\bar{\eta}^{\infty})^{-1} \\
&= \widetilde{\exp}(-e_\chi b) \\
&= \widetilde{\exp}\left(|D_p|^{-1}\left(1 - \frac{1}{p}\right)^{-1} \bar{\lambda} \cdot e_\chi \xi_{N_0}\right),
\end{aligned}$$

and hence, using (32), that

$$
\begin{aligned}
&\widetilde{\exp}^{-1}(\beta_{N_0}^{\infty} \otimes_{\Lambda_{\mathfrak{q}_\chi}} 1)\\
=&|D_p|^{-1}(1-\frac{1}{p})^{-1}e_\chi(\xi_{N_0})\\
=&|D_p|^{-1}(1-\frac{1}{p})^{-1}\chi(u')\prod_{\ell|N_0,\ell\nmid f_\chi}\frac{1}{\ell-1}\prod_{\ell|N_0}e_\ell(-\mathrm{Fr}_\ell^{-1})e_\chi\zeta_{f_\chi}\\
=&|D_p|^{-1}(1-\frac{1}{p})^{-1}\chi(u')\prod_{\ell|N_0,\ell\nmid f_\chi}\frac{1}{\ell-1}\prod_{\ell|N_0}e_\ell(-\mathrm{Fr}_\ell^{-1})[K:\mathbb{Q}(\zeta_{f_\chi})]e_\chi\beta_N\\
=&|D_p/J_p|^{-1}(1-\frac{1}{p})^{-1}\chi(u')\frac{[\mathbb{Q}(\zeta_{N_0}):\mathbb{Q}(\zeta_{f_{\chi,0}})]}{f_{\chi,0}^{r-1}\prod_{\ell|N_0,\ell\nmid f_\chi}(\ell-1)}\prod_{\ell|N_0}e_\ell(-\mathrm{Fr}_\ell^{-1})e_\chi\beta_N
\end{aligned}
$$

where the second equality follows from Lemma 6.1, the third from Proposition 4.1 and the fourth from the fact that $r=1$, $f_\chi = f_{\chi,0}$ and

$$\frac{[K:\mathbb{Q}(\zeta_{f_\chi})]}{|J_p|\prod_{\ell|N_0,\ell\nmid f_\chi}(\ell-1)} = \frac{[\mathbb{Q}(\zeta_{N_0}):\mathbb{Q}(\zeta_{f_{\chi,0}})]}{f_{\chi,0}^{r-1}\prod_{\ell|N_0,\ell\nmid f_\chi}(\ell-1)}.$$

The required equality (27) is now obtained by comparing the above formula for $\widetilde{\exp}^{-1}(\beta_{N_0}^{\infty}\otimes_{\Lambda_{\mathfrak{q}_\chi}} 1)$ to the definition of u in (24).
This completes our proof of Theorem 1.1.

7. Some remarks concerning $T\Omega^{\mathrm{loc}}(M,\mathfrak{A})$

In this section we prove two results that were used in the proof of Theorem 1.1 but which are most naturally formulated in a more general setting. In particular, these results extend the computations made in [9, §5].
We henceforth fix notation as in §2. Thus, we stress, M is no longer assumed to be a Tate motive and the (finite dimensional semisimple) $\mathbb{Q}$-algebra A is not assumed to be either commutative or a group ring.

7.1. The contribution from primes $\ell \neq p$. We first recall the following basic fact about the cohomology of the profinite group $\hat{\mathbb{Z}}$ (for distinction we shall denote the canonical generator $1\in\hat{\mathbb{Z}}$ by σ). Let R be either a pro-p ring, or a localization of such a ring, and let C be a perfect complex of R-modules with a continuous action of $\hat{\mathbb{Z}}$. Then

$$R\Gamma(\hat{\mathbb{Z}},C)\cong \mathrm{Tot}(C\xrightarrow{1-\sigma}C)$$

is a perfect complex of R-modules where 'Tot' denotes the total complex of a double complex. The identity map of C induces a morphism

$$\mathrm{id}_{C,\mathrm{triv}}:\mathbf{1}_R\cong \mathrm{Det}_R^{-1}R\Gamma(\hat{\mathbb{Z}},C)$$

in $V(R)$ which is functorial for exact triangles in the variable C and also commutes with scalar extension.

PROPOSITION 7.1. *For a prime number $\ell \neq p$ we let σ_ℓ denote the Frobenius automorphism in* $\mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{ur}}/\mathbb{Q}_\ell)$. *If*

$$\theta_p^{\ell\text{-}part} : \mathbf{1}_{A_p} \cong \mathrm{Det}_{A_p}^{-1} R\Gamma(\mathbb{Q}_\ell, V_p)$$

denotes the morphism in $V(A_p)$ which occurs in [9, (67)], *then*

$$\mathrm{Det}_{A_p}(-\sigma_\ell \ell^{-1}|(V_p)_{I_\ell})^{-1}\ \theta_p^{\ell\text{-}part}$$

is induced by a morphism $\mathbf{1}_{\mathfrak{A}_p} \cong \mathrm{Det}_{\mathfrak{A}_p}^{-1} R\Gamma(\mathbb{Q}_\ell, T_p)$ in $V(\mathfrak{A}_p)$.

Proof. Recall the exact triangle of complexes of A_p-modules

$$R\Gamma_f(\mathbb{Q}_\ell, V_p) \to R\Gamma(\mathbb{Q}_\ell, V_p) \to R\Gamma_{/f}(\mathbb{Q}_\ell, V_p) \tag{35}$$

from [9, (18)] as well as the isomorphism

$$\mathrm{AV} : R\Gamma_{/f}(\mathbb{Q}_\ell, V_p) \cong R\Gamma_f(\mathbb{Q}_\ell, V_p^*(1))^*[-2]$$

from [9, Lem. 12a)]. The triangle (35) is obtained by applying $R\Gamma(\hat{\mathbb{Z}}, -)$ to the exact triangle

$$H^0(I_\ell, V_p) \to R\Gamma(I_\ell, V_p) \to H^1(I_\ell, V_p)[-1] \tag{36}$$

together with the isomorphism

$$R\Gamma(\hat{\mathbb{Z}}, R\Gamma(I_\ell, V_p)) \cong R\Gamma(\mathbb{Q}_\ell, V_p).$$

According to the convention [9, (19)] the generator σ we use here is σ_ℓ^{-1}. The isomorphism AV is more explicitly given by the diagram

$$\begin{array}{ccc}
(V_p)_{I_\ell}(-1) & \xrightarrow{1-\sigma_\ell^{-1}} & (V_p)_{I_\ell}(-1) \\
\| & & \downarrow{\scriptstyle -\sigma_\ell} \\
(V_p)_{I_\ell}(-1) & \xrightarrow{1-\sigma_\ell} & (V_p)_{I_\ell}(-1) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
((V_p^*)^{I_\ell}(1))^* & \xrightarrow{1-\sigma_\ell^{-1}} & ((V_p^*)^{I_\ell}(1))^*.
\end{array}$$

Note here that $H^1(I_\ell, V_p)$ is naturally isomorphic to $(V_p)_{I_\ell}(-1)$ and that in the isomorphism $((V_p^*)^{I_\ell})^* \cong (V_p)_{I_\ell}$ the first dual is the contragredient σ_ℓ-representation whereas the second is simply the dual. From this last diagram we deduce

$$\mathrm{id}_{(V_p)_{I_\ell}(-1),\mathrm{triv}} = \mathrm{Det}_{A_p}(-\sigma_\ell \ell^{-1}|(V_p)_{I_\ell})\, \mathrm{id}_{((V_p^*)^{I_\ell}(1))^*,\mathrm{triv}}$$

and by the discussion above with $R = A_p$ the exact triangle (36) gives

$$\mathrm{id}_{V_p^{I_\ell},\mathrm{triv}} \otimes \mathrm{id}_{(V_p)_{I_\ell}(-1),\mathrm{triv}}^{-1} = \mathrm{id}_{R\Gamma(I_\ell, V_p),\mathrm{triv}}\,.$$

By the definition of [9, (67)] the morphism $\theta_p^{\ell\text{-part}}$ is induced by the triangle (35), the isomorphism AV and the morphisms $\mathrm{id}_{V_p^{I_\ell},\mathrm{triv}}$ and $\mathrm{id}_{((V_p^*)^{I_\ell}(1))^*,\mathrm{triv}}$. Hence

$$\mathrm{Det}_{A_p}(-\sigma_\ell \ell^{-1}|(V_p)_{I_\ell})^{-1}\ \theta_p^{\ell\text{-part}}$$

is the scalar extension of the morphism $\mathrm{id}_{R\Gamma(I_\ell,T_p),\mathrm{triv}}$ in $V(\mathfrak{A}_p)$ and this finishes the proof of the Proposition. □

7.2. Artin-Verdier Duality. In this subsection we extend [9, Lem. 14] to include the case $p=2$ and hence resolve the issue raised in [9, Rem. 16].
Before stating the main result we recall that [9, (78)] defines a morphism in $V(A_p)$ of the form

$$\theta'_p : \left(\bigotimes_{\ell\in S_{p,f}} \mathrm{Det}_{A_p}^{-1} R\Gamma(\mathbb{Q}_\ell, V_p)\right) \otimes_{A_p} \mathrm{Det}_{A_p}^{-1}(V_p) \cong A_p \otimes_{\mathfrak{A}_p} \Lambda_p(S,T_p) \tag{37}$$

where

$$\Lambda_p(S,T_p) := \mathrm{Det}_{\mathfrak{A}_p} C(\mathbb{Q},T_p)$$

with $C(\mathbb{Q},T_p)$ a certain canonical perfect complex of $\mathfrak{A}_p$-modules (as occurs in the diagram (39) below with $E=\mathbb{Q}$).
We set

$$\epsilon_{V_p} := \mathrm{Det}_{A_p}(2|V_p^+)\,\mathrm{Det}_{A_p}(2|V_p^-)^{-1} \in K_1(A_p).$$

Proposition 7.2. *The morphism $\epsilon_{V_p}\cdot\theta'_p$ is induced by a morphism in $V(\mathfrak{A}_p)$ of the form*

$$\left(\bigotimes_{\ell\in S_{p,f}} \mathrm{Det}_{\mathfrak{A}_p}^{-1} R\Gamma(\mathbb{Q}_\ell, T_p)\right) \otimes_{\mathfrak{A}_p} \mathrm{Det}_{\mathfrak{A}_p}^{-1}(T_p) \cong \Lambda_p(S,T_p). \tag{38}$$

The proof of this result will occupy the remainder of this subsection.
We note first that if p is odd, then $\epsilon_{V_p}\in \mathrm{im}(K_1(\mathfrak{A}_p)\to K_1(A_p))$ and so the above claim is equivalent to asserting that θ'_p itself is induced by a morphism in $V(\mathfrak{A}_p)$ of the form (38). Since this is precisely the statement of [9, Lem. 14] we shall assume henceforth that $p=2$.
Now if E is any number field, then [9, (81)] gives a true nine term diagram

$$\begin{array}{ccccc}
\bigoplus_{v\in S_\infty} R\Gamma_\Delta(E_v,T_p^*(1))^*[-4] & = & \bigoplus_{v\in S_\infty} R\Gamma_\Delta(E_v,T_p^*(1))^*[-4] & & \\
\downarrow & & {\scriptstyle\alpha(E)}\downarrow & & \\
\tilde{R\Gamma}_c(\mathcal{O}_{E,S_p},T_p^*(1))^*[-4] & \longrightarrow & {}_3L(S_p,T_p)[-1] & \longrightarrow & R\Gamma_c(\mathcal{O}_{E,S_p},T_p) \\
\downarrow & & \downarrow & & \| \\
R\Gamma_c(\mathcal{O}_{E,S_p},T_p^*(1))^*[-4] & \longrightarrow & C(E,T_p) & \longrightarrow & R\Gamma_c(\mathcal{O}_{E,S_p},T_p)
\end{array} \tag{39}$$

where the complex ${}_3L(S_p,T_p)$ is endowed with a natural quasi-isomorphism

$$\beta(E) : {}_3L(S_p,T_p) \cong \bigoplus_{v\in S_p} R\Gamma(E_v,T_p).$$

To prove the Proposition we shall make an explicit study of the composite morphism $\beta(\mathbb{Q})\circ\alpha(\mathbb{Q})$. To do this we observe that if E is any Galois extension of $\mathbb{Q}$ with group Γ, then (39), resp. $\beta(E)$, is a true nine-term diagram, resp. quasi-isomorphism, of complexes of $\mathfrak{A}_p[\Gamma]$-modules and the same arguments as used in [8, Lem. 11] show that application of $R\operatorname{Hom}_{\mathbb{Z}_p[\Gamma]}(\mathbb{Z}_p,-)$ to (39), resp. $\beta(E)$, renders a diagram which is naturally isomorphic to the corresponding diagram for $E=\mathbb{Q}$, resp. a quasi-isomorphism which identifies naturally with $\beta(\mathbb{Q})$.

We now fix E to be an imaginary quadratic field and set $\Gamma := \mathrm{Gal}(E/\mathbb{Q})$ and $R\Gamma_{\mathrm{Tate}}(E_v,-) := R\Gamma(E_v,-)$ for each non-archimedean place v. Then for each $v_0\in S$ one has a natural morphism $R\Gamma_{\mathrm{Tate}}(E_{v_0},-)\to R\Gamma(E_{v_0},-)$ and we let $\gamma_{v_0}(E)$ denote the following composite morphism in $D(\mathfrak{A}_p[\Gamma])$

$$R\Gamma_\Delta(E_\infty,T_p^*(1))^*[-3] \xrightarrow{\beta(E)\circ\alpha(E)[1]} \bigoplus_{v\in S_p} R\Gamma(E_v,T_p)\to R\Gamma(E_{v_0},T_p).$$

Now if v_0 is non-archimedean, then $\gamma_{v_0}(E)$ is equal to the composite

$$R\Gamma_\Delta(E_\infty,T_p^*(1))^*[-3]\to\bigoplus_{v\in S_p} R\Gamma_{\mathrm{Tate}}(E_v,T_p)\to R\Gamma(E_{v_0},T_p), \tag{40}$$

where the first arrow denotes the diagonal morphism in the following commutative diagram in $D(\mathfrak{A}_p[\Gamma])$

$$\begin{tikzcd}
R\Gamma_\Delta(E_\infty,T_p^*(1))^*[-3] \arrow[r] \arrow[d, equal] & \tilde{R\Gamma}_c(\mathcal{O}_{E,S_p},T_p^*(1))^*[-3] \arrow[d] & R\Gamma(\mathcal{O}_{E,S_p},T_p) \arrow[l, "\mathrm{AV}"'] \arrow[d] \\
R\Gamma_\Delta(E_\infty,T_p^*(1))^*[-3] \arrow[r] & \bigoplus_{v\in S_p} R\Gamma_{\mathrm{Tate}}(E_v,T_p^*(1))^*[-2] & \bigoplus_{v\in S_p} R\Gamma_{\mathrm{Tate}}(E_v,T_p) \arrow[l, "\oplus\,\mathrm{AV}_v"']
\end{tikzcd}$$

in which the left, resp. right, hand square comes directly from the definition of $R\Gamma_\Delta(E_\infty,T_p^*(1))$ in [9, (80)], resp. from the compatibility of local and global Artin-Verdier duality as in [9, Lem. 12]. Since (v_0 is assumed for the moment to be non-archimedean and) the image of the lower left hand arrow in this diagram is contained in the summand $R\Gamma_{\mathrm{Tate}}(E_\infty,T_p^*(1))^*[-2]$ it is therefore clear that (40) is the zero morphism. Hence, there exists a natural isomorphism in $D(\mathfrak{A}_p[\Gamma])$ of the form

$$C(E,T_p)\cong C_\infty(E,T_p)[-1]\oplus\bigoplus_{v\in S_{p,f}} R\Gamma(E_v,T_p)[-1]$$

where $C_\infty(E,T_p)$ is a complex which lies in an exact triangle in $D(\mathfrak{A}_p[\Gamma])$ of the form

$$R\Gamma_\Delta(E_\infty,T_p^*(1))^*[-3]\xrightarrow{\gamma_\infty(E)} R\Gamma(E_\infty,T_p)\to C_\infty(E,T_p)\to . \tag{41}$$

Now, via the canonical identifications $R\Gamma_\Delta(E_\infty,T_p^*(1))^*[-3]\cong T_p(-1)[-3]$ and $R\Gamma(E_\infty,T_p)\cong T_p[0]$, we may regard $\gamma_\infty(E)$ as an element of

$$\operatorname{Hom}_{D(\mathfrak{A}_p[\Gamma])}(T_p(-1)[-3],T_p[0])\cong \operatorname{Ext}^3_{\mathfrak{A}_p[\Gamma]}(T_p(-1),T_p).$$

With respect to this identification, $C_\infty(E,T_p)$ represents $\gamma_\infty(E)$ viewed as a Yoneda 3-extension and so can be obtained via a push-out diagram of $\mathfrak{A}_p[\Gamma]$-modules of the form

$$\begin{array}{ccccccccccccc} 0 & \to & T_p & \to & T_p[\Gamma] & \xrightarrow{1-c} & T_p[\Gamma] & \xrightarrow{1+c} & T_p[\Gamma] & \to & T_p(-1) & \to & 0 \\ & & {\scriptstyle\mu}\downarrow & & \downarrow & & \| & & \| & & \| & & \\ 0 & \to & T_p & \to & B_\mu & \to & T_p[\Gamma] & \xrightarrow{1+c} & T_p[\Gamma] & \to & T_p(-1) & \to & 0. \end{array} \tag{42}$$

Here we write c for the natural diagonal action of the generator τ of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, the second arrow in the upper row is the map $t \mapsto t + \tau(t)\cdot\gamma$ where γ is the generator of Γ and the fifth arrow in both rows is the map $t + t'\cdot\gamma \mapsto (t-\tau(t'))\otimes\xi^{-1}$ with $\xi := (\zeta_{p^n})_{n\geq 1}$ (regarded as a generator of $\mathbb{Z}_p(1)$).
For any $\mathfrak{A}_p[\Gamma]$-module X the above diagram induces a commutative diagram of the form

$$\begin{array}{ccc} \mathrm{Ext}^i_{\mathfrak{A}_p[\Gamma]}(T_p,X) & \longrightarrow & \mathrm{Ext}^{i+3}_{\mathfrak{A}_p[\Gamma]}(T_p(-1),X) \\ {\scriptstyle\mu^{*,i}}\uparrow & & \| \\ \mathrm{Ext}^i_{\mathfrak{A}_p[\Gamma]}(T_p,X) & \longrightarrow & \mathrm{Ext}^{i+3}_{\mathfrak{A}_p[\Gamma]}(T_p(-1),X). \end{array}$$

But $C_\infty(E,T_p)$ belongs to $D^{\mathrm{perf}}(\mathfrak{A}_p[\Gamma])$ (since the lower row of (39) belongs to $D^{\mathrm{perf}}(\mathfrak{A}_p[\Gamma])$) and so the projective dimension of the $\mathfrak{A}_p[\Gamma]$-module B_μ is finite and therefore at most 1. This implies that the upper (resp. lower) horizontal map in the last diagram is bijective for $i\geq 2$ and surjective for $i=1$ (resp. bijective for $i\geq 1$). The map $\mu^{*,i}$ is therefore bijective for each $i\geq 2$ and surjective for $i=1$ and so a result of Holland [23, Th. 3.1] implies that there exists an automorphism $\alpha\in\mathrm{Aut}_{\mathfrak{A}_p[\Gamma]}(T_p)$ and a projective $\mathfrak{A}_p[\Gamma]$-module P such that $\mu-\alpha$ is equal to a composite of the form $T_p\to P\to T_p$. Now the Γ-module

$$\mathrm{Hom}_{\mathfrak{A}_p}(T_p(-1),P)\cong\mathrm{Hom}_{\mathfrak{A}_p}(T_p(-1),\mathfrak{A}_p)\otimes_{\mathfrak{A}_p}P =: T^*\otimes_{\mathfrak{A}_p}P$$

is cohomologically trivial (indeed, it suffices to check this for $P=\mathfrak{A}_p[\Gamma]$ in which case $T^*\otimes_{\mathfrak{A}_p}\mathfrak{A}_p[\Gamma] = T^*\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\Gamma]\cong\mathbb{Z}_p[\Gamma]^d$ with $d=\mathrm{rank}_{\mathbb{Z}_p}(T^*)$) and so $\mathrm{Ext}^3_{\mathfrak{A}_p[\Gamma]}(T_p(-1),P)\cong H^3(\Gamma,\mathrm{Hom}_{\mathfrak{A}_p}(T_p(-1),P))=0$. In the diagram (42) we may therefore assume that $\mu\in\mathrm{Aut}_{\mathfrak{A}_p[\Gamma]}(T_p)$ and hence can use this diagram to identify $C_\infty(E,T_p)$ with the complex

$$T_p[\Gamma]\xrightarrow{1-c}T_p[\Gamma]\xrightarrow{1+c}T_p[\Gamma],$$

where the first term is placed in degree 0 (and the cohomology is computed via the maps in upper row of (42)). Writing $C_\infty(T_p)$ for the complex

$$T_p\xrightarrow{1-c}T_p\xrightarrow{1+c}T_p$$

(where the first term is placed in degree 0), we may thus deduce the existence of a composite isomorphism in $D(\mathfrak{A}_p)$ of the form

$$\begin{aligned}
&C_\infty(T_p)[-1] \oplus \bigoplus_{\ell \in S_{p,f}} R\Gamma(\mathbb{Q}_\ell, T_p)[-1] \\
&\cong R\operatorname{Hom}_{\mathbb{Z}_p[\Gamma]}(\mathbb{Z}_p, C_\infty(E, T_p)[-1] \oplus \bigoplus_{v \in S_{p,f}} R\Gamma(E_v, T_p)[-1]) \\
&\cong R\operatorname{Hom}_{\mathbb{Z}_p[\Gamma]}(\mathbb{Z}_p, C(E, T_p)) \\
&\cong C(\mathbb{Q}, T_p).
\end{aligned}$$

When taken in conjunction with the natural morphism

$$\begin{aligned}
j(T_p) : \operatorname{Det}_{\mathfrak{A}_p} C_\infty(T_p)[-1] &= \operatorname{Det}^{-1}_{\mathfrak{A}_p}(T_p) \otimes_{\mathfrak{A}_p} (\operatorname{Det}_{\mathfrak{A}_p}(T_p) \otimes_{\mathfrak{A}_p} \operatorname{Det}^{-1}_{\mathfrak{A}_p}(T_p)) \\
&\cong \operatorname{Det}^{-1}_{\mathfrak{A}_p}(T_p) \otimes_{\mathfrak{A}_p} \mathbf{1}_{\mathfrak{A}_p} = \operatorname{Det}^{-1}_{\mathfrak{A}_p}(T_p)
\end{aligned}$$

the above composite isomorphism induces a morphism in $V(\mathfrak{A}_p)$ of the form

$$\theta_p'' : \left(\bigotimes_{\ell \in S_{p,f}} \operatorname{Det}^{-1}_{\mathfrak{A}_p} R\Gamma(\mathbb{Q}_\ell, T_p) \right) \otimes_{\mathfrak{A}_p} \operatorname{Det}^{-1}_{\mathfrak{A}_p}(T_p) \cong \operatorname{Det}_{\mathfrak{A}_p} C(\mathbb{Q}, T_p) =: \Lambda_p(S, T_p).$$

Now $A_p \otimes_{\mathfrak{A}_p} \theta_p''$ differs from the morphism θ_p' in (37) only in the following respect: in place of $A_p \otimes_{\mathfrak{A}_p} j(T_p)$ the morphism θ_p' involves the composite morphism

$$\begin{aligned}
j(V_p) : A_p \otimes_{\mathfrak{A}_p} &\operatorname{Det}_{\mathfrak{A}_p} C_\infty(T_p)[-1] \cong \\
&\operatorname{Det}^{-1}_{A_p}(A_p \otimes_{\mathfrak{A}_p} H^0(C_\infty(T_p))) \otimes_{A_p} \operatorname{Det}^{-1}_{A_p}(A_p \otimes_{\mathfrak{A}_p} H^2(C_\infty(T_p))) \\
&\cong \operatorname{Det}^{-1}_{A_p} V_p
\end{aligned}$$

where the first morphism is the canonical 'passage to cohomology' map and the second is induced by combining the isomorphisms $A_p \otimes_{\mathfrak{A}_p} H^0(C_\infty(T_p)) \cong V_p^+$ and $A_p \otimes_{\mathfrak{A}_p} H^2(C_\infty(T_p)) \cong V_p(-1)^+$ that are induced by the upper row of (42) with the isomorphism $V_p^+ \oplus V_p(-1)^+ \cong V_p^+ \oplus V_p^- = V_p$ (where the second component of the first map sends each element v of $V_p(-1)^+$ to $v \otimes \xi \in V_p^-$). But the complex $A_p \otimes_{\mathfrak{A}_p} C_\infty(T_p)$ identifies with

$$V_p^+ \oplus V_p^- \xrightarrow{(0,2)} V_p^+ \oplus V_p^- \xrightarrow{(2,0)} V_p^+ \oplus V_p^-$$

and so, by an explicit computation, one has $A_p \otimes_{\mathfrak{A}_p} j(T_p) = \epsilon_{V_p} \cdot j(V_p)$ where $\epsilon_{V_p} := \operatorname{Det}_{A_p}(2|V_p^+) \operatorname{Det}_{A_p}(2|V_p^-)^{-1} \in K_1(A_p)$. The induced equality

$$A_p \otimes_{\mathfrak{A}_p} \theta_p'' = \epsilon_{V_p} \cdot \theta_p'$$

then completes the proof of the Proposition.

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