Foreword

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I first met John Coates during my first year as a graduate student at Cambridge. John was about to move back to Cambridge where he had been a graduate student himself. It was at a point in his career when he was starting a whirlwind of moves. Coming from Stanford he spent two years in Cambridge, and one in Australia before making a longer stop in Paris at Orsay. Mathematically however he was just settling down to what has become his most serious and dedicated study of the last thirty years, the arithmetic of elliptic curves. Needless to say for those who have devoted some time to this subject, it is so full of fascinating problems that it is hard to turn from this to anything else. The conjecture of Birch and Swinnerton-Dyer, by then fifteen years old, had made the old subject irresistible.

In the two years he was at Cambridge we wrote four papers on elliptic curves, culminating in the proof of a part of the conjecture for elliptic curves with complex multiplication which are defined over the rationals. When John had been at Cambridge previously as a graduate student of Alan Baker he had worked on questions about the bounding of integral points on curves. Siegel's proof of the finiteness of the number of integral points on curves of genus at least one was not effective. Work of John's, in collaboration with Baker, had given the first proof of an effective bound on the size of the integral solutions of a genus one curve. During his time in the U.S. John had been much influenced by the work of Tate and of Iwasawa. The key insight of Iwasawa had been to see how to translate the theorems of Weil, which related the characteristic polynomial of Frobenius in certain l-adic representations to the zeta function, from the function field case to the number field case. Of course this involved the p-adic zeta function and not the classical one and even then only became a translation from a theorem to a conjecture, but it became a guiding principle in the study of the special values of the zeta function and has remained so to this day. Tate had been studying the relation of K_2 of the ring of integers of a number field to Galois cohomology groups. Together with Lichtenbaum and Sinnott John had developed and examined these conjectures about K-groups using some of the ideas of Iwasawa.

When he returned to Cambridge John and I set about exploring how Iwasawa's approach would work in the case of elliptic curves with complex multiplication. It worked wonderfully well! Although at that time Iwasawa's main conjecture seemed quite out of reach, even in the basic cyclotomic case, one could develop enough using the methods of Iwasawa to get the first real theorems on the Birch and Swinnerton-Dyer conjecture. Of course the search for a solution to this conjecture remains elusive to this day but the progress has been enormous. The theory of complex multiplication has to a large extent ceded its place to the theory of modular forms but the basic idea has largely remained intact, namely to relate the special values of L-functions to the points on the elliptic curve via the class field theory of the division fields of those points.

The original work was all in the context of ordinary primes, these being primes where the reduction of the elliptic curve is ordinary. Subsequently John and his students have extended the study to try to understand first the supersingular case, but still assuming the curve has complex multiplication, and then the more general case where no complex multiplication is assumed. Meanwhile the new ideas of Kolyvagin and of Gross and Zagier have to a large extent brought the general case into line with the complex multiplication case. In the general case where the curves are not assumed to have complex multiplication the fields of division points are no longer abelian over a finite extension of the rationals. To study these fields John and his coauthors have developed a non-abelian version of Iwasawa theory.

This volume contains many papers on these and related topics. However no tribute to John Coates could be complete without a testament to his continuing generosity and skill as a teacher. Cambridge number theory seemed strongest in bringing out the problem solver but one had a sense that in terms of modern developments it was a little isolated. John's arrival brought these two worlds together, and made Cambridge and my own arrival in mathematics more exciting than I could ever have anticipated. John's return to Cambridge in 1986 has cemented his role as a teacher and inspiration to many more generations of Cambridge number theorists, many of whom were present at his 60^{th} birthday celebrations in January of 2005.