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# Cohomological Invariants for $G$-Galois Algebras and Self-Dual Normal Bases

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Abstract. We define degree two cohomological invariants for $G$-Galois algebras over fields of characteristic not 2, and use them to give necessary conditions for the existence of a self–dual normal basis. In some cases (for instance, when the field has cohomological dimension $\leq 2$) we show that these conditions are also sufficient.

## Introduction

Let $k$ be a field of characteristic $\neq 2$, and let $L$ be a finite degree Galois extension of $k$. Let $G = \mathrm{Gal}(L/k)$. The *trace form* of $L/k$ is by definition the quadratic form $q_L : L \times L \to k$ defined by $q_L(x,y) = \mathrm{Tr}_{L/k}(xy)$. Note that $q_L$ is a *$G$-quadratic form*, in other words we have $q_L(gx, gy) = q_L(x,y)$ for all $x, y \in L$. A normal basis $(gx)_{g \in G}$ of $L$ over $k$ is said to be *self-dual* if $q_L(gx, gx) = 1$ and $q_L(gx, hx) = 0$ if $g \neq h$. It is natural to ask which extensions have a self-dual normal basis. This question is investigated in several papers (see for instance [BL 90], [BSe 94], [BPS 13]). It is necessary to work in a more general context than the one of Galois extensions, namely that of *$G$-Galois algebras* (see for instance [BSe 94], §1); one advantage being that this category is stable by base change of the ground field; the notion of a self-dual normal basis is defined in the same way.

If $k$ is a global field, then the *Hasse principle* holds : a $G$-Galois algebra has a self-dual normal basis over $k$ if and only if such a basis exists everywhere locally (see [BPS 13]). The present paper completes this result by giving necessary and sufficient conditions for the existence of a self-dual normal basis when $k$ is a local field (cf. §7). The conditions are given in terms of cohomological invariants defined over the ground field $k$ constructed in §3 and §4.

For an arbitrary ground field $k$, we start with the $H^1$-invariants defined in [BSe 94], §2. Recall from [BSe 94] that the vanishing of these invariants is a necessary condition for the existence of a self-dual normal basis; it is also sufficient in the case of fields of cohomological dimension 1 (see [BSe 94], Corollary 2.2.2 and Proposition 2.2.4).

Let $k[G]$ be the group algebra of $G$ over $k$, and let $J$ be its radical; the quotient $k[G]^s = k[G]/J$ is a semisimple $k$-algebra. Let $\sigma : k[G] \to k[G]$ be the $k$-linear involution sending $g$ to $g^{-1}$; it induces an involution $\sigma^s : k[G]^s \to k[G]^s$. The algebra $k[G]^s$ splits as a product of simple algebras. If $A$ is a $\sigma^s$-stable simple algebra which is a factor of $k[G]^s$, we denote by $\sigma_A$ the restriction of $\sigma^s$ to $A$, and by $E_A$ the subfield of the center of $A$ fixed by $\sigma_A$. We say that $A$ is *orthogonal* if $\sigma_A$ is the identity on the center of $A$, and if over a separable closure of $k$ it is induced by a symmetric form, and *unitary* if $\sigma_A$ is not the identity on the center of $A$ (see 1.3 for details).

Let $L$ be a $G$-Galois algebra over $k$, and let us assume that its $H^1$-invariants are trivial. We then define, for every orthogonal or unitary $A$ as above, cohomology classes in $H^2(k, Z/2Z)$, denoted by $c_A(L)$ in the orthogonal case and by $d_A(L)$ in the unitary case (see §3 and §4). They are invariants of the $G$-Galois algebra $L$. They also provide necessary conditions for the existence of a self-dual normal basis (this involves restriction to certain finite degree extensions of $k$, namely, the extensions $E_A/k$; see Propositions 3.5 and 4.7 for precise statements). If moreover $k$ has cohomological dimension $\leq 2$, then these conditions are also sufficient (Theorem 5.3.). Finally, if $k$ is a local field, then the conditions can be expressed in terms of the invariants $c_A(L)$ and $d_A(L)$, without passing to finite degree extensions (Theorem 7.1). Section 8 applies the results of §7 and the Hasse principle of [BSP 13] to give necessary and sufficient conditions for the existence of a self-dual normal basis when $k$ is a global field (Theorem 8.1).

Section 6 deals with the case of cyclic groups of order a power of 2 over arbitrary fields. We show that at most one of the unitary components $A$ gives rise to a non-trivial invariant $d_A(L)$ (Proposition 6.4 (i)), and that this invariant provides a necessary and sufficient condition for the existence of a self-dual normal basis (Corollary 6.5).

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## §1. Definitions, notation and basic facts

### 1.1. Galois cohomology

We use standard notation in Galois cohomology. If $K$ is a field, we denote by $K_s$ a separable closure of $K$, and by $\Gamma_K$ the Galois group $\mathrm{Gal}(K_s/K)$. For any discrete $\Gamma_K$-module $C$, set $H^i(K,C) = H^i(\Gamma_K, C)$. If $\Gamma$ is a finite or profinite group, set $H^i(\Gamma) = H^i(\Gamma, Z/2Z)$. If $U$ is a $K$-group scheme, we denote by $H^1(K,U)$ the pointed set $H^1(\Gamma_K, U(K_s))$.

### 1.2. Algebras with involution and unitary groups

Let $K$ be a field of characteristic $\neq 2$, and let $R$ be a finite dimensional algebra over $K$. An *involution* of $R$ is a $K$-linear anti-automorphism $\sigma : R \to R$ such that $\sigma^2$ is the identity.

Let us denote by $\mathrm{Comm}_K$ the category of commutative $K$-algebras, and by Group the category of groups. If $(R,\sigma)$ is an algebra with involution, the functor $\mathrm{Comm}_K \to \mathrm{Group}$ given by $S \mapsto \{x \in R \otimes_K S \mid x\sigma(x) = 1\}$ is the functor of points of a scheme over $\mathrm{Spec}(K)$; we denote it by $U_{R,K}$.

Let $h = \langle 1 \rangle$ be the rank one unit hermitian form over $(R,\sigma)$, given by $h(x,y) = x\sigma(y)$ for all $x, y \in R$. Then $U_{R,K}$ is the scheme of automorphisms of the hermitian form $h$. This is a smooth, finitely presented affine group scheme over $\mathrm{Spec}(K)$ (see for instance [BF 15], Appendix A). Moreover, $H^1(K, U_{R,K})$ is in natural bijection with the set of isomorphism classes of rank one hermitian forms over $(R,\sigma)$ that become isomorphic to $h$ over $K_s$ (see [Se 64], chap. III, §1).

If $F$ is a subfield of $K$, then $U_{R,F} = \mathrm{R}_{K/F}(U_{R,K})$, where $\mathrm{R}_{K/F}$ denotes Weil restriction of scalars relative to the extension $K/F$.

Let $Z$ be the center of $R$, and assume that $R$ is a simple algebra. We say that $(R,\sigma)$ is a *central simple algebra with involution over* $K$ if the fixed field of $\sigma$ in $Z$ is equal to $K$. If $(R,\sigma)$ is central simple algebra with involution over $K$, we set $U_R = U_{R,K}$.

### 1.3. Dévissage

Let $G$ be a finite group and let $k[G]$ be its group algebra over $k$. The *canonical involution* of $k[G]$ is the $k$-linear involution $\sigma : k[G] \to k[G]$ such that $\sigma(g) = g^{-1}$ for all $g \in G$. Let $J$ be the radical of $k[G]$, and set $k[G]^s = k[G]/J$; it is a semisimple $k$-algebra. Since $J$ is stable by $\sigma$, we obtain an involution $\sigma^s : k[G]^s \to k[G]^s$. Set $U_G = U_{k[G],k}$ and $U_G^r = U_{k[G]^s,k}$. Let $N$ be the kernel of the natural surjection $U_G \to U_G^r$. Let us define group schemes $N_i$ by setting $N_i(S) = \{x \in N(S) \mid x \equiv 1 \bmod J^i \otimes_k S\}$. Then $1 = N_m \subset N_{m-1} \subset \cdots \subset N_1 = N$, where $m$ is an integer such that $J^m = 0$. Note that $J^i/J^{i+1}$ is a module over the semisimple algebra $k[G]^s$, hence $N_i/N_{i-1}$ is isomorphic to a finite product of additive groups $G_a$; therefore $N$ is a split unipotent group. This implies that $H^1(k, U_G) = H^1(k, U_G^r)$ (see for instance [Sa 81], Lemme 1.13).

The semisimple algebra $k[G]^s$ is known to be a direct product of simple algebras. Note that $k[G]$ comes by scalar extension from $k_0[G]$ for $k_0 = Q$ or $F_p$, hence the centers of the factors of $k[G]^s$ are abelian Galois extensions of $k$ of finite degree; some are stable under $\sigma^s$ (we call them $A$), and others come in pairs, interchanged by $\sigma^s$ (we call them $B$).

If $A$ is a $\sigma^s$-stable simple factor of $k[G]^s$, we denote by $\sigma_A$ the restriction of $\sigma^s$ to $A$, by $F_A$ the center of $A$, and by $E_A$ the subfield of $\sigma_A$-invariant elements of $F_A$. Note that $U_A$ is a group scheme over $\mathrm{Spec}(E_A)$. Similarly, if $B$ is the product of two simple algebras interchanged by $\sigma^s$, we denote by $E_B$ the subfield of the center of $B$ fixed by the involution; $U_{B,E_B}$ is a group scheme over $\mathrm{Spec}(E_B)$.

We have $U_G^r \simeq \prod_A \mathrm{R}_{E_A/k}(U_A) \times \prod_B \mathrm{R}_{E_B/k}(U_{B,E_B})$, hence

$$H^1(k, U_G^r) = \prod_A H^1(k, \mathrm{R}_{E_A/k}(U_A)) \times \prod_B H^1(k, \mathrm{R}_{E_B/k}(U_{B,E_B})).$$

Note that $H^1(k, \mathrm{R}_{E_B/k}(U_{B,E_B})) = H^1(E_B, U_{B,E_B}) = 0$ (see for instance [KMRT 98], (29.2)), that $H^1(k, \mathrm{R}_{E_A/k}(U_A)) = H^1(E_A, U_A)$ (see for instance [O 84], 2.3), and that $H^1(k, U_G) = H^1(k, U_G^r)$ (see above). Therefore we have

$$H^1(k, U_G) = \prod_A H^1(E_A, U_A).$$

The algebras with involution $(A, \sigma_A)$ appearing in this product are of three types :

(a) The involution $\sigma_A : A \to A$ is not the identity on the center $F_A$ of $A$. Hence $F_A/E_A$ is a quadratic extension. Such an algebra with involution is called *unitary*; the group scheme $U_A$ is of Dynkin type A.

(b) The involution $\sigma_A : A \to A$ is the identity on $F_A$ (which is then equal to $E_A$), and, over a separable closure of $E_A$, the involution is induced by a symmetric form. Such an algebra with involution is called *orthogonal*; the group scheme $U_A$ is of Dynkin type B or D.

(c) The involution $\sigma_A : A \to A$ is the identity on $F_A$ (which is then equal to $E_A$), and, over a separable closure of $E_A$, the involution is induced by a skew-symmetric form. Such an algebra with involution is called *symplectic*; the group scheme $U_A$ is of Dynkin type C.

### 1.4. $G$-quadratic forms

A *$G$-quadratic form* is a pair $(M, q)$, where $M$ is a $k[G]$-module that is a finite dimensional $k$-vector space, and $q : M \times M \to k$ is a non-degenerate symmetric bilinear form such that

$$q(gx, gy) = q(x, y)$$

for all $x, y \in M$ and all $g \in G$. We say that two $G$-quadratic forms $(M, q)$ and $(M', q')$ are *isomorphic* if there exists an isomorphism of $k[G]$-modules

$f : M \to M'$ such that $q'(f(x), f(y)) = q(x,y)$ for all $x, y \in M$. If this is the case, we write $(M,q) \simeq_G (M',q')$, or $q \simeq_G q'$. It is well-known that $G$-quadratic forms correspond bijectively to non-degenerate hermitian forms over $(k[G], \sigma)$ (see for instance [BPS 13], 2.1, Example on page 441). The *unit $G$-form* is by definition the pair $(k[G], q_0)$, where $q_0$ is the $G$-form characterized by $q(g,g) = 1$ and $q(g,h) = 0$ if $g \neq h$, for $g, h \in G$.

### 1.5. Trace forms of $G$-Galois algebras

If $L$ is an étale $k$-algebra, we denote by

$$q_L : L \times L \to k, \quad q_L(x,y) = \mathrm{Tr}_{L/k}(xy),$$

its trace form. Then $q_L$ is a non-degenerate quadratic form over $k$; if moreover $L$ is a $G$-Galois algebra, then $q_L$ is a $G$-quadratic form.

Let $L$ be a $G$-Galois algebra; then $L$ has a self-dual normal basis over $k$ if and only if $q_L$ is isomorphic to $q_0$ as a $G$-quadratic form. Let $\phi : \Gamma_k \to G$ be a continuous homomorphism corresponding to $L$ (see for instance [BSe 94], 1.3). Recall that $\phi$ is unique up to conjugation. The composition

$$\Gamma_k \xrightarrow{\phi} G \to U_G(k) \to U_G(k_s)$$

is a 1-cocycle $\Gamma_k \to U_G(k_s)$. Let $u(L)$ be its class in the cohomology set $H^1(k, U_G)$; it does not depend on the choice of $\phi$. The $G$-Galois algebra $L$ has a self-dual normal basis over $k$ if and only if $u(L) = 0$, cf. [BSe 94], Corollaire 1.5.2.

Recall from 1.3 that we have

$$H^1(k, U_G) = \prod_A H^1(E_A, U_A).$$

Let $u_A(L)$ be the image of $u(L)$ in $H^1(E_A, U_A)$; note that $L$ has a self-dual normal basis if and only if $u_A(L) = 0$ for every $A$.

Let $A$ be as above. Composing the injection $G \to U_G(k)$ with the natural map $U_G(k) \to U_G^r(k) \to \mathrm{R}_{E_A/k}(U_A)(k) = U_A(E_A)$, we obtain a homomorphism $G \to U_A(E_A)$, denoted by $i_A$.

Let $\phi_A : \Gamma_{E_A} \to \Gamma_k \to G$ be the composition of $\phi : \Gamma_k \to G$ with the inclusion of $\Gamma_{E_A}$ in $\Gamma_k$. Composing $\phi_A$ with the map $i_A : G \to U_A(E_A)$ defined above we obtain a 1-cocycle $\Gamma_{E_A} \to U_A(k_s)$. The class of this 1-cocycle in $H^1(E_A, U_A)$ is equal to $u_A(L)$.

## §2. The $H^1$-condition

Let $L$ be a $G$-Galois algebra over $k$, and let $\phi : \Gamma_k \to G$ be a homomorphism corresponding to $L$. Let $n$ be an integer $\geq 1$. Then $\phi$ induces a homomorphism

$$\phi^* : H^n(G) \to H^n(k, Z/2Z).$$

Note that $\phi^*$ is independent of the choice of $\phi$ in its conjugacy class (see [Se 68], chap. VII, proposition 3). For all $x \in H^n(G)$, set $x_L = \phi^*(x)$.

Proposition 2.1. *If $L$ has a self-dual normal basis over $k$, then for all $x \in H^1(G)$ we have $x_L = 0$.*

Proof. See [BSe 94], Corollaire 2.2.2.

If $\mathrm{cd}_2(\Gamma_k) \leq 1$, then $L$ has a self-dual normal basis over $k$ if and only if $x_L = 0$ for all $x \in H^1(G)$, see [BSe 94], Proposition 2.2.4.

We say that *the $H^1$-condition is satisfied* if $x_L = 0$ for all $x \in H^1(G)$. Let $G^2$ be the subgroup of $G$ generated by the squares of elements of $G$. Note that $G/G^2$ is an elementary abelian 2-group, and that the $H^1$-condition means that the homomorphism $\Gamma_k \to G \to G/G^2$ induced by $\phi$ is trivial, i.e. $\phi(\Gamma_k) \subset G^2$.

## §3. Orthogonal invariants

We keep the notation of the previous sections. In particular, $G$ is a finite group, $L$ is a $G$-Galois algebra, and $\phi : \Gamma_k \to G$ is a homomorphism corresponding to $L$. Let us suppose that the $H^1$-condition is satisfied.

Let $A$ be an *orthogonal* $\sigma^s$-stable central simple factor of $k[G]^s$ (see 1.3), and recall that the center of $A$ is denoted by $E_A$. Let us denote by $\langle A\rangle$ the subgroup of $\mathrm{Br}(E_A)$ generated by the class of the algebra $A$. Note that since $\sigma_A : A \to A$ is an orthogonal involution, this class has order at most 2, hence $\langle A\rangle$ is a subgroup of $\mathrm{Br}_2(E_A)$.

The aim of this section is to define two invariants : an invariant $c_A(L) \in H^2(k)$ of the $G$–Galois algebra $L$, and an invariant $\mathrm{clif}_A(q_L) \in \mathrm{Br}_2(E_A)/\langle A\rangle$ of the $G$-form $q_L$. We shall compare these two invariants (cf. Theorem 3.3), and give a necessary condition for the existence of self-dual normal bases (Corollary 3.5).

Let $U_A^0$ be the connected component of the identity in $U_A$. Let $i_A : G \to U_A(E_A)$ be the homomorphism defined in 1.5, and let $\pi : U_A(E_A) \to U_A(E_A)/U_A^0(E_A)$ be the projection. Since $U_A(E_A)/U_A^0(E_A)$ is of order $\leq 2$, we have $\pi(i_A(G^2)) = 0$; i.e. $i_A(G^2) \subset U_A^0(E_A)$.

Let $\tilde{U}_A$ be the Spin group of $(A, \sigma)$; note that if $\dim_k(A) \geq 3$, then $\tilde{U}_A$ is the universal cover of $U_A^0$. Let $s : \tilde{U}_A \to U_A^0$ be the covering map. We have an exact sequence of algebraic groups over $E_A$

$$1 \to Z/2Z \to \tilde{U}_A \xrightarrow{s} U_A^0 \to 1.$$

Let us consider the associated cohomology exact sequence

$$\tilde{U}_A(E_A) \xrightarrow{s} U_A^0(E_A) \xrightarrow{\delta} H^1(E_A).$$

LEMMA 3.1. *We have $i_A(G^2) \subset s(\tilde{U}_A(E_A))$.*

PROOF. In view of the above exact sequence, it suffices to prove that $\delta(i_A(G^2)) = 0$. In order to prove this, let us first assume that $A$ is not split. Then we have $U_A(E_A) = U_A^0(E_A)$ (cf. [K 69], Lemma 1 b, see also [B 94], cor. 2). Since $H^1(E_A)$ is a 2-torsion group and since $i_A(G^2) \subset U_A^0(E_A)$, this implies that $\delta(i_A(G^2)) = 0$, as claimed. Assume now that $A$ is split. Then $U_A$ is the orthogonal group of a quadratic form $q$; let $\mathrm{sn} : U_A(E_A) \to H^1(E_A)$ be the associated spinor norm, and note that sn is a group homomorphism (see for instance [L 05], Chapter 5, Theorem 1.13). The homomorphism sn depends on the choice of the quadratic form $q$ with orthogonal group $U_A$, but its restriction to $U_A^0$ does not depend on this choice. Note that $\delta : U_A^0(E_A) \to H^1(E_A)$ is the restriction of sn to $U_A^0(E_A)$. Therefore for all $g \in G$, we have $\delta(i_A(g^2)) = \mathrm{sn}(i_A(g))^2$, and since $H^1(E_A)$ is a 2-torsion group, this implies that $\delta(i_A(G^2)) = 0$. This completes the proof of the lemma.

Let $H$ be a subgroup of $G^2$. By Lemma 3.1, we have $i_A(H) \subset s(\tilde{U}_A(E_A))$. Let

$$V_A^H = \tilde{U}_A(E_A) \times_{U_A^0(E_A)} H$$

be the fibered product of $s : \tilde{U}_A(E_A) \to U_A^0(E_A)$ and $i_A : H \to U_A^0(E_A)$. Therefore we have a central extension

$$1 \to Z/2Z \to V_A^H \xrightarrow{p} H \to 1,$$

where $p$ is the projection to the factor $H$. Note that the surjectivity of $p$ follows from the fact that by Lemma 3.1 every element of $i_A(H)$ has a preimage in $\tilde{U}_A(E_A)$.

Let us denote by

$$e_A^H \in H^2(H)$$

the cohomology class corresponding to the extension $V_A^H$. If $\phi(\Gamma_k) \subset H$, we denote by

$$\phi^* : H^2(H) \to H^2(k)$$

the homomorphism induced by $\phi : \Gamma_k \to H$.

PROPOSITION 3.2. *Let $\psi : \Gamma_k \to G$ be another continuous homomorphism corresponding to the $G$–Galois algebra $L$. Set $H_\phi = \phi(\Gamma_k)$ and $H_\psi = \psi(\Gamma_k)$. Then we have*

$$\phi^*(e_A^{H_\phi}) = \psi^*(e_A^{H_\psi}) \quad in \quad H^2(k).$$

Proof. We have $\psi = \mathrm{Int}(g) \circ \phi$ for some $g \in G$. Note that $i_A(g) \in U_A(E_A)$, and that $\mathrm{Int}(i_A(g))$ is an automorphism of $U_A^0(E_A)$. Any automorphism of $U_A^0(E_A)$ can be lifted to an automorphism of $\tilde{U}_A(E_A)$; indeed, such a lift exists over a separable closure, and is unique, hence defined over the ground field. Let $f : \tilde{U}_A(E_A) \to \tilde{U}_A(E_A)$ be a lift of $\mathrm{Int}(i_A(g))$. Then $f$ induces an isomorphism $V_A^{H_\phi} \to V_A^{H_\psi}$, which sends $H_\phi$ to $H_\psi$, and is the identity on $Z/2Z$. This implies that $\phi^*(e_A^{H_\phi}) = \psi^*(e_A^{H_\psi})$ in $H^2(k)$.

The invariant $c_A(L)$

Recall that we assume that the $H^1$-condition is satisfied. We now choose for $H$ the image $\phi(\Gamma_k)$ of $\Gamma_k$ in $G$, and set $V_A = V_A^H$, $e_A = e_A^H$. We denote by $c_A(L)$ the class of $\phi^*(e_A)$ in $H^2(k)$; Proposition 3.2 shows that this class does not depend on the choice of $\phi : \Gamma_k \to G$ defining the $G$–Galois algebra $L$. Since $H^2(k) \simeq \mathrm{Br}_2(k)$, we can also consider $c_A(L)$ as an element of $\mathrm{Br}_2(k)$.

Recall that the $G$-trace form $q_L$ determines a rank one hermitian form over $(A, \sigma_A)$. We want to relate $c_A(L)$ to the Clifford invariant of this hermitian form.

The invariant $\mathrm{clif}_A(q_L)$

The map $i_A : H \to U_A^0(E_A)$ induces a map of pointed sets

$$i_A : H^1(E_A, H) \to H^1(E_A, U_A^0).$$

Let $u_A^0(L)$ be the image of $[\phi_A] \in H^1(E_A, H)$ by this map. Then the element $u_A(L)$ defined in 1.5 is the image of $u_A^0(L)$ under the further composition with the map $H^1(E_A, U_A^0) \to H^1(E_A, U_A)$.

Let us consider the exact sequence $1 \to Z/2Z \to \tilde{U}_A \to U_A^0 \to 1$, and let $\delta$ be the connecting map $H^1(E_A, U_A^0) \to H^2(E_A) \simeq \mathrm{Br}_2(E_A)$ of the associated cohomology exact sequence. Recall that $\langle A \rangle$ is the subgroup of $\mathrm{Br}_2(E_A)$ generated by the class of the algebra $A$. The *Clifford invariant* of $q_L$ at $A$ is by definition the image of $\delta(u_A^0(L))$ in $\mathrm{Br}_2(E_A)/\langle A \rangle$. Let us denote it by $\mathrm{clif}_A(q_L)$.

Theorem 3.3. *The image of* $\mathrm{Res}_{E_A/k}(c_A(L))$ *in* $\mathrm{Br}_2(E_A)/\langle A \rangle$ *is equal to* $\mathrm{clif}_A(q_L)$.

We need the following lemma :

Lemma 3.4. *Let $K$ be a field, let $C$ be a finite group, and let $f : \Gamma_K \to C$ be a continuous homomorphism. Let us denote by $[f] \in H^1(K, C)$ the corresponding cohomology class. Let*

$$1 \to Z/2Z \to V \to C \to 1$$

*be a central extension with trivial $\Gamma_K$-action. Let $[e] \in H^2(C)$ be the class of a 2-cocycle $e : C \times C \to Z/2Z$ representing this extension. Let $\partial : H^1(K,C) \to H^2(K)$ be the connecting map associated to the above exact sequence, and let $f^* : H^2(C) \to H^2(K)$ be the map induced by $f$. Then*

$$f^*([e]) = \partial([f]).$$

PROOF. This follows from a direct computation. For all $\sigma, \tau \in \Gamma_K$, we have $f^*(e)(\sigma,\tau) = e(f(\sigma), f(\tau)) = x_{f(\sigma)} x_{f(\tau)} x^{-1}_{f(\sigma\tau)}$, where $x : C \to V$ is a section. On the other hand, $(\partial f)(\sigma,\tau) = x_{f(\sigma)}{}^{f(\sigma)}(x_{f(\tau)}) x^{-1}_{f(\sigma\tau)}$, and this is equal to $x_{f(\sigma)} x_{f(\tau)} x^{-1}_{f(\sigma\tau)}$, since the action of $\Gamma_k$ on $V$ is trivial.

PROOF OF THEOREM 3.3. Let $\partial : H^1(E_A, H) \to H^2(E_A)$ be the connecting map of the cohomology exact sequence associated to the exact sequence

$$1 \to Z/2Z \to V_A \to H \to 1$$

with all the groups having trivial $\Gamma_{E_A}$-action. Recall that $\phi_A : \Gamma_{E_A} \to \Gamma_k \to H$ is the composition of $\phi : \Gamma_k \to H$ with the inclusion of $\Gamma_{E_A}$ into $\Gamma_k$. By Lemma 3.4 we have $\partial([\phi_A]) = \phi_A^*(e_A) = \mathrm{Res}_{E_A/k}(\phi^*(e_A)) = \mathrm{Res}_{E_A/k}(c_A(L))$. In view of the commutative diagram of $\Gamma_{E_A}$-groups

$$\begin{array}{ccccccccc} 1 & \to & Z/2Z & \to & \tilde{U}_A(k_s) & \to & U_A^0(k_s) & \to & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \to & Z/2Z & \to & V_A & \to & H & \to & 1 \end{array}$$

we have $\delta(u_A^0(L)) = \partial([\phi_A])$. Therefore we obtain $\mathrm{Res}_{E_A/k}(c_A(L)) = \delta(u_A^0(L))$. Since the class of $\delta(u_A^0(L))$ in $\mathrm{Br}_2(E_A)/\langle A\rangle$ is equal to $\mathrm{clif}_A(q_L)$ by definition, this completes the proof of the theorem.

PROPOSITION 3.5. *If $L$ has a self-dual normal basis over $k$, then $\mathrm{Res}_{E_A/k}(c_A(L))$ is trivial in $\mathrm{Br}_2(E_A)/\langle A\rangle$.*

PROOF. Since $L$ has a self-dual normal basis over $k$, the class $u_A(L)$ corresponds to the class of the rank one unit hermitian form $\langle 1\rangle$ in $H^1(E_A, U_A)$. As $\langle 1\rangle$ corresponds to the trivial cocycle in $Z^1(E_A, U_A^0)$, its Clifford invariant is trivial, in other words, $\mathrm{clif}_A(q_L)$ is trivial. By Theorem 3.3 the image of $\mathrm{Res}_{E_A/k}(c_A(L))$ in $\mathrm{Br}_2(E_A)/\langle A\rangle$ is equal to $\mathrm{clif}_A(q_L)$, hence the proposition is proved.

We conclude this section with an example where $c_A(L) \neq 0$, but $\mathrm{Res}_{E_A/k}(c_A(L)) = 0$ (and hence $\mathrm{clif}_A(q_L) = 0$) :

EXAMPLE 3.6. Let $G = A_5$, the alternating group, and assume that $k = Q$. Let $A$ be a factor of $k[G]$ corresponding to a degree 3 orthogonal representation of $G$; then $A = M_3(E_A)$ with $E_A = k(\sqrt{5})$, and the involution $\sigma_A$ is induced by the unit form $\langle 1,1,1\rangle$. Let $\epsilon \in G$ be a product of two disjoint transpositions.

Let $z \in k^\times$, and let $\psi : \Gamma_k \to \{1, \epsilon\}$ be the corresponding quadratic character. Let $\phi : \Gamma_k \to G$ be given by $\phi = \iota \circ \psi$, where $\iota : \{1,\epsilon\} \to G$ is the inclusion. Let $L$ be the $G$-Galois algebra corresponding to $\phi$. Set $H = \{1, \epsilon\}$, and note that the image of $\phi$ is contained in $H$. Set $N = k[X]/(X^2 - z)$; then we have $L = \mathrm{Ind}_H^G(N)$.

Note that $\epsilon$ lifts to an element of order 4 in $\tilde{A}_5$, hence also in $\tilde{U}_A(E_A)$. Therefore the extension $1 \to Z/2Z \to V_A^H \to H \to 1$ is not trivial; the group $V_A^H$ is cyclic of order 4. Recall that $e_A$ is the class of this extension in $H^2(H)$; hence $e_A$ is the only non-trivial element of $H^2(H)$. By definition, we have $c_A(L) = \phi^*(e_A)$, and this is equal to the cup product $(z)(z) = (-1)(z)$ in $H^2(k)$.

Set $z = 11$. Then $c_A(L) = (-1)(11)$ is not trivial in $H^2(k)$. On the other hand, since $E_A = k(\sqrt{5})$, we have $\mathrm{Res}_{E_A/k}(c_A(L)) = 0$ in $H^2(E_A)$. Note that the subgroup $\langle A\rangle$ of $\mathrm{Br}_2(E_A)$ is trivial, and recall that $\mathrm{clif}_A(q_L) = \mathrm{Res}_{E_A/k}(c_A(L))$ in $\mathrm{Br}_2(E_A) \simeq H^2(E_A)$ by Theorem 3.3; therefore we have $\mathrm{clif}_A(q_L) = 0$.

## §4. Unitary invariants

We keep the notation of the previous sections : $G$ is a finite group, $L$ is a $G$–Galois algebra, and $\phi : \Gamma_k \to G$ is a homomorphism associated to $L$. We suppose that the $H^1$-condition is satisfied by $\phi : \Gamma_k \to G$, hence $\phi(\Gamma_k)$ is a subgroup of $G^2$. Let $A$ be a *unitary* $\sigma^s$-stable central simple factor of $k[G]^s$ (see 1.3). We denote by $F_A$ be the center of $A$; note that $F_A$ is a quadratic extension of $E_A$.

Using the same strategy as in §3, we first define an element of $H^2(k)$ which is an invariant of the $G$-Galois algebra $L$. We then consider the hermitian form $h_A$ over $(A, \sigma)$ determined by $q_L$, and recall the definition of the discriminant of this form, thereby obtaining an element of $\mathrm{Br}_2(E_A)$. This is an invariant of the hermitian form $h_A$, and hence also of the $G$–form $q_L$. We then show that the restriction of the first invariant to $H^2(E_A)$ is equal to the second one (see Theorem 4.5).

We start by recording some facts from Galois cohomology.

Let $E$ be a field of characteristic $\neq 2$, and let $E_s$ be a separable closure of $E$. Let $F$ be a quadratic extension of $E$, let $x \mapsto \overline{x}$ the non-trivial automorphism of $F$ over $E$, and let $F^{\times 1}$ be the subgroup of $F^\times$ consisting of the $x \in F$ such that $x\overline{x} = 1$. Let $\mathrm{N} : F \to E$, given by $\mathrm{N}(x) = x\overline{x}$, be the norm map. We denote by $[F]$ the class of the quadratic extension $F/E$ in $H^1(E)$. For all $x \in E^\times$, we denote by $(x)$ the class of $x$ in $E^\times/E^{\times 2}$, and by $[x]$ the class of $x$ in $E^\times/\mathrm{N}(F^\times)$.

Lemma 4.1. (a) *The connecting homomorphism* $E^\times \to H^1(E, \mathrm{R}^1_{F/E}G_m)$ *associated to the exact sequence* $1 \to \mathrm{R}^1_{F/E}G_m \to \mathrm{R}_{F/E}G_m \xrightarrow{\mathrm{N}} G_m \to 1$ *induces an isomorphism* $\alpha : E^\times/\mathrm{N}(F^\times) \to H^1(E, \mathrm{R}^1_{F/E}G_m)$.
(b) *Let* $x \in E^\times$, *and let* $f_x : \Gamma_E \to \mathrm{R}^1_{F/E}G_m(E_s)$ *be defined by* $f_x(\gamma) = y^{-1}\gamma(y)$, *where* $y \in (F \otimes_E E_s)^\times$ *is such that* $\mathrm{N}(y) = x$. *Then we have* $\alpha((x)) = [f_x]$.

Proof. (a) follows from Hilbert's theorem 90, and (b) from the definition of the connecting homomorphism.

¿From now on, we identify $E^\times/\mathrm{N}(F^\times)$ and $H^1(E, \mathrm{R}^1_{F/E}G_m)$ via the isomorphism $\alpha$.

Lemma 4.2. *Let* $1 \to Z/2Z \to \mathrm{R}^1_{F/E}G_m \xrightarrow{s} \mathrm{R}^1_{F/E}G_m \to 1$ *be the exact sequence of linear algebraic groups with* $s$ *the squaring map. Let* $\delta : H^1(E, \mathrm{R}^1_{F/E}G_m) \to H^2(E)$ *be the connecting homomorphism associated to this exact sequence. Identifying* $H^1(E, \mathrm{R}^1_{F/E}G_m)$ *with* $E^\times/\mathrm{N}(F^\times)$ *via* $\alpha$, *we have*

$$\delta([x]) = (x)[F] \in H^2(E)$$

*for all* $x \in E^\times$, *where* $(x)[F]$ *denotes the cup product of* $(x), [F] \in H^1(E)$.

Proof. A 2-cocycle associated to $(x)[F] \in H^2(E)$ is given by $f(\sigma, \tau)$ such that $f(\sigma, \tau) = 1$ if the restriction of $\sigma$ to $E(\sqrt{x})$ is the identity, or if the restriction of $\tau$ to $F$ is the identity, and $f(\sigma, \tau) = -1$ otherwise. Let us check that the cohomology class of $f$ in $H^2(E)$ is equal to $\delta([x])$. Let $y \in (F \otimes_E E_s)^\times$ be such that $\mathrm{N}_{F\otimes_E E_s/E_s}(y) = y\overline{y} = x$. A 1-cocycle $g$ in $Z^1(E, \mathrm{R}^1_{F/E}G_m)$ associated to $[x]$ is given by $g(\sigma) = y^{-1}\sigma(y)$ for $\sigma \in \Gamma_E$. For all $\tau \in \Gamma_E$, set $z_\tau = y^{-1}\sqrt{x}$ if the restriction of $\tau$ to $F$ is not the identity, and $z_\tau = 1$ otherwise. Then $\mathrm{N}_{F\otimes_E E_s/E_s}(z_\tau) = z_\tau\overline{z_\tau} = (y^{-1}\sqrt{x})(\overline{y}^{-1}\sqrt{x})$ if the restriction of $\tau$ to $F$ is not the identity. Since $y\overline{y} = x$, we have $z_\tau \in \mathrm{R}^1_{F/E}G_m(E_s)$. Further, $s(z_\tau) = y^{-2}x = y^{-1}\tau(y)$ if the restriction of $\tau$ to $F$ is not the identity, and $s(z_\tau) = 1 = y^{-1}\tau(y)$ otherwise. Thus $\delta(g)(\sigma, \tau) = z_\sigma{}^\sigma(z_\tau)z_{\sigma\tau}^{-1}$. It is straightforward to check that $\delta(g)(\sigma, \tau) = 1$ if the restriction of $\sigma$ to $E(\sqrt{x})$ is the identity, or the restriction of $\tau$ to $F$ is the identity, and that $\delta(g)(\sigma, \tau) = -1$ otherwise. This is precisely the cocycle $f$, hence we have $\delta([x]) = (x)[F]$ in $H^2(E)$. This concludes the proof of the lemma.

Lemma 4.3. *We have an injective homomorphism* $E^\times/\mathrm{N}(F^\times) \to \mathrm{Br}_2(E)$ *defined by* $[x] \mapsto (x, F/E)$.

Proof. Indeed, the class of the quaternion algebra $(x, F/E)$ is trivial in $\mathrm{Br}_2(E)$ if and only if $x \in \mathrm{N}(F^\times)$.

We now define an invariant $d_A(L) \in H^2(k, Z/2Z)$ of the $G$-Galois algebra $L$.

The invariant $d_A(L)$

Recall that $F_A^{\times 1}$ is the subgroup of $F_A^\times$ consisting of the $x \in F_A$ such that $x\sigma_A(x) = 1$; in other words, $F_A^{\times 1} = \mathrm{R}^1_{F_A/E_A}G_m(E_A)$. We denote by

$s : \mathrm{R}^1_{F_A/E_A} G_m \to \mathrm{R}^1_{F_A/E_A} G_m$ the squaring map, and by $n : U_A \to \mathrm{R}^1_{F_A/E_A} G_m$ the reduced norm. Recall that $i_A : G \to U_A(E_A)$ is the homomorphism defined in 1.5; we have $n(i_A(G^2)) \subset s(F_A^{\times 1})$.

Let $H$ be a subgroup of $G^2$. Let $V_A^H = F_A^{\times 1} \times_{F_A^{\times 1}} H$ be the fibered product of $s : F_A^{\times 1} \to F_A^{\times 1}$ and $n \circ i_A : H \to F_A^{\times 1}$. Then the sequence

$$1 \to Z/2Z \to V_A^H \to H \to 1$$

is exact. Note that the surjectivity follows from the fact that $n(i_A(H)) \subset s(F_A^{\times 1})$. Therefore $V_A^H$ is a central extension of $H$ by $Z/2Z$. Recall that the $H^1$-condition implies that $\phi(\Gamma_k) \subset G^2$.

Proposition 4.4. *Let $\psi : \Gamma_k \to G$ be another continuous homomorphism corresponding to the $G$–Galois algebra $L$. Set $H_\phi = \phi(\Gamma_k)$ and $H_\psi = \psi(\Gamma_k)$. Then we have*

$$\phi^*(e_A^{H_\phi}) = \psi^*(e_A^{H_\psi}) \quad in \quad H^2(k).$$

Proof. We have $\psi = \mathrm{Int}(g) \circ \phi$ for some $g \in G$. The map $F_A^{\times 1} \times_{F^{\times 1}} H_\phi \to F_A^{\times 1} \times_{F^{\times 1}} H_\psi$, given by $(x, y) \to (x, gyg^{-1})$, gives rise to an isomorphism $V_A^{H_\phi} \to V_A^{H_\psi}$ that is the identity on $Z/2Z$ and sends $H_\phi$ to $H_\psi$. This implies that $\phi^*(e_A^{H_\phi}) = \psi^*(e_A^{H_\psi})$ in $H^2(k)$.

We now choose for $H$ the image $\phi(\Gamma_k)$ of $\Gamma_k$ in $G$, and set $V_A = V_A^H$, $e_A = e_A^H$.

Notation. Let us denote by $d_A(L)$ the class of $\phi^*(e_A)$ in $H^2(k)$; Proposition 4.4 shows that this class is independent of the choice of $\phi : \Gamma_k \to G$ defining the $G$–Galois algebra $L$.

We define the discriminant of the $G$-form $q_L$ at $A$, and compare it with the cohomology class $d_A(L)$.

The invariant $\mathrm{disc}_A(q_L)$

Recall that composing $\phi_A : \Gamma_{E_A} \to H$ with the map $i_A : H \to U_A(k_s)$ we obtain a 1-cocycle $\Gamma_{E_A} \to U_A(k_s)$, the class of which in $H^1(E_A, U_A)$ is $u_A(L)$. The reduced norm $n : U_A \to \mathrm{R}^1_{F_A/E_A} G_m$ induces a map $n : H^1(E_A, U_A) \to E_A^\times / \mathrm{N}(F_A^\times)$.

Notation. Set $\mathrm{disc}_A(q_L) = (n(u_A(L)), F_A/E_A)$ in $\mathrm{Br}_2(E_A)$.

Note that this is well–defined by Lemma 4.3. Since we have $\mathrm{Br}_2(E_A) \simeq H^2(E_A)$, we can also consider $\mathrm{disc}_A(q_L)$ as an element of $H^2(E_A)$. Then $\mathrm{disc}_A(q_L)$ is given by the cup product $n(u_A(L)).[F_A]$ in $H^2(E_A)$. This invariant is related to the previously defined invariant $d_A(L)$ as follows :

Theorem 4.5. *We have $\mathrm{disc}_A(q_L) = \mathrm{Res}_{E_A/k}(d_A(L))$ in $H^2(E_A)$.*

Proof. Let $\partial : H^1(E_A, H) \to H^2(E_A)$ be the connecting map of the exact sequence

$$1 \to Z/2Z \to V_A \to H \to 1$$

with all the groups having trivial $\Gamma_{E_A}$-action. By Lemma 3.4 we have

$$\partial([\phi_A]) = \phi_A^*(e_A) = \mathrm{Res}_{E_A/k}(\phi^*(e_A)) = \mathrm{Res}_{E_A/k}(d_A(L)).$$

We have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & Z/2Z & \to & V_A & \to & H & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & Z/2Z & \to & \mathrm{R}^1_{F_A/E_A} G_m(k_s) & \xrightarrow{s} & \mathrm{R}^1_{F_A/E_A} G_m(k_s) & \to & 1 \end{array}$$

where the second vertical map is the projection on the first factor, and the third one is $H \xrightarrow{i_A} U_A(E_A) \xrightarrow{n} \mathrm{R}^1_{F_A/E_A} G_m(E_A)$.

Let $\delta : H^1(E, \mathrm{R}^1_{F_A/E_A} G_m) \to H^2(E_A)$ be the connecting homomorphism associated to the exact sequence

$$1 \to Z/2Z \to \mathrm{R}^1_{F_A/E_A} G_m \xrightarrow{s} \mathrm{R}^1_{F_A/E_A} G_m \to 1.$$

By the commutativity of the above diagram, we have $\delta([n(u_A(L))]) = \partial([\phi_A])$. Hence we have $\mathrm{Res}_{E_A/k}(d_A(L)) = \delta([n(u_A(L)]))$. We have $\delta([n(u_A(L))]) = (n(u_A(L))).[F_A]$ by Lemma 4.2 and hence $\mathrm{Res}_{E_A/k}(d_A(L)) = \mathrm{disc}_A(q_L)$, as claimed.

Lemma 4.6. *If $q_L$ corresponds to the hermitian form $\langle z_A \rangle$ over $(A, \sigma_A)$, then we have*

$$\mathrm{disc}_A(q_L) = (n(z_A), F_A/E_A) \quad in \quad \mathrm{Br}_2(E_A).$$

Proof. Set $z = z_A$. Let $z = w\sigma_A(w)$ with $w \in A \otimes_{E_A} k_s$. The cocycle $\tau \mapsto w^{-1}\tau(w)$ represents the class of the hermitian form $\langle z \rangle$ in $H^1(E_A, U_A)$. Let us denote this class by $u_z \in H^1(E_A, U_A)$, and note that we have $u_z = u_A(L)$ by definition. The cocycle $\tau \mapsto n(w)^{-1}\tau(n(w))$ represents the class $n(u_z) \in H^1(E_A, \mathrm{R}^1_{F_A/E_A} G_m)$. By Lemma 4.1 this class is mapped by $\alpha^{-1}$ to $[n(z)] \in E_A^\times/\mathrm{N}(F_A^\times)$. Therefore we have $(n(z), F_A/E_A) = (n(u_A(L)), F_A/E_A) = \mathrm{disc}_A(q_L)$, as claimed.

Proposition 4.7. *If $L$ has a self-dual normal basis over $k$, then $\mathrm{Res}_{E_A/k}(d_A(L))$ is trivial in $\mathrm{Br}_2(E_A)$.*

Proof. Since $L$ has a self-dual normal basis, $q_L$ corresponds to the hermitian form $\langle 1 \rangle$ over $(A, \sigma_A)$. By Lemma 4.6 this implies that $\mathrm{disc}_A(q_L)$ is trivial. Since by Theorem 4.5 we have $\mathrm{disc}_A(q_L) = \mathrm{Res}_{E_A/k}(d_A(L))$, the Proposition is proved.

Remark. There are examples where $d_A(L) \neq 0$ but $\mathrm{Res}_{E_A/k}(d_A(L)) = 0$ (hence also $\mathrm{disc}_A(q_L) = 0$); see for instance Example 5.2 (i).

## §5. Self-dual normal bases

We keep the notation of the previous sections. In particular, $G$ is a finite group, $L$ is a $G$-Galois algebra over $k$, and $\phi : \Gamma_k \to G$ is a homomorphism associated to $L$. We now apply the results of the previous sections to give necessary conditions for the existence of a self-dual normal basis, and to show that these are also sufficient when $k$ has cohomological dimension $\leq 2$, see Proposition 5.1 and Theorem 5.3.

Putting together the results of §2 - §4, we have the following :

Proposition 5.1. *Suppose that $L$ has a self-dual normal basis over $k$. Then the $H^1$-condition is satisfied, and*

(i) *For all orthogonal $\sigma^s$-stable central simple factors $A$ of $k[G]^s$, we have*

$$\mathrm{Res}_{E_A/k}(c_A(L)) = 0 \quad \textit{in } \mathrm{Br}_2(E_A)/\langle A\rangle.$$

(ii) *For all unitary $\sigma^s$-stable central simple factors $A$ of $k[G]^s$, we have*

$$\mathrm{Res}_{E_A/k}(d_A(L)) = 0 \quad \textit{in } \mathrm{Br}_2(E_A).$$

Proof. This follows from Propositions 2.1, 3.5 and 4.7.

Example 5.2. (i) The aim of this example is to reinterpret and complete Exemple 10.2 of [BSe 94] using the results of the present paper. Assume that $G$ is cyclic of order 8, and let $s$ be a generator of $G$; let $\epsilon = s^4$ be the element of order 2 of $G$. Let $z \in k^\times$, and let $\sigma : \Gamma_k \to \{1, \epsilon\}$ be the corresponding quadratic character. Let $\phi : \Gamma_k \to G$ be given by $\phi = \iota \circ \sigma$, where $\iota : \{1, \epsilon\} \to G$ is the inclusion. Let $L$ be the $G$-Galois algebra corresponding to $\phi$. Set $H = \{1, \epsilon\}$, and note that the image of $\phi$ is contained in $H$. Set $N = k[X]/(X^2 - z)$; then we have $L = \mathrm{Ind}_H^G(N)$. Set $A = k[X]/(X^4+1)$, and let us write $k[G] = A' \times A$. It is easy to see that the image of $H$ in $A'$ is trivial. The involution $\sigma_A$ sends the class of $X$ to the class of $X^{-1}$. If $k$ contains the 4th roots of unity, then $A$ is a product of two factors exchanged by the involution, hence there $k[G]$ has no involution invariant factor in which the image of $H$ is non trivial. In this case, $L$ has a self-dual normal basis. Assume that $k$ does not contain the 4th roots of unity. Then $A$ is a field; we have $F_A = A$, and $E_A = k[X]/(X^2 - 2)$. Note that $A$ is unitary. We have $i_A(\epsilon) = -1$, hence $i_A(H) = \{1, -1\}$.

Let $i \in F_A$ be a primitive 4th root of unity. By the definition of the extension

$$1 \to Z/2Z \to V_A \to H \to 1$$

(cf. §4), we see that $V_A = \{(1,1), (-1,1), (i,\epsilon), (-i,\epsilon)\}$, a cyclic group of order 4. Recall that $e_A$ is the class of this extension in $H^2(H)$; hence $e_A$ is the only non-trivial element of $H^2(H)$. We have $d_A(L) = \phi^*(e_A) = (z,z) = (z,-1)$, and $\mathrm{Res}_{E_A/k}(d_A(L)) = (z,-1)_E = (z, F_A/E_A)$. Therefore we have

$$d_A(L) = 0 \iff z \text{ is a sum of two squares in } k,$$

and

$$\mathrm{Res}_{E_A/k}(d_A(L)) = 0 \iff z \text{ is a sum of two squares in } E_A = k(\sqrt{2}).$$

It is easy to find examples where $d_A(L) \neq 0$ and $\mathrm{Res}_{E_A/k}(d_A(L)) = 0$; for instance, we can take $k = Q$ and $z = 3$.

By Proposition 5.1 the existence of a self-dual normal basis implies that we have $\mathrm{Res}_{E_A/k}(d_A(L)) = 0$. On the other hand, in [BSe 94], Exemple 10.2 it is checked by direct computation that if $z$ is a sum of two squares in $k(\sqrt{2})$, then $L$ has a self-dual normal basis. Hence we have

$$L \text{ has a self-dual normal basis over } k \iff z \text{ is a sum of two squares in } k(\sqrt{2}).$$

(ii) Assume that $G = D_4$, the dihedral group of order 8. Then a $G$-Galois algebra $L$ has a self-dual normal basis if and only if either $L$ is split or $L = \mathrm{Ind}_H^G(N)$ with $H$ of order 2, and $N = k[X]/(X^2 - z)$ for some $z \in k^\times$ such that z is a sum of two squares in $k$.

Indeed, let $\phi : \Gamma_k \to G$ be a homomorphism associated to $L$. Note that $G^2$ is of order 2, hence the $H^1$-condition holds if and only if the image of $\phi$ is trivial, or equal to $G^2$; in other words, $L$ is split, or induced from a $G^2$-Galois algebra. If $L$ is split, then $L$ has a self-dual normal basis. Set $H = G^2$, and assume that $L = \mathrm{Ind}_H^G(N)$, with $N = k[X]/(X^2 - z)$ for some $z \in k^\times$. It remains to show that $L$ has a self-dual normal basis if and only if $z$ is a sum of two squares in $k$.

The group $G$ has one degree 2 and four degree 1 orthogonal representations. Since the $H^1$-condition holds, the image of $G$ is trivial in the factors of $k[G]$ corresponding to the degree 1 representations. Let $A = M_2(k)$, and let $\sigma_A$ be the involution induced by the 2-dimensional unit form; then the factor of $k[G]$ corresponding to the degree 2 orthogonal representation of $G$ is equal to $A$.

Let $q_A(L)$ be the 2-dimensional quadratic form corresponding to the cohomology class $u_A(L)$. Note that $L$ has a self-dual normal basis if and only if $q_A \simeq \langle 1,1 \rangle$; this is equivalent with $q_A$ having trivial determinant and trivial Hasse-Witt invariant. Recall that the $H^1$-condition is satisfied by hypothesis; hence we have $u_A(L) \in H^1(k, U_A^0)$, and this implies that $\det(q_A(L)) = 1$ in $k^\times/k^{\times 2}$. Since $A$ is a matrix algebra over $k$, we have $w_2(q_A(L)) = \mathrm{clif}(q_A(L))$. By Theorem 3.3, this implies that $w_2(q_A(L)) = c_A(L)$; hence it remains to prove that $c_A(L) = 0$ if and only if $z$ is a sum of two squares in $k$.

If $k$ contains the 4-th roots of unity, then $U_A^0 = \tilde{U}_A = G_m$. If $k$ does not contain the 4-th roots of unity, then $U_A^0 = \tilde{U}_A = R^1_{K/k} G_m$, where $K = k[X]/(X^2+1)$.

In both cases, $s : \tilde{U}_A \to U_A^0$ is the squaring map. Using this, we see that the extension $1 \to Z/2Z \to V_A \to H \to 1$ is non-trivial, and that $c_A(L) = (z, -1)$. Therefore $c_A(L) = 0$ if and only if $z$ is a sum of two squares in $k$, and hence

$$L \text{ has a self-dual normal basis over } k \iff z \text{ is a sum of two squares in } k.$$

(iii) Let $G = A_4$, the alternating group of order 12, and assume for simplicity that $char(k) \neq 3$ and that $k$ contains the third roots of unity. Then $k[G] = k \times k \times k \times M_3(k)$, where the first factor corresponds to the unit representation, the second and the third to the two representations of degree 1 with image of order 3, and the fourth one to the irreducible representation of degree 3. Let $A = M_3(k)$ be the fourth factor, and note that the restriction of $\sigma$ to $A$ is induced by the 3-dimensional unit form. The extension $1 \to Z/2Z \to V_A \to G \to 1$ defined in §3 is

$$1 \to Z/2Z \to \tilde{A}_4 \to A_4 \to 1,$$

corresponding to the unique non-trivial element $e \in H^2(A_4)$ (see [Se 84], 2.3). Let $L$ be a $G$-Galois algebra, and note that the $H^1$-condition is satisfied, since $G$ has no quotient of order 2. Let $E$ be the subalgebra of $L$ fixed by the subgroup $A_3$ of $G = A_4$; then $E$ is an étale algebra of rank 4. Let $\phi : \Gamma_k \to A_4$ be a homomorphism corresponding to $L$. By [Se 84], Theorem 1 we have $\phi^*(e) = w_2(q_E)$, the Hasse-Witt invariant of the quadratic form $q_E$; hence the invariant $c_A(L)$ is equal to $w_2(q_E)$. Let $q_A(L)$ be the 3-dimensional quadratic form corresponding to the cohomology class $u_A(L)$. Then $q_E \simeq q_A(L) \oplus \langle 1 \rangle$, and it is easy to check that $q_A(L) \simeq \langle 1, 1, 1 \rangle \iff w_2(q_E) = 0$, hence $u_A(L) = 0 \iff w_2(q_E) = 0$. Therefore we have

$$L \text{ has a self-dual normal basis over } k \iff w_2(q_E) = 0,$$

recovering a result of [BSe 94] (see [BSe 94], Exemple 1.6).

The case of cyclic groups of order a power of 2 is further developed in §6; we now look at fields of low cohomological dimension. Recall that the *2-cohomological dimension* of $\Gamma_k$, denoted by $\mathrm{cd}_2(\Gamma_k)$, is the smallest integer $d$ such that $H^i(k, C) = 0$ for all $i > d$ and for every finite 2-primary $\Gamma_k$-module $C$. For fields of cohomological dimension $\leq 1$, the question of existence of self-dual normal bases is settled in [BSe 94], 2.2.

Theorem 5.3. *Assume that* $\mathrm{cd}_2(\Gamma_k) \leq 2$. *Then* $L$ *has a self-dual normal basis over* $k$ *if and only if the* $H^1$*-condition is satisfied, and the conditions* (i) *and* (ii) *below hold :*

(i) *For all orthogonal* $\sigma^s$*-stable central simple factors* $A$ *of* $k[G]^s$*, we have*

$$\mathrm{Res}_{E_A/k}(c_A(L)) = 0 \quad in \ \mathrm{Br}_2(E_A)/\langle A \rangle.$$

(ii) *For all unitary* $\sigma^s$*-stable central simple factors* $A$ *of* $k[G]^s$*, we have*

$$\mathrm{Res}_{E_A/k}(d_A(L)) = 0 \quad in \ \mathrm{Br}_2(E_A).$$

Proof. If $L$ has a self-dual normal basis over $k$, then by Proposition 5.1 the $H^1$-condition, as well as conditions (i) and (ii) are satisfied. Conversely, let us assume that the $H^1$-condition, as well as conditions (i) and (ii) hold. Since the $H^1$-condition holds, we can define $c_A(L)$ and $d_A(L)$, cf. §3 and §4. By Theorems 3.3 and 4.5 we have $\mathrm{clif}_A(q_L) = \mathrm{Res}_{E_A/k}(c_A(L))$ and $\mathrm{disc}_A(q_L) = \mathrm{Res}_{E_A/k}(d_A(L))$. Therefore, conditions (i) and (ii) imply that $\mathrm{clif}_A(q_L)$ is trivial for all orthogonal factors $A$, and $\mathrm{disc}_A(q_L)$ is trivial for all unitary factors $A$. Let us prove that $L$ has a self-dual normal basis over $k$. Let us denote by $h_A$ the hermitian form over $(A, \sigma_A)$ corresponding to $u_A(L)$. It is enough to show that for all factors $A$, the class $u_A(L)$ is trivial; this is equivalent with saying that the hermitian form $h_A$ is isomorphic to the unit form $1_A$ over $(A, \sigma_A)$. By Witt cancellation (see for instance [BPS 13], Theorem 2.5.2) this in turn is equivalent to saying that $h_A \oplus -1_A$ is hyperbolic. Let us prove this successively for symplectic, orthogonal and unitary characters.

Assume first that $A$ is symplectic. Then by [BP 95], Theorem 4.3.1 every even dimensional non-degenerate hermitian form over a central simple algebra with involution is hyperbolic. This implies that $h_A \oplus -1_A$ is hyperbolic. Assume now that $A$ is orthogonal, and note that the $H^1$-condition implies that $u_A(L)$ is the image of a class $u^0_A(L)$ of $H^1(E_A, U^0_A)$. This implies that $h_A$ has trivial discriminant. As we saw above, $\mathrm{clif}_A(q_L)$ is trivial, hence the form $h_A \oplus -1_A$ has trivial Clifford invariant. By [BP 95], Theorem 4.4.1 every even dimensional non-degenerate hermitian form over a central simple algebra having trivial discriminant and trivial Clifford invariant is hyperbolic, hence $h_A \oplus -1_A$ is hyperbolic. Assume finally that $A$ is a unitary character. We have seen above that $\mathrm{disc}_A(q_L)$ is trivial, therefore the form $h_A \oplus -1_A$ has trivial discriminant. By [BP 95], Theorem 4.2.1 every even dimensional non-degenerate hermitian form over a central simple algebra having trivial discriminant is hyperbolic, hence $h_A \oplus -1_A$ is hyperbolic.

This implies that $L$ has a self-dual normal basis over $k$, hence the theorem is proved.

Recall that $\phi : \Gamma_k \to G$ is a homomorphism associated to the $G$-Galois algebra $L$, and that for all $x \in H^n(G)$, we denote by $x_L$ the image of $x$ by $\phi^* : H^n(G) \to H^n(k)$. Let $H = \phi(\Gamma_k)$. For $n = 2$, we also need the image of $x$ by the homomorphism $\phi^* : H^n(H) \to H^n(k)$; we denote this image by $x^H_L$.

Corollary 5.4. *Assume that* $\mathrm{cd}_2(\Gamma_k) \leq 2$*, that the* $H^1$*-condition is satisfied, and that we have* $x^H_L = 0$ *for all* $x \in H^2(H)$*. Then* $L$ *has a self-dual normal basis over* $k$*.*

Proof. This follows immediately from Theorem 5.3. Indeed, the $H^1$-condition is satisfied by hypothesis. Moreover, the classes $c_A(L)$ and $d_A(L)$ are by definition in the image of $\phi^* : H^2(H) \to H^2(k)$, hence the hypothesis $x^H_L = 0$ for all $x \in H^2(H)$ implies that $c_A(L) = 0$ for all orthogonal factors $A$, and $d_A(L) = 0$ for all unitary factors $A$. Therefore conditions (i) and (ii) of Theorem 5.3 are satisfied, and hence $L$ has a self-dual normal basis over $k$.

Remarks. (i) Corollary 5.4 suggests the following question : Assume that $\mathrm{cd}_2(\Gamma_k) \le 2$, and that the $H^1$-condition is satisfied. If $x_L = 0$ for all $x \in H^2(G)$, does it follow that $L$ has a self-dual normal basis over $k$ ? This follows from Corollary 5.4 when $L$ is a field extension, in other words, when $\phi$ is surjective : indeed, then $H = G$.

(ii) The question above (see (i)) has a negative answer for fields of higher cohomological dimensions. Indeed, by [BSe 94], III. 10.1, there exist examples of $G$-Galois algebras $L$ over fields of cohomological dimension 3 such that for all $n > 0$ we have $x_L = 0$ for all $x \in H^n(G)$, but $L$ does not have a self-dual normal basis over $k$.

(iii) The converse of the question raised in (i) also has a negative answer : indeed, there exist examples of $G$-Galois algebras $L$ over $Q$ having a self-dual normal basis such that there exists $x \in H^2(G)$ with $x_L \neq 0$ (see [BSe 94], III. 10.2).

The following result was proved in [BSe 94], Corollaire 3.2.2 in the case where $k$ is an imaginary number field; the proof also applies for fields of cohomological dimension $\le 2$, using the results of [BP 95]. We give here an alternative proof.

Corollary 5.5. *Assume that* $\mathrm{cd}_2(\Gamma_k) \le 2$*, and that*

$$H^1(G) = H^2(G) = 0.$$

*Then $L$ has a self-dual normal basis over $k$.*

Proof. Since $H^1(G) = 0$, we have $G = G^2$. Let $A$ be orthogonal or unitary, and let us construct a central extension $V'_A$ of $G$ by $Z/2Z$, as follows. If $A$ is orthogonal, set $V'_A = V_A^G = \tilde{U}_A(E_A) \times_{U_A^0(E_A)} G$, with the notation of §3. If $A$ is unitary, then we set $V'_A = V_A^G = F_A^{\times 1} \times_{F_A^{\times 1}} G$, the notation being as in §4. In each case, we get a central extension $V'_A$ of $G$ by $Z/2Z$. Since $H^2(G) = 0$, this extension is split. Note that the central extension $V_A$ of $H$ by $Z/2Z$ constructed in §3 and §4 is a subgroup of $V'_A$, and that the restriction of the projection $V'_A \to G$ is the projection $V_A \to H$. Hence the extension $V_A$ is also split. This implies that we have $c_A(L) = 0$ for every orthogonal $A$, and $d_A(L) = 0$ for every unitary $A$. By Theorem 5.3 this implies that $L$ has a self-dual normal basis over $k$.

## §6. Cyclic groups of 2-power order

In this section, $G$ is assumed to be *cyclic of order* $2^n$, with $n \ge 2$. We start by giving necessary and sufficient conditions for two $G$–Galois algebras to have isomorphic trace forms in terms of cohomological invariants of degree 1 and 2 (see Proposition 6.2), namely the degree 1 invariants introduced in [BSe 94], and the discriminants of the hermitian forms at the unitary factors (see §4).

We then use the invariants defined in the first part of §4 to give necessary and sufficient conditions for the existence of a self–dual normal basis. We start with settling the case where $k$ contains the 4th roots of unity :

Proposition 6.1. *Assume that $k$ contains the 4th roots of unity. Let $L$ and $L'$ be two $G$–Galois algebras. Then $q_L \simeq_G q_{L'}$ if and only if $x_L = x_{L'}$ for all $x \in H^1(G)$.*

Proof. The algebra $k[G]$ has two orthogonal factors $k$; since $k$ contains the 4th roots of unity, there are no other involution invariant factors. Therefore $u(L) = u(L')$ if and only if the cohomology classes $u$ associated to the two degree 1 orthogonal factors coincide, and this is equivalent with the condition $x_L = x_{L'}$ for all $x \in H^1(G)$. Hence, by [BSe 94], Proposition 1.5.1, we have $q_L \simeq_G q_{L'}$.

More generally, we have :

Proposition 6.2. *Let $L$ and $L'$ be two $G$–Galois algebras. Then $q_L \simeq_G q_{L'}$ if and only if the following conditions hold :*

(i) $x_L = x_{L'}$ *for all* $x \in H^1(G)$.

(ii) $\mathrm{disc}_A(q_L) = \mathrm{disc}_A(q_{L'})$ *for all unitary factors $A$ of $k[G]$.*

Before proving Proposition 6.2, note that when $k$ contains the 4th roots of unity, then Proposition 6.2 follows from Proposition 6.1. Hence we only need to prove the proposition when $k$ does not contain the 4th roots of unity.

From now on, we assume that $k$ *does not contain the 4th roots of unity.* We start by introducing some notation. Set $A(i) = k[X]/(X^{2^{i-1}} + 1)$, for $i = 1, \dots, n$; then the factors of $k[G]$ are $k$, and $A(1), \dots, A(n)$. Note that $k$ and $A(1)$ are orthogonal, and $A(2), \dots, A(n)$ are unitary. For $i = 2, \dots, n$, we have $A(i) = F_{A(i)}$.

Proof of Proposition 6.2. Recall that we are assuming that $k$ does not contain the 4th roots of unity (otherwise, the proposition follows from Proposition 6.1). For all factors $A$ of $k[G]$, let us denote by $h_A$, respectively $h'_A$, the hermitian form over $(A, \sigma_A)$ determined by $q_L$, respectively $q_{L'}$.

Assume that $q_L \simeq_G q_{L'}$. Then (i) holds by [BSe 94], Proposition 2.2.1. Let $A$ be a unitary factor; then the hermitian forms $h_A$ and $h'_A$ are isomorphic. Since $\mathrm{disc}_A(q_L)$ and $\mathrm{disc}_A(q_{L'})$ are invariants of these hermitian forms, condition (ii) holds as well.

Conversely, suppose that (i) and (ii) hold. Let us show that $u_A(L) = u_A(L')$ for all factors $A$. Condition (i) implies that this is true for $A = k$ and $A = A(1)$; indeed, in both cases the group $U_A$ is of order 2. Let us assume that $A$ is a unitary factor, that is, $A = A(i)$ for some $i = 2, \dots, n$. Note that $A = F_A$, hence the hermitian forms $h_A$ and $h'_A$ are one dimensional hermitian forms over the commutative field $F_A$. Such a form is determined up to isomorphism by its discriminant; hence condition (ii) implies that $h_A \simeq h'_A$. Therefore we have $u_A(L) = u_A(L')$ for all factors $A$, hence $u(L) = u(L')$, and by [BSe 94], Proposition 1.5.1 we have $q_L \simeq_G q_{L'}$. This completes the proof of the Proposition.

Let us recall a notation from [Se 84], 1.5 or [Se 92], 9.1.3 : if $m$ is an integer, $m \geq 1$, we denote by $s_m \in H^2(S_m)$ the element of $H^2(S_m)$ corresponding to the central extension

$$1 \to Z/2Z \to \tilde{S}_m \to S_m \to 1$$

which is characterized by the properties :

1. A transposition in $S_m$ lifts to an element of order 2 in $\tilde{S}_m$.
2. A product of two disjoint transpositions lifts to an element of order 4 in $\tilde{S}_m$.

Note that $s_m = 0$ if and only if $m \leq 3$ (see [Se 84], 1.5).

If $m$ is a power of 2, $m \geq 2$, let us denote by $C_m$ the cyclic group of order $m$, and by $e_m$ be the unique non-trivial element of $H^2(C_m)$. Sending a generator of $C_m$ to an $m$-cycle of $S_m$ defines an injective homomorphism $f : C_m \to S_m$; we denote by $f^* : H^2(S_m) \to H^2(C_m)$ the homomorphism induced by $f$.

If $q$ is a quadratic form over $k$, we denote by $w_2(q)$ its Hasse-Witt invariant (see for instance [Se 84], 1.2 or [Se 92], 9.1.2); it is an element of $H^2(k)$.

Lemma 6.3. *Let $m$ be a power of 2.*

(i) *We have $f^*(s_m) = e_m$ in $H^2(C_m)$.*

(ii) *Let $\psi : \Gamma_k \to C_m$ be a continuous homomorphism, and let $K$ be the étale algebra over $k$ corresponding to $\phi$. Then the obstruction to the lifting of $\phi$ to a homomorphism $\Gamma_k \to C_{2m}$ is*

$$w_2(q_K) + (2)(D_K)$$

*where $D_K$ is the discriminant of $K$, and $(2)(D_K)$ denotes the cup product of the elements $(2)$ and $(D_K)$ of $H^1(k)$.*

Proof. (i) Let $\tilde{C}_m$ be the inverse image of $C_m$ in $\tilde{S}_m$; it suffices to show that $\tilde{C}_m \simeq C_{2m}$, in other words that $\tilde{C}_m$ is a non-trivial extension of $C_m$. Raising an $m$-cycle of $S_m$ to the $\frac{m}{2}$-th power yields a product of $\frac{m}{2}$ disjoint transpositions, and the inverse image of such an element in $\tilde{S}_m$ is of order 4. Hence $\tilde{C}_m$ is a non-trivial extension of $C_m$.

(ii) The obstruction to the lifting of $\psi$ is $\psi^*(e_m) \in H^2(k)$. Since $f^*(s_m) = e_m$ by (i), we have

$$(f \circ \psi)^*(s_m) = \psi^*(e_m).$$

On the other hand, $(f \circ \psi)^*(s_m) = w_2(q_K) + (2)(D_K)$ by [Se 84], Theorem 1.

Proposition 6.4. *Let $L$ be a $G$-Galois algebra, and assume that the $H^1$–condition holds. Then we have*

(i) *Let $A$ be a unitary factor of $k[G]$. If $A \neq A(n)$, then $d_A(L) = 0$.*

(ii) *Let $L = K \times \cdots \times K$, where $K$ is a field extension of $k$. Then*

$$d_{A(n)}(L) = w_2(q_K) + (2)(D_K).$$

Proof. Let $\phi : \Gamma_k \to G$ be a homomorphism associated to $L$, let $H = \phi(\Gamma_k)$, and let us denote by $|H|$ its order. Recall from §4 that the extension

$$(*) \qquad 1 \to Z/2Z \to V_A \to H \to 1$$

is defined by $V_A = \{(x, h) \in F_A^{\times 1} \times H \mid x^2 = i_A(h)\}$. Let us show that this extension is split if $A \neq A(n)$. Note that the group $V_A$ is abelian, and hence $(*)$ is not split if and only if $V_A$ is a cyclic group of order $2|H|$. On the other hand, if $A \neq A(n)$, then the order of $i_A(H)$ is strictly less than $|H|$, hence the group $V_A$ does not have any elements of order $2|H|$. Therefore the extension $(*)$ is split, and hence $d_A(L) = 0$; this completes the proof of (i).

Let us prove (ii). If $L$ is split, then (ii) obviously holds, hence we may assume that $|H| \geq 2$. If $A = A(n)$, then the group $V_A$ is cyclic of order $2|H|$, and the extension $(*)$ is not split. Recall that we denote by $e_A \in H^2(H)$ the class of this extension, and that $d_A = \phi^*(e_A) \in H^2(k)$. Note that $\phi^*(e_A)$ is also the obstruction for the lifting of $\phi : \Gamma_k \to H$ to a continuous homomorphism $\Gamma_k \to V_A$; by Lemma 6.3 (ii) this obstruction is equal to $w_2(q_K) + (2)(D_K)$, hence (ii) is proved.

Corollary 6.5. *Let $L$ be a $G$-Galois algebra, and assume that the $H^1$–condition holds. Then $L$ has a self–dual normal basis if and only if* $\mathrm{Res}_{E_{A(n)}/k}(d_{A(n)}(L)) = 0$ *in* $\mathrm{Br}_2(E_{A(n)})$.

Proof. Proposition 6.2 implies that $L$ has a self-dual normal basis if and only if the $H^1$-condition holds and if $\mathrm{disc}_A(q_L) = 0$ for all unitary factors $A$ of $k[G]$. By Theorem 4.5 we have $\mathrm{Res}_{E_A/k}(d_A(L)) = \mathrm{disc}_A(q_L)$, and Proposition 6.4 (i) implies that $d_A(L) = 0$ if $A \neq A(n)$. This completes the proof of the corollary.

Corollary 6.6. *Let $L$ be a $G$-Galois algebra, and assume that the $H^1$–condition holds. Let $L = K \times \cdots \times K$, where $K$ is a field extension of $k$, with $Gal(K/k)$ cyclic of order $m$. If $K$ can be embedded in a Galois extension of $k$ with cyclic Galois group of order $2m$, then $L$ has a self-dual normal basis.*

Proof. Assume that $K$ can be embedded in a Galois extension of $k$ with cyclic Galois group of order $2m$. Then by Lemma 6.3 (ii) we have $w_2(q_K) + (2)(D_K) = 0$. By Proposition 6.4 (ii), this implies that $d_{A(n)}(L) = 0$, and hence by Corollary 6.5 the $G$-Galois algebra $L$ has a self-dual normal basis.

Example 6.7. Assume that $G$ is of order 8. Let $a, b, c, \epsilon \in k$ with $a^2 - b^2\epsilon = c^2\epsilon$; assume $c$ non-zero, and $\epsilon$ not a square. Set $x = \sqrt{\epsilon}$, and let $K = k(\sqrt{a + bx})$; note that $D_K = \epsilon$, and that $K/k$ is a cyclic extension of degree 4 (see for instance [Se 92], Theorem 1.2.1). Let $L$ be the $G$-Galois algebra induced from $K$. Let us prove that

$$L \text{ has a self-dual normal basis} \iff a \text{ is a sum of two squares in } k(\sqrt{2}).$$

Indeed, set $A = A(3)$; by Corollary 6.5 the $G$-Galois algebra $L$ has a self-dual normal basis if and only if $\mathrm{Res}_{E_A/k}(d_A(L)) = 0$. We have $d_A(L) = w_2(q_K) + (2)(\epsilon)$ by Proposition 6.4 (ii).

Let us show that $w_2(q_K) = (-1)(a)$. Set $y = \sqrt{a + bx}$. Then $\{1, x, y, xy\}$ is a basis of $K$ over $k$, and in this basis the quadratic form $q_K$ is the orthogonal sum of the diagonal form $\langle 1, \epsilon\rangle$ and of the quadratic form $q$ given by $aX^2 + 2b\epsilon XY + a\epsilon Y^2$. The form $q$ represents $a$, and its determinant is $\epsilon(a^2 - b^2\epsilon) = c^2\epsilon^2$, hence $\det(q) = 1$ in $k^2/k^{\times 2}$. This implies that $q \simeq \langle a, a\rangle$, hence $q_K \simeq \langle 1, \epsilon, a, a\rangle$, and $w_2(q_K) = (a)(a) = (-1)(a)$.

Therefore $d_A(L) = (-1)(a) + (2)(\epsilon)$. Note that $E_A = k(\sqrt{2})$; hence $\mathrm{Res}_{A/k}(d_A(L)) = \mathrm{Res}_{k(\sqrt{2})/k}((-1)(a))$, and this element is 0 if and only if $a$ is a sum of two squares in $k(\sqrt{2})$.

Note that combining this example with Example 5.2 (i) we get a necessary and sufficient condition for a $C_8$-Galois algebra to have a self-dual normal basis.

## §7. Self-dual normal bases over local fields

We keep the notation of the previous sections, and assume that $k$ is a (non-archimedean) local field. The aim of this section is to give a necessary and sufficient condition for the existence of self-dual normal bases in terms of invariants defined over $k$.

We say that $A$ is *split* if it is a matrix algebra over its center.

Theorem 7.1. *The $G$-Galois algebra $L$ has a self-dual normal basis if and only if the $H^1$-condition holds, and*

(i) *For all orthogonal $A$ such that $[E_A : k]$ is odd and $A$ is split, we have $c_A(L) = 0$ in $\mathrm{Br}_2(k)$.*

(ii) *For all unitary $A$ such that $[E_A : k]$ is odd, we have $d_A(L) = 0$ in $\mathrm{Br}_2(k)$.*

Proof. Assume that the $H^1$-condition is satisfied and that (i) and (ii) hold. Note that if $A$ is not split, then we have $\mathrm{Br}_2(E_A)/\langle A\rangle = 0$, and that if $[E_A : k]$ is even, then the map $\mathrm{Res}_{E_A/k} : \mathrm{Br}_2(k) \to \mathrm{Br}_2(E_A)$ is trivial. Therefore for all orthogonal $A$ we have $\mathrm{Res}_{E_A/k}(c_A(L)) = 0$ in $\mathrm{Br}_2(E_A)/\langle A\rangle$, and for all unitary $A$ we have $\mathrm{Res}_{E_A/k}(d_A(L)) = 0$ in $\mathrm{Br}_2(E_A)$. By Theorem 5.3, this implies that $L$ has a self-dual normal basis.

Conversely, suppose that $L$ has a self-dual normal basis. Then the $H^1$-condition holds by Proposition 2.1. By Theorem 5.1 we have $\mathrm{Res}_{E_A/k}(c_A(L)) = 0$ in $\mathrm{Br}_2(E_A)/\langle A \rangle$ for all orthogonal $A$. Since $\mathrm{Res}_{E_A/k} : \mathrm{Br}_2(k) \to \mathrm{Br}_2(E_A)$ is injective if $[E_A : k]$ is odd, condition (i) holds. Moreover, Theorem 5.1 implies that if $A$ is unitary, then $\mathrm{Res}_{E_A/k}(d_A(L)) = 0$ in $\mathrm{Br}_2(E_A)$. Applying again the injectivity of $\mathrm{Res}_{E_A/k}$ when $[E_A : k]$ is odd, we obtain condition (ii). This completes the proof of the theorem.

## §8. Self-dual normal bases over global fields

We keep the notation of the previous sections. Assume that $k$ is a global field, and let $\Omega_k$ be the set of places of $k$. For all $v \in \Omega_k$, we denote by $k_v$ the completion of $k$ at $v$. For all $k$-algebras $R$, set $R^v = R \otimes_k k_v$. We say that a $G$-Galois algebra is *split* if it is isomorphic to a direct product of copies of $k$ permuted by $G$. We now apply the Hasse principle of [BPS 13] together with Theorem 7.1 above to give necessary and sufficient conditions for the existence of a self-dual normal basis over $k$.

Note that the fields $E_A$ are abelian Galois extensions of $k$ (cf. 1.2).

For all finite places $v$, let us write $E_A^v = K_A(v) \times \cdots \times K_A(v)$, where $K_A(v)$ is a field extension of $k_v$. Set $n_A^v = [K_A(v) : k_v]$.

We need additional notation in the case when $A$ is unitary. Note that while $A$ is a central simple algebra over $F_A$, and $F_A/E_A$ is a quadratic extension, for some places $v \in \Omega_k$ we may have $F_A^v = E_A^v \times E_A^v$ with $\sigma_A$ permuting the components, and $A^v = B \times B$ for some $k_v$-algebra $B$. In order to take this into account, we set $\epsilon_A^v = 0$ if $F_A^v = E_A^v \times E_A^v$, and $\epsilon_A^v = 1$ otherwise.

Theorem 8.1. *The $G$-Galois algebra $L$ has a self-dual normal basis if and only if the $H^1$-condition holds, if $L^v$ is split for all real places $v$, and if for all finite places $v$ we have*

(i) *For all orthogonal $A$ such that $n_A^v$ is odd and $A^v$ is split, we have $c_A(L) = 0$ in $\mathrm{Br}_2(k_v)$.*

(ii) *For all unitary $A$ such that $n_A^v$ is odd and $\epsilon_A^v = 1$, we have $d_A(L) = 0$ in $\mathrm{Br}_2(k_v)$.*

Proof. If $L$ has a self-dual normal basis, then $L^v$ is split for all real places $v$ by [BSe 94], Corollaire 3.1.2, and conditions (i) and (ii) hold for all finite places $v$ by Theorem 7.1. Conversely, assume that $L^v$ is split for all real places $v$, and that for all finite places $v$ conditions (i) and (ii) hold. Then [BSe 94], Corollary 3.1.2 (for real places) and Theorem 7.1 (for finite places) imply the existence of a self-dual normal basis for $L^v$, for all $v \in \Omega_k$. By the Hasse principle result of [BPS 13], Theorem 1.3.1, the $G$-Galois algebra $L$ has a self-dual normal basis over $k$.

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# Concordance Invariance of Levine-Tristram Signatures of Links

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Abstract. We determine for which complex numbers on the unit circle the Levine-Tristram signature and the nullity give rise to link concordance invariants.

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## 1. Introduction

Let $L \subset S^3$ be an $m$-component oriented link in the 3-sphere. Each connected, oriented Seifert surface $F$ for $L$ has a bilinear Seifert form defined by

$$\begin{aligned} V\colon H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) &\to \mathbb{Z} \\ (p[x], q[y]) &\mapsto pq\,\mathrm{lk}(x^-, y), \end{aligned}$$

where $p, q \in \mathbb{Z}$, $x, y$ are simple closed curves on $F$ with associated homology classes $[x], [y]$, and $x^-$ is a push-off of $x$ in the negative normal direction of $F$. Given a unit modulus complex number $z \in S^1 \smallsetminus \{1\}$, choose a basis for $H_1(F;\mathbb{Z})$ and define the hermitian matrix

$$B(z) := (1-z)V + (1-\overline{z})V^T.$$

The *Levine-Tristram signature* $\sigma_L(z)$ of $L$ at $z$ is defined to be the signature of $B(z)$, namely the number of positive eigenvalues minus the number of negative eigenvalues. The *nullity* $\eta_L(z)$ of $L$ at $z$ is the dimension of the null space of $B(z)$. Both quantities can be shown to be invariants of the $S$-equivalence class of the Seifert matrix, and are therefore link invariants [Lev69, Tri69].

We say that two oriented $m$-component links $L$ and $J$ are *concordant* if there is a flat embedding into $S^3 \times I$ of a disjoint union of $m$ annuli $A \subset S^3 \times I$, such that the oriented boundary of $A$ satisfies

$$\partial A = -L \sqcup J \subset -S^3 \sqcup S^3 = \partial(S^3 \times I).$$

An $m$-component link $L$ is *slice* if it is concordant to the $m$-component unlink. The purpose of this paper is to answer the following question: *for which values of $z$ are $\sigma_L(z)$ and $\eta_L(z)$ link concordance invariants?* We work in the topological category, in order to obtain the strongest possible results. In order to state our main theorem, we need one more definition.

Definition 1.1. A complex number $z \in S^1 \smallsetminus \{1\}$ is a *Knotennullstelle* if there exists a Laurent polynomial $p(t) \in \mathbb{Z}[t, t^{-1}]$ with $p(1) = \pm 1$ and $p(z) = 0$.

Note that a complex number $z \in S^1 \smallsetminus \{1\}$ is a Knotennullstelle if and only if there exists a knot $K$ whose Alexander polynomial $\Delta_K$ has the property that $\Delta_K(z) = 0$. This follows from the fact that all Laurent polynomials $q \in \mathbb{Z}[t, t^{-1}]$ with $q(1) = \pm 1$ and $q(t) = q(t^{-1})$ can be realised as Alexander polynomials of knots [BZ03, Theorem 8.13]. Here is our main theorem.

Theorem 1.2. *The link invariants $\sigma_L(z)$ and $\eta_L(z)$ are concordance invariants if and only if $z \in S^1 \smallsetminus \{1\}$ does not arise as a Knotennullstelle.*

*Discussion of previously known results.* The first point to note is that, due to J. C. Cha and C. Livingston [CL04], when $z$ is a Knotennullstelle neither $\sigma_L(z)$ nor $\eta_L(z)$ are link concordance invariants.

Theorem 1.3 (Cha, Livingston). *For any Knotennullstelle $z \in S^1 \smallsetminus \{1\}$, there exists a slice knot $K$ with $\sigma_K(z) \neq 0$ and $\eta_K(z) \neq 0$.*

Given a polynomial $p(t)$ with $p(1) = \pm 1$ and $p(z) = 0$, Cha and Livingston construct a matrix $V$ with $V - V^T$ nonsingular, with $\det(tV - V^T)$ equal to $p(t)p(t^{-1})$, such that the upper left half-size block contains only zeroes, and such that $\sigma(B(z)) \neq 0$. Such a matrix can easily be realised as the Seifert matrix of a slice knot.

Some positive results on concordance invariance are also known. For $z$ a prime power root of unity, $\sigma_L(z)$ and $\eta_L(z)$ are concordance invariants; see [Mur65], [Tri69] and [Kau78]. D. Cimasoni and V. Florens [CF08] dealt with multivariable signature and nullity concordance invariants, but again only at prime power roots of unity.

For the signature and nullity at algebraic numbers away from prime power roots of unity, we could not find any statements or results in the literature pertaining to our question. Levine [Lev07] studied the question in terms of $\rho$-invariants, but only discussed concordance invariance away from the roots of the Alexander polynomial.

By changing the rules slightly, one can obtain a concordance invariant for all $z$. The usual method is to define a function that is the average of the two one-sided limits of the Levine-Tristram signature function. Let $z = e^{i\theta} \in S^1$, and consider:

$$\overline{\sigma}_L(z) := \frac{1}{2}\Big(\lim_{\omega\to\theta_+} \sigma(B(e^{i\omega})) + \lim_{\omega\to\theta_-} \sigma(B(e^{i\omega}))\Big).$$

Since prime power roots of unity are dense in $S^1$, this averaged signature function yields a concordance invariant at every $z \in S^1$. The earliest explicit observation of this that we could find was by Gordon in the survey article [Gor78]. One can also consider the averaged nullity function, to which similar remarks apply:

$$\overline{\eta}_L(z) := \frac{1}{2}\Big(\lim_{\omega\to\theta_+} \eta(B(e^{i\omega})) + \lim_{\omega\to\theta_-} \eta(B(e^{i\omega}))\Big).$$

In particular this is also a link concordance invariant.

Note that the function $\sigma_L \colon S^1 \smallsetminus \{1\} \to \mathbb{Z}$ is continuous away from roots of the Alexander polynomial $\det(tV - V^T)$ of $L$. More generally one can consider the torsion Alexander polynomial $\Delta_L^{\mathrm{Tor}}$ of $L$, which by definition is the greatest common divisor of the $(n-r)\times(n-r)$ minors of $tV - V^T$, where $n$ is the size of $V$ and $r$ is the minimal nonnegative integer for which the set of minors contains a nonzero polynomial. The function $\sigma_L$ is continuous away from the roots of the torsion Alexander polynomial $\Delta_L^{\mathrm{Tor}}$, by [GL15, Theorem 2.1] (their $A_L$ is our $\Delta_L^{\mathrm{Tor}}$).

Thus if $z$ is not a root of the torsion Alexander polynomial of *any* link, the signature cannot jump at that value, and the signature function $\sigma_L(z)$ equals the averaged signature function $\overline{\sigma}_L(z)$ there. Since the averaged function is known to be a concordance invariant, the non-averaged function is also an invariant when $z$ is not the root of any link's Alexander polynomial. The excitement happens when $z$ is the root of the Alexander polynomial of some link, but is not the root of an Alexander polynomial of any knot. The averaged and non-averaged signature functions can differ at such $z$, but nevertheless *both* are concordance invariants. In Section 2 we will give an example which illustrates this difference, and gives an instance where the non-averaged function is more powerful. Similar examples were given in [GL15], but only with jumps occurring at prime power roots of unity.

Finally we remark that our proof of Theorem 1.2 covers the previously known cases of prime power roots of unity and transcendental numbers, as well as the new cases.

*Organisation of the paper.* The rest of the paper is organised as follows. In Section 2, we give an example of two links that are not concordant, where we use the signature and nullity functions at a root of their Alexander polynomials, which is not a prime power root of unity, to detect this fact. Section 4 proves that the nullity is a concordance invariant, and the corresponding fact for signatures is proven in Sections 5 and 6.

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## 2. An application

In the introduction, for a link $L$ we defined the signature function $\sigma_L(z)$ and the nullity function $\eta_L(z)$, for each $z \in S^1 \smallsetminus \{1\}$. From the characterisation in Theorem 1.2, one easily finds new values $z$ for which it was not previously known that $\sigma(z)$ and $\eta(z)$ are concordance invariants. In Proposition 2.3, by exhibiting the obligatory explicit example, we show that these values give obstructions to concordance that are independent from previously known obstructions coming from the signature and nullity functions. We finish the section by constructing, in Proposition 2.5, a family of such examples for any algebraic number on $S^1$. Before the construction, we collect some facts on the set of roots of Alexander polynomials of links. We say that a complex number $z \in S^1 \smallsetminus \{1\}$ is a *Linknullstelle* if $z$ is a root of a non-vanishing single variable Alexander polynomial of some link. We have the following inclusions:

$$\begin{array}{ccccc} \{\text{Knotennullstellen}\} & \subset & \{\text{Linknullstellen}\} & \subset & S^1 \smallsetminus \{1\} \\ & & \cup & & \\ & & \left\{ \begin{array}{c} \text{prime power} \\ \text{roots of } 1 \end{array} \right\} & & \end{array}$$

We will see that these inclusions are strict. The two subsets of the set of Linknullstellen are disjoint, since no prime power root of unity can be a root of a polynomial that augments to $\pm 1$, because the corresponding cyclotomic polynomial augments to the prime. Moreover, the union of the Knotennullstellen and the prime power roots of unity is not exhaustive.

Lemma 2.1.

*(1) The set of Linknullstellen coincides with the set of algebraic numbers in $S^1 \smallsetminus \{1\}$.*

*(2) The number $z_0 = \frac{3+4i}{5} \in S^1$ is an algebraic number, which is neither a Knotennullstelle nor a root of unity.*

*Proof.* Let $z \in S^1 \smallsetminus \{1\}$ be an algebraic number, so that $p(z) = 0$ for some $p \in \mathbb{Z}[t]$. Let

$$q(t) := (t-1)^3 p(t) p(t^{-1}) \in \mathbb{Z}[t, t^{-1}].$$

We claim that there is a link $L$ with single variable Alexander polynomial $\Delta_L(t) = q(t)$. Choose a 2-variable polynomial $P(x,y) \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ with $P(t,t) = p(t)$. Let

$$Q(x,y) := (x-1)(y-1)P(x,y)P(x^{-1}, y^{-1}).$$

A corollary [Hil12, Corollary 8.4.1] to Bailey's theorem [Bai77] states that any polynomial $Q(x,y)$ in $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$, with $Q = \overline{Q}$ up to multiplication by $\pm x^k y^\ell$, and such that $(x-1)(y-1)$ divides $Q$, is the Alexander polynomial of some 2-component link of linking number zero. Thus there exists a 2-component link $L$ with 2-variable Alexander polynomial $Q(x,y)$.

The single variable Alexander polynomial $\Delta_L(t)$ is obtained from the 2-variable Alexander polynomial of a 2-component link $Q(x,y)$ as $(t-1)Q(t,t)$ [BZ03, Remark 9.18]. But

$$(t-1)Q(t,t) = (t-1)^3 P(t,t)P(t^{-1},t^{-1}) = (t-1)^3 p(t)p(t^{-1}) = q(t).$$

This completes the proof of the claim and therefore of (1): the set of Linknullstellen is the set of algebraic numbers lying on $S^1 \smallsetminus \{1\}$.
For (2), first observe that the complex number $z_0 := \frac{3+4i}{5}$ has unit modulus and that $z_0$ is a zero of the polynomial

$$p(t) := 5t^2 - 6t + 5,$$

and therefore is an algebraic number. Note that no cyclotomic polynomial divides the polynomial $p(t)$. This can be checked for the first six by hand, and the rest have degree larger than 2. From Abel's irreducibility theorem, we learn that $z_0$ is not a zero of a cyclotomic polynomial and thus is not a root of unity. Since $p(1) = 4$ and $p(t)$ is irreducible over $\mathbb{Z}[t]$, $z_0$ is not the root of any polynomial that augments to $\pm 1$. As a result, $z_0$ is not a Knotennullstelle. $\square$

Next we describe links $L$ and $L'$ whose signature and nullity functions are equal everywhere on $S^1 \smallsetminus \{1\}$ apart from at $z_0$, which will be a root of the Alexander polynomials of $L$ and $L'$. We find these links by realising suitable Seifert forms.

EXAMPLE 2.2. Consider the following Seifert matrix:

$$V := \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This matrix represents the Seifert form of the 3-component link $L$ given by the boundary of the Seifert surface shown in Figure 1. As usual, a box with $n \in \mathbb{Z}$ inside denotes $n$ full right-handed twists between two bands, made without introducing any twists into the individual bands. To see what we mean, observe that there are three instances in the figure of one full left-handed twist, otherwise known as $-1$ full right-handed twists. The left-most twist is between the bands labelled $e_1$ and $e_5$. To obtain the Seifert matrix, note that the beginning of each of the eight bands is labelled $e_i$, for $i = 1, \dots, 8$. Orient the bands clockwise and compute using $V_{ij} = \operatorname{lk}(e_i^-, e_j)$, where the picture is understood to show the positive side of the Seifert surface.
Produce a link $L'$ from $L$ by removing the single twist in the right-most band, labelled $e_8$ in Figure 1. This gives rise to a Seifert matrix $V'$ for $L'$ which is the same as $V$, except that the bottom right entry is a 0 instead of a 1.

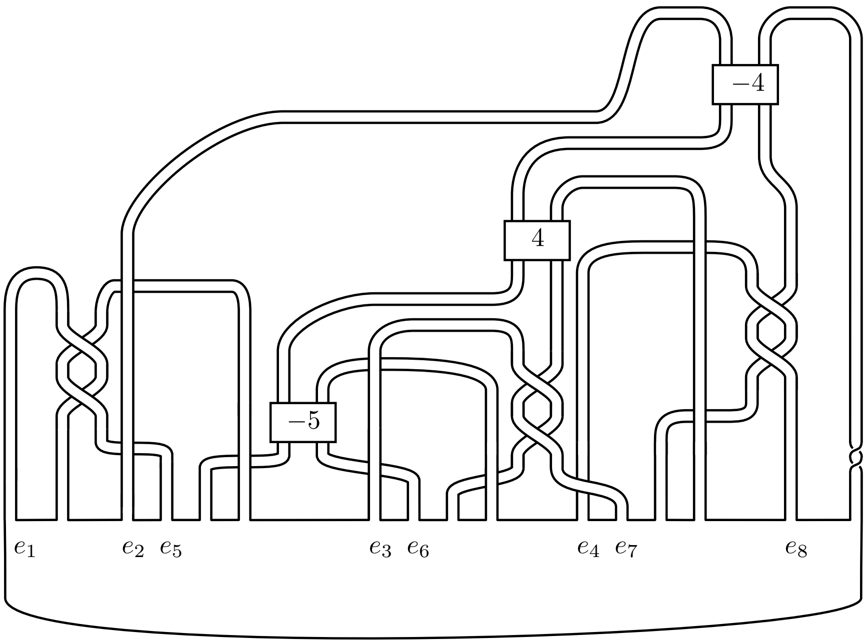


FIGURE 1. Realisation of the Seifert form $V$.

Consider the sesquilinear form $B$ over $\mathbb{Q}[t^{\pm 1}]$ determined by the matrix

$$(1-t)V + (1-t^{-1})V^T.$$

The form $B$ splits into a direct sum of sesquilinear forms. For a Laurent polynomial $p(t) \in \mathbb{Q}[t^{\pm 1}]$, abbreviate the form given by the $2 \times 2$ matrix

$$\begin{pmatrix} 0 & p(t) \\ p(t^{-1}) & 0 \end{pmatrix}.$$

by $[p(t)]$. A calculation shows that $B$ is congruent to the form

$$[t-1] \oplus [t-1] \oplus [t-1] \oplus \begin{pmatrix} 0 & q(t) \\ q(t^{-1}) & -t^{-1}+2-t \end{pmatrix},$$

where the polynomial $q(t)$ is

$$q(t) = t^{-1} \cdot (t-1)^3 \cdot (5t^2 - 6t + 5).$$

On the other hand the corresponding sesquilinear form $B'$ over $\mathbb{Q}[t^{\pm 1}]$ for $L'$ is equivalent to

$$[t-1] \oplus [t-1] \oplus [t-1] \oplus [q(t)].$$

PROPOSITION 2.3. *Let $z_0$ denote the algebraic number $\frac{3+4i}{5}$. The links $L$ and $L'$ constructed in Example 2.2 have the following properties.*

*(1) If $z$ is a root of unity, then $\sigma_L(z) = \sigma_{L'}(z)$ and $\eta_L(z) = \eta_{L'}(z)$.*

(2) *The averaged signature and nullity functions agree, i.e.*

$$\overline{\sigma}_L(z) = \overline{\sigma}_{L'}(z) \text{ and } \overline{\eta}_L(z) = \overline{\eta}_{L'}(z)$$

*for all* $z \in S^1 \smallsetminus \{1\}$.

(3) *The signatures and nullities of* $L$ *and* $L'$ *at* $z_0$ *differ:*

$$\sigma_L(z_0) \neq \sigma_{L'}(z_0) \text{ and } \eta_L(z_0) \neq \eta_{L'}(z_0),$$

*and so* $L$ *is not concordant to* $L'$.

*Proof.* Note that for any $z \in \mathbb{C} \smallsetminus \{0, 1\}$ with $q(z) \neq 0$, the form $B(z)$ over $\mathbb{C}$ is nonsingular and metabolic. The same holds for $B'(z)$. This implies that the signatures $\operatorname{sign} B(z)$ and $\operatorname{sign} B'(z)$ vanish. The nullities $\eta_L(z), \eta_{L'}(z)$ are also both zero. Since the roots of $q(z)$ are exactly $z_0$ and $\overline{z_0}$, which are not roots of unity by Lemma 2.1, we obtain the first statement of the proposition. We also see that the averaged signature function on $S^1 \smallsetminus \{1\}$ and the averaged nullity function are identically zero, so we obtain the second statement.
From Lemma 2.1, we know that $z_0 := \frac{3+4i}{5}$ is not a Knotennullstelle, and

$$\operatorname{sign} B(z_0) = \operatorname{sign} \begin{pmatrix} 0 & 0 \\ 0 & \frac{4}{5} \end{pmatrix} = 1.$$

Thus $\sigma_L(z_0) = 1 = \eta_L(z_0)$. On the other hand, for $L'$ the matrix $B'(z_0)$ is a $2 \times 2$ zero matrix, so we have that $\sigma_{L'}(z_0) = 0$ and $\eta_{L'}(z_0) = 2$. Both signatures and the nullities at $z_0$ differ, so $L$ and $L'$ are not concordant by Theorem 1.2. $\square$

*Remark* 2.4. One can also see that $L$ and $L'$ are not concordant using linking numbers.

A more systematic study of the construction of the example above leads to the following proposition.

Proposition 2.5. *Let* $q(t) \in \mathbb{Z}[t]$ *be a polynomial. Then there exists a natural number* $k > 0$ *and a link* $L$ *with Alexander polynomial* $\Delta_L(t) \doteq q(t^{-1})q(t)(t-1)^k$ *up to units in* $\mathbb{Z}[t, t^{-1}]$ *such that*

(1) *the form* $B(z)$ *of* $L$ *is metabolic and nonsingular for all* $z \in S^1 \smallsetminus \{1\}$ *which are not roots of* $q(t)$, *so* $\sigma_L(z) = 0$.
(2) *if* $z_0 \neq 1$ *is a root of* $q(t)$ *of unit modulus, then* $\sigma_L(z_0) \neq 0$.

The proof of this proposition is based on ideas from [CL04].

*Proof.* Consider the size $n+1$ square matrix $P$ with entries in $\mathbb{Z}[y]$ given by

$$P(y) := \begin{pmatrix} 1 & y & 0 & & & ya_1 \\ 0 & 1 & y & & & ya_2 \\ \vdots & & \ddots & \ddots & & \vdots \\ & & & 1 & y & ya_{n-1} \\ 0 & & & 0 & 1 & ya_n \\ y & 0 & \dots & 0 & 0 & 0 \end{pmatrix},$$

with $a_i$ integers. Over $\mathbb{Z}[y^{\pm 1}]$, the matrix $P$ can be transformed via invertible row operations and column operations to the matrix

$$A(y) = \begin{pmatrix} 1 & 0 & 0 & & & p(y) \\ 0 & 1 & 0 & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ & & & 1 & 0 & 0 \\ 0 & & & 0 & 1 & 0 \\ y & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

with $p(y) = b_1(y)$ where $b_k(y) \in \mathbb{Z}[y]$ is defined by the recursion $b_{k-1}(y) := y \cdot (a_k - b_k(y))$ and $b_n(y) := y \cdot a_n$. Notice that, up to units, we can arrange $p(y)$ to be any polynomial in $\mathbb{Z}[y^{\pm 1}]$ by choosing $n$ sufficiently large and then suitable entries $a_k \in \mathbb{Z}$. That is, multiply by $y^\ell$ so that the lowest order term is the linear term, and take $(-1)^i a_i$ to be the coefficient of $y^{i-1}$ in $p(y)$, for $i = 2, \dots, n+1$.
Pick the entries $a_k$ so that if we evaluate $p(y)$ at $(t-1)$ we get the equality $p(t-1) = q(t)(t-1)^k$ for a suitable integer $k$. Now consider the block matrix

$$V := \begin{pmatrix} 0 & V^u \\ V^b & Q(1) \end{pmatrix}$$

with

$$V^u = \begin{pmatrix} 0 & 1 & 0 & & & a_1 \\ 0 & 0 & 1 & & & a_2 \\ \vdots & & \ddots & \ddots & & \vdots \\ & & & 0 & 1 & a_{n-1} \\ 0 & & & 0 & 0 & a_n \\ 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \qquad V^b = \begin{pmatrix} -1 & 0 & 0 & & & 1 \\ 1 & -1 & 0 & & & 0 \\ 0 & 1 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & -1 & 0 & 0 \\ 0 & & & 1 & -1 & 0 \\ a_1 & a_2 & \dots & a_{n-1} & a_n & 0 \end{pmatrix}$$

and

$$Q(y) = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & y \end{pmatrix}.$$

The matrix $V$ is the Seifert matrix of a link as $V - V^T$ is the intersection form of a genus $n$ oriented surface with three boundary components. Let $L$ be such a link, necessarily a 3-component link. We remark in passing that the matrix $V$ from Example 2.2 is not a special case of the matrix $V$ defined in the current proof, although it is close to being so.
Recall that $B(z) = (1-z)V + (1-\overline{z})V^T = (\overline{z}-1) \cdot \left(zV - V^T\right)$. The matrix $V$ was constructed in such a way that

$$B(z) = \begin{pmatrix} 0 & (\overline{z}-1) \cdot P(z-1) \\ (z-1) \cdot P^T(\overline{z}-1) & Q(-\overline{z}-z+2) \end{pmatrix}.$$

Using the transformations associated to the above row and column operations, we see that $B(z)$ is congruent to

$$B(z) \sim \begin{pmatrix} 0 & (\overline{z}-1)\cdot A(z-1) \\ (z-1)\cdot A^T(\overline{z}-1) & Q(-\overline{z}-z+2) \end{pmatrix}.$$

Note that the matrix $Q$ is unchanged by this congruency, because in the corresponding sequence of row and column operations, it never happens that the last row or column is added to another row or column.

We complete the proof of the proposition by showing that indeed the link $L$ has the required properties. If $z \in S^1 \smallsetminus \{1\}$ is not a zero of $q(t)$, then also $p(z) \neq 0$. Consequently, the form $B(z)$ is nonsingular and metabolic. On the other hand, if $z \in S^1 \smallsetminus \{1\}$ is a root of $q(t)$, then also $p(z) = 0$. In this case the Levine-Tristram form $B(z)$ is a sum

$$B(z) = M \oplus \begin{pmatrix} 0 & 0 \\ 0 & -\overline{z}-z+2 \end{pmatrix}$$

with $M$ nonsingular and metabolic. Thus $\sigma_L(z) = 1$. □

*Remark* 2.6. Replace $Q(1)$ with $Q(0)$ in the construction of the matrix $V$ in the proof of Proposition 2.5, to obtain a matrix $V'$. Using the same construction as in Example 2.2, the matrices $V$ and $V'$ give rise to links $L$ and $L'$ respectively, such that

$$\eta_L(z) = \eta_{L'}(z) \text{ and } \sigma_L(z) = \sigma_{L'}(z)$$

for every $z \in S^1$ that is not a root of $q(t)$. Analogously to Example 2.2, $L$ and $L'$ are not concordant, but again this can also be seen using linking numbers. This leads to the following question. Does there exist a pair of links $L$ and $L'$, with the same pairwise linking numbers, whose signature and nullity functions can only tell the concordance classes of the links apart at an isolated algebraic numbers $z, \bar{z} \in S^1$ that are roots of the Alexander polynomial $\Delta_L = \Delta_{L'}$.

## 3. Twisted homology and integral homology isomorphisms

Now we begin working towards the proof of Theorem 1.2. Fix $z \in S^1 \smallsetminus \{1\}$ to be a unit complex number that is not the root of any polynomial $p(t) \in \mathbb{Z}[t]$ with $p(1) = \pm 1$ i.e. $z$ is not a Knotennullstelle. We denote the classifying space for the integers $\mathbb{Z}$ by $B\mathbb{Z}$, which has the homotopy type of the circle $S^1$. Given a CW complex $X$, a map $X \to B\mathbb{Z}$ induces a homomorphism $\pi_1(X) \to \mathbb{Z}$. This determines a representation

$$\alpha\colon \mathbb{Z}[\pi_1(X)] \to \mathbb{Z}[\mathbb{Z}] \xrightarrow{\mathrm{ev}_z} \mathbb{C}$$

of the group ring of the fundamental group of $X$, with respect to which we can consider the twisted homology

$$H_i(X; \mathbb{C}^\alpha) := H_i\left(\mathbb{C} \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X})\right).$$

Let $\Sigma \subset \mathbb{Z}[\mathbb{Z}]$ be the multiplicative subset of polynomials that map to $\pm 1$ under the augmentation $\varepsilon\colon \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}$, that is $\Sigma = \{p(t) \in \mathbb{Z}[\mathbb{Z}] \,|\, |p(1)| = 1\}$.

By inverting this subset we obtain the localisation $\Sigma^{-1}\mathbb{Z}[\mathbb{Z}]$ of the Laurent polynomial ring. This has the following properties.

(i) The canonical map $\mathbb{Z}[\mathbb{Z}] \to \Sigma^{-1}\mathbb{Z}[\mathbb{Z}]$ is an inclusion, since $\mathbb{Z}[\mathbb{Z}]$ is an integral domain.

(ii) For any $\mathbb{Z}[\mathbb{Z}]$-module morphism $f\colon M \to N$ of finitely generated free $\mathbb{Z}[\mathbb{Z}]$-modules such that the augmentation
$$\varepsilon(f) = \operatorname{Id} \otimes f\colon \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}]} M \to \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}]} N$$
is an isomorphism, we have that
$$\operatorname{Id} \otimes f\colon \Sigma^{-1}\mathbb{Z}[\mathbb{Z}] \otimes_{\mathbb{Z}[\mathbb{Z}]} M \to \Sigma^{-1}\mathbb{Z}[\mathbb{Z}] \otimes_{\mathbb{Z}[\mathbb{Z}]} N$$
is also an isomorphism.

The second property can be reduced to the following. Assume $A$ is a matrix over $\mathbb{Z}[\mathbb{Z}]$ such that $\varepsilon(A)$ is invertible. Consequently, we have $\det(\varepsilon(A)) = \pm 1$ and as $\varepsilon(\det(A)) = \det(\varepsilon(A))$, we deduce that $\det(A) \in \Sigma$. Therefore, the determinant $\det(A)$ is invertible in the localisation $\Sigma^{-1}\mathbb{Z}[\mathbb{Z}]$ and so is the matrix $A$ over $\Sigma^{-1}\mathbb{Z}[\mathbb{Z}]$.
As the unit modulus complex number $z$ that we have fixed is not a Knotennullstelle, the representation $\alpha$ defined above factors through the localisation, i.e. evaluation at $z$ determines a ring homomorphism $\Sigma^{-1}\mathbb{Z}[\mathbb{Z}] \xrightarrow{\Sigma^{-1}\operatorname{ev}_z} \mathbb{C}$ such that the ring homomorphisms $\mathbb{Z}[\mathbb{Z}] \xrightarrow{\operatorname{ev}_z} \mathbb{C}$ and
$$\mathbb{Z}[\mathbb{Z}] \to \Sigma^{-1}\mathbb{Z}[\mathbb{Z}] \xrightarrow{\Sigma^{-1}\operatorname{ev}_z} \mathbb{C}$$
coincide.

LEMMA 3.1. *Let $f\colon X \to Y$ be a map of finite CW complexes over $S^1$, that is there are maps $g\colon X \to S^1$ and $h\colon Y \to S^1$ such that $h \circ f = g$, and suppose that*
$$f_*\colon H_i(X;\mathbb{Z}) \xrightarrow{\cong} H_i(Y;\mathbb{Z})$$
*is an isomorphism for all $i$. Then*
$$f_*\colon H_i(X;\mathbb{C}^\alpha) \xrightarrow{\cong} H_i(Y;\mathbb{C}^\alpha)$$
*is also an isomorphism for all $i$.*

The lemma follows [COT03, Proposition 2.10]. The difference is that we use the well-known refinement that one does not need to invert all nonzero elements. We give the proof for the convenience of the reader. This is adapted from the proof given in [FP12].

*Proof.* The algebraic mapping cone $D_* := \mathscr{C}(f_*\colon C_*(X;\mathbb{Z}) \to C_*(Y;\mathbb{Z}))$ has vanishing homology, and comprises finitely generated free $\mathbb{Z}$-modules. Therefore it is chain contractible. We claim that the chain contraction can be lifted to a chain contraction for $\mathscr{C}(f_*\colon C_*(X;\Sigma^{-1}\mathbb{Z}[\mathbb{Z}]) \to C_*(Y;\Sigma^{-1}\mathbb{Z}[\mathbb{Z}]))$, the mapping cone over the localisation $\Sigma^{-1}\mathbb{Z}[\mathbb{Z}]$.
To see this, let $s\colon D_* \to D_{*+1}$ be a chain contraction, that is we have that $\partial s_i + s_{i-1}\partial = \operatorname{Id}_{D_i}$ for each $i$. Define $\widetilde{D}_* := \mathscr{C}(f_*\colon C_*(X;\mathbb{Z}[\mathbb{Z}]) \to C_*(Y;\mathbb{Z}[\mathbb{Z}]))$

and consider $\varepsilon\colon \widetilde{D}_* \to D_* = \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}]} \widetilde{D}_*$, induced by the augmentation map. Denote $E_* := \mathscr{C}(f_*\colon C_*(X;\Sigma^{-1}\mathbb{Z}[\mathbb{Z}]) \to C_*(Y;\Sigma^{-1}\mathbb{Z}[\mathbb{Z}]))$ and note that there is an inclusion $\widetilde{D}_i \to E_i = \Sigma^{-1}\mathbb{Z}[\mathbb{Z}] \otimes_{\mathbb{Z}[\mathbb{Z}]} \widetilde{D}_i$, induced by the localisation. Lift $s$ to a map $\widetilde{s}\colon \widetilde{D}_* \to \widetilde{D}_{*+1}$, as in the next diagram

$$\begin{tikzcd} \widetilde{D}_* \arrow[r, dashed, "\widetilde{s}"] \arrow[d, "\varepsilon"] & \widetilde{D}_{*+1} \arrow[d, "\varepsilon"] \\ D_* \arrow[r, "s"] & D_{*+1}. \end{tikzcd}$$

The lifts exist since all modules are free and $\varepsilon$ is surjective. But then we have that

$$f := d\widetilde{s} + \widetilde{s}d\colon \widetilde{D}_* \to \widetilde{D}_*$$

is a morphism of free $\mathbb{Z}[\mathbb{Z}]$-modules whose augmentation $\varepsilon(f)$ is an isomorphism. Thus by property (ii) of $\Sigma^{-1}\mathbb{Z}[\mathbb{Z}]$, $f$ is also an isomorphism over $\Sigma^{-1}\mathbb{Z}[\mathbb{Z}]$, and so $\widetilde{s}$ determines a chain contraction for $E_*$. We therefore have that $E_* = C_*(Y,X;\Sigma^{-1}\mathbb{Z}[\mathbb{Z}]) \simeq 0$ as claimed.
Next, tensor $E_*$ with $\mathbb{C}$ over the representation $\alpha$, to get that

$$\mathbb{C}^\alpha \otimes_{\Sigma^{-1}\mathbb{Z}[\mathbb{Z}]} C_*(Y,X;\Sigma^{-1}\mathbb{Z}[\mathbb{Z}]) = C_*(Y,X;\mathbb{C}^\alpha) \simeq 0.$$

Thus $H_i(Y,X;\mathbb{C}^\alpha) = 0$ for all $i$ and so $f_*\colon H_i(X;\mathbb{C}^\alpha) \xrightarrow{\cong} H_i(Y;\mathbb{C}^\alpha)$ is an isomorphism for all $i$ as desired. □

## 4. Concordance invariance of the nullity

In this section we show concordance invariance of the nullity function away from the set of Knotennullstellen.

Definition 4.1 (Homology cobordism). A cobordism $(W^{n+1};M^n,N^n)$ between $n$-manifolds $M$ and $N$ is said to be a *$\mathbb{Z}$-homology cobordism* if the inclusion induced maps $H_i(M;\mathbb{Z}) \to H_i(W;\mathbb{Z})$ and $H_i(N;\mathbb{Z}) \to H_i(W;\mathbb{Z})$ are isomorphisms for all $i \in \mathbb{Z}$.

Theorem 4.2. *Suppose that oriented $m$-component links $L$ and $J$ are concordant and that $z \in S^1 \smallsetminus \{1\}$ is not a Knotennullstelle. Then $\eta_L(z) = \eta_J(z)$.*

*Proof.* As in the statement suppose that $z \in S^1 \smallsetminus \{1\}$ is not a Knotennullstelle. Denote the exterior of the link $L$ by $X_L := S^3 \smallsetminus \nu L$. As above, let $V$ be a matrix representing the Seifert form of $L$ with respect to a Seifert surface $F$ and a basis for $H_1(F;\mathbb{Z})$.
We assert that the matrix $zV - V^T$ presents the homology $H_1(X_L;\mathbb{C}^\alpha)$. This can be seen as follows. Consider the infinite cyclic cover $\overline{X}_L$ corresponding to the kernel of the homomorphism $\pi_1(X_L) \to \mathbb{Z}$, defined as the composition of the abelianisation $\pi_1(X_L) \to H_1(X_L;\mathbb{Z}) \cong \mathbb{Z}^m$, followed by the map $(x_1,\dots,x_m) \mapsto \sum_{i=1}^m x_i$ i.e. each oriented meridian is sent to $1 \in \mathbb{Z}$. A decomposition of $\overline{X}_L$ and the associated Mayer-Vietoris sequence [Lic97, Theorem

6.5] give rise the following presentation

$$\mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}} H_1(F;\mathbb{C}) \xrightarrow{tV-V^T} \mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}} H_1(F;\mathbb{C})^\vee \to H_1(\overline{X}_L;\mathbb{C}) \to 0,$$

where $H_1(F;\mathbb{C})^\vee$ is the dual module $\operatorname{Hom}_{\mathbb{C}}(H_1(F;\mathbb{C}),\mathbb{C})$. Apply the right-exact functor $\mathbb{C}^\alpha \otimes_{\mathbb{C}[t^{\pm 1}]}$ to this sequence, to obtain the sequence

$$\mathbb{C}^\alpha \otimes_{\mathbb{C}} H_1(F;\mathbb{C}) \xrightarrow{zV-V^T} \mathbb{C}^\alpha \otimes_{\mathbb{C}} H_1(F;\mathbb{C})^\vee \to \mathbb{C}^\alpha \otimes_{\mathbb{C}[t^{\pm 1}]} H_1(\overline{X}_L;\mathbb{C}) \to 0.$$

As $H_0(\overline{X}_L;\mathbb{C}) \cong \mathbb{C}$, we have that $\operatorname{Tor}_1^{\mathbb{C}[t^{\pm 1}]}(H_0(\overline{X}_L;\mathbb{C}),\mathbb{C}^\alpha) = 0$ by the projective resolution

$$0 \to \mathbb{C}[t^{\pm 1}] \xrightarrow{\cdot(1-t)} \mathbb{C}[t^{\pm 1}] \to \mathbb{C} \to 0$$

and $z \neq 1$. Since $\mathbb{C}[t^{\pm 1}]$ is a principal ideal domain, we can apply the universal coefficient theorem for homology to deduce that $\mathbb{C}^\alpha \otimes_{\mathbb{C}[t^{\pm 1}]} H_1(\overline{X}_L;\mathbb{C}) = H_1(X_L;\mathbb{C}^\alpha)$. This completes the proof of the assertion that $zV - V^T$ presents the homology $H_1(X_L;\mathbb{C}^\alpha)$.

Next observe that $(\overline{z}-1)(zV - V^T) = (1-z)V + (1-\overline{z})V^T$ presents the same module as $zV - V^T$, since $\overline{z}-1$ is nonzero. The dimension of $H_1(X_L;\mathbb{C}^\alpha)$ therefore coincides with the nullity $\eta_L(z)$, which is by definition the nullity of the matrix $(1-z)V + (1-\overline{z})V^T$.

Now, let $A \subset S^3 \times I$ be a union of annuli giving a concordance between $L$ and $J$, and let $W := S^3 \times I \smallsetminus \nu A$. Then $W$ is a $\mathbb{Z}$-homology bordism between $X_L$ and $X_J$; this is a straightforward computation with Mayer-Vietoris sequences or with Alexander duality; see for example [FP14, Lemma 2.4]. Thus by two applications of Lemma 3.1, with $Y = W$ and $X = X_L$ and $X = X_J$ respectively, we see that $H_1(X_L;\mathbb{C}^\alpha) \cong H_1(W;\mathbb{C}^\alpha) \cong H_1(X_J;\mathbb{C}^\alpha)$, and so the nullities of $L$ and $J$ agree. We need that $z$ is not a Knotennullstelle in order to apply Lemma 3.1. □

## 5. Identification of the signature with the signature of a 4-manifold

In the proof of Theorem 4.2, a key step was to reexpress the nullity $\eta(z)$ of the form $B(z)$ as a topological invariant of a 3-manifold, and then to use the bordism constructed from a concordance to relate the invariants. An analogous approach is used here to obtain the corresponding statement for the signature. Everything in this section is independent of whether $z$ is a Knotennullstelle.

Recall that we fixed an oriented $m$-component link $L \subset S^3$, and that we picked a connected Seifert surface $F$ for $L$. Denote the link complement by $X_L := S^3 \smallsetminus \nu L$. First note that the fundamental class $[F] \in H_2(F,\partial F;\mathbb{Z})$ of the Seifert surface $F$ is independent of the choice of $F$. This follows from the fact that its Poincaré dual is characterised as the unique cohomology class $\xi \in H^1(X_L;\mathbb{Z})$ mapping each meridian $\mu$ to $\xi(\mu) = 1$.

The boundary of $F \subset S^3 \smallsetminus \nu L$ is a collection of embedded curves in the boundary tori that we refer to as the *attaching curves*. The attaching curves together with the meridians determine a framing of each boundary torus of $X_L$. Also,

this framing depends solely on $[F]$, since the connecting homomorphism of the pair $(X_L, \partial X_L)$ maps $\partial[F] = [\partial F]$.
With respect to this framing, we can consider the Dehn filling of slope zero, resulting in the closed 3-manifold $M_L$. By definition, to obtain $M_L$ attach a disc to each of the attaching curves, and then afterwards fill each of the resulting boundary spheres with a 3-ball.

Definition 5.1. The framing of the boundary tori of $X_L$ constructed above is called the *Seifert framing*. The *Seifert surgery* on $L$ is the 3-manifold $M_L$ constructed above.

*Remark* 5.2. For links there is no reason for this framing to agree with the zero-framing of each individual component.

Collapsing the complement of a tubular neighbourhood of the Seifert surface $F$ gives rise to map $S^3 \smallsetminus \nu L \to S^1 = B\mathbb{Z}$, which extends to a map from the Seifert surgery $\phi\colon M_L \to B\mathbb{Z}$. To see this in more detail, parametrise a regular neighbourhood of $F$ as $F \times [-1,1]$, with $F$ as $F \times \{0\}$. The intersection of this parametrised neighbourhood with each component of $\partial F$ determines a parametrised subset $S^1 \times [-1,1] \subset S^1 \times S^1 \subseteq \partial F$. Extend this to a subset $D^2 \times [-1,1] \subset D^2 \times S^1$ for each of the Dehn filling solid tori $D^2 \times S^1$ in $M_L$. Now define

$$\begin{array}{rcl}\phi\colon M_L & \to & S^1 = B\mathbb{Z} \\ x & \mapsto & \begin{cases} e^{\pi i t} & x = (f,t) \in \left(F \cup \bigsqcup^m D^2\right) \times [-1,1] \\ -1 & \text{otherwise.} \end{cases}\end{array}$$

The map $\phi$ classifies the image of the fundamental class of the capped-off Seifert surface in $M_L$, in the sense that $[\phi]$ maps to $[F \cup \bigsqcup^m D^2]$ under $[M_L, S^1] \xrightarrow{\cong} H^1(M_L;\mathbb{Z}) \xrightarrow{\cong} H_2(M_L;\mathbb{Z})$. Recall that the homology class $[F \cup \bigsqcup^m D^2] \in H_2(M_L;\mathbb{Z})$ only depends on the isotopy class of $L$ and so also the homotopy class of $\phi$ does not depend on the Seifert surface $F$. The manifold $M_L$ together with the map $\phi$ defines an element $[(M_L, \phi)] \in \Omega_3(B\mathbb{Z})$, where $\Omega_k(X)$ denotes the bordism group of oriented, topological $k$-dimensional manifolds with a map to $X$. Recall that cobordism is a generalised homology theory fulfilling the suspension axiom, see e.g. [tD08, Chapter 21] and [May99, Section 14.4]. As a consequence, we obtain

$$\widetilde{\Omega}_3(B\mathbb{Z}) = \widetilde{\Omega}_3(S^1) = \widetilde{\Omega}_3(\Sigma S^0) \cong \widetilde{\Omega}_2(S^0) = \Omega_2(\mathrm{pt}) = 0.$$

Thus $\Omega_3(B\mathbb{Z}) \cong \Omega_3(\mathrm{pt}) = 0$ [Roh53].
The group $\Omega_3(B\mathbb{Z}) \cong \Omega_3 \oplus \Omega_2 = 0 \oplus 0 = 0$ is trivial, and we can make use of this fact to define a signature defect invariant, as follows.
For any oriented 3-manifold $M$ with a map $\phi\colon M \to B\mathbb{Z}$, we will define an integer for each complex number $z \in S^1$. Since $\Omega_3(B\mathbb{Z}) = 0$, there exists a 4-manifold $W$ with boundary $M$ and a map $\Phi\colon W \to B\mathbb{Z}$ extending the map $M \to B\mathbb{Z}$ on the boundary. Similarly to before, an element $z \in S^1$ determines a representation

$$\alpha\colon \mathbb{Z}[\pi_1(W)] \xrightarrow{\Phi} \mathbb{Z}[\mathbb{Z}] \xrightarrow{t \mapsto z} \mathbb{C}.$$

Consider the twisted homology $H_i(W;\mathbb{C}^\alpha)$, and consider the intersection form $\lambda_\alpha(W)$ on the quotient $H_2(W;\mathbb{C}^\alpha)/\operatorname{im} H_2(M;\mathbb{C}^\alpha)$. Define the promised integer

$$\sigma(M,\phi,z) := \sigma(\lambda_\alpha(W)) - \sigma(W),$$

where $\sigma(W)$ is the ordinary signature of the intersection form on $W$.
The proof of the following proposition is known for the coefficient system $\mathbb{Q}(t)$, e.g. [Pow16]. For the convenience of the reader, we sketch the key steps for an adaptation to $\mathbb{C}^\alpha$.

Proposition 5.3.

*(i) The intersection form $\lambda_\alpha(W)$ is nonsingular.*

*(ii) The signature defect $\sigma(M,\phi,z)$ is independent of the choice of 4-manifold $W$.*

*Proof.* The long exact sequence of the pair $(W,\partial W) = (W,M)$ gives rise to the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_2(\partial W;\mathbb{C}^\alpha) \arrow[r] & H_2(W;\mathbb{C}^\alpha) \arrow[r] \arrow[rdd] & H_2(W,\partial W;\mathbb{C}^\alpha) \arrow[r] \arrow[d, "\mathrm{PD}_W^{-1}"] & \cdots \\
 & & & H^2(W;\mathbb{C}^\alpha) \arrow[d, "\kappa"] & \\
 & & & \left(H_2(W;\mathbb{C}^\alpha)\right)^\vee , &
\end{tikzcd}$$

where for a $\mathbb{C}$-module $P$ we denote its dual module by $P^\vee := \operatorname{Hom}_{\mathbb{C}}(P,\mathbb{C})$. Since Poincaré-Lefschetz duality $\mathrm{PD}_W$ and the Kronecker pairing $\kappa$ are isomorphisms, we obtain an injective map $H_2(W;\mathbb{C}^\alpha)/\operatorname{im} H_2(M;\mathbb{C}^\alpha) \to H_2(W;\mathbb{C}^\alpha)^\vee$. This map descends to

$$\lambda_\alpha\colon H_2(W;\mathbb{C}^\alpha)/\operatorname{im} H_2(M;\mathbb{C}^\alpha) \to \left(H_2(W;\mathbb{C}^\alpha)/\operatorname{im} H_2(\partial W;\mathbb{C}^\alpha)\right)^\vee ,$$

so that the diagram below commutes:

$$\begin{tikzcd}
H_2(W;\mathbb{C}^\alpha)/\operatorname{im} H_2(M;\mathbb{C}^\alpha) \arrow[r, hook] \arrow[rd, "\lambda_\alpha"'] & H_2(W;\mathbb{C}^\alpha)^\vee \quad . \\
 & \left(H_2(W;\mathbb{C}^\alpha)/\operatorname{im} H_2(\partial W;\mathbb{C}^\alpha)\right)^\vee . \arrow[u]
\end{tikzcd}$$

Consequently, the form $\lambda_\alpha$ is nondegenerate, and so it is nonsingular since it is a form over the field $\mathbb{C}$.
We proceed with the second statement of the proposition, namely independence of $\sigma(M,\phi,z)$ on the choice of $W$. Suppose that we are given two 4-manifolds $W^+, W^-$, both with boundary $\partial W^\pm = M$, and a map $\Phi^\pm\colon W^\pm \to B\mathbb{Z}$ extending $\phi\colon M \to B\mathbb{Z}$. Temporarily, define the signature defects arising from the two choices to be

$$\sigma(W^\pm,\Phi^\pm,z) := \sigma(\lambda_\alpha(W^\pm)) - \sigma(W^\pm).$$

We will show that $\sigma(W^+,\Phi^+,z)=\sigma(W^-,\Phi^-,z)$, and thus that $\sigma(M,\phi,z)$ is a well-defined integer, so our original notation was justified.
Glue $W^+$ and $\overline{W^-}$ together along $M$, to obtain a closed manifold $U$, together with a map $\Phi\colon U\to B\mathbb{Z}$. By Novikov additivity, we learn that

$$\sigma_z(U,\Phi):=\sigma(\lambda_\alpha(U))-\sigma(U)=\sigma(W^+,\Phi^+,z)-\sigma(W^-,\Phi^-,z).$$

This defect $\sigma_z(U,\Phi)$ can be promoted to a bordism invariant $\sigma_z\colon\Omega_4(B\mathbb{Z})\to\mathbb{Z}$, see e.g. [Pow16, Proof of Lemma 3.2] and replace $\mathbb{Q}(t)$ coefficients with $\mathbb{C}^\alpha$ coefficients.

*Claim.* The map $\sigma_z\colon\Omega_4(B\mathbb{Z})\to\mathbb{Z}$ is the zero map.

Let $U$ be a closed 4-manifold together with a map $\Phi\colon U\to S^1$, representing an element of $\Omega_4(B\mathbb{Z})$. By the axioms of generalised homology theories, we have

$$\widetilde{\Omega}_4(S^1)=\widetilde{\Omega}_4(\Sigma S^0)\cong\widetilde{\Omega}_3(S^0)=\Omega_3(\mathrm{pt})=0.$$

Thus an inclusion pt $\to S^1$ induced an isomorphism $\Omega_4(\mathrm{pt})\xrightarrow{\cong}\Omega_4(S^1)$. So $(U,\Phi)$ is bordant over $S^1$ to a 4-manifold $U'$ with a null-homotopic map $\Phi'$ to $S^1$. In this case the local coefficient system $\mathbb{C}^\alpha$ is just the trivial representation $\mathbb{C}$. Consequently, we have $\lambda_\alpha(U')=\lambda(U')$, so $\sigma_z(U',\Phi')=0$. By bordism invariance, $\sigma_z(U,\Phi)=0$, which completes the proof of the claim.
Now the independence of $\sigma(M,\Phi,z)$ on the choice of $W$ follows from

$$0=\sigma_z(U,\Phi)=\sigma(W^+,\Phi^+,z)-\sigma(W^-,\Phi^-,z).$$

□

Now that we have constructed an invariant, we need to relate it to the Levine-Tristram signatures. Recall that $L$ is an oriented link, that $M_L$ is the Seifert surgery, and that we constructed a canonical map $\phi\colon M_L\to S^1$, well-defined up to homotopy.

Lemma 5.4. *Suppose that $z\in S^1\smallsetminus\{1\}$ and let $\phi\colon M_L\to S^1$ be the map defined at the beginning of this section. Then we have*

$$\sigma(M_L,\phi,z)=\sigma_L(z).$$

*Proof.* Construct a 4-manifold with boundary $M_L$ as follows. Let $F$ be a connected Seifert surface for $L$. Push the Seifert surface into $D^4$ and consider its complement $V_F:=D^4\smallsetminus\nu F$. Note that if we cap $F$ off with $m$ 2-discs, we obtain a closed surface. Let $H$ be a 3-dimensional handlebody whose boundary is this surface. Note that $\partial V_F=X_L\cup F\times S^1$. Then define

$$W_F:=V_F\cup_{F\times S^1}H\times S^1.$$

Note that $\partial W_F=M_L$. By [Ko89, pp. 538-9] and [COT04, Lemma 5.4], we have that $\sigma(W_F)=0$ and $\lambda_z(W_F)=(1-z)V+(1-\overline{z})V^T$. Therefore

$$\sigma(\lambda_z(W_F))-\sigma(W_F)=\sigma((1-z)V+(1-\overline{z})V^T)=\sigma(B(z)).$$

□

## 6. Concordance invariance of the signature

We start with a straightforward lemma, then we prove the final part of the main theorem. Recall that the complement $X_L$ and the Seifert surgery $M_L$ are both equipped with a homotopy class of a map to $S^1$, or equivalently with a cohomology class. For the link complement $X_L$, this class $\xi_L \in H^1(X_L;\mathbb{Z})$ is characterised by the property that it sends each oriented meridian to 1.

Lemma 6.1. *Let $L$ and $J$ be concordant links. Their Seifert surgeries $M_L$ and $M_J$ are homology bordant over $S^1$.*

*Proof.* Denote the maps to $S^1$ by $\phi_L \colon M_L \to S^1$ and $\phi_J \colon M_L \to S^1$, and denote the corresponding cohomology classes by $\xi_L \in H^1(M_L;\mathbb{Z})$ and $\xi_J \in H^1(M_J;\mathbb{Z})$. Define $X_L := S^3 \smallsetminus \nu L$ and $X_J := S^3 \smallsetminus \nu J$. Let $A \subset S^3 \times I$ be an embedding of a disjoint union of annuli giving a concordance between $L$ and $J$.
Fix a tubular neighbourhood $\nu A = A \times D^2$ of the annulus $A$ with a trivialisation. Denote $W_A := S^3 \times I \smallsetminus \nu A$, whose boundary consists of the union of $X_L$, $X_J$, and a piece identified with the total space of the unit sphere bundle $A \times S^1$ of $\nu A$. As usual, we refer to a representative $\{\mathrm{pt}\} \times S^1$ for the $S^1$ factor in $A \times S^1$ as a *meridian* of $A$. Note that the inclusions $X_L \subset W_A$ and $X_J \subset W_A$ map the meridians in the link complements to the meridians in $W_A$.

*Claim.* There exists a cohomology class $\xi_A \in H^1(W_A;\mathbb{Z})$ mapping each meridian $\mu_A$ of $A$ to 1.

This can be seen by the Mayer-Vietoris sequence

$$H^1(\nu A;\mathbb{Z}) \oplus H^1(W_A;\mathbb{Z}) \to H^1(\partial \nu A;\mathbb{Z}) \to H^2(S^3 \times I;\mathbb{Z}) = 0,$$

in which the map $H^1(\nu A;\mathbb{Z}) \cong \mathbb{Z}^m \to H^1(\partial\nu A;\mathbb{Z}) \cong (\mathbb{Z} \oplus \mathbb{Z})^m$ is given by $1 \mapsto (1,0)$ on each of the $m$ summands. That is, the homology classes of the meridians of $\partial \nu A \cong A \times S^1$ do not lie in the image of this surjective map, so they must lie in the image of $H^1(W_A;\mathbb{Z})$. This completes the proof of the claim.
It follows that $\xi_A$ is pulled back to the unique classes $\xi_L$ and $\xi_J$ that map the meridians in the link complements to 1. Using the natural isomorphism between the functors $[-, S^1]$ and $H^1(-;\mathbb{Z})$, find a map $\phi_W \colon W_A \to S^1$ that restricts to the prescribed map $\phi_L \sqcup \phi_J \colon X_L \sqcup X_J \to S^1$ on the boundary.
Up to isotopy, there is a unique product structure on an annulus $A = S^1 \times I$. Having fixed such a structure, we consider the manifold

$$Y := W_A \cup_{A \times S^1} \bigsqcup^m (D^2 \times S^1 \times I).$$

The gluing is done in such a way as to restrict on $\bigsqcup^m S^1 \times S^1 \times \{i\}$, for $i = 0, 1$, to the gluing of the Seifert surgery on $X_L$ and $X_J$. By construction, this gives a bordism between $M_L$ and $M_J$.
Note that the map $\phi_W$ and the projection $A \times S^1 \to S^1$ glue together to give a map $\phi_Y : Y \to S^1$. Equipped with this map, $(Y, \phi_Y)$ is an $S^1$-bordism between $(M_L, \phi_L)$ and $(M_J, \phi_J)$.

Finally, we assert that $Y$ is a homology bordism. To see this, first observe, as in the proof of Theorem 4.2, that $W_A$ is a homology bordism from $X_L$ to $X_J$. Flagrantly, $A \times S^1$ is a homology bordism from $S^1 \times S^1$ to itself, and $\bigsqcup^m (D^2 \times S^1 \times I)$ is a homology bordism from $\bigsqcup^m D^2 \times S^1$ to itself. Gluing two homology bordisms together along a homology bordism, with the same maps on homology induced by the gluings for $M_L$, $M_J$ and $Y$, it follows easily from the Mayer-Vietoris sequence and the five lemma that $Y$ is a homology bordism. □

Theorem 6.2. *Suppose that oriented $m$-component links $L$ and $J$ are concordant and that $z \in S^1 \smallsetminus \{1\}$ is not a Knotennullstelle. Then $\sigma_L(z) = \sigma_J(z)$.*

*Proof.* As in the statement of the theorem, suppose that $z \in S^1 \smallsetminus \{1\}$ is not a Knotennullstelle. Let $W_{LJ}$ be a homology bordism between the Seifert surgeries $M_L$ and $M_J$, whose existence is guaranteed by Lemma 6.1. Let $W_J$ be a 4-manifold that gives a nullbordism of $M_J$ over $B\mathbb{Z}$, and define $W_L := W_{LJ} \cup_{M_J} W_J$.

The signature of the intersection form on $H_2(W_L; \mathbb{C}^a)/H_2(M_L; \mathbb{C}^\alpha)$, together with the ordinary signature over $\mathbb{Z}$, determines the signature $\sigma_L(z)$ by Section 5. Similarly, the signature of the intersection form on the quotient $H_2(W_J; \mathbb{C}^\alpha)/H_2(M_J; \mathbb{C}^\alpha)$ and the ordinary signature of $W_J$ determine the signature $\sigma_J(z)$. By Lemma 3.1, we have homology isomorphisms

$$H_2(M_L; \mathbb{C}^\alpha) \xrightarrow{\cong} H_2(W_{LJ}; \mathbb{C}^\alpha) \text{ and } H_2(M_J; \mathbb{C}^\alpha) \xrightarrow{\cong} H_2(W_{LJ}; \mathbb{C}^\alpha).$$

It follows that every class in $H_2(W_L; \mathbb{C}^\alpha)$ has a representative in $W_J$, that

$$H_2(W_L; \mathbb{C}^a)/H_2(M_L; \mathbb{C}^\alpha) \cong H_2(W_J; \mathbb{C}^a)/H_2(M_J; \mathbb{C}^\alpha),$$

and that this isomorphism induces an isometry of the intersection forms. Thus the twisted signatures of both intersection forms are equal. We needed that $z$ is not a Knotennullstelle in order to apply Lemma 3.1 in the preceding argument. The same argument over $\mathbb{Z}$ implies that the ordinary signatures also coincide, that is $\sigma(W_L) = \sigma(W_J)$. Therefore $\sigma(M_L, \phi_L, z) = \sigma(M_J, \phi_J, z)$, and so $\sigma_L(z) = \sigma_J(z)$ by Lemma 5.4. Thus the Levine-Tristram signature at $z$ is a concordance invariant, as desired. □

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# Picard Groups, Weight Structures, and (noncommutative) Mixed Motives

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Abstract. We develop a general theory which, under certain assumptions, enables the computation of the Picard group of a symmetric monoidal triangulated category equipped with a weight structure in terms of the Picard group of the associated heart. As an application, we compute the Picard group of several categories of motivic nature – mixed Artin motives, mixed Artin-Tate motives, bootstrap motivic spectra, noncommutative mixed Artin motives, noncommutative mixed motives of central simple algebras – as well as the Picard group of certain derived categories of symmetric ring spectra.

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## 1. Introduction and statement of results

The computation of the Picard group $\mathrm{Pic}(\mathcal{T})$ of a symmetric monoidal (triangulated) category $\mathcal{T}$ is, in general, a very difficult task. The goal of this article is to explain how the theory of weight structures allows us to greatly simplify this task.

Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a symmetric monoidal triangulated category equipped with a weight structure $w = (\mathcal{T}^{w\geq 0}, \mathcal{T}^{w\leq 0})$; consult §3 for details. Assume that the symmetric monoidal structure $-\otimes-$ (as well as the $\otimes$-unit $\mathbf{1}$) restricts to the heart $\mathcal{H} := \mathcal{T}^{w\geq 0} \cap \mathcal{T}^{w\leq 0}$ of the weight structure. We say that the

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category $\mathcal{T}$ has the *w-Picard property* if the group homomorphism $\mathrm{Pic}(\mathcal{H})\times\mathbb{Z}\to\mathrm{Pic}(\mathcal{T}), (a,n)\mapsto a[n]$, is invertible. Our first main result provides sufficient conditions for this property to hold:

Theorem 1.1. *Assume that the weight structure $w$ on $\mathcal{T}$ is* bounded, *i.e.* $\mathcal{T}=\cup_{n\in\mathbb{Z}}\mathcal{T}^{w\geq 0}[-n]=\cup_{n\in\mathbb{Z}}\mathcal{T}^{w\leq 0}[-n]$, *and that there exists a full, additive, conservative, symmetric monoidal functor from $\mathcal{H}$ into a symmetric monoidal semi-simple abelian category* A *which is moreover* local *in the sense that if $a\otimes b=0$ then $a=0$ or $b=0$. Under these assumptions, the category $\mathcal{T}$ has the $w$-Picard property.*

As explained in [7, §4.3], every bounded weight structure is uniquely determined by its heart. Concretely, given any additive subcategory $\mathcal{H}'\subset\mathcal{T}$ which generates $\mathcal{T}$ and for which we have $\mathrm{Hom}_{\mathcal{H}'}(a,b[n])=0$ for every $n>0$ and $a,b\in\mathcal{H}'$, there exists a unique bounded weight structure on $\mathcal{T}$ with heart the Karoubi-closure of $\mathcal{H}'$ in $\mathcal{T}$. Roughly speaking, the construction of a bounded weight structure on a triangulated category amounts simply to the choice of an additive subcategory with trivial positive Ext-groups.
Our second main result formalizes the conceptual idea that the $w$-Picard property satisfies a "global-to-local" descent principle:

Theorem 1.2. *Assume the following:*

- (A1) *The heart $\mathcal{H}$ of the bounded weight structure $w$ is essentially small and $R$-linear for some commutative indecomposable Noetherian ring $R$. Moreover,* $\mathrm{Hom}_{\mathcal{H}}(a,b)$ *is a finitely generated flat $R$-module for any two objects $a,b\in\mathcal{H}$;*
- (A2) *For every residue field $\kappa(\mathfrak{p})$, with $\mathfrak{p}\in\mathrm{Spec}(R)$, there exists a symmetric monoidal triangulated category $(\mathcal{T}_{\kappa(\mathfrak{p})},\otimes,\mathbf{1})$ equipped with a weight structure $w_{\kappa(\mathfrak{p})}$ and with a weight-exact symmetric monoidal functor $\iota_{\kappa(\mathfrak{p})}\colon\mathcal{T}\to\mathcal{T}_{\kappa(\mathfrak{p})}$. Moreover, the functor $\iota_{\kappa(\mathfrak{p})}$ induces an equivalence of categories between the Karoubization of $\mathcal{H}\otimes_R\kappa(\mathfrak{p})$ and $\mathcal{H}_{\kappa(\mathfrak{p})}$.*

*Under assumptions (A1)-(A2), if the categories $\mathcal{T}_{\kappa(\mathfrak{p})}$ have the $w_{\kappa(\mathfrak{p})}$-Picard property, then the category $\mathcal{T}$ has the $w$-Picard property.*

*Remark* 1.3.
- (i) At assumption (A1) we can consider more generally the case where $R$ is possibly decomposable; consult Remark 5.3(i).
- (ii) As it will become clear from the proof of Theorem 1.2, at assumption (A2) it suffices to consider the residue fields $\kappa(\mathfrak{m})$ associated to the maximal and minimal prime ideals of $R$; consult Remark 5.3(ii).

Due to their generality and simplicity, we believe that Theorems 1.1-1.2 will soon be part of the toolkit of every mathematician interested in Picard groups of triangulated categories. In the next section, we illustrate the usefulness of these results by computing the Picard group of several important categories of motivic nature; consult also §2.6 for a topological application.

## 2. Applications

Let $k$ be a base field, which we assume perfect, and $R$ a commutative ring of coefficients, which we assume indecomposable and Noetherian. Voevodsky's category of geometric mixed motives $\mathrm{DM}_{\mathrm{gm}}(k;R)$ (see [14, 24]), Morel-Voevodsky's stable $\mathbb{A}^1$-homotopy category $\mathrm{SH}(k)$ (see [26, 28, 40]), and Kontsevich's category of noncommutative mixed motives $\mathrm{KMM}(k;R)$ (see [19, 20, 21, 34]), play nowadays a central role in the motivic realm. A major challenge, which seems completely out of reach at the present time, is the computation of the Picard group of these symmetric monoidal triangulated categories[2]. In what follows, making use of Theorems 1.1-1.2, we achieve this goal in the case of certain important subcategories.

2.1. Mixed Artin motives. The category of *mixed Artin motives* $\mathrm{DMA}(k;R)$ is defined as the thick triangulated subcategory of $\mathrm{DM}_{\mathrm{gm}}(k;R)$ generated by the motives $M(X)_R$ of zero-dimensional smooth $k$-schemes $X$. The smallest additive, Karoubian, full subcategory of $\mathrm{DMA}(k;R)$ containing the objects $M(X)_R$ identifies with the (classical) category of Artin motives $\mathrm{AM}(k;R)$.

Theorem 2.1. *When the degrees of the finite separable field extensions of $k$ are invertible in $R$, we have* $\mathrm{Pic}(\mathrm{DMA}(k;R)) \simeq \mathrm{Pic}(\mathrm{AM}(k;R)) \times \mathbb{Z}$.

*Example* 2.2. Theorem 2.1 holds, in particular, in the following cases:

(i) The field $k$ is arbitrary and $R$ is a $\mathbb{Q}$-algebra;
(ii) The field $k$ is formally real (*e.g.* $k = \mathbb{R}$) and $1/2 \in R$;
(iii) Let $p$ be a (fixed) prime number, $l$ a perfect field, and $H$ a Sylow pro-$p$-subgroup of $\mathrm{Gal}(\bar{l}/l)$. Theorem 2.1 also holds with $k := \bar{l}^H$ and $1/p \in R$.

Whenever $R$ is a field, the $R$-linearized Galois-Grothendieck correspondence induces a symmetric monoidal equivalence of categories between $\mathrm{AM}(k;R)$ and the category $\mathrm{Rep}_R(\Gamma)$ of continuous finite dimensional $R$-linear representations of the absolute Galois group $\Gamma := \mathrm{Gal}(\overline{k}/k)$. Since the $\otimes$-invertible objects of $\mathrm{Rep}_R(\Gamma)$ are the 1-dimensional $\Gamma$-representations, $\mathrm{Pic}(\mathrm{AM}(k;R)) \simeq \mathrm{Pic}(\mathrm{Rep}_R(\Gamma))$ identifies with the group of continuous characters from $\Gamma^{\mathrm{ab}}$ to $R^\times$. In the particular case where $k = \mathbb{Q}$, the profinite group $\Gamma^{\mathrm{ab}}$ agrees with $\widehat{\mathbb{Z}}^\times$. Consequently, all the elements of $\mathrm{Rep}_R(\Gamma)$ can be represented by Dirichlet characters. Moreover, in the cases where $\mathrm{char}(k) \neq 2$ and $R = \mathbb{Q}$, we have the following computation

$$k^\times/(k^\times)^2 \xrightarrow{\simeq} \mathrm{Pic}(\mathrm{Rep}_{\mathbb{Q}}(\Gamma)) \qquad \lambda \mapsto (\Gamma \twoheadrightarrow \mathrm{Gal}(k(\sqrt{\lambda})/k) \xrightarrow{\sigma \mapsto -1} \mathbb{Q}^\times)\,,$$

where $\sigma$ stands for the generator of the Galois group $\mathrm{Gal}(k(\sqrt{\lambda})/k) \simeq \mathbb{Z}/2\mathbb{Z}$; see Peter [30, pages 340-341]. A similar computation holds in characteristic 2 with $k^\times/(k^\times)^2$ replaced by $k/\{\lambda + \lambda^2 \mid \lambda \in k\}$.

[2]Consult Bachmann [4], resp. Hu [17], for the construction of $\otimes$-invertible objects in the motivic category $\mathrm{DM}_{\mathrm{gm}}(k;\mathbb{Z}/2\mathbb{Z})$, resp. $\mathrm{SH}(k)$, associated to quadrics.

Now, let $\mathcal{A}(k;R)$ be an additive, Karoubian, symmetric monoidal, full subcategory of $\mathrm{AM}(k;R)$, and $\mathrm{D}\mathcal{A}(k;R)$ the thick triangulated subcategory of $\mathrm{DMA}(k;R)$ generated by the motives associated to the objects of $\mathcal{A}(k;R)$. Under these notations, Theorem 2.1 admits the following generalization:

Theorem 2.3. *Assume that there exists a set of finite separable field extensions $l_i/k, i \in I$, such that the following two conditions hold:*

- (B1) *Every object of the category $\mathcal{A}(k;R)$ is isomorphic to a summand of a finite direct sum of the motives associated to the field extensions $l_i/k, i \in I$;*
- (B2) *For each $i \in I$, the degree of the finite field extension $l_i/k$ is invertible in $R$.*

*Under assumptions (B1)-(B2), we have* $\mathrm{Pic}(\mathrm{D}\mathcal{A}(k;R)) \simeq \mathrm{Pic}(\mathcal{A}(k;R)) \times \mathbb{Z}$.

*Example* 2.4 (Mixed Dirichlet motives). Let $R$ be a field. Following Wildeshaus [41, Def. 3.4], a *Dirichlet motive* is an Artin motive for which the corresponding $\Gamma$-representation factors through an abelian (finite) quotient. Take $\mathcal{A}(k;R)$ to be the category of Dirichlet motives. In this case, the associated symmetric monoidal triangulated category $\mathrm{D}\mathcal{A}(k;R)$ is called the category of *mixed Dirichlet motives*. Since the $\otimes$-invertible objects of $\mathrm{Rep}_R(\Gamma)$ are the 1-dimensional representations, and all these representations factor through an abelian (finite) quotient, the inclusion of categories $\mathcal{A}(k;R) \subset \mathrm{AM}(k;R)$ yields an isomorphism $\mathrm{Pic}(\mathcal{A}(k;R)) \simeq \mathrm{Pic}(\mathrm{AM}(k;R))$. Consequently, in the case where $R$ is of characteristic zero, Theorem 2.3 implies that $\mathrm{Pic}(\mathrm{D}\mathcal{A}(k;R)) \simeq \mathrm{Pic}(\mathrm{AM}(k;R)) \times \mathbb{Z}$. Intuitively speaking, the difference between (mixed) Dirichlet motives and (mixed) Artin motives is not detected by the Picard group.

2.2. Mixed Artin-Tate motives. The category $\mathrm{DMAT}(k;R)$ of *mixed Artin-Tate motives* is defined as the thick symmetric monoidal triangulated subcategory of $\mathrm{DM}_{\mathrm{gm}}(k;R)$ generated by the motives $M(X)_R$ of zero-dimensional smooth $k$-schemes $X$ and by the Tate motives $R(m), m \in \mathbb{Z}$.

Theorem 2.5. *When the degrees of the finite separable field extensions of $k$ are invertible in $R$, we have* $\mathrm{DMAT}(k;R) \simeq \mathrm{Pic}(\mathrm{AM}(k;R)) \times \mathbb{Z} \times \mathbb{Z}$.

Now, let $\mathcal{A}(k;R)$ be an additive, Karoubian, symmetric monoidal, full subcategory of $\mathrm{AM}(k;R)$, and $\mathrm{D}\mathcal{A}\mathrm{T}(k;R)$ the thick symmetric monoidal triangulated subcategory of $\mathrm{DMAT}(k;R)$ generated by the motives associated to the objects of $\mathcal{A}(k;R)$ and by the Tate motives $R(m), m \in \mathbb{Z}$. Theorem 2.5 admits the following generalization:

Theorem 2.6. *Assume that there exists a set of field extensions $l_i/k, i \in I$, as in Theorem 2.3. Under these assumptions, we have* $\mathrm{Pic}(\mathrm{D}\mathcal{A}\mathrm{T}(k;R)) \simeq \mathrm{Pic}(\mathcal{A}(k;R)) \times \mathbb{Z} \times \mathbb{Z}$.

*Example* 2.7 (Mixed Tate motives). Take $\mathcal{A}(k;R)$ to be the smallest additive, Karoubian, full subcategory of $\mathrm{AM}(k;R)$ containing the $\otimes$-unit. In this case, the associated symmetric monoidal triangulated category $\mathrm{D}\mathcal{A}\mathrm{T}(k;R)$ is called the category of *mixed Tate motives*. Since $\mathcal{A}(k;R)$ identifies with the category of finitely generated projective $R$-modules[3], we conclude from Theorem 2.6 that the Picard group of $\mathrm{D}\mathcal{A}\mathrm{T}(k;R)$ is isomorphic to $\mathrm{Pic}(R)\times\mathbb{Z}\times\mathbb{Z}$. Note that we are not imposing the invertibility of any integer in $R$.

*Example* 2.8 (Mixed Dirichlet-Tate motives). Take $\mathcal{A}(k;R)$ to be the category of Dirichlet motives. In this case, the associated symmetric monoidal triangulated category $\mathrm{D}\mathcal{A}\mathrm{T}(k;R)$ is called the category of *mixed Dirichlet-Tate motives*. Recall from Example 2.4 that the Picard group of $\mathcal{A}(k;R)$ is isomorphic to the Picard group of $\mathrm{AM}(k;R)$. Consequently, in the case where $R$ is of characteristic zero, Theorem 2.6 implies that $\mathrm{Pic}(\mathrm{D}\mathcal{A}\mathrm{T}(k;R))\simeq\mathrm{Pic}(\mathrm{AM}(k;R))\times\mathbb{Z}\times\mathbb{Z}$.

2.3. Motivic spectra. The *bootstrap category* $\mathrm{Boot}(k)$ is defined as the thick triangulated subcategory of $\mathrm{SH}(k)$ generated by the $\otimes$-unit $\Sigma^\infty(\mathrm{Spec}(k)_+)$. The former category contains a lot of information. For example, as proved by Levine in [22, Thm. 1], whenever $k$ is algebraically closed and of characteristic zero, the category $\mathrm{Boot}(k)$ identifies with the homotopy category of finite spectra $\mathcal{SH}_c$. In particular, we have non-trivial negative Ext-groups

$$\mathrm{Hom}_{\mathrm{Boot}(k)}(\Sigma^\infty(\mathrm{Spec}(k)_+),\Sigma^\infty(\mathrm{Spec}(k)_+)[-n])\simeq\pi_n(\mathbb{S})\qquad n>0\,, \tag{2.9}$$

where $\mathbb{S}$ stands for the sphere spectrum. Moreover, as proved by Morel in [25, Thm. 6.2.2], whenever $k$ is of characteristic $\neq 2$, we have a ring isomorphism

$$\mathrm{End}_{\mathrm{Boot}(k)}(\Sigma^\infty(\mathrm{Spec}(k)_+))\simeq GW(k)\,, \tag{2.10}$$

where $GW(k)$ stands for the Grothendieck-Witt ring of $k$.

Theorem 2.11. *Assume that* $\mathrm{char}(k)\neq 2$ *and that* $GW(k)$ *is Noetherian. Under these assumptions, we have* $\mathrm{Pic}(\mathrm{Boot}(k))\simeq\mathrm{Pic}(GW(k))\times\mathbb{Z}$.

*Remark* 2.12. The ring $GW(k)$ is Noetherian if and only if $k^\times/(k^\times)^2$ is finite.

*Example* 2.13. Theorem 2.11 holds, in particular, in the following cases:

(i) The field $k$ is quadratically closed (*e.g.* $k$ is algebraically closed or the field of constructible numbers). In this case, we have $GW(k)\simeq\mathbb{Z}$;
(ii) The field $k$ is the field of real numbers $\mathbb{R}$. In this case, we have $GW(\mathbb{R})\simeq\mathbb{Z}[C_2]$, where $C_2$ stands for the cyclic group of order 2;
(iii) The field $k$ is the finite field $\mathbb{F}_q$ with $q$ odd. In this case, $k^\times/(k^\times)^2=C_2$.

Intuitively speaking, Theorem 2.11 shows that none of the motivic spectra which are built using the non-trivial Ext-groups (2.9) is $\otimes$-invertible!

[3] Recall that the Picard group $\mathrm{Pic}(R)$ of a Dedekind domain $R$ is its ideal class group $C(R)$.

2.4. Noncommutative mixed Artin motives. The category of *noncommutative mixed Artin motives* $\mathrm{NMAM}(k;R)$ is defined as the thick triangulated subcategory of $\mathrm{KMM}(k;R)$ generated by the noncommutative motives $U(l)_R$ of finite separable field extensions $l/k$. The smallest additive, Karoubian, full subcategory of $\mathrm{NMAM}(k;R)$ containing the objects $U(l)_R$ identifies with $\mathrm{AM}(k;R)$.

The category of noncommutative mixed Artin motives is in general much richer than the category of mixed Artin motives. For example, whenever $R$ is a $\mathbb{Q}$-algebra, $\mathrm{DMA}(k;R)$ identifies with the category $\mathrm{Gr}_{\mathbb{Z}}\mathrm{AM}(k;R)$ of $\mathbb{Z}$-graded objects in $\mathrm{AM}(k;R)$; see [39, page 217]. This implies that $\mathrm{DMA}(k;R)$ has trivial higher Ext-groups. On the other hand, given any two finite separable field extensions $l_1/k$ and $l_2/k$, we have non-trivial negative Ext-groups (see [33, §4])

$$\mathrm{Hom}_{\mathrm{NMAM}(k;R)}(U(l_1)_R, U(l_2)_R[-n]) \simeq K_n(l_1 \otimes_k l_2)_R \qquad n > 0\,, \tag{2.14}$$

where $K_n(l_1 \otimes_k l_2)$ stands for the $n^{\mathrm{th}}$ algebraic $K$-theory group of $l_1 \otimes_k l_2$. Roughly speaking, $\mathrm{NMAM}(k;R)$ contains not only $\mathrm{AM}(k;R)$ but also all the higher algebraic $K$-theory groups of finite separable field extensions. For example, given a number field $\mathbb{F}$, we have the following computation (due to Borel [12, §12])

$$\mathrm{Hom}_{\mathrm{NMAM}(\mathbb{Q};\mathbb{Q})}(U(\mathbb{Q})_{\mathbb{Q}}, U(\mathbb{F})_{\mathbb{Q}}[-n]) \simeq \begin{cases} \mathbb{Q}^{r_2} & n \equiv 3 \pmod 4 \\ \mathbb{Q}^{r_1+r_2} & n \equiv 1 \pmod 4 \\ 0 & \text{otherwise} \end{cases} \qquad n \geq 2\,,$$

where $r_1$ (resp. $r_2$) stands for the number of real (resp. complex) embeddings of $\mathbb{F}$.

Theorem 2.15. *When the degrees of the finite separable field extensions of $k$ are invertible in $R$, we have* $\mathrm{Pic}(\mathrm{NMAM}(k;R)) \simeq \mathrm{Pic}(\mathrm{AM}(k;R)) \times \mathbb{Z}$.

*Example* 2.16. Theorem 2.15 holds in the cases (i)-(iii) of Example 2.2.

Theorem 2.15 shows that although the category $\mathrm{NMAM}(k;R)$ is much richer than $\mathrm{DMA}(k;R)$, this richness is not detected by the Picard group.

Now, let $\mathcal{A}(k;R)$ be an additive, Karoubian, symmetric monoidal, full subcategory of $\mathrm{AM}(k;R)$, and $\mathrm{NM}\mathcal{A}(k;R)$ the thick triangulated subcategory of $\mathrm{NMAM}(k;R)$ generated by the noncommutative motives associated to the objects of $\mathcal{A}(k;R)$. Theorem 2.15 admits the following generalization:

Theorem 2.17. *Assume that there exists a set of field extensions $l_i/k, \in I$, as in Theorem 2.3. Under these assumptions, we have* $\mathrm{Pic}(\mathrm{NM}\mathcal{A}(k;R)) \simeq \mathrm{Pic}(\mathcal{A}(k;R)) \times \mathbb{Z}$.

*Example* 2.18 (Noncommutative mixed Dirichlet motives). Take $\mathcal{A}(k;R)$ to be the category of Dirichlet motives. In this case, the associated symmetric monoidal triangulated category $\mathrm{NM}\mathcal{A}(k;R)$ is called the category of *noncommutative mixed Dirichlet motives*. Recall from Example 2.4 that the Picard group of $\mathcal{A}(k;R)$ is isomorphic to $\mathrm{Pic}(\mathrm{AM}(k;R))$. Consequently, in the case where $R$ is of characteristic zero, Theorem 2.15 implies that $\mathrm{Pic}(\mathrm{NM}\mathcal{A}(k;R)) \simeq \mathrm{Pic}(\mathrm{AM}(k;R)) \times \mathbb{Z}$. Roughly speaking, the difference between mixed Dirichlet motives and noncommutative mixed Dirichlet motives is not detected by the Picard group.

*Example* 2.19 (Bootstrap category). Take $\mathcal{A}(k;R)$ to be the smallest additive, Karoubian, full subcategory of $\mathrm{AM}(k;R)$ containing the $\otimes$-unit. In this case, the associated symmetric monoidal triangulated category $\mathrm{NM}\mathcal{A}(k;R)$ is called the *bootstrap category*. Since $\mathcal{A}(k;R)$ identifies with the category of finitely generated projective $R$-modules, we conclude from Theorem 2.17 that $\mathrm{Pic}(\mathrm{NM}\mathcal{A}(k;R)) \simeq \mathrm{Pic}(R) \times \mathbb{Z}$. Similarly to Example 2.7, we are not imposing the invertibility of any integer in $R$.

2.5. Noncommutative mixed motives of central simple algebras. Let us denote by $\mathrm{NMCSA}(k;R)$ the thick triangulated subcategory of $\mathrm{KMM}(k;R)$ generated by the noncommutative motives $U(A)_R$ of central simple $k$-algebras $A$. In the same vein, let $\mathrm{CSA}(k;R)$ be the smallest additive, Karoubian, full subcategory of $\mathrm{NMCSA}(k;R)$ containing the objects $U(A)_R$. As proved in [35, Thm. 9.1], given any two central simple $k$-algebras $A$ and $B$, we have the following equivalence

$$U(A)_{\mathbb{Z}} \simeq U(B)_{\mathbb{Z}} \Leftrightarrow [A] = [B] \in \mathrm{Br}(k)\,, \tag{2.20}$$

where $\mathrm{Br}(k)$ stands for the Brauer group of $k$. Intuitively speaking, (2.20) shows that the noncommutative motive $U(A)_{\mathbb{Z}}$ and the Brauer class $[A]$ contain exactly the same information. We have moreover non-trivial negative Ext-groups:

$$\mathrm{Hom}_{\mathrm{NMCSA}(k;\mathbb{Z})}(U(A)_{\mathbb{Z}}, U(B)_{\mathbb{Z}}[-n]) \simeq \pi_n(K(A^{\mathrm{op}} \otimes_k B) \wedge H\mathbb{Z})\ n > 0\,, \tag{2.21}$$

where $H\mathbb{Z}$ stands for the Eilenberg-MacLane spectrum of $\mathbb{Z}$. Roughly speaking, the category $\mathrm{NMCSA}(k;\mathbb{Z})$ contains information not only about the Brauer group but also about all the higher algebraic $K$-theory of central simple algebras.

Theorem 2.22. *The following holds:*

(i) *We have an isomorphism* $\mathrm{Pic}(\mathrm{NMCSA}(k;R)) \simeq \mathrm{Pic}(\mathrm{CSA}(k;R)) \times \mathbb{Z}$*;*
(ii) *We have an isomorphism* $\mathrm{Pic}(\mathrm{CSA}(k;\mathbb{Z})) \simeq \mathrm{Br}(k)$*.*

*Remark* 2.23. Let $R$ be a field. As explained in Remark 10.6, the Picard group of the category $\mathrm{Pic}(\mathrm{CSA}(k;R))$ is trivial when $\mathrm{char}(R) = 0$ and isomorphic to $\mathrm{Br}(k)\{p\}$ when $\mathrm{char}(R) = p > 0$.

Intuitively speaking, Theorem 2.22 shows that none of the noncommutative mixed motives which are built using the non-trivial negative Ext-groups (2.21) is $\otimes$-invertible!

2.6. A topological application. Let $E$ be a commutative symmetric ring spectrum and $\mathcal{D}_c(E)$ the associated derived category of compact $E$-modules; see [15, 31].

Theorem 2.24. *Assume that the ring spectrum $E$ is* connective, *i.e.* $\pi_n(E) = 0$ *for every $n < 0$, and that $\pi_0(E)$ is an indecomposable Noetherian ring. Under these assumptions, we have* $\mathrm{Pic}(\mathcal{D}_c(E)) \simeq \mathrm{Pic}(\pi_0(E)) \times \mathbb{Z}$.

*Example* 2.25 (Finite spectra). Let $E$ be the sphere spectrum $\mathbb{S}$. In this case, the category $\mathcal{D}_c(\mathbb{S})$ is equivalent to the homotopy category of finite spectra $\mathcal{SH}_c$ and $\pi_0(\mathbb{S}) \simeq \mathbb{Z}$. Consequently, we obtain $\mathrm{Pic}(\mathcal{SH}_c) \simeq \mathbb{Z}$. This computation was originally established by Hopkins-Mahowald-Sadofsky in [16] using different tools. Note that this computation may be understood as a particular case of Theorem 2.11.

*Example* 2.26 (Ordinary rings). Let $E$ be the Eilenberg-MacLane spectrum $HR$ of a commutative indecomposable Noetherian ring $R$. In this case, $\mathcal{D}_c(HR) \simeq \mathcal{D}_c(R)$ and $\pi_0(HR) \simeq R$. Consequently, we obtain $\mathrm{Pic}(\mathcal{D}_c(R)) \simeq \mathrm{Pic}(R) \times \mathbb{Z}$; consult Remark 5.3(i) for the case where $R$ is decomposable. This computation was originally established in [13]. Although Fausk did not use weight structures, one observes that by applying our arguments (see §5) to the triangulated category $D_c(R)$, equipped with the weight structure whose heart consists of the finitely generated projective $R$-modules, one obtains a reasoning somewhat similar to his one.

## 3. Weight structures

In this section we briefly review the theory of weight structures. This will give us the opportunity to fix some notations that will be used throughout the article.

*Definition* 3.1. (see [7, Def. 1.1.1]) A *weight structure* $w$ on a triangulated category $\mathcal{T}$, also known as a *co-t-structure* in the sense of Pauksztello [29], consists of a pair of additive subcategories $(\mathcal{T}^{w\geq 0}, \mathcal{T}^{w\leq 0})$ satisfying the following conditions[4]:

(i) The categories $\mathcal{T}^{w\geq 0}$ and $\mathcal{T}^{w\leq 0}$ are closed under taking summands in $\mathcal{T}$;
(ii) We have inclusions of categories $\mathcal{T}^{w\geq 0} \subset \mathcal{T}^{w\geq 0}[1]$ and $\mathcal{T}^{w\leq 0}[1] \subset \mathcal{T}^{w\leq 0}$;
(iii) For every $a \in \mathcal{T}^{w\geq 0}$ and $b \in \mathcal{T}^{w\leq 0}[1]$, we have $\mathrm{Hom}_{\mathcal{T}}(a, b) = 0$;
(iv) For every $a \in \mathcal{T}$ there exists a distinguished triangle $c[-1] \to a \to b \to c$ in $\mathcal{T}$ with $b \in \mathcal{T}^{w\leq 0}$ and $c \in \mathcal{T}^{w\geq 0}$.

Given an integer $n \in \mathbb{Z}$, let $\mathcal{T}^{w\geq n} := \mathcal{T}^{w\geq 0}[-n]$, $\mathcal{T}^{w\leq n} := \mathcal{T}^{w\leq 0}[-n]$, and $\mathcal{T}^{w=n} := \mathcal{T}^{w\geq n} \cap \mathcal{T}^{w\leq n}$. The objects belonging to $\cup_{n\in\mathbb{Z}} \mathcal{T}^{w=n}$ are called *$w$-pure* and the additive subcategory $\mathcal{H} := \mathcal{T}^{w=0}$ is called the *heart* of the weight structure. Finally, a weight structure $w$ is called *bounded* if $\mathcal{T} = \cup_{n\in\mathbb{Z}} \mathcal{T}^{w\geq n} = \cup_{n\in\mathbb{Z}} \mathcal{T}^{w\leq n}$.

Assumption: Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a symmetric monoidal triangulated category

[4]Following [7], we will use the so-called *cohomological convention* for weight structures. This differs from the homological convention used in [8, 10, 11, 41].

equipped with a weight structure $w$. Throughout the article, we will always assume that the symmetric monoidal structure is *$w$-pure* in the sense that the tensor product $-\otimes-$ (as well as the $\otimes$-unit $\mathbf{1}$) restricts to the heart $\mathcal{H}$.

*Remark* 3.2 (Self-duality). The notion of weight structure is (categorically) self-dual. Given a triangulated category $\mathcal{T}$ equipped with a weight structure $w$, the opposite triangulated category $\mathcal{T}^{\mathrm{op}}$ inherits the opposite weight structure $w^{\mathrm{op}}$ with $(\mathcal{T}^{\mathrm{op}})^{w^{\mathrm{op}}\leq 0} := \mathcal{T}^{w\geq 0}$ and $(\mathcal{T}^{\mathrm{op}})^{w^{\mathrm{op}}\geq 0} := \mathcal{T}^{w\leq 0}$.

*Definition* 3.3. An exact functor $F\colon \mathcal{T}\to\mathcal{T}'$ between triangulated categories equipped with weight structures $w$ and $w'$, respectively, is called *weight-exact* if $F(\mathcal{T}^{w\leq 0})\subseteq \mathcal{T}'^{w'\leq 0}$ and $F(\mathcal{T}^{w\geq 0})\subseteq \mathcal{T}'^{w'\geq 0}$.

*Remark* 3.4. Whenever the weight structure $w$ is bounded, an exact functor $F\colon\mathcal{T}\to\mathcal{T}'$ is weight-exact if and only if $F(\mathcal{T}^{w=0})\subseteq\mathcal{T}^{w'=0}$; see [10, Prop. 1.2.3(5)].

3.1. Weight complexes. Let $\mathcal{T}$ be a triangulated category equipped with a weight structure $w$. Following [7, Def. 2.2.1] (see also [8, §2.2]), we can assign to every object $a\in\mathcal{T}$ a certain (cochain) weight $\mathcal{H}$-complex $t(a):\cdots\to a^{m-1}\to a^m\to a^{m+1}\to\cdots$. For example, if $a\in\mathcal{T}^{w=0}$, then we can take for $t(a)$ the complex $\cdots\to 0\to a\to 0\to\cdots$ supported in degree 0. As explained in *loc. cit.*, the assignment $a\mapsto t(a)$ is well-defined only up to homotopy equivalence. Nevertheless, we will use the notation $a^p$ for the $p^{\mathrm{th}}$ term of some choice of a weight $\mathcal{H}$-complex $t(a)$. This is justified by the next result:

Proposition 3.5. *(see* [10, Prop. 1.4.2(6)-(7)]*)*

(i) *Let $F\colon\mathcal{T}\to\mathcal{T}'$ be a weight-exact functor as in Definition 3.3. If $t(a)$ is a weight $\mathcal{H}$-complex for $a$, then $F(t(a))$ is a weight $\mathcal{H}'$-complex for $F(a)$;*

(ii) *Given an additive functor $G\colon\mathcal{H}\to\mathrm{A}$, with values in an abelian category, the assignment $a\mapsto H^0(G(t(a)))$ yields a well-defined (i.e. independent of the choice of $t(a)$) homological functor*[5] $\mathrm{H}_0\colon\mathcal{T}\to\mathrm{A}$. *Moreover, the assignment $G\mapsto\mathrm{H}_0$ is natural in the functor $G$.*

We denote by $\mathrm{H}_n$ the precomposition of $\mathrm{H}_0$ with the $n^{\mathrm{th}}$ suspension functor of $\mathcal{T}$.

*Remark* 3.6. Note that if $a\in\mathcal{T}^{w=m}$, then $\mathrm{H}_n(a)=0$ for every $n\neq m$.

*Remark* 3.7. Following the referee's suggestion, we recall here in an informal way the construction of weight complexes. Let $\mathcal{T}$ be a triangulated category equipped with a weight structure $w$. Given $a\in\mathcal{T}$ and $m\in\mathbb{Z}$, the axiom (iv) of Definition 3.1 implies the existence of a distinguished triangle $b^m\to a\to c^m\to b^m[1]$ with $b^m\in\mathcal{T}^{w\geq m}$ and $c^m\in\mathcal{T}^{w\leq m-1}$. These triangles are *not* determined (up to isomorphism) by the couple $(a,m)$. Nevertheless, given a morphism $g\colon a\to a'$ and an integer $m'\leq m$, we can extend $g$ to a morphism

[5] The homological functors obtained this way are called *pure* due to their relation with Deligne's theory of weights on cohomology; see [8, Rk. 2.4.5(5)].

between the corresponding triangles; this extension is unique whenever $m' < m$. This fact, applied to a fixed object $a$ and to all integers $m$, yields connecting morphisms $\partial^m\colon b^{m+1} \to b^m$. If one shifts the cone of $\partial^m$ by $[m]$, we then obtain a sequence of objects $a^m$ in $\mathcal{T}^{w=0}$. Moreover, the corresponding triangles give rise to connecting morphisms which yield a weight complex for $a$. The above considerations show that weight complexes are naturally "respected" by weight-exact functors. This naturality easily carries over to the pure functors considered in the above Proposition 3.5(ii). However, these pure functors do not depend on any choices up to canonical isomorphisms.

3.2. Karoubization. Given a category $\mathcal{C}$, let us write $\mathrm{Kar}(\mathcal{C})$ for its Karoubization. Recall that the objects of $\mathrm{Kar}(\mathcal{C})$ are the pairs $(a, e)$, with $a \in \mathcal{C}$ and $e$ an idempotent of the ring of endomorphisms $\mathrm{End}_{\mathcal{C}}(a, a)$. The morphisms are given by $\mathrm{Hom}_{\mathrm{Kar}(\mathcal{C})}((a, e), (b, e')) := e \circ \mathrm{Hom}_{\mathcal{C}}(a, b) \circ e'$. By construction, $\mathrm{Kar}(\mathcal{C})$ comes equipped with the canonical functor $\mathcal{C} \to \mathrm{Kar}(\mathcal{C}), a \mapsto (a, \mathrm{id})$. Whenever $\mathcal{C}$ is symmetric monoidal, resp. triangulated, the category $\mathrm{Kar}(\mathcal{C})$ is also symmetric monoidal, resp. triangulated; see [6, Thm. 1.5]. Moreover, the canonical functor $\mathcal{C} \to \mathrm{Kar}(\mathcal{C})$ becomes symmetric monoidal, resp. exact.
The following result relates Karoubian categories to bounded weight structures.

Proposition 3.8. *Let $\mathcal{T}$ be a Karoubian triangulated category. Assume that there exists a full additive subcategory $\mathcal{H}' \subset \mathcal{T}$ that generates*[6] *$\mathcal{T}$ and which is* negative *in $\mathcal{T}$ in the sense that there are no $\mathcal{T}$-extensions of positive degrees between objects of $\mathcal{H}'$. Under these assumptions, there exists a unique bounded weight structure $w$ on $\mathcal{T}$ such that its heart $\mathcal{H}$ contains $\mathcal{H}'$. Moreover, $\mathcal{H}$ is equivalent to* $\mathrm{Kar}(\mathcal{H}')$.

*Proof.* The proof is an immediate consequence of [7, Thm. 4.3.2 II and Prop. 5.2.2]; consult also [11, Cor. 2.1.2] for the generalization of this statement to the case where $\mathcal{T}$ is not necessarily Karoubian. □

## 4. Proof of Theorem 1.1

We start with the following auxiliary result:

Proposition 4.1. *A symmetric monoidal triangulated category $(\mathcal{T}, \otimes, \mathbf{1})$, equipped with a weight structure $w$, has the $w$-Picard property (see §1) if and only if all its $\otimes$-invertible objects are $w$-pure.*

*Proof.* Let $(a, n), (b, m) \in \mathrm{Pic}(\mathcal{H}) \times \mathbb{Z}$. On the one hand, when $n = m$, we have $a[n] \simeq b[m]$ in $\mathcal{T}$ if and only if $a \simeq b$ in $\mathcal{H}$. This follows from the fact that the suspension functor is an auto-equivalence of $\mathcal{T}$. On the other hand, when $n \neq m$, we have $a[n] \not\simeq b[m]$ in $\mathcal{T}$. This follows from the fact that $\mathrm{Hom}_{\mathcal{T}}(a[n], b[m])$, resp. $\mathrm{Hom}_{\mathcal{T}}(b[m], a[n])$, is zero whenever $m < n$, resp. $n < m$; see Definition 3.1(iii). This implies that the canonical group homomorphism

$$\mathrm{Pic}(\mathcal{H}) \times \mathbb{Z} \longrightarrow \mathrm{Pic}(\mathcal{T}) \qquad (a, n) \mapsto a[n] \tag{4.2}$$

[6] *i.e.* the smallest thick triangulated subcategory of $\mathcal{T}$ containing $\mathcal{H}'$ is $\mathcal{T}$ itself.

is injective. Consequently, we conclude that the category $\mathcal{T}$ has the $w$-Picard property if and only if (4.2) is surjective. In other words, $\mathcal{T}$ has the $w$-Picard property if and only if all its $\otimes$-invertible objects are $w$-pure. $\square$

*Remark* 4.3. Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a symmetric monoidal triangulated category equipped with a weight structure $w$. The arguments used in the proof of Proposition 4.1 allow us to conclude that if by hypothesis $a[n] \otimes b[m] \simeq \mathbf{1}$ for certain objects $a, b \in \mathcal{H}$ and integers $n, m \in \mathbb{Z}$, then $n = -m$ and $a$ is the $\otimes$-inverse of $b$.

Let us now prove Theorem 1.1. Let $b \in \mathcal{T}$ be a (fixed) $\otimes$-invertible object. Thanks to Proposition 4.1, it suffices to prove that $b$ is $w$-pure. By assumption, there exists a full, additive, conservative, symmetric monoidal functor $G \colon \mathcal{H} \to \mathrm{A}$ into a symmetric monoidal semi-simple abelian category which is moreover local. Proposition 3.5(ii) applied to this functor $G$ yields well-defined homological functors $\mathrm{H}_n \colon \mathcal{T} \to \mathrm{A}, n \in \mathbb{Z}$.
Consider the homological functor $\mathcal{T} \to \mathrm{A}, a \mapsto \mathrm{H}_0(a \otimes b)$. Since by assumption the weight structure $w$ is bounded, [7, Thm. 2.3.2] applied to the preceding homological functor yields a convergent Künneth spectral sequence

$$E_1^{pq} = \mathrm{H}_q(a^p \otimes b) \Rightarrow \mathrm{H}_{p+q}(a \otimes b)\,. \tag{4.4}$$

The object $a^p$ belongs to the heart $\mathcal{H}$ and the functor $a^p \otimes - \colon \mathcal{T} \to \mathcal{T}$ is weight-exact in the sense of Definition 3.3. Using the fact that $t(b)$ is a weight $\mathcal{H}$-complex for $b$, we conclude from Proposition 3.5(i) that $a^p \otimes t(b)$ is a weight $\mathcal{H}$-complex for $a^p \otimes b$. Therefore, the complex computing $\mathrm{H}_*(a^p \otimes b)$ can be obtained from the complex computing $\mathrm{H}_*(b)$ by tensoring with $G(a^p)$ (recall that $G$ is symmetric monoidal). Since the category A is semi-simple, it follows then that $\mathrm{H}_q(a^p \otimes b) \simeq G(a^p) \otimes \mathrm{H}_q(b)$. Furthermore, the functoriality of the assignment $G \mapsto \mathrm{H}_0$ mentioned in Proposition 3.5(ii) implies that the differential $E_1^{pq} \to E_1^{(p+1)q}$ equals the corresponding morphism induced by the boundary $a^p \to a^{p+1}$ (tensored with $b$). Making use once again of the semi-simplicity of A, we conclude that $E_2^{pq} \simeq \mathrm{H}_p(a) \otimes \mathrm{H}_q(b)$. Recall from [7, Thm. 2.3.2] that, in contrast with the $E_1$-terms, the $E_2$-terms are essentially independent of the choice of (the terms of) the weight complex $t(a)$. Let us denote by $m_a$, resp. $m'_a$, the smallest, resp. largest, integer such that $\mathrm{H}_n(a) = 0$ for every $n < m_a$, resp. $n > m'_a$; the existence of such integers follows from the fact that the weight structure $w$ is bounded. Similarly, let $m_b$, resp. $m'_b$, be the smallest, resp. largest, integer such that $\mathrm{H}_n(b) = 0$ for every $n < m_b$, resp. $n > m'_b$. Since by assumption the category A is local, we have $\mathrm{H}_{m_a}(a) \otimes \mathrm{H}_{m_b}(b) \neq 0$ and $\mathrm{H}_{m'_a}(a) \otimes \mathrm{H}_{m'_b}(b) \neq 0$. Using the second page of the spectral sequence (4.4), we conclude that

$$\mathrm{H}_{m_a + m_b}(a \otimes b) \neq 0 \quad \text{and} \quad \mathrm{H}_{m'_a + m'_b}(a \otimes b) \neq 0\,. \tag{4.5}$$

Now, recall that $b$ is a $\otimes$-invertible object. Therefore, by definition, we have $a \otimes b \simeq \mathbf{1}$ for some ($\otimes$-invertible) object $a \in \mathcal{T}$. Since $\mathrm{H}_n(a \otimes b) \simeq \mathrm{H}_n(\mathbf{1}) = 0$ for every $n \neq 0$, we conclude from (4.5) that $m_b = m'_b$, $m_a = m'_a$, and $m_a = -m_b$.

Thanks to Proposition 4.6 below, this implies that $b \in \mathcal{T}^{w=m_b}$. In particular, the object $b$ is $w$-pure, and so the proof is finished.

Proposition 4.6. *(Conservativity I) Let $\mathcal{T}$ be a triangulated category equipped with a bounded weight structure $w$. Assume that there exists a full, additive, conservative functor $G\colon \mathcal{H} \to \mathrm{A}$ from the heart of $w$ into a semi-simple abelian category. Under this assumption, an object $b \in \mathcal{T}$ belongs to $\mathcal{T}^{w=m}$ if and only if $\mathrm{H}_n(b) = 0$ for every $n \neq m$.*

*Proof.* Consult [8, Cor. 2.3.5]. □

*Remark* 4.7 (Künneth spectral sequence). (i) Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a symmetric monoidal triangulated equipped with a bounded weight structure $w$, and $G\colon \mathcal{H} \to \mathrm{A}$ a symmetric monoidal additive functor. Consider the associated homological functors $\mathrm{H}_n\colon \mathcal{T} \to \mathrm{A}, n \in \mathbb{Z}$. The arguments used in the proof of Theorem 1.1 allow us to conclude that there exists a convergent Künneth spectral sequence

$$E_1^{pq} = \mathrm{H}_q(a^p \otimes b) \Rightarrow \mathrm{H}_{p+q}(a \otimes b)\,.$$

Assume that the (abelian) category A is moreover semi-simple and local. Then, given any $\otimes$-invertible object $b \in \mathcal{T}$, there exists an integer $m_b$ such that $\mathrm{H}_n(b) = 0$ for every $n \neq m_b$ and $\mathrm{H}_{m_b}(b) \in \mathrm{A}$ is $\otimes$-invertible.
(ii) Given non-zero objects $a$ and $b$ as in item (i), Proposition 4.6 yields the existence of integers $m_a$ and $m_b$ satisfying the conditions described in the proof of Theorem 1.1. This implies that $\mathrm{H}_{m_a+m_b}(a \otimes b) \neq 0$, and consequently that $a \otimes b \neq 0$. In particular, $\mathcal{T}$ is *local* in the sense of [5, §4]; consult Proposition 4.2 from *loc. cit.*

## 5. Proof of Theorem 1.2

Let $b \in \mathcal{T}$ be a $\otimes$-invertible object. Thanks to Proposition 4.1, it suffices to prove that $b$ is $w$-pure. Since the functors $\iota_{\kappa(\mathfrak{p})}\colon \mathcal{T} \to \mathcal{T}_{\kappa(\mathfrak{p})}$ are symmetric monoidal, and by assumption the categories $\mathcal{T}_{\kappa(\mathfrak{p})}$ have the $w_{\kappa(\mathfrak{p})}$-Picard property, the objects $\iota_{\kappa(\mathfrak{p})}(b)$ are $w_{\kappa(\mathfrak{p})}$-pure. Concretely, $\iota_{\kappa(\mathfrak{p})}(b)$ belongs to $\mathcal{T}_{\kappa(\mathfrak{p})}^{w=m_{\kappa(\mathfrak{p})}}$ for some integer $m_{\kappa(\mathfrak{p})} \in \mathbb{Z}$. Our goal is to prove that all the integers $m_{\kappa(\mathfrak{p})}$, with $\mathfrak{p} \in \mathrm{Spec}(R)$, are equal and that the object $b$ belongs to $\mathcal{T}^{w=m_{k(\mathfrak{p})}}$. We start by addressing the first goal. Since by assumption the commutative ring $R$ is indecomposable, its spectrum $\mathrm{Spec}(R)$ is connected. Hence, it suffices to verify that $m_{\kappa(\mathfrak{p})} = m_{\kappa(\mathfrak{P})}$ for every $\mathfrak{p} \in \mathrm{Spec}(R)$ belonging to the closure of a prime ideal $\mathfrak{P} \in \mathrm{Spec}(R)$; in the particular case where $R$ is moreover an integral domain we can simply take $\mathfrak{P} = \{0\}$. Note that the assumptions of Theorem 1.2, as well as the definition of the integers $m_{\kappa(\mathfrak{p})}$ and $m_{\kappa(\mathfrak{P})}$, are (categorically) self-dual; see Remark 3.2. Therefore, it is enough to verify the inequalities $m_{\kappa(\mathfrak{p})} \geq m_{\kappa(\mathfrak{P})}$.
Given an $R$-algebra $S$, consider the abelian category $\mathrm{PShv}^S(\mathcal{H})$ of $R$-linear functors from $\mathcal{H}^{\mathrm{op}}$ to the category of $S$-modules. Note that the Yoneda functor

$$\mathcal{H} \longrightarrow \mathrm{PShv}^S(\mathcal{H}) \qquad a \mapsto (c \mapsto \mathrm{Hom}_{\mathcal{H}}(c, a) \otimes_R S) \tag{5.1}$$

induces a fully faithful embedding of $\mathcal{H} \otimes_R S$ into the full subcategory of $\mathrm{PShv}^S(\mathcal{H})$ consisting of projective objects; see [24, Lem. 8.1]. Note also that every $R$-algebra homomorphism $S \to S'$ gives rise to a functor $- \otimes_S S' \colon \mathrm{PShv}^S(\mathcal{H}) \to \mathrm{PShv}^{S'}(\mathcal{H})$. Since $\mathrm{PShv}^S(\mathcal{H})$ is abelian, Proposition 3.5(ii) yields a homological functor

$$\mathrm{H}_0^S \colon \mathcal{T} \longrightarrow \mathrm{PShv}^S(\mathcal{H}) \qquad a \mapsto \big(c \mapsto H^0(\mathrm{Hom}_{\mathcal{H}}(c, t(a)) \otimes_R S)\big) \; .$$

Recall from assumption (A2) that the functor $\iota_{\kappa(\mathfrak{p})}$ induces a $\otimes$-equivalence of categories $\mathrm{Kar}(\mathcal{H} \otimes_R \kappa(\mathfrak{p})) \simeq \mathcal{H}_{\kappa(\mathfrak{p})}$. This implies that $\mathrm{H}_0^{\kappa(\mathfrak{p})}$ factors through $\iota_{\kappa(\mathfrak{p})}$. Consequently, thanks to Remark 3.6, we have $\mathrm{H}_n^{\kappa(\mathfrak{p})}(b) = 0$ for every $n \neq m_{\kappa(\mathfrak{p})}$.

Let us denote by $Q$ the localization of $R/\mathfrak{P}$ at the prime ideal $\mathfrak{p}$. Note that $Q$ is a local Noetherian integral domain with fraction field $\kappa(\mathfrak{P})$. Recall from assumption (A1) that the commutative ring $R$ is Noetherian and that the $R$-modules of morphisms of the heart $\mathcal{H}$ are finitely generated and flat. Thanks to the universal coefficients theorem, this implies that $\mathrm{H}_l^Q(b) \otimes_Q \kappa(\mathfrak{p}) = \mathrm{H}_l^{\kappa(\mathfrak{p})}(b)$, with $l$ being the largest integer such that $\mathrm{H}_l^Q(b) \neq 0$. Consequently, by applying the Nakayama lemma to the local ring $Q$ and to the (objectwise) finitely generated $Q$-module $\mathrm{H}_l^Q(b)$, we conclude that $\mathrm{H}_l^{\kappa(\mathfrak{p})}(b) \neq 0$. Hence, the equality $m_{\kappa(\mathfrak{p})} = l$ holds. Now, since $\kappa(\mathfrak{P})$ is a flat $Q$-module, the universal coefficients theorem yields that $\mathrm{H}_n^{\kappa(\mathfrak{P})}(b) = 0$ for every $n > l$. This allows us to conclude that $l = m_{k(\mathfrak{p})} \geq m_{k(\mathfrak{P})}$.

Let us now address the second goal, *i.e.* prove that $b \in \mathcal{T}^{w=m}$ with $m := m_{k(\mathfrak{p})}$. Making use of Remark 3.2 once again, we observe that it suffices to prove that $b \in \mathcal{T}^{w \leq m}$. Thanks to Proposition 5.2 below, it is enough to verify that $\mathrm{H}_n^R(b) = 0$ for every $n > m$. Let us denote by $l$ the largest integer such that $\mathrm{H}_l^R(b) \neq 0$. An argument similar to the one used in the preceding paragraph, implies that $\mathrm{H}_l^R(b) \otimes_R \kappa(\mathfrak{p}) = \mathrm{H}_l^{\kappa(\mathfrak{p})}(b)$ for every $\mathfrak{p} \in \mathrm{Spec}(R)$. Since $\mathrm{H}_n^{\kappa(\mathfrak{p})}(b) = 0$ for all $n > m$ and $\mathfrak{p} \in \mathrm{Spec}(R)$, we then conclude that $\mathrm{H}_n^R(b) = 0$ for every $n > m$. This finishes the proof.

Proposition 5.2 (Conservativity II). *Let $\mathcal{T}$ be a triangulated category equipped with a bounded weight structure $w$ whose heart $\mathcal{H}$ is $R$-linear and small. Consider the associated homological functors $\mathrm{H}_n^R \colon \mathcal{T} \to \mathrm{PShv}^R(\mathcal{H}), n \in \mathbb{Z}$. Under these assumptions, an object $b \in \mathcal{T}$ belongs to $\mathcal{T}^{w \leq m}$ if and only if $\mathrm{H}_n^R(b) = 0$ for every $n > m$.*

*Proof.* Combine [8, Prop. 2.3.4] with [8, Rk. 2.3.6(2)]. □

*Remark* 5.3. (i) Suppose that in Theorem 1.2 the commutative ring $R$ is of the form $\Pi_{j=1}^r R_j$, with $R_j$ an indecomposable Noetherian ring. In this case, the corresponding idempotents $e_j \in R$ give naturally rise to categorical decompositions $\mathcal{T} \simeq \Pi_{j=1}^r \mathcal{T}_j$ and $\mathcal{H} \simeq \Pi_{j=1}^r \mathcal{H}_j$. By applying Theorem 1.2 to each one of the categories $\mathcal{T}_j$, we conclude that

$$\mathrm{Pic}(\mathcal{T}) \simeq \Pi_{j=1}^r \mathrm{Pic}(\mathcal{T}_j) \simeq \Pi_{j=1}^r (\mathrm{Pic}(\mathcal{H}_j) \times \mathbb{Z}) \simeq \mathrm{Pic}(\mathcal{H}) \times \mathbb{Z}^r$$

whenever all the triangulated categories $\mathcal{T}_j$ are non-zero;

(ii) At assumption (A2) of Theorem 1.2, instead of working with all prime ideals $\mathfrak{p} \in \mathrm{Spec}(R)$, note that it suffices to consider any connected subset of $\mathrm{Spec}(R)$ that contains all maximal ideals of $R$. For example, in the particular case where $R$ is local, it suffices to consider the (unique) closed point $\mathfrak{p}_o$ of $\mathrm{Spec}(R)$.

## 6. Proof of Theorem 2.3

Recall from [24, Part 4 and Lecture 20][39] the construction of the symmetric monoidal triangulated category $\mathrm{DM}_{\mathrm{gm}}(k;R)$. Given any two zero-dimensional smooth $k$-schemes $X$ and $Y$, we have trivial positive Ext-groups:

$$\mathrm{Hom}_{\mathrm{DMA}(k;R)}(M(X)_R, M(Y)_R[n]) = 0 \qquad n > 0\,.$$

This implies that the subcategory $\mathrm{AM}(k;R) \subset \mathrm{DMA}(k;R)$ is *negative* in the sense of Proposition 3.8. Consequently, the subcategory $\mathcal{A}(k;R) \subset \mathrm{D}\mathcal{A}(k;R)$ is also negative. Making use of Proposition 3.8, we then conclude that the $\mathrm{D}\mathcal{A}(k;R)$ carries a bounded weight structure $w_R$ with heart $\mathcal{A}(k;R)$.

Let us now show that the category $\mathrm{D}\mathcal{A}(k;R)$ has the $w_R$-Picard property; note that this automatically concludes the proof. By construction, $\mathcal{A}(k;R)$ is essentially small. Moreover, we have natural isomorphisms

$$\mathrm{Hom}_{\mathrm{D}\mathcal{A}(k;R)}(M(X)_R, M(Y)_R) \simeq CH^0(X \times Y)_R\,.$$

Since the $R$-modules $CH^0(X\times Y)_R$ are free, assumptions (A1) of Theorem 1.2 are verified. In what concerns assumptions (A2), take for $\mathcal{T}_{\kappa(\mathfrak{p})}$ the category $\mathrm{D}\mathcal{A}(k;\kappa(\mathfrak{p}))$ and for $\iota_{\kappa(\mathfrak{p})}$ the functor $-\otimes_R \kappa(\mathfrak{p})\colon \mathrm{D}\mathcal{A}(k;R) \to \mathrm{D}\mathcal{A}(k;\kappa(\mathfrak{p}))$. By construction, the latter functor is weight-exact (see Remark 3.4), symmetric monoidal, and induces an equivalence of symmetric monoidal categories between $\mathrm{Kar}(\mathcal{A}(k;R)\otimes_R \kappa(\mathfrak{p}))$ and $\mathcal{A}(k;\kappa(\mathfrak{p}))$. This shows that assumptions (A2) are also verified.

Let us now prove that the categories $\mathrm{D}\mathcal{A}(k;\kappa(\mathfrak{p}))$ have the $w_{\kappa(\mathfrak{p})}$-Picard property; thanks to Theorem 1.2 this implies that $\mathrm{D}\mathcal{A}(k;R)$ has the $w_R$-Picard property. In order to do so, we will make use of Theorem 1.1. Concretely, we will prove that the categories $\mathcal{A}(k;\kappa(\mathfrak{p}))$ are abelian semi-simple and local. Let us write $L$ for the composite of the finite separable field extensions $l_i/k, i \in I$, inside $\overline{k}$, $\mathrm{G}$ for the profinite Galois group $\mathrm{Gal}(L/k)$, and $\mathrm{G}_i$ for the finite Galois group $\mathrm{Gal}(l_i/k)$. Thanks to assumption (B1), there is a $\otimes$-equivalence between $\mathcal{A}(k;\kappa(\mathfrak{p}))$ and the category of finite dimensional $\kappa(\mathfrak{p})$-linear continuous $\mathrm{G}$-representations $\mathrm{Rep}_{\kappa(\mathfrak{p})}(\mathrm{G})$. Consequently, since $\mathrm{G} \simeq \lim_{i\in I}\mathrm{G}_i$, we conclude that $\mathcal{A}(k;\kappa(\mathfrak{p})) \simeq \mathrm{colim}_{i\in I}\mathrm{Rep}_{\kappa(\mathfrak{p})}(\mathrm{G}_i)$. Now, since the group $\mathrm{G}_i$ is finite, the category $\mathrm{Rep}_{\kappa(\mathfrak{p})}(\mathrm{G}_i)$ may be identified with the category of finitely generated (right) $\kappa(\mathfrak{p})[\mathrm{G}_i]$-modules. Thanks to assumption (B2), the degree of the field extension $l_i/k$ is invertible in $R$ and hence in $\kappa(\mathfrak{p})$. The (classical) Maschke theorem then implies that the category $\mathrm{Rep}_{\kappa(\mathfrak{p})}(\mathrm{G}_i)$ is abelian semi-simple. Note that this category is moreover local since the tensor product is defined on the underlying $\kappa(\mathfrak{p})$-vector spaces. The proof follows now automatically from the above description of the categories $\mathcal{A}(k;\kappa(\mathfrak{p}))$.

## 7. PROOF OF THEOREM 2.6

Let us denote by $\mathcal{A}\mathrm{T}(k;R)$ the smallest additive, Karoubian, full subcategory of $\mathrm{D}\mathcal{A}\mathrm{T}(k;R)$ containing the objects $M(X)_R(m)[2m]$, with $M(X)_R \in \mathcal{A}$ and $m \in \mathbb{Z}$. Under these notations, we have trivial positive Ext-groups:

$$\mathrm{Hom}_{\mathrm{D}\mathcal{A}\mathrm{T}(k;R)}(M(X)_R(m)[2m], M(Y)_R(m')[2m'][n]) = 0 \qquad n > 0\,.$$

This implies that the subcategory $\mathcal{A}\mathrm{T}(k;R) \subset \mathrm{D}\mathcal{A}\mathrm{T}(k;R)$ is negative in the sense of Proposition 3.8. The motives of the zero-dimensional smooth $k$-schemes, as well as the Tate motives, are stable under tensor product. Therefore, $\mathcal{A}\mathrm{T}(k;R)$ generates[7] the triangulated category $\mathrm{D}\mathcal{A}\mathrm{T}(k;R)$. Making use of Proposition 3.8 once again, we then conclude that $\mathrm{D}\mathcal{A}\mathrm{T}(k;R)$ carries a bounded weight structure $w_R$ with heart $\mathcal{A}\mathrm{T}(k;R)$. Thanks to the equivalence of categories

$$\mathrm{Gr}_{\mathbb{Z}}\mathcal{A}(k;R) \xrightarrow{\simeq} \mathcal{A}\mathrm{T}(k;R) \qquad \{M(X_m)\}_{m\in\mathbb{Z}} \mapsto \bigoplus_{m\in\mathbb{Z}} M(X_m)(m)[2m]\,,$$

an argument similar to the one of the proof of Theorem 2.3 implies that the category $\mathrm{D}\mathcal{A}\mathrm{T}(k;R)$ has the $w_R$-Picard property. Consequently, we have $\mathrm{Pic}(\mathrm{D}\mathcal{A}\mathrm{T}(k;R)) \simeq \mathrm{Pic}(\mathcal{A}\mathrm{T}(k;R)) \times \mathbb{Z}$. The proof follows now from the natural isomorphisms

$$\mathrm{Pic}(\mathcal{A}\mathrm{T}(k;R)) \simeq \mathrm{Pic}(\mathrm{Gr}_{\mathbb{Z}}\mathcal{A}(k;R)) \simeq \mathrm{Pic}(\mathcal{A}(k;R)) \times \mathbb{Z}\,.$$

## 8. PROOF OF THEOREM 2.11

Recall from Ayoub [2, §4][3, §2.1.1] the construction of the symmetric monoidal triangulated category $\mathbf{DA}(k;\mathbb{Z})$ (with respect to the Nisnevich topology); in what follows, we write $\mathrm{Boot}(k;\mathbb{Z})$ for the thick triangulated subcategory generated by the $\otimes$-unit $\Sigma^\infty(\mathrm{Spec}(k)_+)_{\mathbb{Z}}$. By construction, we have an exact symmetric monoidal functor $(-)_{\mathbb{Z}}\colon \mathrm{SH}(k) \to \mathbf{DA}(k;\mathbb{Z})$ which restricts to the bootstrap categories. Let $\mathrm{P}(k)$, resp. $\mathrm{P}(k;\mathbb{Z})$, be the smallest additive, Karoubian, full subcategory of $\mathrm{Boot}(k)$, resp. $\mathrm{Boot}(k;\mathbb{Z})$, containing the $\otimes$-unit $\Sigma^\infty(\mathrm{Spec}(k)_+)$, resp. $\Sigma^\infty(\mathrm{Spec}(k)_+)_{\mathbb{Z}}$. We have trivial positive Ext-groups (see [40, Thm. 4.14]):

$$\mathrm{Hom}_{\mathrm{Boot}(k)}(\Sigma^\infty(\mathrm{Spec}(k)_+), \Sigma^\infty(\mathrm{Spec}(k)_+)[n]) = 0 \quad n > 0\,;$$

similarly for $\mathrm{Boot}(k;\mathbb{Z})$. This implies that the subcategory $\mathrm{P}(k) \subset \mathrm{Boot}(k)$, resp. $\mathrm{P}(k;\mathbb{Z}) \subset \mathrm{Boot}(k;\mathbb{Z})$, is negative in the sense of Proposition 3.8. Making use of this latter proposition, we then conclude that the category $\mathrm{Boot}(k)$, resp. $\mathrm{Boot}(k;\mathbb{Z})$, carries a bounded weight structure $w$, resp. $w_{\mathbb{Z}}$, with heart $\mathrm{P}(k)$, resp. $\mathrm{P}(k;\mathbb{Z})$.

[7] *i.e.* the smallest thick triangulated subcategory containing $\mathcal{A}\mathrm{T}(k;R)$ is $\mathrm{D}\mathcal{A}\mathrm{T}(k;R)$.

Let us now show that the category $\mathrm{Boot}(k)$ has the $w$-Picard property. Thanks to the ring isomorphism (2.10), $\mathrm{P}(k)$ identifies with the category $\mathrm{Proj}(GW(k))$ of finitely generated projective $GW(k)$-modules. Moreover, the functor $(-)_{\mathbb{Z}}$ restricts to an equivalence of categories $\mathrm{P}(k) \stackrel{\simeq}{\to} \mathrm{P}(k;\mathbb{Z})$; this is an immediate consequence of [9, Prop. 2.3.7] (this equivalence also follows easily from [27, Thm. 6.37]). Consequently, since the Grothendieck-Witt ring $GW(k)$ is indecomposable (see [18, Prop. 2.2]), all the assumptions (A1) of Theorem 1.2 (with $R = GW(k)$) are verified. In what concerns assumptions (A2), take for $\mathcal{T}_{\kappa(\mathfrak{p})}$ the bounded derived category $\mathcal{D}^b(\kappa(\mathfrak{p}))$ of finite dimensional $\kappa(\mathfrak{p})$-vector spaces $\mathrm{Vect}(\kappa(\mathfrak{p}))$ and for $\iota_{\kappa(\mathfrak{p})}$ the composed functor:

$$\mathrm{Boot}(k) \xrightarrow{(-)_{\mathbb{Z}}} \mathrm{Boot}(k;\mathbb{Z}) \xrightarrow{t(-)} K^b(\mathrm{Proj}(GW(k))) \xrightarrow{-\otimes_{GW(k)}\kappa(\mathfrak{p})} \mathcal{D}^b(\kappa(\mathfrak{p}))\,. \tag{8.1}$$

Some explanations are in order: since the category $\mathbf{DA}(k;\mathbb{Z})$ is defined as the localization of a certain category of complexes, it admits a tensor differential graded (=dg) enhancement. Making use of [4, Lem. 18], we then conclude that the weight complex construction gives rise to an exact symmetric monoidal functor $t(-)$ with values in the bounded homotopy category of $\mathrm{Proj}(GW(k))$. By construction, the composed functor (8.1) is weight-exact, symmetric monoidal, and induces a $\otimes$-equivalence of categories between $\mathrm{Kar}(\mathrm{P}(k)\otimes_{GW(k)}\kappa(\mathfrak{p}))$ and $\mathrm{Vect}(\kappa(\mathfrak{p}))$. This shows that the assumptions (A2) are also verified. Finally, since the categories $\mathcal{D}^b(\kappa(\mathfrak{p}))$ clearly have the $w_{\kappa(\mathfrak{p})}$-Picard property, we conclude from Theorem 1.2 that $\mathrm{Boot}(k)$ has the $w$-Picard property. This finishes the proof.

## 9. Proof of Theorem 2.17

Recall from [34, §9][33, §4] the construction of the symmetric monoidal triangulated category $\mathrm{KMM}(k;R)$. Given any two finite separable field extensions $l_1/k$ and $l_2/k$, we have trivial positive Ext-groups (see [33, Prop. 4.4]):

$$\mathrm{Hom}_{\mathrm{NMAM}(k;R)}(U(l_1)_R, U(l_2)_R[n]) \simeq \pi_{-n}(K(l_1\otimes_k l_2)\wedge HR) = 0 \qquad n>0\,.$$

This implies that the subcategory $\mathrm{AM}(k;R)\subset \mathrm{NMAM}(k;R)$ is negative in the sense of Proposition 3.8. Consequently, the subcategory $\mathcal{A}(k;R)\subset \mathrm{NM}\mathcal{A}(k;R)$ is also negative. Making use of Proposition 3.8, we then conclude that the category $\mathrm{NM}\mathcal{A}(k;R)$ carries a bounded weight structure[8] $w_R$ with heart $\mathcal{A}(k;R)$. Now, a proof similar to the one of Theorem 2.3, with $\mathrm{D}\mathcal{A}(k;R)$ and $\mathrm{D}\mathcal{A}(k;\kappa(\mathfrak{p}))$ replaced by $\mathrm{NM}\mathcal{A}(k;R)$ and $\mathrm{NM}\mathcal{A}(k;\kappa(\mathfrak{p}))$, respectively, allows us to conclude that the category $\mathrm{NM}\mathcal{A}(k;R)$ has the $w_R$-Picard property. Consequently, we have $\mathrm{Pic}(\mathrm{NM}\mathcal{A}(k;R))\simeq \mathrm{Pic}(\mathcal{A}(k;R))\times\mathbb{Z}$.

[8] A bounded weight structure on the category of noncommutative mixed motives was originally constructed in [36, Thm. 1.1].

## 10. Proof of Theorem 2.22

Item (i). Similarly to the proof of Theorem 2.17, given any two central simple $k$-algebras $A$ and $B$, we have trivial positive Ext-groups (see [33, Prop. 4.4]):

$$\mathrm{Hom}_{\mathrm{NMCSA}(k;R)}(U(A)_R, U(B)_R[n]) \simeq \pi_{-n}(K(A^{\mathrm{op}} \otimes_k B) \wedge HR) = 0 \quad n > 0\,.$$

This implies that the subcategory $\mathrm{CSA}(k;R) \subset \mathrm{NMCSA}(k;R)$ is negative in the sense of Proposition 3.8. Making use of this latter proposition, we then conclude that $\mathrm{NMCSA}(k;R)$ carries a bounded weight structure $w_R$ with heart $\mathrm{CSA}(k;R)$.

Let us now show that the category $\mathrm{NMCSA}(k;R)$ has the $w_R$-Picard property. By construction, the category $\mathrm{CSA}(k;R)$ is essentially small. Moreover, since the $K$-theory spectrum $K(A^{\mathrm{op}} \otimes_k B)$ is connective, we have natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathrm{CSA}(k;R)}(U(A)_R, U(B)_R) &\simeq \pi_0(K(A^{\mathrm{op}} \otimes_k B) \wedge HR) \\ &\simeq \pi_0(K(A^{\mathrm{op}} \otimes_k B)) \otimes_{\mathbb{Z}} R \qquad (10.1) \\ &\simeq K_0(A^{\mathrm{op}} \otimes_k B) \otimes_{\mathbb{Z}} R \simeq R\,, \end{aligned}$$

where (10.1) follows from the stable Hurewicz theorem. This implies, in particular, that the assumptions (A1) of Theorem 1.2 are verified. In what concerns assumptions (A2), take for $\mathcal{T}_{\kappa(\mathfrak{p})}$ the category $\mathrm{NMCSA}(k;\kappa(\mathfrak{p}))$ and for $\iota_{\kappa(\mathfrak{p})}$ the functor $-\otimes_R \kappa(\mathfrak{p})\colon \mathrm{NMCSA}(k;R) \to \mathrm{NMCSA}(k;\kappa(\mathfrak{p}))$. By construction, the latter functor is weight-exact (see Remark 3.4), symmetric monoidal, and induces an equivalence of symmetric monoidal categories between $\mathrm{Kar}(\mathrm{CSA}(k;R) \otimes_R \kappa(\mathfrak{p}))$ and $\mathrm{CSA}(k;\kappa(\mathfrak{p}))$. This shows that the assumptions (A2) are also verified.

We now claim that the categories $\mathrm{NMCSA}(k;\kappa(\mathfrak{p}))$ have the $w_{\kappa(\mathfrak{p})}$-Picard property; thanks to Theorem 1.2 this implies that the category $\mathrm{NMCSA}(k;R)$ has the $w_R$-Picard property. Since the categories of finite dimensional (graded) vector spaces are local, our claim follows then from the combination of Theorem 1.1 with the following general result (with $R = \kappa(\mathfrak{p})$):

Proposition 10.2. *Let $R$ be a field.*

(a) *When* $\mathrm{char}(R) = 0$*, the category* $\mathrm{CSA}(k;R)$ *is* $\otimes$*-equivalent to the category of finite dimensional $R$-vector spaces* $\mathrm{vect}(R)$*;*

(b) *When* $\mathrm{char}(R) = p > 0$*, there exists a full, additive, conservative, symmetric monoidal functor from* $\mathrm{CSA}(k;R)$ *into the category of finite dimensional* $\mathrm{Br}(k)\{p\}$*-graded $R$-vector spaces* $\mathrm{Gr}_{\mathrm{Br}(k)\{p\}}\mathrm{vect}(R)$*.*

*Proof.* Given an $R$-linear, additive, Karoubian, rigid[9] symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$, with $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) \simeq R$, recall from [1, §1.4.1 and §1.7.1] the construction of the following categorical ideals

$$\begin{aligned} \mathcal{N}(a,b) &:= \{f\colon a \to b \,|\, \forall g\colon b \to a \;\; \mathrm{tr}(g \circ f) = 0\} \\ \mathcal{R}(a,b) &:= \{f\colon a \to b \,|\, \forall g\colon b \to a \;\; \mathrm{id}_a - (g \circ f) \text{ is invertible}\}\,, \end{aligned}$$

[9] Recall that a symmetric monoidal category is called *rigid* if all its objects are dualizable.

where $\mathrm{tr}(g \circ f)$ stands for the categorical trace of the endomorphism $g \circ f$. As explained in *loc. cit.*, the categorical ideal $\mathcal{N}$ is moreover symmetric monoidal.
Item (a). As proved in [38, Thm. 2.1], we have $U(A)_R \simeq U(k)_R$ for every central simple $k$-algebra $A$. Using the natural ring isomorphism $\mathrm{End}(U(k)_R) \simeq R$, we then conclude that the category $\mathrm{CSA}(k;R)$ is $\otimes$-equivalent to the category of finite dimensional $R$-vector spaces $\mathrm{vect}(R)$.
Item (b). By construction, the category $\mathrm{CSA}(k;R)$ is $R$-linear, additive, and symmetric monoidal. Moreover, all its objects are dualizable and $\mathrm{End}(U(k)_R) \simeq R$; see [34, §1.7.1]. As proved in [32, Prop. 6.11], the quotient $\mathrm{CSA}(k;R)/\mathcal{N}$ is $\otimes$-equivalent to the category $\mathrm{Gr}_{\mathrm{Br}(k)\{p\}}\mathrm{vect}(R)$. Consequently, we have an induced full, additive, and symmetric monoidal functor

$$\mathrm{CSA}(k;R) \longrightarrow \mathrm{Gr}_{\mathrm{Br}(k)\{p\}}\mathrm{vect}(R)\,. \tag{10.3}$$

It remains then only to prove that the functor (10.3) is moreover conservative. In order to do so, we will show the inclusion $\mathcal{N} \subseteq \mathcal{R}$. Thanks to [1, Prop. 7.1.6], this implies that the quotient functor (10.3) is conservative. By definition, the categorical ideals $\mathcal{N}$ and $\mathcal{R}$ are compatible with direct sums and summands. Hence, given central simple $k$-algebras $A$ and $B$, it suffices to show that the inclusion $\mathcal{N}(U(A)_R, U(B)_R) \subseteq \mathcal{R}(U(A)_R, U(B)_R)$ holds. This inclusion follows now from the combination of the definitions of $\mathcal{N}$ and $\mathcal{R}$ with Lemma 10.4 below. □

Lemma 10.4. *Given a central simple $k$-algebra $A$, the following morphism*

$$\mathrm{End}_{\mathrm{CSA}(k;R)}(U(A)_R) \longrightarrow \mathrm{End}_{\mathrm{CSA}(k;R)}(U(k)_R) \simeq R \quad h \mapsto \mathrm{tr}(h)\,, \tag{10.5}$$

*induced by the categorical trace construction, is invertible.*

*Proof.* By construction, the induced morphism (10.5) is $R$-linear. Therefore, thanks to the natural isomorphism $\mathrm{End}(U(A)_R) \simeq R$, (10.5) is completely determined by the image of the identity of $U(A)_R$. In other words, (10.5) reduces to the morphism $R \to R, r \mapsto r \cdot \chi(U(A)_R)$, where $\chi(U(A)_R)$ stands for the Euler characteristic of the noncommutative motive $U(A)_R$. As proved in [34, Prop. 2.24], the Euler characteristic $\chi(U(A)_R)$ agrees with the Grothendieck class $[HH(A)]_R \in K_0(k)_R \simeq R$ of the Hochschild homology $HH(A)$ of $A$. Since $HH(A) \simeq A/[A,A] \simeq k$ (see [23, §1.2.12]), we then conclude that (10.5) is the identity. This finishes the proof. □

*Remark* 10.6. It follows from the proof of Proposition 10.2 that the Brauer group of the symmetric monoidal category $\mathrm{CSA}(k;R)$ is trivial when $\mathrm{char}(R) = 0$ and isomorphic to $\mathrm{Br}(k)\{p\}$ when $\mathrm{char}(k) = p > 0$.

Item (ii). Thanks to equivalence (2.20), we have an injective group homomorphism

$$\mathrm{Br}(k) \longrightarrow \mathrm{Pic}(\mathrm{CSA}(k;\mathbb{Z})) \qquad [A] \mapsto U(A)_{\mathbb{Z}}\,. \tag{10.7}$$

It remains then only to prove that (10.7) is moreover surjective. Recall from [34, §9][36] the construction of the symmetric monoidal triangulated category

KMM($k$) and of the full subcategories NMCSA($k$) and CSA($k$). By construction, we have an exact symmetric monoidal functor $(-)_{\mathbb{Z}}\colon \mathrm{KMM}(k) \to \mathrm{KMM}(k;\mathbb{Z})$ which restricts to a $\otimes$-equivalence $\mathrm{CSA}(k) \simeq \mathrm{CSA}(k;\mathbb{Z})$. Therefore, making use [37, Thm. 2.20(iv)], we observe that the objects $U(A_1)_{\mathbb{Z}} \oplus \cdots \oplus U(A_m)_{\mathbb{Z}}$ of CSA($k;\mathbb{Z}$), with $m > 1$ are *not* $\otimes$-invertible. Since the category CSA($k;\mathbb{Z}$) is Karoubian (see [37, Thm. 2.20(i)]), we then conclude that (10.7) is moreover surjective.

*Remark* 10.8. Given any two central simple $k$-algebras $A$ and $B$, we have

$$\mathrm{Hom}_{\mathrm{NMCSA}(k)}(U(A)_R, U(B)_R[n]) \simeq K_{-n}(A^{\mathrm{op}} \otimes_k B) = 0 \qquad n > 0\,.$$

Therefore, a proof similar to the one of Theorem 2.22, with NMCSA($k;\mathbb{Z}$) replaced by NMCSA($k$), allows us to conclude that $\mathrm{Pic}(\mathrm{NMCSA}(k)) \simeq \mathrm{Br}(k) \times \mathbb{Z}$. In conclusion, although the categories NMCSA($k$) and NMCSA($k;\mathbb{Z}$) are not equivalent, they have nevertheless the same Picard group!

## 11. Proof of Theorem 2.24

Let us denote by P($E$) the smallest additive, Karoubian, full subcategory of $\mathcal{D}_c(E)$ containing the $E$-module $E$. Since by assumption the ring spectrum $E$ is connective, we have trivial positive Ext-groups:

$$\mathrm{Hom}_{\mathcal{D}_c(E)}(E, E[n]) \simeq \pi_{-n}(E) = 0 \qquad n > 0\,.$$

This implies that the subcategory $\mathrm{P}(E) \subset \mathcal{D}_c(E)$ is negative in the sense of Proposition 3.8. Making use of this latter proposition, we then conclude that the category $\mathcal{D}_c(E)$ carries a bounded weight structure $w$ with heart P($E$).
Let us now show that the category $\mathcal{D}_c(E)$ has the $w$-Picard property. By construction, P($E$) identifies with the category of finitely generated projective $\pi_0(R)$-modules. Therefore, by taking $R := \pi_0(E)$, all the assumptions (A1) of Theorem 1.2 are verified. In what concerns assumptions (A2), take for $\mathcal{T}_{k(\mathfrak{p})}$ the category $\mathcal{D}^b(k(\mathfrak{p}))$, equipped with the canonical bounded weight structure with heart Vect($k(\mathfrak{p})$), and for $\iota_{k(\mathfrak{p})}$ the (composed) base-change functor

$$\mathcal{D}_c(E) \xrightarrow{-\wedge_E H\pi_0(E)} \mathcal{D}_c(H\pi_0(E)) \simeq \mathcal{D}_c(R) \xrightarrow{-\otimes_R k(\mathfrak{p})} \mathcal{D}^b(k(\mathfrak{p}))\,.$$

By construction, the latter functor is weight-exact (see Remark 3.4), symmetric monoidal, and induces a $\otimes$-equivalence of categories between $\mathrm{Kar}(\mathrm{P}(E)\otimes_R \kappa(\mathfrak{p}))$ and Vect($k(\mathfrak{p})$). Since the categories $\mathcal{D}^b(k(\mathfrak{p}))$ clearly have the $w_{k(\mathfrak{p})}$-property, we conclude from Theorem 1.2 that $\mathcal{D}_c(E)$ has the $w$-Picard property.
Finally, since the category $\mathcal{D}_c(E)$ has the $w$-Picard property, we have an isomorphism $\mathrm{Pic}(\mathcal{D}_c(E)) \simeq \mathrm{Pic}(\mathrm{P}(E)) \times \mathbb{Z}$. The proof follows now from the fact that $\mathrm{Pic}(\mathrm{P}(E))$ is isomorphic to $\mathrm{Pic}(\pi_0(E))$.

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informing us that some related results, concerning the categories $\mathrm{DM}_{\mathrm{gm}}(k;R)$ and $\mathrm{SH}(k)$ and whose proofs use others methods, will appear in his Ph.D. thesis. Finally, the authors would like to thank the anonymous referee for his/her comments.

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# Large Tilting Sheaves over Weighted Noncommutative Regular Projective Curves

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Abstract. Let $\mathbb{X}$ be a weighted noncommutative regular projective curve over a field $k$. The category $\operatorname{Qcoh}\mathbb{X}$ of quasicoherent sheaves is a hereditary, locally noetherian Grothendieck category. We classify all tilting sheaves which have a non-coherent torsion subsheaf. In case of nonnegative orbifold Euler characteristic we classify all large (that is, non-coherent) tilting sheaves and the corresponding resolving classes. In particular we show that in the elliptic and in the tubular cases every large tilting sheaf has a well-defined slope.

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Contents

## 1. Introduction

Tilting theory is a well-established technique to relate different mathematical theories. An overview of its role in various areas of mathematics can be found in [4]. One of the first results along these lines, due to Beilinson [17], establishes a connection between algebraic geometry and representation theory of finite dimensional algebras. For instance, the projective line $\mathbb{X} = \mathbb{P}_1(k)$ over a field $k$ turns out to be closely related with the Kronecker algebra $\Lambda$, the path algebra of the quiver $\bullet \rightrightarrows \bullet$ over $k$. The connection is provided by the vector bundle $T = \mathcal{O} \oplus \mathcal{O}(1)$, which is a tilting sheaf in $\operatorname{coh} \mathbb{X}$ with endomorphism ring $\Lambda$. The derived Hom-functor $\mathbf{R}\operatorname{Hom}(T, -)$ then defines an equivalence between the derived categories of $\operatorname{Qcoh} \mathbb{X}$ and $\operatorname{Mod} \Lambda$. There are many more such examples, where a noetherian tilting object $T$ in a triangulated category $\mathcal{D}$ provides an equivalence between $\mathcal{D}$ and the derived category of $\operatorname{End}(T)$. We refer to [27, 32, 30], and to [20, 40] for the context of Calabi-Yau and cluster categories.

The weighted projective lines introduced in [27], and their generalizations in [42], called noncommutative curves of genus zero in [38], provide the basic framework for the present article. They are characterized by the existence of a tilting bundle in the category of coherent sheaves $\operatorname{coh} \mathbb{X}$. In this case the corresponding (derived-equivalent) finite-dimensional algebras are the (concealed-) canonical algebras [56, 57, 44], an important class of algebras in representation theory. A particularly interesting and beautiful case is the so-called tubular case. Here every indecomposable coherent sheaf is semistable (with respect to the slope), and the semistable coherent sheaves of slope $q$ form a family of tubes, for every $q$ ([45, 38]). This classification is akin to Atiyah's classification of indecomposable vector bundles over an elliptic curve [12].

The tilting objects mentioned so far are small in the sense that they are noetherian objects, and that their endomorphism rings are finite-dimensional algebras. For arbitrary rings $R$ there is the notion of a (not necessarily noetherian or finitely generated) tilting module $T$, which was extended to Grothendieck categories in [23, 24].

Definition. An object $T$ in a Grothendieck category $\vec{\mathcal{H}}$ is called *tilting* if $T$ generates precisely the objects in $T^{\perp_1} = \{X \in \vec{\mathcal{H}} \mid \operatorname{Ext}^1(T, X) = 0\}$. The class $T^{\perp_1}$ is then called a *tilting class*.

Such "large" tilting objects in general do not produce derived equivalences in the way mentioned above. But they yield recollements of triangulated categories [15, 6, 21], still providing a strong relationship between the derived categories involved.

Large tilting modules occur frequently. For example, they arise when looking for complements to partial tilting modules, or when computing intersections of tilting classes given by classical tilting modules, and they parametrize resolving subcategories of finitely presented modules. We refer to [3] for a survey on these results.

Another reason for the interest in large tilting modules is their deep connection with localization theory. This is best illustrated by the example of a Dedekind domain $R$. The tilting modules over $R$ are parametrized by the subsets $V \subseteq$ Max-Spec $R$, and they arise from localizations at sets of simple modules. More precisely, the universal localization $R \hookrightarrow R_V$ at the simples supported in $V$ yields the tilting module $T_V = R_V \oplus R_V/R$, and the set $V = \emptyset$ corresponds to the regular module $R$, the only finitely generated tilting module [9, Cor. 6.12]. Similar results hold true in more general contexts. Over a commutative noetherian ring, the tilting modules of projective dimension one correspond to categorical localizations in the sense of Gabriel [8]. Over a hereditary ring, tilting modules parametrize universal localizations [2].
An interesting example is provided by the Kronecker algebra $\Lambda$. Here we have a complete analogy to the Dedekind case if we replace the maximal spectrum by the index set $\mathbb{X}$ of the tubular family $\mathbf{t} = \coprod_{x\in\mathbb{X}} \mathcal{U}_x$. Indeed, the infinite dimensional tilting modules are parametrized by the subsets $V \subseteq \mathbb{X}$, and they arise from localizations at sets of simple regular modules. Again, the universal localization $\Lambda \hookrightarrow \Lambda_V$ at the simple regular modules supported in $V$ yields the tilting module $T_V = \Lambda_V \oplus \Lambda_V/\Lambda$, and the set $V = \emptyset$ corresponds to the Lukas tilting module $\mathbf{L}$.
For arbitrary tame hereditary algebras, the classification of tilting modules is more complicated due to the possible presence of finite dimensional direct summands from non-homogeneous tubes. Infinite dimensional tilting modules are parametrized by pairs $(B, V)$ where $B$ is a so-called branch module, and $V$ is a subset of $\mathbb{X}$. The tilting module corresponding to $(B, V)$ has finite dimensional part $B$ and an infinite dimensional part which is of the form $T_V$ inside a suitable subcategory, see [10].

In the present paper, we tackle the problem of classifying large tilting objects in hereditary Grothendieck categories. In particular, we will consider the category Qcoh $\mathbb{X}$ of quasicoherent sheaves over a weighted noncommutative regular projective curve $\mathbb{X}$ over a field $k$, in the sense of [39]. We will discuss how the results described above for tame hereditary algebras extend to this more general setting.
As in module categories, a crucial role will be played by the following notion.

Definition. Let $\vec{\mathcal{H}}$ be a locally coherent Grothendieck category, and let $\mathcal{H}$ the class of finitely presented objects in $\vec{\mathcal{H}}$. We call a class $\mathscr{S} \subseteq \mathcal{H}$ *resolving* if it generates $\vec{\mathcal{H}}$ and has the following closure properties: $\mathscr{S}$ is closed under extensions, direct summands, and $S' \in \mathscr{S}$ whenever $0 \to S' \to S \to S'' \to 0$ is exact with $S$, $S'' \in \mathscr{S}$.

We will use [58] to show the following general existence result for tilting objects.

Theorem 1. [Theorem 4.4] *Let $\vec{\mathcal{H}}$ be a locally coherent Grothendieck category and $\mathscr{S} \subseteq \mathcal{H}$ be resolving with* $\mathrm{pd}(S) \leq 1$ *for all $S \in \mathscr{S}$. Then there is a tilting object $T$ in $\vec{\mathcal{H}}$ with $T^{\perp_1} = \mathscr{S}^{\perp_1}$.*

Tilting classes as above of the form $T^{\perp_1} = \mathscr{S}^{\perp_1}$ for some class $\mathscr{S}$ of finitely presented objects are said to be of *finite type*.

When $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$, the category of finitely presented objects $\mathcal{H} = \operatorname{coh}\mathbb{X}$ is given by the coherent sheaves, and we have

Theorem 2. [Theorem 4.14] *Let $\mathbb{X}$ be a weighted noncommutative regular projective curve and $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$. The assignment $\mathscr{S} \mapsto \mathscr{S}^{\perp_1}$ defines a bijection between*

- *resolving classes $\mathscr{S}$ in $\mathcal{H}$, and*
- *tilting classes $T^{\perp_1}$ of finite type.*

In a module category, all tilting classes have finite type by [16]. In well behaved cases we can import this result to our situation. The complexity of the category $\operatorname{coh}\mathbb{X}$ of coherent sheaves over $\mathbb{X}$ depends on the orbifold Euler characteristic $\chi'_{orb}$. If $\chi'_{orb}(\mathbb{X}) > 0$, then the category $\operatorname{coh}\mathbb{X}$ is of (tame) domestic type, and it is derived-equivalent to the category $\operatorname{mod} H$ for a (finite-dimensional) tame hereditary algebra $H$. In this case, all tilting classes have finite type, and we obtain a complete classification of all large tilting sheaves (Theorem 6.5), which - not surprisingly - is very similar to the classification in [10]. But also in the tubular case, where $\mathbb{X}$ is weighted of orbifold Euler characteristic $\chi'_{orb}(\mathbb{X}) = 0$, tilting classes turn out to always have finite type.

Before we discuss our classification results, let us give some details on the tools we will employ. Our starting point is given by the following property, which is reminiscent of the well-known splitting property (2.1) for $\operatorname{coh}\mathbb{X}$.

Theorem 3. [Theorem 3.8] *Let $T \in \operatorname{Qcoh}\mathbb{X}$ be a sheaf with* $\operatorname{Ext}^1(T,T) = 0$. *Then there is a split exact sequence $0 \to tT \to T \to T/tT \to 0$ where $tT \subseteq T$ denotes the (largest) torsion subsheaf of $T$ and is a direct sum of finite length sheaves and of injective sheaves.*

This result shows that the classification of large (= non-coherent) tilting sheaves splits, roughly speaking, into two steps:

(i) The first is the classification of large tilting sheaves $T$ which are torsionfree (that is, with $tT = 0$). This seems to be a very difficult problem in general, but it turns out that in the cases when $\mathbb{X}$ is a noncommutative curve of genus zero which is of domestic or of tubular type, we get all these tilting sheaves with the help of Theorem 1.

(ii) If, on the other hand, the torsion part $tT$ of a large tilting sheaf $T$ is non-zero, then it is quite straightforward to determine the shape of $tT$; it is a direct sum of Prüfer sheaves and a certain so-called branch sheaf $B$, which is coherent. We can then apply perpendicular calculus to $B$, in order to reduce the problem to the case that $tT$ is a direct sum of Prüfer sheaves, or to $tT = 0$, which is the torsionfree case (i).

If Prüfer sheaves occur in the torsion part, then the corresponding torsionfree part is uniquely determined. This leads to the following, general result:

Theorem 4. [Corollary 4.12] *Let $\mathbb{X}$ be a weighted noncommutative regular projective curve. The tilting sheaves in* $\operatorname{Qcoh}\mathbb{X}$ *which have a non-coherent torsion subsheaf are up to equivalence in bijective correspondence with pairs $(B, V)$, where $V$ is a non-empty subset of $\mathbb{X}$ and $B$ is a branch sheaf.*

We will see in Section 5 that the tilting sheaf corresponding to $(B, V)$ has coherent part $B$ and a non-coherent part $T_V$ formed inside a suitable perpendicular subcategory, the categorical counterpart of universal localization. In particular, the torsionfree part of $T_V$ can be interpreted as a projective generator of the quotient category obtained from $\operatorname{Qcoh}\mathbb{X}$ by localization at the simple objects supported in $V$. Of course, there are also tilting sheaves given by pairs $(B, V)$ with $V = \emptyset$. Here the non-coherent part is the Lukas tilting sheaf inside a suitable subcategory, that is, it is given by the resolving class formed by all vector bundles. Altogether, the pairs $(B, V)$ correspond to Serre subcategories of $\operatorname{coh}\mathbb{X}$, and tilting sheaves are closely related with Gabriel localization, like in the case of tilting modules over commutative noetherian rings, cf. also [7, Sec. 5].
Let us now discuss the tubular case. Following [53], we define for every $w \in \mathbb{R}\cup\{\infty\}$ the class $\mathcal{M}(w)$ of quasicoherent sheaves of slope $w$. Reiten and Ringel have shown [53] that every indecomposable object has a well-defined slope. Our main result is as follows.

Theorem 5. [Theorem 8.6] *Let $\mathbb{X}$ be of tubular type. Then every large tilting sheaf in* $\operatorname{Qcoh}\mathbb{X}$ *has a well-defined slope $w$. If $w$ is irrational, then there is up to equivalence precisely one tilting sheaf of slope $w$. If $w$ is rational or $\infty$, then the large tilting sheaves of slope $w$ are classified like in the domestic case.*

In Section 9, we will briefly discuss the elliptic case, where $\chi'_{orb}(\mathbb{X}) = 0$ and $\mathbb{X}$ is non-weighted. Some of our main results will extend to this situation. In particular, Theorem 9.1 will resemble the tubular case described above. As it turns out, this will be much easier than in the (weighted) tubular case, using an Atiyah [12] type classification, namely, that all coherent sheaves lie in homogeneous tubes.
When the orbifold Euler characteristic $\chi'_{orb}(\mathbb{X}) \geq 0$, our results also yield a classification of certain resolving classes in $\operatorname{coh}\mathbb{X}$ (see Corollaries 6.7 and 8.7 and Theorem 9.1(5)). Furthermore, Theorem 4 enables us to recover and refine some results from [14] on maximal rigid objects in tube categories (Corollary 4.19).
If $\chi'_{orb}(\mathbb{X}) < 0$, then $\operatorname{coh}\mathbb{X}$ is wild. We stress that Theorem 4 also holds in this case, but we have not attempted to classify the torsionfree large tilting sheaves in the wild case.
There is one main difference to the module case. We recall that one of the standard characterising properties of a tilting module $T \in \operatorname{Mod} R$ is the existence of an exact sequence
$$0 \to R \to T_0 \to T_1 \to 0$$
with $T_0, T_1 \in \operatorname{Add}(T)$. In contrast to $\operatorname{Mod} R$, the category $\operatorname{Qcoh}\mathbb{X}$ lacks a projective generator. When $\mathbb{X}$ has genus zero, the replacement for the ring $R$

in our category is a tilting bundle $T_{\text{can}}$ whose endomorphism ring is a canonical algebra. Indeed, for every large tilting sheaf $T$ we can always find such a tilting bundle $T_{\text{can}}$ and a short exact sequence $0 \to T_{\text{can}} \to T_0 \to T_1 \to 0$, even with $T_0$, $T_1 \in \operatorname{add}(T)$. If $T$ has a non-coherent torsion part, then we can even do this with $\operatorname{Hom}(T_1, T_0) = 0$, cf. Theorem 10.1.

Since noncommutative curves of genus zero are derived-equivalent to canonical algebras in the sense of Ringel and Crawley-Boevey [57], our results are closely related to the classification of large tilting modules over canonical algebras. The module case is treated more directly in [7], where we also address the dual concept of cotilting modules and the classification of pure-injective modules.

## 2. Weighted noncommutative regular projective curves

In this section we collect some preliminaries on the category of quasicoherent sheaves we are going to study, and we introduce large tilting sheaves.

The main purpose of noncommutative algebraic geometry is to study abelian categories which have the same formal properties as $\operatorname{coh}(X)$ or $\operatorname{Qcoh}(X)$ for a scheme $X$. These categories are regarded as the geometric objects themselves, based on the Gabriel-Rosenberg reconstruction theorem which tells us that the scheme $X$ can be reconstructed from $\operatorname{coh}(X)$ or $\operatorname{Qcoh}(X)$. By analogy it is then convenient to use a similar terminology as for the objects of classical algebraic geometry. We refer to [64, Ch. 3].

Following this philosophy, we define the class of noncommutative curves which we will study in this paper by the axioms (NC 1) to (NC 5) below; the condition (NC 6) will follow from the others.

The axioms. A noncommutative curve $\mathbb{X}$ is given by a category $\mathcal{H}$ which is regarded as the category $\operatorname{coh}\mathbb{X}$ of *coherent sheaves* over $\mathbb{X}$. Formally it behaves like a category of coherent sheaves over a (commutative) regular projective curve over a field $k$ (we refer to [39]):

(NC 1) $\mathcal{H}$ is small, connected, abelian and every object in $\mathcal{H}$ is noetherian;

(NC 2) $\mathcal{H}$ is a $k$-category with finite-dimensional Hom- and Ext-spaces;

(NC 3) There is an autoequivalence $\tau$ on $\mathcal{H}$, called Auslander-Reiten translation, such that Serre duality

$$\operatorname{Ext}^1_{\mathcal{H}}(X, Y) = \operatorname{D}\operatorname{Hom}_{\mathcal{H}}(Y, \tau X)$$

holds, where $\operatorname{D} = \operatorname{Hom}_k(-, k)$. (In particular $\mathcal{H}$ is then hereditary.)

(NC 4) $\mathcal{H}$ contains an object of infinite length.

Splitting of coherent sheaves. Assume $\mathcal{H}$ satisfies (NC 1) to (NC 4). The following rough picture of the category $\mathcal{H}$ is very useful ([47, Prop. 1.1]). Every indecomposable coherent sheaf $E$ is either of finite length, or it is *torsionfree*, that is, it does not contain any simple sheaf; in the latter case $E$ is also called a (vector) bundle. We thus write

$$\mathcal{H} = \mathcal{H}_+ \vee \mathcal{H}_0, \tag{2.1}$$

with $\mathcal{H}_+ = \operatorname{vect}\mathbb{X}$ the class of vector bundles and $\mathcal{H}_0$ the class of sheaves of finite length; we have $\operatorname{Hom}(\mathcal{H}_0, \mathcal{H}_+) = 0$. Decomposing $\mathcal{H}_0$ in its connected components we have

$$\mathcal{H}_0 = \coprod_{x \in \mathbb{X}} \mathcal{U}_x,$$

where $\mathbb{X}$ is an index set (explaining the terminology $\mathcal{H} = \operatorname{coh}\mathbb{X}$) and every $\mathcal{U}_x$ is a connected uniserial length category.

Weighted noncommutative regular projective curves. Assume that $\mathcal{H}$ is a $k$-category satisfying properties (NC 1) to (NC 4) and the following additional condition.

(NC 5) $\mathbb{X}$ consists of infinitely many points.

Then we call $\mathbb{X}$ (or $\mathcal{H}$) a *weighted* (or *orbifold*) *noncommutative regular projective curve* over $k$. "Regular" can be replaced by "smooth" if $k$ is a perfect field; we refer to [39, Sec. 7]. We refer also to [47]; we excluded certain degenerate cases described therein by our additional axiom (NC 5). It is shown in [39] that a weighted noncommutative regular projective curve $\mathbb{X}$ satisfies automatically also the following condition.

(NC 6) For all points $x \in \mathbb{X}$ there are (up to isomorphism) precisely $p(x) < \infty$ simple objects in $\mathcal{U}_x$, and for almost all $x$ we have $p(x) = 1$.

The numbers $p(x)$ with $p(x) > 1$ are called the *weights*.

The "classical" case $\mathcal{H} = \operatorname{coh} X$ with $X$ a regular projective curve is included in this setting. This classical case is extended into two directions: (1) curves with a noncommutative function field $k(\mathbb{X})$ are allowed; here $k(\mathbb{X})$ is a skew field which is finite dimensional over its centre, which has the form $k(X)$ for a regular projective curve $X$; (2) additionally (a finite number of) weights are allowed.

Weighted noncommutative regular projective curves are *noncommutative smooth proper curves* in the sense of Stafford and van den Bergh [62, Sec. 7] (where $k$ is assumed to be algebraically closed); these categories were classified by Reiten and van den Bergh [52]. Indeed, our axioms (NC 1), (NC 2), (NC 3) are in accordance with the notion in [62]. By assuming additionally (NC 4) we avoid for instance categories which are just tubes.

Genus zero. We consider also the following condition.

(g-0) $\mathcal{H}$ admits a tilting object.

It is shown in [44] that then $\mathcal{H}$ even contains a torsionfree tilting object $T_{\mathrm{can}}$ whose endomorphism algebra is canonical, in the sense of [57]. We call such a tilting object *canonical*, or, by considering the full subcategory formed by the indecomposable summands of $T_{\mathrm{can}}$, *canonical configuration*, cf. 5.11. We recall that $T \in \mathcal{H}$ is called *tilting*, if $\operatorname{Ext}^1(T, T) = 0$, and if for all $X \in \mathcal{H}$ we have $X = 0$ whenever $\operatorname{Hom}(T, X) = 0 = \operatorname{Ext}^1(T, X)$. (This notion will be later generalized to quasicoherent sheaves.) If $\mathcal{H}$ satisfies (NC 1) to (NC 4) and (g-0), then we say that $\mathbb{X}$ is a *noncommutative curve of genus zero*; the condition (NC 5) is then automatically satisfied, we refer to [38]. The weighted projective lines, defined by Geigle-Lenzing [27], are special cases of noncommutative curves of genus zero. We remark that in the classical case $\mathcal{H} = \operatorname{coh}(X)$, where $X$ is a (commutative) regular projective curve with structure sheaf $\mathcal{O}$, the condition (g-0) is equivalent to $\operatorname{Ext}^1(\mathcal{O}, \mathcal{O}) = 0$, which means that $X$ is of (geometric) genus zero in the classical sense; cf. Remark 2.2.

The Grothendieck group and the Euler form. The Grothendieck group $\mathrm{K}_0(\mathcal{H})$ of $\mathcal{H}$ is defined as the factor of the free abelian group on the isomorphism classes on objects of $\mathcal{H}$ modulo the additivity relations on short exact sequences. We write $[X]$ for the class of a coherent sheaf $X$ in the Grothendieck group $\mathrm{K}_0(\mathcal{H})$ of $\mathcal{H}$. The Grothendieck group is equipped with the Euler form, which is defined on classes of objects $X$, $Y$ in $\mathcal{H}$ by

$$\langle [X], [Y] \rangle = \dim_k \operatorname{Hom}(X, Y) - \dim_k \operatorname{Ext}^1(X, Y).$$

We will usually write $\langle X, Y \rangle$, without the brackets.
In case $\mathbb{X}$ is of genus zero, $\mathcal{H}$ admits a tilting object whose endomorphism ring is a finite dimensional algebra, and thus the Grothendieck group $\mathrm{K}_0(\mathcal{H})$ of $\mathcal{H}$ is finitely generated free abelian. (From this it follows more directly that every $\mathbb{X}$ of genus zero satisfies (NC 6).)

*In the following, if not otherwise specified, let $\mathcal{H} = \operatorname{coh} \mathbb{X}$ be a weighted noncommutative regular projective curve.*

Homogeneous and exceptional tubes. For every $x \in \mathbb{X}$ the connected uniserial length categories $\mathcal{U}_x$ are called *tubes*. The number $p(x) \geq 1$ is called the *rank* of the tube $\mathcal{U}_x$. Tubes of rank 1 are called *homogeneous*, those with $p(x) > 1$ *exceptional*. We say that a point $x$ is homogeneous (resp. exceptional) if so is the corresponding tube $\mathcal{U}_x$. If $S_x$ is a simple sheaf in $\mathcal{U}_x$, then $\operatorname{Ext}^1(S_x, S_x) \neq 0$ in the homogeneous case, and $\operatorname{Ext}^1(S_x, S_x) = 0$ in the exceptional case. More generally, a coherent sheaf $E$ is called *exceptional*, if $E$ is indecomposable and $E$ has no self-extensions. It follows then by an argument of Happel and Ringel that $\operatorname{End}(E)$ is a skew field; we refer to [50, 3.2.3]. It is well-known and easy to see that the exceptional sheaves in $\mathcal{U}_x$ are just those indecomposables of length $\leq p(x) - 1$ (which exist only for $p(x) > 1$). In particular there are only finitely many exceptional sheaves of finite length.
If $p = p(x)$, then all simple sheaves in $\mathcal{U}_x$ are given (up to isomorphism) by the Auslander-Reiten orbit $S_x = \tau^p S_x, \tau S_x, \ldots, \tau^{p-1} S_x$.
For the terminology on wings and branches in exceptional tubes we refer to Section 4.7.

Non-weighted curves. By a *(non-weighted) noncommutative regular projective curve* over the field $k$ we mean a category $\mathcal{H} = \operatorname{coh} \mathbb{X}$ satisfying axioms (NC 1) to (NC 5), and additionally

(NC 6') $\operatorname{Ext}^1(S, S) \neq 0$ (equivalently: $\tau S \simeq S$) holds for all simple objects $S \in \mathcal{H}$.

This condition means that all tubes are homogeneous, that is, $p(x) = 1$ for all $x \in \mathbb{X}$; therefore these curves are also called homogeneous in [38]. For a detailed treatment of this setting we refer to [39]. We stress that thus, by abuse of language, non-weighted curves are special cases of weighted curves.

Grothendieck categories with finiteness conditions. Let us briefly recall some notions we will need in the sequel. An abelian category $\mathcal{A}$ is a *Grothendieck category*, if it is cocomplete, has a generator, and direct limits are exact. Every Grothendieck category is also complete and has an injective cogenerator. A Grothendieck category is called *locally coherent* (resp. *locally noetherian*, resp. *locally finite*) if it admits a system of generators which are coherent (resp. noetherian, resp. of finite length). In this case every object in $\mathcal{A}$ is a direct limit of coherent (resp. noetherian, resp. finite length) objects. If $\mathcal{A}$ is locally coherent then the coherent and the finitely presented objects coincide, and the full subcategory $\operatorname{fp}(\mathcal{A})$ of finitely presented objects is abelian. For more details on Grothendieck categories we refer to [26, 63, 31, 34].

The Serre construction. $\mathcal{H} = \operatorname{coh}\mathbb{X}$ is a noncommutative noetherian projective scheme in the sense of Artin-Zhang [11] and satisfies Serre's theorem. This means that there is a positively $H$-graded (not necessarily commutative) noetherian ring $R$ (with $(H, \leq)$ an ordered abelian group of rank one) such that

$$\mathcal{H} = \frac{\operatorname{mod}^H(R)}{\operatorname{mod}_0^H(R)}, \tag{2.2}$$

the quotient category of the category of finitely generated $H$-graded modules modulo the Serre subcategory of those modules which are finite-dimensional over $k$. (We refer to [38, Prop. 6.2.1], [39] and [52, Lem. IV.4.1].) With this description we can define $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$ as the quotient category

$$\vec{\mathcal{H}} = \frac{\operatorname{Mod}^H(R)}{\operatorname{Mod}_0^H(R)}, \tag{2.3}$$

where $\operatorname{Mod}_0^H(R)$ denotes the localizing subcategory of $\operatorname{Mod}^H(R)$ of all $H$-graded torsion, that is, locally finite-dimensional, modules. The category $\vec{\mathcal{H}}$ is hereditary abelian, and a locally noetherian Grothendieck category; every object in $\vec{\mathcal{H}}$ is a direct limit of objects in $\mathcal{H}$ (therefore the symbol $\vec{\mathcal{H}}$). The full abelian subcategory $\mathcal{H}$ consists of the coherent (= finitely presented = noetherian) objects in $\vec{\mathcal{H}}$, we also write $\mathcal{H} = \operatorname{fp}(\vec{\mathcal{H}})$. Every indecomposable coherent sheaf has a local endomorphism ring, and $\mathcal{H}$ is a Krull-Schmidt category.
We remark that $\vec{\mathcal{H}}$ can, by [26, II. Thm. 1], also be recovered from its subcategory $\mathcal{H}$ of noetherian objects as the category of left-exact (covariant) $k$-functors from $\mathcal{H}^{\mathrm{op}}$ to $\operatorname{Mod}(k)$. We also note that our categories $\mathcal{H}$ (resp. $\vec{\mathcal{H}}$) can be described alternatively as categories $\operatorname{coh}(\mathcal{A})$ (resp. $\operatorname{Qcoh}(\mathcal{A})$) of coherent (resp. quasicoherent) modules over certain hereditary orders $\mathcal{A}$; we refer to [39, Thm. 7.11].

Prüfer sheaves. Let $E$ be an indecomposable sheaf in a tube $\mathcal{U}_x$. By the *ray* starting in $E$ we mean the (infinite) sequence of all the indecomposable sheaves in $\mathcal{U}_x$, which contain $E$ as a subsheaf. The corresponding monomorphisms (inclusions) form a direct system. If the socle of $E$ is the simple $S$, then the corresponding direct limit of this system is the *Prüfer sheaf* $S[\infty]$. In other words, $S[\infty]$ is the union of all indecomposable sheaves of finite length containing $S$ (or $E$). Dually we define *corays* ending in $E$ as the sequence of all indecomposable sheaves in $\mathcal{U}_x$ admitting $E$ as a factor.
If $S$ is a simple sheaf, then we denote by $S[n]$ the (unique) indecomposable sheaf of length $n$ with socle $S$. Thus, the collection $S[n]$ ($n \geq 1$) forms the ray starting in $S$, and their union is $S[\infty]$. The Prüfer sheaves form an important class of indecomposable (we refer to [54]), quasicoherent, non-coherent sheaves.

Rank. Line bundles. Let $\mathcal{H}/\mathcal{H}_0$ be the quotient category of $\mathcal{H}$ modulo the Serre category of sheaves of finite length, let $\pi\colon \mathcal{H} \to \mathcal{H}/\mathcal{H}_0$ the quotient functor, which is exact. The abelian category $\mathcal{H}/\mathcal{H}_0$ is, by [47, Prop. 3.4], of the form $\mathcal{H}/\mathcal{H}_0 \simeq \operatorname{mod}(k(\mathcal{H}))$ for a unique skew field $k(\mathcal{H})$, called the *function field* of $\mathcal{H}$ (or $\mathbb{X}$). Then $\vec{\mathcal{H}}/\vec{\mathcal{H}_0} = \operatorname{Mod}(k(\mathcal{H}))$. The $k(\mathcal{H})$-dimension on $\mathcal{H}/\mathcal{H}_0$ induces the *rank* function on $\mathcal{H}$ by the formula $\operatorname{rk}(F) := \dim_{k(\mathcal{H})}(\pi F)$. It is additive on short exact sequences and thus induces a linear form $\operatorname{rk}\colon \mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$. The objects in $\mathcal{H}_0$ are just the objects of rank zero, every non-zero vector bundle has a positive rank, [47, Prop. 1.2]. The vector bundles of rank one are called *line bundles*. A line bundle $L$ is called *special* if for each $x \in \mathbb{X}$ there is (up to isomorphism) *precisely one* simple sheaf $S_x$ concentrated at $x$ with

$$\operatorname{Ext}^1(S_x, L) \neq 0. \tag{2.4}$$

Special line bundles always exist, cf. [39, Prop. 1.1].
Furthermore, every non-zero morphism from a line bundle $L'$ to a vector bundle is a monomorphism, and $\operatorname{End}(L')$ is a skew field, [47, Lem. 1.3]. Every vector bundle has a line bundle filtration, [47, Prop. 1.6].

The sheaf of rational functions. The *sheaf $\mathcal{K}$ of rational functions* is the injective envelope of any line bundle $L$ in the category $\vec{\mathcal{H}}$; this does not depend on the chosen line bundle. Besides the Prüfer sheaves, this is another very important quasicoherent, non-coherent sheaf. It is torsionfree by [36, Lem. 14], and it is a generic sheaf in the sense of [41]; its endomorphism ring is the function field, $\operatorname{End}_{\vec{\mathcal{H}}}(\mathcal{K}) \simeq \operatorname{End}_{\mathcal{H}/\mathcal{H}_0}(\pi L) \simeq k(\mathcal{H})$.

The derived category. Since $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$ is a hereditary category, the derived category

$$\mathcal{D} = \mathcal{D}(\vec{\mathcal{H}}) = \operatorname{Add}\Big(\bigvee_{n\in\mathbb{Z}} \vec{\mathcal{H}}[n]\Big) \tag{2.5}$$

is the repetitive category of $\vec{\mathcal{H}}$. This means: Every object in $\mathcal{D}$ can be written as $\bigoplus_{i\in I} X_i[i]$ for a subset $I \subseteq \mathbb{Z}$ and $X_i \in \vec{\mathcal{H}}$ for all $i$, and for all objects

$X$, $Y \in \vec{\mathcal{H}}$ and all integers $n$, $m$ we have

$$\operatorname{Ext}^{n-m}_{\vec{\mathcal{H}}}(X,Y) = \operatorname{Hom}_{\mathcal{D}}(X[m], Y[n]).$$

The *bounded* derived category $\mathcal{D}^b = \mathcal{D}^b(\vec{\mathcal{H}})$ is the full subcategory of $\mathcal{D}$ with objects those complexes which have bounded cohomology. It has a similar repetitive structure as in (2.5), where Add is replaced by add and the subset $I$ in $\mathbb{Z}$ as above is finite.

Generalized Serre duality. It follows easily from [35, Thm. 4.4] that on $\vec{\mathcal{H}}$ we have Serre duality in the following sense. Let $\tau$ be the Auslander-Reiten translation on $\mathcal{H}$ and $\tau^-$ its (quasi-) inverse. For all $X \in \mathcal{H}$ and all $Y \in \vec{\mathcal{H}}$ we have

$$\operatorname{D}\operatorname{Ext}^1_{\vec{\mathcal{H}}}(X,Y) = \operatorname{Hom}_{\vec{\mathcal{H}}}(Y,\tau X) \quad \text{and} \quad \operatorname{Ext}^1_{\vec{\mathcal{H}}}(Y,X) = \operatorname{D}\operatorname{Hom}_{\vec{\mathcal{H}}}(\tau^- X, Y),$$

with D denoting the duality $\operatorname{Hom}_k(-,k)$.

Purity. The notion of purity is of great importance in our setting. For details we refer to [51, Ch. 5].

(1) A short exact sequence $\eta\colon 0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ in $\vec{\mathcal{H}}$ is called *pure-exact*, if for every $F \in \mathcal{H}$ (that is, $F$ finitely presented) the induced sequence $\operatorname{Hom}(F,\eta) : 0 \to \operatorname{Hom}(F,A) \to \operatorname{Hom}(F,B) \to \operatorname{Hom}(F,C) \to 0$ is exact. In this case $\alpha$ (resp. $\beta$) is called a pure monomorphism (resp. pure epimorphism), and $A$ a pure subobject of $B$.

(2) An object $E \in \vec{\mathcal{H}}$ is called *pure-injective* if for every pure-exact sequence $0 \to A \to B \to C \to 0$ the induced sequence $0 \to \operatorname{Hom}(C,E) \to \operatorname{Hom}(B,E) \to \operatorname{Hom}(A,E) \to 0$ is exact.

(3) An object $E \in \vec{\mathcal{H}}$ is called $\Sigma$-*pure-injective* if the coproduct $E^{(I)}$ is pure-injective for every set $I$.

Lemma 2.1. *Every coherent sheaf $F \in \mathcal{H}$ is pure-injective.*

*Proof.* If $\mu$ is a pure-exact sequence in $\vec{\mathcal{H}}$, then $\operatorname{Hom}_{\vec{\mathcal{H}}}(\tau^- F,\mu)$ is exact. Since $\operatorname{Ext}^2_{\vec{\mathcal{H}}}(-,-)$ vanishes, this amounts to exactness of $\operatorname{Ext}^1_{\vec{\mathcal{H}}}(\tau^- F,\mu)$, and hence of $\operatorname{D}\operatorname{Ext}^1_{\vec{\mathcal{H}}}(\tau^- F,\mu)$, which in turn is equivalent to exactness of $\operatorname{Hom}_{\vec{\mathcal{H}}}(\mu,F)$ by Serre duality. This gives the claim. □

Almost split sequences. Since the objects of $\mathcal{H}$ are pure-injective, it follows directly from [35, Prop. 3.2] that the category $\mathcal{H}$ has almost split sequences which also satisfy the almost split properties in the larger category $\vec{\mathcal{H}}$; more precisely: for every indecomposable $Z \in \mathcal{H}$ there is a non-split short exact sequence

$$0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$$

in $\mathcal{H}$ with $X = \tau Z$ indecomposable such that for every object $Z' \in \vec{\mathcal{H}}$ any morphism $Z' \to Z$ that is not a retraction factors through $\beta$ (and equivalently, for every object $X' \in \vec{\mathcal{H}}$ any morphism $X \to X'$ that is not a section factors through $\alpha$).

Hereditary orders. For the details on notions and results in this and the following subsections we refer to [39]. Let $\mathcal{H}$ be a weighted noncommutative regular projective curve over $k$. Let $\bar{p}$ be the least common multiple of the weights $p(x)$. The centre of the function field $k(\mathcal{H})$ is of the form $k(X)$, the function field of a unique regular projective curve $X$ over $k$. We call $X$ the *centre curve* of $\mathcal{H}$. The dimension $[k(\mathcal{H}) : k(X)]$ is finite, a square number, which we denote by $s(\mathcal{H})^2$. We call $s(\mathcal{H})$ the *skewness* of $\mathcal{H}$ (or $\mathbb{X}$). The (closed) points of $X$ are in one-to-one correspondence to the (closed) points of $\mathbb{X}$. Let $\mathcal{O} = \mathcal{O}_X$ be the structure sheaf of $X$. For every $x \in X$ we have the local rings $(\mathcal{O}_x, \mathfrak{m}_x)$, and the residue class field $k(x) = \mathcal{O}_x/\mathfrak{m}_x$. For all $x \in \mathbb{X}$ there are the ramification indices $e_\tau(x) \geq 1$. There exist only finitely many points $x \in \mathbb{X}$ with $p(x)e_\tau(x) > 1$. By a result of Reiten and van den Bergh [52], [39, Thm. 7.11] the category $\mathcal{H}$ can be realized as $\mathcal{H} = \operatorname{coh}(\mathcal{A})$, the category of coherent $\mathcal{A}$-modules, where $\mathcal{A}$ is a torsionfree coherent sheaf of hereditary $\mathcal{O}$-orders in a full matrix algebra over $k(\mathcal{H})$. Moreover, $\vec{\mathcal{H}} = \operatorname{Qcoh}(\mathcal{A})$.

If $\mathbb{X}$ is weighted then there is an underlying non-weighted curve $\mathbb{X}_{nw}$, which follows from (NC 6) by perpendicular calculus [28], cf. [39, Prop. 1.1]. We have $\bar{p} = 1$ (that is, $\mathbb{X} = \mathbb{X}_{nw}$) if and only if $\mathcal{A}$ is a maximal order.

Structure sheaf. We now define the *structure sheaf* $L$ of $\mathcal{H} = \operatorname{coh}(\mathcal{A})$ to be a line bundle with the following properties: in the non-weighted case ($\bar{p} = 1$) we set $L_{\mathcal{A}} = \mathcal{A}_{\mathcal{A}}$, and in the weighted case ($\bar{p} > 1$) we let $L$ be a special line bundle corresponding to the structure sheaf of the underlying non-weighted curve via perpendicular calculus, cf. [39, Prop. 1.1]. In the following we will always consider the pair $(\mathcal{H}, L)$, that is, $\mathcal{H}$ equipped with structure sheaf $L$. We recall that $k(\mathcal{H}) = \operatorname{End}_{\mathcal{H}/\mathcal{H}_0}(\pi L)$.

Orbifold Euler characteristic and representation type. One defines the *average Euler form* $\langle\langle E, F\rangle\rangle = \sum_{j=0}^{\bar{p}-1} \langle \tau^j E, F\rangle$, and then the normalized *orbifold Euler characteristic* of $\mathcal{H}$ by $\chi'_{orb}(\mathbb{X}) = \frac{1}{s(\mathcal{H})^2 \bar{p}^2} \langle\langle L, L\rangle\rangle$. If $k$ is perfect, one has a nice formula to compute the Euler characteristic:

$$\chi'_{orb}(\mathbb{X}) = \chi'(X) - \frac{1}{2} \sum_x \Big(1 - \frac{1}{p(x)e_\tau(x)}\Big)[k(x) : k]. \tag{2.6}$$

Here, $\chi'(X) = \dim_k \operatorname{Hom}_X(\mathcal{O}, \mathcal{O}) - \dim_k \operatorname{Ext}^1_X(\mathcal{O}, \mathcal{O})$ is the normalized Euler characteristic of the centre curve $X$ (or of $\operatorname{coh}(X)$; cf. also [39, Rem. 13.11 (1)]). If $k$ is not perfect, there is still a similar formula, we refer to [39, Cor. 13.13].

The orbifold Euler characteristic determines the representation type of the category $\mathcal{H} = \operatorname{coh} \mathbb{X}$ (see also Theorem 2.3 below):

- $\mathbb{X}$ is domestic: $\chi'_{orb}(\mathbb{X}) > 0$
- $\mathbb{X}$ is elliptic: $\chi'_{orb}(\mathbb{X}) = 0$, and $\mathbb{X}$ non-weighted ($\bar{p} = 1$)
- $\mathbb{X}$ is tubular: $\chi'_{orb}(\mathbb{X}) = 0$, and $\mathbb{X}$ properly weighted ($\bar{p} > 1$)
- $\mathbb{X}$ is wild: $\chi'_{orb}(\mathbb{X}) < 0$.

In this paper we will prove some general results for all representation types, and we will obtain finer and complete classification results in the cases of non-negative orbifold Euler characteristic.

Remark 2.2. (1) If $\mathbb{X}$ is non-weighted with structure sheaf $L$, then we call the number $g(\mathbb{X}) = [\mathrm{Ext}^1(L,L) : \mathrm{End}(L)]$ the *genus* of $\mathbb{X}$. The condition $g(\mathbb{X}) = 1$ is equivalent to the elliptic case. In case $g(\mathbb{X}) \geq 1$ there does not exist any exceptional object in $\mathcal{H}$; this follows readily from the Riemann-Roch formula [39, Prop. 9.1]. Now it follows with [38, 0.5.4] that the condition $g(\mathbb{X}) = 0$ is equivalent to condition (g-0); actually, in this case there is a tilting bundle of the form $T = L \oplus \overline{L}$ with $\overline{L}$ indecomposable of rank one or two, and $\mathrm{End}(T)$ is a tame hereditary $k$-algebra.

(2) If $\mathbb{X}$ is weighted then $\mathcal{H} = \operatorname{coh}\mathbb{X}$ contains a tilting bundle (that is, $\mathcal{H}$ satisfies (g-0)) if and only if $g(\mathbb{X}_{nw}) = 0$. In other words, $\mathcal{H}$ satisfies (g-0) if the genus, in the non-orbifold sense, is zero. This follows from (1) with [42, Thm. 4.3].

Degree and slope. We define the *degree* function $\deg\colon \mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$, by

$$\deg(F) = \frac{1}{\kappa\varepsilon}\langle\!\langle L, F\rangle\!\rangle - \frac{1}{\kappa\varepsilon}\langle\!\langle L, L\rangle\!\rangle \,\mathrm{rk}(F), \tag{2.7}$$

with $\kappa = \dim_k \mathrm{End}(L)$ and $\varepsilon$ the positive integer such that the resulting linear form $\mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$ becomes surjective. We have $\deg(L) = 0$, and deg is positive and $\tau$-invariant on sheaves of finite length. The *slope* of a non-zero coherent sheaf $F$ is defined as $\mu(F) = \deg(F)/\mathrm{rk}(F) \in \widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. Moreover, $F$ is called *stable* (*semistable*, resp.) if for every non-zero proper subsheaf $F'$ of $F$ we have $\mu(F') < \mu(F)$ (resp. $\mu(F') \leq \mu(F)$).
More details on these numerical invariants will be given in 5.10.

Stability. The stability notions are very useful for the classification of vector bundles (we refer to [27, Prop. 5.5], [47], [38, Prop. 8.1.6], [39]):

Theorem 2.3. *Let $\mathcal{H} = \operatorname{coh}\mathbb{X}$ be a weighted noncommutative regular projective curve over $k$.*

(1) *If $\chi'_{orb}(\mathbb{X}) > 0$ (domestic type), then every indecomposable vector bundle is stable and exceptional. Moreover, $\operatorname{coh}\mathbb{X}$ admits a tilting bundle.*

(2) *If $\chi'_{orb}(\mathbb{X}) = 0$ (elliptic or tubular type), then every indecomposable coherent sheaf is semistable. If $\mathbb{X}$ is tubular (that is, $\bar{p} > 1$), then $\operatorname{coh}\mathbb{X}$ admits a tilting bundle. If $\mathbb{X}$ is elliptic (that is, $\bar{p} = 1$) then every indecomposable coherent sheaf $E$ is non-exceptional and satisfies $\tau E \simeq E$.*

(3) *If $\chi'_{orb}(\mathbb{X}) < 0$, then every Auslander-Reiten component in $\mathcal{H}_+ = \operatorname{vect}\mathbb{X}$ is of type $\mathbb{Z}A_\infty$, and $\mathcal{H}$ is of wild representation type. ($\operatorname{coh}\mathbb{X}$ may or may not satisfy (g-0).)* □

Orthogonal and generated classes. Let $\mathcal{X}$ be a class of objects in $\vec{\mathcal{H}}$. We will use the following notation:

$$\mathcal{X}^{\perp_0} = \{F \in \vec{\mathcal{H}} \mid \operatorname{Hom}(\mathcal{X}, F) = 0\}, \quad \mathcal{X}^{\perp_1} = \{F \in \vec{\mathcal{H}} \mid \operatorname{Ext}^1(\mathcal{X}, F) = 0\},$$
$$^{\perp_0}\mathcal{X} = \{F \in \vec{\mathcal{H}} \mid \operatorname{Hom}(F, \mathcal{X}) = 0\}, \quad ^{\perp_1}\mathcal{X} = \{F \in \vec{\mathcal{H}} \mid \operatorname{Ext}^1(F, \mathcal{X}) = 0\},$$
$$\mathcal{X}^{\perp} = \mathcal{X}^{\perp_0} \cap \mathcal{X}^{\perp_1}, \quad ^{\perp}\mathcal{X} = {^{\perp_0}\mathcal{X}} \cap {^{\perp_1}\mathcal{X}}.$$

Following [28] we call $^{\perp}\mathcal{X}$ (resp. $\mathcal{X}^{\perp}$) the *left-perpendicular* (resp. *right-perpendicular*) category of $\mathcal{X}$. By $\operatorname{Add}(\mathcal{X})$ (resp. $\operatorname{add}(\mathcal{X})$) we denote the class of all direct summands of direct sums of the form $\bigoplus_{i\in I} X_i$, where $I$ is any set (resp. finite set) and $X_i \in \mathcal{X}$ for all $i$. By $\operatorname{Gen}(\mathcal{X})$ we denote the class of all objects $Y$ *generated by* $\mathcal{X}$, that is, such that there is an epimorphism $X \to Y$ with $X \in \operatorname{Add}(\mathcal{X})$ (and similarly $\operatorname{gen}(\mathcal{X})$ with $\operatorname{add}(\mathcal{X})$).

Let $(I, \leq)$ be an ordered set and $\mathcal{X}_i$ classes of objects for all $i \in I$, in any additive category. We write $\bigvee_{i\in I} \mathcal{X}_i$ for $\operatorname{add}(\bigcup_{i\in I} \mathcal{X}_i)$ if additionally $\operatorname{Hom}(\mathcal{X}_j, \mathcal{X}_i) = 0$ for all $i < j$ is satisfied. In particular, notation like $\mathcal{X}_1 \vee \mathcal{X}_2$ and $\mathcal{X}_1 \vee \mathcal{X}_2 \vee \mathcal{X}_3$ makes sense (where $1 < 2 < 3$).

The following induction technique will be very important.

Reduction of weights. Let $S$ be an exceptional simple sheaf. In other words, $S$ lies on the mouth of a tube, with index $x$, of rank $p(x) > 1$. Then the right perpendicular category $S^{\perp}$ is equivalent to $\operatorname{Qcoh}\mathbb{X}'$, where $\mathbb{X}'$ is a curve such that the rank $p'(x)$ of the tube of index $x$ is $p'(x) = p(x) - 1$ and all other weights and all the numbers $e_\tau(y)$ are preserved. We refer to [28] for details. From the formula (2.6) (and [39, Cor. 13.13], which holds over any field) of the orbifold Euler characteristic we see $\chi'(\mathbb{X}') > \chi'(\mathbb{X})$, and we conclude readily that $\mathbb{X}'$ is of domestic type if $\mathbb{X}$ is tubular or domestic. By similar reasons, $\mathbb{X}'$ is of genus zero if so is $\mathbb{X}$.

Tubular shifts. If $x \in \mathbb{X}$ is a point of weight $p(x) \geq 1$, then there is an autoequivalence $\sigma_x$ of $\mathcal{H}$ (which extends to an autoequivalence of $\vec{\mathcal{H}}$), called the *tubular shift* associated with $x$. We refer to [44, (S10)] and [38, Sec. 0.4] for more details. These are generalizations of the tubular mutations [49], and they are also related to the Seidel-Thomas twists [59]; in case $p(x) = 1$ the tubular shift $\sigma_x$ actually agrees with the Seidel-Thomas twist $T_E$ with $E = S_x$ the simple sheaf at $x$, since this is spherical in the sense that $\operatorname{Ext}^1(E, E) \simeq \operatorname{End}(E)$ is a finite dimensional skew field (in [59] only the case $\operatorname{End}(S_x) = k$ is considered). We just recall that for every vector bundle $E$ there is a universal exact sequence

$$0 \to E \to \sigma_x(E) \to E_x \to 0, \tag{2.8}$$

where $E_x = \bigoplus_{j=0}^{p(x)-1} \operatorname{Ext}^1(\tau^j S_x, E) \otimes \tau^j S_x \in \mathcal{U}_x$ with the tensor product taken over the skew field $\operatorname{End}(S_x)$. We also write

$$\sigma_x(E) = E(x) \quad \text{and} \quad (\sigma_x)^n(E) = E(nx),$$

and we will use the more handy notation

$$E_x = \bigoplus_{j=0}^{p(x)-1} (\tau^j S_x)^{e(j,x,E)}$$

with the exponents given by the multiplicities

$$e(j,x,E) = [\mathrm{Ext}^1(\tau^j S_x, E) : \mathrm{End}(S_x)],$$

the $\mathrm{End}(S_x)$-dimension of $\mathrm{Ext}^1(\tau^j S_x, E)$. In the particular case when $E = L$ is the structure sheaf (which is a special line bundle), and $S_x$ is such that $\mathrm{Hom}(L, S_x) \neq 0$, we have $e(j,x,L) = e(x)$ for $j = p(x) - 1$ and $= 0$ otherwise.

Tilting sheaves. Let $\vec{\mathcal{H}}$ be a Grothendieck category, for instance $\vec{\mathcal{H}} = \mathrm{Qcoh}\,\mathbb{X}$.

Definition 2.4. An object $T \in \vec{\mathcal{H}}$ is called a *tilting object* or *tilting sheaf* if $\mathrm{Gen}(T) = T^{\perp_1}$. Then $\mathrm{Gen}(T)$ is called the associated *tilting class*.

This definition is inspired by [23, Def. 2.3], but we dispense with the self-smallness assumption made there. In a module category, we thus recover the definition of a *tilting module* (of projective dimension one) from [25].

We recall that the *projective dimension* $\mathrm{pd}(X)$ of an object $X$ in $\vec{\mathcal{H}}$ is defined to be the smallest integer $n \geq -1$ such that $\mathrm{Ext}^{n+1}(X, -) = 0$ holds, and $\infty$, if no such $n$ exists. Here, Ext-groups are defined via injective resolutions.

Lemma 2.5 ([23, Prop. 2.2]). *An object $T \in \vec{\mathcal{H}}$ is tilting if and only if the following conditions are satisfied:*

(TS0) *$T$ has projective dimension* $\mathrm{pd}(T) \leq 1$.

(TS1) $\mathrm{Ext}^1(T, T^{(I)}) = 0$ *for every cardinal $I$.*

(TS2) $T^{\perp} = 0$*, that is: if $X \in \vec{\mathcal{H}}$ satisfies* $\mathrm{Hom}(T, X) = 0 = \mathrm{Ext}^1(T, X)$*, then* $X = 0$.

We will mostly consider hereditary categories $\vec{\mathcal{H}}$ where (TS0) is automatically satisfied. In case $\vec{\mathcal{H}} = \mathrm{Qcoh}\,\mathbb{X}$ with $\mathbb{X}$ of genus zero, we will also consider the following condition, where $T_{\mathrm{can}} \in \mathcal{H}$ is a tilting bundle such that $\mathrm{End}(T_{\mathrm{can}}) = \Lambda$ is a canonical algebra, that is, $T_{\mathrm{can}}$ is a fixed canonical configuration.

(TS3) There are an autoequivalence $\sigma$ on $\mathcal{H}$ and an exact sequence

$$0 \to \sigma(T_{\mathrm{can}}) \to T_0 \to T_1 \to 0$$

such that $\mathrm{Add}(T_0 \oplus T_1) = \mathrm{Add}(T)$; if this can be realized with the additional property $\mathrm{Hom}(T_1, T_0) = 0$, then we say that $T$ satisfies condition (TS3+).

Since $\sigma(T_{\mathrm{can}})$ is a tilting bundle, (TS3) implies (TS2). As it will turn out, in case of genus zero, all tilting sheaves we construct will satisfy (TS3), and some will even satisfy (TS3+), see Example 4.22, Corollary 8.8, and Section 10.

*Let $\vec{\mathcal{H}}$ additionally be locally coherent with $\mathcal{H} = \mathrm{fp}(\vec{\mathcal{H}})$.*

Lemma 2.6. *Let $T \in \vec{\mathcal{H}}$ be tilting.*

(1) $\mathrm{Gen}(T) = \mathrm{Pres}(T)$*, the class of objects in* $\vec{\mathcal{H}}$ *which are cokernels of morphisms of the form* $T^{(J)} \to T^{(I)}$.

(2) $T^{\perp_1} \cap {}^{\perp_1}(T^{\perp_1}) = \mathrm{Add}(T)$.

(3) *If* $X \in \mathcal{H}$ *is coherent having a local endomorphism ring and* $X \in \mathrm{Add}(T)$*, then* $X$ *is a direct summand of* $T$.

*Proof.* (1) The same proof as in [25, Lemma 1.2] works here.
(2) Is an easy consequence of (1).
(3) Since $X$ is coherent, we get $X \in \mathrm{add}(T)$. Since $X$ has local endomorphism ring, the claim follows. □

Definition 2.7. Two tilting objects $T$, $T' \in \vec{\mathcal{H}}$ are *equivalent*, if they generate the same tilting class. This is equivalent to $\mathrm{Add}(T) = \mathrm{Add}(T')$. A tilting sheaf $T \in \vec{\mathcal{H}}$ is called *large* if it is not equivalent to a coherent tilting sheaf.

*For the rest of this section we assume that* $\mathbb{X}$ *is of genus zero and* $\vec{\mathcal{H}} = \mathrm{Qcoh}\,\mathbb{X}$ *with a fixed special line bundle* $L$.

Tilting bundles and concealed-canonical algebras. We fix a tilting bundle $T_{\mathrm{cc}} \in \mathcal{H}$. Its endomorphism ring $\Sigma$ is a concealed-canonical $k$-algebra. Every concealed-canonical algebra arises in this way, we refer to [44]. Especially for $T_{\mathrm{cc}} = T_{\mathrm{can}}$, a canonical configuration, we get a canonical algebra. We remark that $T_{\mathrm{cc}}$ is in particular a noetherian tilting object in $\vec{\mathcal{H}}$. It is well-known that $T_{\mathrm{cc}}$ is a (compact) generator of $\mathcal{D}$ inducing an equivalence

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{D}}(T_{\mathrm{cc}}, -) \colon \mathcal{D}(\mathrm{Qcoh}\,\mathbb{X}) \longrightarrow \mathcal{D}(\mathrm{Mod}\,\Sigma)$$

of triangulated categories (cf. [18, Prop. 1.5] and [33, Thm. 8.5]). Via this equivalence the module category $\mathrm{Mod}\,\Sigma$ can be identified (like in [43, Thm. 3.2] and [41]) with the full subcategory $\mathrm{Add}(\mathcal{T}_{\mathrm{cc}} \vee \mathcal{F}_{\mathrm{cc}}[1])$ of $\mathcal{D}$, where $(\mathcal{T}_{\mathrm{cc}}, \mathcal{F}_{\mathrm{cc}})$ is the torsion pair in $\vec{\mathcal{H}}$ given by $\mathcal{T}_{\mathrm{cc}} = \mathrm{Gen}(T_{\mathrm{cc}}) = {T_{\mathrm{cc}}}^{\perp_1}$ and $\mathcal{F}_{\mathrm{cc}} = {T_{\mathrm{cc}}}^{\perp_0}$. This torsion pair induces a split torsion pair $(\mathcal{Q}, \mathcal{C}) = (\mathcal{F}_{\mathrm{cc}}[1], \mathcal{T}_{\mathrm{cc}})$ in $\mathrm{Mod}\,\Sigma$. Moreover, $\mathrm{mod}\,\Sigma = (\mathcal{T}_{\mathrm{cc}} \cap \mathcal{H}) \vee (\mathcal{F}_{\mathrm{cc}} \cap \mathcal{H})[1]$.

Correspondences between tilting objects. Following [16], we call a tilting sheaf $T \in \vec{\mathcal{H}}$ of *finite type* if the tilting class $T^{\perp_1}$ is determined by a class of finitely presented objects $\mathscr{S} \subseteq \mathcal{H}$ such that $T^{\perp_1} = \mathscr{S}^{\perp_1}$. If $T$ is of finite type, then $\mathscr{S} := {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$ is the largest such class. We are now going to see that all tilting sheaves lying in $\mathcal{T}_{\mathrm{cc}}$ are of finite type.
We call an object $T$ in the triangulated category $\mathcal{D}^b = \mathcal{D}^b(\mathrm{Qcoh}\,\mathbb{X})$ a *tilting complex* if the following two conditions hold.

(TC1) $\mathrm{Hom}_{\mathcal{D}}(T, T^{(I)}[n]) = 0$ for all cardinals $I$ and all $n \in \mathbb{Z}$, $n \neq 0$.

(TC2) If $X \in \mathcal{D}^b$ satisfies $\mathrm{Hom}_{\mathcal{D}}(T, X[n]) = 0$ for all $n \in \mathbb{Z}$, then $X = 0$.

Proposition 2.8. *The following statements are equivalent for* $T \in \mathcal{T}_{\mathrm{cc}}$ *(viewed as a complex concentrated in degree zero).*

(1) $T$ *is a tilting sheaf in* $\vec{\mathcal{H}}$.

(2) *$T$ is a tilting complex in $\mathcal{D}^b$.*
(3) *$T$ is a tilting module in $\operatorname{Mod}\Sigma$ (of projective dimension at most one).*

*Moreover, every tilting sheaf $T \in \vec{\mathcal{H}}$ lying in $\mathcal{T}_{\mathrm{cc}}$ is of finite type.*

*Proof.* Clearly (2) implies (1) and (3). We show that (1) implies (2). Since $\vec{\mathcal{H}}$ is hereditary, $\operatorname{Ext}^1_{\vec{\mathcal{H}}}(T, T^{(I)}) = 0$ is equivalent to $\operatorname{Hom}_{\mathcal{D}}(T, T^{(I)}[n]) = 0$ for all $n \neq 0$. Let $X = \bigoplus_{i=-s}^{s} X_i \in \mathcal{D}^b$ be such that $X_i \in \vec{\mathcal{H}}[i]$, and assume

$$\operatorname{Hom}_{\mathcal{D}}(T, X_i[n]) = 0 \text{ for all } n \in \mathbb{Z} \text{ and all } i. \tag{2.9}$$

Since $X_i[-i] \in \vec{\mathcal{H}}$, this implies for $n = -i$ and $n = -i+1$ the condition

$$\operatorname{Hom}_{\vec{\mathcal{H}}}(T, X_i[-i]) = 0 = \operatorname{Ext}^1_{\vec{\mathcal{H}}}(T, X_i[-i]).$$

By (1) we conclude $X_i[-i] = 0$, and thus $X_i = 0$. Finally, we conclude $X = 0$.
The proof that (3) implies (2) is similar. We just have to observe that condition (2.9) yields $\operatorname{Ext}^1_{\vec{\mathcal{H}}}(T, X_i[-i]) = 0$, that is, $X_i[-i] \in \operatorname{Gen}(T) \subseteq \mathcal{T}_{\mathrm{cc}}$, and thus $X_i$ is, up to shift in the derived category, a $\Sigma$-module.
Assume that $T$ satisfies condition (1). In order to show that $T$ is of finite type, we set $\mathscr{S} = {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$ and verify $\mathscr{S}^{\perp_1} = T^{\perp_1}$. The inclusion $\mathscr{S}^{\perp_1} \supseteq T^{\perp_1}$ is trivial. Further, since $T \in \mathcal{T}_{\mathrm{cc}}$, we have $T_{\mathrm{cc}} \in \mathscr{S}$, and thus $\mathscr{S}^{\perp_1} \subseteq \mathcal{T}_{\mathrm{cc}}$ consists of $\Sigma$-modules. We view $T$ as a tilting $\Sigma$-module and exploit the corresponding result in $\operatorname{Mod}\Sigma$ from [16]. It states that the tilting class ${T_\Sigma}^{\perp_1} = \{X \in \operatorname{Mod}\Sigma \mid \operatorname{Ext}^1_\Sigma(T, X) = 0\}$ is determined by a class $\widetilde{\mathscr{S}} = {}^{\perp_1}({T_\Sigma}^{\perp_1}) \cap \operatorname{mod}\Sigma$ of finitely presented modules of projective dimension at most one, that is, ${T_\Sigma}^{\perp_1} = \widetilde{\mathscr{S}}^{\perp_1}$. Notice that $\widetilde{\mathscr{S}} \subseteq \mathcal{T}_{\mathrm{cc}}$. Otherwise there would be an indecomposable $F \in \mathcal{F}_{\mathrm{cc}}$ with $F[1] \in \widetilde{\mathscr{S}}$. Then $\operatorname{Ext}^1_{\vec{\mathcal{H}}}(T, \tau F) = \operatorname{D}\operatorname{Hom}_{\vec{\mathcal{H}}}(F, T) = \operatorname{D}\operatorname{Ext}^1_\Sigma(F[1], T) = 0$, that is, $\tau F \in \operatorname{Gen}(T) \subseteq \mathcal{T}_{\mathrm{cc}}$, and $\operatorname{Ext}^1_{\vec{\mathcal{H}}}(T_{\mathrm{cc}}, \tau F) = 0$. But also $\operatorname{Hom}_{\vec{\mathcal{H}}}(T_{\mathrm{cc}}, \tau F) = \operatorname{D}\operatorname{Ext}^1_{\vec{\mathcal{H}}}(F, T_{\mathrm{cc}}) = \operatorname{D}\operatorname{Hom}_{\mathcal{D}}(F[1], T_{\mathrm{cc}}[2]) = \operatorname{D}\operatorname{Ext}^2_\Sigma(F[1], T_{\mathrm{cc}}) = 0$ since $\operatorname{pdim}_\Sigma F[1] \leq 1$, and so $F[1] = 0$, a contradiction.
Now any object $X$ in $\mathcal{T}_{\mathrm{cc}}$ can be viewed both in $\operatorname{Mod}\Sigma$ and $\vec{\mathcal{H}}$, and the functors $\operatorname{Ext}^1_\Sigma(X, -)$ and $\operatorname{Ext}^1_{\vec{\mathcal{H}}}(X, -)$ coincide on $\mathcal{T}_{\mathrm{cc}}$. In particular, $\widetilde{\mathscr{S}} \subseteq \mathscr{S}$, and if $X$ is a sheaf in $\mathscr{S}^{\perp_1}$, then $X$ is a $\Sigma$-module with $\operatorname{Ext}^1_\Sigma(S, X) = 0$ for all $S \in \widetilde{\mathscr{S}}$, hence $\operatorname{Ext}^1_{\vec{\mathcal{H}}}(T, X) = \operatorname{Ext}^1_\Sigma(T, X) = 0$, that is, $X \in T^{\perp_1}$. This finishes the proof. $\square$

We will construct and classify a certain class of large tilting sheaves independently of the representation type, even independently of the genus, namely the tilting sheaves with a large torsion part. A complete classification of all large tilting sheaves will be obtained in the domestic and the tubular (that is: in the non-wild) genus zero cases.

The domestic case is akin to the tame hereditary case:

Tame hereditary algebras. There is a tilting bundle $T_{\mathrm{cc}}$ such that $H = \mathrm{End}(T_{\mathrm{cc}})$ is a tame hereditary algebra if and only if $\mathbb{X}$ is of domestic type. In this case it follows from Proposition 2.8 that the large tilting $H$-modules (of projective dimension at most one), as classified in [10], correspond (up to equivalence) to the large tilting sheaves in $\mathrm{Qcoh}\,\mathbb{X}$. Indeed, recall that $T_{\mathrm{cc}}$ induces a torsion pair $(\mathcal{T}_{\mathrm{cc}}, \mathcal{F}_{\mathrm{cc}})$ in $\mathrm{Qcoh}\,\mathbb{X}$ and a split torsion pair $(\mathcal{Q}, \mathcal{C})$ in $\mathrm{Mod}\,H$. By [10, Thm. 2.7] every large tilting $H$-module lies in the class $\mathcal{C} \subseteq \mathrm{Mod}\,H$, and it will be shown in Proposition 6.3 below that every large tilting sheaf lies in $\mathcal{T}_{\mathrm{cc}}$.

## 3. Torsion, torsionfree, and divisible sheaves

*In this section let $\vec{\mathcal{H}} = \mathrm{Qcoh}\,\mathbb{X}$, where $\mathbb{X}$ is a weighted noncommutative regular projective curve over a field $k$.* Our main aim is to prove that every tilting sheaf splits into a direct sum of indecomposable sheaves of finite length, Prüfer sheaves, and a torsionfree sheaf.

Definition 3.1. Let $V \subseteq \mathbb{X}$ be a subset. A quasicoherent sheaf $F$ is called *$V$-torsionfree* if $\mathrm{Hom}(S_x, F) = 0$ for all $x \in V$ and all simple sheaves $S_x \in \mathcal{U}_x$. In case $V = \mathbb{X}$ the sheaf $F$ is *torsionfree*. We set

$$\mathscr{S}_V = \coprod_{x \in V} \mathcal{U}_x$$

and denote by

$$\mathcal{F}_V = \mathscr{S}_V{}^{\perp_0}$$

the class of $V$-torsionfree sheaves.
Similarly, a quasicoherent sheaf $D$ is called *$V$-divisible* if $\mathrm{Ext}^1(S_x, D) = 0$ for all $x \in V$ and for all simple sheaves $S_x \in \mathcal{U}_x$. In case $V = \mathbb{X}$ we call $D$ just *divisible*. We denote by

$$\mathcal{D}_V = \mathscr{S}_V{}^{\perp_1}$$

the class of $V$-divisible sheaves. It is closed under direct summands, set-indexed direct sums, extensions and epimorphic images. Furthermore, we call $D$ *precisely $V$-divisible* if $D$ is $V$-divisible, and if $\mathrm{Ext}^1(S, D) \neq 0$ for every simple sheaf $S \in \mathscr{S}_{\mathbb{X}\setminus V}$.

Remark 3.2. The class $\mathscr{S}_V$ is a Serre subcategory in $\mathcal{H} = \mathrm{fp}(\vec{\mathcal{H}})$, its direct limit closure $\mathcal{T}_V = \vec{\mathscr{S}_V}$ is a localizing subcategory in $\vec{\mathcal{H}}$ of finite type, and $(\mathcal{T}_V, \mathcal{F}_V)$ is a hereditary torsion pair in $\vec{\mathcal{H}}$. In particular, the canonical quotient functor $\pi\colon \vec{\mathcal{H}} \to \vec{\mathcal{H}}/\mathcal{T}_V$ has a right-adjoint $s\colon \vec{\mathcal{H}}/\mathcal{T}_V \to \vec{\mathcal{H}}$ which commutes with direct limits. The class of $V$-torsionfree and $V$-divisible sheaves

$$\mathscr{S}_V{}^{\perp} = \mathcal{T}_V{}^{\perp} \simeq \vec{\mathcal{H}}/\mathcal{T}_V \tag{3.1}$$

is a full exact subcategory of $\vec{\mathcal{H}}$, that is, the inclusion functor $j\colon \mathscr{S}_V{}^{\perp} \to \vec{\mathcal{H}}$ is exact and induces an isomorphism $\mathrm{Ext}^1_{\mathscr{S}_V{}^{\perp}}(A, B) \simeq \mathrm{Ext}^1_{\vec{\mathcal{H}}}(A, B)$ for all $A$, $B \in \mathscr{S}_V{}^{\perp}$. In particular, $\mathrm{Ext}^1_{\mathscr{S}_V{}^{\perp}}$ is right exact, so that the category $\vec{\mathcal{H}}/\mathcal{T}_V \simeq \mathscr{S}_V{}^{\perp}$

is hereditary. For details we refer to [28, Prop. 1.1, Prop. 2.2, Cor. 2.4], [31, Thm. 2.8], [34, Lem. 2.2, Thm. 2.6, Thm. 2.8, Cor. 2.11].

We note that in case $V = \mathbb{X}$ the subclass $\mathscr{S}_{\mathbb{X}} = \mathcal{H}_0$ of $\mathcal{H}$ is the class of finite length sheaves, $\mathcal{T} = \mathcal{T}_{\mathbb{X}}$ in $\vec{\mathcal{H}}$ forms the class of torsion sheaves, $\mathcal{F} = \mathcal{F}_{\mathbb{X}}$ the class of torsionfree sheaves, and $\mathcal{F} \cap \mathcal{H} = \operatorname{vect}\mathbb{X}$ the class of vector bundles.

Lemma 3.3. *Let $X \in \vec{\mathcal{H}}$. Let $tX$ be the largest subobject of $X$ which lies in $\mathcal{T}$, the* torsion subsheaf *of $X$. Then the quotient $X/tX$ is torsionfree, and the canonical sequence*

$$\eta\colon 0 \to tX \to X \to X/tX \to 0$$

*is pure-exact.*

*Proof.* Clearly, $X/tX$ is torsionfree. Let $F \in \mathcal{H}$. We know that $F = F_+ \oplus F_0$, where $F_+$ is a vector bundle and $F_0$ is of finite length. It follows that $\operatorname{Ext}^1(F, tX) = \operatorname{Ext}^1(F_+, tX) \oplus \operatorname{Ext}^1(F_0, tX)$. The left summand is zero by Serre duality, since every vector bundle is torsionfree. Moreover, $\operatorname{Hom}(F_0, X/tX) = 0$, so $\operatorname{Hom}(F, X) \to \operatorname{Hom}(F, X/tX)$ is surjective. □

Lemma 3.4. *A quasicoherent sheaf is injective if and only if it is divisible.*

*Proof.* Trivially every injective sheaf is divisible. Conversely, every divisible sheaf $Q$ is $L'$-injective for every line bundle $L'$: this means that if $L'' \subseteq L'$ is a sub line bundle of $L'$, then every morphism $f \in \operatorname{Hom}(L'', Q)$ can be extended to $L'$. Indeed, there is commutative diagram with exact sequences

$$\begin{tikzcd}
0 \arrow[r] & L'' \arrow[r] \arrow[d, "f"'] & L' \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Q \arrow[r] & X \arrow[r] & E \arrow[r] & 0
\end{tikzcd}$$

with $E$ of finite length. Since $Q$ is divisible, the lower sequence splits, and it follows that $f$ lifts to $L'$. This shows that $Q$ is $L'$-injective. Since the line bundles form a system of generators of $\vec{\mathcal{H}}$, we obtain by the version [63, V. Prop. 2.9] of Baer's criterion that $Q$ is injective in $\vec{\mathcal{H}}$. □

Remark 3.5. By the closure properties mentioned above, the class $\mathcal{D}$ of divisible sheaves is a torsion class. Given an object $X \in \vec{\mathcal{H}}$, we denote by $dX$ the largest divisible subsheaf of $X$. Since $dX$ is injective,

$$X \simeq dX \oplus X/dX.$$

The sheaves with $dX = 0$, called *reduced*, form the torsion-free class corresponding to the torsion class $\mathcal{D}$.

Proposition 3.6.

(1) *The indecomposable injective sheaves are (up to isomorphism) the sheaf $\mathcal{K}$ of rational functions and the Prüfer sheaves $S[\infty]$ ($S \in \mathcal{H}$ simple).*

(2) *Every torsion sheaf $F$ is of the form*

$$F = \bigoplus_{x \in \mathbb{X}} F_x \quad \text{with} \quad F_x \in \vec{\mathcal{U}}_x \text{ unique,} \tag{3.2}$$

*and there are pure-exact sequences*

$$0 \to E_x \to F_x \to P_x \to 0 \tag{3.3}$$

*in $\vec{\mathcal{U}}_x$ with $E_x$ a direct sum of indecomposable finite length sheaves and $P_x$ a direct sum of Prüfer sheaves (for all $x \in \mathbb{X}$).*

(3) *Every sheaf of finite length is $\Sigma$-pure-injective.*

*Proof.* (1) It is well-known that in a locally noetherian category every injective object is a direct sum of indecomposable injective objects. Every indecomposable injective object has a local endomorphism ring and is the injective envelope of each of its non-zero subobjects. For details we refer to [26].

Let $E$ be an indecomposable injective sheaf. We consider its torsion part $tE$. If $tE \neq 0$, then $E$ has a simple subsheaf $S$. It follows that $E$ is injective envelope of $S$, and thus it contains the direct family $S[n]$ $(n \geq 1)$ and its union $S[\infty]$. We claim that $E = S[\infty]$. Indeed, it is easy to see that $S[\infty]$ is uniserial, with each proper subobject of the form $S[n]$ for some $n \geq 1$. If there were a simple object $U$ with $0 \neq \operatorname{Ext}^1(U, S[\infty]) = \operatorname{D}\operatorname{Hom}(S[\infty], \tau U)$, then there would be a surjective map $S[\infty] \to \tau U$, whose kernel would have to be a (maximal) subobject of $S[\infty]$, hence of the form $S[n]$, which is impossible since $S[\infty]$ has infinite length. It follows that $S[\infty]$ is divisible, thus injective, and we conclude $E = S[\infty]$.

If, on the other hand, $tE = 0$, then $E$ is torsionfree and contains a line bundle $L'$ as a subobject. Then $E$ is the injective envelope of $L'$. In the quotient category $\mathcal{H}/\mathcal{H}_0$ the structure sheaf $L$ and $L'$ become isomorphic ([47]), and thus (by definition of the morphism spaces in the quotient category) there is a third line bundle $L''$ which maps non-trivially to both, $L'$ and $L$. It follows that $L'$ has the same injective envelope as $L$, namely $\mathcal{K}$.

(2) The torsion class $\mathcal{T}$ is a hereditary (cf. [52, Prop. A.2]) locally finite Grothendieck category with injective cogenerator given by the direct sum of all the Prüfer sheaves. We have the coproduct of (locally finite) categories

$$\mathcal{T} = \coprod_{x \in \mathbb{X}} \vec{\mathcal{U}}_x, \tag{3.4}$$

from which we derive (3.2).

In order to proof the existence of a sequence (3.3), we show that $\vec{\mathcal{U}}_x$ *coincides with the category of torsion modules over a certain bounded hereditary noetherian prime ring*, and then we apply the similar result [61, Thm. 1] for modules.

To this end we briefly recall some notions, cf. [64, Ch. 4]: let $M_R$ be a topological module over the topological ring $R$; then $M$ is called pseudo-compact if it is Hausdorff, complete, and its topology is generated by submodules of finite colength; the ring $R$ is called pseudo-compact if $R_R$ is. Moreover, $M_R$ is called discrete if its topology is discrete; this is the case if and only if the right annihilator ideals $\operatorname{Ann}(x)$ are open for every $x \in M$.

Let now $\mathcal{U} = \mathcal{U}_x$ be a tube of rank $p \geq 1$, with simple objects $S, \tau S, \dots, \tau^{p-1}S$, and $E$ the injective cogenerator of $\vec{\mathcal{U}}$ given by $\bigoplus_{j=0}^{p-1} \tau^j S[\infty]$. Its (opposite) endomorphism algebra $R = \operatorname{End}(E)^{\mathrm{op}}$ is a pseudo-compact ring: a basis of a suitable (Gabriel) topology is given by the right ideals $I(U)$ of endomorphisms of $E$ annihilating $U$ (for $U \in \mathcal{U}$). By [26, IV.4. Cor. 1] the category $\vec{\mathcal{U}}$ is dual to $\operatorname{PC}(R)$, the category of pseudo-compact $R$-modules, the duality is given by the functor $X \mapsto \operatorname{Hom}(X, E)$; note that in [26] left modules are considered, whereas we consider right modules, like in [64]. Since $\operatorname{soc}(E) = \bigoplus_{i=0}^{p-1} \tau^i S$, we get $R/\operatorname{rad}(R) \simeq \operatorname{End}(\operatorname{soc}(E)) \simeq D^p$ as $k$-algebras, with $D = \operatorname{End}(\tau^i S)$, by [26, IV.4. Prop. 12]. In particular, the simple $R$-modules are finite dimensional. It follows that $R$ is cofinite in the sense of [64]. From [64, Prop. 4.10] we get that $R^{\mathrm{op}} = \operatorname{End}(E)$ is also pseudo-compact, and $\operatorname{PC}(R)^{\mathrm{op}} \simeq \operatorname{Dis}(R^{\mathrm{op}})$. Thus, $\vec{\mathcal{U}}$ is equivalent to $\operatorname{Dis}(R^{\mathrm{op}})$.

We now show that "discrete module" coincides with "torsion module". Using the special shape of $\vec{\mathcal{U}}$, it follows from [1] (cf. also [39, Prop. 13.4]) that $R^{\mathrm{op}} \simeq H_p(V, \mathfrak{m})$, given by matrices $(a_{ij}) \in \mathrm{M}_p(V)$ with $a_{ij} \in \mathfrak{m}$ for $j > i$; here $V = \operatorname{End}(\tau^i S[\infty])$ is a (noncommutative) complete local principal ideal domain with maximal ideal $\mathfrak{m}$, so that every non-zero one-sided ideal is a power of $\mathfrak{m}$. In particular, $R^{\mathrm{op}}$ is a complete semiperfect, bounded hereditary noetherian prime ring. By [65, Prop. 3.22] the topology on $R^{\mathrm{op}}$ is the $J$-adic one, with $J$ the Jacobson radical, which is generated by a normal and regular element. Since moreover, by the special shape of $R^{\mathrm{op}}$, each non-zero ideal contains a power of $J$, we readily see that $M \in \operatorname{Mod}(R^{\mathrm{op}})$ is discrete if and only if each element in $M$ is annihilated by a power of $J$, or equivalently, each element in $M$ is annihilated by a non-zero ideal. This means that $M$ is torsion in the sense of [55, p. 373]. In particular, then each element in $M$ is annihilated by a regular element. The converse is also true: by [63, Sec. IV.6.3.] each regular element generates an essential right ideal, which, by boundedness, contains a non-zero ideal.

We summarize: *The category $\vec{\mathcal{U}}$ coincides with the category of those $R^{\mathrm{op}}$-modules $M$ which are torsion in the sense that each element of $M$ is annihilated by a regular element.* Now, in the terminology of [61], the sequence (3.3) expresses that $E_x$ is a *basic submodule* of the torsion module $F_x$, and the existence of such a pure submodule is given by [61, Thm. 1].

(3) Each indecomposable $R$-module $F$ of finite length has finite endolength, since it is finite dimensional over $k$, by the argument from the preceding part. From [66, Beisp. 2.6 (1)] we obtain that $F$ is a $\Sigma$-pure-injective $R$-module. Since an object $M$ in a locally noetherian category is pure-injective if and only if the summation map $M^{(I)} \to M$ factors through the canonical embedding

$M^{(I)} \to M^I$ for every $I$ (we refer to [51, Thm. 5.4]), we conclude that $F$ is $\Sigma$-pure-injective also in $\vec{\mathcal{H}}$. $\square$

If $F$ is a torsion sheaf like in (3.2), we call the set of those $x \in \mathbb{X}$ with $F_x \neq 0$ the *support* of $F$. If the support of $F$ is of the form $\{x\}$, we say $F$ is *concentrated* at $x$.

Corollary 3.7. *Let $F \in \vec{\mathcal{H}}$ be a torsion sheaf.*

(1) *There is a pure-exact sequence*

$$0 \to E \xrightarrow{\subseteq} F \to F/E \to 0 \tag{3.5}$$

*such that $E$ is a direct sum of finite length sheaves and $F/E$ is injective.*

(2) *If $F$ has no non-zero direct summand of finite length, then $F$ is a direct sum of Prüfer sheaves.*

(3) *If $F$ is a reduced torsion sheaf and $E_1, \ldots, E_n$ are the only indecomposable direct summands of $F$ of finite length, then $F$ is pure-injective and isomorphic to $\bigoplus_{j=1}^n {E_j}^{(I_j)}$ for suitable sets $I_j$.*

(4) *If $F$ is indecomposable, then $F$ is either of finite length or a Prüfer sheaf.*

*Proof.* (1) The direct sum of all pure-exact sequences (3.3) ($x \in \mathbb{X}$) is pure-exact.
(2) This follows from (1) by purity. (Locally, in $x$, we can also refer to [60, Thm. 10].)
(3) We consider the pure-exact sequence (3.5). By assumption, $E$ must be of the form $\bigoplus_{j=1}^n {E_j}^{(I_j)}$ (indeed, since $E$ is pure in $F$, its direct summands of finite length, being pure-injective, are also direct summands of $F$). Now $E$ is, by part (3) of Proposition 3.6, pure-injective, and thus $F \simeq E \oplus F/E$. Since $F$ is reduced, we conclude $F \simeq E$.
(4) This follows readily from (2). $\square$

The following basic splitting property will be crucial for our treatment of large tilting sheaves.

Theorem 3.8. *Let $T \in \vec{\mathcal{H}}$ be a sheaf such that $\operatorname{Ext}^1(T,T) = 0$ holds.*

(1) *The torsion part $tT$ is a direct sum of Prüfer sheaves and exceptional sheaves of finite length. Accordingly, it is pure-injective.*

(2) *The canonical exact sequence $0 \to tT \to T \to T/tT \to 0$ splits.*

*Proof.* By Lemma 3.3 it suffices to prove part (1). By Lemma 2.1 the assertion is true in case $tT$ is coherent. If $tT$ does not admit any non-zero summand of finite length, then we conclude from Corollary 3.7 (2) that $tT$ is a direct sum of Prüfer sheaves, and then $tT$ is in particular pure-injective. Let now $E$ be an indecomposable summand of $tT$ of finite length. The composition of embeddings $E \to tT \to T$ gives a surjection $\operatorname{Ext}^1(T,T) \to \operatorname{Ext}^1(E,T)$, showing that $\mathrm{Ext}^1(E,T)=0$. Forming the push-out, the projection $tT \to E$ yields the following commutative exact diagram.

$$\begin{tikzcd}
0 \arrow[r] & tT \arrow[r] \arrow[d, two heads] & T \arrow[r] \arrow[d] & T/tT \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E \arrow[r] & T' \arrow[r] & T/tT \arrow[r] & 0.
\end{tikzcd}$$

Using Serre duality $\mathrm{Ext}^1(T/tT,E) = \mathrm{D}\,\mathrm{Hom}(\tau^- E, T/tT) = 0$, the lower sequence splits, showing that there is an epimorphism $T \to E$. This gives a surjective map $\mathrm{Ext}^1(E,T) \to \mathrm{Ext}^1(E,E)$, showing that $\mathrm{Ext}^1(E,E)=0$. Therefore $E$ must belong to an exceptional tube of some rank $p>1$, and has length $<p$. Thus there are only finitely many such $E$. From Corollary 3.7 and Remark 3.5 we conclude that $tT$ is a direct sum of copies of these finitely many exceptionals of finite length and of Prüfer sheaves. This proves the theorem. □

Given a tilting sheaf $T \in \vec{\mathcal{H}}$, we will often write

$$T = T_+ \oplus T_0$$

with $T_0 = tT$ the *torsion* and $T_+ \simeq T/tT$ the *torsionfree part* of $T$. We will say that $T$ has a *large torsion part* if $tT$ is large in the sense that there is no coherent sheaf $E$ such that $\mathrm{Add}(tT) = \mathrm{Add}(E)$.

## 4. Tilting sheaves induced by resolving classes

In this section we introduce the notion of a resolving class, and we employ it to construct the torsionfree Lukas tilting sheaf $\mathbf{L}$ and the tilting sheaves $T_{(B,V)}$. We further classify all tilting sheaves with large torsion part, and we establish a bijection between resolving classes and tilting classes of finite type.

4.1. Let $\vec{\mathcal{H}}$ be a locally coherent Grothendieck category with $\mathcal{H} = \mathrm{fp}(\vec{\mathcal{H}})$. Let $T$ be a tilting object of finite type in $\vec{\mathcal{H}}$, that is,

$$\mathcal{B} := \mathrm{Gen}(T) = T^{\perp_1} = \mathscr{S}^{\perp_1}$$

for some $\mathscr{S} \subseteq \mathcal{H}$, which we choose to be the largest class with this property

$$\mathscr{S} = {}^{\perp_1}\mathcal{B} \cap \mathcal{H}.$$

Applying $\mathrm{Ext}^1(S,-)$ to the sequence

$$0 \to X \to E(X) \to E(X)/X \to 0 \tag{4.1}$$

where $X \in \vec{\mathcal{H}}$ is arbitrary and $E(X)$ is its injective envelope, we see that

(o) $\mathscr{S}$ consists of objects $S$ with $\mathrm{pd}_{\vec{\mathcal{H}}}(S) \le 1$.

We list further properties of $\mathscr{S}$ that can be verified by the reader:

(i) $\mathscr{S}$ is closed under extensions;
(ii) $\mathscr{S}$ is closed under direct summands;
(iii) $S' \in \mathscr{S}$ whenever $0 \to S' \to S \to S'' \to 0$ is exact with $S, S'' \in \mathscr{S}$.

Definition 4.2. Let $\vec{\mathcal{H}}$ be a locally coherent Grothendieck category. We call a class $\mathscr{S} \subseteq \mathcal{H} = \mathrm{fp}(\vec{\mathcal{H}})$ *resolving* if it satisfies (i), (ii), (iii), and generates $\vec{\mathcal{H}}$.

Remark 4.3. A generating system $\mathscr{S} \subseteq \mathcal{H}$ is resolving whenever it is closed under extensions and subobjects. In case $\vec{\mathcal{H}} = \operatorname{Qcoh} \mathbb{X}$ the converse also holds true; we refer to Corollary 4.17 below.

Theorem 4.4. *Let $\vec{\mathcal{H}}$ be locally coherent and $\mathscr{S}$ a resolving class such that $\mathrm{pd}_{\vec{\mathcal{H}}}(S) \leq 1$ for all $S \in \mathscr{S}$. Then there is a tilting object $T$ in $\vec{\mathcal{H}}$ with $T^{\perp_1} = \mathscr{S}^{\perp_1}$.*

*Proof.* The class $\mathcal{B} = \mathscr{S}^{\perp_1}$ is pretorsion, that is, it is closed under direct sums (recall that $\mathscr{S} \subseteq \mathcal{H}$ consists of finitely presented objects) and epimorphic images (here we need the assumption on the projective dimension). Further, it is special preenveloping as $({}^{\perp_1}\mathcal{B}, \mathcal{B})$ is a complete cotorsion pair, see [58, Sec. 1.3 and Cor. 2.15]. By assumption, $\mathscr{S}$ contains a system of generators $(G_i, i \in I)$ for $\vec{\mathcal{H}}$. Set $G = \bigoplus_{i \in I} G_i$, and take a special $\mathcal{B}$-preenvelope of $G$, i.e. a short exact sequence

$$0 \to G \to T_0 \to T_1 \to 0 \tag{4.2}$$

where $T_0 \in \mathcal{B}$ and $T_1 \in {}^{\perp_1}\mathcal{B}$. Since $\mathcal{B}$ is pretorsion, also $T_1 \in \mathcal{B}$, and $T = T_0 \oplus T_1$ satisfies $\mathrm{Gen}(T) \subseteq \mathcal{B}$. We claim that $T$ is the desired tilting object. Indeed, for every $X \in \vec{\mathcal{H}}$ there is a natural isomorphism

$$\mathrm{Ext}^1\Big(\bigoplus_{i \in I} G_i, X\Big) \simeq \prod_{i \in I} \mathrm{Ext}^1(G_i, X). \tag{4.3}$$

(This we get from the natural isomorphism $\mathrm{Hom}(\bigoplus_{i \in I} G_i, X) \simeq \prod_{i \in I} \mathrm{Hom}(G_i, X)$ by applying $\mathrm{Hom}(G_i, -)$ and $\mathrm{Hom}(\bigoplus_{i \in I} G_i, -)$ to the exact sequence (4.1).) Since $G_i \in \mathscr{S}$ for all $i \in I$, we deduce

$$\mathrm{Ext}^1(G, X) = 0 \quad \text{for all } X \in \mathcal{B}. \tag{4.4}$$

Hence $G \in {}^{\perp_1}\mathcal{B}$, and (4.2) shows that $T_0$ and $T$ belong to ${}^{\perp_1}\mathcal{B}$ as well. So

$$\mathrm{Gen}(T) \subseteq \mathcal{B} \subseteq T^{\perp_1}.$$

Let now $X \in T^{\perp_1}$. Since $G$ is a generator, there is an epimorphism $G^{(J)} \to X$ and a commutative exact diagram

$$\begin{tikzcd} 0 \arrow[r] & G^{(J)} \arrow[r] \arrow[d] & (T_0)^{(J)} \arrow[r] \arrow[d] & (T_1)^{(J)} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X \arrow[r] & X' \arrow[r] & (T_1)^{(J)} \arrow[r] & 0. \end{tikzcd}$$

Since $X \in {T_1}^{\perp_1}$ and thus by (4.3) also $X \in ({T_1}^{(J)})^{\perp_1}$, the lower sequence splits. Therefore we get an epimorphism ${T_0}^{(J)} \to X$, showing that $X \in \mathrm{Gen}(T)$. We conclude that $T$ is a tilting object with $\mathrm{Gen}(T) = \mathcal{B}$. $\square$

*Let now $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$, where $\mathbb{X}$ is a weighted noncommutative regular projective curve over a field $k$.* We exhibit two applications of the theorem. The first one is quite easy.

Proposition 4.5. *Let $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$, where $\mathbb{X}$ is a weighted noncommutative regular projective curve. There is a torsionfree large tilting sheaf $\mathbf{L}$, called* Lukas tilting sheaf, *such that $\mathbf{L}^{\perp_1} = (\operatorname{vect}\mathbb{X})^{\perp_1}$.*

*Proof.* The class $\mathscr{S} = \operatorname{vect}\mathbb{X}$ is resolving. By Theorem 4.4 there is a tilting sheaf $\mathbf{L}$ with $(\operatorname{vect}\mathbb{X})^{\perp_1} = \mathbf{L}^{\perp_1}$. We show that $\mathbf{L}$ is torsionfree. Assume that $\mathbf{L}$ has a non-zero torsion part $T_0$. By Theorem 3.8 this is a direct summand of $\mathbf{L}$. Then

$$(\operatorname{vect}\mathbb{X})^{\perp_1} = \mathbf{L}^{\perp_1} \subseteq {T_0}^{\perp_1} \cap (\operatorname{vect}\mathbb{X})^{\perp_1} \subsetneq (\operatorname{vect}\mathbb{X})^{\perp_1},$$

where the last inclusion is proper because there exists a simple sheaf $S$ with $\operatorname{Hom}(S, T_0) \neq 0$ and thus $\tau S \in (\operatorname{vect}\mathbb{X})^{\perp_1} \setminus {T_0}^{\perp_1}$. Thus we get a contradiction. We conclude that $T_0 = 0$. Clearly, $\mathbf{L}$ is then also large. $\square$

We record the following observation for later reference.

Lemma 4.6. *$\mathbf{L}^{\perp_1}$ contains the class $\mathcal{D}_V$ of $V$-divisible sheaves for any $\emptyset \neq V \subseteq \mathbb{X}$.*

*Proof.* With the notation of Definition 3.1, we have ${\mathscr{S}_V}^{\perp_1} = {}^{\perp_0}\mathscr{S}_V$ and $(\operatorname{vect}\mathbb{X})^{\perp_1} = {}^{\perp_0}\operatorname{vect}\mathbb{X}$ by Serre duality. Let $F$ be a sheaf such that there is a non-zero morphism to a vector bundle, and consequently also to a line bundle. Since every non-zero subsheaf of a line bundle is a line bundle again, there is even an epimorphism from $F$ to a line bundle. This line bundle maps onto a simple sheaf concentrated at $x \in V$. We conclude that $F$ is not $V$-divisible. $\square$

The second application is the classification of all tilting sheaves having a large torsion part. We first introduce some terminology.

4.7. Branch sheaves. Let $\mathcal{U} = \mathcal{U}_x$ be a tube of rank $p > 1$. We recall that an indecomposable sheaf $E \in \mathcal{U}$ is exceptional (that is, $\operatorname{Ext}^1(E, E) = 0$) if and only if its length is $\leq p - 1$; in particular, there are only finitely many such $E$. If $E$ is exceptional in $\mathcal{U}$, then we call the collection $\mathcal{W}$ of all the subquotients of $E$ the *wing rooted in* $E$. The set of all simple sheaves in $\mathcal{W}$ is called the *basis* of $\mathcal{W}$. It is of the form $S, \tau^{-}S, \ldots, \tau^{-(r-1)}S$ for an exceptional simple sheaf $S$ and an integer $r$ with $1 \leq r \leq p - 1$ which equals the length of the *root* $E$; we call such a set of simples a *segment* in $\mathcal{U}$, and we say that two wings (or segments) in $\mathcal{U}$ are *non-adjacent* if the segments of their bases (or the segments) are disjoint and their union consists of $< p$ simples and is not a segment [46, Ch. 3].

We remark that the full subcategory $\operatorname{add}\mathcal{W}$ of $\mathcal{H}$ is equivalent to the category of finite-dimensional representations of the linearly oriented Dynkin quiver $\vec{\mathbb{A}}_r$, cf. [46, Ch. 3]. By [56, p. 205] any tilting object $B$ in the category $\operatorname{add}\mathcal{W}$ has precisely $r$ non-isomorphic indecomposable summands $B_1, \ldots, B_r$ forming a so-called *connected branch* $B$ in $\mathcal{W}$: one of the $B_i$ is isomorphic to the root

$E$, and for every $j$ the wing rooted in $B_j$ contains precisely $\ell_j$ indecomposable summands of $B$, where $\ell_j$ is the length of $B_j$. In particular, for every $j$ we have a (full) *subbranch* of $B$ rooted in $B_j$; if $B_j$ is different from the root of $\mathcal{W}$, we call this subbranch *proper*.

Following [46, Ch. 3], we call a sheaf $B$ of finite length a *branch sheaf* if it is a multiplicity free direct sum of connected branches in pairwise non-adjacent wings; it then follows that $\operatorname{Ext}^1(B, B) = 0$.

Every branch sheaf $B$ decomposes into $B = \bigoplus_{x\in\mathbb{X}} B_x$; of course $B_x \neq 0$ only if $x$ is one of the finitely many exceptional points $x_1, \dots, x_t$, and there are only finitely many isomorphism classes of branch sheaves.

Given a non-empty subset $V \subseteq \mathbb{X}$, we can also write

$$B = B_{\mathfrak{i}} \oplus B_{\mathfrak{e}}$$

where $B_{\mathfrak{e}}$ is supported in $\mathbb{X} \setminus V$ and $B_{\mathfrak{i}}$ in $V$. In such case we will say that $B_{\mathfrak{e}}$ is *exterior* and $B_{\mathfrak{i}}$ is *interior* with respect to $V$.

We now turn to the main result of this section. It states that any choice of a non-empty subset $V \subseteq \mathbb{X}$ and a branch sheaf $B$ determines a unique tilting sheaf $T$ with large torsion part, and every such tilting sheaf arises in this way. More precisely, the set $V$ is the support of the non-coherent (Prüfer) summands in the torsion part $tT$ of $T$, while $B$ collects the coherent summands of $tT$. Furthermore, the summand $B_{\mathfrak{i}}$ of $B$ which is interior with respect to $V$ determines the rays contributing a Prüfer summand to $T$.

Theorem 4.8. *Let $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$, where $\mathbb{X}$ is a weighted noncommutative regular projective curve.*

(1) *Let $\emptyset \neq V \subseteq \mathbb{X}$ and $B \in \mathcal{H}_0$ be a branch sheaf. There is, up to equivalence, a unique large tilting sheaf $T = T_+ \oplus T_0$ whose whose torsionfree part $T_+$ is $V$-divisible, and whose torsion part is given by*

$$T_0 = B \oplus \bigoplus_{x\in V} \bigoplus_{j\in\mathcal{R}_x} \tau^j S_x[\infty], \tag{4.5}$$

*where the non-empty sets $\mathcal{R}_x \subseteq \{0, \dots, p(x) - 1\}$ are uniquely determined by $B$, see (4.8).*

(2) *Every tilting sheaf with large torsion part is, up to equivalence, as in (1).*

Notation. Let $\emptyset \neq V \subseteq \mathbb{X}$ and $B = B_{\mathfrak{i}} \oplus B_{\mathfrak{e}}$ be a branch sheaf with interior and exterior part with respect to $V$ given by $B_{\mathfrak{i}}$ and $B_{\mathfrak{e}}$, respectively. The large tilting sheaf from Theorem 4.8 will be denoted by

$$T_{(B,V)} = T_{(B_{\mathfrak{i}},V)} \oplus B_{\mathfrak{e}}. \tag{4.6}$$

For the proof we need several preparations. We start by describing the torsion part of a tilting sheaf.

Lemma 4.9. *Let $T$ be a tilting sheaf and $x$ an exceptional point of weight $p = p(x) > 1$ such that $(tT)_x \neq 0$. There are two possible cases:*

(1) *“Exterior branch”: $(tT)_x$ contains no Prüfer sheaf, but at most $p-1$ indecomposable summands of finite length, which are arranged in connected branches in pairwise non-adjacent wings.*

(2) *“Interior branch”: $(tT)_x$ contains precisely $s$ Prüfer sheaves, where $1 \le s \le p$, and precisely $p-s$ indecomposable summands of finite length. The latter lie in wings of the following form: if $S[\infty]$, $\tau^{-r}S[\infty]$ are summands of $T$ with $2 \le r \le p$, but the Prüfer sheaves $\tau^{-}S[\infty], \dots, \tau^{-(r-1)}S[\infty]$ in between are not, then there is a (unique) connected branch in the wing $\mathcal{W}$ rooted in $S[r-1]$ that occurs as a summand of $T$.*

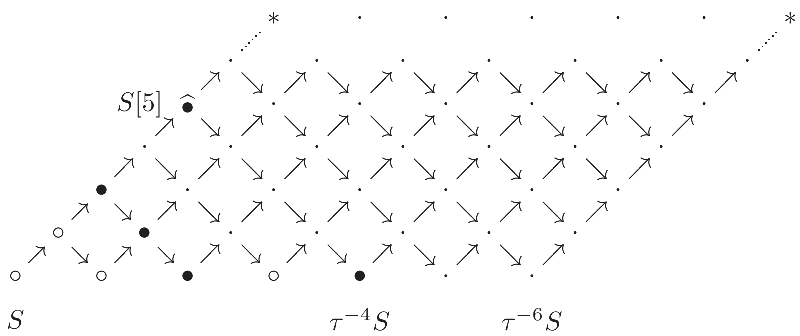


Figure 4.1. Lemma 4.9 (2) with $r=6$, $\bullet$ = a branch $B$, $\widehat{\bullet}$ = its root, $*$ = Prüfer summand of $T$; $\circ$ = undercut $B^{>}$ as in (4.9)

*Proof.* Given a simple object $S \in \mathcal{U}_x$, the corresponding Prüfer sheaf $S[\infty]$ is $S[p]$-filtered, and thus by [58, Prop. 2.12] we have

(4.7) $S[\infty]$ is a summand of $T$ $\Leftrightarrow$ ${}^{\perp_1}(T^{\perp_1})$ contains the ray $\{S[n] \mid n \ge 1\}$.

If no such ray exists, then $(tT)_x$ has at least one indecomposable summand of finite length, and it is well-known that all such summands are arranged in branches in pairwise non-adjacent wings, compare [46, Ch. 3].

Assume now that, say, $S[\infty]$ and $\tau^{-r}S[\infty]$ are summands of $T$, but no Prüfer sheaf “in between” is a summand, where $2 \le r \le p$ (when $r=p$, there is precisely one Prüfer summand). We show that $S[r-1]$ is a summand of $T$. By (4.7) this is equivalent to show $\operatorname{Ext}^1(T, S[r-1]) = 0$. If this is not the case, then $\operatorname{Hom}(\tau^{-}S[r-1], T) \neq 0$, and thus there exists an indecomposable summand $E$ of $T$ lying on a ray starting in $\tau^{-}S[r-1], \dots, \tau^{-(r-2)}S[2]$ or $\tau^{-(r-1)}S$. But for such an $E$ we have $0 \neq \mathrm{D}\operatorname{Hom}(\tau^{-}E, \tau^{-r}S[\infty]) = \operatorname{Ext}^1(\tau^{-r}S[\infty], E)$, contradicting the fact that $T$ has no self-extension. Thus $S[r-1]$ is a direct summand of $T$. The latter argument also shows that every indecomposable summand of $T$ of finite length and lying on a ray starting in $S, \tau S, \dots, \tau^{-(r-1)}S$ actually lies in the wing $\mathcal{W}$ rooted in $S[r-1]$.

We claim that the direct sum $B$ of all indecomposable summands of $T$ lying in $\mathcal{W}$ forms a tilting object in $\operatorname{add}\mathcal{W}$. We have $\operatorname{Ext}^1(B,B)=0$. Assume that $B$ is not a tilting object in $\mathcal{W}$. Then there is an indecomposable $E\in\mathcal{W}$, not a direct summand of $B$, such that $\operatorname{Ext}^1(E\oplus B, E\oplus B)=0$. Let $E'$ be the indecomposable quotient of $S[r-1]$ such that $E$ embeds into $E'$. We have a short exact sequence $0\to F\to S[r-1]\to E'\to 0$ with indecomposable $F\in\mathcal{W}$. Let $T_+$ be the torsionfree part of $T$. Then exactness of $0=\operatorname{Hom}(F,T_+)\to \operatorname{Ext}^1(E',T_+)\to\operatorname{Ext}^1(S[r-1],T_+)=0$ shows $\operatorname{Ext}^1(E',T_+)=0$, and then also $\operatorname{Ext}^1(E,T_+)=0$. Moreover $\operatorname{Ext}^1(T_+,E)=\operatorname{D}\operatorname{Hom}(\tau^- E,T_+)=0$, and since $E\in\mathcal{W}$, there are no extensions between $E$ and Prüfer summands of $T$. We conclude that $E\in T^{\perp_1}\cap{}^{\perp_1}(T^{\perp_1})=\operatorname{Add}(T)$, a contradiction. Thus $B$ is tilting, and it forms a connected branch.

Doing this with every "gap" between Prüfer sheaves in $(tT)_x$, one sees that $(tT)_x$ contains precisely $p-s$ indecomposable summands of finite length. $\square$

Lemma 4.10. *In the preceding lemma, the torsionfree part $T_+$ of $T$ belongs to $\mathcal{W}^{\perp_1}$ for every wing $\mathcal{W}$ occurring in (1) or (2), and it is even $x$-divisible in case (2).*

*Proof.* The first part of the statement is shown as in the preceding proof. In case (2) it then remains to check that $T_+$ has no extensions with the simple objects in $\mathcal{U}_x$ which do not belong to the wings defined by the Prüfer summands of $T$. Let $\mathcal{W}$ be such wing and $E$ such simple object, that is, $E\notin\mathcal{W}$, but $\tau E\in\mathcal{W}$. Assume $0\neq\operatorname{Ext}^1(E,T_+)\simeq\operatorname{D}\operatorname{Hom}(T_+,\tau E)$. Since $\operatorname{Hom}(T_+,\tau\mathcal{W})=0$, repeated application of the almost split property yields an indecomposable object $U$ on the ray starting in $S$ such that $\operatorname{Hom}(T_+,\tau U)\neq 0$. By Serre duality $\operatorname{Ext}^1(U,T_+)\neq 0$, and since $U$ embeds in $S[\infty]$, also $\operatorname{Ext}^1(S[\infty],T_+)\neq 0$, a contradiction. $\square$

As mentioned above, the interior branch sheaves and the Prüfer sheaves occurring in the torsion part of a tilting sheaf are interrelated. In the situation of Lemma 4.9 (2), we denote by $\mathcal{R}_x$ the set of cardinality $s$ of all $j\in\{0,\dots,p(x)-1\}$ such that the Prüfer sheaf $\tau^j S[\infty]$ is a direct summand of $T$. Each such set defines a unique collection

$$\mathcal{W}=\{\tau^j S[\infty]\mid j\in\mathcal{R}_x\}^{\perp_1}\cap\mathcal{U}_x$$

of pairwise non-adjacent wings in the exceptional tube $\mathcal{U}_x$, whereas the branch $B$, viewed as collection of indecomposable sheaves, is given as

$$B=\operatorname{Add}(T)\cap\mathcal{U}_x.$$

In particular, this shows that a tilting sheaf $T'$ with a different branch $B'\neq B$ in $\mathcal{U}_x$ will have $T'^{\perp_1}\neq T^{\perp_1}$, that is, $T$ and $T'$ cannot be equivalent.

Conversely, every non-zero branch sheaf in $\mathcal{U}_x$ – which we will often identify with the set of its indecomposable summands – defines a unique collection $\mathcal{W}$

of pairwise non-adjacent wings in $\mathcal{U}_x$, and this defines uniquely the set $\mathcal{R}_x$; namely, if $S, \tau^- S, \ldots, \tau^{-(r-1)}S$ is a basis of one of the wings in $\mathcal{W}$, we have

$$\mathcal{R}_x = \{j = 0, \ldots, p(x) - 1 \mid \tau^{j+1}S \notin \mathcal{W}\}. \tag{4.8}$$

We now consider a pair $(B, V)$ given by a branch sheaf $B \in \mathcal{H}$ and a subset $V \subseteq \mathbb{X}$, and we associate a resolving class to it. For the moment $V = \emptyset$ is permitted. In case $V \neq \emptyset$, the corresponding tilting sheaf $T$ given by Theorem 4.4 will have the properties required by Theorem 4.8.

The resolving class $\mathscr{S}$ associated to $(B, V)$ will consist of all vector bundles, of the rays given by the sets $\mathcal{R}_x$ in (4.8), and of some objects determined by $B$. Up to $\tau$-shift, these objects will lie in the wings defined by $B$, namely, in the part which lies "under" $B$, in a sense that we are going to explain below.

Let us fix some notation. Recall that $B = \bigoplus_{x\in\mathbb{X}} B_x$ where each $B_x$ is a direct sum of connected branches in pairwise non-adjacent wings in $\mathcal{U}_x$. For every $x$ denote by $\mathcal{W}_x$ the collection of all such wings, and for every $x \in V$ let $\mathcal{R}_x$ be the associated non-empty subset of $\{0, \ldots, p(x) - 1\}$ defined by (4.8).

In order to determine the part of $\mathcal{W}_x$ lying "under" $B_x$, we will have to distinguish two cases. In fact, when $B_x$ is exterior with respect to $V$, it turns out that we have to consider $\tau\mathcal{W}_x$ rather than $\mathcal{W}_x$.

Given a connected branch $C$ with associated wing $\mathcal{W}_C$, let us call the set

$$C^> := \begin{cases} C^{\perp_0} \cap \mathcal{W}_C & \text{if } C \text{ is interior,} \\ C^{\perp_0} \cap \tau\mathcal{W}_C & \text{if } C \text{ is exterior,} \end{cases} \tag{4.9}$$

the *undercut* of $C$. The undercut $B^>$ of the branch sheaf $B$ is the union of the undercuts of all its connected branch components. The undercut is illustrated in Figure 4.1 above. Another example is shown in Figure 10.1.

Lemma 4.11. *Let $V \subseteq \mathbb{X}$ and $B = B_{\mathfrak{i}} \oplus B_{\mathfrak{e}}$ be a branch sheaf.*

(1) *With the notation above, the class*

$$\mathscr{S} = \operatorname{add}\Big(\operatorname{vect}\mathbb{X} \cup \tau^-(B^>) \cup \bigcup_{x\in V} \{\tau^j S_x[n] \mid j \in \mathcal{R}_x,\, n \in \mathbb{N}\}\Big) \tag{4.10}$$

*is resolving.*

(2) *If $T$ is a tilting sheaf with $T^{\perp_1} = \mathscr{S}^{\perp_1}$, then $\mathscr{S} = {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$, the torsionfree part $T_+$ is $V$-divisible, and the torsion part is given by*

$$T_0 = B \oplus \bigoplus_{x\in V} \bigoplus_{j\in\mathcal{R}_x} \tau^j S_x[\infty].$$

*Proof.* (1) The class $\mathscr{S}$ is clearly closed under subobjects. A simple case by case analysis shows that $\mathscr{S}$ is also closed under extensions. For instance, if $0 \to A \to E \to C \to 0$ is a short exact sequence with $A$ a vector bundle and $C \in \mathscr{S}$ indecomposable of finite length, then $E = E_+ \oplus E_0$, with $E_+$ a vector bundle and $E_0$ of finite length; it follows that $E_0$ is isomorphic to a subobject of $C$, and thus $E_0 \in \mathscr{S}$, and then $E \in \mathscr{S}$. Compare also [10, p. 36 from line

-19]. Since $\mathscr{S}$ contains the system of generators $\operatorname{vect}\mathbb{X}$, we conclude that it is resolving.
(2) By Serre duality, an indecomposable coherent sheaf $E \in \mathcal{H}$ belongs to ${}^{\perp_1}(T^{\perp_1})$ if and only if $\tau E \in (T^{\perp_1})^{\perp_0} = (\mathscr{S}^{\perp_1})^{\perp_0} = ({}^{\perp_0}\tau\mathscr{S})^{\perp_0}$. We claim that this is further equivalent to $\tau E \in \tau\mathscr{S}$, that is, $E \in \mathscr{S}$. Indeed, the claim is shown by arguing inside the abelian category $\mathcal{H}$ as in [53, Lem. 1.3], keeping in mind that $\tau\mathscr{S}$ is closed under subobjects and extensions by part (1).
We thus have $\mathscr{S} = {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$. It follows from (4.7) that $T$ has precisely the Prüfer summands $\tau^j S_x[\infty]$ with $x \in V$ and $j \in \mathcal{R}_x$. In particular, $T_+$ is $V$-divisible by Lemma 4.10. Furthermore,

$$\mathscr{S}^{\perp_1} \cap \mathscr{S} = T^{\perp_1} \cap {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H} = \operatorname{Add}(T) \cap \mathcal{H}, \tag{4.11}$$

and we now show that this class further coincides with $\operatorname{add}(B)$.
Let $\mathcal{W}$ be the union of non-adjacent wings associated to $B$, and let $B_1$ and $B_2$ be two indecomposable summands of $B$. Then $0 = \operatorname{Ext}^1(B_1, B_2) = \operatorname{D}\operatorname{Hom}(B_2, \tau B_1)$. Thus $\tau B_1 \in B^{\perp_0}$. If $B_1$ is either exterior, or interior with $\tau B_1 \in \mathcal{W}$, then $\tau B_1 \in B^>$, that is, $B_1 \in \tau^-(B^>) \subseteq \mathscr{S}$. If, on the other hand, $B_1$ is interior with $\tau B_1 \notin \mathcal{W}$, then $B_1 \in \mathscr{S}$ by definition of $\mathcal{R}_x$. Moreover, we have $\operatorname{Ext}^1(\tau^-(B^>), B_1) = \operatorname{D}\operatorname{Hom}(B_1, B^>) = 0$, and then $\operatorname{Ext}^1(\tau^j S_x[n], B_1) = \operatorname{D}\operatorname{Hom}(B_1, \tau^{j+1} S_x[n]) = 0$, for any $x \in V$ and $j \in \mathcal{R}_x$, shows that $B_1 \in \mathscr{S}^{\perp_1}$.
Conversely, let $E \in \mathscr{S} \cap \mathscr{S}^{\perp_1}$ be indecomposable. By (4.11) we have that $E$ is a summand of $T$, in particular $E$ is exceptional and belongs to an exceptional tube. If $E$ is supported in $V$, then it is a summand of $B_{\mathfrak{i}}$ by Lemma 4.9 and the fact that the connected parts of $B$ form tilting objects in the corresponding wings. If $E$ is not supported in $V$, then it belongs to $\tau^-(C^>)$ for a connected branch component $C$ of $B_{\mathfrak{e}}$. Since $\tau^-(C^>) = {}^{\perp_1}C \cap \mathcal{W}_C$ where $\mathcal{W}_C$ is the wing associated to $C$, we infer again that $E$ is a summand of $B_{\mathfrak{e}}$.
We conclude that $T_0$ is given by $B \oplus \bigoplus_{x\in V} \bigoplus_{j\in\mathcal{R}_x} \tau^j S_x[\infty]$, as desired. □

We can now complete our classification of tilting sheaves with large torsion part.

*Proof of Theorem 4.8.* (1) By the preceding lemma there exists a (large) tilting sheaf with the claimed properties.
(2) Let now $T = T_+ \oplus T_0$ be any tilting sheaf with a non-coherent torsion part $T_0$. From Lemma 4.9 we infer that $T_0$ is of the form $B \oplus \bigoplus_{x\in V} \bigoplus_{j\in\mathcal{R}_x} \tau^j S_x[\infty]$. It is sufficient to show that the class $\mathscr{S}$ from (4.10) satisfies $\mathscr{S} = {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$, since this will imply $T^{\perp_1} = \mathscr{S}^{\perp_1}$, as desired.
By Lemma 4.10 the torsionfree part $T_+$ of $T$ is $V$-divisible. From Lemma 4.6 we infer $T_+ \in (\operatorname{vect}\mathbb{X})^{\perp_1}$. Since also $T_0 \in (\operatorname{vect}\mathbb{X})^{\perp_1}$ by Serre duality, we conclude $\operatorname{Ext}^1(X, T) = 0$ for any vector bundle $X$, hence $\operatorname{vect}\mathbb{X} \subseteq {}^{\perp_1}(T^{\perp_1})$.
Next, we show $\tau^-(B^>) \subseteq {}^{\perp_1}(T^{\perp_1})$. If $E \in \tau^-(B_{\mathfrak{i}}{}^>)$, then $\operatorname{Ext}^1(E, B) = \operatorname{D}\operatorname{Hom}(B, \tau E) = 0$ by definition of the undercut. Since $T_+$ and the Prüfer sheaves are $V$-divisible, we get $\operatorname{Ext}^1(E, T) = 0$ and $E \in {}^{\perp_1}(T^{\perp_1})$. If $E \in \tau^-(B_{\mathfrak{e}}{}^>)$, then it belongs to $\tau^-(C^>) = {}^{\perp_1}C \cap \mathcal{W}_C$ for a connected

branch component $C$ of $B_{\mathfrak{e}}$ with associated wing $\mathcal{W}_C$. It follows $\operatorname{Ext}^1(E,B) = \operatorname{D}\operatorname{Hom}(B,\tau E) = 0$, and $\operatorname{Ext}^1(E,T_+) = 0$ by Lemma 4.10, so again $E \in {}^{\perp_1}(T^{\perp_1})$.
Finally, if $E$ belongs to a ray $\{\tau^j S_x[n] \mid n \geq 1\}$ with $x \in V$ and $j \in \mathcal{R}_x$, then $E \in {}^{\perp_1}(T^{\perp_1})$ by (4.7).
Altogether we have shown $\mathscr{S} \subseteq {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$. In order to prove the reverse inclusion, let $E \in \mathcal{H}$ be indecomposable with $E \in {}^{\perp_1}(T^{\perp_1})$. By definition of $\mathscr{S}$, we can assume that $E$ is of finite length, and further, if concentrated at a point $x \in V$, that it has the form $\tau^j S_x[n]$ with $j \notin \mathcal{R}_x$. This means $\tau^j S_x \in \tau^- \mathcal{W}$ by (4.8), so there is a connected branch component $C$ of $B_{\mathfrak{i}}$ with associated wing $\mathcal{W}_C$ such that $\tau^j S_x \in \tau^- \mathcal{W}_C$. Since $C$ is a summand of $T$, we have $E \in {}^{\perp_1}C \cap \tau^- \mathcal{W}_C = \tau^-(C^{>}) \subseteq \mathscr{S}$.
It remains to check the case when $E$ is concentrated at a point $x \notin V$. Notice that $\operatorname{Hom}(T,\tau E) \simeq \operatorname{D}\operatorname{Ext}^1(E,T) = 0$ implies $\operatorname{Ext}^1(T,\tau E) \neq 0$ by condition (TS2). But the latter amounts to $\operatorname{Ext}^1(B_{\mathfrak{e}},\tau E) \neq 0$, or equivalently, $\operatorname{Hom}(E,B_{\mathfrak{e}}) \neq 0$. Let $0 \neq f : E \to B_{\mathfrak{e}}$. If $E$ is simple, $f$ is a monomorphism, and $E \in \mathscr{S}$ because $B_{\mathfrak{e}} \in \tau^-(B_{\mathfrak{e}}{}^{>}) \subseteq \mathscr{S}$ and $\mathscr{S}$ is closed under subjects. If $E$ has length $\ell > 1$, we consider the short exact sequence $0 \to \operatorname{Ker} f \to E \to \operatorname{Im} f \to 0$ where $\operatorname{Im} f$ belongs to $\mathscr{S} \subseteq {}^{\perp_1}(T^{\perp_1})$ and $\operatorname{Ker} f \in {}^{\perp_1}(T^{\perp_1})$. Proceeding by induction on $\ell$ and using that $\mathscr{S}$ is closed under extensions, we conclude that $E \in \mathscr{S}$, which completes the proof. □

Corollary 4.12. *Let $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$ with $\mathbb{X}$ a weighted noncommutative regular projective curve. There is a bijection between the equivalence classes of tilting sheaves in $\vec{\mathcal{H}}$ having a large torsion part, and the set of pairs $(B,V)$ given by a branch sheaf $B \in \mathcal{H}$ and a subset $\emptyset \neq V \subseteq \mathbb{X}$.* □

Remark 4.13. It is well known that the hereditary torsion pairs in $\operatorname{Qcoh}\mathbb{X}$ are in bijection with the Serre subcategories of $\operatorname{coh}\mathbb{X}$. As explained in [7, Sec. 5.2], this bijection restricts to a bijective correspondence between the hereditary torsion pairs $(\mathcal{T}, \mathcal{F})$ with non-trivial $\mathcal{F}$ (or equivalently, such that $\mathcal{F}$ generates $\operatorname{Qcoh}\mathbb{X}$) and the Serre subcategories consisting of finite length objects. Moreover, one easily verifies that the Serre subcategories of $\operatorname{add}\mathcal{H}_0$ are precisely the small additive closures of unions of tubes and pairwise non-adjacent wings. In other words, there is a surjective map from the set of all pairs $(B,V)$ given by a branch sheaf $B$ and a subset $V \subseteq \mathbb{X}$, and the Serre subcategories of $\operatorname{add}\mathcal{H}_0$. This map is not injective in general, because different branch sheaves can give rise to the same wings. In the non-weighted case, however, the parametrization of tilting sheaves reduces to the subsets $V \subseteq \mathbb{X}$, and we obtain a bijection between tilting sheaves and faithful hereditary torsion pairs in $\operatorname{Qcoh}\mathbb{X}$, in perfect analogy with the classification of tilting modules over commutative noetherian rings from [8]. For more details we refer to [7, Sec. 5.2].

A correspondence. Next, we establish an analogue of [5, Thm. 2.2] stating that the resolving subclasses of $\mathcal{H}$ correspond bijectively to tilting classes of finite type. As we will see below, in the domestic and in the tubular cases every tilting class is of finite type.

Theorem 4.14. *Let $\mathbb{X}$ be a weighted noncommutative regular projective curve and $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$. The assignments $\Phi\colon \mathscr{S} \mapsto \mathscr{S}^{\perp_1}$ and $\Psi\colon \mathcal{B} \mapsto {}^{\perp_1}\mathcal{B} \cap \mathcal{H}$ define mutually inverse bijections between*

- *resolving classes $\mathscr{S}$ in $\mathcal{H}$, and*
- *tilting classes $\mathcal{B} = T^{\perp_1}$ with $T \in \vec{\mathcal{H}}$ tilting of finite type.*

For the proof of the Theorem, we need the following observations.

Remark 4.15. In the situation of Lemma 4.9 (2), the right perpendicular category $\mathcal{W}^{\perp}$ of a wing $\mathcal{W}$ rooted in $S[r-1]$ coincides with the right perpendicular category to its basis $S, \tau^{-}S, \dots, \tau^{-(r-2)}S$. If $B$ forms a (connected) branch in $\mathcal{W}$, then also $B^{\perp} = \mathcal{W}^{\perp}$, and when forming this perpendicular category, the $r$ rays starting in the simple objects $S, \tau^{-}S, \dots, \tau^{-(r-2)}S, \tau^{-(r-1)}S$ and the corresponding Prüfer sheaves are turned into a single ray $\tau^{-(r-1)}S[rn]$, $n \geq 1$, and a single Prüfer sheaf $S[\infty]$.

Lemma 4.16 (Perpendicular Lemma). *Let $B \in \mathcal{H}$ be a branch sheaf. Let $T \in \vec{\mathcal{H}}$ be a sheaf such that $T \in B^{\perp}$.*

(1) *We have $B^{\perp} \simeq \operatorname{Qcoh}\mathbb{X}'$, where $\mathbb{X}'$ is a noncommutative regular projective curve with reduced weights $1 \leq p_i' \leq p_i$.*

(2) *$T \oplus B$ is a (large) tilting sheaf in $\vec{\mathcal{H}}$ if and only if $T$ is a (large) tilting sheaf in $\vec{\mathcal{H}}' = \operatorname{Qcoh}\mathbb{X}'$.*

*Proof.* (1) This follows from the preceding remark.
(2) It is clear that $T \oplus B$ satisfies (TS1) if and only if so does $T$. We assume that $T \oplus B$ satisfies (TS2). Let $X \in \vec{\mathcal{H}}'$ such that $\operatorname{Hom}(T, X) = 0 = \operatorname{Ext}^1(T, X)$. Since $\vec{\mathcal{H}}' = B^{\perp}$ we get $\operatorname{Hom}(T \oplus B, X) = 0 = \operatorname{Ext}^1(T \oplus B, X)$, and hence $X = 0$ follows, and $T$ satisfies (TS2). Conversely, let $T$ satisfy (TS2). Let $X \in \vec{\mathcal{H}}$ with $\operatorname{Hom}(T \oplus B, X) = 0 = \operatorname{Ext}^1(T \oplus B, X)$. Then in particular $X \in B^{\perp} = \vec{\mathcal{H}}'$, and also $\operatorname{Hom}(T, X) = 0 = \operatorname{Ext}^1(T, X)$. Then $X = 0$, so that $T \oplus B$ satisfies (TS2). □

*Proof of Theorem 4.14.* $\Phi(\mathscr{S}) = \mathscr{S}^{\perp_1}$ defines a map between the named sets by Theorem 4.4. By the discussion in 4.1 we see that $\mathscr{S} := \Psi(\mathcal{B}) = {}^{\perp_1}\mathcal{B} \cap \mathcal{H}$ satisfies conditions (i), (ii) and (iii) for resolving. Notice that $\mathscr{S}$ is even closed under subobjects since $\operatorname{Qcoh}\mathbb{X}$ is hereditary. We show that $\mathscr{S}$ also generates $\vec{\mathcal{H}}$.

First we show that $\mathscr{S}$ contains a non-zero vector bundle. Let $\mathscr{S}' \subseteq \mathcal{H}$ with $\mathcal{B} = \mathscr{S}'^{\perp_1}$. Then

$$\mathscr{S}' \subseteq {}^{\perp_1}(\mathscr{S}'^{\perp_1}) \cap \mathcal{H} = \mathscr{S}. \tag{4.12}$$

We assume that $\mathscr{S}$ does not contain any non-zero vector bundle, which we will lead to a contradiction. Then $\mathscr{S}' \subseteq \mathcal{H}_0$. Let $T$ be tilting with $\mathcal{B} = T^{\perp_1}$. Since a coherent $X$ lies in ${}^{\perp_1}\mathcal{B}$ if and only if $\operatorname{Ext}^1(X, T) = 0$, we get $\operatorname{Hom}(T, E) \neq 0$ for every non-zero vector bundle $E$. If $T$ is additionally torsionfree, then we infer $\operatorname{Ext}^1(T, F) = 0$ for all finite length sheaves $F$. It follows from (TS2) that $T$ is a generator for $\vec{\mathcal{H}}$, and then also projective. From Serre duality we conclude that there is no non-zero morphism from a vector bundle to $T$, which is impossible. If on the other hand, $T$ has a large torsion part, then by Lemma 4.10 the torsionfree part $T_+$ is $x$-divisible for (at least) one point $x$. But $T$, and then also $T_+$, maps epimorphic to some line bundle $L'$, and $L'$ maps non-trivially to a simple sheaf $S_x$ concentrated at $x$, thus $\operatorname{Hom}(T_+, S_x) \neq 0$, contradicting the $x$-divisibility. The final case to consider is that the torsion part $T_0$ is a branch sheaf $B$. By Lemma 4.16 then $T_+$ is torsionfree tilting in $B^\perp = \operatorname{Qcoh}\mathbb{X}' \subseteq \vec{\mathcal{H}}$. Since $\operatorname{vect}\mathbb{X}' = \operatorname{vect}\mathbb{X} \cap B^\perp$ (the inclusion of the right perpendicular category is rank-preserving, by [28, Prop. 9.6]), we infer that $T_+$ maps non-trivially to any non-zero vector bundle over $\mathbb{X}'$, and we get a contradiction by the torsionfree case treated before. Thus in any case, $\mathscr{S}$ contains a non-zero vector bundle.

Since $\mathscr{S}$ is closed under subobjects, it contains also a line bundle $L'$. By [52, Lem. IV.4.1], [39, Rem. 3.8] there is a suitable product $\sigma$ of tubular shifts such that $(L', \sigma)$ forms an ample pair, and there is a monomorphism $\sigma^{-1}L' \to L'$. We conclude that $\mathscr{S}$ contains the system of generators $\{\sigma^{-n}L' \mid n \geq 0\}$ for $\vec{\mathcal{H}}$.

We have thus shown that $\Phi$ and $\Psi$ define maps between the named sets. Now, from (4.12) we infer $\Psi\Phi(\mathscr{S}) \supseteq \mathscr{S}$. The converse inclusion follows from [53, Lem. 1.3] as in the proof of Lemma 4.11 (2). Thus $\Psi\Phi(\mathscr{S}) = \mathscr{S}$. Moreover, $\Phi\Psi(\mathcal{B}) = ({}^{\perp_1}\mathcal{B} \cap \mathcal{H})^{\perp_1} \supseteq ({}^{\perp_1}\mathcal{B})^{\perp_1} \supseteq \mathcal{B}$. Since $\mathcal{B}$ is of finite type, there is $\mathscr{S}' \subseteq \mathcal{H}$ such that $\mathcal{B} = \mathscr{S}'^{\perp_1}$, and from (4.12) we conclude $\mathscr{S}' \subseteq \Psi(\mathcal{B})$, hence $\Phi\Psi(\mathcal{B}) = \mathcal{B}$. This completes the proof of the theorem. □

Corollary 4.17. *Let $\mathbb{X}$ be a weighted noncommutative regular projective curve and $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$. A generating system $\mathscr{S} \subseteq \mathcal{H}$ is resolving if and only if it is closed under extensions and subobjects.* □

We further have the following immediate consequence of Theorem 4.4.

Corollary 4.18. *Let $\mathbb{X}$ be a weighted noncommutative regular projective curve and $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$. If $\mathscr{S}' \subseteq \mathcal{H}$ is a set containing at least one non-zero vector bundle, then there is a tilting sheaf $T \in \vec{\mathcal{H}}$ with $T^{\perp_1} = \mathscr{S}'^{\perp_1}$.*

*Proof.* Let $\mathcal{B} = \mathscr{S}'^{\perp_1}$. Then $\mathscr{S} := {}^{\perp_1}\mathcal{B} \cap \mathcal{H}$ satisfies $\mathscr{S}^{\perp_1} = \mathcal{B}$, it is closed under extensions and subobjects, and we see as in the proof of Theorem 4.14 that it contains a generating system. Thus $\mathscr{S}$ is resolving, and the claim follows from Theorem 4.4. □

Maximal rigid objects in a (large) tube. Let $\vec{\mathcal{U}}$ be the direct limit closure of a tube $\mathcal{U}$ in $\vec{\mathcal{H}}$. Recall from Section 3 that $\vec{\mathcal{U}}$ is an exact subcategory of $\vec{\mathcal{H}}$, and it is itself a hereditary locally finite Grothendieck category, cf. also [19]. Following [14], we call an object $U$ in $\vec{\mathcal{U}}$ *rigid* if $\operatorname{Ext}^1(U, U) = 0$, and *maximal rigid* if it is rigid and every indecomposable $Y \in \vec{\mathcal{U}}$ satisfying $\operatorname{Ext}^1(U \oplus Y, U \oplus Y) = 0$ is a direct summand of $U$. This definition relies on the fact that

every rigid object $U$ has an indecomposable decomposition. Indeed, up to multiplicities, $U$ is a finite direct sum of indecomposables, which are either Prüfer sheaves or exceptional coherent sheaves, cf. Theorem 3.8. $U$ is said to be of *Prüfer type* if it has a Prüfer summand. Finally, two maximal rigid objects are said to be *equivalent* if they have the same indecomposable direct summands.
As a consequence of the discussion above, we can recover and refine results from [14, Sec. 5].

Corollary 4.19. *Let $\vec{\mathcal{U}}$ be the direct limit closure of a tube $\mathcal{U} = \mathcal{U}_x$ in $\vec{\mathcal{H}}$. The following statements are equivalent for an object $U \in \vec{\mathcal{U}}$.*

1. *$U$ is maximal rigid in $\vec{\mathcal{U}}$.*
2. *$U$ is tilting in $\vec{\mathcal{U}}$.*
3. *$U$ is of Prüfer type and it coincides, up to multiplicities, with the summand $(tT)_x$ supported at $x$ in the torsion part of some large tilting sheaf $T \in \vec{\mathcal{H}}$.*

*Moreover, the map $U \mapsto (\mathcal{T}_U, \mathcal{F}_U)$ where $\mathcal{F}_U := {}^{\perp_1}U \cap \mathcal{U}$ and $\mathcal{T}_U := {}^{\perp_0}\mathcal{F}_U \cap \mathcal{U}$ defines a bijective correspondence between equivalence classes of such objects $U$ and torsion pairs in $\mathcal{U}$ whose torsionfree class generates $\mathcal{U}$. If $B$ is the coherent part of $U$, which is a branch sheaf, and the set $\mathcal{R}_x$ is defined as in (4.8), then the torsion pair corresponding to $U$ is explicitly given as*

$$\mathcal{F}_U = \operatorname{add}\big(\tau^-(B^>) \cup \{\tau^j S_x[n] \mid j \in \mathcal{R}_x,\, n \in \mathbb{N}\}\big) \quad \textit{and} \quad \mathcal{T}_U = \operatorname{gen}(\tau^- B),$$

*and we have*

$$\mathcal{F}_U \cap \mathcal{F}_U{}^{\perp_1} = \operatorname{add}(B). \tag{4.13}$$

*Proof.* The implication (3)⇒(1) follows immediately from Lemma 4.9 (2).
For the implication (3)⇒(2) let $T = T_{(B,\{x\})} = T_+ \oplus U$ be a large tilting sheaf in $\vec{\mathcal{H}}$. In order to prove that $U$ is tilting in $\vec{\mathcal{U}}$, it suffices to verify condition (TS2), that is, to show that any $X \in U^\perp \cap \vec{\mathcal{U}}$ must be zero. Let $E$ be a direct summand of $X$ of finite length. Then also $E \in U^\perp$. Using Serre duality we obtain moreover $E \in T^\perp$, since $T_+$ is torsionfree and $x$-divisible. Thus $E = 0$ since $T$ is tilting. So $X$ does not have any non-zero summand of finite length, hence it is a direct sum of Prüfer sheaves in $\vec{\mathcal{U}}$ by Corollary 3.7. Since $U$ has a Prüfer summand (which maps onto all Prüfer sheaves in $\vec{\mathcal{U}}$), the condition $\operatorname{Hom}(U, X) = 0$ implies $X = 0$, as desired.
We now show that each of (1) or (2) implies (3). Let $U$ be maximal rigid or tilting in $\vec{\mathcal{U}}$, and assume without loss of generality that there are no multiplicities. Then $U = B \oplus U'$ where $U' \neq 0$ is a direct sum of Prüfer sheaves and $B$ is of finite length. If $B \neq 0$, then $U'$ defines a collection $\mathcal{W} = U'^{\perp_1} \cap \mathcal{U}$ of pairwise non-adjacent wings in the exceptional tube $\mathcal{U}$, and we infer as in the proof of Lemma 4.9 (2) that $B$ is a direct sum of connected branches in $\mathcal{W}$. In other words, $B$ is a branch sheaf, and $U$ satisfies (3), being for instance the torsion part of the tilting sheaf $T = T_{(B,\{x\})} = T_+ \oplus U$.

Moreover, by Lemma 4.11, there is a resolving subcategory $\mathscr{S}$ of $\mathcal{H}$ corresponding to $(B, \{x\})$. It has the form $\mathscr{S} = {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H} = {}^{\perp_1}T \cap \mathcal{H}$, and it gives rise to a resolving subcategory $\mathscr{S} \cap \mathcal{U}$ in $\mathcal{U}$, which coincides with $\mathcal{F}_U = {}^{\perp_1}U \cap \mathcal{U}$ because $T_+$ is $x$-divisible. The explicit shape of $\mathcal{F}_U$ is an immediate consequence of (4.10). Moreover, we have $(\mathscr{S} \cap \mathcal{U})^{\perp_1} \cap \mathcal{U} = \mathscr{S}^{\perp_1} \cap \mathcal{U}$ (since $\operatorname{Ext}^1(\mathcal{H}_+, \mathcal{U}) = 0$ by Serre duality), and since $\mathscr{S}^{\perp_1} = \operatorname{Gen}(T)$, we get ${\mathcal{F}_U}^{\perp_1} \cap \mathcal{U} = \operatorname{Gen}(T) \cap \mathcal{U} = \operatorname{gen}(B)$. Thus $\mathcal{T}_U = {}^{\perp_0}\mathcal{F}_U \cap \mathcal{U} = \tau^- {\mathcal{F}_U}^{\perp_1} \cap \mathcal{U} = \operatorname{gen}(\tau^- B)$. By (4.11) we finally obtain $\mathcal{F}_U \cap {\mathcal{F}_U}^{\perp_1} = \mathscr{S} \cap \mathscr{S}^{\perp_1} \cap \mathcal{U} = \operatorname{Add}(T) \cap \mathcal{U} = \operatorname{add}(B)$, which proves (4.13).

It follows readily that $U \mapsto (\mathcal{T}_U, \mathcal{F}_U)$ defines a map between the named sets, and this map is injective since $\mathcal{F}_U$, by (4.13), determines the branch part of $U$, and therefore $U$ itself. This map is also surjective: if $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{U}$ with $\mathcal{F}$ generating, then $\mathcal{F}$ is clearly resolving in $\mathcal{U}$, and we can apply Theorem 4.4 for the hereditary, locally finite Grothendieck category $\vec{\mathcal{U}}$ to obtain a tilting object $U$ in $\vec{\mathcal{U}}$ with $U^{\perp_1} = \mathcal{F}^{\perp_1}$. As in the proof of Lemma 4.11 (2) we get $\mathcal{F} = {}^{\perp_1}U \cap \mathcal{U} = \mathcal{F}_{\mathcal{U}}$, from which the claim follows. □

Genus zero. *For the rest of this section let $\mathbb{X}$ be of genus zero and $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$.* We refine the results above with the following notion.

Definition 4.20. Let $\mathscr{S}$ be a class of objects in $\mathcal{H}$. We call $\mathscr{S}$ *strongly resolving* if it is closed under extensions and subobjects, and if it contains a tilting bundle $T_{\mathrm{cc}}$.

Remark 4.21. Let $\mathscr{S} \subseteq \mathcal{H}$ be a strongly resolving class containing a tilting bundle $T_{\mathrm{cc}}$. Then $\mathscr{S}$ is resolving (this is verified by using that $T_{\mathrm{cc}}(-nx) \subseteq T_{\mathrm{cc}}$ by (2.8) for all $n \geq 0$ and all points $x \in \mathbb{X}$, and that the system $(T_{\mathrm{cc}}(-nx), n \geq 0)$ is generating by [38, Prop. 6.2.1]).

So we can apply Theorem 4.4 to obtain a tilting sheaf $T$ generating the class $\mathcal{B} = \mathscr{S}^{\perp_1}$. More explicitly, any special $\mathcal{B}$-preenvelope

$$0 \to T_{\mathrm{cc}} \to T_0 \to T_1 \to 0 \tag{4.14}$$

of $T_{\mathrm{cc}}$ leads to a tilting sheaf of finite type

$$T = T_0 \oplus T_1$$

with $T^{\perp_1} = \mathcal{B}$ and $T \in \operatorname{Gen}(T_{\mathrm{cc}})$.

Indeed, the exact sequence $\operatorname{Ext}^1(T_1, X) \to \operatorname{Ext}^1(T_0, X) \to \operatorname{Ext}^1(T_{\mathrm{cc}}, X) \to 0$ shows that $X \in T^{\perp_1}$ implies $X \in {T_{\mathrm{cc}}}^{\perp_1} = \operatorname{Gen}(T_{\mathrm{cc}})$, and the claim follows replacing $G$ by $T_{\mathrm{cc}}$ in the proof of Theorem 4.4.

Notice that the sheaves $T_0$ and $T_1$ are $\mathscr{S}$-filtered in the sense of [58, Def. 2.9], and the class ${}^{\perp_1}(T^{\perp_1})$ consists precisely of the direct summands of the $\mathscr{S}$-filtered objects, see [58, Thm. 2.13 and Cor. 2.15].

Example 4.22. (1) The system $\mathscr{S} = \operatorname{vect}\mathbb{X}$ of all vector bundles is strongly resolving, and the Lukas tilting sheaf $\mathbf{L}$ from Proposition 4.5 with $\mathbf{L}^{\perp_1} = \mathscr{S}^{\perp_1}$ is large, torsionfree and satisfies condition (TS3).

(2) Let $T = T_{(B,V)}$ where $\emptyset \neq V \subseteq \mathbb{X}$ and $B$ is a branch sheaf. The class $\mathscr{S} = {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$ is given by (4.10), and it is strongly resolving as $\operatorname{vect} \mathbb{X} \subseteq \mathscr{S}$; we even have $T_{\mathrm{can}} \in \mathscr{S}$. By the preceding discussion $T^{\perp_1} = \mathscr{S}^{\perp_1}$ and $T \in \operatorname{Gen}(T_{\mathrm{can}})$. Sequence (4.14) shows that $T$ satisfies (TS3). In fact, we will see in Theorem 10.1 that $T$ even satisfies condition (TS3+).

## 5. Tilting sheaves under perpendicular calculus

*Throughout this section, $\vec{\mathcal{H}} = \operatorname{Qcoh} \mathbb{X}$ with $\mathbb{X}$ a weighted noncommutative regular projective curve over a field $k$.* We use perpendicular calculus (in particular Lemma 4.16) to reduce some considerations to tilting sheaves $T_V = T_{(0,V)}$ with trivial branch sheaf $B = 0$. This will allow us to obtain an explicit description of the torsionfree part $T_+$ of any tilting sheaf $T_{(B,V)}$ and an alternate method to determine the Prüfer summands in the torsion part.

Remark 5.1. The Perpendicular Lemma 4.16 has several applications.

(1) Let $B \in \mathcal{H}$ be a branch sheaf. Let $T \in \vec{\mathcal{H}}$ be a sheaf such that $tT$ and $B$ have disjoint supports and $\operatorname{Ext}^1(B, T) = 0$ holds. Then $T \in B^{\perp}$. (This follows by applying $\operatorname{Hom}(B, -)$ to the canonical exact sequence $0 \to tT \to T \to T/tT \to 0$.) Thus we can use Lemma 4.16 to reduce our considerations to tilting sheaves with trivial exterior branch part $B_{\mathfrak{e}}$.

(2) Let $\mathbb{X}$ be a noncommutative regular projective curve of weight type $(p_1, \ldots, p_t)$ (with $p_i \geq 2$), and assume that $\mathbb{X}'$ arises from $\mathbb{X}$ by reduction of some weights, so that $\mathbb{X}'$ is of weight type $(p_1', \ldots, p_t')$, with $1 \leq p_i' \leq p_i$. Then the classification of (large) tilting sheaves in $\operatorname{Qcoh} \mathbb{X}$ is at least as complicated as the classification in $\operatorname{Qcoh} \mathbb{X}'$. Indeed, if $T'$ is a (large) tilting sheaf in $\operatorname{Qcoh} \mathbb{X}'$, then we can find a branch sheaf $B \in \operatorname{coh} \mathbb{X}$ such that $T = T' \oplus B$ is (large) tilting in $\operatorname{Qcoh} \mathbb{X}$: namely, we have $\operatorname{Qcoh} \mathbb{X}' \simeq \mathcal{E}^{\perp} \subseteq \operatorname{Qcoh} \mathbb{X}$ for a finite set $\mathcal{E}$ of exceptional simple sheaves; we can then take any branch sheaf $B$ whose components lie in the wings whose bases belong to $\mathcal{E}$; then $B^{\perp} = \mathcal{E}^{\perp}$ and $T' \in B^{\perp}$. Clearly, if $T_1'$ and $T_2'$ are not equivalent, then $T_1' \oplus B$ and $T_2' \oplus B$ are also not equivalent.

(3) In particular: if $\mathbb{X}$ is a weighted projective line of wild type (in the sense of [27]), then $\operatorname{Qcoh} \mathbb{X}$ contains all large tilting sheaves coming from a suitable weighted projective line $\mathbb{X}'$ of tubular type.

*Let us now assume* that $V \neq \emptyset$ and $B_{\mathfrak{e}} = 0$. Then all the branches of $B = B_{\mathfrak{i}}$ are interrelated with Prüfer summands of $T_{(B,V)}$ as described in Lemma 4.9 (2). Let $\vec{\mathcal{H}}' = (\tau^- B)^{\perp} = \operatorname{Qcoh} \mathbb{X}'$ and $i \colon \vec{\mathcal{H}}' \to \vec{\mathcal{H}}$ the inclusion. If we define, in analogy of Definition 3.1, the class $\mathscr{S}_V'$ and its direct limit closure $\mathcal{T}_V' = \vec{\mathscr{S}}_V'$ in $\vec{\mathcal{H}}'$, then it is easy to see that we have

$$\vec{\mathcal{H}}'/\mathcal{T}_V' \simeq {\mathscr{S}_V'}^{\perp} = (\tau^- B)^{\perp} \cap (i\mathscr{S}_V')^{\perp} = {\mathscr{S}_V}^{\perp} \simeq \vec{\mathcal{H}}/\mathcal{T}_V.$$

LEMMA 5.2. *Let $T = T_{(B,V)}$ be the tilting sheaf in $\vec{\mathcal{H}}$ given by (4.6) with torsionfree part $T_+$. We assume $B_{\mathfrak{c}} = 0$. Then*

$$T_V := T_{(0,V)} = T_+ \oplus \bigoplus_{x \in V} \bigoplus_{j \in \mathcal{R}_x} \tau^j S_x[\infty] \tag{5.1}$$

*is a large tilting sheaf in $\vec{\mathcal{H}}'$.*

*Proof.* It is sufficient to show that $T_{(0,V)}$ lies in the right-perpendicular category $(\tau^- B)^\perp$. By the definition of $\mathcal{R}_x$, and since the $\tau^j S_x[\infty]$ are injective, this is true for the direct sum of the Prüfer summands. Since $T_+$ is $V$-divisible, this also holds for $T_+$. □

We conclude

COROLLARY 5.3. *$T_{(B,V)} = T_{(B_{\mathfrak{i}},V)} \oplus B_{\mathfrak{c}}$ and $T_{(0,V)}$ have the same torsionfree part.* □

We will now deal with $T_V = T_{(0,V)}$. Its torsion part consists of Prüfer sheaves only. We consider $T_V$ as object in $\vec{\mathcal{H}}' = \operatorname{Qcoh} \mathbb{X}' = (\tau^- B)^\perp$, and we exhibit the following explicit construction.

Let $\Lambda'$ be a finite direct sum of indecomposable vector bundles $F_j$ in $\vec{\mathcal{H}}' = \operatorname{Qcoh} \mathbb{X}'$ such that $\Lambda'$ maps onto each simple sheaf in $\vec{\mathcal{H}}'$. For instance,

- by [39, Prop. 1.1], we can always find special line bundles $F_j$ with this property (by applying suitable tubular shifts to the structure sheaf $L$); or
- in case $\mathbb{X}$ is of genus zero, we can take alternatively $\Lambda' = T'_{\mathrm{can}}$, a canonical configuration in $\vec{\mathcal{H}}'$. (See Remark 5.12.)

We denote by $e(j,x) = e(j,x,\Lambda')$ the $\operatorname{End}(S_x)$-dimension of $\operatorname{Ext}^1(\tau^j S_x, \Lambda')$, by $p'(x)$ the weight of $x$ in $\mathbb{X}'$, and consider the universal sequence in $\mathcal{H}'$

$$0 \to \Lambda' \to \Lambda'(x) \to \bigoplus_{j=0}^{p'(x)-1} (\tau^j S_x)^{e(j,x)} \to 0 \tag{5.2}$$

where the $\tau^j S_x$ are the simple sheaves in $\mathcal{H}'$ concentrated at $x$. Since the inclusion $S_x \to S_x[\infty]$ yields a surjection $\operatorname{Ext}^1(S_x[\infty], \Lambda') \to \operatorname{Ext}^1(S_x, \Lambda')$, this induces a short exact sequence in $\vec{\mathcal{H}}' \subseteq \vec{\mathcal{H}}$

$$\eta_x \colon 0 \to \Lambda' \to \Lambda'_x \to \bigoplus_{j=0}^{p'(x)-1} (\tau^j S_x[\infty])^{e(j,x)} \to 0. \tag{5.3}$$

Note that $\tau^j S_x[\infty]$ are also Prüfer sheaves in $\vec{\mathcal{H}}$. For $x \in V$ these short exact sequences are spliced together via

$$\operatorname{Ext}^1\big(\bigoplus_{y \in V} \tau^j S_y[\infty], \Lambda'\big) \simeq \prod_{y \in V} \operatorname{Ext}^1(\tau^j S_y[\infty], \Lambda'), \tag{5.4}$$

which defines

$$\eta_V\colon 0\to\Lambda'\to\Lambda'_V\to\bigoplus_{x\in V}\bigoplus_{j=0}^{p'(x)-1}(\tau^j S_x[\infty])^{e(j,x)}\to 0. \tag{5.5}$$

Lemma 5.4. *$\Lambda'_V$ is torsionfree and precisely $V$-divisible.*

*Proof.* That $\Lambda'_V$ is torsionfree and $V$-divisible can be shown as in the proof of [55, Prop. 5.2]. Let $y\in\mathbb{X}\setminus V$ and $S\in\mathcal{U}_y$ be simple. By applying $\operatorname{Hom}(S,-)$ to sequence (5.5) we get $\operatorname{Ext}^1(S,\Lambda'_V)\simeq\operatorname{Ext}^1(S,\Lambda')\neq 0$. Thus $\Lambda'_V$ is precisely $V$-divisible. □

We now adopt the notation from Section 3 and interpret the sequence $\eta_V$ in (5.5) in terms of localization theory.

Lemma 5.5. *Assume $V\neq\emptyset$ and $B_{\mathfrak{e}}=0$. Let $\pi=\pi_V\colon\vec{\mathcal{H}}\to\vec{\mathcal{H}}/\mathcal{T}_V$ be the canonical quotient functor.*

(1) *In ${\mathscr{S}_V}^{\perp}\simeq\vec{\mathcal{H}}/\mathcal{T}_V$ we have $\pi\Lambda'\simeq\pi(\Lambda'_V)$.*

(2) *$\pi\Lambda'$ is a finitely presented projective generator in ${\mathscr{S}_V}^{\perp}\simeq\vec{\mathcal{H}}/\mathcal{T}_V$.*

(3) *The functor $X\mapsto\operatorname{Hom}_{\vec{\mathcal{H}}/\mathcal{T}_V}(\pi\Lambda',X)$ yields an equivalence*

$$\vec{\mathcal{H}}/\mathcal{T}_V\simeq\operatorname{Mod}(\operatorname{End}_{\vec{\mathcal{H}}/\mathcal{T}_V}(\pi\Lambda')).$$

*In particular, ${\mathscr{S}_V}^{\perp}$ is locally noetherian.*

*Proof.* (1) This is clear by the exact sequence (5.5).
(2) Let $x\in V$. Then $\Lambda'$ and $\Lambda'(nx)$ become isomorphic in $\vec{\mathcal{H}}/\mathcal{T}_V$ for all $n\in\mathbb{Z}$, which follows from (5.2). We note that every short exact sequence in $\vec{\mathcal{H}}/\mathcal{T}_V$ is isomorphic to the image of a short exact sequence in $\vec{\mathcal{H}}$ under the quotient functor $\pi$. If $A\in\mathcal{H}$, then, by [38, 0.4.6], [39], for sufficiently large $n>0$ we have $\operatorname{Ext}^1(\Lambda'(-nx),A)=0$, which shows that $\pi\Lambda'\simeq\pi(\Lambda'(-nx))$ is projective with respect to images of coherent objects. Since the class $\operatorname{Ker}\operatorname{Ext}^1(\pi\Lambda',-)$ is closed under direct limits, it follows that $\pi\Lambda'$ is projective. Since also, again by [38, 0.4.6], for sufficiently large $n>0$ we have $\operatorname{Hom}(\Lambda'(-nx),A)\neq 0$, we get $\operatorname{Hom}(\pi\Lambda',\pi A)\neq 0$ for every $A\in\mathcal{H}$, and it follows easily that $\pi\Lambda'$ is a generator in the quotient category. It is finitely presented because $\operatorname{Hom}(\Lambda',-)$ and hence $\operatorname{Hom}(\pi\Lambda',-)$ preserve direct limits (we refer to Remark 3.2 and [34, Lem. 2.5]).
(3) This is a well-known result by Gabriel-Mitchell, we refer to [13, II.1]. For the last statement, recall that $\Lambda'$ is noetherian, and so is $\operatorname{End}_{\vec{\mathcal{H}}/\mathcal{T}_V}(\pi\Lambda')$. □

As an additional information on $\Lambda'_V$ we exhibit its minimal injective resolution. We recall that the sheaf $\mathcal{K}$ of rational functions is the injective envelope of the structure sheaf $L$.

Proposition 5.6. *Let $\emptyset \neq V \subseteq \mathbb{X}$. There is a short exact sequence*

$$0 \to \Lambda'_V \to \Lambda'_{\mathbb{X}} \to \bigoplus_{y\in\mathbb{X}\setminus V} \bigoplus_{j=0}^{p'(y)-1} (\tau^j S_y[\infty])^{e(j,y)} \to 0. \tag{5.6}$$

*This is the minimal injective resolution of $\Lambda'_V$. Moreover, $\Lambda'_{\mathbb{X}} \simeq \mathcal{K}^n$ with $n = \operatorname{rk}(\Lambda')$.*

*Proof.* Via the identity (5.4) we have $\eta_V = (\eta_y)_{y\in V}$ and $\eta_{\mathbb{X}} = (\eta_x)_{x\in\mathbb{X}}$. Thus inclusion $\iota\colon \bigoplus_{y\in V}\bigoplus_j(\tau^j S_y[\infty])^{e(j,y)} \to \bigoplus_{x\in\mathbb{X}}\bigoplus_j(\tau^j S_x[\infty])^{e(j,x)}$ induces a map on the $\operatorname{Ext}^1$-spaces, which on the products induces projection onto the components of $V$, and thus maps $\eta_{\mathbb{X}}$ to $\eta_V$. Thus there is a pull-back diagram

$$\begin{tikzcd}
\eta_V\colon 0 \arrow[r] & \Lambda' \arrow[r] \arrow[d, equal] & \Lambda'_V \arrow[r] \arrow[d] & \bigoplus_{y\in V} \bigoplus_{j=0}^{p'(y)-1} (\tau^j S_y[\infty])^{e(j,y)} \arrow[r] \arrow[d, "\iota"] & 0\,, \\
\eta_{\mathbb{X}}\colon 0 \arrow[r] & \Lambda' \arrow[r] & \Lambda'_{\mathbb{X}} \arrow[r] & \bigoplus_{x\in \mathbb{X}} \bigoplus_{j=0}^{p'(x)-1} (\tau^j S_x[\infty])^{e(j,x)} \arrow[r] & 0
\end{tikzcd}$$

that is, $\eta_V = \eta_{\mathbb{X}}\cdot\iota$. Now we get sequence (5.6) with the snake lemma. The sequence (5.5) is, for $V=\mathbb{X}$, the minimal injective resolution of $\Lambda'$; this follows from the construction of $\Lambda'_{\mathbb{X}}$ like in [53, Thm. 4.1]. Therefore $\Lambda'_{\mathbb{X}} \simeq \mathcal{K}^n$ with $n = \operatorname{rk}(\Lambda')$. From the monomorphisms $\Lambda' \to \Lambda'_V \to \Lambda'_{\mathbb{X}}$ it is then clear that the sequence (5.6) is the minimal injective resolution of $\Lambda'_V$. □

Since the sequence (5.6) lies in ${\mathscr{S}_V}^{\perp} = \operatorname{Mod}(\operatorname{End}_{\vec{\mathcal{H}}/\mathcal{T}_V}(\pi\Lambda'))$, it is also the minimal injective resolution of the projective generator $\pi\Lambda'_V$.

The main result about the torsionfree part interprets $T_+$ as a projective generator in the localization of $\vec{\mathcal{H}}$ (or $\vec{\mathcal{H}}'$) at $V$.

Proposition 5.7. $\operatorname{Add}(T_+) = \operatorname{Add}(\Lambda'_V)$.

*Proof.* Invoking the uniqueness statement of Theorem 4.8 it is sufficient to show that $Q = Q_+ \oplus Q_0$ with $Q_+ = \Lambda'_V$ and $Q_0 = T_0 = \bigoplus_{x\in V}\bigoplus_{j=0}^{p'(x)-1}\tau^j S_x[\infty]$ is a tilting object in $\vec{\mathcal{H}}'$. From Lemma 5.5 we deduce $\operatorname{Ext}^1(Q_+, {Q_+}^{(I)}) = 0$, and using the sequence (5.6) we see that $\operatorname{Ext}^1(Q, Q^{(I)}) = 0$ for each set $I$. Let $X \in \vec{\mathcal{H}}'$. We conclude that $X \in \operatorname{Gen}(Q)$ implies $X \in Q^{\perp_1}$. We have to show that the converse also holds. So, let now $X \in Q^{\perp_1}$. In particular, $X \in {Q_0}^{\perp_1}$. The embeddings $S_y \to S_y[\infty] \to Q_0$ give rise to epimorphisms $\operatorname{Ext}^1(Q_0, X) \to \operatorname{Ext}^1(S_y, X)$ for all $y \in V$, and hence $X$ is $V$-divisible. Consider the short exact sequences $0 \to K \to {Q_+}^{(I)} \to B \to 0$ and $0 \to B \to X \to C \to 0$, where $I = \operatorname{Hom}(Q_+, X)$, so that $B$ is the trace of $Q_+$ in $X$. It is sufficient to show that $C = 0$. Since $X$ is $V$-divisible, the same holds for $C$. Moreover $\operatorname{Hom}(Q_+, C) = 0$. We show, that $C$ is $V$-torsionfree. Assume, this is not the case. Then there is $y \in V$ such that $\operatorname{Hom}(S_y, C) \neq 0$. Since $C$ (and thus also $tC$ and $(tC)_y$) is $y$-divisible, we get $S_y[\infty] \subseteq (tC)_y \subseteq C$. Since $S_y[\infty]$ is injective, there is a surjection $\operatorname{Hom}(Q_+, S_y[\infty]) \to \operatorname{Hom}(\Lambda', S_y[\infty]) \neq 0$, and $\operatorname{Hom}(Q_+, C) \neq 0$ follows, a contradiction. Thus, $C \in \mathscr{S}_V{}^{\perp}$, and since $\operatorname{Hom}(Q_+, C) = 0$, we get $C = 0$ by Lemma 5.5. This finishes the proof. □

The following is a reformulation of Theorem 4.8.

Theorem 5.8. *Let $\mathbb{X}$ be a weighted noncommutative regular projective curve. The tilting sheaves in $\vec{\mathcal{H}}$ having a large torsion part are, up to equivalence, the sheaves of the form*

$$T_{(B,V)} = T_V \oplus B$$

*with a subset $\emptyset \neq V \subseteq \mathbb{X}$, a branch sheaf $B = B_{\mathfrak{i}} \oplus B_{\mathfrak{e}}$ with interior and exterior part $B_{\mathfrak{i}}$ and $B_{\mathfrak{e}}$, respectively, and a tilting sheaf $T_V$ in the category $\operatorname{Qcoh}\mathbb{X}' = (B_{\mathfrak{e}} \oplus \tau^- B_{\mathfrak{i}})^{\perp} \subseteq \vec{\mathcal{H}}$, given as the direct sum of the middle term and the end term of the sequence (5.5).* □

Corollary 5.9. *Let $\mathbb{X}$ be a (non-weighted) noncommutative regular projective curve. The tilting sheaves in $\vec{\mathcal{H}}$ having a large torsion part are, up to equivalence, the sheaves $T_V$ with $\emptyset \neq V \subseteq \mathbb{X}$.* □

Genus zero. Before we specialize the above construction to the genus zero case in Remark 5.12 below, we need to explain some notations and concepts, which will also be used in later sections.

5.10. Numerical invariants. Each noncommutative curve of genus zero $\mathbb{X}$ has a so-called underlying tame bimodule, which is either of dimension type $(2,2)$ or $(1,4)$. In the first case we have $\varepsilon = 1$, in the second $\varepsilon = 2$. We recall that the structure sheaf $L$ has the property that for every point $x \in \mathbb{X}$ there is precisley one simple $S_x \in \mathcal{U}_x$ with $\operatorname{Hom}(L, S_x) \neq 0$, and $\operatorname{End}(L)$ is a skew field. One then defines $\kappa = [\operatorname{End}(L) : k]$ and for every point $x$

$$f(x) = \frac{1}{\varepsilon}[\operatorname{Hom}(L, S_x) : \operatorname{End}(L)], \ e(x) = [\operatorname{Hom}(L, S_x) : \operatorname{End}(S_x)].$$

For an exceptional point $x_i$ one writes $f_i = f(x_i)$ and $e_i = e(x_i)$. We have

$$\deg(S_x) = \frac{\bar{p}}{p(x)} f(x).$$

If $k$ is algebraically closed, then all the numbers $\varepsilon$, $\kappa$, $e(x)$, $f(x)$ are equal to 1. We refer to [44], [42] and [38] for details.

5.11. Canonical configuration. Let $\mathbb{X}$ again be of genus zero and of arbitrary weight type. Let $L$ be the structure sheaf, which is of degree 0 and hence of slope 0. Let $S_1, \ldots, S_t$ be the simple exceptional sheaves such that $\operatorname{Hom}(L, S_i) \neq 0$. The exceptional vector bundles $L_i(j)$ are defined [44, Sec. 5] as the middle terms of the $\operatorname{add}(L)$-couniversal sequences

$$0 \to L^{\varepsilon f_i} \to L_i(j) \to \tau^- S_i[j] \to 0, \tag{5.7}$$

for $i = 1, \ldots, t$ and $j = 1, \ldots, p_i - 1$. Similarly, $\overline{L}$ is defined as the middle term of the $\mathrm{add}(L)$-couniversal sequence

$$0 \to L^{\varepsilon} \to \overline{L} \to S \to 0, \tag{5.8}$$

where $S$ is a simple sheaf concentrated at a point $x_0$ with $p(x_0) = 1$ and $f(x_0) = 1$. The vector bundle $\overline{L}$ is exceptional, has rank $\varepsilon \in \{1, 2\}$ and slope $\bar{p}/\varepsilon$. From (5.8) we deduce that $\overline{L}$, like $L$, satisfies

$$\mathrm{Hom}(\overline{L}, \tau^j S_i) \neq 0 \quad \text{if and only if} \quad j \equiv 0 \bmod p_i. \tag{5.9}$$

The collection of all vector bundles $L$, $\overline{L}$ and the $L_i(j)$ yields the canonical configuration (5.10) (associated with $L$), which we denote by $T_{\mathrm{can}}$. Its endomorphism ring is a canonical algebra, cf. [44, Prop. 5.5]. Considered as full subcategory of $\mathcal{H}$ it has the following form:

(5.10)

$$\begin{tikzcd}
 & L_1(1) \arrow[r] & L_1(2) \arrow[r] & \cdots \arrow[r] & L_1(p_1-2) \arrow[r] & L_1(p_1-1) \arrow[ddr] & \\
 & L_2(1) \arrow[r] & L_2(2) \arrow[r] & \cdots \arrow[r] & L_2(p_2-2) \arrow[r] & L_2(p_2-1) \arrow[dr] & \\
L \arrow[uur] \arrow[ur] \arrow[dr] & \vdots & \vdots & & & \vdots & \overline{L} \\
 & L_t(1) \arrow[r] & L_t(2) \arrow[r] & \cdots \arrow[r] & L_t(p_t-2) \arrow[r] & L_t(p_t-1) \arrow[ur] &
\end{tikzcd}$$

By [44, 5.4 and 5.5] there are short exact sequences

$$0 \to L^{\varepsilon f_i} \to L_i(1) \to \tau^- S_i \to 0 \tag{5.11}$$

$$0 \to L_i(j-1) \to L_i(j) \to \tau^{-j} S_i \to 0 \tag{5.12}$$

$$0 \to L^{\varepsilon} \to \overline{L} \to S \to 0 \tag{5.13}$$

$$0 \to L_i(j) \to \overline{L}^{f_i} \to \tau^{-j} S_i[p_i - j] \to 0. \tag{5.14}$$

Remark 5.12. Let $\mathbb{X}$ be of genus zero and consider the tilting sheaf $T_{(B,V)}$ in $\vec{\mathcal{H}} = \mathrm{Qcoh}\,\mathbb{X}$. Let $\Lambda = T_{\mathrm{can}}$ be the canonical configuration (5.10). *We can choose $\Lambda'$ from above as the canonical configuration $T'_{\mathrm{can}}$ in the category $\vec{\mathcal{H}}' = \mathrm{Qcoh}\,\mathbb{X}'$.*

Indeed, if a branch sheaf $B = B_{\mathfrak{i}} \oplus B_{\mathfrak{e}}$ is given, we can assume, by applying suitable tubular shifts (associated to the exceptional points) to $\Lambda$, that we have $\mathrm{Hom}(L, B_{\mathfrak{i}}) = 0 = \mathrm{Hom}(\overline{L}, B_{\mathfrak{i}})$ and $\mathrm{Hom}(L, \tau B_{\mathfrak{e}}) = 0 = \mathrm{Hom}(\overline{L}, \tau B_{\mathfrak{e}})$. Then those direct summands of $\Lambda$ lying in $(B_{\mathfrak{e}} \oplus \tau^- B_{\mathfrak{i}})^{\perp} \simeq \mathrm{Qcoh}\,\mathbb{X}'$ form a canonical configuration $\Lambda' = T'_{\mathrm{can}}$ in $\mathrm{Qcoh}\,\mathbb{X}'$, containing $L$ and $\overline{L}$; it arises from $\Lambda$ by removing some "non-adjacent segments" $L_i(j), L_i(j+1), \ldots, L_i(j+r-2)$ from the inner parts of the arms. (Compare also [46, Thm. 3.1].)

## 6. The domestic case

*In this section let $\mathbb{X}$ be a noncommutative curve of genus zero.* Assume that $\mathbb{X}$ is of domestic type, that is, the normalized orbifold Euler characteristic $\chi'_{orb}(\mathbb{X})$

is positive. This means, for the degree of the line bundle $\tau L = L \otimes_{\mathcal{A}} \boldsymbol{\omega}_{\mathcal{A}} = L(\omega)$ (with $\boldsymbol{\omega}_{\mathcal{A}}$ the *dualizing sheaf* in $\mathcal{H} = \operatorname{coh}(\mathcal{A})$) we have

$$\delta(\omega) := \deg(\tau L) = -\frac{2\bar{p}s^2}{\kappa\varepsilon} \cdot \chi'_{orb}(\mathbb{X}) < 0.$$

Here, $\bar{p}$ is the least common multiple of the weights $p_1, \ldots, p_t$, moreover $\kappa = \dim_k \operatorname{End}(L)$ and $s = s(\mathcal{H}) = [k(\mathcal{H}) : k(X)]^{1/2}$ the skewness. For every indecomposable vector bundle $E$ one has the following slope formula

$$\mu(\tau E) = \mu(E) + \delta(\omega).$$

We recall the main features of the domestic case:

- (D1) All indecomposable vector bundles are stable and exceptional.
- (D2) If $E$ and $F$ are indecomposable vector bundles, then $\operatorname{Hom}(E, F) = 0$ if $\mu(E) > \mu(F)$.
- (D3) If $E$ is an indecomposable vector bundle then $\mu(\tau E) < \mu(E)$.
- (D4) The collection $\mathcal{F}$ of indecomposable vector bundles $F$ such that $0 \leq \mu(F) < -\delta(\omega)$ forms a slice in the sense of [56, 4.2], and $T_{\mathrm{her}} := \bigoplus_{F \in \mathcal{F}} F$ is a tilting bundle having a tame hereditary algebra as endomorphism ring. We refer to [47, Prop. 6.5] (the result there is in a more general context).
- (D5) There are only finitely many Auslander-Reiten orbits of vector bundles. (From (D3) it follows that $\mathcal{F}$ contains precisely one indecomposable from each Auslander-Reiten orbit, the finiteness follows from (D4).)

Lemma 6.1. *Let $\mathbb{X}$ be domestic. Let $T$ be a torsionfree tilting sheaf. Then there is $m \in \mathbb{Z}$ such that $\operatorname{Hom}(T, E) = 0$ for every indecomposable vector bundle $E$ with $\mu(E) < m$.*

*Proof.* The simple idea is the following: if $T$ would map non-trivially to vector bundles of arbitrarily small slopes, then, using line bundle filtrations, $T$ would be a generator for the class of all vector bundles. But by the tilting property, torsionfreeness and Serre duality we then get $\operatorname{Hom}(F, T) = 0$ for all coherent sheaves $F$, which is impossible. Filling this idea with details for a formal proof is quite straightforward in case of a weighted projective line, but slightly technical in the general case; we postpone these details to the appendix, cf. Lemma A.8. □

Lemma 6.2. *Assume that $\mathbb{X}$ is domestic, and that $T \in \vec{\mathcal{H}}$ is a large tilting object which is torsionfree. Then there is no non-zero morphism from $T$ to a vector bundle.*

*Proof.* By the previous lemma, let $m$ be an integer such that $\operatorname{Hom}(T, F) = 0$ for all vector bundles $F$ with $\mu(F) < m$. Let $\mathcal{F}$ be a set of representatives of indecomposable vector bundles $F$ with $m + \delta(\vec{\omega}) \leq \mu(F) < m$. By property (D4) the bundle $T_{\mathrm{her}} = \bigoplus_{F \in \mathcal{F}} F$ is tilting and its endomorphism ring is a tame hereditary algebra $H$ such that $\operatorname{Ext}^1(T_{\mathrm{her}}, T) = 0$. Thus, by Proposition 2.8, $T$ can be identified with an $H$-module.

We assume that there is a vector bundle $E$ with $\mathrm{Hom}(T,E)\neq 0$. Our aim is to get a contradiction. By the previous lemma we can assume $T$ does not map non-trivially to any predecessor of $E$ (since they have smaller slopes by stability). Then every non-zero morphism $T\to E$ must be a split epimorphism, by the almost split property. Thus, $T$ is a tilting $H$-module having a finite dimensional indecomposable preprojective module $P$ (corresponding to $E$) as a direct summand, and then $T$ is equivalent to a finite dimensional tilting module $T'$ by [10, Thm. 2.7]. In other words, $\mathrm{Add}(T)=\mathrm{Add}(T')$ in $\mathrm{Mod}\,H$, and then also in $\vec{\mathcal{H}}$, where $T'$ is a coherent tilting sheaf. Since $T$ is large, this gives the desired contradiction and proves the lemma. $\square$

Proposition 6.3. *Let $\mathbb{X}$ be a domestic curve and $T\in\vec{\mathcal{H}}$ a large tilting sheaf. Then $T\in\mathrm{Gen}(T_{\mathrm{cc}})$ for every tilting bundle $T_{\mathrm{cc}}$. In particular, $T$ is of finite type.*

*Proof.* For $T=T_{(B,V)}$ this was already shown in Remark 4.21. Therefore we can assume that $T$ is torsionfree. By the preceding lemma we have $\mathrm{Ext}^1(T_{\mathrm{cc}},T)=\mathrm{D}\,\mathrm{Hom}(T,\tau T_{\mathrm{cc}})=0$, that is, $T\in\mathrm{Gen}(T_{\mathrm{cc}})$. The last statement then follows from Proposition 2.8. $\square$

Proposition 6.4. *Assume that $\mathbb{X}$ is domestic, and that $T\in\vec{\mathcal{H}}$ is a large tilting sheaf which is torsionfree. Then $T$ is equivalent to the Lukas tilting sheaf $\mathbf{L}$.*

*Proof.* Since $T$ is torsionfree, $T^{\perp_1}$ contains the class of torsion sheaves $\vec{\mathscr{S}}_{\mathbb{X}}$ by Serre duality. Then ${}^{\perp_1}(T^{\perp_1})\cap\mathrm{coh}\,\mathbb{X}\subseteq\mathrm{vect}\,\mathbb{X}$, and by Lemma 6.2 we even have equality. Now Proposition 6.3 yields $\mathrm{Gen}(T)=\mathrm{Gen}(\mathbf{L})$, compare also Theorem 4.14. $\square$

The main result of this section summarizes the discussions above:

Theorem 6.5. *Let $\mathbb{X}$ be a domestic curve.*

(1) *The large tilting sheaves in $\vec{\mathcal{H}}$ are, up to equivalence, the sheaves of the form*
$$T_{(B,V)}=T_{(B_{\mathfrak{i}},V)}\oplus B_{\mathfrak{e}}$$
*with a subset $V\subseteq\mathbb{X}$, a branch sheaf $B=B_{\mathfrak{i}}\oplus B_{\mathfrak{e}}$ with interior and exterior part $B_{\mathfrak{i}}$ and $B_{\mathfrak{e}}$, respectively, and a tilting sheaf $T_{(B_{\mathfrak{i}},V)}$ in the category ${B_{\mathfrak{e}}}^{\perp}=\mathrm{Qcoh}\,\mathbb{X}'$; here $T_{(B_{\mathfrak{i}},V)}$ with $V\neq\emptyset$ is given by Theorems 4.8 and 5.8, and $T_{(B_{\mathfrak{i}},\emptyset)}=T_{(0,\emptyset)}=\mathbf{L}'$ is the Lukas tilting sheaf in ${B_{\mathfrak{e}}}^{\perp}$.*

(2) *There is a bijection between the set of equivalence classes of large tilting sheaves in $\vec{\mathcal{H}}$ and the set of pairs $(B,V)$ given by a branch sheaf $B\in\mathcal{H}$ and a subset $V\subseteq\mathbb{X}$. Moreover, every large tilting object is uniquely determined (up to equivalence) by its torsion part.* $\square$

As a special case we get:

Corollary 6.6. *Let $\mathbb{X}$ be a non-weighted noncommutative curve of genus zero. The large tilting sheaves in $\vec{\mathcal{H}}$ are, up to equivalence, the sheaves of the form $T_V$ with $\emptyset \neq V \subseteq \mathbb{X}$ defined in (5.1), and the Lukas tilting sheaf $\mathbf{L}$.* $\square$

For completeness, we record the corresponding classification of resolving classes (compare Theorem 4.14 and Lemma 4.11).

Corollary 6.7. *Let $\mathbb{X}$ be a domestic curve. The complete list of the resolving classes $\mathscr{S} \subseteq \mathcal{H}$ containing $\operatorname{vect}\mathbb{X}$ is given by*

$$\operatorname{add}\big(\operatorname{vect}\mathbb{X} \cup \tau^{-}(B^{>}) \cup \bigcup_{x\in V} \{\tau^j S_x[n] \mid j \in \mathcal{R}_x,\, n \in \mathbb{N}\}\big)$$

*with $V \subseteq \mathbb{X}$ and $B$ a branch sheaf.* $\square$

## 7. Semistability in Euler characteristic zero

*Throughout this section let $\mathbb{X}$ be a weighted noncommutative projective curve of orbifold Euler characteristic zero, and $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$.*

The main feature of the case $\chi'_{orb}(\mathbb{X}) = 0$ is that *every indecomposable coherent sheaf is semistable*, cf. Theorem 2.3. We collect here some basic properties which essentially follow from semistability and thus hold both in the tubular and in the elliptic case. Later, in the next two sections, we will have to distinguish the two cases. For general information on the tubular case we refer to [45], [41], [53, Ch. 13], [38, Ch. 8] and [39, Sec. 13], on the elliptic case to [39, Sec. 9].

Let us recall some notation. We write $\bar{p}$ for the least common multiple of the weights $p_1, \ldots, p_t$, that is, $\bar{p} = 1$ if $\mathbb{X}$ is elliptic, and $\bar{p} > 1$ if $\mathbb{X}$ is tubular. Further, the slope of a non-zero object $E \in \mathcal{H}$ is defined by $\mu(E) = \frac{\deg(E)}{\operatorname{rk}(E)} \in \widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$, with $\deg(E) = \frac{1}{\kappa\varepsilon}\langle\langle L, E\rangle\rangle$, cf. (2.7).

By semistability we have the following result, similar to Atiyah's classification [12].

Theorem 7.1 ([38, Prop. 8.1.6], [39, Thm. 9.7]). *For every $\alpha \in \widehat{\mathbb{Q}}$ the full subcategory $\mathbf{t}_\alpha$ of $\mathcal{H}$ formed by the semistable sheaves of slope $\alpha$ is a non-trivial abelian uniserial category whose connected components form stable tubes; the tubular family $\mathbf{t}_\alpha$ is parametrized again by a weighted noncommutative regular projective curve $\mathbb{X}_\alpha$ over $k$ which satisfies $\chi'_{orb}(\mathbb{X}_\alpha) = 0$ and is derived-equivalent to $\mathbb{X}$.* $\square$

We can thus write

$$\mathcal{H} = \bigvee_{\alpha\in\widehat{\mathbb{Q}}} \mathbf{t}_\alpha.$$

In particular, $\mathbf{t}_\infty$ consists of the finite length sheaves.

We will need the following important application of the Riemann-Roch formula from [39, Thm. 13.8].

Lemma 7.2. *If $X$, $Y \in \mathcal{H}$ are indecomposable with $\mu(X) < \mu(Y)$, then there exists $j$ with $0 \leq j \leq \bar{p} - 1$ such that $\operatorname{Hom}(X, \tau^j Y) \neq 0$.* $\square$

Quasicoherent sheaves having a real slope. For $w \in \widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ we define

$$\mathbf{p}_w = \bigcup_{\alpha < w} \mathbf{t}_\alpha \qquad \mathbf{q}_w = \bigcup_{w < \beta} \mathbf{t}_\beta,$$

where $\alpha$, $\beta \in \widehat{\mathbb{Q}}$. Accordingly, $\mathcal{H} = \mathbf{p}_w \vee \mathbf{t}_w \vee \mathbf{q}_w$ if $w$ is rational, and $\mathcal{H} = \mathbf{p}_w \vee \mathbf{q}_w$ if $w$ is irrational. Moreover, let

$$\mathcal{C}_w = \mathbf{q}_w{}^{\perp_0} = {}^{\perp_1}\mathbf{q}_w \qquad \mathcal{B}_w = {}^{\perp_0}\mathbf{p}_w = \mathbf{p}_w{}^{\perp_1}$$

and

$$\mathcal{M}(w) = \mathcal{B}_w \cap \mathcal{C}_w.$$

The sheaves in $\mathcal{M}(w)$ are said to have *slope* $w$. Clearly, for coherent sheaves this definition of slope is equivalent to the former one, and for irrational $w$ there are only non-coherent sheaves in $\mathcal{M}(w)$.
For $v \leq w \leq \infty$ we have $\mathcal{C}_v \subseteq \mathcal{C}_w$ and $\mathcal{B}_v \supseteq \mathcal{B}_w$. Moreover,

$$\bigcap_{w \in \widehat{\mathbb{R}}} \mathcal{C}_w = 0 \quad \text{and} \quad \bigcup_{w \in \widehat{\mathbb{R}}} \mathcal{C}_w = \mathcal{C}_\infty = \vec{\mathcal{H}},$$

and

$$\bigcap_{w \in \widehat{\mathbb{R}}} \mathcal{B}_w = \mathcal{B}_\infty = {}^{\perp_0} \operatorname{vect} \mathbb{X} \quad \text{and} \quad \mathcal{H} \cap \bigcup_{w \in \widehat{\mathbb{R}}} \mathcal{B}_w = \mathcal{H}.$$

We note that for example $\bigoplus_{\alpha \in \widehat{\mathbb{Q}}} S_\alpha$ with $S_\alpha \in \mathbf{t}_\alpha$ quasisimple is not in $\bigcup_{w \in \widehat{\mathbb{R}}} \mathcal{B}_w$. Let $X \in \vec{\mathcal{H}}$ be a non-zero object. Let $v = \sup\{r \in \widehat{\mathbb{R}} \mid X \in \mathcal{B}_r\} \in \widehat{\mathbb{R}} \cup \{-\infty\}$ and $w = \inf\{r \in \widehat{\mathbb{R}} \mid X \in \mathcal{C}_r\} \in \widehat{\mathbb{R}}$. Since $X \neq 0$ we have $v \leq w$.
In the special case, when $w = \infty$, a sheaf $X \in \vec{\mathcal{H}}$ has slope $\infty$ if and only if $X \in {}^{\perp_0} \operatorname{vect} \mathbb{X} = (\operatorname{vect} \mathbb{X})^{\perp_1}$. (This, as a definition, makes also sense for other representation types; *in the domestic case*, we have seen that *every large tilting sheaf has slope* $\infty$.)

Interval categories. The following technique is very useful in the tubular or elliptic setting. Let $\alpha \in \widehat{\mathbb{Q}}$. Denote by $\mathcal{H}\langle\alpha\rangle$ the full subcategory of $\mathcal{D}^b(\mathcal{H})$ defined by

$$\bigvee_{\beta > \alpha} \mathbf{t}_\beta[-1] \vee \bigvee_{\gamma \leq \alpha} \mathbf{t}_\gamma.$$

The abelian category $\mathcal{H}\langle\alpha\rangle$ is a HRS-tilt of $\mathcal{H}$ in $\mathcal{D}^b(\mathcal{H})$ with respect to the split torsion pair $(\mathcal{T}_\alpha, \mathcal{F}_\alpha)$ in $\mathcal{H}$ given by $\mathcal{T}_\alpha = \bigvee_{\beta > \alpha} \mathbf{t}_\beta$ and $\mathcal{F}_\alpha = \bigvee_{\gamma \leq \alpha} \mathbf{t}_\gamma$, see [29, I. Thm. 3.3] and [48, Prop. 2.2]. By [38, Prop. 8.1.6], [39, Thm. 9.7] we have $\mathcal{H}\langle\alpha\rangle = \operatorname{coh} \mathbb{X}_\alpha$ for some curve $\mathbb{X}_\alpha$ with $\chi'_{orb}(\mathbb{X}_\alpha) = 0$ and being derived-equivalent to $\mathbb{X}$. (If $k$ is algebraically closed, then $\mathbb{X}_\alpha$ is isomorphic to $\mathbb{X}$; but this is not true in general.) The rank function on $\mathcal{H}\langle\alpha\rangle$ defines a linear form $\operatorname{rk}_\alpha \colon \mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$. A sequence $\eta \colon 0 \to E' \xrightarrow{u} E \xrightarrow{v} E'' \to 0$ with objects $E', E, E''$ in $\mathcal{H} \cap \mathcal{H}\langle\alpha\rangle$ is exact in $\mathcal{H}$ if and only if it is exact in $\mathcal{H}\langle\alpha\rangle$; indeed, both conditions are equivalent to $E' \xrightarrow{u} E \xrightarrow{v} E'' \xrightarrow{\eta} E'[1]$ being a triangle in $\mathcal{D}^b(\mathcal{H})$.

Lemma 7.3 (Reiten-Ringel). *For every $w \in \widehat{\mathbb{R}}$ the pair $(\mathrm{Gen}(\mathbf{q}_w), \mathcal{C}_w)$ is a torsion pair, which is split in case $w \in \widehat{\mathbb{Q}}$.*

*Proof.* As in [53, Lem. 1.4] one shows that $\mathrm{Gen}(\mathbf{q}_w)$ is extension-closed; the same proof works in the locally noetherian category $\vec{\mathcal{H}}$, replacing "finite length" by "finitely presented". Then $\mathrm{Gen}(\mathbf{q}_w) = {}^{\perp_0}({\mathbf{q}_w}^{\perp_0}) = {}^{\perp_0}\mathcal{C}_w$ follows like in [53, Lem. 1.3], and thus $(\mathrm{Gen}(\mathbf{q}_w), \mathcal{C}_w)$ is a torsion pair. For the splitting property in case $w = \alpha \in \widehat{\mathbb{Q}}$ we have to show that every short exact sequence $\eta\colon 0 \to X \to Y \to Z \to 0$ with $X \in \mathrm{Gen}(\mathbf{q}_\alpha)$ and $Z \in \mathcal{C}_\alpha$ splits. We may assume that $X$ is a subobject of $Y$ and $Z = Y/X$. If $Z$ is finitely presented, this follows from Serre duality. For general $Z \in \mathcal{C}_\alpha$, we consider the set of subobjects $U$ of $Y$ such that $U \cap X = 0$ and $Y/(X+U) \in \mathcal{C}_\alpha$. Like in [53, Prop. 1.5(b)] one has a maximal such $U$, and as in [53, Prop. 1.5(a)] one shows $Y = X \oplus U$, so that $\eta$ splits. (If one assumes that the inclusion $X + U \subsetneq Y$ is proper, then $\vec{\mathcal{H}}$ being locally noetherian allows to find $Y'$ with $X + U \subsetneq Y' \subseteq Y$ with $Y'/(X+U)$ finitely presented. Then we proceed like in [53]. We remark that an analogue of condition (F) therein can be proved along the same lines by exploiting the fact that an indecomposable $E \in \mathcal{H}$ belongs to $\mathbf{q}_\alpha$ if and only if $\delta(E) > 0$, where $\delta = -\,\mathrm{rk}_\alpha$.) □

Let $\alpha \in \widehat{\mathbb{Q}}$. By $\vec{\mathcal{H}}\langle\alpha\rangle$ we denote the direct limit closure of $\mathcal{H}\langle\alpha\rangle$ in $\mathcal{D}^b(\vec{\mathcal{H}})$. We have $\vec{\mathcal{H}}\langle\alpha\rangle = \mathrm{Qcoh}\,\mathbb{X}_\alpha$. If $X \in \vec{\mathcal{H}}$ has a rational slope $\alpha$, then clearly $X \in \vec{\mathcal{H}} \cap \vec{\mathcal{H}}\langle\alpha\rangle$ where the intersection is formed in $\mathcal{D}^b(\vec{\mathcal{H}}) = \mathcal{D}^b(\vec{\mathcal{H}}\langle\alpha\rangle)$; in $\vec{\mathcal{H}}\langle\alpha\rangle$ then $T$ has slope $\infty$. Clearly, $\mathcal{C}_\alpha = \vec{\mathcal{H}}\langle\alpha\rangle \cap \vec{\mathcal{H}}$.

Lemma 7.4. *Let $\alpha \in \widehat{\mathbb{Q}}$. For an object $T$ in $\vec{\mathcal{H}}$ lying in $\mathcal{C}_\alpha$, the following conditions are equivalent:*

(i) *$T$ is a tilting sheaf in $\vec{\mathcal{H}}$;*
(ii) *$T$ is a tilting complex in $\mathcal{D}^b(\vec{\mathcal{H}})$;*
(iii) *$T$ is a tilting sheaf in $\vec{\mathcal{H}}\langle\alpha\rangle$.*

*Proof.* This is shown like in Proposition 2.8. □

Remark 7.5. There is an interesting class of locally coherent categories which are derived-equivalent to $\vec{\mathcal{H}}$: If $w$ is irrational, then we define $\mathcal{H}\langle w\rangle = \bigvee_{\beta>w} \mathbf{t}_\beta[-1] \vee \bigvee_{\gamma<w} \mathbf{t}_\gamma$ and $\vec{\mathcal{H}}\langle w\rangle$ similarly as above. It is easy to see that $\mathcal{H}\langle w\rangle$ is hereditary and does not contain any simple object. Accordingly, $\vec{\mathcal{H}}\langle w\rangle$ is a Grothendieck category (we refer to [7, Sec. 2.4+2.5]) which is locally coherent but *not* locally noetherian. Moreover, $\vec{\mathcal{H}}\langle w\rangle$ is derived-equivalent to $\vec{\mathcal{H}}$, and in the tubular case it contains a finitely presented tilting object $T_{\mathrm{can}}$ whose endomorphism ring is a tubular canonical algebra. It is not difficult to show that there are only countably many irrational $w'$ such that the category $\vec{\mathcal{H}}\langle w'\rangle$ (resp. $\mathcal{H}\langle w'\rangle$) is equivalent to $\vec{\mathcal{H}}\langle w\rangle$ (resp. $\mathcal{H}\langle w\rangle$). It would be of interest to get a better understanding of the "geometric meaning" of these categories.

Indecomposable quasicoherent sheaves. The following statement reflects the importance of the concept of slope in the tubular/elliptic case, also for quasicoherent sheaves.

Theorem 7.6 (Reiten-Ringel). (1) *$\operatorname{Hom}(\mathcal{M}(w'),\mathcal{M}(w))=0$ for $w<w'$.*
(2) *Every indecomposable sheaf has a well-defined slope $w\in\widehat{\mathbb{R}}$.*

*Proof.* (1) This follows like in [53, Thm. 13.1].
(2) We transfer the original proof for modules over a tubular algebra in [53, Thm. 13.1] to $\operatorname{Qcoh}\mathbb{X}$; we need a slight modification. Let $X\in\vec{\mathcal{H}}$ be indecomposable. Then $0\neq X\in\bigcup_{w\in\widehat{\mathbb{R}}}\mathcal{C}_w\setminus\bigcap_{w\in\widehat{\mathbb{R}}}\mathcal{C}_w$. Let $w\in\widehat{\mathbb{R}}$ be the infimum of all $\alpha\in\widehat{\mathbb{Q}}$ such that $X\in\mathcal{C}_\alpha$. Since $\mathbf{q}_w=\bigcup_{w<\alpha}\mathbf{q}_\alpha$, we have $\operatorname{Hom}(\mathbf{q}_w,X)=0$, that is, $X\in\mathcal{C}_w$.
We now show that $X\in\mathcal{B}_w={}^{\perp_0}\mathbf{p}_w$. We observe that

$$\mathcal{B}_w=\bigcap_{\alpha<w}{}^{\perp_0}\mathbf{t}_\alpha$$

and $\operatorname{Gen}(\mathbf{q}_\alpha)\subseteq{}^{\perp_0}\mathbf{t}_\alpha$. Hence, if $X\notin\mathcal{B}_w$, then there is a rational $\beta<w$ with $X\notin\operatorname{Gen}(\mathbf{q}_\beta)$. But $(\operatorname{Gen}(\mathbf{q}_\beta),\mathcal{C}_\beta)$ is a split torsion pair, and since $X$ is indecomposable, we get $X\in\mathcal{C}_\beta$. Since $\beta<w$ this gives a contradiction to the choice of $w$. □

Remark 7.7. If $T$ is a *noetherian* tilting object in $\vec{\mathcal{H}}$ (that is, $T\in\mathcal{H}$ (which exists if and only if $\bar{p}>1$)), then $T$ does *not* have any slope. In fact, if $T=T_1\oplus\ldots\oplus T_n$ with pairwise nonisomorphic indecomposable $T_i$, then $n$ coincides with the rank of the Grothendieck group $\mathrm{K}_0(\mathcal{H})$. If $T$ would have a slope $\alpha$, then each summand $T_i$ would be exceptional of slope $\alpha$, hence lying in one of the finitely many exceptional tubes of slope $\alpha$. If such a tube has rank $p>1$, then there are at most $p-1$ indecomposable summands of $T$ lying in this tube. If $p_1,\ldots,p_t$ are the weights of $\mathbb{X}$, then $n=|\mathrm{K}_0(\mathcal{H})|=\sum_{i=1}^t(p_i-1)+2>\sum_{i=1}^t(p_i-1)\geq n$, giving a contradiction.

Proposition 7.8. *Let $w\in\widehat{\mathbb{R}}$. There is a large tilting sheaf $\mathbf{L}_w$ of slope $w$.*

*Proof.* Applying Theorem 4.4 to the strongly resolving subcategory $\operatorname{add}(\mathbf{p}_w)$, we get a tilting sheaf $T$ with $\operatorname{Gen}(T)=\mathscr{S}^{\perp_1}={\mathbf{p}_w}^{\perp_1}=\mathcal{B}_w$. Moreover, by the tilting property clearly $T\in{}^{\perp_1}\mathcal{B}_w$, which is a subclass of $\mathcal{C}_w$. By the preceding remark, $T$ is large. □

Let $T\in\vec{\mathcal{H}}$ be a tilting object of rational slope $\alpha$. Then in $\vec{\mathcal{H}}\langle\alpha\rangle$ the splitting property of Theorem 3.8 holds, that is, the canonical exact sequence $0\to t_\alpha(T)\to T\to T/t_\alpha(T)\to 0$ in $\vec{\mathcal{H}}\langle\alpha\rangle$ splits, where $t_\alpha(T)$ denotes the torsion subsheaf of $T$ in $\vec{\mathcal{H}}\langle\alpha\rangle$.

Definition 7.9. Let $T$ be a tilting sheaf of slope $w$. We call $T$ a *torsionfree* tilting sheaf, if either $w$ is irrational, or if $w=\alpha\in\widehat{\mathbb{Q}}$ and $t_\alpha(T)=0$.

Lemma 7.10. *For every $w\in\widehat{\mathbb{R}}$ the tilting sheaf $\mathbf{L}_w$ is torsionfree.*

*Proof.* For irrational $w$ there is nothing to show. Switching to the category $\vec{\mathcal{H}}\langle\alpha\rangle$ if $w = \alpha$ is rational, we can assume without loss of generality that $w = \infty$. Then the claim follows from Proposition 4.5. $\square$

We will now treat the elliptic case and the tubular case separately, starting with the tubular case.

## 8. The tubular case

*Throughout this section let $\mathbb{X}$ be a tubular noncommutative curve of genus zero and $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$.*

Lemma 8.1. *Let $\alpha \in \widehat{\mathbb{Q}}$. Let $T \in \vec{\mathcal{H}}$ be a large tilting sheaf with $T \in \mathcal{C}_\alpha$ and $t_\alpha(T) \neq 0$. Then $T$ has slope $\alpha$.*

*Proof.* Switching to the category $\vec{\mathcal{H}}\langle\alpha\rangle = \operatorname{Qcoh}\mathbb{X}_\alpha$, we can assume without loss of generality that $\alpha = \infty$. (This will just simplify the notation.) If $tT$ contains a non-coherent summand, then with Theorem 5.8 we get that $T$ has slope $\infty$, since $T \in \mathcal{B}_\infty$ follows from 4.6. If, on the other hand, $tT$ only consists of coherent summands (necessarily only finitely many indecomposables) then $T/tT$ is a torsionfree tilting sheaf in

$$\operatorname{Qcoh}\mathbb{X}' = tT^{\perp} \subseteq \vec{\mathcal{H}},$$

where $\mathbb{X}'$ is a curve with reduced weights, thus of domestic type. By [28, Prop. 9.6] the induced inclusion $\operatorname{coh}\mathbb{X}' \subseteq \mathcal{H}$ is rank-preserving. The torsionfree sheaf $T/tT$ is equivalent to the Lukas tilting sheaf $\mathbf{L}' \in \operatorname{Qcoh}\mathbb{X}'$ by Proposition 6.4. We show that $\mathbf{L}'$, as object in $\vec{\mathcal{H}}$, has slope $\infty$. We assume this is not the case. Then there is $\beta < \infty$ with $\operatorname{Hom}(\mathbf{L}', \mathbf{t}_\beta) \neq 0$. Since in $\vec{\mathcal{H}}$ every vector bundle has a line bundle filtration, it follows that there is a line bundle $L$ with $\operatorname{Hom}(\mathbf{L}', L) \neq 0$. Since non-zero subobjects of line bundles are line bundles, we can assume without loss of generality that there is an epimorphism $f\colon \mathbf{L}' \to L$. Let $E$ be an indecomposable vector bundle over $\mathbb{X}'$, considered as object in $\vec{\mathcal{H}}$. Let $x_0 \in \mathbb{X}$ be a homogeneous point. The support of $tT$ is disjoint from $x_0$, and thus the associated tubular shift automorphism $\sigma_0$ fixes $tT$. Then $E(nx_0) \in \operatorname{vect}\mathbb{X}'$ for all $n > 0$: indeed, by definition of the tubular shift there is an exact sequence $0 \to E \to E(nx_0) \to C \to 0$ in $\vec{\mathcal{H}}$ with $C$ lying in the tube $\mathcal{U}_{x_0}$; then $E, C \in tT^{\perp}$ implies $E(nx_0) \in tT^{\perp}$, having the same rank as $E$. By [44, (S15)], for $n \gg 0$ we have $\operatorname{Hom}(L, E(nx_0)) \neq 0$. We get $\operatorname{Hom}(\mathbf{L}', E(nx_0)) \neq 0$, which also holds in the full subcategory $\operatorname{Qcoh}\mathbb{X}'$, and gives a contradiction since in $\operatorname{Qcoh}\mathbb{X}'$ one has $\mathbf{L}' \in {}^{\perp_0}\operatorname{vect}\mathbb{X}'$. Thus $\mathbf{L}'$ has slope $\infty$, and so has $T$, which is equivalent to $\mathbf{L}' \oplus tT$. $\square$

In the tubular case, the tilting bundle $T_{\mathrm{cc}}$ can be chosen such that its indecomposable summands have arbitrarily small slopes. This will imply that tilting sheaves have finite type. The following statement is crucial.

Lemma 8.2. *For any large tilting sheaf $T \in \vec{\mathcal{H}}$ there is $\alpha \in \widehat{\mathbb{Q}}$ with $T \in \mathcal{B}_\alpha$.*

*Proof.* If $T$ has a non-trivial torsion part, then $T$ has slope $\infty$ by Lemma 8.1. Thus we will assume in the following that $T$ is torsionfree.
Let $\mathcal{B} = \operatorname{Gen}(T)$ and $\mathscr{S} = {}^{\perp_1}\mathcal{B} \cap \mathcal{H}$. Suppose there is *no* $\alpha$ with $T \in \mathcal{B}_\alpha$. We will lead this to a contradiction.

(1) *There are infinitely many and arbitrarily small* $\alpha$ *with* $\operatorname{Hom}(T, \mathbf{t}_\alpha) \neq 0$. Indeed, otherwise there would be some $\alpha$ with $\mathbf{p}_\alpha \subseteq \mathscr{S}$, and then

$$T \in \mathcal{B} \subseteq ({}^{\perp_1}\mathcal{B})^{\perp_1} \subseteq \mathscr{S}^{\perp_1} \subseteq \mathbf{p}_\alpha{}^{\perp_1} = \mathcal{B}_\alpha,$$

which is not the case by our assumption.

(2) *There is no* $\alpha$ *such that* $\mathscr{S} \cap \mathbf{p}_\alpha = \emptyset$. Indeed, if there were such $\alpha$, then $\operatorname{Hom}(T, X) \neq 0$ for all $X \in \mathbf{p}_\alpha$. So, for any line bundle $L$ in $\mathbf{p}_\alpha$, the trace $L'$ of $T$ in $L$ would be a non-zero line bundle again. Applying $\operatorname{Ext}^1(T, -)$ to the short exact sequence $0 \to L' \to L \to C \to 0$ would give $\operatorname{Ext}^1(T, L) = 0$, as $T$ is torsionfree and $C$ has finite length. Then, given a point $x \in \mathbb{X}$ and an integer $n \geq 1$, we would infer $\operatorname{Ext}^1(T, L(nx)) = 0$ from the exact sequence $0 \to L \to L(nx) \to F \to 0$ with $F$ of finite length. Now, since any line bundle $L$ in $\mathcal{H}$ satisfies $L(-nx) \in \mathbf{p}_\alpha$ for $n \gg 0$, we would conclude that $\operatorname{Hom}(L, T) = \operatorname{D}\operatorname{Ext}^1(T, \tau L) = 0$ holds for all line bundles, and using line bundle filtrations, even for all vector bundles. But this is clearly impossible, since $T \neq 0$, as torsionfree object, is a direct limit of vector bundles. This proves (2).

Thus $\mathscr{S} \cap \mathbf{t}_\alpha \neq \emptyset$ for infinitely many and arbitrarily small $\alpha$.

(3) *Every indecomposable* $X \in \mathscr{S} \cap \mathbf{t}_\alpha$ *is exceptional.* Indeed, let $X \in \mathscr{S} \cap \mathbf{t}_\alpha$ be indecomposable, and let $\beta < \alpha$ with $\operatorname{Hom}(T, \mathbf{t}_\beta) \neq 0$. Considering images, there is an indecomposable $B \in \mathcal{H}$ with $B \in \operatorname{Gen}(T)$ and slope $\mu(B) < \alpha$. By Lemma 7.2 we have $\operatorname{Hom}(B, \tau^j X) \neq 0$ for some integer $j$. If we assume that $X$ is not exceptional, we can even show $\operatorname{Hom}(B, \tau X) \neq 0$. Indeed, this is clear if $X$ lies in a homogeneous tube, which means $\tau X = X$, while for $X$ lying in an exceptional tube of rank $p > 1$ we know from Lemma 7.2 that $B$ maps non-trivially into a quasisimple object of the tube, and by the almost split property it follows inductively that $B$ maps non-trivially into each object from the tube which has quasilength $\geq p$, so in particular into $\tau X$. Now we get $\operatorname{Ext}^1(X, T) = \operatorname{D}\operatorname{Hom}(T, \tau X) \neq 0$, which is a contradiction to $X \in \mathscr{S} \subseteq {}^{\perp_1}\mathcal{B}$. This shows (3).

We now fix an indecomposable, exceptional $X \in \mathscr{S} \cap \mathbf{t}_\alpha$ lying in a tube of rank $p > 1$.

(4) *There is an indecomposable* $Y$ *in the same tube and of the same quasi-length as* $X$ *such that* $\operatorname{Hom}(T, Y) \neq 0$. In order to show this, we start with an arbitrary indecomposable $Z$ of quasi-length $p$ in the same tube. Since $\tau^- Z$ is not exceptional, and thus not in $\mathscr{S}$, we have $\operatorname{Hom}(T, Z) \neq 0$. Then, considering the almost split sequences, we get inductively that $T$ maps non-trivially to an object of quasi-length $\ell$ for any $\ell < p$: given $0 \neq f \in \operatorname{Hom}(T, Z)$ where $Z$ is indecomposable of quasilength $\ell$, with $2 \leq \ell \leq p$, there is an irreducible monomorphism $\iota$ ending in $Z$ and an irreducible epimorphism $\pi$ starting in $Z$,

and either $\pi f \neq 0$, or $f$ factors through $\iota$; in both cases $T$ maps non-trivially to an object in the tube of quasilength $\ell - 1$. This shows (4).

(5) *There is an indecomposable coherent direct summand $B$ of $T$ of slope $\mu(B) \leq \alpha$.* Indeed, since for the fixed $X$ as above $\operatorname{Hom}(T, \tau X) = 0$, we can assume that the object $Y$ from (4) satisfies $\operatorname{Hom}(T, Y) \neq 0$ and $\operatorname{Hom}(T, \tau Y) = 0$. We conclude $\operatorname{Ext}^1(Y, T) = \operatorname{D}\operatorname{Hom}(T, \tau Y) = 0$, thus $Y \in \mathscr{S}$. Let $B$ be an indecomposable summand of the trace of $T$ in $Y$. Since $B \subseteq Y$, we conclude $\operatorname{Ext}^1(B, T) = 0$, hence $B \in \mathscr{S}$. Thus $B \in \mathcal{B} \cap {}^{\perp_1}\mathcal{B}$, and by Lemma 2.6, $B$ is a direct summand of $T$, of slope $\mu(B) \leq \alpha$.

Repeating these arguments for slope smaller than $\mu(B)$ we get inductively an infinite sequence of indecomposable coherent sheaves $B_1, B_2, B_3, \dots$ lying in $\operatorname{add}(T)$, and with slopes $\mu(B_1) > \mu(B_2) > \mu(B_3) > \dots$. We conclude $\operatorname{Ext}^1(B_i, B_j) = 0$ for all $i$, $j$ and $\operatorname{Hom}(B_i, B_j) = 0$ for all $i < j$. So, for all $n$, the sequence $(B_n, \dots, B_2, B_1)$ is exceptional in $\mathcal{H}$. This gives our desired contradiction, since the length of exceptional sequences in $\mathcal{H}$ is bounded by the (finite!) rank of the Grothendieck group $\operatorname{K}_0(\mathcal{H})$. □

Proposition 8.3. *Let $\mathbb{X}$ be tubular. Every tilting sheaf $T \in \vec{\mathcal{H}}$ is of finite type.*

*Proof.* By Lemma 8.2 there is $\alpha \in \widehat{\mathbb{Q}}$ with $T \in \mathcal{B}_\alpha = \mathbf{p}_\alpha{}^{\perp_1}$. Then $\operatorname{Ext}^1(\mathbf{p}_\alpha, T) = 0$, and choosing a tilting bundle $T_{cc} \in \mathbf{p}_\alpha$, we get $\operatorname{Ext}^1(T_{cc}, T) = 0$. Now we can apply Proposition 2.8. □

The proof above also shows that $\mathscr{S} = {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$ is a strongly resolving subcategory of $\mathcal{H}$ such that $\operatorname{Gen}(T) = \mathscr{S}^{\perp_1}$. Now let us start conversely with a strongly resolving subcategory.

Lemma 8.4. *Let $\alpha \in \widehat{\mathbb{Q}}$ and $\mathscr{S} \subseteq \mathcal{C}_\alpha \cap \mathcal{H}$ be strongly resolving.*

(1) *There is a tilting sheaf $T \in \vec{\mathcal{H}}$ with $T \in \mathcal{C}_\alpha$ and $T^{\perp_1} = \mathscr{S}^{\perp_1}$. Moreover:*
(2) *If $\mathscr{S} \cap \mathbf{t}_\alpha \neq \emptyset$, then $t_\alpha(T) \neq 0$.*
(3) *If $\mathscr{S} \cap \mathbf{t}_\alpha = \emptyset$, then $t_\alpha(T) = 0$.*

*Proof.* (1) By Theorem 4.4 there is a tilting sheaf $T \in \vec{\mathcal{H}}$ with $\mathscr{S}^{\perp_1} = T^{\perp_1}$. Moreover, there is an exact sequence (4.14) with $T = T_0 \oplus T_1$, and by Remark 4.21 the summands $T_0$ and $T_1$ are $\mathscr{S}$-filtered. Since $\mathcal{C}_\alpha = {}^{\perp_1}\mathbf{q}_\alpha$ is closed under filtered direct limits (which follows from [58, Prop. 2.12]), we get $T_0$, $T_1 \in \mathcal{C}_\alpha$, thus $T$ is in $\mathcal{C}_\alpha$.

(2) Assume that $t_\alpha(T) = 0$. Let $S \in \mathscr{S} \cap \mathbf{t}_\alpha$ be indecomposable. Then $\operatorname{Ext}^1(T, S) = \operatorname{D}\operatorname{Hom}(\tau^- S, T) = 0$, that is, $S \in T^{\perp_1}$. For every $X \in T^{\perp_1} = \mathscr{S}^{\perp_1}$ we have $\operatorname{Ext}^1(S, X) = 0$, and thus $S \in {}^{\perp_1}(T^{\perp_1})$. Since, by Lemma 2.6, $T^{\perp_1} \cap {}^{\perp_1}(T^{\perp_1}) = \operatorname{Add}(T)$ we get $S \in \operatorname{Add}(T)$, and then $S$ is a summand of $t_\alpha(T)$, contradiction. Thus $t_\alpha(T) \neq 0$.

(3) Assume that $t_\alpha(T) \neq 0$. By Lemma 8.1 then $T$ has slope $\alpha$, so $\operatorname{Gen}(T) \subseteq \mathcal{B}_\alpha$, and we even have equality since $\mathscr{S} \subseteq \mathbf{p}_\alpha$. So $T$ is torsionfree by Lemma 7.10, contradiction. □

The main result of this section is the following.

Theorem 8.5. *Let $\mathbb{X}$ be tubular. Every large tilting sheaf $T$ has a slope $w \in \widehat{\mathbb{R}}$.*

*Proof.* Let $\mathcal{B} = \mathrm{Gen}(T) = T^{\perp_1}$ and $\mathscr{S} = {}^{\perp_1}\mathcal{B} \cap \mathcal{H}$. Define $w = \inf\{r \in \widehat{\mathbb{R}} \mid T \in \mathcal{C}_r\} \in \widehat{\mathbb{R}}$. This is well-defined. We show that $T$ has slope $w$. By properties of the infimum we have $T \in \mathcal{C}_w$, but $T \notin \mathcal{C}_v$ for all $v < w$. We have to show that $T \in \mathcal{B}_w$. By the preceding lemma $T$ is of finite type, in other words, $T^{\perp_1} = \mathscr{S}^{\perp_1}$. For every rational number $\alpha < w$ let

$$\mathscr{S}_\alpha = \mathscr{S} \cap \mathcal{C}_\alpha.$$

Since $\mathscr{S}$ is strongly resolving by Lemma 8.2, the same holds for $\mathscr{S}_\alpha$. Since $T \notin \mathcal{C}_\alpha$, the set of all rational numbers $\alpha < w$ with $\mathscr{S} \cap \mathbf{t}_\alpha \neq \emptyset$ is not bounded by a smaller number than $w$; this follows from Lemma 8.4 and since $T$ is determined by $\mathscr{S}$. Thus there is a sequence of rational numbers

$$\alpha_1 < \alpha_2 < \alpha_3 < \cdots < w$$

with $\lim_{i\to\infty} \alpha_i = w$ and

$$\mathscr{S} \cap \mathbf{t}_{\alpha_i} \neq \emptyset. \tag{8.1}$$

By Lemma 8.4 there is a tilting object $T_i$ with $T_i^{\perp_1} = \mathscr{S}_{\alpha_i}{}^{\perp_1}$ and $T_i \in \mathcal{C}_{\alpha_i}$ and with $t_{\alpha_i}(T_i) \neq 0$. Now, by Lemma 8.1 the tilting object $T_i$ has slope $\alpha_i$. Then we get $\mathrm{Gen}(T_i) \subseteq \mathrm{Gen}(\mathbf{L}_{\alpha_i}) = \mathcal{B}_{\alpha_i}$ (the largest tilting class of slope $\alpha_i$). Since $\mathscr{S}_{\alpha_i} \subseteq \mathscr{S}$, we get

$$\mathcal{B}_{\alpha_i} \supseteq \mathscr{S}_{\alpha_i}{}^{\perp_1} \supseteq \mathscr{S}^{\perp_1} \ni T$$

for all $i$, and thus $T \in \bigcap_{i\geq 1} \mathcal{B}_{\alpha_i} = \mathcal{B}_w$. □

Theorem 8.6. *Let $\mathbb{X}$ be a noncommutative curve of genus zero of tubular type.*

(1) *The sheaves $\mathbf{L}_w$ with $w \in \widehat{\mathbb{R}}$ are, up to equivalence, the unique torsionfree large tilting sheaves (in the sense of Definition 7.9).*

(2) *The equivalence classes of large non-torsionfree tilting sheaves are in bijective correspondence with triples $(\alpha, B, V)$, where $\alpha \in \widehat{\mathbb{Q}}$, $V \subseteq \mathbb{X}_\alpha$ and $B \in \mathrm{add}\,\mathbf{t}_\alpha$ is a branch sheaf, and $(B, V) \neq (0, \emptyset)$.*

*Proof.* (1) Let $T$ be a torsionfree tilting sheaf of slope $w$. Then $T^{\perp_1} \subseteq \mathcal{B}_w = \mathbf{L}_w{}^{\perp_1}$. Hence we have ${}^{\perp_1}(\mathbf{L}_w{}^{\perp_1}) \cap \mathcal{H} = \mathrm{add}(\mathbf{p}_w) = {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$; the last equality follows, since $T$ generates every sheaf of finite length. Now $T^{\perp_1} = \mathbf{L}_w{}^{\perp_1}$ follows from Proposition 8.3.

(2) Every large non-torsionfree tilting sheaf $T$ has a slope $\alpha \in \widehat{\mathbb{Q}}$. By Lemma 7.4, $T$ is a large tilting sheaf in $\vec{\mathcal{H}}\langle\alpha\rangle$, having a non-zero torsion part $t_\alpha(T)$. We now apply Theorems 5.8 and 6.5 to the category $\vec{\mathcal{H}}\langle\alpha\rangle$. The non-torsionfree tilting sheaves of slope $\alpha$ are given by

- $T_{(B,V)}$ with $\emptyset \neq V \subseteq \mathbb{X}$ (here $t_\alpha(T)$ is non-coherent);
- $\mathbf{L}' \oplus B$, with $0 \neq B \in \mathrm{add}\,\mathbf{t}_\alpha$ a branch sheaf and $\mathbf{L}' \in B^\perp = \mathrm{Qcoh}\,\mathbb{X}'_\alpha$ the Lukas tilting sheaf over the domestic curve $\mathbb{X}'_\alpha$ (here $t_\alpha(T) = B$ is coherent).

This finishes the proof. □

We say that a resolving class $\mathscr{S} \subseteq \mathcal{H}$ *has slope* $w$ if $\mathbf{p}_w \subseteq \mathscr{S}$ and $\mathscr{S}$ does not contain any indecomposable of slope $\beta > w$.

Corollary 8.7. *For a tubular curve $\mathbb{X}$, the complete list of the resolving classes $\mathscr{S}$ in $\mathcal{H} = \operatorname{coh} \mathbb{X}$ having a slope is given by*

- $\operatorname{add} \mathbf{p}_w$ *with* $w \in \widehat{\mathbb{R}}$*; and*
- $\operatorname{add}\big(\mathbf{p}_\alpha \cup \tau^-(B^>) \cup \bigcup_{x \in V}\{\tau^j S_x[n] \mid j \in \mathcal{R}_x, n \in \mathbb{N}\}\big)$ *with* $\alpha \in \widehat{\mathbb{Q}}$, $V \subseteq \mathbb{X}_\alpha$*,* $B \in \operatorname{add} \mathbf{t}_\alpha$ *a branch sheaf, and* $(B, V) \neq (0, \emptyset)$.

*Proof.* By Theorem 8.6, the list contains precisely the resolving classes corresponding to the large tilting sheaves under the bijection of Theorem 4.14, and they all have a slope. Conversely, let $\mathscr{S}$ be resolving having a slope $w$ and $T$ a tilting sheaf such that $T^{\perp_1} = \mathscr{S}^{\perp_1}$. If $w$ is irrational, then $\mathscr{S} = \operatorname{add} \mathbf{p}_w$. If, on the other hand, $w = \alpha \in \widehat{\mathbb{Q}}$, then $\operatorname{Add}(T) \cap \mathcal{H} = \mathscr{S} \cap \mathscr{S}^{\perp_1} \subseteq \operatorname{add}(\mathbf{p}_\alpha \cup \mathbf{t}_\alpha) \cap {\mathbf{p}_\alpha}^{\perp_1} \subseteq \operatorname{add} \mathbf{t}_\alpha$, that is, all coherent summands of $T$ belong to the same tubular family, and therefore $T$ cannot be coherent. □

Corollary 8.8 (Property (TS3)). *Let $T_{\text{can}}$ be the canonical tilting bundle. Let $T \in \vec{\mathcal{H}}$ be a large tilting sheaf. Then for any homogeneous point $x_0$ and $n \gg 0$ there is a short exact sequence*

$$0 \to T_{\text{can}}(-nx_0) \to T_0 \to T_1 \to 0$$

*with* $\operatorname{add}(T_0 \oplus T_1) = \operatorname{add}(T)$.

*Proof.* If $T$ has slope $w$, choose $n \gg 0$ such that all indecomposable summands of $T_{\text{can}}(-nx_0)$ have slope smaller than $w$. □

## 9. The elliptic case

The tubular case, where all indecomposable coherent sheaves lie in tubes, is very similar (but weighted) to Atiyah's classification of indecomposable vector bundles over elliptic curves [12]. There are even more affinities between elliptic and tubular curves, see [22]. It is thus natural to expect a similar classification of large tilting sheaves as in the tubular case. But there are some technical differences: Since these curves are non-weighted, that is, do not have exceptional tubes, there is *no* coherent tilting sheaf. Reduction arguments using perpendicular calculus as in the proof of Lemma 8.1 are not possible. Moreover, the Grothendieck groups of elliptic curves are *not* finitely generated abelian, hence the (last part of the) proof of the crucial Lemma 8.2 does not work in the elliptic case. Additionally, we do not know whether in the elliptic case all tilting sheaves are of finite type. On the other hand, because all tubes are homogeneous, some arguments are even easier. For instance, in the elliptic case Lemma 7.2 has the stronger form

$$X, Y \in \mathcal{H} \text{ indecomposable}, \ \mu(X) < \mu(Y) \quad \Rightarrow \quad \operatorname{Hom}(X, Y) \neq 0. \tag{9.1}$$

Examples are the "classical" (commutative) elliptic curves over an algebraically closed field, and the real elliptic curves: the Klein bottle, the annulus, the Möbius band and the elliptic Witt curves, [39, Ex. 12.2].

For every rational $\alpha$ and each $\vec{\mathcal{H}}\langle\alpha\rangle$, Corollary 5.9 yields tilting sheaves $T_{\alpha,V}$ of slope $\alpha$ with non-zero torsion part supported in $\emptyset \neq V \subseteq \mathbb{X}_\alpha$.

THEOREM 9.1. *Let $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$ be the category of quasicoherent sheaves over a noncommutative elliptic curve.*

(1) *Every tilting sheaf in $\vec{\mathcal{H}}$ has a slope $w \in \widehat{\mathbb{R}}$.*
(2) *For every $w \in \widehat{\mathbb{R}}$ there is a tilting sheaf $\mathbf{L}_w$ with ${\mathbf{L}_w}^{\perp_1} = \mathcal{B}_w$ which is torsionfree (in the sense of Definition 7.9).*
(3) *For every $\alpha \in \widehat{\mathbb{Q}}$ and every non-empty $V \subseteq \mathbb{X}_\alpha$ there is, up to equivalence, precisely one tilting sheaf $T$ of slope $\alpha$ with $t_\alpha(T)$ supported in $V$, namely $T = T_{\alpha,V}$.*
(4) *Every tilting sheaf of finite type is equivalent to one listed in (2) or (3).*
(5) *The resolving subclasses of $\mathcal{H}$ are given precisely by $\operatorname{add}\mathbf{p}_w$ with $w \in \widehat{\mathbb{R}}$, and $\operatorname{add}\big(\mathbf{p}_\alpha \cup \bigcup_{x\in V}\mathbf{t}_{\alpha,x}\big)$ with $\alpha \in \widehat{\mathbb{Q}}$ and $\emptyset \neq V \subseteq \mathbb{X}_\alpha$.*

*Proof.* (1), (2), (3) We show that every tilting sheaf $T \in \vec{\mathcal{H}}$ has a slope $w \in \widehat{\mathbb{R}}$. To this end, let $w = \inf\{r \in \widehat{\mathbb{R}} \mid T \in \mathcal{C}_r\} \in \widehat{\mathbb{R}}$. We assume first that $w$ is rational; then without loss of generality $w = \infty$. If $tT \neq 0$, then $T$ is by Corollary 5.9 of the form $T_V$ with $\emptyset \neq V \subseteq \mathbb{X}$ (in particular, we also have uniqueness in this case). Let now $T$ be torsionfree. Then $\operatorname{vect}\mathbb{X} \subseteq {}^{\perp_1}T$. Indeed, otherwise one finds a line bundle $L'$, say of slope $\alpha < \infty$, such that $T$ maps onto $L'$. By (9.1), $L'$ maps non-trivially to *each* vector bundle of slope $> \alpha$. Since, by torsionfreeness, all simple sheaves lie in $\operatorname{Gen}(T)$, it follows that all line bundles of slope $> \alpha$ lie in $\operatorname{Gen}(T)$. Let $E$ be an indecomposable vector bundle of slope $> \alpha$. Then $L'$ is a subsheaf of $E$, and we find a line bundle $L''$ with $L' \subseteq L'' \subseteq E$ such that $E' = E/L''$ is torsionfree, thus a line bundle. Since $\operatorname{rk}(E') = \operatorname{rk}(E) - 1$ and $\mu(E') \geq \mu(E) > \alpha$ we see by induction that every indecomposable vector bundle of slope $> \alpha$ lies in $\operatorname{Gen}(T) = T^{\perp_1}$. By Serre duality $\operatorname{Hom}(\mathbf{q}_\beta, T) = 0$ for all rational $\beta$ with $\alpha < \beta < \infty$. But then $T \in \mathcal{C}_\alpha$, which gives a contradiction to the choice of $w$ $(= \infty)$. It follows that $T$ has slope $\infty$, moreover $\mathscr{S} := {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H} = \operatorname{vect}\mathbb{X}$.
Let now $w$ be irrational. We have to show $T \in {\mathbf{p}_w}^{\perp_1}$. Otherwise, there is a rational $\alpha < w$ such that $\operatorname{Hom}(T, \mathbf{t}_\alpha) \neq 0$. Considering images, we can assume with loss of generality that there is an epimorphism in this set. Then it is easy to see that there is $x \in \mathbb{X}_\alpha$ such that $T$ generates a tube $\mathbf{t}_{\alpha,x}$. Then it follows like in [53, Rem. 13.3], that $T$ generates all coherent objects $E$ of all rational slopes $\beta$ with $\alpha < \beta \leq \infty$. But this means, by Serre duality, that for all those $E$ we have $\operatorname{Hom}(E, T) = 0$, and thus $T \in \mathcal{C}_\alpha$. This is a contradiction to the choice of $w$. We conclude $T \in {\mathbf{p}_w}^{\perp_1} = \mathcal{B}_w$, and $T$ has slope $w$. (We remark that this argument for irrational $w$ also applies to the torsionfree case when $w$ is rational.)
Finally, for every $w \in \widehat{\mathbb{R}}$ there is a torsionfree tilting sheaf $\mathbf{L}_w$. Indeed, $\mathscr{S} = \operatorname{add}\mathbf{p}_w$ generates $\vec{\mathcal{H}}$ and is thus resolving. The claim now follows from Theorem 4.4.

(4) Let $T$ be tilting of finite type, $T^{\perp_1} = \mathscr{S}^{\perp_1}$ for some $\mathscr{S} \subseteq \mathcal{H}$ which we choose as $\mathscr{S} = {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$. By (1), $T$ has a slope $w$. If $T$ has a non-trivial torsion part, then $T$ is equivalent to a tilting sheaf in (3) by Corollary 5.9. So we assume that $T$ is torsionfree. Since a coherent object $X$ is in $\mathscr{S}$ if and only if $\operatorname{Ext}^1(X,T)=0$, we have $\mathbf{p}_w \subseteq \mathscr{S}$: indeed, $\operatorname{Ext}^1(\mathbf{t}_\alpha, T) = \operatorname{D}\operatorname{Hom}(T, \mathbf{t}_\alpha) = 0$ for all rational $\alpha < w$ by slope reasons. Furthermore, if $X \in \mathbf{q}_w$, then $\operatorname{Ext}^1(T, \tau X) = 0$ as $T \in \mathcal{C}_w = {}^{\perp_1}\mathbf{q}_w$, so $\operatorname{Ext}^1(X,T) \simeq \operatorname{D}\operatorname{Hom}(T,\tau X) \neq 0$, and $X \notin \mathscr{S}$. Finally, in case $w \in \widehat{\mathbb{Q}}$, it follows as in Lemma 8.4 that $\mathscr{S} \subseteq \mathcal{C}_w \cap \mathcal{H}$ satisfies $\mathscr{S} \cap \mathbf{t}_w = \emptyset$. We thus conclude $\mathscr{S} = \operatorname{add}\mathbf{p}_w$, and $T$ is equivalent to the tilting sheaf $\mathbf{L}_w$ from (2).

(5) Using (4), the claim follows from Theorem 4.14 and Lemma 4.11. $\square$

## 10. Combinatorial descriptions and an example

*Let $\mathbb{X}$ be a noncommutative curve of genus zero, of arbitrary weight type.* In this section we further investigate the large tilting sheaves $T_{(B,V)}$ with $V \neq \emptyset$. We already know that they are of finite type and satisfy condition (TS3). We give an explicit construction for the sequence in (TS3), and we verify the stronger property (TS3+).

We denote by $\Lambda$ a canonical tilting bundle $T_{\text{can}}$, as in Remark 5.12. By copresenting each indecomposable summand of $T_{\text{can}}$ by summands of $T_{(B,V)}$ we will prove the following.

Theorem 10.1. *Let $\mathbb{X}$ be of genus zero and $T = T_{(B,V)}$ with $V \neq \emptyset$ as in (4.6). The canonical configuration $T_{\text{can}} = \Lambda$ has an $\operatorname{add}(T)$-copresentation as follows:*

$$0 \to T_{\text{can}} \to T_0' \oplus B_0 \to T_1' \oplus B_1 \to 0 \tag{10.1}$$

*with $T_0' \in \operatorname{add}(\Lambda_V')$ torsionfree, $T_1' \in \operatorname{add}(\bigoplus_{x \in V} \bigoplus_{j \in \mathcal{R}_x} \tau^j S_x[\infty])$ and $B_0$, $B_1 \in \operatorname{add}(B)$ such that $\operatorname{Hom}(B_1, B_0) = 0$; moreover, in $T_1'$ all Prüfer summands $\tau^j S_x[\infty]$ of $T$ occur and $\operatorname{add}(B_0 \oplus B_1) = \operatorname{add}(B)$.*

As a first preparation we have the following simple fact.

Lemma 10.2. *Let $B$ be a connected branch and $B'$ a proper subbranch of $B$, rooted in $Z \in B$. Then one of the following two cases holds.*

(1) *There is an epimorphism $X \to Z$ with $X \in B \setminus B'$, and then there is no non-zero morphism from $B'$ to $B \setminus B'$.*
(2) *There is a monomorphism $Z \to Y$ with $Y \in B \setminus B'$, and then there is no non-zero morphism from $B \setminus B'$ to $B'$.*

*Proof.* Since $B'$ is proper, it is clear that there is either an epimorphism $X \to Z$ or a monomorphism $Z \to Y$ with $X$ or $Y$ in $B \setminus B'$, respectively. Let $\mathcal{W}'$ be the wing rooted in $Z$. Since $B'$ forms a tilting object in $\mathcal{W}'$, it is clear, that $\mathcal{W}'$ is disjoint with $B \setminus B'$. Let $U \in B'$ and $V \in B \setminus B'$. Assume the first case, and $\operatorname{Hom}(U,V) \neq 0$. Then $V$ lies on a ray starting in the basis of $\mathcal{W}'$, but not in $\mathcal{W}'$. We then get $\operatorname{Hom}(X, \tau V) \neq 0$. By Serre duality we get $\operatorname{Ext}^1(V,X) \neq 0$, which gives a contradiction because of $\operatorname{Ext}^1(B,B)=0$. The second case follows similarly. $\square$

10.3. Let $T=T_{(B,V)}$ be a given large tilting sheaf with $V\neq\emptyset$. For the moment we assume, for notational simplicity, that $B$ is an *inner* branch sheaf. Let us explain the strategy we are going to pursue for the proof of the theorem.

Step 1: *Initial step.* We start with the canonical configuration $\Lambda=T_{\mathrm{can}}$ in $\vec{\mathcal{H}}=\operatorname{Qcoh}\mathbb{X}$, which consists of arms between $L$ and $\overline{L}$, compare (5.10). By applying suitable tubular shifts to $\Lambda$, we can assume without loss of generality that $\operatorname{Hom}(L,B)=0=\operatorname{Hom}(\overline{L},B)$. We then form $\vec{\mathcal{H}}'=\operatorname{Qcoh}\mathbb{X}'=(\tau^{-}B)^{\perp}$. Then the subconfiguration $\Lambda'$ of indecomposable summands of $\Lambda$ lying in $(\tau^{-}B)^{\perp}$ forms a canonical configuration $\Lambda'=T'_{\mathrm{can}}$ in $\vec{\mathcal{H}}'$, containing $L$ and $\overline{L}$, compare Remark 5.12. For $\Lambda'$ we have the copresentation

$$0\to\Lambda'\to\Lambda'_V\to\bigoplus_{x\in V}\bigoplus_{j=0}^{p'(x)-1}(\tau^{j}S_x[\infty])^{e(j,x)}\to 0. \tag{10.2}$$

from (5.5), which is already of the desired form with respect to the theorem we want to prove; by construction, it gives an $\operatorname{add}(T)$-copresentation of each indecomposable summand of $\Lambda'$. It remains to compute suitable copresentations for each indecomposable summand of $\Lambda$ not in $\Lambda'$, and then to take the direct sum of all of these sequences with (10.2). This will be done inductively working in each connected branch component, starting with the root of that component. Let us consider one such component lying in a wing $\mathcal{W}$ rooted in, say, $S[r-1]$ with $2\leq r\leq p$, concentrated at a point $x\in V$. We will call $S[\infty]$ the Prüfer sheaf *above* $\mathcal{W}$.

Step 2: *Induction start with root.* Note that $S[r]\in(\tau^{-}B)^{\perp}\simeq\operatorname{Qcoh}\mathbb{X}'$ becomes simple. The basis of $\mathcal{W}$ is given by the simple sheaves $S,\tau^{-}S,\dots,\tau^{-(r-2)}S$. This segment of simples corresponds to a segment of direct summands of $\Lambda'$ lying in the inner of one arm. We denote this segment by $L(1),\dots,L(r-1)$, so that there are epimorphisms

$$L(i)\twoheadrightarrow\tau^{-i+1}S\qquad i=1,\dots,r-1. \tag{10.3}$$

(We will do this for every branch component, and then we will need, of course, a shift of indices. In the notation of (5.10) the segment $L(1),\dots,L(r-1)$ is $L_i(j),\dots,L_i(j+r-2)$ for some arm-index $i$ and some $j$.) With this "calibration" the sequence (5.7) becomes

$$0\to L(0)\to L(r-1)\to S[r-1]\to 0 \tag{10.4}$$

where $L(0)$ is a predecessor of $L(1)$, either still in the inner of the same arm, or $L(0)=L^{\varepsilon f(x)}$; in any case $L(0)\in\operatorname{add}(\Lambda')$. This means that for $L(0)$ we already have a copresentation. Using Lemma 10.5 below, we will get a copresentation for $L(r-1)$, which will be compatible with the statement of our theorem.

We will then proceed in a similar way with the other members of the connected branch $B$, going down the branch inductively, as described in the next step.

Step 3: *Induction step.* We introduce further notation. We define

$$W_{ij}=S[i]/S[i-j]\in\mathcal{W}$$

for $i = 1, \dots, r-1$; $j = 1, \dots, i$, where $S[0] = 0$. We call $W_{ij}$ *wing objects*, and the pair of indices $(i, j)$ *wing pairs*. The length of $W_{ij}$ is $j$; we say that $i$ is the *level* and $i - j$ the *colevel* of $W_{ij}$. So $W_{ij}$ is uniquely determined by its level and colevel, which fix the ray and coray $W_{ij}$ belongs to. Applying the construction of an $\operatorname{add}(L)$-couniversal extension to the short exact sequences $0 \to W_{jj} \to W_{ii} \to W_{i,i-j} \to 0$, and recalling that we have $\operatorname{Hom}(L, \mathcal{W}) = 0$, we deduce from [44, Prop. 5.1] that there are short exact sequences

$$0 \to L(j) \to L(i) \to W_{i,i-j} \to 0 \tag{10.5}$$

for $1 \leq j < i$. We assume now that $W_{i,i-j}$ be part of $B$. The (direct) neighbours of smaller length in the same component of the branch might be

$$\begin{tikzcd} & W_{i,i-j} \arrow[ddr] & \\ W_{i-\ell,i-j-\ell} \arrow[ur] & & \\ & & W_{i,i-j-s} \end{tikzcd}$$

where $W_{i-\ell,i-j-\ell} \to W_{i,i-j}$ denotes a composition of $\ell$ irreducible monomorphisms and $W_{i,i-j} \to W_{i,i-j-s}$ a composition of $s$ irreducible epimorphisms. In this situation we compute an $\operatorname{add}(T)$-copresentation of $L(i-\ell)$ and $L(j+s)$, respectively, if $\operatorname{add}(T)$-copresentations of $L(j)$ or $L(i)$, respectively, are already known. In other words: having already exploited $W_{i,i-j}$ for computing a suitable copresentation of an indecomposable summand of $\Lambda$, we will then use its lower neighbours for computing copresentations for further summands. The two different kinds of neighbours are reflected by the following two lemmas. Roughly speaking, the first lemma (treating the epimorphism case) adds the branch summand $W_{i,i-j-s}$ to the end term, the second (treating the monomorphism case) the branch summand $W_{i-\ell,i-j-\ell}$ to the middle term in the copresentation of $\Lambda$.

Lemma 10.4. *Let $(i, i-j)$ and $(i, i-j-s)$ be wing pairs and assume that $W_{i,i-j}$ and $W_{i,i-j-s}$ are summands of $B$. Assume there is an exact sequence*

$$0 \to L(i) \to \mathcal{G} \oplus B_0 \to P \oplus B_1 \to 0$$

*such that*

(i) $B_0$, $B_1 \in \operatorname{add}(B)$ *are disjoint with the subbranch rooted in* $W_{i,i-j-s}$*;*
(ii) $\operatorname{Hom}(B_1, B_0) = 0$*;*
(iii) $\mathcal{G}$ *is torsionfree and $x$-divisible;*
(iv) *$P$ is a direct sum of copies of the Prüfer sheaf $S[\infty]$ above the wing $\mathcal{W}$.*

*Then there is an exact sequence*

$$0 \to L(j+s) \to \mathcal{G} \oplus B_0 \to P \oplus W_{i,i-j-s} \oplus B_1 \to 0$$

*with* $\operatorname{Hom}(W_{i,i-j-s}, B_0) = 0$.

*Proof.* The sequence

$$0 \to L(j+s) \to L(i) \to W_{i,i-j-s} \to 0$$

together with the given sequence yields the exact commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L(j+s) \arrow[r] \arrow[d, equal] & L(i) \arrow[r] \arrow[d] & W_{i,i-j-s} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L(j+s) \arrow[r] & \mathcal{G} \oplus B_0 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & & P \oplus B_1 \arrow[r, equal] \arrow[d] & P \oplus B_1 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

The right column splits, since $W_{i,i-j-s}$ and $B_1$ as summands of the branch $B$ are Ext-orthogonal, and since $\operatorname{Ext}^1(P, W_{i,i-j-s}) = \operatorname{D}\operatorname{Hom}(\tau^- W_{i,i-j-s}, P) = 0$. Because of (i) we get $\operatorname{Hom}(W_{i,i-j-s}, B_0) = 0$ from Lemma 10.2. □

Lemma 10.5. *Let $(i, i-j)$ and $(i-\ell, i-j-\ell)$ be wing pairs such that $W_{i,i-j}$ and $W_{i-\ell,i-j-\ell}$ are summands of $B$ (the case $\ell = 0$ is permitted). Assume there is an exact sequence*

$$0 \to L(j) \to \mathcal{G} \oplus B_0 \to P \oplus B_1 \to 0$$

*such that $B_0, B_1 \in \operatorname{add}(B)$ are disjoint from the subbranch rooted in $W_{i-\ell,i-j-\ell}$, $\operatorname{Hom}(B_1, B_0) = 0$, $\mathcal{G}$ is torsionfree and $x$-divisible, and $P$ is a direct sum of copies of the Prüfer sheaf above the wing $\mathcal{W}$. Then there is an exact sequence*

$$0 \to L(i-\ell) \to \mathcal{G} \oplus B_0 \oplus W_{i-\ell,i-j-\ell} \to P \oplus B_1 \to 0,$$

*and* $\operatorname{Hom}(B_1, W_{i-\ell,i-j-\ell}) = 0$.

*Proof.* There is the push-out diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & L(j) \arrow[r] \arrow[d] & \mathcal{G} \oplus B_0 \arrow[r] \arrow[d] & P \oplus B_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L(i-\ell) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & P \oplus B_1 \arrow[r] & 0 \\
 & W_{i-\ell,i-j-\ell} \arrow[r, equal] \arrow[d] & W_{i-\ell,i-j-\ell} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Now, since $\mathcal{G}$ is $x$-divisible and $W_{i-\ell,i-j-\ell}$ and $B_0$ as summands of the branch $B$ are Ext-orthogonal, the middle column splits. Moreover, $\operatorname{Hom}(B_1, W_{i-\ell,i-j-\ell}) = 0$ follows again from Lemma 10.2. □

10.6. Let now $B$ be an *exterior* branch part, the inner branch parts already treated. We proceed similarly to the inner case. We briefly explain the differences. By applying suitable tubular shifts to $\Lambda$, we can assume without loss of generality that $\mathrm{Hom}(L,\tau B)=0=\mathrm{Hom}(\overline{L},\tau B)$. We then form $\vec{\mathcal{H}}'=\operatorname{Qcoh}\mathbb{X}'=B^{\perp}$. Then the subconfiguration $\Lambda'$ of indecomposable summands of $\Lambda$ lying in $B^{\perp}$ forms a canonical configuration $\Lambda'=T'_{\mathrm{can}}$ in $\vec{\mathcal{H}}'$, containing $L$ and $\overline{L}$. Note that $\tau S[r]\in B^{\perp}\simeq\operatorname{Qcoh}\mathbb{X}'$ becomes simple. The basis of a wing $\mathcal{W}$ corresponding to a connected component of $B$ is given by the simple sheaves (concentrated at $x$) $S,\tau^{-}S,\dots,\tau^{-(r-2)}S$. This segment of simples corresponds to a segment of direct summands of $\Lambda'$ lying in the inner of one arm. We denote this segment by $L(1),\dots,L(r-1)$, so that there are epimorphisms

$$L(i)\twoheadrightarrow\tau^{-i+2}S\qquad i=1,\dots,r-1. \tag{10.6}$$

This yields a short exact sequence

$$0\to L(1)\to L(r)\to S[r-1]\to 0 \tag{10.7}$$

where $L(r)$ is either in the inner of the same arm, or $L(r)=\overline{L}^{f(x)}$. (We refer to the diagram in [44, p. 536].) Thus the desired copresentation of $L(r)$ is already given. Then, for $L(1)$ and for the induction step we have modified versions of Lemma 10.4 and 10.5, just taking into account the different notation (10.6).

*Proof of Theorem 10.1.* Let $\Lambda$ be a given canonical configuration, considered as full subcategory of $\mathcal{H}$. As usual we write $B=B_{\mathfrak{i}}\oplus B_{\mathfrak{e}}$ with respect to $V$. We can assume that the canonical configuration $\Lambda'$ in $(\tau^{-}B_{\mathfrak{i}}\oplus B_{\mathfrak{e}})^{\perp}\simeq\operatorname{Qcoh}\mathbb{X}'$ is a subconfiguration of $\Lambda$, containing $L$ and $\overline{L}$. Recall that $B$ decomposes into $B=\bigoplus_{i=1}^{t}B_{x_i}$ over the exceptional points $x_1,\dots,x_t$, and each $B_{x_i}$ (in case it is nonzero) is a direct sum of finitely many connected branches in non-adjacent wings. Then

$$\Lambda=\Lambda'\oplus\bigoplus_{i=1}^{t}\bigoplus_{\ell}L_i(\ell)$$

for suitable $\ell$, forming finitely many non-adjacent segments in $\{1,\dots,p_i-1\}$, corresponding to the connected branches as described above.
Step 1 yields the $\mathrm{add}(T)$-copresentation

$$0\to\Lambda'\to T'_0\to T'_1\to 0 \tag{10.8}$$

of $\Lambda'$, given by (10.2). We then have to compute suitable copresentations for the $L_i(\ell)$. By forming the direct sum we will get the desired copresentation for $\Lambda$. This can be done separately by performing Step 2 and Step 3 for every connected branch (using Lemma 10.4 and 10.5 and keeping in mind the modifications in 10.6).

We still have to show that in this way we obtain $\mathrm{add}(T)$-copresentations of *all* indecomposable summands of $\Lambda$. It is enough to do this for every single wing $\mathcal{W}$ involved, say $\mathcal{W}$ is rooted in $S[r-1]$, and the corresponding summands of $\Lambda$ are given by $L(1),\dots,L(r-1)$. (So this notation applies to the inner case, the exterior is treated similarly.) The kernel of the epimorphism $L(r-1) \to S[r-1] = W_{r-1,r-1}$ is (a power of) an indecomposable summand of $\Lambda'$, and from Lemma 10.5 (case $\ell = 0$) we get an add$(T)$-copresentation of $L(r-1)$. Let $W_{ij}$ be a summand of $B$, different from the root $S[r-1]$. Then $W_{ij}$ has a unique upper neighbour $Z$ in $B$. There are two cases:

(a) There is an epimorphism $Z \to W_{ij}$. Then Lemma 10.4 gives a copresentation of $L(i-j)$ where $i-j$ is the colevel of $W_{ij}$.

(b) There is a monomorphism $W_{ij} \to Z$. Then Lemma 10.5 gives a copresentation of $L(i)$ where $i$ is the level of $W_{ij}$.

So either the level or the colevel determines the index of the summand of $\Lambda$ we can treat with the help of $W_{ij}$. In both cases the obtained index lies between 1 and $r-2$. Assume now that there are two different summands $W_{ij}$ and $W_{k\ell}$ of $B$, which are also different from the root of $\mathcal{W}$, and which yield the *same* index under the procedure above. We consider the upper neighbours of $U$ of $W_{ij}$ and $V$ of $W_{k\ell}$. If there are epimorphisms $U \to W_{ij}$ and $V \to W_{k\ell}$, then we conclude that the colevels of $W_{ij}$ and $W_{k\ell}$ coincide; similarly if there are monomorphisms $W_{ij} \to U$ and $W_{k\ell} \to V$, then the levels of both coincide. In the mixed case, when there is a monomorphism $W_{ij} \to U$ and an epimorphism $V \to W_{k\ell}$, then the level of $W_{ij}$ is the colevel of $W_{k\ell}$. In all these cases it is easy to see that there are non-zero extensions between one of these objects and the other or the upper neighbour of the other, which gives a contradiction. Indeed, if $W_{ij}$ and $W_{k\ell}$ have the same colevel, they belong to the same ray and $i \neq k$, say $i < k$. Then $\operatorname{Ext}^1(W_{kl}, U) = \operatorname{D}\operatorname{Hom}(U, \tau W_{kl}) \neq 0$. The level case is similar. In the mixed case, let $c$ be the level of $W_{ij} = W_{cj}$ and the colevel of $W_{k\ell} = W_{k,k-c}$. Then $W_{ij}$ lies on the coray ending in $W_{c,1}$ and $\tau W_{k\ell} = W_{k-1,\ell}$ lies on the ray starting in $W_{c,1}$, so $\operatorname{Ext}^1(W_{kl}, W_{ij}) = \operatorname{D}\operatorname{Hom}(W_{ij}, \tau W_{k\ell}) \neq 0$.

It follows that the $r-1$ summands of the branch $B$ yield copresentations for $r-1$ distinct indecomposable summands of $\Lambda$, which are then necessarily given by $L(1), \dots, L(r-1)$. $\square$

We now illustrate the procedure, which can be done for each involved exceptional tube separately. In the following example we have two wings in the same tube to consider. (Note that compared with Lemmas 10.4 and 10.5 by a matter of notation there are unavoidable shifts of indices.)

Example 10.7. In the following we will use the numerical invariants from 5.10 and the short exact sequences from 5.11, which are the building blocks of the canonical configuration (5.10). Let $\Lambda$ be a canonical algebra of weight type given by the sequence (11), the only exceptional point given by $x$, let $V = \{x\}$ and $e = e(x)$, $f = f(x)$, $d = ef$ and $\varepsilon \in \{1, 2\}$ be the numerical type of $\mathbb{X}$. Then $\Lambda$ is realized as canonical configuration

$$L \to L(1) \to L(2) \to L(3) \to L(4) \to \dots \to L(9) \to L(10) \to \overline{L}$$

in $\mathcal{H}$. Let

$$B = S[4] \oplus \tau^{-2}S[2] \oplus \tau^{-2}S \oplus S \oplus S'[3] \oplus S'[2] \oplus \tau^{-}S'$$

be a branch, where we assume that $S$ is simple with $\operatorname{Hom}(L, \tau^2 S) \neq 0$ and $S' = \tau^{-6} S$. Then $\operatorname{Hom}(L(i+2), \tau^{-i} S) \neq 0$ for $i = -1, 0, \ldots, 8$. There are two connected components of $B$, lying in the wings rooted in $S[4]$ and $S'[3]$, respectively. The situation is illustrated in Figure 10.1, where the indecomposable summands of the branch $B$ are denoted by $\bullet$, the roots of the two wings by $\widehat{\bullet}$. The two vertical lines indicate the identification by the $\tau$-period. We also exhibit the undercuts by $\circ$, and the four Prüfer sheaves belonging to $T_{(B,V)}$ by the symbol $*$ over the corresponding ray. We have

$$\Lambda' = L \oplus L(1) \oplus L(6) \oplus L(7) \oplus \overline{L} \in (\tau^- B)^{\perp}.$$

There are the universal exact sequences in $(\tau^- B)^{\perp} = \operatorname{Qcoh} \mathbb{X}'$ (where the only weight of $\mathbb{X}'$ is given by $p' = 5$)

$$0 \to L \to \mathcal{G} \to \tau S[\infty]^e \to 0 \tag{10.9}$$

$$0 \to \overline{L} \to \mathcal{G}^{\varepsilon} \to \tau S[\infty]^{\varepsilon e} \to 0 \tag{10.10}$$

$$0 \to L(i+2) \to \mathcal{G}_i^{\varepsilon f} \to \tau^{-(i+1)} S[\infty]^{\varepsilon d} \to 0 \quad \text{for } i = -1,\, 4,\, 5. \tag{10.11}$$

with torsionfree, indecomposable $\mathcal{G}$, $\mathcal{G}_i$; note that $\mathcal{G}$, $\mathcal{G}_i \in (\tau^- B)^{\perp}$, and thus these objects are $x$-divisible. Their direct sum gives the short exact sequence

$$0 \to \Lambda' \to \Lambda'_V \to \tau S[\infty]^{(1+\varepsilon)e} \oplus S[\infty]^{\varepsilon d} \oplus \tau^{-5} S[\infty]^{\varepsilon d} \oplus S'[\infty]^{\varepsilon d} \to 0$$

where $\Lambda'_V = \mathcal{G}^{1+\varepsilon} \oplus \mathcal{G}_{-1}^{\varepsilon f} \oplus \mathcal{G}_4^{\varepsilon f} \oplus \mathcal{G}_5^{\varepsilon f}$. This was Step 1.

We now treat the first branch. This corresponds to the segment $L(2)$, $L(3)$, $L(4)$, $L(5)$ of $\Lambda$. Step 2: Applying Lemma 10.5 (to the sequence $0 \to L(1) \to L(5) \to S[4] \to 0$) gives the exact sequence

$$0 \to L(5) \to \mathcal{G}_{-1}^{\varepsilon f} \oplus S[4] \to S[\infty]^{\varepsilon d} \to 0. \tag{10.12}$$

Step 3: Applying Lemma 10.4 again yields

$$0 \to L(3) \to \mathcal{G}_{-1}^{\varepsilon f} \oplus S[4] \to \tau^{-2} S[2] \oplus S[\infty]^{\varepsilon d} \to 0. \tag{10.13}$$

Now applying Lemma 10.5 two times yields

$$0 \to L(4) \to \mathcal{G}_{-1}^{\varepsilon f} \oplus S[4] \oplus \tau^{-2} S \to \tau^{-2} S[2] \oplus S[\infty]^{\varepsilon d} \to 0 \tag{10.14}$$

and

$$0 \to L(2) \to \mathcal{G}_{-1}^{\varepsilon f} \oplus S \to S[\infty]^{\varepsilon d} \to 0. \tag{10.15}$$

The second branch corresponds to the segment $L(8)$, $L(9)$, $L(10)$ of $\Lambda$. Step 2, and then Step 3, which is applying Lemma 10.5 two times and then Lemma 10.4, yields the exact sequences

$$0 \to L(10) \to \mathcal{G}_5^{\varepsilon f} \oplus S'[3] \to S'[\infty]^{\varepsilon d} \to 0, \tag{10.16}$$

then

$$0 \to L(9) \to \mathcal{G}_5^{\varepsilon f} \oplus S'[3] \oplus S'[2] \to S'[\infty]^{\varepsilon d} \to 0 \tag{10.17}$$

and finally

$$0 \to L(8) \to \mathcal{G}_5^{\varepsilon f} \oplus S'[3] \oplus S'[2] \to \tau^- S' \oplus S'[\infty]^{\varepsilon d} \to 0 \tag{10.18}$$

Forming the direct sum of all 12 short exact sequences (10.9)–(10.18) we get the add$(T)$-copresentation of $\Lambda$ as in Theorem 10.1.

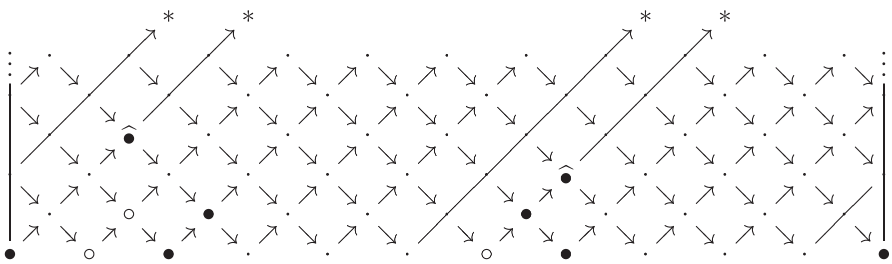

Figure 10.1. The branches of Example 10.7

## Appendix A. Slope arguments in the domestic case

In this appendix we complement the arguments given in the proof of Lemma 6.1 by more details. Most of them are well-established for weighted projective lines, see for example [43, Thm. 2.7]. Here we see that in general we have to be careful with the special line bundles.

Let $\mathbb{X}$ be an *arbitrary* noncommutative curve of genus zero. Recall that a line bundle $L'$ is called *special* if for every exceptional point $x_i$ there is *precisely one* simple sheaf $S_i$ concentrated at $x_i$ with $\operatorname{Hom}(L', S_i) \neq 0$. Every autoequivalence $\sigma$ of $\mathcal{H}$ induces an autoequivalence of $\mathcal{H}_0$ and thus of $\mathcal{H}/\mathcal{H}_0$, and is therefore rank-preserving. Hence, if $L'$ is a special line bundle, then so is $\sigma L'$. For Geigle-Lenzing weighted projective lines [27] it is well-known (see [43, 2.1]) that for each degree $\vec{x}$ we have

$$\operatorname{Hom}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{\omega}+\vec{c})) \neq 0 \text{ if } \operatorname{Hom}(\mathcal{O}, \mathcal{O}(\vec{x})) = 0.$$

Since $\operatorname{Hom}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{\omega}+\vec{c})) = \operatorname{D}\operatorname{Ext}^1(\mathcal{O}(\vec{c}), \mathcal{O}(\vec{x}))$, when we write $L = \mathcal{O}$ and $\overline{L}$ replaces $\mathcal{O}(\vec{c})$, the following statement is the generalization of this to noncommutative curves of genus zero (of arbitrary weight type).

Lemma A.1. *Let $\mathbb{X}$ be a noncommutative curve of genus zero and $X$ be an indecomposable vector bundle. Then* $\operatorname{Hom}(L, X) \neq 0$ *or* $\operatorname{Ext}^1(\overline{L}, X) \neq 0$ *holds.*

In the domestic case this is a special case of [47, Prop. 4.1].

*Proof.* Assume that $\operatorname{Hom}(L, X) = 0 = \operatorname{Ext}^1(\overline{L}, X)$. We now apply $\operatorname{Hom}(-, X)$ to several of the exact sequences above. Sequence (5.13) gives

$$0 \to \operatorname{Hom}(S, X) \to \operatorname{Hom}(\overline{L}, X) \to \operatorname{Hom}(L^{\varepsilon}, X) \to$$
$$\to \operatorname{Ext}^1(S, X) \to \operatorname{Ext}^1(\overline{L}, X) \to \operatorname{Ext}^1(L^{\varepsilon}, X) \to 0$$

and from the assumptions we conclude that all terms are zero. Applying then $\operatorname{Hom}(-, X)$ to (5.14) shows $\operatorname{Ext}^1(L_i(j), X) = 0$. Similarly, (5.11) and (5.12) inductively yield $\mathrm{Hom}(L_i(j), X) = 0$. Altogether this gives that $\mathrm{Hom}(T_{\mathrm{can}}, X) = 0 = \mathrm{Ext}^1(T_{\mathrm{can}}, X)$, and since $T_{\mathrm{can}}$ is a tilting object we get $X = 0$, a contradiction. □

Lemma A.2. *Let $\mathbb{X}$ be domestic. Let $L$ be a special line bundle. Let $F$ be an indecomposable vector bundle of slope $\mu(F) - \mu(L) > \bar{p}/\varepsilon + \delta(\omega)$. Then* $\mathrm{Hom}(L, F) \neq 0$.

*Proof.* For every special line bundle $L$ we can form a canonical configuration like (5.10), see [44]. Then $L$ does not necessarily have degree zero, but still $\mu(\overline{L}) - \mu(L) = \bar{p}/\varepsilon$.
We assume $\mathrm{Hom}(L, F) = 0$. Then by Lemma A.1 $\mathrm{Ext}^1(\overline{L}, F) \neq 0$, and by Serre duality, $\mathrm{Hom}(F, \tau\overline{L}) \neq 0$. But by assumption

$$\mu(F) > \bar{p}/\varepsilon + \delta(\omega) + \mu(L) = \mu(\overline{L}) + \delta(\omega) = \mu(\tau\overline{L}),$$

which contradicts the stability of indecomposable vector bundles in the domestic case (Theorem 2.3). □

Remark A.3. In the domestic case every indecomposable vector bundle is exceptional. In particular this is true for every line bundle. But there are domestic cases where not every line bundle is special. Take for example the domestic symbol $\binom{2}{2}$. It tells us that there is precisely one exceptional point $x$, and this point satisfies $p(x) = 2$, $f(x) = 1$ and $e(x) = 2$. (For the general definition of a symbol of a genus zero curve we refer to [37].) Let now $L$ be a special line bundle which maps onto the simple $S_x$. The kernel then is a line bundle $L'$. One verifies that $[L']$ is a 1-root in $\mathrm{K}_0(\mathbb{X})$ and that $\mathrm{Hom}(L', S_x) \neq 0$ and $\mathrm{Hom}(L', \tau S_x) \neq 0$. Hence $L'$ is not special.

Lemma A.4. *Let $\mathbb{X}$ be domestic. Let $T$ be a torsionfree tilting sheaf. Assume that for every $n \in \mathbb{Z}$ there is a* special *line bundle $L_n$ with $\mu(L_n) < n$ such that $\mathrm{Hom}(T, L_n) \neq 0$. If $L'$ is an arbitrary line bundle, then also* $\mathrm{Hom}(T, L') \neq 0$.

*Proof.* Let $L'$ be a line bundle. Choose $n \in \mathbb{Z}$ such that $n < \mu(L') - \bar{p}/\varepsilon - \delta(\omega)$. Then $\mu(L') - \mu(L_n) > \bar{p}/\varepsilon + \delta(\omega)$, and by Lemma A.2, since $L_n$ is special, we have $\mathrm{Hom}(L_n, L') \neq 0$. Hence there is a monomorphism $L_n \to L'$. Since $\mathrm{Hom}(T, L_n) \neq 0$ we get $\mathrm{Hom}(T, L') \neq 0$ as well. □

In order to remove the word "special" from the preceding lemma, we use the Riemann-Roch formula (see [39])

$$\frac{1}{\bar{p}\kappa}\langle\!\langle X, Y\rangle\!\rangle = -\frac{\varepsilon}{2}\delta(\omega)\cdot \mathrm{rk}(X)\cdot\mathrm{rk}(Y) + \frac{\varepsilon}{\bar{p}}\begin{vmatrix}\mathrm{rk}(X) & \mathrm{rk}(Y)\\ \deg(X) & \deg(Y)\end{vmatrix} \tag{A.1}$$

where

$$\langle\!\langle X, Y\rangle\!\rangle = \sum_{j=0}^{\bar{p}-1}\langle X, \tau^{-j}Y\rangle$$

is the average Euler form. In particular, if $L'$ and $L$ are line bundles with $\mu(L) = \deg(L) \geq \deg(L') = \mu(L')$ then (by $\delta(\omega) < 0$) we have $\langle\!\langle L', L\rangle\!\rangle >$

0. Since, by stability, $\operatorname{Ext}^1(L',\tau^{-j}L) = \operatorname{D}\operatorname{Hom}(\tau^{-j-1}L,L') = 0$, and thus $\langle L',\tau^{-j}L\rangle = \dim_k \operatorname{Hom}(L',\tau^{-j}L)$, we obtain $\operatorname{Hom}(L',\tau^{-j}L) \neq 0$ for some $j \in \{0,\dots,\bar{p}-1\}$. It follows that there is a monomorphism $L' \to \tau^{-j}L$, where $\mu(\tau^{-j}L) = \mu(L) - j\cdot\delta(\omega) \leq \mu(L) - (\bar{p}-1)\cdot\delta(\omega)$.
If $L$ is a special line bundle, then also $\tau^n L$ is special for every integer $n$, and has slope $\mu(\tau^n L) = \mu(L)+n\cdot\delta(\omega)$. So, if $L'$ is a given line bundle, then there is a *special* line bundle $L$ of slope $\mu(L)$ in the interval $[\mu(L'),\mu(L')-\delta(\omega)[$. With the preceding paragraph we obtain $j\in\{0,\dots,\bar{p}-1\}$ and a monomorphism $L'\to\tau^{-j}(L)$, for which $\mu(\tau^{-j}L)\leq\mu(L)-(\bar{p}-1)\cdot\delta(\omega)<\mu(L')-\bar{p}\cdot\delta(\omega)$ holds. To summarize:

Lemma A.5. *Let $\mathbb{X}$ be domestic. For every line bundle $L'$ there is a* special *line bundle $L$ with a monomorphism $L'\to L$, so that the distance of slopes*

$$0\leq\mu(L)-\mu(L')<-\bar{p}\cdot\delta(\omega)$$

*is bounded by a constant.* □

We then have: if $L'$ is such that $\operatorname{Hom}(T,L')\neq 0$, then, since there is a monomorphism $L'\to L$, also $\operatorname{Hom}(T,L)\neq 0$. As a consequence we get: if we find line bundles $L'$ of arbitrarily small slope with $\operatorname{Hom}(T,L')\neq 0$, then we also find *special* line bundles $L$ of arbitrarily small slope with $\operatorname{Hom}(T,L)\neq 0$. Therefore we now have the stronger version of Lemma A.4.

Lemma A.6. *Let $\mathbb{X}$ be domestic. Let $T$ be a torsionfree tilting sheaf. Assume that for every $n\in\mathbb{Z}$ there is a line bundle $L_n$ with $\mu(L_n)<n$ such that* $\operatorname{Hom}(T,L_n)\neq 0$. *If $L'$ is an arbitrary line bundle, then also* $\operatorname{Hom}(T,L')\neq 0$. □

Lemma A.7. *Let $\mathbb{X}$ be domestic. Let $T$ be a torsionfree tilting sheaf. Then there is $n_0\in\mathbb{Z}$ such that* $\operatorname{Hom}(T,L')=0$ *for every line bundle $L'$ with $\mu(L')<n_0$.*

*Proof.* Otherwise we would have $\operatorname{Hom}(T,L')\neq 0$ for all line bundles $L'$ by the preceding lemma. As in the proof of Lemma 8.2, we see that for a line bundle $L'$ the condition $\operatorname{Hom}(T,L')\neq 0$ amounts to $L'\in\operatorname{Gen}(T)$, and since every vector bundle has a line bundle filtration, we infer that all vector bundles lie in $\operatorname{Gen}(T)=T^{\perp_1}$. By Serre duality we get that *no* vector bundle (even no coherent sheaf) maps to the torsionfree sheaf $T$, which is a contradiction, since $\vec{\mathcal{H}}$ is locally noetherian. □

We conclude with the desired result.

Lemma A.8 (Lemma 6.1). *Let $\mathbb{X}$ be domestic. Let $T$ be a torsionfree tilting sheaf. Then there is $m\in\mathbb{Z}$ such that* $\operatorname{Hom}(T,E)=0$ *for every indecomposable vector bundle $E$ with $\mu(E)<m$.*

*Proof.* Let $\mathcal{F}$ be the set of indecomposable vector bundles $F$ with $0\leq\mu(F)<-\delta(\omega)$. This is a finite set by (D5), and every indecomposable vector bundle is of the form $\tau^n F$ for some $F\in\mathcal{F}$ and some $n\in\mathbb{Z}$. For every $F\in\mathcal{F}$ we fix a line bundle filtration, which altogether form a finite collection $\mathcal{L}$ of line bundles. We denote by $\alpha = \alpha(\mathcal{F})$ the maximum of slopes of the objects in $\mathcal{L}$. Then $\alpha(\tau^m\mathcal{F}) = \alpha + m\delta(\omega)$. With the bound $n_0$ from Lemma A.7, for all $m \in \mathbb{Z}$ such that $\alpha + m\delta(\omega) < n_0$, we get $\mathrm{Hom}(T, \tau^m\mathcal{L}) = 0$, and thus $\mathrm{Hom}(T, \tau^m\mathcal{F}) = 0$. It follows that $\mathrm{Hom}(T, E) = 0$ for every indecomposable vector bundle $E$ with $\mu(E) < m\delta(\omega)$. □

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# Modular Equalities for Complex Reflection Arrangements

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Abstract. We compute the combinatorial Aomoto–Betti numbers $\beta_p(\mathcal{A})$ of a complex reflection arrangement. When $\mathcal{A}$ has rank at least 3, we find that $\beta_p(\mathcal{A}) \leqslant 2$, for all primes $p$. Moreover, $\beta_p(\mathcal{A}) = 0$ if $p > 3$, and $\beta_2(\mathcal{A}) \neq 0$ if and only if $\mathcal{A}$ is the Hesse arrangement. We deduce that the multiplicity $e_d(\mathcal{A})$ of an order $d$ eigenvalue of the monodromy action on the first rational homology of the Milnor fiber is equal to the corresponding Aomoto–Betti number, when $d$ is prime. We give a uniform combinatorial characterization of the property $e_d(\mathcal{A}) \neq 0$, for $2 \leqslant d \leqslant 4$. We completely describe the monodromy action for full monomial arrangements of rank 3 and 4. We relate $e_d(\mathcal{A})$ and $\beta_p(\mathcal{A})$ to multinets, on an arbitrary arrangement.

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Contents



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## 1. Introduction and statement of results

1.1. Milnor fibration and monodromy. The complement $M$ of a degree $n$ complex hypersurface in $\mathbb{C}^l$, $\{f = 0\}$, and the associated Milnor fibration, $f : M \to \mathbb{C}^\times$, first analysed by Milnor in his seminal book [15], attracted a lot of attention over the years. Multiplication by $\exp(\frac{2\pi\sqrt{-1}}{n})$ induces the geometric monodromy action on the associated *Milnor fiber* $F = f^{-1}(1)$, $h : F \to F$, and the *algebraic monodromy action*, $\{h_i : H_i(F,\mathbb{Q}) \to H_i(F,\mathbb{Q})\}$.
Computing $h_i$ is a major problem in the field, when $f$ has a non-isolated singularity at 0. Even for the defining polynomial of a (central) complex hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^l$ and $i = 1$, the answer is far from being clear. This case was tackled in the recent literature by many authors, who used a variety of tools; see for instance [20] for a brief survey. In this paper, we focus on *reflection arrangements*, associated to finite complex reflection groups.
It is well-known that every arrangement complement $M_{\mathcal{A}}$ has the homotopy type of a connected, finite CW-complex with torsion-free homology, whose first integral homology group, $H_1(M_{\mathcal{A}},\mathbb{Z}) = \mathbb{Z}^n$, comes endowed with a natural basis, given by meridian loops around the hyperplanes.
It is also well-known that, for an arbitrary arrangement $\mathcal{A}$, $h_1$ induces a $\mathbb{Q}[\mathbb{Z}]$-module decomposition,

$$H_1(F_{\mathcal{A}},\mathbb{Q}) = \bigoplus_d (\mathbb{Q}[t]/\Phi_d(t))^{e_d(\mathcal{A})}, \tag{1}$$

where $\Phi_d$ is the $d$-th cyclotomic polynomial, $e_d(\mathcal{A}) = 0$ if $d \nmid n$, and $e_1(\mathcal{A}) = n - 1$. See for instance [14, (1.1)].
A pleasant feature of hyperplane arrangements is the rich combinatorial structure encoded by the associated intersection lattice, $\mathcal{L}_\bullet(\mathcal{A})$, whose elements are the intersections of hyperplanes from $\mathcal{A}$, ranked by codimension and ordered by inclusion. In this context, the open monodromy action problem takes the following more precise form: are the multiplicities $e_d(\mathcal{A})$ combinatorial? If so, give a formula involving only $\mathcal{L}_\bullet(\mathcal{A})$.

1.2. Characteristic and resonance varieties. Our approach to decomposition (1) is topological, based on two types of jump loci, associated to CW-complexes having the properties recalled for $M_{\mathcal{A}}$, and the interplay between them.
The complex characteristic variety $\mathcal{V}^i_q(M)$, sitting inside the character torus $\mathbb{T}(M) := \mathrm{Hom}(H_1(M,\mathbb{Z}),\mathbb{C}^\times) = (\mathbb{C}^\times)^n$ is the locus of those $\rho \in \mathbb{T}(M)$ for which $\dim_{\mathbb{C}} H_i(M,\mathbb{C}_\rho) \geqslant q$, where $\mathbb{C}_\rho$ denotes the associated rank 1 local system on $M$.
Note that $\mathrm{Hom}(H_1(M,\mathbb{Z}),\mathbb{C}) = H^1(M,\mathbb{C}) = \mathbb{C}^n$, and denote by $\exp : H^1(M,\mathbb{C}) \twoheadrightarrow \mathbb{T}(M)$ the natural exponential map. For an integer $d \geqslant 1$, let $\rho_d \in \mathbb{T}(M)$ be the exponential of the diagonal cohomology class equal to $\frac{2\pi\sqrt{-1}}{d}$, with respect to the distinguished $\mathbb{Z}$-basis. When $M = M_{\mathcal{A}}$ is an arbitrary arrangement complement and $d > 1$, it is well-known that

$$e_d(\mathcal{A}) = \dim_{\mathbb{C}} H_1(M_{\mathcal{A}},\mathbb{C}_{\rho_d})\,. \tag{2}$$

See for instance [14, §2.3], and also [3, Theorem 2.5 and Corollary 6.4] for more general results of this type.

The resonance variety $\mathscr{R}^i_q(M,\Bbbk)$ over a field $\Bbbk$, sitting inside $H^1(M,\Bbbk)$, is the locus of those $\sigma \in H^1(M,\Bbbk)$ for which $\dim_{\Bbbk} H^i(H^\bullet(M,\Bbbk),\sigma *) \geqslant q$, where $\sigma *$ denotes the left multiplication by $\sigma$ in the cohomology ring. When $\Bbbk$ is the prime field $\mathbb{F}_p$, denote by $\sigma_p \in H^1(M,\mathbb{F}_p)$ the diagonal cohomology class equal to 1, with respect to the distinguished $\mathbb{Z}$-basis, and define the modulo $p$ *Aomoto-Betti number* by

$$\beta_p(M) := \dim_{\mathbb{F}_p} H^1(H^\bullet(M,\mathbb{F}_p),\sigma_p *) \tag{3}$$

When $M = M_{\mathcal{A}}$ is an arrangement complement, we will replace $M$ by $\mathcal{A}$ in the notation.

By a celebrated theorem of Orlik and Solomon [16], the cohomology ring of $M_{\mathcal{A}}$ is combinatorial. More precisely, $\beta_p(\mathcal{A})$ may be computed from $\mathcal{L}_{\leqslant 2}(\mathcal{A})$, as well as $\mathscr{R}^1_q(\mathcal{A},\Bbbk)$, for all $q$ and $\Bbbk$.

1.3. MODULAR BOUNDS. It follows from Theorem 11.3 in [19] that

$$e_{p^s}(\mathcal{A}) \leqslant \beta_p(\mathcal{A}), \text{for all } s \geqslant 1\,, \tag{4}$$

when $\mathcal{A}$ is an arbitrary arrangement.

Actually, the above modular bound holds for all CW-complexes considered in §1.2, with the multiplicity replaced by the value from equality (2), and is in general strict, in the broader context. Our first main result says that the modular bound (4) becomes an equality, for reflection arrangements and $s = 1$.

THEOREM 1.1. *Let $\mathcal{A}$ be a complex reflection arrangement. Then $e_p(\mathcal{A}) = \beta_p(\mathcal{A})$, for all primes $p$. In particular, $e_p(\mathcal{A})$ is determined by $\mathcal{L}_{\leqslant 2}(\mathcal{A})$.*

1.4. AOMOTO-BETTI NUMBERS FOR REFLECTION ARRANGEMENTS. Reflection arrangements have a distinguished history, going back as far as Jordan's work from 1878 on the symmetry group of the famous Hessian configuration. A related open problem is whether the *Hessian arrangement* is the only arrangement supporting a 4-net. In Theorem 1.2(2) below, we solve this problem for reflection arrangements.

Finite complex reflection groups have been classified by Shephard and Todd [22] (see also [2], [17]). Each such group $G$ gives rise to the complex reflection arrangement $\mathcal{A}(G)$, consisting of the fixed points of all reflections in $G$. Among them, we have the monomial arrangements $\mathcal{A}(m,m,l)$ in $\mathbb{C}^l$ $(l \geqslant 2)$, with defining polynomials $\Pi_{1\leqslant i<j\leqslant l}\,(z_i^m - z_j^m)$ $(m \geqslant 1)$, and full monomial arrangements $\mathcal{A}(m,1,l)$ in $\mathbb{C}^l$ $(l \geqslant 2)$, defined by $z_1 \dots z_l \cdot \Pi_{1\leqslant i<j\leqslant l}\,(z_i^m - z_j^m)$ $(m \geqslant 2)$. We may now state our second main result.

THEOREM 1.2. *For a complex reflection arrangement $\mathcal{A}$ of rank at least* 3*, the following hold.*

(1) *If $p > 3$, then $\beta_p(\mathcal{A}) = 0$.*
(2) *$\beta_2(\mathcal{A}) \neq 0 \Leftrightarrow \beta_2(\mathcal{A}) = 2 \Leftrightarrow \mathcal{A}$ supports a* 4*-net $\Leftrightarrow \mathcal{A}$ is the Hessian arrangement.*

(3) *The only cases when* $\beta_3(\mathcal{A}) \neq 0$ *are:* $\mathcal{A}(m,1,3)$ *with* $m \equiv 1$ (*mod* 3)*, where* $\beta_3 = 1$*;* $\mathcal{A}(m,m,3)$ *with* $m \geqslant 2$*, where* $\beta_3 = 1$ *if* $m \not\equiv 0$ (*mod* 3) *and otherwise* $\beta_3 = 2$*;* $\mathcal{A}(m,m,4)$*, where* $\beta_3 = 1$.
(4) *In particular,* $\beta_p(\mathcal{A}) \leqslant 2$*, for all primes* $p$.

We imposed the rank condition since the Aomoto-Betti numbers for arrangements of rank at most 2 are known (see for instance [14]). Moreover, when $\mathcal{A} = \mathcal{A}(m,m,2)$ with $m \equiv 0 \pmod p$ it is easy to see that $\beta_p(\mathcal{A}) = m-2$. Thus, the conclusion of Theorem 1.2(4) no longer holds, for $m > 4$.
The resonance varieties $\mathscr{R}^1_q(\mathcal{A}, \mathbb{C})$ are quite well-understood, due to work by Falk, Libgober, Marco-Buzunáriz and Yuzvinsky; see [10], [13]. There are results in positive characteristic which show a different qualitative behaviour of resonance in this case; see for instance Falk [9]. The complete picture over $\mathbb{F}_p$ largely remains a mystery. Our Theorems 1.2 and 1.1, together with recent vanishing results due to Dimca and Sticlaru ([6], [7]), verify the strong modular conjecture from [20], for the important class of complex reflection arrangements.

1.5. A combinatorial non-triviality test. Dimca, Ibadula and Măcinic asked in [4] the following natural question: if $d > 1$ and $e_d(\mathcal{A}) \neq 0$, does this imply that $\rho_d \in \exp \mathscr{R}^1_1(\mathcal{A}, \mathbb{C})$? A positive answer (for all $d$) would imply that the non-triviality of $h_1$ is combinatorial, since the converse implication is known, for all values of $d$.

Theorem 1.3. *If* $\mathcal{A}$ *is a complex reflection arrangement, then the above question has a positive answer, for* $2 \leqslant d \leqslant 4$.

We derive Theorem 1.3 and Theorem 1.1 from Theorem 1.2 with the aid of a general result (proved in Theorem 4.3) that relates combinatorial structures on arrangements satisfying the key multinet axiom introduced by Falk-Yuzvinsky in [10] to the algebraic monodromy action and the Aomoto-Betti numbers of an arrangement. The tools from our paper also enable us to give a complete, combinatorial description in Proposition 4.4 for the monodromy action on $H_1(F_{\mathcal{A}}, \mathbb{Q})$, in the case of full monomial arrangements of rank 3 and 4. Related results may be found in [12] and [20]. By Theorems 1.1, 1.2 and [6, 7], this complete, combinatorial description holds for arbitrary complex reflection arrangements.

## 2. Non-exceptional reflection arrangements

We first compute the Aomoto-Betti numbers of monomial and full monomial arrangements.

2.1. The classification. (cf. [22], [2], [17])
A finite reflection group $G$ decomposes as a product of irreducible factors of the same kind. At the level of arrangements, $\mathcal{A}(G \times G')$ is the product $\mathcal{A}(G) \times \mathcal{A}(G')$, and the corresponding complements satisfy $M(G \times G') = M(G) \times M(G')$. (Whenever convenient, we will abbreviate notation and replace $\mathcal{A}(G)$ by $G$, when speaking about associated objects.)

The irreducible reflection arrangements of rank at least 3 comprise the monomial and full monomial arrangements, $\mathcal{A}(m,m,l)$ ($m \geqslant 2, l \geqslant 3$ or $m = 1, l \geqslant 4$) and $\mathcal{A}(m,1,l)$ ($l \geqslant 3$), plus the exceptional arrangements $\mathcal{A}(G_{23}) - \mathcal{A}(G_{37})$. The Hessian arrangement is $\mathcal{A}(G_{25})$. See for instance [17] for the notations.

2.2. Vanishing criteria. Given an arbitrary arrangement $\mathcal{A}$ and an $r$–flat $X \in \mathcal{L}_r(\mathcal{A})$, set $\mathcal{A}_X \coloneqq \{H \in \mathcal{A} | H \supseteq X\}$, and define the multiplicity of $X$ to be equal to $|\mathcal{A}_X|$.
Using the distinguished $\mathbb{Z}$-basis, we identify an element $\eta \in H^1(M_\mathcal{A}, \mathbb{F}_p)$ with the family $\{\eta_H \in \mathbb{F}_p\}_{H \in \mathcal{A}}$. We denote by $Z_p(\mathcal{A}) \subseteq H^1(M_\mathcal{A}, \mathbb{F}_p)$ the kernel of $\sigma_p*$.
Definition (3) implies that $\beta_p(\mathcal{A}) = \dim_{\mathbb{F}_p} Z_p(\mathcal{A}) - 1$. Our computations are based on the following well-known result (see e.g. [20]).

Lemma 2.1. *An element $\eta$ belongs to $Z_p(\mathcal{A})$ if and only if, for any $X \in \mathcal{L}_2(\mathcal{A})$,*

$$\begin{cases} \sum_{H \in \mathcal{A}_X} \eta_H = 0 & \text{if} \quad |\mathcal{A}_X| \equiv 0 \quad (\text{mod}\, p) \\ \eta_H = \eta_{H'}, \forall\, H, H' \in \mathcal{A}_X & \text{if} \quad |\mathcal{A}_X| \not\equiv 0 \quad (\text{mod}\, p) \end{cases}$$

Clearly, $\beta_p(\mathcal{A}) = 0$ if and only if $\eta \in Z_p(\mathcal{A})$ implies that $\eta$ is constant. A first useful vanishing criterion is due to Yuzvinsky.

Lemma 2.2. *([25]) If $|\mathcal{A}| \not\equiv 0$ (mod $p$), then $\beta_p(\mathcal{A}) = 0$.*

A second convenient situation is the following.

Lemma 2.3. *Assume that $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$.*

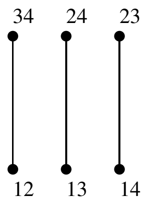
(1) *$H_1(M_\mathcal{A}, \mathbb{C}_{\rho_d}) = 0$, for all $d > 1$.*
(2) *$\beta_p(\mathcal{A}) = 0$, for all primes $p$.*

*Proof.* Assuming the contrary, we infer from [18, Proposition 13.1] that $\rho_d \in \mathcal{V}_1^1(M_{\mathcal{A}_1} \times M_{\mathcal{A}_2}) = \{1\} \times \mathcal{V}_1^1(M_{\mathcal{A}_2}) \cup \mathcal{V}_1^1(M_{\mathcal{A}_1}) \times \{1\}$, respectively $\sigma_p \in \mathcal{R}_1^1(M_{\mathcal{A}_1} \times M_{\mathcal{A}_2}, \mathbb{F}_p) = \{0\} \times \mathcal{R}_1^1(M_{\mathcal{A}_2}, \mathbb{F}_p) \cup \mathcal{R}_1^1(M_{\mathcal{A}_1}, \mathbb{F}_p) \times \{0\}$, in contradiction with the fact that all coordinates of $\rho_d$ (respectively $\sigma_p$) are different from 1 (respectively 0). □

By Lemma 2.3(2), we only need to compute $\beta_p(G)$ for an irreducible complex reflection group $G$.
To state the third vanishing criterion, we need to introduce certain simple graphs, associated to an arrangement $\mathcal{A}$ and an integer $k \geqslant 2$, with vertex set $\mathcal{A}$. The edges of $\Gamma_k(\mathcal{A})$ are defined by the condition $|\mathcal{A}_{H \cap H'}| \not\equiv 0$ (mod $k$), for $k > 2$, and by $|\mathcal{A}_{H \cap H'}|$ is either odd or equal to 2, for $k = 2$. The defining property for $\Gamma_{(k)}(\mathcal{A})$ is $|\mathcal{A}_{H \cap H'}| = k$. Note that $\Gamma_{(2)}(\mathcal{A})$ is a subgraph of $\Gamma_p(\mathcal{A})$, for all primes $p$. The equivalence relation on $\mathcal{A}$ associated to the edge paths of $\Gamma_k(\mathcal{A})$, respectively $\Gamma_{(k)}(\mathcal{A})$, will be denoted by $\sim_k$, respectively $\sim_{(k)}$.

Example 2.4. Let $\mathcal{A} = \mathcal{A}(1,1,l)$ be the braid arrangement in $\mathbb{C}^l$, with hyperplanes labeled by the two-element subsets of $\{1, \ldots, l\}$. For $ij \neq st$, the multiplicity of the 2–flat determined by the hyperplanes $z_i = z_j$ and $z_s = z_t$ is 2 if $|\{i, j, s, t\}| = 4$ and 3 if $|\{i, j, s, t\}| = 3$. Let $p$ be a prime. It follows from the above definition that $\Gamma_p(\mathcal{A})$ is a complete graph (with full edge set) if $p \neq 3$. For $p = 3$, $ij$ and $st$ are connected by an edge of $\Gamma_3(\mathcal{A})$ if and only if $|\{i, j, s, t\}| = 4$. We infer that the graph $\Gamma_3(\mathcal{A})$ is discrete (with no edges) when $l \leqslant 3$, has 3 connected components when $l = 4$ (see the picture of the corresponding graph below), and is connected when $l \geqslant 5$.

$\Gamma_3(\mathcal{A}(1,1,4))$:




We obtain the following immediate consequences of Lemma 2.1.

LEMMA 2.5. *Each of the properties below implies that* $\beta_p(\mathcal{A}) = 0$.

(1) *The graph* $\Gamma_p(\mathcal{A})$ *is connected.*
(2) *The graph* $\Gamma_{(2)}(\mathcal{A})$ *is connected.*
(3) *For all* $X \in \mathcal{L}_2(\mathcal{A})$, $p \nmid |\mathcal{A}_X|$.

2.3. INTERSECTION LATTICES. The Aomoto-Betti numbers for $\mathcal{A}(1,1,l)$ were computed in [14]. They verify all statements from Theorems 1.2 and 1.1. Hence, we may suppose from now on that $m \geqslant 2$.

We need to describe $\mathcal{L}_{\leqslant 2}(\mathcal{A})$. It will be convenient to label the various hyperplanes as follows. Set $\omega = \exp(\frac{2\pi\sqrt{-1}}{m})$, $(H_i)\ z_i = 0$ for all $1 \leqslant i \leqslant l$, and $(H_{ij^\alpha})\ z_i - \omega^\alpha z_j = 0$, for $1 \leqslant i < j \leqslant l$ and $\alpha \in \mathbb{Z}/m\mathbb{Z}$. We go on by listing the 2-flats (identified with the corresponding subarrangements $\mathcal{A}_X$).

CASE I: $\mathcal{A}(m,1,l), l \geqslant 4$ :
$\mathrm{I_a}$: $\{H_i, H_j, H_{ij^\alpha}(\alpha \in \mathbb{Z}/m\mathbb{Z})\}$, with multiplicity $m+2$;
$\mathrm{I_b}$: $\{H_{ij^\alpha}, H_{jk^\beta}, H_{ik^{\alpha+\beta}}\}$, with multiplicity 3;
$\mathrm{I_c}$: $\{H_{ij^\alpha}, H_{kh^\beta}\}$, with multiplicity 2;
$\mathrm{I_d}$: $\{H_i, H_{jk^\alpha}\}$, with multiplicity 2.
CASE II: $\mathcal{A}(m,m,l), l \geqslant 4$ : types $\mathrm{I_b}$ and $\mathrm{I_c}$, plus $\{H_{ij^\alpha}(\alpha \in \mathbb{Z}/m\mathbb{Z})\}$, with multiplicity $m$.
CASE III: $\mathcal{A}(m,1,3)$ : types $\mathrm{I_a}$, $\mathrm{I_b}$ and $\mathrm{I_d}$.
CASE IV: $\mathcal{A}(m,m,3)$ : types $\mathrm{I_b}$ and II.

2.4. $\beta_p$-VANISHING. We will use Lemma 2.5 to treat the cases when $\beta_p(\mathcal{A}) = 0$ in the non-exceptional families. To simplify things, we suppress $H$ from notation and identify the hyperplanes with their labels, $i$ and $ij^\alpha$.

We claim that, for $\mathcal{A} = \mathcal{A}(m,1,l)$ with $l \geqslant 4, \Gamma_{(2)}(\mathcal{A})$ is connected. Indeed, given $i < j$ we may find $h < k$ with $i,j,h,k$ distinct. Hence, $i \sim_{(2)} hk^0 \sim_{(2)} j$ and $ij^\alpha \sim_{(2)} k$, which proves connectivity. Similar arguments lead to the following conclusions. If $\mathcal{A} = \mathcal{A}(m,1,l)$ with $l = 3$ and $p \neq 3$, then $\Gamma_p(\mathcal{A})$ is connected; for $p = 3$ and $m \not\equiv 1 \pmod 3, \Gamma_3(\mathcal{A})$ is connected. The remaining full monomial Aomoto-Betti numbers, $\beta_3(m,1,3)$ with $m \equiv 1 \pmod 3$, will be computed later on.

If $\mathcal{A} = \mathcal{A}(m,m,l)$ with $l \geqslant 5$, then $\Gamma_{(2)}(\mathcal{A})$ is connected. For $l = 3,4$ and $p \neq 3, \Gamma_p(\mathcal{A})$ is connected. So, for monomial arrangements, only $\beta_3$ in ranks 3 and 4 remains to be calculated.

2.5. The remaining non-exceptional cases. A mod 3 cocycle $\eta \in Z_3(\mathcal{A})$ is a family of elements of $\mathbb{F}_3$, $\eta_k$ and $\eta_{ij^\alpha}$, satisfying the equations from Lemma 2.1, for any $X \in \mathcal{L}_2(\mathcal{A})$.
Case $\mathcal{A} = \mathcal{A}(m, 1, 3)$ with $m \equiv 1 (\text{mod } 3)$.
The equations coming from 2-flats of type $\mathrm{I_d}$ say that $\eta_{jk^\alpha} = \eta_i$, where $i$ is the third element of $\{1, 2, 3\}$. The equations of type $\mathrm{I_b}$ become equivalent to $\eta_1 + \eta_2 + \eta_3 = 0$, while type $\mathrm{I_a}$ equations say that $\eta_i + \eta_j + m\eta_k = 0$, for all $i \neq j \neq k$. We infer that $\beta_3(m, 1, 3) = 1$, as asserted in Theorem 1.2.
Case $\mathcal{A} = \mathcal{A}(m, m, 4)$.
The equations of type $\mathrm{I_c}$ say that $\eta_{ij^\alpha} = \eta_{ij}$, and $\eta_{34} = \eta_{12}, \eta_{24} = \eta_{13}, \eta_{23} = \eta_{14}$. Type $\mathrm{I_b}$ equations reduce then to $\eta_{12} + \eta_{13} + \eta_{14} = 0$, while type II conditions follow from $\eta_{ij^\alpha} = \eta_{ij}$. Again, $\beta_3(m, m, 4) = 1$, as claimed.
Case $\mathcal{A} = \mathcal{A}(m, m, 3)$ with $m \not\equiv 0 (\text{mod } 3)$.
The equations of type II say that $\eta_{ij^\alpha} = \eta_{ij}$, and the type $\mathrm{I_b}$ conditions then reduce to $\eta_{12} + \eta_{23} + \eta_{13} = 0$. This shows that $\beta_3(m, m, 3) = 1$, as asserted.
Case $\mathcal{A} = \mathcal{A}(m, m, 3)$ with $m = 3n$.
Set $\eta_{12^\alpha} = a_\alpha, \eta_{23^\alpha} = b_\alpha, \eta_{13^\alpha} = c_\alpha$. With this notation, the equations of type $\mathrm{I_b}$ are equivalent to the system

$$a_\alpha + b_\beta + c_{\alpha+\beta} = 0, \forall\, \alpha, \beta \in \mathbb{Z}/m\mathbb{Z}, \tag{5}$$

while the conditions of type II read

$$\sum a_\alpha = \sum b_\beta = \sum c_\gamma = 0. \tag{6}$$

We first solve the system (5), as follows. It implies that $a_\alpha + b_\beta = a_{\alpha'} + b_{\beta'}$, if $\alpha+\beta = \alpha'+\beta'$, in particular $a_\alpha - a_0 = b_\alpha - b_0 \coloneqq d_\alpha$, for all $\alpha$, and $d_\alpha + d_\beta = d_{\alpha'} + d_{\beta'}$, if $\alpha+\beta = \alpha'+\beta'$. We infer that $d_\alpha = \alpha d_1$, for all $\alpha$. Hence $a_\alpha = a_0 + \alpha d_1, b_\alpha = b_0 + \alpha d_1$ and $c_\alpha = -a_0 - b_0 - \alpha d_1$, which solves the system (5). In particular, its solution space is 3-dimensional.
Finally, it is an easy matter to check that (5) $\Rightarrow$ (6), since $m = 3n$. Therefore, $\beta_3(3n, 3n, 3) = 2$, as asserted.
This proves Theorem 1.2 for non-exceptional reflection arrangements.

## 3. Exceptional reflection arrangements

We finish the proof of Theorem 1.2, by computing the Aomoto-Betti numbers of the exceptional complex reflection arrangements of rank at least 3, $G_{23} - G_{37}$.

3.1. The groups $G_{31}$, $G_{32}$, $G_{33}$. Case $\mathcal{A} = \mathcal{A}(G_{31})$. The hyperplanes of $\mathcal{A}$ live in $\mathbb{C}^4$. Their defining equations are as follows (see [11]). Set $\omega = \exp(\frac{2\pi\sqrt{-1}}{4})$. The hyperplanes of $\mathcal{A}$ are:

- $(H_i)$ $z_i = 0$ $(1 \leqslant i \leqslant 4)$;
- $(H_{ij^\beta})$ $z_i - \omega^\beta z_j = 0$ $(1 \leqslant i < j \leqslant 4,\ \beta \in \mathbb{Z}/4\mathbb{Z})$;
- $(H_{\overline{\alpha}})$ $z_1 + \sum_{2\leqslant i\leqslant 4} \omega^{\alpha_i} z_i = 0$ $(\overline{\alpha} = (\alpha_2, \alpha_3, \alpha_4) \in (\mathbb{Z}/4\mathbb{Z})^3, \alpha_2 + \alpha_3 + \alpha_4 \equiv 0 \ (\text{mod } 2))$.

By Lemma 2.5(2), it is enough to show that $\Gamma_{(2)}(G_{31})$ is connected. This can be seen as follows. Clearly, the 2-flat $H_k \cap H_{ij^\beta}$ has multiplicity 2, when $i \neq j \neq k$. This implies that $i \sim_{(2)} j \sim_{(2)} kh^\beta$, for all $1 \leqslant i < j \leqslant 4$, $1 \leqslant k < h \leqslant 4$ and $\beta \in \mathbb{Z}/4\mathbb{Z}$. Given any $H_{\overline{\alpha}}$, it is not hard to see that the multiplicity of $H_{\overline{\alpha}} \cap H_{12^{\alpha_2}}$ is 2. This proves connectivity, as claimed.

Case $\mathcal{A} = \mathcal{A}(G_{32})$. Set $\omega = \exp(\frac{2\pi\sqrt{-1}}{3})$. The arrangement $\mathcal{A}$ consists of the following hyperplanes in $\mathbb{C}^4$ (see [24]):

- $(H_i)$ $z_i = 0$ $(1 \leqslant i \leqslant 4)$;
- $(H_{1^{\alpha\beta}})$ $z_2 + \omega^\alpha z_3 + \omega^\beta z_4 = 0$;
- $(H_{2^{\alpha\beta}})$ $z_1 + \omega^\alpha z_3 - \omega^\beta z_4 = 0$;
- $(H_{3^{\alpha\beta}})$ $z_1 - \omega^\alpha z_2 + \omega^\beta z_4 = 0$;
- $(H_{4^{\alpha\beta}})$ $z_1 + \omega^\alpha z_2 - \omega^\beta z_3 = 0$ $(\alpha, \beta \in \mathbb{Z}/3\mathbb{Z})$.

Clearly, the 2-flats $H_i \cap H_j$ $(i \neq j)$ and $H_i \cap H_{i^{\alpha\beta}}$ have multiplicity 2. This shows that $\Gamma_{(2)}(G_{32})$ is connected and we are done.

Case $\mathcal{A} = \mathcal{A}(G_{33})$. Here, $\omega = \exp(\frac{2\pi\sqrt{-1}}{3})$ and the hyperplanes (in $\mathbb{C}^6$) are as follows (see [2, 21]):

- $(H_{ij^\beta})$ $z_i - \omega^\beta z_j = 0$ $(1 \leqslant i < j \leqslant 4,\ \beta \in \mathbb{Z}/3\mathbb{Z})$;
- $(H_{\overline{\alpha}})$ $\sum_{1 \leqslant i \leqslant 4} \omega^{\alpha_i} z_i + z_5 + z_6 = 0$ $(\overline{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in (\mathbb{Z}/3\mathbb{Z})^4,\ \sum \alpha_i = 0)$.

Plainly, $ij^\beta \sim_{(2)} kh^0 \sim_{(2)} ij^{\beta'}$, for all $\beta, \beta'$ (where $i \neq j \neq k \neq h$). Like in the Case $G = G_{31}$, it can be checked that $H_{\overline{\alpha}} \cap H_{ij^\beta}$ has multiplicity 2, if $\beta = \alpha_j - \alpha_i$. We infer that $\Gamma_{(2)}(G_{33})$ is connected, and we are done.

3.2. More vanishing criteria. We will no longer need defining equations to settle the remaining cases. We will use instead a couple of new vanishing arguments. For the beginning, let us recall from [17, pp. 224-225] the following very useful properties of reflection groups and arrangements, derived from a key result of Steinberg [23]. For any complex reflection group $G$ and any $X \in \mathcal{L}_r(\mathcal{A}(G))$, the fixer subgroup $G_X := \{g \in G \mid gx = x,\ \forall x \in X\}$ is a reflection group, and $\mathcal{A}(G)_X = \mathcal{A}(G_X)$ is again a reflection arrangement, of rank $r$. By the construction of $\mathcal{A}(G)$, the group $G$ acts on the arrangement $\mathcal{A}(G)$, hence on the intersection lattice $\mathcal{L}_\bullet(G)$. Let us denote by $\mathcal{O}_X$ the $G$-orbit of $X \in \mathcal{L}(G)$. Let Type$(X)$ be the isomorphism type of the reflection group $G_X$. It follows from [17, Lemma 6.88] that the type is constant on each orbit $\mathcal{O}_X$. Moreover, Table C from [17] gives $|\mathcal{A}(G)|$ and the orbit partition of $\mathcal{L}_\bullet(G)$ for all exceptional groups, in terms of types of orbits.

This leads to the quick computation of the sets

$$\mathscr{P}(G) := \{p \text{ prime} \mid \exists\, X \in \mathcal{L}_2(G) \text{ such that } |\mathcal{A}(G_X)| \equiv 0 \ (\mathrm{mod}\ p)\}$$

In particular, $\mathscr{P}(G) \subseteq \{2, 3, 5\}$, for every exceptional arrangement of rank at least 3. We infer from Lemma 2.5(3) that $\beta_p(G) = 0$, if $p > 5$. Hence, we may suppose from now on that $p \leqslant 5$.

For an arbitrary arrangement $\mathcal{A}$, $H \in \mathcal{A}$ and a prime $p$, we define

$$m_p(H) = 1 + \sum (|\mathcal{A}_X| - 1), \tag{7}$$

where the sum is taken over those $X \in \mathcal{L}_2(\mathcal{A})$ such that $X \subseteq H$ and $|\mathcal{A}_X| \not\equiv 0 \pmod{p}$.

The numbers from (7) may be extracted from Table C in [17], for exceptional reflection arrangements of rank at least 3. This is based on the following fact, valid for an arbitrary reflection group $G$. For $X, Y \in \mathcal{L}(G)$, let $u(X, Y)$ be the number of $Z \in \mathcal{L}(G)$ such that $Z \in \mathcal{O}_Y$ and $Z \subseteq X$. Clearly, this number depends only on $\mathcal{O}_X$ and $\mathcal{O}_Y$. The values $u(H, X)$ may be found in Table C, for all orbit types corresponding to $H \in \mathcal{L}_1(G)$ and $X \in \mathcal{L}_2(G)$.

Lemma 3.1. *For an arrangement $\mathcal{A}$ and a prime $p$, the following hold.*

(1) *If $m_p(H) > \frac{|\mathcal{A}|}{2}$ for all $H \in \mathcal{A}$, then $\beta_p(\mathcal{A}) = 0$.*
(2) *If $m_p(H) > \frac{|\mathcal{A}|}{3}$ for all $H \in \mathcal{A}$ and $\mathcal{A}$ has no rank 2 flats of multiplicity $p \cdot r$ with $r > 1$, then $\beta_p(\mathcal{A}) = 0$.*

*Proof.* If $\beta_p(\mathcal{A}) \neq 0$, there is a non-constant function $\eta \in \mathbb{F}_p^{\mathcal{A}}$ satisfying all equations from Lemma 2.1. Fix $H \in \mathcal{A}$ and set $\eta_H = \alpha$. We claim that $|\{\eta = \alpha\}| \geqslant m_p(H)$. Indeed, if $X \in \mathcal{L}_2(\mathcal{A})$ is contained in $H$ and $|\mathcal{A}_X| \not\equiv 0 \pmod{p}$, then $\eta$ must have the constant value $\alpha$ on $\mathcal{A}_X$. An easy count of all these hyperplanes gives the claimed inequality.

In Part (1), this implies that $\eta$ must be constant, a contradiction. In Part (2), we infer that $\eta$ has only two distinct values. By adding constant functions and multiplying by non-zero elements in $\mathbb{F}_p$, we may assume that these values are $\eta_H = 0$ and $\eta_{H'} = 1$. By Lemma 2.1, the flat $X = H \cap H'$ has multiplicity $p \cdot r$, imposing the condition $\sum_{K \in \mathcal{A}_X} \eta_K = 0$. Since necessarily $r = 1$, we arrive again at a contradiction. □

3.3. Induction on rank. We start with a couple of general considerations. A subarrangement $\mathcal{B} \subseteq \mathcal{A}$ is called *line-closed in* $\mathcal{A}$ (see the first definition from [8, Definition 1.1]) if $\mathcal{B}_X = \mathcal{A}_X$, for all $X \in \mathcal{L}_2(\mathcal{B})$. This property implies that the restriction map, $\mathbb{F}_p^{\mathcal{A}} \longrightarrow \mathbb{F}_p^{\mathcal{B}}$, sends $Z_p(\mathcal{A})$ into $Z_p(\mathcal{B})$. Clearly, $\mathcal{A}_Y$ is line-closed in $\mathcal{A}$, for any $Y \in \mathcal{L}(\mathcal{A})$.

Lemma 3.2. *Let $G$ be a complex reflection group. Assume that $2 \leqslant r < \operatorname{rank}(\mathcal{A}(G))$ and $\beta_p(\mathcal{A}(G')) = 0$, for all irreducible groups $G' \in \operatorname{Type}(\mathcal{L}_r(\mathcal{A}(G)))$. Then $\beta_p(\mathcal{A}(G)) = 0$.*

*Proof.* Assuming the contrary, there exist $\eta \in Z_p(G)$ and $H_1, H_2 \in \mathcal{A}(G)$ such that $\eta_{H_1} \neq \eta_{H_2}$. From our assumption on $r$, we may find $H_3, \dots, H_r \in \mathcal{A}(G)$ such that $X = H_1 \cap H_2 \cdots \cap H_r \in \mathcal{L}_r(G)$. Set $\mathcal{B} = \mathcal{A}(G)_X = \mathcal{A}(G_X)$. We deduce that $\beta_p(G_X) \neq 0$. If $G_X$ is reducible, this contradicts Lemma 2.3. Otherwise, our second assumption is violated. □

3.4. The rank 3 case. The rank 3 exceptional groups are $H_3, G_{24}, G_{25}, G_{26}$ and $G_{27}$. Table C from [17] provides the following information on each group:

- $\mathscr{P}(G) = \{2,3,5\};\ \{2,3\};\ \{2\};\ \{2,5\};\ \{2,3,5\};$
- $|\mathcal{A}(G)| = 15; 21; 12; 21; 45.$

By Lemma 2.2 and Lemma 2.5(3), the only primes $p$ which might give $\beta_p(G) \neq 0$ are as follows:

$$3, 5\ (H_3);\ 3\ (G_{24});\ 2\ (G_{25});\ -\ (G_{26});\ 3, 5\ (G_{27}).$$

Using Lemma 3.1(1), we obtain $\beta_3(H_3) = \beta_3(G_{24}) = \beta_3(G_{27}) = \beta_5(G_{27}) = 0$, and Lemma 3.1(2) gives $\beta_5(H_3) = 0$. Finally, $\beta_2(G_{25}) = 2$, cf. [20]. Thus, Theorem 1.2 is proved in this case. Indeed, the Hessian arrangement $\mathcal{A}(G_{25})$ supports a 4-net (see e.g. [26]). This implies that $\beta_2 \neq 0$, by [20]; the other implications from Theorem 1.2(2) are obvious.
Applying Lemma 3.2 for $r = 3$ and $p = 5$, we also infer that Theorem 1.2(1) holds for all complex reflection arrangements of rank at least 3. Thus, we only need to show that $\beta_2(G) = \beta_3(G) = 0$, when $G$ is exceptional of rank at least 4, in order to complete the proof of Theorem 1.2.

3.5. The remaining cases. The only rank 5 exceptional arrangement is $G_{33}$, for which we know from §3.1 that all $\beta_p$ vanish. By the computations from Section 2, the same thing happens for non-exceptional irreducible arrangements of rank 5. Lemma 3.2, applied for $r = 5$, guarantees then that we may reduce our proof to the rank 4 case. Here, the list is $G = F_4, G_{29}, H_4, G_{31}, G_{32}$, and the last two groups were treated in §3.1.
Case $G = F_4$. The irreducible rank 3 types are listed in Table C from [17]: $B_3$ and $C_3$; in both cases, the arrangement $\mathcal{A}(G')$ is $\mathcal{A}(2, 1, 3)$, for which all $\beta_p$ vanish, cf. Section 2. We may conclude by resorting to Lemma 3.2 for $r = 3$.
Case $G = G_{29}$. The list of irreducible rank 3 types is: $G' = A_3, B_3, G(4, 4, 3)$. Taking $r = 3$ and $p = 2$ in Lemma 3.2, we deduce from Section 2 that $\beta_2(G_{29}) = 0$. Since $|\mathcal{A}(G_{29})| = 40, \beta_3(G_{29}) = 0$, by Lemma 2.2.
Case $G = H_4$. The irreducible types of $\mathcal{L}_3(G)$ are: $G' = A_3, H_3$. Again by Lemma 3.2 and previous computations, $\beta_2(H_4) = 0$. Finally, $\beta_3(H_4) = 0$, as follows from Lemma 3.1(1).
The proof of Theorem 1.2 is complete.

## 4. Multinets and jump loci

In this section we prove Theorems 1.1 and 1.3. Along the way, we establish a useful general result that relates combinatorial structures on arrangements satisfying the main multinet axiom to the algebraic monodromy of its Milnor fibration and its Aomoto-Betti numbers.

4.1. Multinets and weighted partitions. The work of Falk and Yuzvinsky from [10] gives, among other things, a description of the resonance variety of an arrangement $\mathcal{A}$, $\mathscr{R}^1_1(\mathcal{A}, \mathbb{C})$, in terms of *multinets* on the associated matroid. A $k-$multinet on $\mathcal{L}_{\leqslant 2}(\mathcal{A})$ is a partition $\Pi$ with $k \geqslant 3$ non-empty blocks, $\mathcal{A} = \bigsqcup_{\alpha \in [k]} \mathcal{A}_\alpha$, together with a function, $\mathfrak{m} : \mathcal{A} \to \mathbb{Z}_{>0}$, satisfying certain axioms. The most important is the following:
For any $H \in \mathcal{A}_\alpha$ and $H' \in \mathcal{A}_\beta$ with $\alpha \neq \beta$, and every $\gamma \in [k]$,

$$n_X := \sum_{K \in \mathcal{A}_\gamma \cap \mathcal{A}_X} \mathfrak{m}_K \tag{8}$$

is independent of $\gamma$, where $X = H \cap H' \in \mathcal{L}_2(\mathcal{A})$.
Let $\Pi$ be a partition of $\mathcal{A}$ with $k \geqslant 3$ non-empty blocks, as above. Let $\mathfrak{m} : \mathcal{A} \to \mathbb{Z}$ be an assignment of arbitrary integer weights to the hyperplanes of $\mathcal{A}$.

Definition 4.1. The pair $\mathcal{N} = (\Pi, \mathfrak{m})$ is a *weighted k-partition* if axiom (8) is satisfied. We will say that $\mathcal{N}$ is *h-reduced* ($h \geqslant 1$) if $\mathfrak{m}_K \equiv 1 (\mathrm{mod}\ h)$, for all $K \in \mathcal{A}$.

The underlying partition of a $k$-multinet, together with its positive weight function, is a weighted $k$-partition. Moreover, the usual notion of reduced multinet corresponds to $h = 1$.
We will need a result from [20] related to axiom (8). To recollect it, we start with a few notations. Set $H_{\mathcal{A}} := H_1(M_{\mathcal{A}}, \mathbb{Z})$ and denote by $\{a_H\}_{H \in \mathcal{A}}$ the distinguished $\mathbb{Z}$-basis. Let $S$ be $\mathbb{CP}^1 \backslash \{k \text{ points}\}$ and set $H_S := H_1(S, \mathbb{Z}) = \mathbb{Z} - \mathrm{span}\langle c_\alpha | \alpha \in [k] \rangle / \sum_{\alpha \in [k]} c_\alpha$, where $c_\alpha$ is the class of a small loop in $S$ around the point $\alpha$.
Let $\cup_{\mathcal{A}} : \bigwedge^2 H^1(M_{\mathcal{A}}, \mathbb{Z}) \to H^2(M_{\mathcal{A}}, \mathbb{Z})$ be the cup product. Recalling from [16] that $H^\bullet(M_{\mathcal{A}}, \mathbb{Z})$ has no torsion, we denote by $\nabla_{\mathcal{A}} : H_2(M_{\mathcal{A}}, \mathbb{Z}) \to \bigwedge^2 H_{\mathcal{A}}$ the $\mathbb{Z}$-dual comultiplication map.
The next result improves Theorem 2.4 from [10], in several ways. The hypothesis of Proposition 4.2 is reduced to the key axiom (*iii*) from [10, Definition 2.1]. The weight function $\mathfrak{m}$ may take arbitrary integer values, while in [10] positivity plays a crucial role. The conclusion in [10] is that $\mathrm{im}(\phi \otimes \Bbbk)^* \subseteq H^1(M_{\mathcal{A}}, \Bbbk)$ is an isotropic subspace, for a characteristic 0 field $\Bbbk$ (in positive characteristic, an additional condition on $\mathfrak{m}$ is needed in [10]), while Proposition 4.2 gives the conclusion over $\mathbb{Z}$. For the reader's convenience, we include the proof.

Proposition 4.2 ([20]). *Let $\mathcal{N} = (\Pi, \mathfrak{m})$ be a weighted k-partition. Then $\bigwedge^2 \phi \circ \nabla_{\mathcal{A}} = 0$, where $\phi : H_{\mathcal{A}} \to H_S$ sends $a_H$ to $\mathfrak{m}_H c_\alpha$, for $H \in \mathcal{A}_\alpha$. Therefore, $\cup_{\mathcal{A}} \circ \bigwedge^2 \phi^* = 0$, by taking $\mathbb{Z}$-duals.*

*Proof.* Let $\mathfrak{h}(\mathcal{A})$ be the $\mathbb{Z}$-form of the holonomy Lie algebra of $\mathcal{A}$, appearing in Proposition 5.2 from [20]. By definition, this is the graded $\mathbb{Z}$-Lie algebra quotient of the free $\mathbb{Z}$-Lie algebra generated by $H_{\mathcal{A}}$, $\mathbb{L}^\bullet(H_{\mathcal{A}})$, graded by bracket length, by the graded Lie ideal generated by $\mathrm{im}(\nabla_{\mathcal{A}}) \subseteq \bigwedge^2 H_{\mathcal{A}}$, where $\bigwedge^2 H_{\mathcal{A}}$ is identified with $\mathbb{L}^2(H_{\mathcal{A}})$ via the Lie bracket. Let $\mathbb{L}^\bullet(\phi) : \mathbb{L}^\bullet(H_{\mathcal{A}}) \to \mathbb{L}^\bullet(H_S)$ be the graded $\mathbb{Z}$-Lie algebra map extending $\phi$. Our claim says that $\mathbb{L}^\bullet(\phi)$ factors through $\mathfrak{h}(\mathcal{A})$.
To check this, we recall from [20, (30)] that the defining Lie relations of $\mathfrak{h}(\mathcal{A})$ are:

$$\sum_{K \in \mathcal{A}_X} [a_K, a_L], \quad \text{for} \quad X \in \mathcal{L}_2(\mathcal{A}) \quad \text{and} \quad L \in \mathcal{A}_X .$$

Thus, we have to show that $[\sum_{K \in \mathcal{A}_X} \phi(a_K), \phi(a_L)] = 0 \in \mathbb{L}^2(H_S)$.
There are two cases to consider. When $X$ is mono-coloured, i.e., $\mathcal{A}_X \subseteq \mathcal{A}_\alpha$ for some $\alpha \in [k]$, by construction $\phi(a_K) \in \mathbb{Z} \cdot c_\alpha$, for all $K \in \mathcal{A}_X$, and we are done. Otherwise, $X = H \cap H'$, with $H \in \mathcal{A}_\alpha$, $H' \in \mathcal{A}_\beta$ and $\alpha \neq \beta$. Again by construction, $\sum_{K \in \mathcal{A}_X} \phi(a_K) = \sum_{\gamma \in [k]} (\sum_{K \in \mathcal{A}_\gamma \cap \mathcal{A}_X} \mathfrak{m}_K) c_\gamma$, which equals $n_X(\sum_{\gamma \in [k]} c_\gamma)$, by axiom (8). This implies that $\sum_{K \in \mathcal{A}_X} \phi(a_K) = 0 \in \mathbb{L}^1(H_S)$, by the definition of $H_S$, which completes the proof. $\square$

4.2. Relating weighted partitions to jump loci. We are now ready to state our result, keeping the previous notation.

Theorem 4.3. *Assume that* $\mathcal{N} = (\Pi, \mathfrak{m})$ *is a k-reduced weighted k-partition. Then the following hold, for all divisors p, d of k with p prime and* $d > 1$.

(1) $\rho_d(\mathcal{A}) \in \exp \mathscr{R}^1_1(\mathcal{A}, \mathbb{C})$, *in particular* $e_d(\mathcal{A}) > 0$.
(2) $\beta_p(\mathcal{A}) \neq 0$.

*Proof.* Part (2). Since the weighted partition is $k$-reduced and $p|k$, $\phi \otimes \Bbbk$ is surjective, by construction, for $\Bbbk = \mathbb{F}_p$ and $\mathbb{C}$. It follows from Proposition 4.2 that $\mathrm{im}(\phi \otimes \Bbbk)^* \subseteq H^1(M_{\mathcal{A}}, \Bbbk)$ is a $(k-1)$-dimensional subspace, isotropic with respect to the cup product. The linear map sending each $c_\alpha$ to $1 \in \mathbb{F}_p$ defines an element of $H^1(S, \mathbb{F}_p)$, denoted $\sigma_p(S)$. Clearly, $\phi^*(\sigma_p(S)) = \sigma_p(\mathcal{A})$, since $\mathcal{N}$ is in particular $p$-reduced. By definition (3), $\beta_p(\mathcal{A}) \neq 0$ as claimed, since $k \geqslant 3$.
Part (1). Note that $\phi : H_{\mathcal{A}} \to H_S$ induces homomorphisms $\phi^* : \mathbb{T}(S) \to \mathbb{T}(M_{\mathcal{A}})$ and $\phi^* : H^1(S, \mathbb{C}) \hookrightarrow H^1(M_{\mathcal{A}}, \mathbb{C})$, compatible with the surjective exponential maps of $S$ and $M_{\mathcal{A}}$. The map sending each $c_\alpha$ to $\exp(\frac{2\pi\sqrt{-1}}{k})$ defines an element of the character torus, $\rho_k(S) \in \mathbb{T}(S)$. Plainly, $\phi^*(\rho_k(S)) = \rho_k(\mathcal{A})$, since $\mathcal{N}$ is $k$-reduced. We also have $(\rho_k)^{k/d} = \rho_d$, for both $S$ and $\mathcal{A}$, since we assumed that $d$ divides $k$. Hence, $\phi^*(\rho_d(S)) = \rho_d(\mathcal{A}) \in \exp\ \phi^*(H^1(S, \mathbb{C})) \subseteq \exp\ \mathscr{R}^1_1(\mathcal{A}, \mathbb{C})$, where the last inclusion follows from the argument in Part (2). Indeed, we know that $\phi^*(H^1(S, \mathbb{C}))$ is an isotropic subspace in $H^1(M_{\mathcal{A}}, \mathbb{C})$, of dimension at least 2, and we may simply use the definition of $\mathscr{R}^1_1$. The conclusion $e_d(\mathcal{A}) > 0$ is a direct consequence of equality (2), since it is well-known that $\exp\ \mathscr{R}^1_1(\mathcal{A}, \mathbb{C}) \subseteq \mathscr{V}^1_1(M_{\mathcal{A}})$; see e.g. [5, Theorem D] for a more general result. □

4.3. Reduced weighted partitions on complex reflection arrangements. We begin the proof of Theorem 1.3. Let $\mathcal{A}$ be a complex reflection arrangement and assume $e_d(\mathcal{A}) > 0$, with $2 \leqslant d \leqslant 4$. We will show that $\rho_d(\mathcal{A}) \in \exp\ \mathscr{R}^1_1(\mathcal{A}, \mathbb{C})$ with the aid of Theorem 4.3(1), which requires the existence of a certain weighted partition on $\mathcal{A}$.
An easy preliminary remark is that the question from [4] always has a positive answer, for any arrangement $\mathcal{A}$ of rank at most 2. To see this, note first that the assumption $e_d(\mathcal{A}) \neq 0$ is equivalent to $\rho_d \in \mathscr{V}^1_1(M_{\mathcal{A}})$, by equality (2). When $\mathrm{rank}(\mathcal{A}) \leqslant 2$, it is known that $\mathscr{V}^1_1(M_{\mathcal{A}}) = \exp\ \mathscr{R}^1_1(\mathcal{A}, \mathbb{C})$, so the conclusion follows trivially. Consequently, we may also suppose that the rank is at least 3. On the other hand, $e_d(\mathcal{A}) > 0$ and $d = p^s$ together imply, via the modular bound (4), that $\beta_p(\mathcal{A}) > 0$. Therefore, $\mathcal{A}$ must be either the Hessian arrangement, or one of the arrangements from Theorem 1.2(3). To apply Theorem 4.3(1), we need to describe suitable weighted partitions on these arrangements, in each case.
The Hessian arrangement supports a reduced 4-multinet (actually, a 4-net). The monomial arrangement $\mathcal{A}(m, m, 3)$ has a reduced 3-multinet (in fact, a 3-net), as noted in [10].
A (non-reduced) 3-multinet on the full monomial arrangement $\mathcal{A}(m, 1, 3)$ was constructed in [10]. It is immediate to check that the weighted partition associated to this multinet is 3-reduced, when $m \equiv 1 (\mathrm{mod}\ 3)$.

The last case is $\mathcal{A} = \mathcal{A}(m,m,4)$, with hyperplanes $(H_{ij^\mu})$ $z_i - \omega^\mu z_j = 0$, where $1 \leqslant i < j \leqslant 4$, $\mu \in \mathbb{Z}/m\mathbb{Z}$ and $\omega = \exp(\frac{2\pi\sqrt{-1}}{m})$. We define a partition $\Pi$ with three blocks, $\{H_{ij^\mu}, H_{kh^\nu} \mid \mu, \nu \in \mathbb{Z}/m\mathbb{Z}\}$, and set $\mathfrak{m} \equiv 1$ on $\mathcal{A}$. It is straightforward to verify axiom (8) by using the description of 2-flats given in §2.3. (Actually, this is a 3-net on $\mathcal{L}_{\leqslant 2}(\mathcal{A})$.)

4.4. Proof of Theorem 1.3 completed. In case $\mathcal{A} = \mathcal{A}(G_{25})$, $d$ must be 2 or 4. We may take $k = 4$ in Theorem 4.3 to obtain the desired conclusion. The remaining cases, described in Theorem 1.2(3), lead to $d = 3$. Taking $k = 3$, we conclude as before. $\square$

4.5. Proof of Theorem 1.1. We may suppose that $\operatorname{rank}(\mathcal{A}) \geqslant 3$, since otherwise the conclusion is known (see [14, p. 773]). By the modular bound (4) and Theorem 1.2, $e_p(\mathcal{A}) = \beta_p(\mathcal{A})$, when $\mathcal{A}$ is not $\mathcal{A}(G_{25})$ or one of the arrangements listed in Theorem 1.2(3). Moreover, we have to verify the conclusion only for $p = 2$ (in case $\mathcal{A}(G_{25})$) or for $p = 3$ (in the remaining cases).
The equality $e_2(G_{25}) = \beta_2(G_{25}) = 2$ is well-known (see e.g. [20]). When $\mathcal{A}$ is not $\mathcal{A}(m,m,3)$ with $m \equiv 0 \pmod 3$, we know that $\beta_3(\mathcal{A}) = 1$. In these cases, we may use the 3-reduced weighted 3-partitions from §4.3, for $d = k = 3$, exactly as in §4.4, to obtain that $e_3(\mathcal{A}) > 0$. Now we are done, since the modular bound (4) implies that $e_3(\mathcal{A}) \leqslant 1$.
The last case, $\mathcal{A} = \mathcal{A}(m,m,3)$ with $m = 3n$ and $p = 3$, when $\beta_3(\mathcal{A}) = 2$, requires a more careful treatment.
The (relabeled) hyperplanes of $\mathcal{A}$ are $\{H_{12^\alpha},\ H_{23^\beta},\ H_{13^{-\gamma}} \mid \alpha, \beta, \gamma \in \mathbb{Z}/m\mathbb{Z}\}$. We recall from §2.3 the two types of 2-flats: $\{H_{ij^\alpha} \mid \alpha \in \mathbb{Z}/m\mathbb{Z}\}$ and $\{H_{12^\alpha},\ H_{23^\beta},\ H_{13^{-\gamma}}\}$ with $\alpha + \beta + \gamma = 0$.
We have to show that $e_3(\mathcal{A}) \geqslant 2$, in order to finish the proof. To this end, we need two reduced weighted 3-partitions on $\mathcal{A}$. The first one, $\mathscr{N}$, is constructed in [10]. The blocks of the partition $\Pi$ are given by $\{H_{ij^\alpha} \mid \alpha \in \mathbb{Z}/m\mathbb{Z}\}_{1 \leqslant i < j \leqslant 3}$. The blocks of the second one, $\mathscr{N}'$, are defined by $\{H_{ij^\alpha} \mid 1 \leqslant i < j \leqslant 3, \alpha \equiv \tau (\mathrm{mod}\ 3)\}_{\tau \in \mathbb{F}_3}$, with $\alpha$ replaced by $-\alpha$ when $\{i, j\} = \{1, 3\}$.
Let us check axiom (8) for $\mathscr{N}'$. The 2-flats $X$ appearing in axiom (8) clearly coincide with those $\mathcal{A}_X$ that contain two hyperplanes with different colours with respect to $\Pi'$. For $X = \{H_{ij^\alpha} \mid \alpha \in \mathbb{Z}/m\mathbb{Z}\}$ we find that $n_X = n$. For $X = \{H_{12^\alpha},\ H_{23^\beta},\ H_{13^{-\gamma}}\}$ with $\alpha + \beta + \gamma = 0$, the condition on colours translates to $\alpha \not\equiv \beta \not\equiv \gamma (\mathrm{mod}\ 3)$, and implies that $n_X = 1$. Hence $\Pi'$ defines a reduced weighted 3-partition $\mathscr{N}'$.
Consider the two (surjective) homomorphisms from Proposition 4.2, $\phi, \phi' : H_{\mathcal{A}} \to H_S$. Clearly, $\phi^*\mathbb{T}(S) = \exp\ \phi^* H^1(S, \mathbb{C})$ is a positive dimensional subtorus of $\mathbb{T}(M_{\mathcal{A}})$, and similarly for $\phi'$. Moreover, $\phi^*\mathbb{T}(S) \subseteq \mathscr{V}_1^1(M_{\mathcal{A}})$, since $\phi^* H^1(S, \mathbb{C}) \subseteq H^1(M_{\mathcal{A}}, \mathbb{C})$ is isotropic of dimension 2, hence contained in $\mathscr{R}_1^1(\mathcal{A}, \mathbb{C})$, and likewise for $\phi'$. Therefore, we may find two irreducible components of $\mathscr{V}_1^1(M_{\mathcal{A}})$, $W$ and $W'$, such that $\phi^*\mathbb{T}(S) \subseteq W$ and $\phi'^*\mathbb{T}(S) \subseteq W'$. On the other hand, $\rho_3(\mathcal{A}) \in W \cap W'$, by the argument from the proof of Theorem 4.3(1).

In this situation, it follows from a result of Artal Bartolo, Cogolludo and Matei [1, Proposition 6.9] that $\rho_3(\mathcal{A}) \in \mathcal{V}_2^1(M_{\mathcal{A}})$, if $W \neq W'$. Hence, $e_3(\mathcal{A}) \geqslant 2$, by equality (2), and we are done.
Suppose then that $W = W'$. We know from [10] that actually $W = \phi^* \mathbb{T}(S)$, since $\phi$ comes from a 3-net. Taking tangent spaces at the origin $1 \in \mathbb{T}(M_{\mathcal{A}})$, we infer that $\phi'^* H^1(S, \mathbb{C}) \subseteq \phi^* H^1(S, \mathbb{C})$.
We identify $H^1(M_{\mathcal{A}}, \mathbb{C})$ with $\mathbb{C}^{\mathcal{A}}$, using the distinguished $\mathbb{Z}$-basis. In this way, the subspace $\phi^* H^1(S, \mathbb{C})$ (respectively $\phi'^* H^1(S, \mathbb{C})$) is identified with the subset of those elements $\eta \in \mathbb{C}^{\mathcal{A}}$ (respectively $\eta' \in \mathbb{C}^{\mathcal{A}}$) taking the constant values $a, b, c$ (respectively $a', b', c'$) on the blocks of $\Pi$ (respectively $\Pi'$), where $a+b+c = a'+b'+c' = 0$. Now, it is an easy matter to check that $\phi^* H^1(S, \mathbb{C}) \cap \phi'^* H^1(S, \mathbb{C}) = 0$. This contradiction finishes the proof of Theorem 1.1. □

4.6. Full monodromy action. It follows from decomposition (1) that the characteristic polynomial $\Delta_{\mathcal{A}}(t) = (t-1)^{|\mathcal{A}|-1} \Pi_{1<d|n} \Phi_d(t)^{e_d(\mathcal{A})}$ encodes the full monodromy action on $H_1(F_{\mathcal{A}}, \mathbb{Q})$.
The approach via modular bounds works only for prime power monodromy multiplicities, $e_{p^s}(\mathcal{A})$. One way to avoid this inconvenience is to impose restrictions on multiplicities of 2-flats, like in [20] for instance, to arrive at full monodromy computations. Unfortunately, as we saw in §2.3, arbitrarily high flat multiplicities may appear for non-exceptional complex reflection arrangements.
Even in this kind of situation, there is hope related to the following well-known vanishing criterion (see e.g. [14]): if $d \nmid |\mathcal{A}_X|$ for any $X \in \mathcal{L}_2(\mathcal{A})$, then $e_d(\mathcal{A}) = 0$. It turns out that this works for full monomial arrangements of small rank. The result below verifies in particular the strong form of the conjecture from [20].

Proposition 4.4. *For $\mathcal{A} = \mathcal{A}(m, 1, l)$, with $l = 3$ or $4$, $\Delta_{\mathcal{A}}(t) = (t-1)^{|\mathcal{A}|-1}(t^2 + t + 1)$, if $l = 3$ and $m \equiv 1 (mod\ 3)$, and $\Delta_{\mathcal{A}}(t) = (t-1)^{|\mathcal{A}|-1}$, otherwise.*

*Proof.* We have to compute $e_d(\mathcal{A})$ for all divisors $d > 1$ of $|\mathcal{A}|$. If $d$ is prime, this was done in Theorem 1.1 and Theorem 1.2. It follows from §2.3 that the 2-flats of $\mathcal{A}$ have multiplicities $2, 3$ or $m+2$, $|\mathcal{A}| = 3(m+1)$ for $l = 3$ and $|\mathcal{A}| = 2(3m+2)$ for $l = 4$. If $d$ is not prime and $e_d(\mathcal{A}) \neq 0$, the vanishing criterion forces $m \equiv -2 \pmod{d}$. Writing that $|\mathcal{A}| \equiv 0 \pmod{d}$, we obtain for $l = 3$ that $3 \equiv 0 \pmod{d}$, a contradiction, and $8 \equiv 0 \pmod{d}$, for $l = 4$. In the second case, the modular bound implies that $\beta_2(\mathcal{A}) > 0$, contradicting Theorem 1.2(2). □

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# Algebraic Subellipticity and Dominability of Blow-Ups of Affine Spaces

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Abstract. Little is known about the behaviour of the Oka property of a complex manifold with respect to blowing up a submanifold. A manifold is of Class $\mathscr{A}$ if it is the complement of an algebraic subvariety of codimension at least 2 in an algebraic manifold that is Zariski-locally isomorphic to $\mathbb{C}^n$. A manifold of Class $\mathscr{A}$ is algebraically subelliptic and hence Oka, and a manifold of Class $\mathscr{A}$ blown up at finitely many points is of Class $\mathscr{A}$. Our main result is that a manifold of Class $\mathscr{A}$ blown up along an arbitrary algebraic submanifold (not necessarily connected) is algebraically subelliptic. For algebraic manifolds in general, we prove that strong algebraic dominability, a weakening of algebraic subellipticity, is preserved by an arbitrary blow-up with a smooth centre. We use the main result to confirm a prediction of Forster's famous conjecture that every open Riemann surface may be properly holomorphically embedded into $\mathbb{C}^2$.

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## 1. Introduction and Results

Modern Oka theory has evolved from Gromov's seminal work on the Oka principle [10]. (The monograph [6] is a comprehensive reference on Oka theory. See also the surveys [7] and [8].) Oka theory may be viewed as the study of approximation and interpolation problems for holomorphic maps from Stein spaces into suitable complex manifolds. The goal, for suitable targets, is to show that such a problem can be solved as soon as there is no topological obstruction to its solution. The suitable targets turn out to be the so-called Oka manifolds. From another point of view, Oka theory is the study of complex manifolds that are the targets of *many* holomorphic maps from Stein spaces, with the word *many* interpreted homotopically. The fundamental result in this direction is that every continuous map from a Stein space to an Oka manifold can be deformed to a holomorphic map. From a third point of view, Oka theory is seen as an answer to the question: What is a good definition of *anti-hyperbolicity* for complex manifolds?

The prototypical examples of Oka manifolds are complex Lie groups and their homogeneous spaces. Among other known examples are manifolds of the so-called Class $\mathscr{A}$. A manifold is of Class $\mathscr{A}$ if it is the complement of an algebraic subvariety of codimension at least 2 in an algebraic manifold[2] that is Zariski-locally isomorphic to $\mathbb{C}^n$. (A similar class was introduced in [10, §3.5.D].) The subclass $\mathscr{A}_0$ of algebraic manifolds Zariski-locally isomorphic to $\mathbb{C}^n$ contains, for example, $\mathbb{C}^n$ itself, complex projective spaces, all Grassmannians, all compact rational surfaces, all smooth complete toric varieties, and any vector bundle over a manifold in $\mathscr{A}_0$. (Our definitions of $\mathscr{A}_0$ and $\mathscr{A}$ are more general than [6, Definition 6.4.5]; see Remark 3.) For more examples of manifolds of class $\mathscr{A}_0$, see [1, Section 4] (where the term *A-covered* is used).

A challenging open question in basic Oka theory is whether the Oka property for, say, projective manifolds is a birational invariant. In other words, how can you say what it means for a complex manifold to be bimeromorphically equivalent to an Oka manifold $Y$ without mentioning $Y$? We do not know. Our understanding of the interaction of the Oka property with the operation of blowing up a submanifold, even just a point, is still very limited. The following result is due to Gromov ([10, §3.5.D"]; see also [6, Proposition 6.4.7] and [1, Section 4, Statement (9)]).

THEOREM (Gromov). *A manifold of Class $\mathscr{A}$ blown up at finitely many points is of Class $\mathscr{A}$ and hence Oka.*

[2]An algebraic manifold is a smooth algebraic variety over $\mathbb{C}$, by definition quasi-compact in the Zariski topology. We take a subvariety to be closed and not necessarily irreducible.

Forstnerič proved that $\mathbb{C}^n$ blown up at each point of a tame discrete set is Oka [6, Proposition 6.4.11]. It follows that a complex torus of dimension at least 2, blown up at finitely many points, is Oka [6, Corollary 6.4.12]. We are not aware of any other previous results about blow-ups of Oka manifolds being Oka.

Our main result is a strengthening of Gromov's theorem.

Main Theorem. *A manifold of Class $\mathscr{A}$ blown up along any algebraic submanifold (not necessarily connected) is Oka.*

We do not tackle the Oka property directly, but instead verify a geometric sufficient condition for it to hold, called algebraic subellipticity. (This is how manifolds of Class $\mathscr{A}$ are shown to be Oka.) An algebraic manifold is algebraically subelliptic if it has a finite dominating family of algebraic sprays [6, Definition 5.5.11]. Algebraic subellipticity is a very interesting property for the following reasons.

- It is (obviously) a purely algebraic property, but ...
- ... it has massive analytic consequences (namely the Oka property).
- It satisfies a localisation principle (due to Gromov [10, §3.5.B]; see also [6, Proposition 6.4.2]), which sometimes offers the only way to the Oka property, for example here and in [11, Proposition 4.10]. There is no known holomorphic analogue of this principle.
- It implies several algebraic Oka-type properties [6, Sections 7.8 and 7.10]. For example, if $X$ is an affine algebraic variety and $Y$ is an algebraically subelliptic manifold, then a holomorphic map $X \to Y$ is approximable by regular maps, uniformly on compact subsets of $X$, if and only if it is homotopic to a regular map.

The bulk of this paper is devoted to the proof of the following result.

Theorem 1. *Let $S$ be an algebraic subvariety of $\mathbb{C}^n$, $n \geq 2$, of codimension at least 2. The blow-up of $\mathbb{C}^n \setminus S$ along an algebraic submanifold is algebraically subelliptic.*

By localisation of algebraic subellipticity, the following corollary is immediate, and implies our main theorem.

Corollary 2. *The blow-up of a manifold of class $\mathscr{A}$ along an algebraic submanifold is algebraically subelliptic.*

Remark 3. In Forstnerič's monograph, the localisation principle for algebraic subellipticity is proved under the assumption that the algebraic manifold $Y$ in question is quasi-projective [6, Proposition 6.4.2]. This assumption is only used to ensure that for every point $y \in Y$ and every algebraic subvariety $Z$ of $Y$ with $y \notin Z$, there is an algebraic hypersurface $H$ in $Y$ with $Z \subset H$ but

$y \notin H$. By [4, Theorem 4.1], every algebraic manifold has this property, so the quasi-projectivity assumption is not needed.

Next we present two corollaries of the fact that $\mathbb{C}^n$ blown up along an algebraic submanifold is Oka.

The first result confirms a prediction of the conjecture that every open Riemann surface may be properly holomorphically embedded into $\mathbb{C}^2$. This is the remaining unresolved case of Forster's famous conjecture [5, p. 183]. Let $A$ be an open Riemann surface embedded in $\mathbb{C}^n$ (such an embedding exists for every $n \geq 3$). If there is an embedding $f : A \to \mathbb{C}^2$, then $f$ extends to a holomorphic map $F : \mathbb{C}^n \to \mathbb{C}^2$, and $F^{-1}(f(A))$ either is, or (if $F^{-1}(f(A)) = \mathbb{C}^n$) contains, a hypersurface in $\mathbb{C}^n$ containing $A$ that retracts holomorphically onto $A$. When $A$ is algebraic, Corollary 4 below confirms that $A$ is indeed a hypersurface retract.

By [9, proof of Proposition 12 and Remark 13], if $A$ is a connected analytic submanifold of $\mathbb{C}^n$, every holomorphic vector bundle over $A$ is holomorphically trivial, the blow-up $B$ of $\mathbb{C}^n$ along $A$ is Oka, and every continuous map $A \to B$ is null-homotopic, then $A$ is a holomorphic retract of a smooth analytic hypersurface in $\mathbb{C}^n$. This result, Theorem 1, and the observation that $B$ is simply connected yield the following corollary.

COROLLARY 4. *Let $A$ be a connected algebraic submanifold of $\mathbb{C}^n$. If $A$ is a curve or $A$ is contractible, then $A$ is a holomorphic retract of a smooth analytic hypersurface in $\mathbb{C}^n$.*

As far as we know, there are contractible affine algebraic manifolds $A$ that are not known to be a hypersurface, for example Ramanujam's surface $R$ and products such as $R \times R$ and $R \times \mathbb{C}^k$. For such $A$, the corollary is nontrivial.

One of the dozen or more nontrivially equivalent formulations of the Oka property says that a complex manifold $Y$ is Oka if for every Stein manifold $X$ with a subvariety $S$, a holomorphic map $S \to Y$ has a holomorphic extension $X \to Y$ if it has a continuous extension. The second result follows from Theorem 1 and the universal property of the blow-up; the details are given in Section 3.

COROLLARY 5. *Let $A$ be an algebraic submanifold of $\mathbb{C}^n$, $n \geq 2$, $A \neq \mathbb{C}^n$, and let $T$ be a discrete subset of $\mathbb{C}^m$, $m \geq 1$, or a smooth analytic curve in $\mathbb{C}^m$, $m \geq 2$. Let $f : T \to \mathbb{C}^n$ be holomorphic (an arbitrary map if $T$ is discrete). Then $f$ extends to a holomorphic map $F : \mathbb{C}^m \to \mathbb{C}^n$ such that $F^{-1}(A)$ is a hypersurface.*

We interpret the corollary to mean that there are *many* holomorphic maps $\mathbb{C}^m \to \mathbb{C}^n$ that pull $A$ back to a hypersurface.

We now turn to a weaker, simpler property for which we can obtain stronger results. An algebraic manifold $X$ is said to be algebraically dominable at a point $x$ in $X$ if there is a regular map $f : \mathbb{C}^n \to X$ such that $f(0) = x$ and $f$ is a local isomorphism at 0. We say that $X$ is algebraically dominable if it is algebraically dominable at some point, and that $X$ is strongly algebraically dominable if it is dominable at every point.

We use the technology of composed sprays and the Quillen-Suslin theorem to prove the following result.

Proposition 6. *An algebraically subelliptic manifold is strongly algebraically dominable.*

The next corollary is then immediate.

Corollary 7. *The blow-up of a manifold of class $\mathscr{A}$ along an algebraic submanifold is strongly algebraically dominable.*

Note that if a projective manifold is algebraically dominable, then it is unirational and hence rationally connected. We do not know any examples of algebraic manifolds that are dominable but not algebraically subelliptic, but it seems unlikely that the two properties are equivalent. Strong dominability is not known to imply the Oka property.

Using Theorem 1 and Proposition 6, we establish the following result.

Proposition 8. *The blow-up of $\mathbb{C}^n$, $n \geq 2$, along a closed subscheme $A$ is algebraically dominable at every point over the complement of the singular locus of $A$.*

A closed subscheme of $\mathbb{C}^n$ is nothing but an ideal in the coordinate ring $\mathbb{C}[x_1, \ldots, x_n]$.

Finally, we are able to show that algebraic dominability is preserved by an arbitrary blow-up with a smooth centre. The analogous result for algebraic subellipticity is beyond our reach for now.

Theorem 9. *Let $B$ be the blow-up of an algebraic manifold $X$ along an algebraic submanifold. If $X$ is algebraically dominable at a point $x$, then $B$ is algebraically dominable at every point over $x$. Hence, if $X$ is algebraically dominable, so is $B$, and if $X$ is strongly algebraically dominable, so is $B$.*

Let us mention the related result that if $X$ is uniformly rational (meaning that $X$ is covered by open sets isomorphic to open subsets of affine space), then so is $B$ ([10, §3.5.E], [3, Proposition 2.6]).

In the next section we prove Theorem 1. In the final section we prove Corollary 5, Proposition 6, Proposition 8, and Theorem 9.

## 2. Proof of Theorem 1

2.1. This section is devoted to the proof of our main result, Theorem 1. We start by proving the theorem in case $S = \varnothing$. Let $B$ be the blow-up of $\mathbb{C}^n$, $n \geq 2$, along an algebraic submanifold $A$ of $\mathbb{C}^n$ (not necessarily connected) with exceptional divisor $E \subset B$. Write $\pi$ for the projection $B \to \mathbb{C}^n$. Without loss of generality we may assume that each component of $A$ has codimension at least 2. We will show that $B$ is algebraically subelliptic. By Gromov's localisation principle, it suffices to show that $B$ can be covered by Zariski-open sets $U$ carrying regular sprays $\mathbb{C}^s \times U \to B$ that together dominate at each point $b$ of $B$. Now $B \setminus E$ is isomorphic to $\mathbb{C}^n \setminus A$, which, as shown by Gromov ([10, §0.5.B(iii)], [6, Proposition 5.5.14]), is algebraically elliptic (with some high value of $s$). Thus we take $b \in E$. The sprays constructed below all have $s = 1$.

Let $a = \pi(b) \in A$. We may take $a$ to be the origin in $\mathbb{C}^n$. Viewing $E$ as the projectivised normal bundle of $A$, we can represent $b$ by a vector $v \in T_a\mathbb{C}^n \setminus T_aA$. The kernel of the tangent map $d_b\pi : T_bB \to T_a\mathbb{C}^n$ is the subspace $T_b\pi^{-1}(a)$ of dimension $\operatorname{codim}_a A - 1$. The image of $d_b\pi$ is $\mathbb{C}v \oplus T_aA$. We first construct sprays that span the kernel. Then we give a different construction of sprays that span some vector (that we have not tried to pin down) over a generic vector in the image. This suffices to prove the theorem.

Let $r = \operatorname{codim}_a A \geq 2$. After a linear change of coordinates, $T_aA \subset T_a\mathbb{C}^n \cong \mathbb{C}^n$ is given by the equations $x_1, \ldots, x_r = 0$. Then, in a Zariski neighbourhood $U$ of $a$ in $\mathbb{C}^n$, $A$ is the common zero locus of polynomials $u_1, \ldots, u_r$ with $u_j(x) = x_j +$ higher order terms. We can take $\mathbb{C}^n \setminus U$ to consist of the components of $A$ other than the component $A_0$ containing $a$ (call their union $A_1$) and of the common zeros of $u_1, \ldots, u_r$ other than $A_0$. By removing from $U$ a subvariety of $A_0$ not containing $a$, we may assume that $d_xu_1, \ldots, d_xu_r$ are linearly independent for all $x \in A \cap U$. We view $\pi^{-1}(U) \subset B$ as the closure in $U \times \mathbb{P}^{r-1}$ of the set

$$\{(x, \lambda) \in (U \setminus A) \times \mathbb{P}^{r-1} : \lambda = [u_1(x), \ldots, u_r(x)]\}.$$

In other words, $\pi^{-1}(U)$ is the graph of the rational map $[u_1, \ldots, u_r] : U \to \mathbb{P}^{r-1}$. The map $\pi$ is the projection onto the first factor. Note that $\pi^{-1}(U)$ is covered by $r$ affine Zariski-open sets of the same form, one of which is

$$Y = \{(x, \lambda) \in U \times \mathbb{C}^{r-1} : u_j(x) = \lambda_j u_r(x), j = 1, \ldots, r-1\}.$$

Note also that $u_r \circ \pi$ is a defining function for $E \cap Y$ as a submanifold of $Y$. We may assume that $b \in Y$. Let $\tilde{B}$ be the graph of the rational map $[u_1, \ldots, u_r] : \mathbb{C}^n \to \mathbb{P}^{r-1}$ and $\tilde{\pi} : \tilde{B} \to \mathbb{C}^n$ be the projection. The projection $\tilde{\pi}^{-1}(\mathbb{C}^n \setminus A_1) \to \pi^{-1}(\mathbb{C}^n \setminus A_1)$ is an isomorphism over $U$.

2.2. To produce the first type of spray, we make use of the complete regular flows on $\mathbb{C}^n$ fixing $A$ pointwise, and therefore restricting to complete flows on $\mathbb{C}^n \setminus A$, that appear in Gromov's proof that $\mathbb{C}^n \setminus A$ is algebraically elliptic. Define

$$\phi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n, \quad \phi(t, x) = x + th(\tau(x))\zeta,$$

where $\tau : \mathbb{C}^n \to \mathbb{C}^{n-1}$ is a surjective linear projection such that $\tau|A$ is proper, $\zeta \neq 0$ is in the kernel of $\tau$, and $h : \mathbb{C}^{n-1} \to \mathbb{C}$ is a polynomial which vanishes on the subvariety $\tau(A)$. For a generic choice of $h$, $\tau$, $\zeta$, and $\xi \in T_bB$, we have:

- $\eta = d_b\pi(\xi) \notin T_aA$.
- $\zeta \notin \mathbb{C}\eta + T_aA$.
- $d_bu_r(\eta) \neq 0$.
- $(d_{\tau(a)}h \circ d_a\tau)(\eta) \neq 0$.

Extend $\xi$ to a vector field (with the same name) on a small enough neighbourhood of $b$ in $E \cap Y$ that the above properties hold with $b$ replaced by a nearby $y \in E \cap Y$ and $a$ replaced by $\pi(y)$.

Define a regular map $f : \mathbb{C} \times Y \to \mathbb{C}^n \setminus A_1$ by the formula

$$f(t, y) = \phi(t, \pi(y)) = th(\tau(x))\zeta + x.$$

If $y = (x, \lambda) \in E \cap Y$, then $u_j(f(t, y)) = u_j(x) = 0$, so there are regular functions $\lambda_1, \ldots, \lambda_r$ on $\mathbb{C} \times Y$ such that $u_j(f(t, y)) = u_r(x)\lambda_j(t, y)$ for $j = 1, \ldots, r$ and $(t, y) \in \mathbb{C} \times Y$. The map $f$ lifts to a rational map $F : \mathbb{C} \times Y \to \tilde{\pi}^{-1}(\mathbb{C}^n \setminus A_1) \subset \tilde{B}$ with

$$F(t, y) = (f(t, y), [\lambda_1(t, y), \ldots, \lambda_r(t, y)]).$$

We claim that $F$ is regular on $\mathbb{C} \times V$ for some Zariski neighbourhood $V \subset Y$ of $b$.

First, it is clear that $F$ is regular on $\mathbb{C} \times (Y \setminus E)$. Next, for $F$ to be regular on $\mathbb{C} \times \{b\}$, we require $(\lambda_1(t, b), \ldots, \lambda_r(t, b)) \neq (0, \ldots, 0)$ for all $t \in \mathbb{C}$. Differentiating the identity $u_j(f(t, y)) = u_r(x)\lambda_j(t, y)$ with respect to $y$ at $(t, b)$ and evaluating the tangent maps at $\xi$ gives

$$(1) \qquad d_au_j\big(t(d_{\tau(a)}h \circ d_a\tau \circ d_b\pi)(\xi)\zeta + d_b\pi(\xi)\big) = \lambda_j(t, b)d_b(u_r \circ \pi)(\xi).$$

The common kernel of $d_au_1, \ldots, d_au_r$ is $T_aA$, so our requirement is met if

$$t(d_{\tau(a)}h \circ d_a\tau \circ d_b\pi)(\xi)\zeta + \eta \notin T_aA$$

for all $t \in \mathbb{C}$. This holds since $\zeta \notin \mathbb{C}\eta + T_aA$ and $\eta \notin T_aA$. Finally, we show that $F$ is regular on $\mathbb{C} \times \{y\}$ for $y \in E \cap Y$ sufficiently close to $b$. Otherwise, there is a sequence $((t_\nu, y_\nu))$ with $y_\nu \in E \cap Y$, $y_\nu \to b$, and $\lambda_j(t_\nu, y_\nu) = 0$ for $j = 1, \ldots, r$. We may assume that $t_\nu \to \infty$, for otherwise the inequality $(\lambda_1(t, b), \ldots, \lambda_r(t, b)) \neq (0, \ldots, 0)$ for all $t \in \mathbb{C}$ is contradicted. Now (1) holds with $b$ replaced by $y_\nu$ and $a$ by $\pi(y_\nu) \in A$, and $t = t_\nu$. Letting $\nu \to \infty$, we conclude that $d_a u_j\big((d_{\tau(a)}h \circ d_a\tau)(\eta)\zeta\big) = 0$ for $j = 1, \ldots, r$, that is, $(d_{\tau(a)}h \circ d_a\tau)(\eta)\zeta \in T_aA$, which is ruled out by the generic choices made above.

Thus, postcomposing $F$ with the projection onto $\pi^{-1}(\mathbb{C}^n \setminus A_1)$, which is an isomorphism over $U$, yields a regular spray $G$ on $V \subset \pi^{-1}(U)$ with values in $\pi^{-1}(\mathbb{C}^n \setminus A_1) \subset B$. Now $\dfrac{\partial f}{\partial t}(0, b) = 0$, so $\dfrac{\partial G}{\partial t}(0, b)$ must lie in $\operatorname{Ker} d_b\pi = T_b\pi^{-1}(a)$. Differentiating (1) with respect to $t$ at $(0, b)$ gives

$$\frac{\partial \lambda_j}{\partial t}(0, b) d_a u_r(\eta) = (d_{\tau(a)}h \circ d_a\tau)(\eta) d_a u_j(\zeta).$$

By the choice of $u_1, \ldots, u_r$, $d_a u_j(\zeta) = \zeta_j$. Hence the derivative at 0 of the lifting $\mathbb{C} \to \mathbb{C}^r \setminus \{0\}$, $t \mapsto (\lambda_1(t, b), \ldots, \lambda_r(t, b))$, is

$$\frac{(d_{\tau(a)}h \circ d_a\tau)(\eta)}{d_a u_r(\eta)}(\zeta_1, \ldots, \zeta_r).$$

This shows that we can produce $r - 1$ sprays that span all of $T_b\pi^{-1}(a)$.

2.3. We now turn to a different construction of sprays that span some vector over a generic vector in the image $\mathbb{C}v \oplus T_aA$ of $d_b\pi$.

It is well known that every algebraic subvariety of $\mathbb{C}^n$ is a rational hypersurface retract. Here, we restrict a linear projection $L : \mathbb{C}^n \to \mathbb{C}^{n-r+1}$ to $A_0$ and let $W = L^{-1}(L(A_0))$. (Recall that $r = \operatorname{codim}_a A$.) For generic $L$, the regular map $A_0 \to L(A_0)$ is biregular at $a$, the hypersurface $W$ in $\mathbb{C}^n$ is smooth at $a$, and we have a rational retraction $W \to L(A_0) \to A_0$. Thus, possibly after shrinking $U$, there is a hypersurface $W$ in $\mathbb{C}^n$ containing $A_0$ and smooth at $a$, with a regular retraction $\rho : W \cap U \to A \cap U$. We may assume that any one of the polynomials $u_1, \ldots, u_r$, say $u_r$, is a defining function for $W$. Let $V$ be the hypersurface $(W \cap U) \times \mathbb{C}^{r-1}$ in $U \times \mathbb{C}^{r-1}$.

Now $V$ is defined by the equation $u_r = 0$ and $Y$ is defined by the equations $u_j = \lambda_j u_r$, $j = 1, \ldots, r - 1$. Thus $V \cap Y = E \cap Y$. Since $d_x u_1, \ldots, d_x u_r$ are linearly independent for all $x \in A \cap U$, we see that $V$ and $Y$ intersect transversely over $A \cap U$.

It is well known that the Zariski topology of a smooth algebraic variety has a basis consisting of open sets that are isomorphic to closed affine hypersurfaces ([2, Theorem 5.7], [13, Theorem 2.5]). We need a variant of this fact.

CLAIM. There is a Zariski neighbourhood $Z$ of $b$ in $U \times \mathbb{C}^{r-1}$ and a regular embedding $\gamma$ of $(V \cup Y) \cap Z$ as a closed subvariety of $\mathbb{C}^m$, $m = n + r - 1$.

We take the claim for granted for now and prove it in the next subsection. Write $V' = V \cap Z$ and $Y' = Y \cap Z$. Because $\gamma(V')$ and $\gamma(Y')$ intersect transversely, the well-defined map $\gamma(V' \cup Y') \to \mathbb{C}^n$ defined on $\gamma(V')$ as $\rho \circ \pi \circ \gamma^{-1}$, and on $\gamma(Y')$ as $\pi \circ \gamma^{-1}$, is regular. We extend this map to a regular map $\phi : \mathbb{C}^m \to \mathbb{C}^n$. Then $\gamma(E \cap Y') \subset \gamma(V') \subset \phi^{-1}(A)$.

Let $I(A)$ be the defining ideal of $A$. Next we show that $\phi^* I(A)$ is principal near $\gamma(b)$. Let $p$ be a defining polynomial for $\gamma(V')$. Then there are polynomials $q_1, \dots, q_r$ such that

$$u_j \circ \phi = p\, q_j, \qquad j = 1, \dots, r.$$

It suffices to show that $\gamma(E \cap Y') \cap \{q_1, \dots, q_r = 0\}$ is empty (so $\phi^{-1}(A) = \gamma(V')$ near $\gamma(E \cap Y')$). For this, it is enough to find a tangent vector $w \in T_{\gamma(b)}\mathbb{C}^m$ such that

$$q_j(\gamma(b)) d_{\gamma(b)} p(w) + p(\gamma(b)) d_{\gamma(b)} q_j(w) = d_{\gamma(b)}(u_j \circ \phi)(w) \neq 0$$

for some $j \in \{1, \dots, r\}$, since then $q_j(\gamma(b)) \neq 0$. Thus we need $d_{\gamma(b)}\phi(w) \notin T_a A$. Now $d_b \pi(T_b Y)$ is larger than $T_a A$, so there is $w \in T_{\gamma(b)}\gamma(Y')$ with $d_{\gamma(b)}(\pi \circ \gamma^{-1})(w) \notin T_a A$. Since $\phi = \pi \circ \gamma^{-1}$ on $\gamma(Y')$, we have $d_{\gamma(b)}\phi(w) = d_{\gamma(b)}(\pi \circ \gamma^{-1})(w)$.

Take $\zeta$ in $T_{\gamma(b)}\mathbb{C}^m$ (identified with $\mathbb{C}^m$ itself) and define a regular map

$$f : \mathbb{C} \times Y' \to \mathbb{C}^n, \quad f(t, y) = \phi(\gamma(y) + t\zeta),$$

with $f(0, \cdot) = \pi$ on $Y'$. Since $f^* I(A)$ is principal near $(0, b)$, the rational lifting $F : \mathbb{C} \times Y' \to B$ of $f$ is regular near $(0, b)$. In fact, for generic $\zeta \in \mathbb{C}^m$, $F$ is regular on the product of $\mathbb{C}$ and some Zariski neighbourhood of $b$ in $Y'$. Namely, let $Q$ be the subvariety of $\mathbb{C}^m$ where $\phi^* I(A)$ is not principal. We need the line $\gamma(b) + \mathbb{C}\zeta$ to avoid $Q$, also at infinity in $\mathbb{P}^m$. Since $\operatorname{codim} Q \geq 2$, this holds for generic $\zeta$.

Now $\dfrac{\partial f}{\partial t}(0, b) = d_{\gamma(b)}\phi(\zeta)$. Since $d_{\gamma(b)}\phi(T_{\gamma(b)}\gamma(V')) = T_a A$, we have

$$d_{\gamma(b)}\phi(T_{\gamma(b)}\mathbb{C}^m) = d_b \pi(T_b Y).$$

Hence we obtain local sprays $F$ such that $\dfrac{\partial F}{\partial t}(0, b)$ lies over a generic vector in $d_b \pi(T_b Y)$, as desired.

2.4. We conclude the proof of Theorem 1 in case $S = \varnothing$ by proving the claim. Our argument is based on Jelonek's proof of [13, Theorem 2.5].

Let $\overline{V} = W \times \mathbb{C}^{r-1}$ and $\overline{Y}$ be the closure of $V$ and $Y$ in $\mathbb{C}^m$, respectively. Then $R = (\overline{V} \cup \overline{Y}) \setminus U$ is a subvariety of codimension at least 2 in $\mathbb{C}^m$. Let $T$ be the union of $\overline{V}$ and a hypersurface containing $\overline{Y}$. Then $T$ is a hypersurface in $\mathbb{C}^m$ with $b \in T$. We will show that $b$ has a Zariski neighbourhood $Z$ in $\mathbb{C}^m$, disjoint from $R$, such that $T \cap Z$ embeds as a closed subvariety of $\mathbb{C}^m$.

After a generic change of coordinates of the form $x_j \mapsto x_j + a_j x_m$, $j = 1, \dots, m-1$, $x_m \mapsto x_m$, $T$ has a defining polynomial of the form

$$x_m^k + \sum_{j=0}^{k-1} a_j(x_1, \dots, x_{m-1}) x_m^j = 0.$$

Let $p : \mathbb{C}^m \to \mathbb{C}^{m-1}$ be the projection $(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_{m-1})$. Then $p(R)$ is contained in a hypersurface in $\mathbb{C}^{m-1}$ defined by a polynomial $h$. Let $H = \{x \in \mathbb{C}^m : x_m = 0\}$ and $N = \{x \in \mathbb{C}^m : h(x_1, \ldots, x_{m-1}) = 0\}$. We may assume that $0 \notin T \cup N$ and $b \notin H \cup N$. Let $R' = T \cap (H \cup N)$. Then $R \subset R'$ and $Z = \mathbb{C}^m \setminus R'$ is a Zariski neighbourhood of $b$. Define

$$F : \mathbb{C}^m \to \mathbb{C}^m, \quad (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_{m-1}, h(x_1, \ldots, x_{m-1})x_m).$$

Clearly, $F$ restricts to an automorphism of $\mathbb{C}^m \setminus N$. Using the form of the defining polynomial of $T$, it is easy to show that

$$\overline{F(T)} \cap N \subset H \cap N.$$

It follows that $\overline{F(T)} \setminus H = F(T) \setminus H$. Since $F(N) \subset H$, we have

$$F(T) \setminus H = F(T \setminus N) \setminus H \subset \mathbb{C}^m \setminus N.$$

Hence $\overline{F(T)} \setminus H$ is isomorphic to

$$F^{-1}(F(T \setminus N) \setminus H) = T \setminus (H \cup N) = T \cap Z.$$

Now define

$$\sigma : \mathbb{C}^m \to \mathbb{C}^m, \quad (x_1, \ldots, x_m) \mapsto (x_1 x_m, \ldots, x_{m-1} x_m, x_m).$$

Then $\sigma$ is an automorphism of $\mathbb{C}^m \setminus H$ and $\sigma^{-1}(H) = \sigma^{-1}(0) = H$. Since $0 \notin T \cup N$, we have $0 \notin \overline{F(T)}$, so

$$\sigma^{-1}(\overline{F(T)}) = \sigma^{-1}(\overline{F(T)} \setminus \{0\}) = \sigma^{-1}(\overline{F(T)}) \setminus H = \sigma^{-1}(\overline{F(T)} \setminus H).$$

We conclude that $T \cap Z$ is isomorphic to the closed subvariety $\sigma^{-1}(\overline{F(T)})$ of $\mathbb{C}^m$.

2.5. Now let $S$ be an algebraic subvariety of $\mathbb{C}^n$, $n \geq 2$, of codimension at least 2, and $A$ be an algebraic submanifold of $\mathbb{C}^n \setminus S$. Let $B$ be the blow-up of $\mathbb{C}^n \setminus S$ along $A$. We indicate how the proof above can be modified so as to show that $B$ is algebraically subelliptic.

We include $S$ in $\mathbb{C}^n \setminus U$. In the definition of the map $\phi$ in the construction of the first type of spray, we replace $A$ by the union of $S$ and the closure of $A$ in $\mathbb{C}^n$. The map $f$ then takes values in $\mathbb{C}^n \setminus (A_1 \cup S)$ and the construction goes through.

In the definition of the map $f$ in the construction of the second type of spray, we replace $\gamma(y) + t\zeta$ by a flow that avoids $\phi^{-1}(S)$. To obtain such a flow we need $\operatorname{codim} \phi^{-1}(S) \geq 2$, which must be built into the construction of $\phi$ as an extension. To this end we use the following corollary of a theorem of Jelonek.

Proposition 10. *Let $m \geq n$, $X$ be an algebraic subvariety of $\mathbb{C}^m$, and $f : X \to \mathbb{C}^n$ be a polynomial map. Then there is a polynomial map $F : \mathbb{C}^m \to \mathbb{C}^n$ extending $f$ such that* $\dim F^{-1}(z) \setminus X \leq m - n$ *for all $z \in \mathbb{C}^n$.*

*Proof.* Embed $\mathbb{C}^n$ as $\mathbb{C}^n \times \{0\}$ in $\mathbb{C}^m$. Then $f$ induces a map $\tilde{f} : X \to \mathbb{C}^m$, which extends to a polynomial map $\tilde{F} : \mathbb{C}^m \to \mathbb{C}^m$ such that $\tilde{F}|\mathbb{C}^m \setminus X$ has finite fibres [12, Theorem 3.9]. Let $\pi : \mathbb{C}^m \to \mathbb{C}^n$, $(z_1, \dots, z_m) \mapsto (z_1, \dots, z_n)$. Then $F = \pi \circ \tilde{F}$ is the desired map. □

## 3. Other Proofs

*Proof of Corollary 5.* Let $\pi : B \to \mathbb{C}^n$ be the blow-up along $A$ and let $f : T \to \mathbb{C}^n$ be holomorphic. First note that $f$ factors through $\pi$ by a holomorphic map $g : T \to B$. This is clear if $T$ is discrete, so suppose that $T$ is a smooth analytic curve. If $f(T) \not\subset A$, then the preimage of $A$ by $f$, as a complex subspace of $T$, is locally principal since $\dim T = 1$, so by the universal property of the blow-up, $f$ factors through $\pi$. If $f(T) \subset A$, we use the geometric construction of the blow-up. The pullback by $f$ of the normal bundle of $A$ in $\mathbb{C}^n$ is holomorphically trivial, again since $\dim T = 1$, and a nowhere-vanishing section of the pullback bundle over $T$ defines $g$.

Next we need an extension of $g : T \to B$ to a continuous map $\mathbb{C}^m \to B$. If $T$ is discrete, this is elementary. For example, take an injection $g_1 : T \to \mathbb{R}$ and a continuous map $g_2 : \mathbb{R} \to B$ such that $g = g_2 \circ g_1$, and extend $g_1$ to a continuous map $\mathbb{C}^m \to \mathbb{R}$. If $T$ is a smooth analytic curve, since $B$ is simply connected and $T$ is homotopy equivalent to a disjoint union of bouquets of circles, $g$ is homotopic to a continuous map $\tilde{g} : T \to B$ with a countable image. It is easy to see that $\tilde{g}$ extends continuously to $\mathbb{C}^m$ (for example by factoring $\tilde{g}$ through $\mathbb{R}$ as above), so $g$ does as well.

Since $B$ is Oka, $g$ has a holomorphic extension $h : \mathbb{C}^m \to B$. Let $F = \pi \circ h : \mathbb{C}^m \to \mathbb{C}^n$. Then $F$ is a holomorphic extension of $f$ and $F^{-1}(A) = h^{-1}(\pi^{-1}(A))$ is a hypersurface – except that $F^{-1}(A)$ might be empty or all of $\mathbb{C}^m$. To avert the former, add an extra point or component to $T$ and let $f$ map it into $A$. To avert the latter, add an extra point or component to $T$ and let $f$ map it outside of $A$. □

*Proof of Proposition 6.* We refer to [6, Section 6.3] for Gromov's theory of composed sprays. Let $X$ be an algebraic manifold with a dominating family of algebraic sprays $(E_j, \pi_j, s_j)$, $j = 1, \dots, m \geq 2$ (if $m = 1$, there is nothing to prove). The composed spray $(E_1 * E_2, \pi_1 * \pi_2, s_1 * s_2)$ is defined as the pullback

$$E_1 * E_2 = \{(e_1, e_2) \in E_1 \times E_2 : s_1(e_1) = \pi_2(e_2)\}$$

with

$$\pi_1 * \pi_2(e_1, e_2) = \pi_1(e_1), \quad s_1 * s_2(e_1, e_2) = s_2(e_2).$$

Then $E_1 * E_2$ is a vector bundle over $E_1$, and it has a natural zero-section over $X$, but we do not know whether it is a vector bundle, even holomorphically, over $X$. Otherwise it is a spray over $X$ in the usual sense. With that same proviso, we have a composed spray bundle $E = (\cdots(E_1 * E_2) * \cdots) * E_m$, which is dominating over $X$. Now $E$ is a vector bundle over a vector bundle over ... a vector bundle over $X$, so each fibre of $E$ is a vector bundle over a vector bundle over ... an affine space. (Up to this point, the theory of composed sprays is the same in the algebraic category and the holomorphic category.) We now invoke the Quillen-Suslin theorem, which states that every algebraic vector bundle over an affine space is algebraically trivial, and conclude that each fibre of $E$ is isomorphic to an affine space, which implies that $X$ is strongly algebraically dominable. $\square$

*Proof of Proposition 8.* Let $A$ be a closed subscheme of $\mathbb{C}^n$, $n \geq 2$. The defining ideal of $A$ is generated by polynomials $h_1, \ldots, h_m$ with greatest common divisor $h$. The blow-up of $\mathbb{C}^n$ along $A$ is the same as the blow-up of $\mathbb{C}^n$ along the subscheme defined by the ideal generated by $h_1/h, \ldots, h_m/h$. Thus we may assume that $A$ has codimension at least 2. In particular, the singular locus $Z$ of $A$ has codimension at least 2. By Theorem 1, the blow-up of $\mathbb{C}^n \setminus Z$ along $A \setminus Z$ is algebraically subelliptic and hence strongly algebraically dominable by Proposition 6. $\square$

*Proof of Theorem 9.* Let $B$ be the blow-up of an algebraic manifold $X$ along an algebraic submanifold $A$. Suppose that $X$ is algebraically dominable at a point $x$ and let $y \in B$ lie over $x$. Let $f : \mathbb{C}^n \to X$ be a regular map that takes 0 to $x$ and is a local isomorphism at 0. Let $\widehat{\mathbb{C}}^n$ be the blow-up of $\mathbb{C}^n$ along the subscheme $f^*A$. Then 0 is not a singular point of $f^*A$. Denote the blow-up projections by $\pi : B \to X$ and $p : \widehat{\mathbb{C}}^n \to \mathbb{C}^n$. Let $F : \widehat{\mathbb{C}}^n \to B$ be the regular lifting of $f \circ p$ by $\pi$, taking a point $z$ over 0 to $y$. Then $F$ is a local isomorphism at $z$, so it suffices to show that $\widehat{\mathbb{C}}^n$ is dominable at $z$, but this follows from Proposition 8. $\square$

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