Singularities of Moduli of Curves with a Universal Root

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Abstract. In a series of recent papers, Chiodo, Farkas and Ludwig carry out a deep analysis of the singular locus of the moduli space of stable (twisted) curves with an ℓ-torsion line bundle. They show that for $\ell \leq 6$ and $\ell \neq 5$ pluricanonical forms extend over any desingularization. This opens the way to a computation of the Kodaira dimension without desingularizing, as done by Farkas and Ludwig for $\ell = 2$, and by Chiodo, Eisenbud, Farkas and Schreyer for $\ell = 3$. Here we treat roots of line bundles on the universal curve systematically: we consider the moduli space of curves C with a line bundle L such that $L^{\otimes \ell} \cong \omega_C^{\otimes k}$. New loci of canonical and non-canonical singularities appear for any $k \notin \ell\mathbb{Z}$ and $\ell > 2$, we provide a set of combinatorial tools allowing us to completely describe the singular locus in terms of dual graphs. We characterize the locus of non-canonical singularities, and for small values of ℓ we give an explicit description.

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The moduli space $\mathcal{M}_g$ of smooth curves of genus g, alongside with its compactification $\overline{\mathcal{M}}_g$ in the sense of Deligne-Mumford, is one of the most fascinating and largely studied objects in algebraic geometry. It is also the subject of wide open questions; the one that motivates our work here is inspired by Harris and Mumford [12], they prove that for $g > 23$, $\mathcal{M}_g$ is of general type. The question on how and precisely where the transition to general type varieties happens remains open and highly intertwined with several other open questions in this field. However, a new line of attack seems to stand out clearly. It appears that the study of the finite covers of moduli of curves and roots of a universal bundle is more manageable. A series of papers by Farkas and Farkas-Verra (see [9] and [11]) provide a complete description for spin structures. Chiodo, Eisenbud, Farkas and Schreyer show in [7] that for curves with a 3-torsion bundle, the transition happens before $g = 12$. The previous results use a compactification

of the moduli spaces via twisted curves following from [3, 1, 2, 6], and are based on a preliminary analysis by Chiodo, Farkas and Ludwig of the singular locus of the moduli space of curves carrying a torsion line bundle, or level curves: in particular in [8] they characterize the singular locus and show that for ℓ-torsion bundles with $\ell \leq 6$ and $\ell \neq 5$, pluricanonical forms could be extended over any desingularization using a method developed by Harris and Mumford.
The purpose of this paper is to identify and analyze the locus of singularities for every line bundle on the universal curve. If $\mathcal{U}_g$ is the universal curve on $\mathcal{M}_g$, we can focus on roots of any order of the powers of the relative dualizing sheaf $\omega = \omega_{\mathcal{U}_g/\mathcal{M}_g}$. Indeed, these are all possible bundles on $\mathcal{U}_g$, up to pullbacks from the moduli space $\mathcal{M}_g$, as discussed for example in [14, 4]. We consider the moduli space $\mathcal{R}^k_{g,\ell}$ parametrizing ℓth roots of $\omega^{\otimes k}$, *i.e.* triples (C, L, ϕ) where C is a smooth curve, L a line bundle over C and ϕ an isomorphism $L^{\otimes \ell} \to \omega_C^{\otimes k}$. In the case $k = 0$, our analysis matches with Chiodo and Farkas work on level curves [8]. Furthermore, we give a more general description of the singularities for all k, in particular on those cases where it is not possible to extend everywhere the pluricanonical forms using the known techniques.
To compactify $\mathcal{R}^k_{g,\ell}$ we use the theory of stack-theoretic stable curves: we consider the moduli stack $\overline{\mathsf{R}}^k_{g,\ell}$ sending a scheme S to a triple $(\mathsf{C} \to S, \mathsf{L}, \phi)$ where C is a stack-theoretic curve with a stable curve C as coarse space and possibly non-trivial stabilizers on its nodes, L is a line bundle over C and ϕ an isomorphism $\mathsf{L}^{\otimes \ell} \to \omega_{\mathsf{C}}^{\otimes k}$. This moduli is represented by a Deligne-Mumford stack. The open dense substack $\mathsf{R}^k_{g,\ell}$, which admits only smooth curves C, has $\mathcal{R}^k_{g,\ell}$ as coarse space. The stack structure extends the covering $\mathcal{R}^k_{g,\ell} \to \mathcal{M}_g$ on all $\overline{\mathcal{M}}_g$.

In this paper we focus on the singular locus of $\overline{\mathcal{R}}^k_{g,\ell}$, called $\operatorname{Sing} \overline{\mathcal{R}}^k_{g,\ell}$, and on the locus of non-canonical singularities, also called non-canonical locus and noted $\operatorname{Sing}^{\mathrm{nc}} \overline{\mathcal{R}}^k_{g,\ell}$. The singular locus of $\overline{\mathcal{M}}_g$ is characterized in [12] by introducing the concept of elliptic tail quasireflection (see Definition 2.11): $[C] \in \overline{\mathcal{M}}_g$ is in $\operatorname{Sing} \overline{\mathcal{M}}_g$ if and only if there exists $\alpha \in \operatorname{Aut}(C)$ not generated by elliptic tail quasireflections. This locus admits a natural lift to $\overline{\mathcal{R}}^k_{g,\ell}$ by introducing the group $\operatorname{Aut}'(C)$ of automorphisms on C which preserve the root structure. However, the stack structure of a curve comes equipped with new *ghost* automorphisms, *i.e.* automorphisms acting trivially on the coarse space of the curve, but possibly non-trivially on the additional structure. Because of such ghosts, there is a new locus of singularities which can be studied using new tools in graph theory.
First, to each point $[\mathsf{C}, \mathsf{L}, \phi] \in \overline{\mathcal{R}}^k_{g,\ell}$ we attach its dual graph $\Gamma(\mathsf{C})$: its vertex set is, as usual, the set of irreducible components of C, and its edge set is the set of nodes of C. Moreover, we introduce a multiplicity index M (see Definition 3.1), which is a function on the edges of the dual graph and depends on the line bundle L (it maps each node with a privileged branch to the character $\mathsf{L}|_{\mathsf{n}}$).

Finally, for any prime p in the factorization of ℓ, if e_p is the p-adic valuation power of ℓ, we note $\Gamma_p(\mathsf{C})$ the graph obtained contracting those edges of the dual graph where M is divisible by p^{e_p}. A graph is tree-like if it has no circuits except for loops. Generalizing previous results from [13] and [8], we prove the following (see Theorem 4.6).

Theorem. *For any $g \geq 4$ and ℓ positive integer, the point $[\mathsf{C}, \mathsf{L}, \phi] \in \overline{\mathcal{R}}^k_{g,\ell}$ is smooth if and only if the two following conditions are satisfied:*

i. the image $\operatorname{Aut}'(C)$ of the coarsening morphism $\underline{\operatorname{Aut}}(\mathsf{C}, \mathsf{L}, \phi) \to \operatorname{Aut}(C)$ is generated by elliptic tail quasireflections;
ii. the dual graph contractions $\Gamma(\mathsf{C}) \to \Gamma_p(\mathsf{C})$ yield a tree-like graph for all prime numbers p dividing ℓ.

This is a generalization of Chiodo and Farkas [8, Theorem 2.28]. In the case $k = 0$ treated by them, the dual graph contractions must be *bouquet* graphs, *i.e.* graphs with only one vertex. We spell out this generalization on Remark 4.7.
The role of dual graph structure in characterizing the geometry of $\overline{\mathcal{R}}^k_{g,\ell}$ goes even deeper. Consider the contraction $\Gamma(\mathsf{C}) \to \Gamma_0(\mathsf{C})$ of the edges where M vanishes. The graph $\Gamma_0(\mathsf{C})$ and the multiplicity index M completely describe the group of ghost automorphisms, and as a consequence they carry all the information about the new singularities. We have that the non-canonical locus is of the form

$$\operatorname{Sing}^{\mathrm{nc}} \overline{\mathcal{R}}^k_{g,\ell} = T^k_{g,\ell} \cup J^k_{g,\ell},$$

where $T^k_{g,\ell} \subset \pi^{-1} \operatorname{Sing}^{\mathrm{nc}} \overline{\mathcal{M}}_g$ is the analogue of old non-canonical singularities, and $J^k_{g,\ell}$ is the new part of the non-canonical locus. For the T-locus the generalization of the technique developed in [12] allows the extension of pluricanonical forms. Unfortunately, it is unknown if this extension is possible over the J-locus. This is the main obstacle toward the analysis of birational geometry $\overline{\mathcal{R}}^k_{g,\ell}$. One of the aims of this paper is precisely to describe $J^k_{g,\ell}$ further. In Section 4.1 we introduce a stratification of the boundary $\overline{\mathcal{R}}^k_{g,\ell} \backslash \mathcal{R}^k_{g,\ell}$ labelled by decorated graphs, *i.e.* pairs $(\Gamma_0(\mathsf{C}), M)$: we will decompose the J-locus using this stratification.
In the last section we work out the cases $\ell = 2,\ 3,\ 5$ and 7. Using the age criterion, we develop a machinery to study the ghosts group through graph properties, we conclude in particular that in many cases the J-locus is the union of strata labelled only by *vine* graphs, *i.e.* graphs with only two vertices. The first exception comes out with $\ell = 7$, when a 3-vertices graph stratum is needed to complete the description.

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1. Preliminaries on curves and their dual graphs

In this paper ℓ is always a positive integer. Every curve is an algebraic curve of genus $g \geq 0$ over an algebraically closed field k whose characteristic is 0 or prime to ℓ. By S-curve we mean a family of curves on a scheme S, *i.e.* a flat morphisms $C \to S$ such that every geometric fibre is a curve of given genus.
Given a line bundle F on a curve C, an *ℓth root* of F on C is a pair $(L; \phi)$, where L is a line bundle on C and ϕ is an isomorphism of line bundles $L^{\otimes \ell} \to F$ is an isomorphism. A morphism between two ℓth roots (L, ϕ), (L', ϕ'), is a morphism $\rho\colon L \to L'$ of line bundles on C such that $\phi' = \phi \circ \rho^{\otimes \ell}$.
If C is a smooth curve and F is a line bundle on C and ℓ divides $\deg F$, F has at least one ℓth root, and in this case the number of ℓth roots is exactly ℓ^{2g}. This property fails for general singular curves. Replacing scheme theoretical curves by twisted curves, that we define just below, we still get the "right" number of ℓth roots.
In the last part of this section we will introduce some basic tools on graph theory.

1.1. Twisted curves and ℓth roots.

We recall the definition of twisted curve.

Definition 1.1 (twisted curve). A twisted curve C is a Deligne-Mumford stack whose coarse space is a curve with nodal singularities, such that its smooth locus is represented by a scheme, and the local pictures of the singularities are given by $[\{xy = 0\}/\boldsymbol{\mu}_r]$, r positive integer, with any primitive rth root of unity ζ acting as $\zeta \cdot (x, y) = (\zeta x, \zeta^{-1} y)$. In this case we say that the node has a $\boldsymbol{\mu}_r$ stabilizer.

If n is a node of C with non-trivial stabilizer $\boldsymbol{\mu}_r$, the local picture of the curve at n is $xy = 0$ with the action of $\boldsymbol{\mu}_r$ described above. Given a line bundle $\mathsf{F} \to \mathsf{C}$ at n, its local picture at the node is $\mathbb{A}^1 \times \{xy = 0\} \to \{xy = 0\}$ with any primitive root $\zeta \in \boldsymbol{\mu}_r$ acting as

$$\zeta \cdot (t, x, y) = (\zeta^m t,\ \zeta x,\ \zeta^{-1} y), \quad \text{with} \quad m \in \mathbb{Z}/r, \tag{1}$$

on $\mathbb{A}^1 \times \{xy = 0\}$. The index $m \in \mathbb{Z}/r$, called *local multiplicity*, is uniquely determined when we assign a privileged choice of a branch of n where ζ acts as $x \mapsto \zeta x$. If we switch the privileged branch, the local multiplicity changes to its opposite $-m$.
A line bundle L on C is *faithful* if the associated morphism $\mathsf{C} \to \mathsf{B}\mathbb{G}_m$ is representable. As pointed out in [8, Remark 1.4], a line bundle on a twisted

curve is faithful if and only if the local multiplicity index m at every node n is coprime with the order r of the local stabilizer.

DEFINITION 1.2. An ℓth root of the line bundle F on the twisted curve C, is a pair (L, ϕ), where L is a faithful line bundle on C and $\phi \colon \mathsf{L}^{\otimes \ell} \to \mathsf{F}$ is an isomorphism between $\mathsf{L}^{\otimes \ell}$ and F.

Fixing a primitive rth root of the unity ζ will not affect our results and will permit us to identify (non-canonically) $\boldsymbol{\mu}_r$ and $\mathbb{Z}/r$. In what follows we chose the primitive root $\xi_r = \exp(2i\pi/r)$ for every node.

1.2. THE DUAL GRAPH. If a curve is not smooth, it may have several irreducible components. The information about the relative crossing of this components is encoded in the dual graph.

DEFINITION 1.3 (dual graph). Given a curve C (scheme theoretic or twisted), the dual graph $\Gamma(C)$ has the set of irreducible components of C as vertex set, and the set of nodes of C as edge set E. The edge associated to the node n links the vertices associated to the components where the branches of n lie.

We introduce some graph theory which will be important to study the moduli spaces of twisted curves equipped with a root.

1.2.1. *Cochains over a graph.* Consider a connected graph Γ with vertex set V and edge set E, we call *loop* an edge that starts and ends on the same vertex, we call *separating* an edge e such that the graph with vertex set V and edge set $E\backslash\{e\}$ is disconnected. We note by E_{sep} the set of separating edges. We note by $\mathbb{E}$ the set of oriented edges: the elements of this set are edges in E equipped with an orientation. In particular for every edge $e \in \mathbb{E}$ we note e_+ the head vertex and e_- the tail, and there is a 2-to-1 projection $\mathbb{E} \to E$. We also introduce a conjugation in $\mathbb{E}$, such that for each $e \in \mathbb{E}$, the conjugated edge $\bar{e}$ is obtained by reversing the orientation, in particular $(\bar{e})_+ = e_-$.

REMARK 1.4. In the dual graph setting, where the vertices are the components of a curve C and the edges are the nodes, the oriented edge set $\mathbb{E}$ could be seen as the set of branches at the curve nodes. Indeed, every edge e, equipped with an orientation, is bijectively associated to the branch it is pointing at.

DEFINITION 1.5. The group of 0-cochains is the group of $\mathbb{Z}/\ell$-valued functions on V

$$C^0(\Gamma; \mathbb{Z}/\ell) := \{a \colon V \to \mathbb{Z}\} = \bigoplus_{v \in V} \mathbb{Z}/\ell.$$

The group of 1-cochains is the group of antisymmetric functions on $\mathbb{E}$

$$C^1(\Gamma; \mathbb{Z}/\ell) := \{b \colon \mathbb{E} \to \mathbb{Z}/\ell |\ b(\bar{e}) = -b(e)\}.$$

After assigning an orientation to every edge $e \in E$, we may identify $C^1(\Gamma) = \bigoplus_{e \in E} \mathbb{Z}/\ell$, but we prefer working without any prescribed choice of orientation.

These spaces are equipped with two non-degenerate bilinear $\mathbb{Z}/\ell$-valued forms

$$\langle a_1, a_2\rangle := \sum_{v\in V} a_1(v)a_2(v) \qquad \langle b_1, b_2\rangle := \frac{1}{2}\sum_{e\in\mathbb{E}} b_1(e)b_2(e)$$

for all $a_1, a_2 \in C^0(\Gamma)$ and $b_1, b_2 \in C^1(\Gamma)$. Consider the operator $\partial\colon C^1(\Gamma) \to C^0(\Gamma)$ such that

$$\partial b(v) := \sum_{e_+=v} b(e), \quad \forall b \in C^1(\Gamma)\ \forall v \in V.$$

Definition 1.6. We introduce $G(\Gamma; \mathbb{Z}/\ell) := (\ker\partial)^\perp$, the orthogonal complement with respect to the form just introduced.

We also introduce the exterior differential $\delta\colon C^0(\Gamma) \to C^1(\Gamma)$ such that

$$\delta a(e) := a(e_+) - a(e_-), \quad \forall a \in C^0(\Gamma)\ \forall e \in \mathbb{E}.$$

Proposition 1.7 (see [8, p.9]). *The adjoint of ∂ is δ, and $G(\Gamma; \mathbb{Z}/\ell) = \operatorname{Im}\delta$.* □

The exterior differential fits into an useful exact sequence.

$$0 \to \mathbb{Z}/\ell \xrightarrow{i} C^0(\Gamma; \mathbb{Z}/\ell) \xrightarrow{\delta} C^1(\Gamma; \mathbb{Z}/\ell).$$

Where the injection i sends $m \in \mathbb{Z}$ on the cochain constantly equal to m. This sequence gives the dimension of $\operatorname{Im}\delta$.

$$\operatorname{Im}\delta \cong (\mathbb{Z}/\ell)^{\#V-1}. \tag{2}$$

1.2.2. *Construction of a basis for $C^1(\Gamma)$.*

Definition 1.8 (cuts, paths and circuits). A cut is a 1-cochain $b\colon \mathbb{E} \to \mathbb{Z}$ such that there exists a non-empty subset $W \subset V$ and the values of b are determined in the following way: $b(e) = 1$ if the head of e is in W and the tail in $V\backslash W$, $b(e) = -1$ if the head is in $V\backslash W$ and the tail in W, $b(e) = 0$ elsewhere. The support of b in $\mathbb{E}$ is sometimes called *cut-set* of b.
A path in a graph Γ is a sequence $e_1, e_2, \ldots, e_k$ of edges in $\mathbb{E}$ overlying k distinct non-oriented edges in E, and such that the head of e_i is the tail of e_{i+1} for all $i = 1, \ldots, k$.
A circuit is a closed path, *i.e.* a path $e_1, \ldots, e_k$ such that the head of e_k is the tail of e_1. We often refer to a circuit by referring to its characteristic function, *i.e.* the cochain b such that $b(e_i) = 1$ for all i, $b(\bar{e}_i) = -1$ and $b(e) = 0$ if e is not on the circuit.

Remark 1.9. Any cut b is an element of $\operatorname{Im}\delta$. Indeed, $b = \delta a$ if a is the characteristic function of W.

Moreover, the group $\operatorname{Im}\delta$ is sometimes called *cuts space* because it is generated by cuts (see Proposition 1.14). We also have that $\operatorname{Ker}\partial$ is generated by circuits and this implies the following fact.

PROPOSITION 1.10. *A 1-cochain b is in $G(\Gamma;\mathbb{Z}/\ell) = \operatorname{Im}\delta$ if and only if, for every circuit $K = (e_1, \dots, e_k)$ in $\mathbb{E}$, we have*

$$b(K) = \sum_{i=1}^{k} b(e_i) = 0.$$

TERMINOLOGY (graphs tree-like and trees). A tree-like graph is a connected graph whose only circuits are loops. A tree is a graph that does not contain any circuit.

REMARK 1.11. For every connected graph Γ, the first Betti number $b_1(\Gamma) = \#E - \#V + 1$ is the dimension rank of the homology group $H_1(\Gamma;\mathbb{Z})$. Note that, b_1 being positive,

$$\#E \geq \#V - 1$$

with equality if and only if Γ is a tree.

For every connected graph Γ with vertex set V and edge set E, we can choose a connected subgraph T with the same vertex set and edge set $E_T \subset E$ such that T is a tree. The graph T is a *spanning tree* of Γ and has several interesting properties. We call $\mathbb{E}_T$ the set of oriented edges of the spanning tree T. Here we notice that E contains a distinguished subset of edges E_{sep} whose size is *smaller* than V-1

LEMMA 1.12. *If $E_{\text{sep}} \subset E$ is the set of edges in Γ that are separating, then $\#E_{\text{sep}} \leq \#V - 1$ with equality if and only if Γ is tree-like.*

Proof. If T is a spanning tree for Γ and E_T its edge set, then $E_{\text{sep}} \subset E_T$. Indeed, an edge $e \in E_{\text{sep}}$ is the only path between its two extremities, therefore, since T is connected, e must be in E_T. Thus $\#E_{\text{sep}} \leq \#E_T = \#V - 1$, with equality if and only if all the edges of Γ are loops or separating edges, *i.e.* if Γ is a tree-like graph. □

LEMMA 1.13 (see [5, Lemma 5.1]). *If T is a spanning tree for Γ, for every oriented edge $e \in \mathbb{E}_T$ there is a unique cut $b \in C^1(\Gamma;\mathbb{Z}/\ell)$ such that $b(e) = 1$ and the elements of the support of b other than e and $\bar{e}$ are all in $\mathbb{E}\backslash\mathbb{E}_T$.* □

We call $\operatorname{cut}_\Gamma(e;T)$ the unique cut b resulting from the lemma for all $e \in \mathbb{E}_T$.

PROPOSITION 1.14. *If T is a spanning tree for Γ, then the elements $\operatorname{cut}_\Gamma(e;T)$, with e varying on E_T, form a basis of $\operatorname{Im}(\delta)$.*

For a proof of this and a deeper analysis of the image of δ, see [5, Chap.5].

1.2.3. *Definition and basic properties of edge contraction.* Another tool in graph theory is edge contraction. If we have a graph Γ with vertex set V and edge set E, we can choose a subset $F \subset E$. Contracting edges in F means taking the graph Γ_0 such that:

(1) the edge set is $E_0 := E\backslash F$;
(2) given the relation in V, $v \sim w$ if v and w are linked by an edge $e \in F$, the vertex set is $V_0 := V/\sim$.

The construction of Γ_0 follows and we have a natural morphism $\Gamma \to \Gamma_0$ called *contraction* of F. Edge contraction will be useful, in particular we will consider the image of the exterior differential δ and its restriction over contraction of a given graph. If Γ_0 is a contraction of Γ, then $E(\Gamma_0)$ is canonically a subset of $E(\Gamma)$. As a consequence, cochains over Γ_0 are cochains over Γ with the additional condition that the values on $E(\Gamma)\backslash E(\Gamma_0)$ are all 0. Then we have a natural immersion

$$C^i(\Gamma_0;\mathbb{Z}/\ell) \hookrightarrow C^i(\Gamma;\mathbb{Z}/\ell).$$

Consider the two operators

$$\delta\colon C^0(\Gamma;\mathbb{Z}/\ell) \to C^1(\Gamma;\mathbb{Z}/\ell) \quad \text{and} \quad \delta_0\colon C^0(\Gamma_0;\mathbb{Z}/\ell) \to C^1(\Gamma_0;\mathbb{Z}/\ell).$$

Clearly δ_0 is the restriction of δ on $C^0(\Gamma_0;\mathbb{Z})$. ¿From this observation we have the following.

Proposition 1.15.

$$\operatorname{Im}\delta_0 = C^1(\Gamma_0;\mathbb{Z}) \cap \operatorname{Im}\delta.$$

Remark 1.16. We make an observation about separating edges of a graph after contraction. Given a graph contraction $\Gamma \to \Gamma_0$, the separating edges who are not contracted remain separating. Moreover, an edge that is not separating cannot become separating.

2. The moduli space of ℓth roots of $\omega^{\otimes k}$

We start by considering the moduli stack $\overline{\mathsf{M}}_g$. Its objects are flat morphisms $C \to S$ such that every geometric fiber is a stable genus g curve, *i.e.* a curve with at most nodal singularities and an ample canonical line bundle. The $\overline{\mathsf{M}}_g$ coarse space is the moduli space of stable curves $\overline{\mathcal{M}}_g$. Harris and Mumford proved in [12] that $\overline{\mathcal{M}}_g$ is of general type for $g > 23$, and classified its singularities. We observe that the moduli stack M_g of smooth curves is an open dense substack of $\overline{\mathsf{M}}_g$.

Despite the construction of the moduli space of ℓth roots over smooth curves is quite natural, its extension over $\overline{\mathcal{M}}_g$ requires some twisted curve machinery. More precisely, we consider the stack $\mathsf{R}^k_{g,\ell}$ whose objects are smooth curves C with a line bundle L and an isomorphism $L^{\otimes \ell} \cong \omega_C^{\otimes k}$. This stack comes with a natural proper forgetful functor $\mathsf{R}^k_{g,\ell} \to \mathsf{M}_g$. To extend this forgetful morphism over $\overline{\mathsf{M}}_g$, we will introduce ℓth roots for any stable curve.

Definition 2.1. Given a stable curve C of genus g and a line bundle F on it, an ℓth root of F on C is a triple $(\mathsf{C}, \mathsf{L}, \phi)$ such that C is a twisted curve, $u\colon \mathsf{C} \to C$ its coarsening, L is a faithful line bundle on C and $\phi\colon \mathsf{L}^{\otimes \ell} \to u^*F$ an isomorphism.

Remark 2.2. There exists such a root if and only if ℓ divides $\deg F$, and in this case the number of ℓth roots is ℓ^{2g}.

Definition 2.3. The stack $\overline{\mathsf{R}}^k_{g,\ell}$ has for objects the triples $(\mathsf{C} \to S, \mathsf{L}, \phi)$ where $\mathsf{C} \to S$ is a twisted S-curve whose coarse space is a stable curve of genus g, L is a faithful line bundle over C and

$$\phi\colon \mathsf{L}^{\otimes \ell} \to \omega_{\mathsf{C}}^{\otimes k}$$

is an isomorphism between $\mathsf{L}^{\otimes \ell}$ and the kth power of the canonical bundle ω_{C}.

Remark 2.4. The canonical bundle ω_{C} is the pullback of ω_C via the coarsening morphism. In other words, the objects of $\overline{\mathsf{R}}^k_{g,\ell}$ are stable curves with an ℓth root of their canonical bundle.

Remark 2.5. Because of Remark 2.2, we need that ℓ divides $\deg \omega_C^{\otimes k} = k \cdot (2g-2)$ to make $\mathsf{R}^k_{g,\ell}$ non-empty.

A triple $(\mathsf{C}, \mathsf{L}, \phi)$ as above will be improperly called a *rooted curve* when there is no risk of confusion. The stack $\mathsf{R}^k_{g,\ell}$ is an open dense substack of $\overline{\mathsf{R}}^k_{g,\ell}$. Moreover, we note $\pi\colon \overline{\mathsf{R}}^k_{g,\ell} \to \overline{\mathsf{M}}_g$ the extended forgetful functor sending $(\mathsf{C} \to S, \mathsf{L}, \phi)$ to $C \to S$ where C is the coarse space of C. By Remark 2.2, every π fiber has the same length ℓ^{2g}.

2.1. Local structure of the roots moduli. We consider the stack $\overline{\mathsf{R}}^k_{g,\ell}$, with $g \geq 2$ and $0 \leq k < \ell$, of rooted curves of genus g. We denote by $\overline{\mathcal{R}}^k_{g,\ell}$ its associated coarse space.

We are interested in the singularities of this moduli space, in particular we want a characterization of the singular locus of $\overline{\mathcal{R}}^k_{g,\ell}$. If we note $\widetilde{\mathsf{M}}_g$ the moduli stack of twisted curves, the forgetful morphism $\overline{\mathsf{R}}^k_{g,\ell} \to \widetilde{\mathsf{M}}_g$ is étale, as showed in [6]. At the same time the composition morphism $\overline{\mathsf{R}}^k_{g,\ell} \to \widetilde{\mathsf{M}}_g \to \overline{\mathsf{M}}_g$ is ramified, as showed by Chiodo and Farkas in [8]. However, the local structure of $\overline{\mathsf{R}}^k_{g,\ell}$ is very similar to the one of $\overline{\mathsf{M}}_g$, and we will follow the same approach developed by Chiodo and Farkas for the case $k = 0$. We denote by $[\mathsf{C}, \mathsf{L}, \phi]$ the point of $\overline{\mathsf{R}}^k_{g,\ell}$ associated to the rooted curve $(\mathsf{C}, \mathsf{L}, \phi)$, then the local picture of the stack $\overline{\mathsf{R}}^k_{g,\ell}$ at $[\mathsf{C}, \mathsf{L}, \phi]$ is

$$[\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)/\operatorname{Aut}(\mathsf{C}, \mathsf{L}, \phi)]\,,$$

where the universal deformation $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ is a smooth scheme of dimension $3g-3$, and

$$\operatorname{Aut}(\mathsf{C}, \mathsf{L}, \phi) = \left\{(\mathsf{s}, \rho) |\ \mathsf{s} \in \operatorname{Aut}(\mathsf{C}) \text{ and } \rho\colon \mathsf{s}^*\mathsf{L} \xrightarrow{\cong} \mathsf{L} \text{ such that } \phi \circ \rho^{\otimes \ell} = \mathsf{s}^*\phi\right\}$$

is the automorphism group of $(\mathsf{C}, \mathsf{L}, \phi)$. This implies that the local picture of the moduli space $\overline{\mathcal{R}}^k_{g,\ell}$ is the classical quotient $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)/\operatorname{Aut}(\mathsf{C}, \mathsf{L}, \phi)$. The deformation space $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ is canonically isomorphic to $\mathrm{Def}(\mathsf{C})$ via the étale forgetful functor $(\mathsf{C}, \mathsf{L}, \phi) \mapsto \mathsf{C}$. Also we see that the action of $\operatorname{Aut}(\mathsf{C}, \mathsf{L}, \phi)$ on

$\mathrm{Def}(\mathsf{C})$ is not faithful. In particular the quasi-trivial automorphisms $(\mathrm{id}_{\mathsf{C}}, \zeta)$ with $\zeta \in \boldsymbol{\mu}_\ell$, whose action scale the fibers, have trivial action. Thus it becomes natural to consider the group

$$\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi) := \mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)/\{(\mathrm{id}_{\mathsf{C}}; \zeta)|\ \zeta \in \boldsymbol{\mu}_\ell\} = \{\mathsf{s} \in \mathrm{Aut}(\mathsf{C})|\ \mathsf{s}^*\mathsf{L} \cong \mathsf{L}\}.$$

The local picture of $\overline{\mathcal{R}}^k_{g,\ell}$ at $(\mathsf{C}, \mathsf{L}, \phi)$ could be rewritten as

$$\mathrm{Def}(\mathsf{C})/\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi).$$

Definition 2.6. A quasireflection is an element $q \in \mathrm{GL}(\mathbb{C}^m)$ such that its fixed space is a hyperplane of $\mathbb{C}^m$.

The smoothness of quotient singularities depends on the quotient group to be generated by quasireflections (see [15]). The following fact will permit us to classify moduli space singularities.

Fact 2.7. The scheme theoretic quotient $\mathrm{Def}(\mathsf{C})/\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ is smooth if and only if $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ is spanned by elements acting as the identity or as quasireflectons.

Before studying the action of $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$, we point out the structure of the universal deformation $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi) = \mathrm{Def}(\mathsf{C})$.

Remark 2.8. We will follow the usual construction developed in [12]. We note C the coarse space of C and by $N := \{\mathsf{n}_1, \dots \mathsf{n}_k\}$ the node set of C. If $\{\mathsf{C}_1, \dots, \mathsf{C}_m\}$ is the set of irreducible components of C, we denote by $\overline{\mathsf{C}}_i$ the normalizations of C_i, by g_i the genus of $\overline{C}_i$ and by $\mathsf{nor}\colon \bigsqcup_{i=1}^m \overline{\mathsf{C}}_i \to \mathsf{C}$ the normalization morphism of C. We observe that the curves $\overline{\mathsf{C}}_i$ are scheme theoretic smooth curves. The divisor D_i on $\overline{\mathsf{C}}_i$ is the preimage of N by the restriction of nor on this component. Following [8] we call $\mathrm{Def}(\mathsf{C}, \mathrm{Sing}\,\mathsf{C})$ the universal deformation of C alongside with its nodes. Then we have the canonical splitting

$$\mathrm{Def}(\mathsf{C}, \mathrm{Sing}\,\mathsf{C}) = \bigoplus_{i=1}^m H^1(\overline{\mathsf{C}}_i, T_{\overline{\mathsf{C}}_i}(-D_i)),$$

where $T_{\overline{\mathsf{C}}_i}$ is the tangent bundle to $\overline{\mathsf{C}}_i$ and $H^1(\overline{\mathsf{C}}_i, T_{\overline{\mathsf{C}}_i}(-D_i)) = \mathrm{Def}(\overline{\mathsf{C}}_i, D_i)$ parametrizes deformations of curve $\overline{\mathsf{C}}_i$ with a marking on the points of divisor D_i.

Once we mod out $\mathrm{Def}(\mathsf{C}, \mathrm{Sing}\,\mathsf{C})$, we have another canonical splitting:

$$\mathrm{Def}(\mathsf{C})/\,\mathrm{Def}(\mathsf{C}, \mathrm{Sing}\,\mathsf{C}) = \bigoplus_{i=1}^k R_i,$$

where every R_i is unidimensional. The non-canonical choice of a coordinate t_i over R_i corresponds to fixing a smoothing of node n_i.

Given the coarse space C of C, we consider its universal deformation $\mathrm{Def}(C)$ and its universal deformation alongside with its nodes $\mathrm{Def}(C, \mathrm{Sing}\, C)$. We have another canonical splitting $\mathrm{Def}(C)/\,\mathrm{Def}(C, \mathrm{Sing}\, C) = \bigoplus N_i$, where the N_i are unidimensional and there are natural coverings $R_i \to N_i$ of order $r(\mathsf{n}_i)$ ramified at the origin, where $r(\mathsf{n}_i)$ is the order of the stabilizer at n_i.

REMARK 2.9. We point out a classical formula relating the genus of C and the genera of the curves C_i.

$$g(\mathsf{C}) = \sum_{i=1}^{\nu} g_i + b_1(\Gamma) = \sum_{i=1}^{\nu} g_i + \#E(\Gamma) - \#V(\Gamma) + 1,$$

where $\Gamma = \Gamma(\mathsf{C})$ is the dual graph of C.

The coarsening $\mathsf{C} \to C$ induces a group homomorphism

$$\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi) \to \mathrm{Aut}(C).$$

We note the kernel and the image of this morphism by $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ and $\mathrm{Aut}'(C)$ (see also [8, chap. 2]). They fit into the following short exact sequence,

$$1 \to \underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) \to \underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi) \to \mathrm{Aut}'(C) \to 1. \tag{3}$$

DEFINITION 2.10. Within a stable curve C, an elliptic tail is an irreducible component of geometric genus 1 that meets the rest of the curve in only one point called an elliptic tail node. Equivalently, T is an elliptic tail if and only if its algebraic genus is 1 and $T \cap \overline{C \backslash T} = \{\mathsf{n}\}$.

DEFINITION 2.11. An element $i \in \mathrm{Aut}(C)$ is an elliptic tail automorphism if there exists an elliptic tail T of C such that i fixes T and his restriction to $\overline{C \backslash T}$ is the identity. An elliptic tail automorphism of order 2 is called an elliptic tail quasireflection (ETQR). In the literature ETQRs are called elliptic tail involutions (or ETIs), we changed this convention in order to generalize the notion.

REMARK 2.12. Every curve of algebraic genus 1 with one marked point has exactly one involution i. Then there is a unique ETQR associated to every elliptic tail.
More precisely an elliptic tail E could be of two types. The first type is a smooth curve of geometric genus 1 with one marked point, *i.e.* an elliptic curve: in this case we have $E = \mathbb{C}/\Lambda$, for Λ integral lattice of rank 2, the marked point is the origin, and the only involution is the map induced by $x \mapsto -x$ on $\mathbb{C}$. The second type is the rational line with one marked point and an autointersection point: in this case we can write $E = \mathbb{P}^1/\{1 \equiv -1\}$, the marked point is the origin, and the only involution is the map induced by $x \mapsto -x$ on $\mathbb{P}^1$.

THEOREM 2.13 (See [12, theorem 2]). *Consider a stable curve C of genus $g \geq 4$. An element of $\mathrm{Aut}(C)$ acts as a quasireflection on $\mathrm{Def}(C)$ if and only if it is an ETQR. In particular, if $\eta \in \mathrm{Aut}(C)$ is an ETQR acting non trivially on the tail T with elliptic tail node n, then η acts on $\mathrm{Def}(C)$ as $t_{\mathsf{n}} \mapsto -t_{\mathsf{n}}$ on the coordinate associated to n, and as $t \mapsto t$ on the remaining coordinates.*

REMARK 2.14. Consider a (stack-theoretic) curve E of genus 1 with one marked point. We call E its coarse space. In the case of an elliptic tail of a curve C, the marked point is the point of intersection between E and $\overline{\mathsf{C}\backslash\mathsf{E}}$.
If E is an elliptic curve, then $\mathsf{E} = E$ and the curve has exactly one involution i_0. In case E is rational, its normalization is the stack $\overline{\mathsf{E}} = [\mathbb{P}^1/\boldsymbol{\mu}_r]$, with $\boldsymbol{\mu}_r$ acting by multiplication, and $\mathsf{E} = \overline{\mathsf{E}}/\{0 \equiv \infty\}$. There exists a canonical involution i_0 in this case too: we consider the pushforward of the involution on $\mathbb{P}^1$ such that $z \mapsto 1/z$.
Given any twisted curve C with an elliptic tail E whose elliptic tail node is n, we found a canonical involution i_0 on E up to non-trivial action on n.

DEFINITION 2.15. We generalize the notion of ETQR to rooted curves. An element $\mathsf{i} \in \mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$ is an ETQR if there exists an elliptic tail E of C with elliptic tail node n, such that the action of i on $\mathsf{C}\backslash\mathsf{E}$ is trivial, and the action on E, up to non-trivial action on n, is the canonical involution i_0.

3. THE AUTOMORPHISMS OF ROOTED CURVES

In this section we will characterize smooth and singular points of $\overline{\mathcal{R}}^k_{g,\ell}$ using the dual graph of curves and their multiplicity indices. For simplicity, we will focus on the case ℓ prime. See Remark 3.14 for a discussion on the generalization to composite level ℓ.
Given a rooted curve $(\mathsf{C}, \mathsf{L}, \phi)$, we come back to the local multiplicity on a node n whose local picture is $[\{xy = 0\}/\boldsymbol{\mu}_r]$ with r positive integer. As in equation (1), once we choose a privileged branch, the action on the bundle fiber near the node is $\xi_r(t, x, y) = (\xi_r^m t, \xi_r x, \xi_r^{-1} y)$. We observe that the canonical line bundle ω_{C} is the pullback of the canonical line bundle over the coarse space C, and this, with the isomorphism $\mathsf{L}^{\otimes \ell} \cong \omega_{\mathsf{C}}$, implies that $(\xi_r^m)^\ell = 1$. So ℓm is a multiple of r. As a consequence of the faithfulness of L, the order r equals 1 or ℓ.

DEFINITION 3.1. The multiplicity index of $(\mathsf{C}, \mathsf{L}, \phi)$ is the cochain $M \in C^1(\Gamma; \mathbb{Z}/\ell)$ such that, for all $e \in \mathbb{E}$ oriented edge of the dual graph $\Gamma(\mathsf{C})$, $M(e) = m$ the local multiplicity with respect to the privileged branch associated to e (see Remark 1.4).

REMARK 3.2. If the node has trivial stabilizer, *i.e.* $r = 1$, we pose $M(e) := 0$.

We show a restating of Remark 2.2.

DEFINITION 3.3. Consider a curve C with dual graph Γ and line bundle F on C. The multidegree cochains is the function $\underline{\deg}(F, C) \in C^0(\Gamma; \mathbb{Z}/\ell)$ such that

$$\underline{\deg}(F, C)(v) := \deg F|_v \mod \ell \quad \forall v \in V(\Gamma)$$

where by $F|_v$ we mean the restriction of F on the component associated to vertex v.

PROPOSITION 3.4. *Consider a stable curve C with dual graph Γ and consider a cochain M in $C^1(\Gamma; \mathbb{Z}/\ell)$. Also consider the differential $\partial\colon C^1(\Gamma; \mathbb{Z}/\ell) \to$*

$C^0(\Gamma;\mathbb{Z}/\ell)$ (see Section 1.2.1). There exists an ℓth root of the line bundle F on C with multiplicity index M, if and only if

$$\partial M \equiv \underline{\deg}(F, C) \mod \ell.$$

Proof. If $(\mathsf{C}, \mathsf{L}, \phi)$ is any curve with an ℓth root of F, then we obtain the result simply verifying a degree condition on every irreducible component of C. Consider the setting introduced in Remark 2.8: we call v_i the Γ vertex associated to the irreducible components C_i and g_i the genus of the normalization $\overline{\mathsf{C}}_i$. Then we have

$$\deg \mathsf{L}|_{v_i} = r_i + \sum_{e_+=v_i} \frac{M(e)}{\ell},$$

with r_i an integer for all i. Knowing that $(\mathsf{L}|_{v_i})^{\otimes \ell} \cong F|_{v_i}$ and multiplying by ℓ the previous equality, we obtain, as we wanted,

$$\deg F|_{v_i} \equiv \sum_{e_+=v_i} M(e) \mod \ell.$$

To prove the other implication we will show that the multidegree condition implies $\deg F \equiv 0 \mod \ell$, and conclude by Remark 2.2. Indeed, $\deg F = \sum_{v_i \in V(\Gamma)} \deg F|_{v_i}$. Using the condition we obtain

$$\deg F \equiv \sum_{v_i \in V(\Gamma)} \sum_{e_+=v_i} M(e) \equiv \sum_{e \in \mathbb{E}(\Gamma)} M(e) \equiv 0 \mod \ell.$$

□

Consider a rooted curve $(\mathsf{C}, \mathsf{L}, \phi)$ such that the coarse space of C is C. Starting from the dual graph $\Gamma(\mathsf{C})$ and the multiplicity index M of $(\mathsf{C}, \mathsf{L}, \phi)$, consider the new contracted graph $\Gamma_0(\mathsf{C})$ defined by

(1) the vertex set $V_0 = V(C)/\sim$, defined by modding out the relation

$$(e_+ \sim e_- \text{ if } M(e) \equiv 0);$$

(2) the edge set $E_0 = \{e \in E(C) \mid M(e) \not\equiv 0\}$.

Remark 3.5. The graph Γ_0 is obtained by contracting the edges of Γ where the function M vanishes.

Definition 3.6. The pair $(\Gamma_0(\mathsf{C}), M)$, where M is the restriction of the multiplicity index on the contracted edge set, is called *decorated graph* of the curve $(\mathsf{C}, \mathsf{L}, \phi)$. If the cochain M is clear from context, we will refer also to $\Gamma_0(\mathsf{C})$ or Γ_0 alone as the decorated graph.

To study $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ we start from a bigger group, the group $\mathrm{Aut}_C(\mathsf{C})$ containing automorphisms of C fixing the coarse space C. Consider a node n of C whose local picture is $[\{xy = 0\}/\boldsymbol{\mu}_r]$. Consider an automorphism $\eta \in \mathrm{Aut}_C(\mathsf{C})$. The local action of η at n is $(x, y) \mapsto (\zeta x, y) = (x, \zeta y)$, with $\zeta \in \boldsymbol{\mu}_r$. As a consequence of the definition of $\mathrm{Aut}_C(\mathsf{C})$, the action of η outside the C nodes is trivial. Then the whole group $\mathrm{Aut}_C(\mathsf{C})$ is generated by automorphisms of the form $(x, y) \mapsto (\zeta x, y)$ on a node and trivial elsewhere.

We are interested in representing $\mathrm{Aut}_C(\mathsf{C})$ as acting on the edges of the dual graph, thus we introduce the group of functions $\mathbb{E} \to \mathbb{Z}/\ell$ that are even with respect to conjugation

$$S(\Gamma; \mathbb{Z}/\ell) := \{b\colon \mathbb{E} \to \mathbb{Z}/\ell \mid b(\bar{e}) \equiv b(e)\}.$$

We have a canonical identification sending the function $b \in S(\Gamma_0(\mathsf{C}); \mathbb{Z}/\ell)$ to the automorphism η with local action $(x, y) \mapsto (\xi_\ell^{b(e)} x, y)$ on the node associated to the edge e if $M(e) \not\equiv 0$. Therefore the decorated graph encodes the automorphisms acting trivially on the coarse space. We can write

$$\mathrm{Aut}_C(\mathsf{C}) \cong S(\Gamma_0(\mathsf{C}); \mathbb{Z}/\ell) \cong (\mathbb{Z}/\ell)^{E_0(\Gamma)}. \tag{4}$$

We already saw that elements of $C^1(\Gamma; \mathbb{Z}/\ell)$ are odd functions from $\mathbb{E}$ to $\mathbb{Z}/\ell$. Given $g \in S(\Gamma; \mathbb{Z}/\ell)$ and $N \in C^1(\Gamma; \mathbb{Z}/\ell)$, their natural product gN is still an odd function, thus an element of $C^1(\Gamma; \mathbb{Z}/\ell)$.
Given the normalization morphism $\mathsf{nor}\colon C_{\mathsf{nor}} \to C$, consider the short exact sequence of sheaves over C

$$1 \to \mathbb{Z}/\ell \to \mathsf{nor}_* \mathsf{nor}^* \mathbb{Z}/\ell \xrightarrow{t} \mathbb{Z}/\ell|_{\mathrm{Sing}\, C} \to 1.$$

The sections of the central sheaf are $\mathbb{Z}/\ell$-valued functions over C_{nor}. Moreover, the image of a section s by t is the function that assigns to each node the difference between the two values of s on the preimages. The cohomology of this sequence gives the following long exact sequence
(5)

$$1 \to \mathbb{Z}/\ell \to C^0(\Gamma; \mathbb{Z}/\ell) \xrightarrow{\delta} C^1(\Gamma; \mathbb{Z}/\ell) \xrightarrow{\tau} \mathrm{Pic}(C)[\ell] \xrightarrow{\mathsf{nor}^*} \mathrm{Pic}(C_{\mathsf{nor}})[\ell] \to 1.$$

Here, $\mathrm{Pic}(C)[\ell] = H^1(C, \mathbb{Z}/\ell)$ is the subgroup of $\mathrm{Pic}(C)$ of elements of order dividing ℓ, *i.e.* of ℓth roots of the trivial bundle.
We know that

$$\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) = \{[\mathsf{s}, \rho] \in \underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi) |\ \mathsf{s} \in \mathrm{Aut}_C(\mathsf{C})\} = \{\mathsf{s} \in \mathrm{Aut}_C(\mathsf{C}) |\ \mathsf{s}^* \mathsf{L} \cong \mathsf{L}\}.$$

As showed in [6], we have

$$\mathsf{a}^* \mathsf{L} = \mathsf{L} \otimes \tau\left(\mathsf{a} M\right).$$

This means that, if a is an automorphism in $\mathrm{Aut}_C(\mathsf{C})$, the pullback $\mathsf{a}^*\mathsf{L}$ is totally determined by the product $\mathsf{a}M \in \mathbb{C}^1(\Gamma_0; \mathbb{Z}/\ell)$, where a is seen as an element of $S(\Gamma_0; \mathbb{Z}/\ell)$, and M is the multiplicity index of $(\mathsf{C}, \mathsf{L}, \phi)$.
As a consequence we have the following theorem.

THEOREM 3.7. *An element* $\mathsf{a} \in \mathrm{Aut}_C(\mathsf{C})$*, lifts to* $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ *if and only if*

$$\mathsf{a} M \in \mathrm{Ker}(\tau) = \mathrm{Im}\, \delta.$$

We recall the subcomplex $C^i(\Gamma_0(\mathsf{C}); \mathbb{Z}/\ell) \subset C^i(\Gamma(\mathsf{C}); \mathbb{Z}/\ell)$. Moreover, if δ_0 is the δ operator on $C^i(\Gamma_0)$, *i.e.* the restriction of the δ operator to this space, from Proposition 1.15 we know that $\mathrm{Im}(\delta_0) = C^1(\Gamma_0) \cap \mathrm{Im}\, \delta$.

Remark 3.8. As a consequence of Theorem 3.7, we get the canonical identification

$$\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) = \mathrm{Im}(\delta_0)$$

via the multiplication $\mathsf{a} \mapsto \mathsf{a}M$. Because of Proposition 1.7 we also have $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) = M \cdot G(\Gamma_0; \mathbb{Z}/\ell)$.

Remark 3.9. In the first section we obtained a characterization of the cochains in $\mathrm{Im}(\delta)$ that we could restate in our new setting. Indeed, because of Proposition 1.10, an automorphism $\mathsf{a} \in S(\Gamma_0; \mathbb{Z}/\ell)$ is an element of $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ if and only if for every circuit $(e_1, \dots, e_k)$ in Γ_0 we have $\sum_{i=1}^k \mathsf{a}(e) \equiv 0 \mod \ell$.

We can characterize the automorphisms in $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ that are quasireflections. We will regard every automorphism $\mathsf{a} \in \underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ as the corresponding element in $S(\Gamma_0; \mathbb{Z}/\ell)$. This representation is good to investigate the action of a on $\mathrm{Def}(\mathsf{C})$. The action of a is non-trivial only on those coordinates of $\mathrm{Def}(\mathsf{C})$ associated to the nodes of C with non-trivial stabilizer. In particular, if t_{n} is the coordinate associated to the node n and e is the edge of $\Gamma_0(\mathsf{C})$ associated to n, then

$$\mathsf{a}\colon t_{\mathsf{n}} \mapsto \xi_\ell^{\mathsf{a}(e)} \cdot t_{\mathsf{n}}.$$

Proposition 3.10. *An automorphism $\mathsf{a} \in \mathrm{Aut}_C(\mathsf{C})$ is a quasireflection in $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ if and only if $\mathsf{a}(e) \equiv 0$ for all edges but one that is a separating edge of $\Gamma_0(\mathsf{C})$.*

Proof. If a is a quasireflection, the action on all but one of the coordinates must be trivial. Therefore $\mathsf{a}(e) \equiv 0$ on all the edges but one, say e_1. If e_1 is in any circuit $(e_1, \dots, e_k)$ of Γ_0 with $k \geq 1$, we have, by Remark 3.9, that $\sum \mathsf{a}(e_i) \equiv 0$. As $\mathsf{a}(e_1) \not\equiv 0$, there exists $i > 1$ such that $\mathsf{a}(e_i) \not\equiv 0$, contradiction. Thus e_1 is not in any circuit, then it is a separating edge.
Reciprocally, consider an automorphism $\mathsf{a} \in S(\Gamma_0; \mathbb{Z}/\ell)$ such that there exists an oriented separating edge e_1 with the property that $\mathsf{a}(e) \equiv 0$ for all $e \in \mathbb{E}\backslash\{e_1, \bar{e}_1\}$ and $\mathsf{a}(e_1) \not\equiv 0$. Then for every circuit $(e'_1, \dots, e'_k)$ we have $\sum \mathsf{a}(e'_i) \equiv 0$ and so a is in $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$. □

Definition 3.11. We call $\mathrm{QR}(\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi))$ or simply $\mathrm{QR}(\mathsf{C}, \mathsf{L}, \phi)$ the group generated by quasireflections automorphism of the curve $(\mathsf{C}, \mathsf{L}, \phi)$. We call $\mathrm{QR}(\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi))$ or simply $\mathrm{QR}_C(\mathsf{C}, \mathsf{L}, \phi)$ the group generated by ghost quasireflections.

Remark 3.12. If we note $E_{\mathrm{sep}} \subset E_0$ the subset of separating edges of Γ_0, Proposition 3.10 gives a simple description of the group $\mathrm{QR}_C(\mathsf{C}, \mathsf{L}, \phi)$.

$$\mathrm{QR}_C(\mathsf{C}, \mathsf{L}, \phi) \cong (\mathbb{Z}/\ell)^{E_{\mathrm{sep}}} \subset S(\Gamma_0; \mathbb{Z}/\ell).$$

Theorem 3.13. *The group $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ is generated by quasireflections if and only if the graph Γ_0 is tree-like.*

Proof. After the previous remark,

$$\mathrm{QR}_C(\mathsf{C},\mathsf{L},\phi) \cong (\mathbb{Z}/\ell)^{E_{\mathrm{sep}}} \subset \mathrm{Im}(\delta_0) \cong \underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\phi).$$

We know that $\mathrm{Im}(\delta_0) \cong \mathbb{Z}/\ell^{\#V_0-1}$ and thus the inclusion is an equality if and only if $\#E_{\mathrm{sep}} = \#V_0 - 1$. By Lemma 1.12 we conclude. □

REMARK 3.14. We show a generalization of the previous result to every ℓ, and to do this we update our tools following [8, §2.4]. First of all we generalize the notion of multiplicity index: given the local multiplicity $m \in \mathbb{Z}/r$ at a certain node n, we know that r divides $m\ell$, we define $M(e) := m\ell/r \in \mathbb{Z}/\ell$, where $e \in \mathbb{E}$ is an oriented edge. If ℓ is prime, $M \in C^1(\Gamma;\mathbb{Z}/\ell)$ coincide with the previous definition. We also generalize contracted graphs. Consider the factorization of ℓ in prime numbers, $\ell = \prod p^{e_p}$, where $e_p = \nu_p(\ell)$ is the p-adic valuation of ℓ. Given the dual graph $\Gamma(\mathsf{C}) = \Gamma$, for every prime p we define $\Gamma(\nu_p^k)$ contracting every edge e of Γ such that p^k divides $M(e)$. We will note $\Gamma_p(\mathsf{C}) := \Gamma(\nu_p^{e_p})$. We have the chain of contraction already introduced in [8]:

$$\Gamma \to \Gamma_p = \Gamma(\nu_p^{e_p}) \to \Gamma(\nu_p^{e_p-1}) \to \cdots \to \Gamma(\nu_p^1) \to \{\cdot\}.$$

Moreover, we introduce $S_d(\Gamma;\mathbb{Z}/\ell) \subset S(\Gamma;\mathbb{Z}/\ell)$, the group of even functions $f\colon \mathbb{E} \to \mathbb{Z}/\ell$ such that $f(e) = f(\bar{e}) \in \mathbb{Z}/r(e)$, and $C^1_d(\Gamma;\mathbb{Z}/\ell) \subset C^1(\Gamma;\mathbb{Z}/\ell)$, the group of odd functions $N\colon \mathbb{E} \to \mathbb{Z}/\ell$ such that $N(e) = -N(\bar{e}) \in \mathbb{Z}/r(e)$. We point out that if r divides ℓ, we will look at $\mathbb{Z}/r$ as immersed in $\mathbb{Z}/\ell$, indeed $\mathbb{Z}/r = (\ell/r)\cdot\mathbb{Z}/r \subset \mathbb{Z}/\ell$. In particular for every prime p, $\mathbb{Z}/p \subset \mathbb{Z}/p^2 \subset \mathbb{Z}/p^3 \subset \cdots$.

THEOREM 3.15. *Consider a rooted curve $(\mathsf{C},\mathsf{L},\phi)$. The groups $\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\phi)$ is generated by quasireflections if and only if the graphs $\Gamma_p(\mathsf{C})$ are tree-like for every prime p dividing ℓ.*

Proof. As in the case with ℓ prime, $S_d(\Gamma;\mathbb{Z}/\ell)$ is canonically identified with $\mathrm{Aut}_C(\mathsf{C}) = \bigoplus_{e\in E}\mathbb{Z}/r(e)$. Given the multiplicity index M of $(\mathsf{C},\mathsf{L},\phi)$, as before we have that a lifts to $\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\phi)$ if and only if $\mathsf{a}M$ is in $\mathrm{Im}\,\delta$. Thus $\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\phi)$ is canonically identified with $C^1_d(\Gamma;\mathbb{Z}/\ell) \cap \mathrm{Im}\,\delta$.
We introduce the p-adic valuation on edges as in [8, §2.4.2]: given $e \in \mathbb{E}$, $\nu_p(e) := \mathrm{val}_p(M(e) \mod p^{e_p})$. We consider also the function $\bar{\nu}_p$ such that $\bar{\nu}_p(e) := \min(e_p, \nu_p(e))$. We observe that, by definition M, for every oriented edge $e \in \mathbb{E}(\Gamma)$ the local stabilizer order is

$$r(e) = \prod_{p|\ell} p^{e_p - \bar{\nu}_p(e)}.$$

We define one last new object. We have a canonical immersion $C^i(\Gamma(\nu_p^k);\mathbb{Z}/p^{e_p-k+1}) \hookrightarrow C^i(\Gamma;\mathbb{Z}/\ell)$ for $i = 0,1$. We define δ_p^k as the restriction of the delta operator on the group $C^0(\Gamma(\nu_p^k);\mathbb{Z}/p^{e_p-k+1})$ for all p in the factorization of ℓ and for all k between 1 and e_p.

As in Lemma 3.10, in the composite case $\mathsf{a} \in \mathrm{Aut}_C(\mathsf{C})$ is a quasireflection in $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ if and only if $\mathsf{a}(e) \equiv 0$ for all edges but one that is a separating edge. This allows the following decomposition

$$\mathrm{QR}_C(\mathsf{C}, \mathsf{L}, \phi) = \bigoplus_{e \in E_{\mathrm{sep}}(\Gamma)} \mathbb{Z}/r(e) = \bigoplus_{e \in E_{\mathrm{sep}}} \bigoplus_{p|\ell} \mathbb{Z}/p^{e_p - \bar{\nu}_p(M(e))} = \bigoplus_{p|\ell} \bigoplus_{k=1}^{e_p} (\mathbb{Z}/p^k)^{\beta_p^k},$$

where $\beta_p^k := \#E_{\mathrm{sep}}\left(\Gamma(\nu_p^{e_p-k+1})\right) - \#E_{\mathrm{sep}}\left(\Gamma(\nu_p^{e_p-k})\right)$ if $k < e_p$ and $\beta_p^{e_p} := \#E_{\mathrm{sep}}\left(\nu_p^1\right)$. Following [8, Lemma 2.22], we have a similar decomposition for $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$:

$$\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) = C_d^1(\Gamma; \mathbb{Z}/\ell) \cap \mathrm{Im}\,\delta = \bigoplus_{p|\ell} \sum_{k=1}^{e_p} \mathrm{Im}\,\delta_p^k = \bigoplus_{p|\ell} \bigoplus_{k=1}^{e_p} (\mathbb{Z}/p^k)^{\alpha_p^k},$$

where $\alpha_p^k := \#V\left(\Gamma(\nu_p^{e_p-k+1})\right) - \#V\left(\Gamma(\nu_p^{e_p-k})\right)$ $\forall k \geq 0$. We observe that $\alpha_p^k \geq \beta_p^k$ for all p dividing ℓ and $k \geq 0$. Moreover, $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ coincide with $\mathrm{QR}_C(\mathsf{C}, \mathsf{L}, \phi)$ if and only if $\alpha_p^k = \beta_p^k$ for all p and k. Fixing p, this is equivalent to impose $\sum_k \beta_p^k = \sum_k \alpha_p^k$. In the previous expression the left hand side is $\#E_{\mathrm{sep}}(\Gamma_p)$ and the right hand side is $\#V(\Gamma_p) - 1$, we saw in Lemma 1.12 that the equality is achieved if and only if Γ_p is tree-like. $\square$

4. The singular locus of $\overline{\mathcal{R}}_{g,\ell}^k$

After the analysis of quasireflections on $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$, we complete the description of quasireflections on the whole automorphism groups. What follows is true for every ℓ.

Lemma 4.1. *Consider an element* q *of* $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$. *It acts as a quasireflection on* $\mathrm{Def}(\mathsf{C})$ *if and only if one of the following is true:*

1. *the automorphism* q *is a ghost quasireflection, i.e. an element of* $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ *which moreover operates as a quasireflection;*
2. *the automorphism* q *is an ETQR, using the generalized Definition 2.15.*

Proof. We first prove the "only if" part. Consider $\mathsf{q} \in \underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$, we call q its coarsening. If q acts trivially on certain coordinates of $\mathrm{Def}(\mathsf{C})$, *a fortiori* we have that q acts trivially on the corresponding coordinates of $\mathrm{Def}(C)$. So q acts as the identity or as a quasireflection. In the first case, q is a ghost automorphism and we are in case (1). If q acts as a quasireflection, then it is a classical ETQR as we pointed out on Theorem 2.13, and it acts non-trivially on the elliptic tail node n associated to an elliptic tail. It remains to know the action of q on the nodes, other than n, with non-trivial stabilizer. The action in these nodes must be trivial, because every non-trivial action induces a non-trivial action also on the associated coordinate of the node. Therefore, the q restriction to the elliptic tail has to be the canonical involution i_0 (see Remark 2.14). By Definition 2.15 this implies that q is an ETQR of $(\mathsf{C}, \mathsf{L}, \phi)$.

For the "if" part, we observe that a ghost quasireflection is automatically a quasireflection. It remains to prove the point (2). By definition of ETQR its action can be non-trivial only on the separating node of the tail. The local coarse picture of the node is $\{xy = 0\}$, where $y = 0$ is the branch lying on the elliptic tail. Then the action of i on the coarse space is $(x, y) \mapsto (-x, y)$. Therefore the action is *a fortiori* non trivial on the coordinate associated to the stack node n. $\square$

Lemma 4.2. *If* $\mathrm{QR}(\mathrm{Aut}(C))$ *(also called* $\mathrm{QR}(C)$*) is the group generated by ETQRs inside* $\mathrm{Aut}(C)$*, then any element* $q \in \mathrm{QR}(C)$ *which could be lifted to* $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$*, has a lifting in* $\mathrm{QR}(\mathsf{C}, \mathsf{L}, \phi)$*, too.*

Proof. By definition, $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ is the set of automorphisms $\mathsf{s} \in \mathrm{Aut}(\mathsf{C})$ such that $\mathsf{s}^*\mathsf{L} \cong \mathsf{L}$. Consider $q \in \mathrm{QR}(C)$ such that its decomposition in quasireflections is $q = i_0 i_1 \cdots i_m$, and i_k is an ETQR acting non-trivially on an elliptic tail E_k. Any lifting of q is in the form $\mathsf{q} = \mathsf{i}_0\mathsf{i}_1 \cdots \mathsf{i}_m \cdot \mathsf{a}$, where i_k is a (generalized) ETQR acting non-trivially on E_k, and a is a ghost acting non-trivially only on nodes outside the tails E_k. We observe that every i_k is a lifting in $\mathrm{Aut}(\mathsf{C})$ of i_k, we are going to prove that moreover $\mathsf{i}_k \in \underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$. By construction, $\mathsf{q}^*\mathsf{L} \cong \mathsf{L}$ if and only if $\mathsf{i}_k^*\mathsf{L} \cong \mathsf{L}$ for all k and $\mathsf{a}^*\mathsf{L} \cong \mathsf{L}$. This implies that every i_k lives in $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$, then $\mathsf{q}\mathsf{a}^{-1}$ is a lifting of q living in $\mathrm{QR}(\mathsf{C}, \mathsf{L}, \phi)$. $\square$

Remark 4.3. We recall the short exact sequence

$$1 \to \underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) \xrightarrow{h} \underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi) \xrightarrow{p} \mathrm{Aut}'(C) \to 1$$

and introduce the group $\mathrm{QR}'(C) \subset \mathrm{Aut}'(C)$, generated by liftable quasireflections. Knowing that $p(\mathrm{QR}(\mathsf{C}, \mathsf{L}, \phi)) \subset \mathrm{QR}'(C) \subset \mathrm{Aut}'(C) \cap \mathrm{QR}(C)$, the previous lemma shows that $\mathrm{QR}'(C) = p(\mathrm{QR}(\mathsf{C}, \mathsf{L}, \phi))$. Using also Lemma 4.1, we obtain that the following is a short exact sequence

$$1 \to \mathrm{QR}_C(\mathsf{C}, \mathsf{L}, \phi) \to \mathrm{QR}(\mathsf{C}, \mathsf{L}, \phi) \to \mathrm{QR}'(C) \to 1.$$

Theorem 4.4. *The group* $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ *is generated by quasireflections if and only if both* $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ *and* $\mathrm{Aut}'(C)$ *are generated by quasireflections.*

Proof. After the previous remark, the following is a short exact sequence

$$1 \to \underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)/\, \mathrm{QR}_C(\mathsf{C}, \mathsf{L}, \phi) \to$$
$$\to \underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)/\, \mathrm{QR}(\mathsf{C}, \mathsf{L}, \phi) \to \mathrm{Aut}'(C)/\, \mathrm{QR}'(C) \to 1.$$

The theorem follows. $\square$

After Theorem 4.4 and 3.13, we have the following characterization of the singular locus inside $\overline{\mathcal{R}}^k_{g,\ell}$.

Theorem 4.5. *Let* $g \geq 4$ *and* ℓ *a prime number. Given a rooted curve* $(\mathsf{C}, \mathsf{L}, \phi)$*, with* C *coarse space of* C*, the point* $[\mathsf{C}, \mathsf{L}, \phi] \in \overline{\mathcal{R}}^k_{g,\ell}$ *is smooth if and only if*

$\mathrm{Aut}'(C)$ *is generated by ETQRs of* C *and the contracted graphs* $\Gamma_0(\mathsf{C})$ *is tree-like.*

After Remark 3.14, we can generalize the previous theorem to all ℓ. It suffices to consider contracted graphs $\Gamma_p(\mathsf{C})$ for every prime p dividing ℓ

Theorem 4.6. *For any* $g \geq 4$ *and* ℓ *positive integer. The point* $[\mathsf{C}, \mathsf{L}, \phi]$ *is smooth if and only if* $\mathrm{Aut}'(C)$ *is generated by ETQRs of* C *and the* $\Gamma_p(\mathsf{C})$ *are tree-like.*

Remark 4.7. The theorem above is a generalization of [8, Theorem 2.28]. In particular Chiodo and Farkas proved that in the case $k = 0$, the moduli of level curves, the contracted graphs Γ_p are bouquet if $[\mathsf{C}, \mathsf{L}, \phi]$ is smooth. A bouquet is a graph with only one vertex, or equivalently a tree-like graph with no non-loop edges. This happens because in the case $k = 0$, every separating node must have trivial stabilizer, *i.e.* the M cochain must be 0 and therefore they disappear after the contraction. In the general case, M could be non-zero on separating edges too, and the theorem is still true using the more general notion of tree-like graph.

4.1. The locus of singular points via a new stratification. As we saw, the information about the automorphism group of a certain rooted curve $(\mathsf{C}, \mathsf{L}, \phi)$ is coded by its dual decorated graph $(\Gamma_0(\mathsf{C}), M)$. It is therefore quite natural to introduce a stratification of $\overline{\mathcal{R}}^k_{g,\ell}$ using this notion. For this, we extend the notion of graph contraction: if $\Gamma'_0 \to \Gamma'_1$ is a usual graph contraction, the ring $C^1(\Gamma'_1; \mathbb{Z}/\ell)$ is naturally immersed in $C^1(\Gamma'_0; \mathbb{Z}/\ell)$, then the contraction of a pair (Γ'_0, M'_0) is the pair (Γ'_1, M'_1) where the cochain M'_1 is the restriction of M'_0. If it is clear from the context, we could refer to the decorated graph restriction simply with the graph contraction $\Gamma'_0 \to \Gamma'_1$.

Definition 4.8. Given a decorated graph (Γ, M) with $M \in C^1(\Gamma; \mathbb{Z}/\ell)$, consider this locus of $\overline{\mathcal{R}}^k_{g,\ell}$:

$$\mathcal{S}_{(\Gamma,M)} := \Big\{ [\mathsf{C}, \mathsf{L}, \phi] \in \overline{\mathcal{R}}^k_{g,\ell} : \ \Gamma_0(\mathsf{C}) = \Gamma, \\ \text{and } M \text{ is the multiplicity index of } (\mathsf{C}, \mathsf{L}, \phi) \Big\}.$$

These loci are the strata of our stratification. We can find a first link between the decorated graph and geometric properties of the stratum.

Proposition 4.9. *If we consider the codimension of* $\mathcal{S}_{(\Gamma,M)}$ *inside* $\overline{\mathcal{R}}^k_{g,\ell}$*, we have*

$$\operatorname{Codim} \mathcal{S}_{(\Gamma,M)} = \#E(\Gamma).$$

Proof. We take a general point $[\mathsf{C}, \mathsf{L}, \phi]$ of stratum $\mathcal{S}_{(\Gamma,M)}$, it has $\#V(\Gamma)$ irreducible components $\mathsf{C}_1, \dots, \mathsf{C}_{\#V}$. We call g_i the genus of C_i and k_i the number of nodes of C on this component. Then we have $\sum k_i = 2\#E(\Gamma)$. We obtain that the dimension of $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)$ is

$$\dim \mathrm{Def}(\mathsf{C},\mathsf{L},\phi) = \sum_{i=1}^{\#V}(3g_i - 3 + k_i) = 3\sum g_i - 3\#V + 2\#E.$$

Thus, because of Remark 2.9, we have $\dim \mathrm{Def}(\mathsf{C},\mathsf{L},\phi) = 3g-3-\#E$, where g is the genus of C. The result on the codimension follows. $\square$

Using contraction, we have this description of the closed strata.

$$\overline{\mathcal{S}}_{(\Gamma_1',M_1')} = \left\{ [\mathsf{C},\mathsf{L},\phi] \in \overline{\mathsf{R}}^k_{g,\ell} : \begin{array}{l} \text{if } (\Gamma_0, M) \text{ is the decorated graph of } (\mathsf{C},\mathsf{L},\phi), \\ \text{there exists a contraction } (\Gamma_0,M) \to (\Gamma_1',M_1') \end{array} \right\}.$$

We introduce two closed loci of $\overline{\mathcal{R}}^k_{g,\ell}$,

$$N^k_{g,\ell} := \left\{ [\mathsf{C},\mathsf{L},\phi] |\ \mathrm{Aut}'(C) \text{ is not generated by ETQRs} \right\},$$

$$H^k_{g,\ell} := \{ [\mathsf{C},\mathsf{L},\phi] |\ \underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\phi) \text{ is not generated by quasireflections} \}.$$

Equivalently, $[\mathsf{C},\mathsf{L},\phi] \in H^k_{g,\ell}$ if and only if $\Gamma_0(\mathsf{C})$ is not tree-like. We have by Theorem 4.5 that the singular locus $\mathrm{Sing}\,\overline{\mathcal{R}}^k_{g,\ell}$ is their union

$$\mathrm{Sing}\,\overline{\mathcal{R}}^k_{g,\ell} = N^k_{g,\ell} \cup H^k_{g,\ell}.$$

Remark 4.10. Consider the natural projection $\pi\colon \overline{\mathcal{R}}^k_{g,\ell} \to \overline{\mathcal{M}}_g$, we observe that

$$N^k_{g,\ell} \subset \pi^{-1}\,\mathrm{Sing}\,\overline{\mathcal{M}}_g.$$

Indeed, after Remark 4.3, $\mathrm{QR}'(C) = \mathrm{Aut}'(C) \cap \mathrm{QR}(C)$ and therefore $\mathrm{Aut}(C) = \mathrm{QR}(C)$ if and only if $\mathrm{Aut}'(C) = \mathrm{QR}'(C)$. This implies that $\left(\pi^{-1}\,\mathrm{Sing}\,\overline{\mathcal{M}}_g\right)^c \subset (N^k_{g,\ell})^c$, and taking the complementary we have the result.

The stratification introduced is particularly useful in describing the “new” locus $H^k_{g,\ell}$. We recall the definition of *vine* curves.

Definition 4.11. We note $\Gamma_{(2,n)}$ a graph with two vertices linked by n edges. A curve C is an n-vine curve if $\Gamma(\mathsf{C})$ contracts to $\Gamma_{(2,n)}$ for some $n \geq 2$. Equivalently an n-vine curve has coarse space $C = \mathsf{C}_1 \cup C_2$ which is the union of two curves intersecting each other n times.

The next step shows that every H-curve has a dual graph which contracts to $\Gamma_{(2,n)}$.

Lemma 4.12. *If $[\mathsf{C},\mathsf{L},\phi] \in \mathrm{Sing}\,\overline{\mathcal{R}}^k_{g,\ell}$ is a point in $H^k_{g,\ell}$ whose decorated graph is (Γ_0, M), then there exists $n \geq 2$ and a graph contraction $\Gamma_0 \to \Gamma_{(2,n)}$. Equivalently C is an n-vine curve for some $n \geq 2$.*

Proof. If $[\mathsf{C},\mathsf{L},\phi] \in H^k_{g,\ell}$, by definition Γ_0 contains a circuit that is not a loop. We will show an edge contraction $\Gamma_0 \to \Gamma_{(2,n)}$ for some $n \geq 2$.
Consider two different vertices v_1 and v_2 that are consecutive on a non-loop circuit $K \subset \Gamma_0$. Now consider a partition $V = V_1 \bigsqcup V_2$ of the vertex set such

that $v_1 \in V_1$ and $v_2 \in V_2$. This defines an edge contraction $\Gamma_0 \to \Gamma_{(2,n)}$ where $e \in E(\Gamma)$ is contracted if its two extremities lie in the same V_i. As K is a circuit, necessarily $n \geq 2$ and the theorem is proved. $\square$

We conclude that $H^k_{g,\ell}$ is the union of the closed strata associated to vine graphs.

$$H^k_{g,\ell} = \bigcup_{\substack{n\geq 2 \\ M\in C^1(\Gamma_{(2,n)};\mathbb{Z}/\ell)}} \overline{\mathcal{S}}_{(\Gamma_{(2,n)},M)}.$$

5. Non-canonical singularities of $\overline{\mathcal{R}}^k_{g,\ell}$

After the description of the singular locus of $\overline{\mathcal{R}}^k_{g,\ell}$, in this section we find the locus of non-canonical singularities $\operatorname{Sing}^{\mathrm{nc}} \overline{\mathcal{R}}^k_{g,\ell}$. To do this we introduce the age function and point out some basic facts about quotient singularities: the characterization of the non-canonical locus will follow from the age criterion 5.3. Also, for some small values of ℓ, we will develop a description of $\operatorname{Sing}^{\mathrm{nc}} \overline{\mathcal{R}}^k_{g,\ell}$ in terms of the stratification introduced above.

5.1. The age criterion. If G is a finite group, the age is an additive positive function from the representations ring of G to rational numbers,

$$\text{age}\colon \operatorname{Rep}(G) \to \mathbb{Q}.$$

First consider the group $\mathbb{Z}/r$ for any r positive integer. Given the character $\boldsymbol{k}$ such that $1 \mapsto k \in \mathbb{Z}/r$, we define $\text{age}(\boldsymbol{k}) = k/r$. These characters are a basis for $\operatorname{Rep}(\mathbb{Z}/r)$, then we can extend age over all the representation ring.
Consider a G-representation $\rho\colon G \to \mathrm{GL}(V)$, the age function could be defined on any injection $i\colon \mathbb{Z}/r \hookrightarrow G$ simply composing the injection with ρ.

$$\text{age}_V\colon i \mapsto \text{age}(\rho \circ i).$$

Age can finally be defined on the group G by

$$G \xrightarrow{f} \bigsqcup_{r\geq 1} \{i |\ i\colon \boldsymbol{\mu}_r \hookrightarrow G\} \xrightarrow{\text{age}_V} \mathbb{Q},$$

where f is the set bijection sending $g \in G$, element of order r, to the injection obtained by mapping $1 \in \mathbb{Z}/r$ to g.

Definition 5.1 (junior and senior groups). A finite group $G \subset \mathrm{GL}(\mathbb{C}^m)$ that contains no quasireflections is called junior if the image of the age function intersects the open interval $]0,1[$,

$$\text{age}\, G \cap\ ]0,1[\ \neq \varnothing,$$

and senior otherwise.

Remark 5.2. All the functions we have introduced are canonically defined except for the identification f, that depends on the choice of the primitive root $\xi_r = \exp(2\pi i/r)$. Moreover, the image of age: $G \to \mathbb{Q}$ does not depend on this choice, then our previous choice of a root system does not affect the study of junior and senior groups and will continue unchanged.

Theorem 5.3 (age criterion, [16]). *If $G \subset \mathrm{GL}(\mathbb{C}^m)$ is a finite subgroup without quasireflections, the singularity $\mathbb{C}^m/G$ is canonical if and only if G is a senior group.*

Remark 5.4. To see the age explicitly, for $g \in G \subset \mathrm{GL}(\mathbb{C}^m)$ with G finite subgroup and $\mathrm{ord}(g) = r$, consider a basis of $\mathbb{C}^m$ such that $g = \mathrm{Diag}(\xi_r^{a_1}, \dots, \xi_r^{a_m})$. In this setting $\mathrm{age}(g) = \frac{1}{r}\sum a_i$.

5.2. The non-canonical locus. We know that any point $[\mathsf{C}, \mathsf{L}, \phi] \in \overline{\mathcal{R}}^k_{g,\ell}$ has a neighborhood isomorphic to the quotient $\mathrm{Def}(\mathsf{C})/\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$, then after the age criterion 5.3 we are searching for junior automorphism in $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$. We also need the denominator group to be quasireflections free to use the criterion, so we will repeatedly use the following result.

Proposition 5.5 (see [15]). *Consider the finite subgroup $G \subset \mathrm{GL}(\mathbb{C}^m)$ and the group $\mathrm{QR}(G)$ generated by G quasireflections. There exists an isomorphism $\varphi\colon \mathbb{C}^m/\mathrm{QR}(G) \to \mathbb{C}^m$ and a subgroup $K \subset \mathrm{GL}(\mathbb{C}^m)$ isomorphic to $G/\mathrm{QR}(G)$ such that the following diagram is commutative.*

$$\begin{array}{ccccc}
\mathbb{C}^m & \longrightarrow & \mathbb{C}^m/\mathrm{QR}(G) & \xrightarrow{\varphi} & \mathbb{C}^m \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{C}^m/G & \xrightarrow{\cong} & (\mathbb{C}^m/\mathrm{QR}(G))/(G/\mathrm{QR}(G)) & \xrightarrow{\cong} & \mathbb{C}^m/K
\end{array}$$

We introduce two closed loci which are central in our description.

Definition 5.6 (T-curves). A rooted curve $(\mathsf{C}, \mathsf{L}, \phi)$ is a T-curve if there exists an automorphism $\mathsf{a} \in \underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ such that its coarsening a is an elliptic tail automorphism of order 6. The locus of T-curves in $\overline{\mathcal{R}}^k_{g,\ell}$ is noted $T^k_{g,\ell}$.

Definition 5.7 (J-curves). A rooted curve $(\mathsf{C}, \mathsf{L}, \phi)$ is a J-curve if the group

$$\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)/\mathrm{QR}_C(\mathsf{C}, \mathsf{L}, \phi),$$

which is the group of ghosts quotiented by its subgroup of quasireflections, is junior. The locus of J-curve in $\overline{\mathcal{R}}^k_{g,\ell}$ is noted $J^k_{g,\ell}$.

Theorem 5.8. *For $g \geq 4$, the non-canonical locus of $\overline{\mathcal{R}}^k_{g,\ell}$ is formed by T-curves and J-curves, i.e. it is the union*

$$\mathrm{Sing}^{\mathrm{nc}}\,\overline{\mathcal{R}}^k_{g,\ell} = T^k_{g,\ell} \cup J^k_{g,\ell}.$$

Remark 5.9. We observe that Theorem 2.44 of Chiodo and Farkas [8], affirms exactly that in the case $k = 0$, $\ell \leq 6$ and $\ell \neq 5$, the J-locus $J^0_{g,\ell}$ is empty for every genus g, and therefore $\mathrm{Sing}^{\mathrm{nc}}\,\overline{\mathcal{R}}^0_{g,\ell}$ coincides with the T-locus for these values of ℓ.

To show Theorem 5.8 we will prove a stronger proposition.

Proposition 5.10. *Given a rooted curve* $(\mathsf{C},\mathsf{L},\phi)$ *of genus* $g\geq 4$ *which is not a* J*-curve, if* $\mathsf{a}\in\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L},\phi)/\,\mathrm{QR}(\mathsf{C},\mathsf{L},\phi)$ *is a junior automorphism, then its coarsening* a *is an elliptic tail automorphism of order* 3 *or* 6*.*

Proof. We introduce the notion of $\star$-smoothing, following [12] and [13].

Definition 5.11. Consider a rooted curve $(\mathsf{C},\mathsf{L},\phi)$, suppose $\mathsf{a}\in\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L},\phi)$ is an automorphisms such that there exists a cycle of m non-separating nodes $\mathsf{n}_0,\dots,\mathsf{n}_{m-1}$, *i.e.* we have $\mathsf{a}(\mathsf{n}_i)=\mathsf{n}_{i+1}$ for all $i=0,1\dots,m-2$ and $\mathsf{a}(\mathsf{n}_{m-1})=\mathsf{n}_0$. The triple is $\star$-smoothable if and only if the action of a^m over the coordinate associated to every node is trivial. This is equivalent to ask $\mathsf{a}^m(t_{\mathsf{n}_i})=t_{\mathsf{n}_i}$ for any $i=0,1,\dots,m-2$, where t_{n_i} is the coordinate associated to node n_i.

If $(\mathsf{C},\mathsf{L},\phi)$ is $\star$-smoothable, there exists a deformation of $(\mathsf{C},\mathsf{L},\phi)$ together with an automorphism a and smoothing the m nodes of the cycle. Moreover, this deformation preserves the age of the a-action on $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)/\,\mathrm{QR}$. Indeed, the eigenvalues of a are a discrete and locally constant set, thus constant by deformation. As the T-locus and the J-locus are closed by $\star$-smoothing, we can suppose, as an additional hypothesis of Theorem 5.8, that our curves are $\star$-*rigid*, *i.e.* non-$\star$-smoothable. ¿From this point we suppose that $(\mathsf{C},\mathsf{L},\phi)$ is $\star$-rigid.

We will show in eight steps that if the group

$$\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L},\phi)/\,\mathrm{QR}\,(\mathsf{C},\mathsf{L},\phi)$$

is junior, and $(\mathsf{C},\mathsf{L},\phi)$ is not a J-curve, then it is a T-curve. After the age criterion 5.3 and Proposition 5.5, this will prove Theorem 5.8. ¿From now on we work under the hypothesis that $\mathsf{a}\in\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\phi)/\,\mathrm{QR}$ is a non-trivial automorphism aged less than 1, and $(\mathsf{C},\mathsf{L},\phi)$ is not a J-curve.

Step 1. Consider the decorated graph (Γ_0,M) of $(\mathsf{C},\mathsf{L},\phi)$. As before, we call E_{sep} the set of separating edges of Γ_0. Then, following Remark 2.8, we can estimate the age by a splitting of the form

$$\bigoplus_{e\in E_{\mathrm{sep}}}\mathbb{A}_{t_e}\oplus\bigoplus_{e'\in E\setminus E_{\mathrm{sep}}}\mathbb{A}_{t_{e'}}\oplus\bigoplus_{i=1}^{m}H^1(\overline{\mathsf{C}}_i,T_{\overline{\mathsf{C}}_i}(-D_i)),$$

where t_e is a coordinate parametrizing the smoothing of the node associated to edge e, and the curves $\overline{\mathsf{C}}_i$ are the normalizations of the irreducible components of C.

Every automorphism in $\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L},\phi)$ fixes the three summand in the sum above. Moreover, every quasireflection acts only on the first summand, then, by Propostion 5.5, the group $\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L},\phi)/\,\mathrm{QR}$ acts on

$$\left(\frac{\bigoplus_{e\in E_{\mathrm{sep}}}\mathbb{A}_{t_e}}{\mathrm{QR}(\mathsf{C},\mathsf{L},\phi)}\right)\oplus\bigoplus_{e'\in E\setminus E_{\mathrm{sep}}}\mathbb{A}_{t_{e'}}\oplus\bigoplus_{i=1}^{m}H^1(\overline{\mathsf{C}}_i,T_{\overline{\mathsf{C}}_i}(-D_i)).\tag{6}$$

Every quasireflection acts on exactly one coordinate t_e with $e \in E_{\text{sep}}$. We rescale all the coordinates t_e by the action of $\text{QR}(\mathsf{C}, \mathsf{L}, \phi)$. We call $\tilde{t}_e$, for $e \in E(\Gamma_0)$, the new set of coordinates. Obviously $\tilde{t}_{e'} = t_{e'}$ if $e' \in E(\Gamma_0) \backslash E_{\text{sep}}$.

Step 2. We show two lemmata about the age contribution of the a-action on nodes, which we will call *aging* on nodes.

Definition 5.12 (coarsening order). If $\mathsf{a} \in \underline{\text{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ and a is its coarsening, then we define

$$\text{c-ord}\,\mathsf{a} := \text{ord}\,a.$$

The coarsening order is the least integer m for which a^m is a ghost automorphism.

Lemma 5.13. *Suppose that $\mathsf{Z} \subset \mathsf{C}$ is a subcurve of C such that $\mathsf{a}(\mathsf{Z}) = \mathsf{Z}$ and $\mathsf{n}_0, \dots \mathsf{n}_{m-1}$ is a cycle, by a, of nodes in Z. Then we have the following inequalities:*

(1) $\text{age}(\mathsf{a}) \geq \frac{m-1}{2}$*;*
(2) if the nodes are non-separating, $\text{age}(\mathsf{a}) \geq \frac{m}{\text{ord}(\mathsf{a}|_{\mathsf{Z}})} + \frac{m-1}{2}$*;*
(3) if $\mathsf{a}^{\text{c-ord}\,\mathsf{a}}$ is a senior ghost, we have $\text{age}(\mathsf{a}) \geq \frac{1}{\text{c-ord}(\mathsf{a})} + \frac{m-1}{2}$.

Proof. We call $\tilde{t}_0, \tilde{t}_1, \dots, \tilde{t}_{m-1}$ the coordinates associated respectively to nodes $\mathsf{n}_0, \dots, \mathsf{n}_{m-1}$. By hypothesis, $\mathsf{a}(\tilde{t}_0) = c_1 \cdot \tilde{t}_1$ and $\mathsf{a}^i(\tilde{t}_0) = c_i \cdot \tilde{t}_i$ for all $i = 2, \dots, m-1$, where c_i are complex numbers. If $n = \text{ord}(\mathsf{a}|_{\mathsf{Z}})$, we have $\mathsf{a}^m(\tilde{t}_0) = \xi_n^{um} \cdot \tilde{t}_0$ where ξ_n is the primitive nth root of unity and $0 \leq u < n/m$. We call u exponent of the cycle. Observe that $\mathsf{a}(\tilde{t}_{i-1}) = (c_i/c_{i-1}) \cdot \tilde{t}_i$ and $\mathsf{a}^m(\tilde{t}_i) = \xi_n^{um} \cdot \tilde{t}_i$ for all i.
We can explicitly write the eigenvectors for the action of a on the coordinates $\tilde{t}_0, \dots, \tilde{t}_{m-1}$. Set $d := n/m$ and $b := sd + u$ with $0 \leq s < m$, and consider the vector

$$v_b := (\tilde{t}_0 = 1,\ \tilde{t}_1 = c_1 \cdot \xi_n^{-b}, \dots,\ \tilde{t}_i = c_i \cdot \xi_n^{-ib}, \dots).$$

Then $\mathsf{a}(v_b) = \xi_n^b \cdot v_b$. The contribution to the age of the eigenvalue ξ_n^b is b/n, thus we have

$$\text{age}\,\mathsf{a} = \sum_{s=0}^{m-1} \frac{sd+u}{n} = \frac{mu}{n} + \frac{m-1}{2},$$

proving point (1).
If the nodes are non-separating, as we are working on a $\star$-rigid curve, we have $u \geq 1$ and the point (2) is proved.
Suppose that the action of a on C has j nodes cycles of order $m_1, m_2, \dots, m_j$, of exponents respectively $u_1, \dots, u_j$. If $k = \text{c-ord}\,\mathsf{a}$, a^k fixes every node and its age is $(\sum m_i u_i k)/n$, which is greater or equal to 1 by hypothesis. By the previous result, the age of a on the ith cycle is bounded by $m_i u_i/n + (m_i - 1)/2$. As a consequence

$$\text{age}\,\mathsf{a} = \sum_{i=1}^{j} \left(\frac{m_i u_i}{n} + \frac{m_i - 1}{2} \right) \geq \frac{1}{k} + \frac{m_1 - 1}{2}.$$

□

Lemma 5.14. *Suppose that* $(\mathsf{C}, \mathsf{L}, \phi)$ *is* $\star$*-rigid and consider a non-separating node* n*. If the coarsening of* a *acts locally exchanging the branches of* n*, then the* a*-action yields an aging of* $1/2$ *on the nodes.*

Proof. The automorphism a induces an obvious automorphism of the decorated graph (Γ_0, M). We will call this automorphism a too, with a little abuse of notation. Thus we have $\mathsf{a}^*M = M$. In particular, if e_{n} is an oriented edge associated to n, then $M(e_{\mathsf{n}}) \equiv \mathsf{a}^*M(e_{\mathsf{n}}) \equiv M(\overline{e}_{\mathsf{n}}) \equiv -M(e_{\mathsf{n}})$. Therefore $M(e_{\mathsf{n}}) \equiv \ell/2$. If ℓ is prime thus necessarily $\ell = 2$. Anyway, also when ℓ is composite, by the generalized definition of M (see Remark 3.14) we have that the order of the local stabilizer is $r = 2$ and $\mathsf{a}(e_{\mathsf{n}}) \in \mathbb{Z}/2$. As $(\mathsf{C}, \mathsf{L}, \phi)$ is $\star$-rigid, $\mathsf{a}(e_{\mathsf{n}}) \not\equiv 0$, therefore the aging is $1/2$. □

Step 3. We observe that all the nodes of C are fixed except at most two of them, who are exchanged. Moreover, if a pair of non-fixed nodes exists, they contribute by at least $1/2$. This fact is a straightforward consequence of the first point in Lemma 5.13.

Step 4. Consider an irreducible component $\mathsf{Z} \subset \mathsf{C}$, then $\mathsf{a}(\mathsf{Z}) = \mathsf{Z}$. To show this, we suppose there exists a cycle of irreducible components $\mathsf{C}_1, \ldots, \mathsf{C}_m$ with $m \geq 2$ such that $\mathsf{a}(\mathsf{C}_i) = \mathsf{C}_{i+1}$ for $i = 1, \ldots, m-1$, and $\mathsf{a}(\mathsf{C}_m) = \mathsf{C}_1$. We call $\overline{\mathsf{C}}_i$ the normalizations of these components, and D_i the preimages of C nodes on $\overline{\mathsf{C}}_i$. We point out that this construction implies that $(\overline{\mathsf{C}}_i, D_i) \cong (\overline{\mathsf{C}}_j, D_j)$ for all i, j. Then, an argument of [12, p.34] shows that the action of a on

$$\bigoplus_{i=1}^{m} H^1(\overline{\mathsf{C}}_i, T_{\overline{\mathsf{C}}_i}(-D_i)) \subset \mathrm{Def}(\mathsf{C})$$

gives a contribution of at least $k \cdot (m-1)/2$ to $\operatorname{age} \mathsf{a}$, where

$$k = \dim H^1(\overline{\mathsf{C}}_i, T_{\overline{\mathsf{C}}_i}(-D_i)) = 3g_i - 3 + \#D_i.$$

This give us two cases for which m could be greater than 1 with still a junior age: $k = 1$ and $m = 2$ or $k = 0$.

If $k = 1$ and $m = 2$, we have $g_i = 0$ or 1 for $i = 1, 2$. Moreover, the aging of at least $1/2$ sums to another aging of $1/2$ if there is a pair of non-fixed nodes. As a is junior, we conclude that $\mathsf{C} = \mathsf{C}_1 \cup \mathsf{a}(\mathsf{C}_1)$ but this implies $g(\mathsf{C}) \leq 3$, contradiction.

If $k = 0$, we have $g_i = 1$ or $g_i = 0$, the first is excluded because it implies $\#D_i = 0$ but the component must intersect the curve somewhere. Thus, for every component in the cycle, the normalization $\overline{\mathsf{C}}_i$ is the projective line $\mathbb{P}^1$ with 3 marked points. We have two cases: the component C_i intersect $\overline{\mathsf{C} \backslash \mathsf{C}_i}$ in 3 points or in 1 point, in the second case C_i has an autointersection node and $\mathsf{C} = \mathsf{C}_1 \cup \mathsf{a}(\mathsf{C}_1)$, which is a contradiction because $g(\mathsf{C}) \geq 4$. It remains the case in the image below.

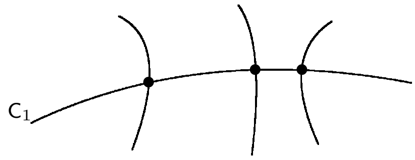


Figure 1. Case with $\mathsf{C}_1 \cong \mathbb{P}^1$ and 3 marked points

As $\mathsf{C}_1, \mathsf{C}_2, \dots, \mathsf{C}_m$ are moved by a, every node on C_1 is transposed with another one or is fixed with its branches interchanged. In both cases the aging is $1/2$, after a straightforward analysis we obtain an age contribution bigger than 1 using Lemmata 5.13 and 5.14.

Step 5. We prove that every node is fixed by a. Consider the normalization $\mathsf{nor}\colon \bigsqcup_i \overline{\mathsf{C}}_i \to \mathsf{C}$ already introduced. If the age of a is lower than 1, *a fortiori* we have $\operatorname{age} \mathsf{a}|_{\overline{\mathsf{C}}_i} < 1$ for all i. In [12, p.28] there is a list of those smooth stable curves for which there exists a non-trivial junior action.

i. The projective line $\mathbb{P}^1$ with $\mathsf{a}\colon z \mapsto (-z)$ or $(\xi_4 z)$;
ii. an elliptic curve with a of order $2,\ 3,\ 4$ or 6;
iii. an hyperelliptic curve of genus 2 or 3 with a the hyperelliptic involution;
iv. a bielliptic curve of genus 2 with a the canonical involution.

We observe that the order of the a-action on these components is always $2,\ 3,\ 4$ or 6. As a consequence, if a is junior, then $n = \text{c-ord}\,\mathsf{a} = 2, 3, 4, 6$ or 12, as it is the greatest common divisor between the $\text{c-ord}\left(\mathsf{a}|_{\overline{\mathsf{C}}_i}\right)$.
First we suppose $\operatorname{ord} \mathsf{a} > \text{c-ord}\,\mathsf{a}$, thus $\mathsf{a}^{\text{c-ord}\,\mathsf{a}}$ is a ghost and it must be senior. Indeed, if $\mathsf{a}^{\text{c-ord}\,\mathsf{a}}$ is aged less than 1, then $(\mathsf{C}, \mathsf{L}, \phi)$ admits junior ghosts, contradicting our assumption. By point (3) of Lemma 5.13, if there exists a pair of non-fixed nodes, we obtain an aging of $1/n + 1/2$ on node coordinates. If $\operatorname{ord} \mathsf{a} = \text{c-ord}\,\mathsf{a}$ the bound is even greater. As every component is fixed by a, the two nodes are non-separating, and by point (2) of Lemma 5.13 we obtain an aging of $2/n + 1/2$.
If $\overline{\mathsf{C}}_i$ admits an automorphism of order $3,\ 4$ or 6, by a previous analysis of Harris and Mumford (see [12] again), this yields an aging of, respectively, $1/3$, $1/2$ and $1/3$ on $H^1(\overline{\mathsf{C}}_i, T_{\overline{\mathsf{C}}_i}(-D_i))$.
These results combined, show that a non-fixed pair of nodes gives an age greater than 1. Thus, if a is junior, every node is fixed.

Step 6. We study the action of a separately on every irreducible component. The a-action is non-trivial on at least one component C_i, and this component must lie in the list above.
In case (i), $\overline{\mathsf{C}}_i$ has at least 3 marked points because of the stability condition. Actions of type $x \mapsto \zeta x$ have two fixed points on $\mathbb{P}^1$, thus at least one of the marked points is non-fixed. A non-fixed preimage of a node has order 2, thus the coarsening a of a is the involution $z \mapsto -z$. Moreover, C_i is the autointersection of the projective line and a exchanges the branches of the node. Because of

Lemma 5.14, a acts non-trivially with order 2 on the node, and its action on the associated coordinate gives an aging of $1/2$.
The analysis for cases (iii) and (iv) is identical to that developed in [12]: the only possibilities of a junior action is the case of an hyperelliptic curve E of genus 2 intersecting $\overline{\mathsf{C}\backslash\mathsf{E}}$ in exactly one point, whose hyperinvolution gives an aging of $1/2$ on $H^1(\overline{\mathsf{C}}_i, T_{\overline{\mathsf{C}}_i}(-D_i))$.
Finally, in case (ii), we use again the analysis of [12]. The elliptic component E has 1 or 2 point of intersection with $\overline{\mathsf{C}\backslash\mathsf{E}}$. If there is 1 point of intersection, elliptic tail case, for a good choice of coordinates the coarsening a acts as $z \mapsto \xi_n z$, where n is $2, 3, 4$ or 6. The aging is, respectively, $0,\ 1/3,\ 1/2,\ 1/3$. If there are 2 points of intersection, elliptic ladder case, the order of a on E must be 2 or 4 and the aging respectively $1/2$ or $3/4$.

Step 7. Resuming what we saw until now, if a is a junior automorphism of $(\mathsf{C}, \mathsf{L}, \phi)$, a its coarsening and C_1 an irreducible component of C, then we have one of the following:

A. component C_1 is an hyperelliptic tail, crossing the curve in one point, with a acting as the hyperelliptic involution and aging $1/2$ on $H^1(\overline{C}_1, T_{\overline{\mathsf{C}}_i}(-D_1))$;
B. component C_1 is a projective line $\mathbb{P}^1$ autointersecting itself, crossing the curve in one point, with a the involution which fixes the nodes, and aging $1/2$;
C. component C_1 is an elliptic ladder, crossing the curve in two points, with a of order 2 or 4 and aging respectively $1/2$ or $3/4$;
D. component C_1 is an elliptic tail, crossing the curve in one point, with a of order $2, 3, 4$ or 6 and aging $0, 1/3, 1/2$ or $1/3$;
E. automorphism a acts trivially on C_1 with no aging.

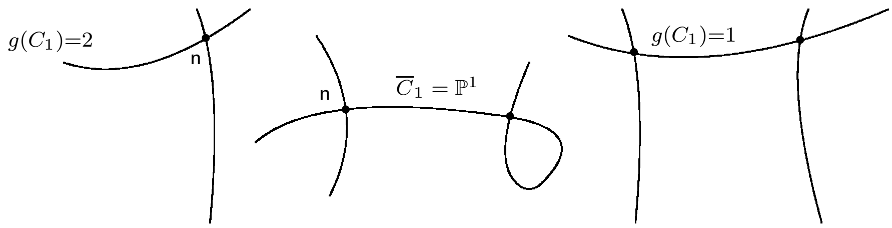


Figure 2. Components of type A, B and C.

We rule out cases (A), (B) and (C). At first we suppose there is a component of type (A) or (B). For genus reasons, the component intersected in both cases must be of type (E). We study the local action on the separating node n. The local picture of n is $[\{xy = 0\}/\boldsymbol{\mu}_r]$ where r is the order of the local stabilizer. The smoothing of the node is given by the stack $[\{xy = t_{\mathsf{n}}\}/\boldsymbol{\mu}_r]$. We observe that there exists a quasireflection of order r acting non-trivially on the node, and it generates quasireflections acting on this node. Thus $\tilde{t}_{\mathsf{n}} = t_{\mathsf{n}}^r$. For a good

choice of coordinates we have that the local picture of the coarse space C at the node is $\{x'y'=0\}$, with $x'=x^r$ and $y'=y^r$. Therefore the local deformation of the coarse space is given by $\{x'y'=\tilde{t}_{\mathsf{n}}\}$. The local action of a on the coarse space is $x'\mapsto -x'$ and $y'\mapsto y'$, thus the action of a lifts to $\tilde{t}_{\mathsf{n}}\mapsto -\tilde{t}_{\mathsf{n}}$. The additional age contribution is $1/2$, ruling out this case.
In case there is a component of type (C), if its nodes are separating, then one of them must intersect a component of type (E) and we use the previous idea. In case nodes are non-separating, we use Lemma 5.13. If $\operatorname{ord}\mathsf{a} > \operatorname{c-ord}\mathsf{a}$, then $\mathsf{a}^{\operatorname{c-ord}\mathsf{a}}$ is a senior ghost because $(\mathsf{C},\mathsf{L},\phi)$ is not a J-curve, thus by point (3) of the lemma there is an aging of $1/\operatorname{c-ord}\mathsf{a}$ on the node coordinates. If $\operatorname{ord}\mathsf{a}=\operatorname{c-ord}\mathsf{a}$, the bound is even greater, as by point (2) we have an aging of $2/\operatorname{c-ord}\mathsf{a}$. We observe that $\operatorname{c-ord}\mathsf{a}=2,4$ or 6, and in case $\operatorname{c-ord}\mathsf{a}=6$ there must be a component of type (E). Using additional contributions listed above we rule out the case (C).

Step 8. We proved that C contains components of type (D) or (E), *i.e.* the automorphism a acts non-trivially only on elliptic tails. If n is the elliptic tail node, there are possibly two quasireflections acting on the coordinate t_{n}: a ghost automorphism associated to this node and the elliptic tail quasireflection. If the order of the local stabilizer is r, then $\tilde{t}_{\mathsf{n}}=t_{\mathsf{n}}^{2r}$.
If $\operatorname{ord}\mathsf{a}=2$ we are in the ETQR case, this action is a quasireflection and it contributes to rescaling the coordinate t_{n}.
If $\operatorname{ord}\mathsf{a}=4$, the action on the (coarse) elliptic tail is $z\mapsto \xi_4 z$. The space $H^1(\overline{\mathsf{C}}_i, T_{\overline{\mathsf{C}}_i}(-D_i))$ is the space of 2-forms $H^0(\overline{\mathsf{C}}_i,2K_i)$: this space is generated by $dz^{\otimes 2}$ and the action of a is $dz^{\otimes 2}\mapsto \xi_4^2 dz^{\otimes 2}$. Moreover, if the local picture of the elliptic tail node is $[\{xy=0\}/\boldsymbol{\mu}_r]$, then $\mathsf{a}\colon (x,y)\mapsto(\zeta x,\zeta' y)$ such that $\zeta^r=\xi_4$ and $(\zeta')^r=1$. As a consequence $\mathsf{a}\colon t_{\mathsf{n}}\mapsto \zeta\zeta' t_{\mathsf{n}}$ and therefore $\tilde{t}_{\mathsf{n}}\mapsto \xi_2\tilde{t}_{\mathsf{n}}$. Then, $\operatorname{age}\mathsf{a}=1/2+1/2$, proving the seniority of a.
If E admits an automorphism a of order 3 or 6, the action on the (coarse) elliptic tail is $\mathsf{a}\colon z\mapsto \xi_6^k z$. Then $dz^{\otimes 2}\mapsto \xi_3^k dz^{\otimes 2}$ and $\tilde{t}_{\mathsf{n}}\mapsto \xi_3^k\tilde{t}_{\mathsf{n}}$. For $k=1,4$ we have age lower than 1.

If $(\mathsf{C},\mathsf{L},\phi)$ is not a J-curve, we have shown that the only case where $\mathsf{a}\in \underline{\operatorname{Aut}}(\mathsf{C},\mathsf{L},\phi)/\operatorname{QR}$ is junior, is when its coarsening a is an elliptic tail automorphism of order 3 or 6. □

The previous theorem reduces the analysis of $\operatorname{Sing}^{\mathrm{nc}}\overline{\mathcal{R}}^k_{g,\ell}$ to two loci. The next section will be devoted to the J-locus. About the T-locus we observe that it is that part of the non-canonical locus "coming" from $\overline{\mathcal{M}}_g$. More formally, if $\pi\colon \overline{\mathcal{R}}^k_{g,\ell}\to\overline{\mathcal{M}}_g$ is the natural projection, we have $T^k_{g,\ell}\subset \pi^{-1}\operatorname{Sing}^{\mathrm{nc}}\overline{\mathcal{M}}_g = T_{g,1,0}$. Harris and Mumford showed that multicanonical forms extend over the T-locus on $\overline{\mathcal{M}}_g$. Their proof could be adapted for $\overline{\mathcal{R}}^k_{g,\ell}$ as shown precisely in [13, Theorem 4.1] for the case $\ell=2$, $k=1$.

In the case of the J-locus it is unknown if the extension of the pluricanonical forms is possible. Harris and Mumford's techniques do not adapt and there is at the moment no global approach to treat this case. In the next section we will give a description of the J-locus based on the dual graph structure. In particular we will show the cases with non-empty J-locus for small values of ℓ.

6. The J-locus

The J-locus is the "new" part of the non-canonical locus which appears passing from $\overline{\mathsf{M}}_g$ to one of its coverings $\overline{\mathsf{R}}^k_{g,\ell}$. For some values of ℓ and k this locus could be empty, however, a consequence of our analysis is that it is actually not empty for any $\ell > 2$ and $k \neq 0$.
We will exhibit an explicit decomposition of $J^k_{g,\ell}$ in terms of the strata $\mathcal{S}_{(\Gamma,M)}$. We point out a significant difference with respect to the description of the singular locus: in the case of $H^k_{g,\ell}$, that we obtained in Section 4.1, we showed a decomposition in terms of loci whose generic point represents a two component curve, *i.e.* a vine curve. For the J-locus we do not have such an elegant minimal decomposition: for values of ℓ large enough, there exists strata representing J-curves with an arbitrary high number of components, such that each one of their smoothing is not a J-curve. Equivalently, there are decorated graphs with an arbitrary high number of vertices and admitting junior automorphisms, but such that each one of its contraction does not admit junior automorphisms.

Remark 6.1. We already observed that a ghost automorphism always acts trivially on loop edges of decorated dual graphs, and that quasireflections only act on separating edges. Thus we can ignore these edges in studying $\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\phi)/\mathrm{QR}_C(\mathsf{C},\mathsf{L},\phi)$. ¿From this point we will automatically contract loops and separating edges as they appear. This is not a big change in our setting, in fact graphs without loops and separating edges are a subset of the graphs we considered until now. Reducing our analysis to this subset is done for the sole purpose of simplifying the notation.

Age is not well-behaved with respect to graph contraction, but there is another invariant which is better behaved: we will define a number associated to every ghost, which respects a super-additive property in the case of strata intersection (see Theorem 6.7). Here we only work under the condition that ℓ be a prime number.
At first consider a decorated graph contraction

$$(\Gamma_0, M) \to (\Gamma_1, M_1).$$

¿From the definition of the stratification, we know that this implies

$$\mathcal{S}_{(\Gamma_1,M_1)} \subset \overline{\mathcal{S}}_{(\Gamma_0,M)}.$$

We know that $\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\phi)$ is canonically isomorphic to the group $G(\Gamma_0;\mathbb{Z}/\ell) \subset S(\Gamma_0;\mathbb{Z}/\ell)$ for every rooted curve $(\mathsf{C},\mathsf{L},\phi)$ in $\mathcal{S}_{(\Gamma_0,M)}$, and moreover $S(\Gamma_0;\mathbb{Z}/\ell)$ is canonically identified with $C^1(\Gamma_0;\mathbb{Z}/\ell)$ via M multiplication. As

there is no risk of confusion, from now on we will not repeat the notation of $\mathbb{Z}/\ell$ on the label of groups G, S and C^1.
Because of contraction, the edge set $E(\Gamma_1)$ is a subset of $E(\Gamma_0)$, so that the group $S(\Gamma_1)$ is naturally immersed in $S(\Gamma_0)$, and $C^1(\Gamma_1)$ in $C^1(\Gamma_0)$. These immersions are compatible with multiplication by M, thus $G(\Gamma_1) = G(\Gamma_0) \cap S(\Gamma_1)$ and we have a natural immersion of the G groups too.

Proposition 6.2. *Given a contraction $(\Gamma_0, M) \to (\Gamma_1, M_1)$, the elements of $G(\Gamma_1)$ are those cochains of $G(\Gamma_0)$ whose support is contained in $E(\Gamma_1)$.*

This gives an interesting correspondence between curve specialization, decorated graph contraction, strata inclusion and canonical immersions between the associated ghost automorphism groups.
¿From the definition of $G(\Gamma_0; \mathbb{Z}/\ell)$ and Proposition 1.14, we know that $G(\Gamma_0) \cong \mathbb{Z}/\ell^{\#V(\Gamma_0)-1}$, and we have an explicit basis for it. We consider a spanning tree T for Γ_0, we call $e_1, e_2, \ldots, e_k$ the edges of T, each one with an orientation, such that $k = \#V(\Gamma_0) - 1$. Then the cuts $\mathrm{cut}_{\Gamma_0}(e_i; T)$ form a basis of $G(\Gamma_0)$. We can also write

$$G(\Gamma_0) = \bigoplus_{i=1}^{k} \left(\mathrm{cut}_{\Gamma_0}(e_i; T) \cdot \mathbb{Z}/\ell\right).$$

For an n-vine decorated graph $(\Gamma_{(2,n)}, M)$, the G-group is cyclic.

Remark 6.3. An element $\mathsf{a} \in G(\Gamma_0)$ could be seen as living on stratum $\mathcal{S}_{(\Gamma_0,M)}$. We observe that if $(\Gamma_0, M) \to (\Gamma_1, M_1)$ is a contraction, then a lies in $G(\Gamma_1)$ by the natural injection. Thus every automorphism living on $\mathcal{S}_{(\Gamma_0,M)}$ lives on all the closure $\overline{\mathcal{S}}_{(\Gamma_0,M)}$.

If a ghost automorphism is junior, it carries a non-canonical singularity that spreads all over the closure of the stratum where the automorphism lives. This informal statement justifies the following definition.

Definition 6.4. The age of a stratum $\mathcal{S}_{(\Gamma_0,M)}$ is the minimum age of a ghost automorphism a in $G(\Gamma_0)$. As in the case of group age, the age of an automorphism depends on the primitive root chosen, but the stratum age does not.

With this new notation, the locus of non-canonical singularities could be written as follows,

$$\mathrm{Sing}^{\mathrm{nc}}\, \overline{\mathcal{R}}^k_{g,\ell} = \bigcup_{\mathrm{age}\, \mathcal{S}_{(\Gamma_0,M)} < 1} \overline{\mathcal{S}}_{(\Gamma_0,M)}.$$

Indeed, if $[\mathsf{C}, \mathsf{L}, \phi] \in \overline{\mathcal{R}}^k_{g,\ell}$ has a junior ghosts group, then this point lies on the closure of a junior stratum. Conversely, every point in the closure of a junior stratum has a junior ghosts group.

Definition 6.5. We say that a set of contractions $\{(\Gamma_0, M) \to (\Gamma_i, M_i)\}$, for $i = 1, \ldots, k$, *covers* Γ_0 if $E(\Gamma_0) = \bigcup_i E(\Gamma_i)$. We observe that if this set is a covering, then $G(\Gamma_i, M_i) \subset G(\Gamma_0, M)$ for all i and

$$\overline{\mathcal{S}}_{(\Gamma_0,M)} \subset \bigcap_{i=1}^{k} \overline{\mathcal{S}}_{(\Gamma_i,M_i)}.$$

Definition 6.6. An automorphism $\mathsf{a} \in G(\Gamma_0)$ is *supported* on stratum $\mathcal{S}_{(\Gamma_0,M)}$ if its support is the whole $E(\Gamma_0)$ set. We observe that this property has an immediate moduli interpretation: an automorphism supported on $\mathcal{S}_{(\Gamma_0,M)}$ appears in the ghost group of every curve in $\overline{\mathcal{S}}_{(\Gamma_0,M)}$ but it does not appear in any other stratum whose closure contains $\mathcal{S}_{(\Gamma_0,M)}$.

Unfortunately, the age of a strata intersection is not bounded by the sum of the ages of strata intersecting. However, there exists another invariant which has a superadditive property with respect to strata intersection. We will pay attention to the new automorphisms that appear at the intersection, *i.e.* those automorphisms supported on the intersection stratum, using the notion just introduced.

Theorem 6.7. *Consider a covering* $(\Gamma_0, M) \to (\Gamma_i, M_i)$*, with* $i = 1, \dots, m$*, such that*

$$G(\Gamma_0) = \sum_{i=1}^{m} G(\Gamma_i).$$

Then for every a *supported on* $\mathcal{S}_{(\Gamma_0,M)}$ *we have*

$$\operatorname{age} \mathsf{a} - \#E(\Gamma_0) \geq \sum_{i=1}^{m} \left(\operatorname{age} \mathcal{S}_{(\Gamma_i,M_i)} - \#E(\Gamma_i)\right).$$

To prove the theorem we need the following lemma

Lemma 6.8. *If* a *is supported on* $\mathcal{S}_{(\Gamma_0,M)}$*, then*

$$\operatorname{age} \mathsf{a} + \operatorname{age} \mathsf{a}^{-1} = \#E(\Gamma_0).$$

Proof. Given an edge $e \in E(\Gamma_0)$, by definition, $\mathsf{a}^{-1}(e), \mathsf{a}(e) \in \mathbb{Z}/\ell$. As a is supported on $\mathcal{S}_{(\Gamma,M)}$, $\mathsf{a}(e) \not\equiv 0$ for all e in $E(\Gamma_0)$, and then this component brings to $\operatorname{age} \mathsf{a}$ and $\operatorname{age} \mathsf{a}^{-1}$ respectively a value of $\mathsf{a}(e)/\ell$ and $(1 - \mathsf{a}(e)/\ell)$. As a consequence we obtain $\operatorname{age} \mathsf{a} + \operatorname{age} \mathsf{a}^{-1} = \#E(\Gamma) = \operatorname{Codim} \mathcal{S}_{(\Gamma_0,M)}$. $\square$

As a direct consequence of the previous lemma, we have $\operatorname{age} \mathsf{a}^{-1} = \#E(\Gamma_0) - \operatorname{age} \mathsf{a}$. By hypothesis we can write,

$$\mathsf{a}^{-1} = \mathsf{a}_1 + \mathsf{a}_2 + \cdots + \mathsf{a}_m,$$

where $\mathsf{a}_i \in G(\Gamma_i, M_i)$ for all i, and we call c_i the cardinality of a_i support. By subadditivity of age, we have $\operatorname{age} \mathsf{a}^{-1} \leq \sum \operatorname{age} \mathsf{a}_i$, then using Lemma 6.8 we obtain

$$\mathsf{a} - \#E \geq \sum_{i=1}^{m} \left(\operatorname{age} \mathsf{a}_i^{-1} - c_i\right).$$

By the fact that $c_i \leq \#E(\Gamma_i)$ for all i, and by the definition of age for the strata, the Theorem is proved. $\square$

We observe that $\#E(\Gamma_i) = \operatorname{Codim} \mathcal{S}_{(\Gamma_i, M_i)}$ by Proposition 4.9, thus we found an inequality about age involving geometric data. There is another formulation of the statement. We already observed that for every graph Γ, $b_1(\Gamma) = \#E - (\#V - 1)$. If the sum in the hypothesis is a direct sum, the rank condition is equivalent to $\#V(\Gamma_0) - 1 = \sum \#V(\Gamma_i) - 1$. Therefore we have the following.

Corollary 6.9. *If the* (Γ_i, M_i) *cover* (Γ_0, M) *and*

$$G(\Gamma_0) = \bigoplus_{i=1}^{m} G(\Gamma_i),$$

then for every automorphism a *supported on* $G(\Gamma_0)$ *we have*

$$\operatorname{age} \mathsf{a} - b_1(\Gamma_0) \geq \sum_{i=1}^{m} \left(\operatorname{age} \mathcal{S}_{(\Gamma_i, M_i)} - b_1(\Gamma_i) \right).$$

In the case of two strata intersecting, a rank condition implies the splitting of $G(\Gamma_0, M)$ in a direct sum.

Lemma 6.10. *Consider the contractions of decorated graphs* $(\Gamma_0, M) \to (\Gamma_i, M_i)$ *for* $i = 1, 2$. *If* Γ_1 *and* Γ_2 *cover* Γ_0, *and moreover*

$$\#V(\Gamma_0) - 1 = (\#V(\Gamma_1) - 1) + (\#V(\Gamma_2) - 1),$$

then we have

$$G(\Gamma_0) = G(\Gamma_1) \oplus G(\Gamma_2).$$

Proof. Before proving it, we point out a useful fact: given any contraction $\Gamma_0 \to \Gamma_i$, the natural injection $G(\Gamma_i) \hookrightarrow G(\Gamma_0)$ sends cuts on cuts. We suppose, without loss of generalities, that $\#V(\Gamma_1) \leq \#V(\Gamma_2)$ and we prove the lemma by induction on $\#V(\Gamma_1)$.
The base case $\#V(\Gamma_1) = 1$ is empty, $\Gamma_0 = \Gamma_2$ and the thesis follows obviously. Now suppose $\#V(\Gamma_1) = q > 1$, then $\#V(\Gamma_2) < \#V(\Gamma_0)$ and so there exists two vertices of Γ_0 connected by edges who lies in $E(\Gamma_1)$ but not in $E(\Gamma_2)$. We call e_1 one of these edges in $E(\Gamma_1)$ and T_1 a spanning tree of Γ_1 containing e_1. If $\operatorname{cut}_{\Gamma_1}(e_1; T_1)$ is the corresponding cut, it is also an element of $G(\Gamma_0)$. By rank conditions it suffices to prove that $G(\Gamma_0) = G(\Gamma_1) + G(\Gamma_2)$. Consider $\mathsf{a} \in G(\Gamma_0)$ which is not a sum of elements in $G(\Gamma_1)$ and $G(\Gamma_2)$. Now we define

$$\mathsf{a}' := \mathsf{a} - k \cdot \operatorname{cut}_{\Gamma_1}(e_1; T_1),$$

where k is the necessary integer such that $\mathsf{a}'(e_1) \equiv 0$. Consider the graphs (Γ_0', M') and (Γ_1', M_1') obtained contracting the edge e_1 in Γ_0 and Γ_1 respectively. By construction the contractions $\Gamma_0' \to \Gamma_1'$ and $\Gamma_0' \to \Gamma_2$ still respect the hypothesis, and $\#V(\Gamma_1') = \#V(\Gamma_1) - 1$. Therefore by induction the automorphism a', which is an element of $G(\Gamma_0', M')$, is a sum

$$\mathsf{a}' = \mathsf{a}_1' + \mathsf{a}_2,$$

with $\mathsf{a}'_1 \in G(\Gamma'_1) \subset G(\Gamma_1)$ and $\mathsf{a}_2 \in G(\Gamma_2)$. Finally $\mathsf{a} = k \cdot \mathrm{cut}_{\Gamma_1}(e; T_1) + \mathsf{a}'_1 + \mathsf{a}_2$, then it is in $G(\Gamma_1) + G(\Gamma_2)$. This is a contradiction and so the lemma is proved. $\square$

In what follows we will find, for some small prime values of ℓ, a description of $J^k_{g,\ell}$ by our stratification. Before starting we point out that our analysis will focus in the cases $J^0_{g,\ell}$ and $J^1_{g,\ell}$.

Proposition 6.11. *If ℓ is prime, we have a natural stack isomorphism $\mathsf{R}^1_{g,\ell} \cong \mathsf{R}^k_{g,\ell}$ for every k between 1 and $\ell - 1$.*

Proof. Consider a scheme S and a triple $(\mathsf{C} \to S, \mathsf{L}, \phi)$ in $\mathsf{R}^1_{g,\ell}(S)$. Consider the map sending it to

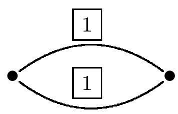
$$(\mathsf{C} \to S, \mathsf{L}^{\otimes k}, \phi^{\otimes k}) \in \mathsf{R}^k_{g,\ell}(S).$$

As ℓ is prime and $k \not\equiv 0 \mod \ell$, this morphism has a canonical inverse, so we obtained an isomorphism of categories. $\square$

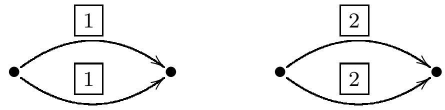
6.1. The locus $J^k_{g,\ell}$ for $\ell = 2$. In this case the J-locus is always empty. Indeed, every automorphisms in $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)/\,\mathrm{QR}$ must have a support of cardinality at least 2, but for every edge in the support, a ghost a has a contribution of 1/2 to its age. Hence there are no junior automorphisms in $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)/\,\mathrm{QR}$. This result was already obtained by Farkas and Ludwig for the Prym space $\overline{\mathcal{R}}^0_{g,2}$ in [10], and by Ludwig for $\overline{\mathcal{R}}^1_{g,2}$ in [13].

6.2. The locus $J^k_{g,\ell}$ for $\ell = 3$. The process of finding the $J^k_{g,\ell}$ decomposition, for specific values of ℓ and k, will always follows three steps.

Step 1. We identify at first the graphs which can support a junior automorphism in $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)/\,\mathrm{QR}$, *i.e.* those graphs with $\#E < \ell$ and no separating edges. If Γ is one of these graphs, we identify the $\mathbb{Z}/\ell$-valued automorphisms supported on Γ, *i.e.* the elements of $\bigoplus_{e \in E(\Gamma)} \mathbb{Z}/\ell$ which are non-trivial on every edge and are junior. These automorphisms are the junior elements in $\mathrm{Aut}_C(\mathsf{C})$ for an ℓ-twisted curve whose dual graph is Γ.
If $\ell = 3$ there is only one junior automorphism which can be supported on a $\mathbb{Z}/3$-valued decorated graph, the one represented in the image below and supported on $\Gamma_{(2,2)}$.



Step 2. For each one of these junior automorphisms, we search for multiplicity cochains that respect the lift condition of Theorem 3.7 on the automorphisms above. In this case the only possibilities are the following cochains



M_1 M_2

In fact, the two decorated graphs $(\Gamma_{(2,2)}, M_1)$ and $(\Gamma_{(2,2)}, M_2)$ are isomorphic by the isomorphism inverting the two vertices. Thus for $\ell = 3$, there is only one class of decorated graphs admitting junior automorphisms.

Step 3. By Proposition 3.4, there is an additional condition that the decorated graph must satisfy: the cochain ∂M_1 have to be the multidegree cochain of $\omega_{\mathsf{C}}^{\otimes k}$. Equivalently

$$(7) \qquad \sum_{e_+ = v} M_1(e) \equiv \deg \omega|_v^{\otimes k} \equiv k \cdot (2g_v - 2 + N_v) \mod \ell \quad \forall v \in V(\Gamma),$$

where g_v is the genus of the component correspondent to vertex v, and N_v is the degree of this vertex, *i.e.* the number of edges touching it.

By Proposition 6.11 we can focus in cases $k = 0$ and 1. If $k = 1$ the condition of (7) is empty, because 2 and 3 are coprime and there always exists a sequence of g_v satisfying the equality. Then we have

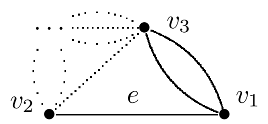
$$J^1_{g,3} = \overline{\mathcal{S}}_{(\Gamma_{(2,2)}, M_1)}.$$

In case $k = 0$, by (7) we have $\sum_{e_+ = v} M_1(e) \equiv 0 \mod 3$ for both vertices, but this condition is not satisfied by $\big(\Gamma_{(2,2)}, M_1\big)$, then

$$J^0_{g,3} = \varnothing.$$

6.3. The locus $J^k_{g,\ell}$ for $\ell = 5$. *Step 1 and 2.* For $\ell = 5$, every graph (Γ, M) such that there exists a junior automorphism $\mathsf{a} \in G(\Gamma, M)$, contracts on a vine stratum. This is a consequence of the following lemma.

Lemma 6.12. *If ℓ is a prime number, consider a decorated graph (Γ, M) such that there exists a vertex $v_1 \in V(\Gamma)$ connected with exactly two vertices v_2, v_3 and such that between v_1 and v_2 there is only one edge called e*



If $\mathcal{S}_{(\Gamma,M)}$ is a junior stratum, then there exists a non-trivial graph contraction $(\Gamma, M) \to (\Gamma_1, M_1)$ such that $\mathcal{S}_{(\Gamma_1,M_1)}$ is also junior.

Remark 6.13. This lemma permits one to simplify the analysis of junior strata. Every stratum labeled with a graph containing the configuration above, can be ignored in the analysis. Indeed, if $\mathcal{S}_{(\Gamma,M)}$ is a subset of the J-locus, there exists a decorated graph (Γ_1, M_1) with less vertices such that $\mathcal{S}_{(\Gamma,M)} \subset \overline{\mathcal{S}}_{(\Gamma_1,M_1)} \subset J^k_{g,\ell}$.

Proof of the Lemma. Consider $\mathsf{a} \in G(\Gamma, M)$ such that $\operatorname{age} \mathsf{a} < 1$. If a is not supported on $\mathcal{S}_{(\Gamma,M)}$, we contract one edge where a acts trivially and the lemma is proved. Thus we suppose a supported on $\mathcal{S}_{(\Gamma,M)}$. We call e' one of the edges connecting v_1 and v_3. We consider a spanning tree T of Γ passing though e' and not passing through e. Then we call Γ_1 and Γ_2 the two contractions of Γ obtained contracting respectively e' and $E_T \backslash \{e'\}$. By Lemma 6.10, we have

$$G(\Gamma, M) = G(\Gamma_1, M_1) \oplus G(\Gamma_2, M_2).$$

Therefore we use Theorem 6.7 to obtain

$$\operatorname{age} \mathsf{a} - \#E(\Gamma) \geq \left(\operatorname{age} \mathcal{S}_{(\Gamma_1,M_1)} - \#E(\Gamma_1)\right) + \left(\operatorname{age} \mathcal{S}_{(\Gamma_2,M_2)} - \#E(\Gamma_2)\right).$$

As $\#E(\Gamma) = \#E(\Gamma_1) + \#E(\Gamma_2) - 1$ by construction, a junior implies $\mathcal{S}_{(\Gamma_1,M_1)}$ or $\mathcal{S}_{(\Gamma_2,M_2)}$ to be junior. $\square$

The configuration of Lemma 6.12 appears in every non-vine graph with less than 5 edges. As a consequence the reduction to vine strata follows.
To summarize these vine strata we introduce a new notation. Consider a k-vine graph, with vertices v_1 and v_2 and edges $e_1, \dots, e_k$ all taken oriented from v_1 to v_2. If the multiplicity index M on the graph take values $M(e_i) = m_i$, $m_i \in \mathbb{Z}/\ell$, we note this decorated graph $(m_1, m_2, \dots, m_k)$. For example the graphs appeared in the precedent paragraph are noted $(1,1)$ and $(2,2)$ (and are isomorphic). We can now state the ten classes of vine graphs which support a junior ghost for $\ell = 5$,

$$(1,1),\ (2,2),\ (1,2),\ (1,3),\ (1,1,1),\ (2,2,2), \\ (1,1,3),\ (2,2,1),\ (1,1,1,1),\ (2,2,2,2).$$

Step 3. For $k = 1$, equation (7) is always respected for some genus labellings of the graph. Therefore we have

$$J^1_{g,5} = \overline{\mathcal{S}}_{(1,1)} \cup \overline{\mathcal{S}}_{(2,2)} \cup \overline{\mathcal{S}}_{(1,2)} \cup \overline{\mathcal{S}}_{(1,3)} \cup \overline{\mathcal{S}}_{(1,1,1)} \cup \\ \cup \overline{\mathcal{S}}_{(2,2,2)} \cup \overline{\mathcal{S}}_{(1,1,3)} \cup \overline{\mathcal{S}}_{(2,2,1)} \cup \overline{\mathcal{S}}_{(1,1,1,1)} \cup \overline{\mathcal{S}}_{(2,2,2,2)}.$$

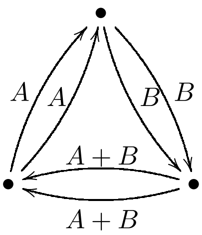
If $k = 0$ the equation is satisfied by two vine graphs, and we obtain the following

$$J^0_{g,5} = \overline{\mathcal{S}}_{(1,1,3)} \cup \overline{\mathcal{S}}_{(2,2,1)}.$$

In particular, this result fills the hole in Chiodo and Farkas analysis in [8]. They proved that the J-locus in the space $\overline{\mathcal{R}}^0_{g,\ell}$ is empty for $\ell \leq 6$ and $\ell \neq 5$.

6.4. The locus $J^k_{g,\ell}$ for $\ell = 7$. *Step 1 and 2.* Using Lemma 6.12, we observe that for $\ell = 7$ there are two kinds of graphs admitting junior automorphisms. The first kind is the usual vine graph, but we can also have a 3-cycle graph such that every pair of vertices is connected by two edges. In this second case, the only possible automorphism with age lower than 1 takes value 1 on every edge. As a consequence the possible decorations are like in figure below.
We define two sets of decorated graphs

$$V_7 := \{\text{vine decorated graphs admitting junior automorphism}\}/\cong$$



$$C_7 := \{\text{graphs decorated as in the figure}\}/\cong .$$

Step 3. If $k = 1$, condition (7) is always verified by some genus labeling. Therefore, we have

$$J^1_{g,7} = \bigcup_{(\Gamma,M)\in V_7} \overline{\mathcal{S}}_{(\Gamma,M)} \cup \bigcup_{(\Gamma,M)\in C_7} \overline{\mathcal{S}}_{(\Gamma,M)}.$$

If $k = 0$, we call V'_7 the subset of V_7 of decorated graphs respecting equation (7). Every graph in C_7 does not respect the equation. Indeed, we must have $4A + 2B \equiv 0 \mod 7$ and $2B - 2A \equiv 0 \mod 7$, therefore $A \equiv B \equiv 0$ which is not allowed. Finally, we have

$$J^0_{g,7} = \bigcup_{(\Gamma,M)\in V'_7} \overline{\mathcal{S}}_{(\Gamma,M)}.$$

In other words, the J-locus of $\overline{\mathcal{R}}^0_{g,7}$ is a union of vine strata.

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