Intrinsic Volumes and Gaussian Polytopes: The Missing Piece of the Jigsaw

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Abstract. The intrinsic volumes of Gaussian polytopes are considered. A lower variance bound for these quantities is proved, showing that, under suitable normalization, the variances converge to strictly positive limits. The implications of this missing piece of the jigsaw in the theory of Gaussian polytopes are discussed.

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1 Introduction and results

Fix a space dimension $d \in \mathbb{N}$ and denote by γ_d the standard Gaussian measure on $\mathbb{R}^d$ with density φ_d equal to

$$\varphi_d(x) := (2\pi)^{-\frac{d}{2}} \exp\left(-\frac{\|x\|^2}{2}\right), \qquad x \in \mathbb{R}^d . \tag{1}$$

Given $n \geq d+1$ let $X_1, \ldots, X_n$ be independent random points that are distributed on $\mathbb{R}^d$ according to the probability measure γ_d. The random convex hull

$$K_n := [X_1, \ldots, X_n]$$

of these points is a *Gaussian polytope*. These random polytopes are central objects considered in stochastic geometry and are also of importance in convex geometric analysis or coding theory. For example, Gluskin [8] has used Gaussian polytopes in his analysis of the diameter of the Minkowski compactum and Gaussian polytopes also arise as lower-dimensional shadows of randomly rotated high-dimensional regular simplices as shown by Baryshnikov and Vitale [4]. We refer to the survey article of Reitzner [12] for further background information and references.

We denote for $\ell \in \{0, \ldots, d\}$ by $V_\ell(K_n)$ the ℓth intrinsic volume of K_n, that is,

$$V_\ell(K_n) = \binom{d}{\ell} \frac{\kappa_d}{\kappa_\ell \kappa_{d-\ell}} \int_{\mathbb{G}(d,\ell)} \mathrm{vol}_\ell(K_n|L)\, \nu_\ell(\mathrm{d}L)\,.$$

Here, $\mathbb{G}(d, \ell)$ is the Grassmannian of ℓ-dimensional linear subspaces of $\mathbb{R}^d$ supplied with the unique Haar probability measure ν_ℓ and $\mathrm{vol}_\ell(K_n|L)$ stands for the ℓ-dimensional Lebesgue measure of the orthogonal projection $K_n|L$ of K_n onto L measured within the subspace L. Moreover, for $j \in \mathbb{N}$, $\kappa_j := \pi^{j/2}\, \Gamma(1 + \frac{j}{2})^{-1}$ denotes the volume of the j-dimensional unit ball. The intrinsic volumes are of outstanding importance in convex geometry, since according to a classical theorem of Hadwiger they form a basis of the vector space of all continuous and rigid-motion invariant real-valued valuations on convex sets, cf. [14]. For example, $V_d(K_n) = \mathrm{vol}_d(K_n)$ is the volume, $2V_{d-1}(K_n)$ coincides with the surface area and $\frac{2\kappa_{d-1}}{d\kappa_d} V_1(K_n)$ corresponds to the mean width of K_n.

It is well known from the work of Affentranger [1] that the expectation $\mathbf{E}[V_\ell(K_n)]$ of $V_\ell(K_n)$ satisfies

$$\lim_{n\to\infty} (\log n)^{-\frac{\ell}{2}}\, \mathbf{E}[V_\ell(K_n)] = \binom{d}{\ell} \frac{\kappa_d}{\kappa_{d-\ell}}\,.$$

More recently, the asymptotic behaviour of the variance $\mathbf{Var}[V_\ell(K_n)]$ of $V_\ell(K_n)$ has moved into the focus of attention. Using the classical Efron-Stein jackknife inequality Hug and Reitzner [10] have obtained a first upper bound of the form $\mathbf{Var}[V_\ell(K_n)] \le c_d (\log n)^{\frac{\ell-3}{2}}$ with a constant $c_d \in (0, \infty)$ only depending on the space dimension d (but not on ℓ). In a remarkable paper of Calka and Yukich [7] the precise variance asymptotic was derived, showing thereby that the upper bound from [10] does not have the right order of magnitude. In fact, [7, Theorem 1.5] says that

$$\lim_{n\to\infty} (\log n)^{\frac{d+3}{2} - \ell}\, \mathbf{Var}[V_\ell(K_n)] = c_{d,\ell}\,, \tag{2}$$

with constants $c_{d,\ell} \in [0, \infty)$ only depending on d and on ℓ. However, using their methods the authors of [7] were not able to exclude the possibility that $c_{d,\ell} = 0$. The aim of the present paper is to fill this gap and to show that, in fact, $c_{d,\ell} > 0$. This answers a question raised at several places in the literature, see [3, Section 14], the comment after [7, Theorem 1.5] or [9, Remark 3.6]. Our result reads as follows:

Theorem 1. *Let $\ell \in \{1, \dots, d\}$ and let K_n be a Gaussian polytope. Then there exists a constant $v_{d,\ell} \in (0,\infty)$ only depending on d and on ℓ such that*

$$\mathbf{Var}[V_\ell(K_n)] \geq v_{d,\ell} \, (\log n)^{-\frac{d+3}{2}+\ell} ,$$

whenever n is sufficiently large.

In particular, Theorem 1 in conjunction with (2) shows that the limit

$$\lim_{n\to\infty} (\log n)^{\frac{d+3}{2}-\ell} \, \mathbf{Var}[V_\ell(K_n)] = c_{d,\ell}$$

exists and takes a strictly positive and finite value.

Remark 2. (i) Let us first comment on the boundary case $\ell = 0$ in Theorem 1. Since $V_0(K) = \mathbf{1}_{\{K \neq \emptyset\}}$ for any convex set $K \subset \mathbb{R}^d$, we have that $V_0(K_n) = 1$ with probability one and hence $\mathbf{Var}[V_0(K_n)] = 0$.

(ii) Since $V_d(K_n)$ is the volume of the Gaussian polytope K_n, the case $\ell = d$ is already covered by Theorem 6.1 in [3], which ensures that $v_{d,d} \in (0,\infty)$. Our proof comprises this situation as a special case.

A random polytope model closely related to K_n can be described as follows. For each $n \in \mathbb{N}$ let η_n be a Poisson point process on $\mathbb{R}^d$ with intensity measure $n\gamma_d$. The convex hull of the points of η_n will be denoted by Π_n and is called the *Gaussian Poisson polytope.* Following the coupling construction in the proof of [3, Lemma 7.1] one easily sees that expectation and variance asymptotic for Π_n are literally the same as for K_n. Moreover, the strict positivity of the constants $v_{d,\ell}$ in Theorem 1 implies that $(\log n)^{\frac{d+3}{2}-\ell}\mathbf{Var}[V_\ell(\Pi_n)]$ converges to a positive and finite limit. We summarize the missing piece in the proof of this result in the following corollary:

Corollary 3. *Let $\ell \in \{1, \dots, d\}$ and let Π_n be the Gaussian Poisson polytope. Then there exists a constant $v_{d,\ell} \in (0,\infty)$ only depending on d and on ℓ such that*

$$\mathbf{Var}[V_\ell(\Pi_n)] \geq v_{d,\ell} \, (\log n)^{\ell-\frac{d+3}{2}} ,$$

whenever n is sufficiently large.

The result of Theorem 1 and Corollary 3 can be regarded as the missing piece of the jigsaw in the theory of Gaussian polytopes. Let us mention some of the implications that are now immediate:

- *Central limit theorems.* As explained in [3, 7], the positivity of the limiting variance is the only missing piece in the proof of the central limit theorem for the normalized intrinsic volumes of Π_n. The result follows by the methods developed in [3, 6, 7]. Moreover, a de-Poissonization argument similar to that in [3] leads to the corresponding result for K_n; we omit the details.

- *Concentration inequalities.* As explained in the recent work [9], the positivity of the limiting variance is the only missing ingredient in the proof of a concentration inequality for $V_\ell(\Pi_n)$. The precise form of such an inequality can now be determined from [9, Theorem 3.1]: For any $\ell \in \{1, \ldots, d\}$ one can find a constant $c \in (0, \infty)$ only depending on d and on ℓ such that

$$\begin{aligned}\mathbf{P}\big(|V_\ell(\Pi_n) - \mathbf{E}[V_\ell(\Pi_n)]| \geq y\,\sqrt{\mathbf{Var}[V_\ell(\Pi_n)]}\,\big) \\ \leq 2\exp\Big(-\frac{1}{4}\min\Big\{\frac{y^2}{2^{2d+\ell+5}}, c\,(\log n)^{\frac{d-1}{4(2d+\ell+5)}}\,y^{\frac{1}{2d+\ell+5}}\Big\}\Big)\end{aligned}$$

  for all $y \geq 0$ and sufficiently large n.

- *Marcinkiewicz-Zygmund-type strong laws of large numbers.* The concentration inequality for $V_\ell(\Pi_n)$ mentioned in the previous paragraph can directly be used to derive Marcinkiewicz-Zygmund-type strong laws of large numbers along the lines of the proof of [9, Theorem 1.3]: For any $\ell \in \{1, \ldots, d\}$ and $p > 1 - \frac{d+3}{\ell}$ one has that

$$\frac{V_\ell(\Pi_n) - \mathbf{E}[V_\ell(\Pi_n)]}{(\log n)^{p\frac{\ell}{2}}} \longrightarrow 0$$

  with probability one, as $n \to \infty$. Using the monotonicity of intrinsic volumes and a simple coupling argument, one easily verifies that the same result also holds with Π_n replaced by K_n. In that form, this refines the ordinary strong law of large numbers from [10, Corollary 1.2], which corresponds to the special case $p = 1$.

- *Moderate deviations.* Moderate deviations for the volume and the face numbers of the Gaussian Poisson polytopes Π_n have also been investigated in [9]. Again, the only missing piece for the extension of these results to the intrinsic volumes is the positivity of the limiting variances; we omit the details.

Remark 4. Let $\lambda > 0$ be an arbitrary real number, let η_λ be a Poisson point process on $\mathbb{R}^d$ with intensity measure $\lambda\gamma_d$ and denote by Π_λ the random convex hull induced by η_λ. Using the monotonicity of intrinsic volumes and a simple coupling argument, one easily verifies that the result of Corollary 3 continues to hold with Π_n and $\log n$ replaced by Π_λ and $\log \lambda$, respectively. The same comment applies to the central limit theorem, the concentration inequalities, the Marcinkiewicz-Zygmund-type strong laws of large numbers and to the moderate deviations mentioned above.

The rest of this paper is structured as follows. In Section 2 we recall the essential steps of a geometric construction from [3] and proof some auxiliary results that are needed in the proof of Theorem 1. The latter is the content of the final Section 3.

2 Preparations

2.1 Notation

The symbols $\|\cdot\|$ and $\langle\,\cdot\,,\,\cdot\,\rangle$ are used for the Euclidean norm and scalar product in $\mathbb{R}^d$, respectively. Moreover, for a set $B \subset \mathbb{R}^d$ we write $[B]$ for the convex hull of B. We denote the d-dimensional unit ball by $\mathbb{B}^d := \{x \in \mathbb{R}^d : \|x\| \le 1\}$ and write $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$ for the corresponding unit sphere. The normalized surface measure on $\mathbb{S}^{d-1}$ is denoted by $\nu_{\mathbb{S}^{d-1}}$. Further, for a point $z \in \mathbb{R}^d \setminus \{0\}$ and $\alpha \in [0, \pi/2]$ we write $C(z,\alpha)$ for the closed circular cone whose axis is the halfline $\{tz : t \ge 0\}$ and whose angle is α. More formally, if $\sphericalangle(z,x)$ stands for the ordinary angle between z and another point $x \in \mathbb{R}^d$, $C(z,\alpha)$ is given by $C(z,\alpha) := \{x \in \mathbb{R}^d : \sphericalangle(x,z) \le \alpha\}$.

Our underlying probability space is $(\Omega, \mathcal{A}, \mathbf{P})$ and we implicitly assume that it is rich enough to carry all the random objects we consider in this paper. By $\mathbf{E}[\,\cdot\,]$ we denote expectation (integration) with respect to $\mathbf{P}$ and $\mathbf{Var}[\,\cdot\,]$ stands for the variance of the argument random variable. The indicator function of an event $A \in \mathcal{A}$ is denoted by $\mathbf{1}_A$.

For two sequences $(a_n : n \in \mathbb{N})$ and $(b_n : n \in \mathbb{N})$ we write $a_n \ll b_n$ (or $a_n \gg b_n$) if we can find a constant $c \in (0,\infty)$ not depending on n and an index $n_0 \in \mathbb{N}$ such that $a_n \le c\, b_n$ (or $a_n \ge c\, b_n$) for all $n \ge n_0$. Finally, $a_n \approx b_n$ means that $a_n \ll b_n \ll a_n$.

In this paper constants are denoted by $c_1, c_2, \ldots$ It is implicitly assumed that these constants are finite and strictly positive, and only depend on the space dimension d, unless otherwise stated.

2.2 A geometric construction

In this section we recall a geometric construction as well as some of the results already obtained [3] that we use below. We define

$$r = r(n) := \sqrt{2\log n - \log\log n}\,, \qquad n \in \mathbb{N}\,,$$

and denote by $\mathbb{S}(r) := \{x \in \mathbb{R}^d : \|x\| = r\}$ the centred sphere of radius r. By $y_1, \ldots, y_m \in \mathbb{S}(r)$ we denote a maximal system of points such that $\|y_i - y_j\| \ge 2c_1$ for some sufficiently large c_1. A simple volume comparison argument provides an estimate for the size of such a set, see [3, Claim 5.1]:

Lemma 5. *One has that* $m \approx (\log n)^{\frac{d-1}{2}}$.

For each $i \in \{1, \ldots, m\}$ define $y_i^0 := (1+r^{-2})y_i$ and notice that $\|y_i - y_i^0\| = r^{-1}$. Let further for $i \in \{1, \ldots, m\}$, $H_i := \{x \in \mathbb{R}^d : \langle x, y_i\rangle = r\}$ be the tangent hyperplane of $\mathbb{S}(r)$ at y_i and fix a regular simplex in H_i whose vertices $y_i^1, \ldots, y_i^d$ are chosen from the $(d-2)$-dimensional sphere $\mathbb{S}^{d-2}(y_i, \sqrt{2})$ of radius $\sqrt{2}$ in H_i centred at y_i. (Thus $\mathbb{S}(r) = \mathbb{S}(0,r)$ but we keep the simpler notation for $\mathbb{S}(r)$.) The simplex $\Delta_i := [y_i^0, y_i^1, \ldots, y_i^d]$ is the convex hull of y_i^0 and the points $y_i^1, \ldots, y_i^d \in H_i$, see Figure 1. It is not difficult to estimate the volume $V_d(\Delta_i)$ and the Gaussian measure $\gamma_d(\Delta_i)$ of these simplices, see [3, Claim 5.2]:

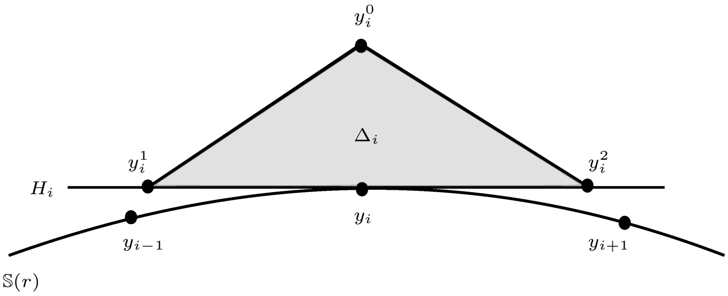

Figure 1: Construction of the simplices Δ_i.

Lemma 6. *For each $i \in \{1,\dots,m\}$ one has that $V_d(\Delta_i) \approx (\log n)^{-\frac{1}{2}}$ and $\gamma_d(\Delta_i) \approx n^{-1}$.*

For each $i \in \{1,\dots,m\}$ and $j \in \{0,\dots,d\}$ we let Δ_i^j be a homothetic copy of Δ_i with y_i^j being the centre of the homothety and the factor being a sufficiently small number c_2, that is, $\Delta_i^j := y_i^j + c_2(\Delta_i - y_i^j)$.
Let D_i be the the cone $D_i := \mathrm{pos}(\{y_i^j - y_i^0 : j \in \{1,\dots,d\}\})$, where we write $\mathrm{pos}(\,\cdot\,)$ for the positive hull of the argument set. This is the internal cone at vertex y_i^0 of the simplex Δ_i, which has a simple structure because its base is a $(d-1)$-dimensional regular simplex and the opposite vertex y_i^0 is at height r^{-1} over this base exactly above its centre. In particular, one can check easily that

$$C\Big(y_i - y_i^0, \arctan\frac{\sqrt{2}\,r}{d-1}\Big) \subset D_i \subset C\big(y_i - y_i^0, \arctan\sqrt{2}\,r\big)\,. \tag{3}$$

Since each Δ_i^j is only a homothetic copy of Δ_i with a scaling factor not depending on n, the following holds by construction:

Lemma 7. *For each $i \in \{1,\dots,m\}$ and $j \in \{0,\dots,d\}$ one has that $V_d(\Delta_i^j) \approx (\log n)^{-\frac{1}{2}}$ and $\gamma_d(\Delta_i^j) \approx n^{-1}$.*

For each $i \in \{1,\dots,m\}$ and $j \in \{0,\dots,d\}$ let z_i^j be an arbitrary point in Δ_i^j and define the cone $C_i := \mathrm{pos}(\{z_i^j - z_i^0 : j \in \{1,\dots,d\}\})$. We recall the following fact about these cones from [3, Lemma 5.4], which ensures a certain independence property used below:

Lemma 8. *One can choose the constant c_1 in the above construction sufficiently large and c_2 sufficiently small such that for each $i \in \{1,\dots,m\}$ the translated cone $z_i^0 + C_i$ contains all simplices Δ_k with $k \in \{1,\dots,m\} \setminus \{i\}$.*

Observe further that the simplices $[z_i^0,\dots,z_i^d]$ and Δ_i are very close to each other if the factor of homothety c_2 is small enough. So relations (3) imply that

$$C_i^1 := C\Big(y_i - z_i^0, \arctan\frac{r}{d-1}\Big) \subset C_i \subset C(y_i - z_i^0, \arctan 2r) =: C_i^2\,. \tag{4}$$

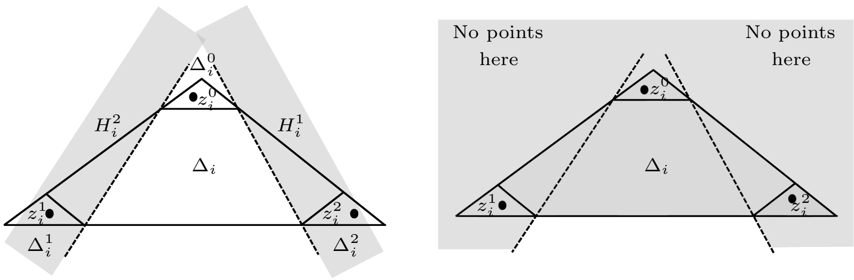

Figure 2: The simplices Δ_i^j, the points z_i^j and the half-spaces H_i^j (left). Illustration of the events A_i (right).

Next, for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, d\}$ we denote by H_i^j the half-space containing Δ_i^k for all $k \in \{0, \dots, d\} \setminus \{0, j\}$, not containing Δ_i^0 and Δ_i^j, and such that the hyperplane bounding H_i^j touches all the simplices $\Delta_i^0, \dots, \Delta_i^d$ except for Δ_i^j, see Figure 2 (left). We are now in the position to define for each $i \in \{1, \dots, m\}$ the event $A_i \in \mathcal{A}$ that precisely one point from the random sample $X_1, \dots, X_n$ is contained in each simplex of the form Δ_i^j and no further point from $X_1, \dots, X_n$ is contained in $H_i^+ \cup H_i^1 \cup \dots \cup H_i^d$, see Figure 2 (right). Here, H_i^+ is the half-space bounded by H_i not containing the origin. The following probability estimate is taken from [3, Lemma 6.2]:

Lemma 9. *There exists a constant $c_3 \in (0, 1)$ such that $\mathbf{P}(A_i) \geq c_3$ for all $i \in \{1, \dots, m\}$.*

The facts summarized so far have been used in [3] to prove a lower variance bound for the volume $V_d(K_n)$ of K_n. Since we are interested in all intrinsic volumes $V_1(K_n), \dots, V_d(K_n)$, a refinement is necessary to obtain such bounds. In fact, we now follow and adapt the method already applied in [2, 5, 11] to handle the more general situation.

2.3 The effect of local perturbations

Let $z \in \mathbb{S}^{d-1}$ and G be a measurable subset of $\mathbb{G}(d, \ell)$ for some $\ell \in \{0, \dots, d\}$. The angle $\sphericalangle(z, G)$ between z and G is defined as $\min\{\sphericalangle(z, x) : x \in L, L \in G\}$, where $\sphericalangle(z, x) = \arccos \frac{\langle x, z \rangle}{\|x\|}$ is the ordinary angle between z and x. We observe the following geometric fact, see also [2, Lemma 1]:

Lemma 10. *Let $z \in \mathbb{S}^{d-1}$ and $\ell \in \{1, \dots, d\}$. One can find a constant $c_4 \in (0, \infty)$ only depending on d and on ℓ such that*

$$\nu_\ell(\{L \in \mathbb{G}(d, \ell) : \sphericalangle(z, L) \leq a\}) \gg a^{d-\ell}$$

for all $0 < a < c_4$.

Proof. For $M \in \mathbb{G}(d, \ell-1)$ we denote by $\mathbb{G}(M,\ell)$ the relative Grassmannian of ℓ-dimensional linear subspaces of $\mathbb{R}^d$ containing M. This space is supplied with a unique Haar probability measure ν_ℓ^M, see Chapter 7.1 in [13]. Similarly, we let $\mathbb{G}(z^\perp, \ell-1)$ be the relative Grassmannian of $(\ell-1)$-dimensional linear subspaces of $\mathbb{R}^d$ that are contained in the hyperplane $z^\perp$ orthogonal to 1-dimensional linear subspace spanned by z. The unique Haar probability measure on $\mathbb{G}(z^\perp, \ell-1)$ is denoted by $\nu_{\ell-1}^{z^\perp}$. For $M \in \mathbb{G}(z^\perp, \ell-1)$ let $u \in \mathbb{S}^{d-1} \cap M^\perp$ be such that $\sphericalangle(z,u) \le a$. It is clear that the ℓ-dimensional linear subspace $\operatorname{span}(M,u)$ spanned by M and u is contained in the set $\{L \in \mathbb{G}(d,\ell) : \sphericalangle(z,L) \le a\}$ we are interested in. Formally, using Fubini's theorem for flag spaces (see [13, Theorem 7.1.1]) in the second step we write

$$\begin{aligned}
&\nu_\ell(\{L \in \mathbb{G}(d,\ell) : \sphericalangle(z,L) \le a\})\\
&\quad= \int_{\mathbb{G}(d,\ell)} \mathbf{1}_{\{\sphericalangle(z,L)\le a\}}\, \nu_\ell(\mathrm{d}L)\\
&\quad= \int_{\mathbb{G}(d,\ell-1)} \int_{\mathbb{G}(M,\ell)} \mathbf{1}_{\{\sphericalangle(z,L)\le a\}}\, \nu_\ell^M(\mathrm{d}L)\nu_{\ell-1}(\mathrm{d}M)\\
&\quad\ge \int_{\mathbb{G}(z^\perp,\ell-1)} \int_{\mathbb{G}(M,\ell)} \mathbf{1}_{\{\sphericalangle(z,L)\le a\}}\, \nu_\ell^M(\mathrm{d}L)\nu_{\ell-1}^{z^\perp}(\mathrm{d}M)\\
&\quad\ge \int_{\mathbb{G}(z^\perp,\ell-1)} \int_{\mathbb{S}^{d-1}\cap M^\perp} \mathbf{1}_{\{\sphericalangle(z,u)\le a\}}\, \nu_{\mathbb{S}^{d-1}\cap M^\perp}(\mathrm{d}u)\nu_{\ell-1}^{z^\perp}(\mathrm{d}M)\\
&\quad= \int_{\mathbb{G}(z^\perp,\ell-1)} \nu_{\mathbb{S}^{d-1}\cap M^\perp}(\{u \in \mathbb{S}^{d-1}\cap M^\perp : \sphericalangle(z,u)\le a\})\, \nu_{\ell-1}^{z^\perp}(\mathrm{d}M)\,.
\end{aligned}$$

Since $M^\perp$ has dimension $d-\ell+1$, the set of points $u \in \mathbb{S}^{d-1} \cap M^\perp$ with $\sphericalangle(z,u) \le a$ forms a spherical cap in the $(d-\ell)$-dimensional subsphere $\mathbb{S}^{d-1}\cap M^\perp$ of $\mathbb{S}^{d-1}$. It has radius of order a and volume of order $a^{d-\ell}$, where by volume we mean here the normalized $(d-\ell)$-dimensional Hausdorff measure $\nu_{\mathbb{S}^{d-1}\cap M^\perp}$ on $\mathbb{S}^{d-1} \cap M^\perp$. Hence, for sufficiently small a, we have

$$\nu_{\mathbb{S}^{d-1}\cap M^\perp}(\{u \in \mathbb{S}^{d-1}\cap M^\perp : \sphericalangle(z,u)\le a\}) \gg a^{d-\ell}$$

and, since $\nu_{\ell-1}^{z^\perp}$ is a probability measure, also

$$\nu_\ell(\{L \in \mathbb{G}(d,\ell) : \sphericalangle(z,L) \le a\}) \gg a^{d-\ell}\,.$$

The proof is complete. □

For $i \in \{1,\ldots,m\}$ put $F_i := [z_i^1,\ldots,z_i^d]$ and define

$$\widetilde{V}_\ell(z;F_i) := \binom{d}{\ell}\frac{\kappa_d}{\kappa_\ell\kappa_{d-\ell}} \int_{\mathbb{G}(d,\ell)} \mathbf{1}_{\{L\cap C_i^2\neq\emptyset\}}\, \mathrm{vol}_\ell([z,F_i]|L)\,\nu_\ell(\mathrm{d}L)\,,\quad z\in\Delta_i^0\,.$$

The next lemma provides a lower bound for the variance of these local functionals.

Lemma 11. *Fix $\ell \in \{1,\dots,d\}$, let $i \in \{1,\dots,m\}$ and let Z_i be a point chosen with respect to the normalized Gaussian measure restricted to Δ_i^0. Then*

$$\mathbf{Var}_i[\widetilde{V}_\ell(Z_i;F_i)] \gg (\log n)^{-(d-\ell+1)},$$

where the notation $\mathbf{Var}_i[\,\cdot\,]$ refers to the variance that is taken with respect to $Z_i \in \Delta_i^0$.

Proof. Denote by w_i the centre of the facet of Δ_i^0 opposite to the vertex y_i^0, and define the points $w_i^1 := \frac{2}{3}y_i^0 + \frac{1}{3}w_i$ and $w_i^2 := \frac{1}{3}y_i^0 + \frac{2}{3}w_i$. Furthermore, the regions $R_i^1, R_i^2 \subset \Delta_i^0$ are given by $R_i^1 := (w_i^1 - C_i^2) \cap \Delta_i^0$ and $R_i^2 := (w_i^2 + C_i^2) \cap \Delta_i^0$. It is crucial to observe that one can find a constant c_7 only depending on d such that $V_d(R_i^k) \geq c_7 V_d(\Delta_i^0)$ for $k = 1$ and $k = 2$. This follows from (4). Together with the first part of Lemma 7 and the fact that the Gaussian density (1) satisfies

$$\varphi_d(x) \approx \frac{\sqrt{\log n}}{n} \qquad \text{for all } x \in \mathbb{R}^d \text{ with } r \leq \|x\| \leq r + \frac{1}{r},$$

we see that the Gaussian measure of R_i^k is

$$\gamma_d(R_i^k) \approx n^{-1}, \qquad k \in \{1,2\}. \tag{5}$$

Next, fix some $L \in \mathbb{G}(d,\ell)$ that intersects the interior of the polar of the cone C_i^2. This condition means that L has an orthonormal basis $e_1,\dots,e_\ell$ such that the hyperplane $H_{i,0} := \{x \in \mathbb{R}^d : \langle x, e_1\rangle = \langle w_i^2, e_1\rangle\}$ has only one point (namely the origin) in common with C_i^2. Let $H_{i,0}^+$ be the half-space bounded by $H_{i,0}$ not containing the origin. Finally, let us define the set $G_i := H_{i,0}^+ \cap (w_i^1 + C_i^2) \subset \Delta_i^0$. We choose points $Z_i^1 \in R_i^1$ and $Z_i^2 \in R_i^2$. The whole construction is illustrated in Figure 3 where L appears translated by w_i^2 (not affecting the $\mathrm{vol}_\ell(G_i|L)$). Observe now that R_i^1 and R_i^2 are separated by the hyperplane $H_{i,0}$. Consequently, we have that $Z_i^2 \in [Z_i^1, F_i]$, which implies the inclusion $[Z_i^2, F_i] \subset [Z_i^1, F_i]$. In addition, $G_i \subset H_{i,0}^+$ and $R_i^2 \cap H_{i,0}^+ = \{w_i^2\}$, which yields $G_i \cap [Z_i^2, F_i] = \{w_i^2\}$. Finally, we observe that the hyperplane parallel to $H_{i,0}$ separates R_i^1 and G_i, whence $G_i \subseteq [Z_i^1, F_i]$. The construction also shows $V_d(G_i) \approx r^{-1} = (\log n)^{-\frac{1}{2}}$ and implies that

$$\mathrm{vol}_\ell(G_i|L) \gg (\log n)^{-\frac{1}{2}}. \tag{6}$$

As result, we arrive at the estimate

$$\mathrm{vol}_\ell([Z_i^1, F_i]|L) - \mathrm{vol}_\ell([Z_i^2, F_i]|L) \geq \mathrm{vol}_\ell(G_i|L).$$

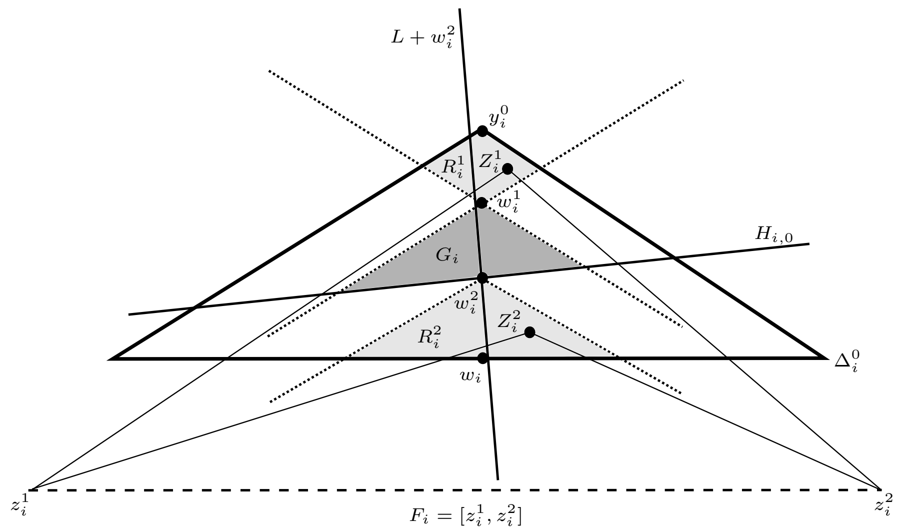

Figure 3: Construction in the proof of Lemma 11.

Hence,

$$\begin{aligned}
&\widetilde{V}_\ell(Z_i^1; F_i) - \widetilde{V}_\ell(Z_i^2; F_i)\\
&= \binom{d}{\ell} \frac{\kappa_d}{\kappa_\ell \kappa_{d-\ell}} \int_{\mathbb{G}(d,\ell)} \mathbf{1}_{\{L \cap C_i^2 \neq \emptyset\}} \big(\mathrm{vol}_\ell([Z_i^1, F_i]|L) - \mathrm{vol}_\ell([Z_i^2, F_i]|L)\big)\, \nu_\ell(\mathrm{d}L)\\
&\geq \binom{d}{\ell} \frac{\kappa_d}{\kappa_\ell \kappa_{d-\ell}} \int_{\mathbb{G}(d,\ell)} \mathbf{1}_{\{L \cap C_i^2 \neq \emptyset\}}\, \mathrm{vol}_\ell(G_i|L)\, \nu_\ell(\mathrm{d}L)\\
&\gg (\log n)^{-\frac{1}{2}}\, \nu_\ell(\{L \in \mathbb{G}(d,\ell) : L \cap C_i^2 \neq \emptyset\})\\
&\gg (\log n)^{-\frac{1}{2}}\, (\log n)^{-\frac{d-\ell}{2}}\\
&= (\log n)^{-\frac{d-\ell+1}{2}},
\end{aligned}$$

where we used (6), the definition of C_i^2, and Lemma 10. Note that the latter can indeed be applied with $a = 1/\log n$, since $1/\log n < c_4$ for sufficiently large n. Selecting now Z_i^k, $k \in \{1, 2\}$, independently at random according to the normalized Gaussian measure restricted to Δ_i^0 (i.e., Z_i^1 and Z_i^2 are independent copies of Z_i), we conclude that

$$\begin{aligned}
\mathbf{Var}[\widetilde{V}_\ell(Z_i; F_i)] &= \frac{1}{2}\mathbf{E}\big[\big(\widetilde{V}_\ell(Z_i^1; F_i) - \widetilde{V}_\ell(Z_i^2; F_i)\big)^2\big]\\
&\geq \frac{1}{2}\mathbf{E}\big[\big(\widetilde{V}_\ell(Z_i^1; F_i) - \widetilde{V}_\ell(Z_i^2; F_i)\big)^2\, \mathbf{1}_{R_i^1}(Z_i^1)\mathbf{1}_{R_i^2}(Z_i^2)\big]\\
&\gg (\log n)^{-(d-\ell+1)}\, \mathbf{P}(Z_i^1 \in R_i^1, Z_i^2 \in R_i^2)\,.
\end{aligned}$$

To obtain a lower bound for $\mathbf{P}(Z_i^1 \in R_i^1, Z_i^2 \in R_i^2)$ we recall (5) and combine this with the second assertion of Lemma 7 as well as with the independence of the random points Z_i^1 and Z_i^2. This implies that

$$\mathbf{P}(Z_i^1 \in R_i^1, Z_i^2 \in R_i^2) = \prod_{k=1}^{2} \mathbf{P}(Z_i^k \in R_i^1) = \prod_{k=1}^{2} \frac{\gamma_d(R_i^k)}{\gamma_d(\Delta_i^0)} \geq c_8^2$$

with a constant $c_8 \in (0, \infty)$ only depending on d. Hence,

$$\mathbf{Var}[\widetilde{V}_\ell(Z; F_i)] \gg (\log n)^{-(d-\ell+1)} ,$$

completing thereby the proof of the lemma. $\square$

3 Proof of Theorem 1

Recall the geometric construction and its properties from the previous section and denote by $\mathcal{F} \subset \mathcal{A}$ the σ-field generated by the random points $X_1, \ldots, X_n$, except those in the simplices Δ_i^0 for which $\mathbf{1}_{A_i} = 1$, $i \in \{1, \ldots, m\}$. The conditional variance formula implies that

$$\begin{aligned}\mathbf{Var}[V_\ell(K_n)] &= \mathbf{E}\big[\mathbf{Var}[V_\ell(K_n)|\mathcal{F}]\big] + \mathbf{Var}\big[\mathbf{E}[V_\ell(K_n)|\mathcal{F}]\big] \\ &\geq \mathbf{E}\big[\mathbf{Var}[V_\ell(K_n)|\mathcal{F}]\big] .\end{aligned}$$

Now, conditioned on $\mathcal{F}$, suppose that $\mathbf{1}_{A_i} = 1$, write Z_i for the (unique) random point in Δ_i^0 and denote by F_i the convex hull of the random points in Δ_i^j with $j \in \{1, \ldots, d\}$. We notice that if $\mathbf{1}_{A_i} = 1$ for each $i \in I$ in a subset $I \subset \{1, \ldots, m\}$, then $(\widetilde{V}_\ell(Z_i; F_i) : i \in I)$ is a family of independent random variables as a consequence of the result of Lemma 8. This independence property implies that

$$\mathbf{Var}[V_\ell(K_n)|\mathcal{F}] = \sum_{\substack{i=1 \\ \mathbf{1}_{A_i}=1}}^{m} \mathbf{Var}_i[V_\ell(K_n)] = \sum_{\substack{i=1 \\ \mathbf{1}_{A_i}=1}}^{m} \mathbf{Var}_i[\widetilde{V}_\ell(Z_i; F_i)] ,$$

where, as in the previous section, the notation $\mathbf{Var}_i[\,\cdot\,]$ refers to the variance that is taken only with respect to the point $Z_i \in \Delta_i^0$ and we only sum over those $i \in \{1, \ldots, m\}$ with the property that $\mathbf{1}_{A_i} = 1$. These variances can be controlled by means of Lemma 11, which implies that

$$\mathbf{Var}[V_\ell(K_n)|\mathcal{F}] \gg (\log n)^{-(d-\ell+1)} \sum_{i=1}^{m} \mathbf{1}_{A_i} .$$

Taking expectations and finally applying Lemma 9 as well as Lemma 5, we arrive at

$$\begin{aligned}\mathbf{Var}[V_\ell(K_n)] &\gg (\log n)^{-(d-\ell+1)} \sum_{i=1}^{m} \mathbf{P}(A_i)\\ &\gg (\log n)^{-(d-\ell+1)} \times (\log n)^{\frac{d-1}{2}}\\ &= (\log n)^{\ell-\frac{d+3}{2}}\,.\end{aligned}$$

This completes the argument and the proof of Theorem 1. □

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