Milne's Correcting Factor and Derived De Rham Cohomology II

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Received: December 1, 2016

Communicated by Stephen Lichtenbaum

Abstract. Milne's correcting factor, which appears in the Zeta-value at $s=n$ of a smooth projective variety X over a finite field $\mathbb{F}_q$, is the Euler characteristic of the derived de Rham cohomology of $X/\mathbb{Z}$ modulo the Hodge filtration F^n. In this note, we extend this result to arbitrary separated schemes of finite type over $\mathbb{F}_q$ of dimension at most d, provided resolution of singularities for schemes of dimension at most d holds. More precisely, we show that Geisser's generalization of Milne's factor, whenever it is well defined, is the Euler characteristic of the eh-cohomology with compact support of the derived de Rham complex relative to $\mathbb{Z}$ modulo F^n.

2010 Mathematics Subject Classification: 14G10, 14F40, 11S40, 11G25

Keywords and Phrases: Zeta functions, Special values, Derived de Rham cohomology, eh-cohomology

1 Introduction

For any separated scheme X of finite type over the finite field $\mathbb{F}_q$, the special values of the zeta function $Z(X,t) := \prod_{x\in X_0}(1-t^{\deg(x)})^{-1}$ are conjecturally given by

$$\lim_{t\to q^{-n}} Z(X,t)\cdot(1-q^nt)^{\rho_n} = \pm\chi(H^*_{W,c}(X,\mathbb{Z}(n)),\cup e)\cdot q^{\chi_c^{eh}(X/\mathbb{F}_q,\mathcal{O},n)}. \quad (1)$$

Here $H^*_{W,c}(X,\mathbb{Z}(n))$ denotes Geisser's "arithmetic cohomology with compact support", $\cup e$ is cup-product with the fundamental class $e\in H^1(W_{\mathbb{F}_q},\mathbb{Z})$ and $q^{\chi_c^{eh}(X/\mathbb{F}_q,\mathcal{O},n)}$ is Geisser's generalization of Milne's correcting factor. The factor $q^{\chi_c^{eh}(X/\mathbb{F}_q,\mathcal{O},n)}$ is well defined under the assumption that resolution of singularities for schemes of dimension $\leq \dim(X)$ holds. The same assumption guaranties that, for X smooth projective, $H^*_{W,c}(X,\mathbb{Z}(n))$ coincides with Weil-étale motivic cohomology and $q^{\chi_c^{eh}(X/\mathbb{F}_q,\mathcal{O},n)}$ coincides with Milne's correcting factor. For arbitrary X, the definitions of $H^*_{W,c}(X,\mathbb{Z}(n))$ and $q^{\chi_c^{eh}(X/\mathbb{F}_q,\mathcal{O},n)}$ involve eh-cohomology with compact support. For instance

$$\chi_c^{eh}(X/\mathbb{F}_q,\mathcal{O},n):=\sum_{i\leq n,j\in\mathbb{Z}}(-1)^{i+j}\cdot(n-i)\cdot\dim_{\mathbb{F}_q}H^j_c(X_{eh},\Omega^i)$$

where $H^i_c(X_{eh},\Omega^i)$ denotes eh-cohomology with compact support of the sheaf of differentials Ω^i. Let $\mathbf{Sch}^d/\mathbb{F}_q$ be the category of separated schemes of finite type over $\mathbb{F}_q$ of dimension at most d. We say that $R(d)$ holds if any $X\in\mathbf{Sch}^d/\mathbb{F}_q$ admits resolution of singularities (see [2] Definition 2.4 for a precise statement). T. Geisser has shown in [2] that, if $R(d)$ holds and if the groups $H^i_W(Y,\mathbb{Z}(n))$ are finitely generated for any smooth projective variety Y of dimension at most d, then $\chi_c^{eh}(X/\mathbb{F}_q,\mathcal{O},n)$ is well defined and (1) holds for any $X\in\mathbf{Sch}^d/\mathbb{F}_q$.
It was pointed out in [6] that, for X smooth projective, Milne's correcting factor is the (multiplicative) Euler-Poincaré characteristic of the derived de Rham cohomology complex $R\Gamma(X_{Zar},L\Omega^*_{X/\mathbb{Z}}/F^n)$ and that (1) can be restated in terms of a certain fundamental line. The aim of this note is to show that this remark applies for arbitrary separated schemes of finite type over $\mathbb{F}_q$. More precisely, we denote by $\mathrm{Sh}_{eh}(\mathbf{Sch}^d/\mathbb{F}_q)$ the category of sheaves of sets on the category $\mathbf{Sch}^d/\mathbb{F}_q$ endowed with the eh-topology. The resulting eh-topos $\mathrm{Sh}_{eh}(\mathbf{Sch}^d/\mathbb{F}_q)$ is endowed with a structure ring $\mathcal{O}^{eh}$, which is defined as the eh-sheafification of the presheaf $X\mapsto\mathcal{O}_X(X)$ on $\mathbf{Sch}^d/\mathbb{F}_q$. We denote by $L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n$ the derived de Rham complex modulo the Hodge filtratio n F^n associated with the morphism of ringed topoi

$$(\mathrm{Sh}_{eh}(\mathbf{Sch}^d/\mathbb{F}_q),\mathcal{O}^{eh})\longrightarrow(\mathrm{Spec}(\mathbb{Z}),\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})})$$

where $\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})}$ is the usual structure sheaf on $\mathrm{Spec}(\mathbb{Z})$. Then we consider its cohomology with compact support $R\Gamma_c(X_{eh},L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n)$. Under the assumption of Theorem 1.1(4) below, one may define the fundamental line

$$\Delta(X/\mathbb{Z},n):=\det{}_{\mathbb{Z}}R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes_{\mathbb{Z}}\det{}_{\mathbb{Z}}R\Gamma_c(X_{eh},L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n)$$

and its trivialization

$$\lambda_X:\mathbb{R}\xrightarrow{\sim}\Delta(X/\mathbb{Z},n)\otimes_{\mathbb{Z}}\mathbb{R}$$

which is induced by the acyclic complex

$$\cdots\xrightarrow{\cup\theta}H^i_{W,c}(X,\mathbb{Z}(n))_{\mathbb{R}}\xrightarrow{\cup\theta}H^{i+1}_{W,c}(X,\mathbb{Z}(n))_{\mathbb{R}}\xrightarrow{\cup\theta}\cdots$$

Here the fundamental class $\theta=\mathrm{Id}_{\mathbb{R}}\in H^1(\mathbb{R},\mathbb{R})=$ "$H^1(W_{\mathbb{F}_1},\mathbb{R})$" is in some sense analogous to $e\in H^1(W_{\mathbb{F}_q},\mathbb{Z})$. We denote by $\zeta^*(X,n)$ the leading coefficient in the Taylor development of $\zeta(X,s)=Z(X,q^{-s})$ near $s=n$.

Theorem 1.1. *Let X be a separated scheme of finite type over $\mathbb{F}_q$ and let $n \in \mathbb{Z}$ be an integer. Assume that X has dimension d and that $R(d)$ holds.*

1. *If X is smooth projective, the canonical map*

$$R\Gamma(X_{Zar}, L\Omega^*_{X/\mathbb{Z}}/F^n) \to R\Gamma_c(X_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n)$$

   is a quasi-isomorphism.

2. *The complex $R\Gamma_c(X_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n)$ is bounded with finite cohomology groups.*

3. *We have*

$$\prod_{i\in\mathbb{Z}} \mid H^i_c(X_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n) \mid^{(-1)^i} \;=\; q^{\chi^{eh}_c(X/\mathbb{F}_q,\mathcal{O},n)}. \tag{2}$$

4. *Assume moreover that for any smooth projective variety Y of dimension $\leq d$, the usual Weil-étale cohomology groups $H^i_W(Y,\mathbb{Z}(n))$ are finitely generated for all i. Then one has*

$$\Delta(X/\mathbb{Z}, n) \;=\; \mathbb{Z}\cdot\lambda_X\left(\zeta^*(X,n)^{-1}\right).$$

In particular, Theorem 1.1(1)–(3) holds (unconditionally) for $\dim(X) \leq 2$ and Theorem 1.1(4) holds for $\dim(X) \leq 1$. This note is organized as follows. We fix some notations and definitions in Section 2. In Section 3, we give the proof of Theorem 1.1, which is based on the following computation of the cohomology sheaves of the complex $L\Lambda^n_{\mathcal{O}^{eh}} L_{\mathcal{O}^{eh}/\mathbb{Z}}$: we define an isomorphism (see Proposition 3.6)

$$\mathcal{H}^{i-n}(L\Lambda^n_{\mathcal{O}^{eh}} L_{\mathcal{O}^{eh}/\mathbb{Z}}) \simeq \Omega^{i\leq n}_{\mathcal{O}^{eh}/\mathbb{F}_q}$$

where $\Omega^{i\leq n} := \Omega^i$ for $i \leq n$ and $\Omega^{i\leq n} := 0$ for $i > n$. This argument also gives a slightly different proof of the main result of [6], see Remark 3.5.

2 Preliminaries

2.1 The derived de Rham complex

Given a ring A and an A-module M, we denote by $\Lambda_A(M)$ (resp. $\Gamma_A(M)$) the exterior A-algebra of M (resp. the divided power algebra of M, see [1] App. A), and by $\Lambda^i_A(M)$ (resp. $\Gamma^i_A(M)$) its submodule of homogeneous elements of degree i. If $(\mathcal{S}, A)$ is a ringed topos and M an A-module, one defines $\Lambda_A(M)$, $\Gamma_A(M)$, $\Lambda^i_A(M)$ and $\Gamma^i_A(M)$ as above, internally in $\mathcal{S}$. Then $\Lambda_A(M)$ (resp. $\Gamma_A(M)$) coincides with the sheafification of $U \mapsto \Lambda_{A(U)}(M(U))$ (resp. $U \mapsto \Gamma_{A(U)}(M(U))$). We denote by $L\Lambda^i_A$ the left derived functor of the (non-additive) exterior power functor Λ^i_A (see [4] I.4.2). We often omit the subscript A and simply write $\Lambda^i M$, $\Gamma^i M$ and $L\Lambda^i M$. Let $A \to B$ be a morphism of rings in $\mathcal{S}$. We denote by $\Omega^1_{B/A}$ the B-module of Kähler differe ntials, we set $\Omega^i_{B/A} := \Lambda^i_B \Omega^1_{B/A}$ and we denote by $\Omega^{<n}_{B/A}$ the complex of A-modules $[\Omega^0_{B/A} \to \Omega^1_{B/A} \to \cdots \to \Omega^{n-1}_{B/A}]$ put in degrees $[0, n-1]$. Let $P_A(B)$ be the standard simplicial free resolution of the A-algebra B (see [4] I.1.5.5.6), and let $L_{B/A}$ be the cotangent complex ([4] II.1). By definition $L_{B/A}$ is the complex of B-modules associated with the simplicial B-module $\Omega^1_{P_A(B)/A} \otimes_{P_A(B)} B$. Similarly we define $L\Lambda^i_B L_{B/A}$ as the (actual) complex of B-modules associated with the simplicial B-module $\Omega^i_{P_A(B)/A} \otimes_{P_A(B)} B$. The derived de Rham complex modulo F^n is defined as the total complex (see [5] VIII.2.1)

$$L\Omega^*_{B/A}/F^n := \mathrm{Tot}(\Omega^{<n}_{P_A(B)/A})$$

which we simply see in this paper as a complex of A-modules. The Hodge filtration on $L\Omega^*_{B/A}/F^n$ satisfies $\mathrm{gr}^p(L\Omega^*_{B/A}/F^n) \simeq L\Lambda^p_B L_{B/A}[-p]$ for $p < n$ and $\mathrm{gr}^p(L\Omega^*_{B/A}/F^n) = 0$ otherwise. For example, if $(X, \mathcal{O}_X)$ is a scheme, then $P_{\mathbb{Z}}(\mathcal{O}_X)$ denotes the standard simplicial free resolution of $\mathbb{Z} \to \mathcal{O}_X$ in the small Zariski topos of the scheme X, and $L_{X/\mathbb{Z}} := L_{\mathcal{O}_X/\mathbb{Z}}$ is the cotangent complex associated with the morphism of schemes $X \to \mathrm{Spec}(\mathbb{Z})$.

If $f : \mathcal{S}' \to \mathcal{S}$ is a morphism of topoi, we write $f^{-1} : \mathcal{S} \to \mathcal{S}'$ for the set-theoretic inverse image functor of f. Let $f : (\mathcal{S}', A') \to (\mathcal{S}, A)$ be a morphism of ringed topoi, i.e. a morphism of topoi $f : \mathcal{S}' \to \mathcal{S}$ together with a morphism of rings $f^{-1}A \to A'$ in $\mathcal{S}'$. One defines

$$L\Omega^*_f/F^n = L\Omega^*_{(\mathcal{S}',A')/(\mathcal{S},A)}/F^n := L\Omega^*_{A'/f^{-1}A}/F^n$$

which is a complex of $f^{-1}A$-modules in $\mathcal{S}'$. We denote by $f^* : \mathrm{Mod}(A) \to \mathrm{Mod}(A')$ the inverse image functor for modules, i.e. $f^*M := f^{-1}M \otimes_{f^{-1}A} A'$, where $\mathrm{Mod}(A)$ (resp. $\mathrm{Mod}(A')$) is the category of A-modules in $\mathcal{S}$ (resp. of A'-modules in $\mathcal{S}'$).

Lemma 2.1. *Let $f : \mathcal{S}' \to \mathcal{S}$ be a morphism of topoi and let $A \to B$ be a morphism of rings in $\mathcal{S}$. Then we have $f^{-1}(P_A(B)) \simeq P_{f^{-1}A}(f^{-1}B)$, $f^{-1}\left(L\Omega^*_{B/A}/F^n\right) \simeq L\Omega^*_{f^{-1}B/f^{-1}A}/F^n$, an isomorphism of $f^{-1}B$-modules $f^{-1}(\Omega^i_{B/A}) \simeq \Omega^i_{f^{-1}B/f^{-1}A}$ and an isomorphism of complexes of $f^{-1}B$-modules $f^{-1}(L\Lambda^i_B L_{B/A}) \simeq L\Lambda^i_{f^{-1}B} L_{f^{-1}B/f^{-1}A}$.*

Proof. The identifications $f^{-1}(P_A(B)) \simeq P_{f^{-1}A}(f^{-1}B)$ and $f^{-1}(\Omega^1_{B/A}) \simeq \Omega^1_{f^{-1}B/f^{-1}A}$ follow from the definitions (see [4] II.1.2.1.4 and [4] II.1.1.4.1). Moreover we have $f^{-1}(\Lambda^i_R(M)) \simeq \Lambda^i_{f^{-1}R}(f^{-1}M)$ for any ring R in $\mathcal{S}$ and any R-module M. The result follows easily. □

2.2 Derived de Rham cohomology with compact support

The following definition is due to Thomas Geisser [2]. Let $\mathbf{Sch}^d/\mathbb{F}_q$ be the category of separated schemes of finite type over $\mathbb{F}_q$ of dimension $\leq d$.

Definition 2.2. *The eh-topology on* $\mathbf{Sch}^d/\mathbb{F}_q$ *is the Grothendieck topology generated by the following coverings:*

- *étale coverings*
- *abstract blow-ups: If we have a cartesian square*

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "f'"] & X' \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

where f is proper, i a closed embedding, and f induces an isomorphism $X' - Z' \xrightarrow{\sim} X - Z$, then $(X' \xrightarrow{f} X, Z \xrightarrow{i} X)$ is a covering.

We denote by $\mathrm{PSh}(\mathbf{Sch}^d/\mathbb{F}_q)$ the category of presheaves of sets on $\mathbf{Sch}^d/\mathbb{F}_q$ and by $\mathrm{Sh}_{eh}(\mathbf{Sch}^d/\mathbb{F}_q)$ the topos of eh-sheaves of sets on $\mathbf{Sch}^d/\mathbb{F}_q$. Note that the functor

$$y : \mathbf{Sch}^d/\mathbb{F}_q \hookrightarrow \mathrm{PSh}(\mathbf{Sch}^d/\mathbb{F}_q) \to \mathrm{Sh}_{eh}(\mathbf{Sch}^d/\mathbb{F}_q),$$

given by composing the Yoneda embedding and eh-sheafification, is not fully faithful. Hence the eh-topology is not subcanonical. For example, if X^{red} denotes the maximal reduced closed subscheme of $X \in \mathbf{Sch}^d/\mathbb{F}_q$, then the induced map $yX^{\mathrm{red}} \to yX$ is an isomorphism. If U is an object of $\mathbf{Sch}^d/\mathbb{F}_q$ and $\mathcal{F}$ an eh-sheaf on $\mathbf{Sch}^d/\mathbb{F}_q$, we choose a Nagata compactification $U \hookrightarrow X$ with closed complement $Z \hookrightarrow X$ (so that X is proper over $\mathbb{F}_q$ and U is open and dense in X), and we define

$$R\Gamma_c(U_{eh}, \mathcal{F}) := \mathrm{Cone}\left(R\Gamma(X_{eh}, \mathcal{F}) \to R\Gamma(Z_{eh}, \mathcal{F})\right)[-1].$$

Here $R\Gamma(X_{eh}, \mathcal{F})$ denotes the cohomology of the slice topos $\mathrm{Sh}_{eh}(\mathbf{Sch}^d/\mathbb{F}_q)/yX$ with coefficients in $\mathcal{F}\times yX \to yX$. Equivalently, $R\Gamma(X_{eh}, -)$ is the total derived functor of the functor $\mathcal{F} \mapsto \mathcal{F}(X)$. It can be shown that $R\Gamma_c(U_{eh}, \mathcal{F})$ does not depend on the compactification (see [2] Proposition 3.2). Then $R\Gamma_c(U_{eh}, \mathcal{F})$ is contravariant for proper maps and covariant for open immersions. For an open-closed decomposition $(U \xrightarrow{j} X \xleftarrow{i} Z)$, there is an exact triangle

$$R\Gamma_c(U_{eh}, \mathcal{F}) \to R\Gamma_c(X_{eh}, \mathcal{F}) \to R\Gamma_c(Z_{eh}, \mathcal{F}) \to$$

Notation 2.3. *The structure ring $\mathcal{O}^{eh}$ on $\mathrm{Sh}_{eh}(\mathbf{Sch}^d/\mathbb{F}_q)$ is the eh-sheaf associated with the presheaf of rings*

$$\begin{array}{cccc} \mathcal{R}: & (\mathbf{Sch}^d/\mathbb{F}_q)^{op} & \longrightarrow & \mathrm{Rings} \\ & X & \longmapsto & \mathcal{O}_X(X) \end{array}.$$

Consider the morphism of ringed topoi

$$\psi : (\mathrm{Sh}_{eh}(\mathbf{Sch}^d/\mathbb{F}_q), \mathcal{O}^{eh}) \longrightarrow (\mathrm{Spec}(\mathbb{Z}), \mathcal{O}_{\mathrm{Spec}(\mathbb{Z})})$$

induced by the evident morphism of sites. Let

$$L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n := L\Omega^*_{\psi}/F^n$$

be the corresponding derived de Rham complex modulo the n^{th}*-step of the Hodge filtration. Derived de Rham cohomology modulo* F^n *with compact support is given by*

$$X \mapsto R\Gamma_c(X_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n)$$

for $X \in \mathbf{Sch}^d/\mathbb{F}_q$*. It is covariantly functorial for open immersions and contravariantly functorial for proper maps.*

We now explain our notation $L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n$. There is a unique morphism of rings $\mathbb{Z}^{eh} \to \mathcal{O}^{eh}$, where $\mathbb{Z}^{eh}$ denotes the constant sheaf of rings associated with $\mathbb{Z}$ on $\mathrm{Sh}_{eh}(\mathbf{Sch}^d/\mathbb{F}_q)$. Let $L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}^{eh}}/F^n$ be the corresponding derived de Rham complex modulo F^n. Then we have

$$L\Omega^*_{\psi}/F^n \simeq L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}^{eh}}/F^n. \tag{3}$$

Indeed, consider the structure sheaf $\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})}$ and the constant sheaf $\mathbb{Z}$ over the small Zariski topos of $\mathrm{Spec}(\mathbb{Z})$. We have $L_{\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})}/\mathbb{Z}} = 0$ (see [4] II.2.3.1 and II.2.3.6), hence $L_{\mathcal{O}^{eh}/\psi^{-1}\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})}} \simeq L_{\mathcal{O}^{eh}/\psi^{-1}\mathbb{Z}} = L_{\mathcal{O}^{eh}/\mathbb{Z}^{eh}}$. We obtain $L\Lambda^* L_{\mathcal{O}^{eh}/\mathbb{Z}^{eh}} \simeq L\Lambda^* L_{\mathcal{O}^{eh}/\psi^{-1}\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})}}$ hence

$$L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}^{eh}}/F^n \simeq L\Omega^*_{\mathcal{O}^{eh}/\psi^{-1}\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})}}/F^n := L\Omega^*_{\psi}/F^n$$

by the Hodge filtration. Finally, we note that $L_{\mathcal{O}^{eh}/\mathbb{Z}}$ and $L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n$ could be left-unbounded (see however Corollary 3.8).

2.3 The fundamental line

For an object C in the derived category of abelian groups such that $H^i(C)$ is finitely generated for all i and $H^i(C) = 0$ for almost all i, we set

$$\det_{\mathbb{Z}}(C) := \bigotimes_{i\in\mathbb{Z}} \det_{\mathbb{Z}}^{(-1)^i} H^i(C).$$

If $H^i(C)$ is moreover finite for all i, then we call the following isomorphism

$$\det_{\mathbb{Z}}(C)\otimes_{\mathbb{Z}}\mathbb{Q} \xrightarrow{\sim} \bigotimes_{i\in\mathbb{Z}} \det_{\mathbb{Q}}^{(-1)^i}\left(H^i(C)\otimes_{\mathbb{Z}}\mathbb{Q}\right) \xrightarrow{\sim} \bigotimes_{i\in\mathbb{Z}} \det_{\mathbb{Q}}^{(-1)^i}(0) \xrightarrow{\sim} \mathbb{Q}$$

the canonical $\mathbb{Q}$-trivialization of $\det_{\mathbb{Z}}(C)$. In this situation, the canonical $\mathbb{Q}$-trivialization $\det_{\mathbb{Z}}(C)\otimes_{\mathbb{Z}}\mathbb{Q} \simeq \mathbb{Q}$ identifies $\det_{\mathbb{Z}}(C)$ with

$$\mathbb{Z}\cdot\left(\prod_{i\in\mathbb{Z}} |H^i(C)|^{(-1)^{i+1}}\right) \subset \mathbb{Q}.$$

For $X \in \mathbf{Sch}^d/\mathbb{F}_q$, one defines [2]

$$R\Gamma_{W,c}(X,\mathbb{Z}(n)) := R\Gamma(W_{\mathbb{F}_q}, R\Gamma_c(X_{\overline{\mathbb{F}}_q,eh}, \rho^{-1}\mathbb{Z}(n)))$$

where $\overline{\mathbb{F}}_q$ is an algebraic closure, $W_{\mathbb{F}_q}$ is the Weil group, ρ is the morphism defined in Lemma 3.1, and the $\mathbb{Z}(n)$ on the right hand side is the motivic complex on $\mathbf{Sm}^d/\mathbb{F}_q$. Assuming that $R\Gamma_{W,c}(X,\mathbb{Z}(n))$ and $R\Gamma_c(X_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n)$ are both well defined and perfect, the fundamental line is defined as follows:

$$\Delta(X/\mathbb{Z},n) := \det{}_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det{}_{\mathbb{Z}} R\Gamma_c(X_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n).$$

Recall that $W_{\mathbb{F}_q} \simeq \mathbb{Z}$ with generator given by the Frobenius F. Consider the map $\mathfrak{f} : W_{\mathbb{F}_q} \to W_{\mathbb{F}_1} := \mathbb{R}$ satisfying $\mathfrak{f}(F) = \log(q)$, and define $\theta = \mathrm{Id}_{\mathbb{R}} \in H^1(\mathbb{R},\mathbb{R})$. Then $\mathfrak{f}^*\theta \in H^1(W_{\mathbb{F}_q},\mathbb{R})$ maps the Frobenius $F \in W_{\mathbb{F}_q}$ to $\log(q) \in \mathbb{R}$, whereas $e \in H^1(W_{\mathbb{F}_q},\mathbb{R})$ maps the Frobenius F to $1 \in \mathbb{R}$. We have

$$R\Gamma_{W,c}(X,\mathbb{Z}(n))_{\mathbb{R}} \simeq R\Gamma(W_{\mathbb{F}_q}, R\Gamma_c(X_{\overline{\mathbb{F}}_q,eh}, \rho^{-1}\mathbb{Z}(n))_{\mathbb{R}}).$$

So cup-product with the class $\mathfrak{f}^*\theta \in H^1(W_{\mathbb{F}_q},\mathbb{R})$ defines a map

$$H^i_{W,c}(X,\mathbb{Z}(n))_{\mathbb{R}} \xrightarrow{\cup\theta} H^{i+1}_{W,c}(X,\mathbb{Z}(n))_{\mathbb{R}}$$

which differs from

$$H^i_{W,c}(X,\mathbb{Z}(n))_{\mathbb{R}} \xrightarrow{\cup e} H^{i+1}_{W,c}(X,\mathbb{Z}(n))_{\mathbb{R}}$$

by the factor $\log(q)$. The complex

$$\cdots \xrightarrow{\cup\theta} H^i_{W,c}(X,\mathbb{Z}(n))_{\mathbb{R}} \xrightarrow{\cup\theta} H^{i+1}_{W,c}(X,\mathbb{Z}(n))_{\mathbb{R}} \xrightarrow{\cup\theta} \cdots$$

is acyclic [2] hence gives a trivialization

$$\lambda_X : \mathbb{R} \xrightarrow{\sim} \det{}_{\mathbb{R}} R\Gamma_{W,c}(X,\mathbb{Z}(n))_{\mathbb{R}} \xrightarrow{\sim} \Delta(X/\mathbb{Z},n) \otimes_{\mathbb{Z}} \mathbb{R}.$$

where the second isomorphism is induced by the canonical $\mathbb{Q}$-trivialization of $\det_{\mathbb{Z}} R\Gamma_c(X_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n)$, whose existence requires that $R\Gamma_c(X_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n)$ is bounded with finite cohomology groups.

3 Proof of Theorem 1.1

We denote by $\mathbf{Sm}^d/\mathbb{F}_q$ the full subcategory of $\mathbf{Sch}^d/\mathbb{F}_q$ consisting of smooth $\mathbb{F}_q$-schemes. We endow $\mathbf{Sm}^d/\mathbb{F}_q$ with the Zariski topology and we denote by $\mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q)$ the corresponding topos.

Recall the following description of the topos $\mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q)$ (see [3] IV.4.10.6). A sheaf $\mathcal{F}$ on $\mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q)$ can be seen as a family of sheaves $\mathcal{F}_X$ on the small Zariski topos X_{Zar} for any $X \in \mathbf{Sm}^d/\mathbb{F}_q$ together with transition maps

$\alpha_f : f^{-1}\mathcal{F}_X \to \mathcal{F}_Y$ for any map $f : Y \to X$ satisfying $\alpha_{f\circ g} = \alpha_g \circ g^{-1}\alpha_f$ and such that α_f is an isomorphism whenever f is an open immersion. A morphism $\mathcal{F} \to \mathcal{G}$ in $\mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q)$ is a given by a family of morphisms $\mathcal{F}_X \to \mathcal{G}_X$ compatible with the transition maps. For any $X \in \mathbf{Sm}^d/\mathbb{F}_q$, the functor

$$\begin{array}{rccc} \mathrm{res}_X : & \mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q) & \longrightarrow & X_{Zar} \\ & \mathcal{F} & \longmapsto & \mathcal{F}_X \end{array},$$

mapping the big Zariski sheaf $\mathcal{F}$ to its restriction $\mathcal{F}_X$ to the small Zariski site of X, commutes with arbitrary small limits and colimits. It is therefore the inverse image of a morphism of topoi

$$s_X : X_{Zar} \longrightarrow \mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q)/X \longrightarrow \mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q).$$

In fact the morphism $X_{Zar} \longrightarrow \mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q)/X$ is a section of the morphism

$$\mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q)/X \simeq \mathrm{Sh}_{Zar}((\mathbf{Sm}^d/\mathbb{F}_q)/X) \longrightarrow X_{Zar} \tag{4}$$

which is induced by the evident morphism of sites. The same description of abelian sheaves on $\mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q)$ is valid. We denote by $\mathcal{O}$ the canonical structure ring on $\mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q)$, i.e. $\mathcal{O}(X) := \mathcal{O}_X(X)$ for any $X \in \mathbf{Sm}^d/\mathbb{F}_q$. We have $\mathrm{res}_X(\mathcal{O}) = \mathcal{O}_X$ where $\mathcal{O}_X$ denotes the usual structure sheaf on the smooth scheme X. As above, a complex of $\mathcal{O}$-modules $\mathcal{F}$ can be seen as family of complexes of $\mathcal{O}_X$-modules $\mathcal{F}_X$ in the small Zariski topos X_{Zar} together with transition maps of complexes of $\mathcal{O}_Y$-modules $\alpha_f : f^*\mathcal{F}_X := f^{-1}\mathcal{F}_X \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y \to \mathcal{F}_Y$ for any map $f : Y \to X$ satisfying $\alpha_{f\circ g} = \alpha_g \circ g^*\alpha_f$, and such that α_f is an isomorphism whenever f is an open immer sion.

We denote by $R(d)$ the condition given in ([2] Definition 2.4). The morphism ρ of the next lemma was defined in ([2] Lemma 2.5), see also ([7] Proposition 5.11).

Lemma 3.1. *Assume that $R(d)$ holds. Then we have a composite morphism of topoi*

$$\rho : \mathrm{Sh}_{eh}(\mathbf{Sch}^d/\mathbb{F}_q) \xrightarrow{\sim} \mathrm{Sh}_{eh}(\mathbf{Sm}^d/\mathbb{F}_q) \longrightarrow \mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q)$$

where the first morphism is an equivalence. Moreover we have

$$\rho^{-1}\mathcal{O} \simeq \mathcal{O}^{eh}. \tag{5}$$

Proof. We consider the topology on $\mathbf{Sm}^d/\mathbb{F}_q$ induced by the eh-topology on $\mathbf{Sch}^d/\mathbb{F}_q$ (see [3] III.3), and we define $\mathrm{Sh}_{eh}(\mathbf{Sm}^d/\mathbb{F}_q)$ as the topos of sheaves on this site. It follows from $R(d)$ and ([2] Lemma 2.2.b) that $\mathbf{Sm}^d/\mathbb{F}_q$ is a topologically generating full subcategory of $\mathbf{Sch}^d/\mathbb{F}_q$ with respect to the eh-topology. By ([3] III.4.1), the first morphism is an equivalence. The inclusion functor $(\mathbf{Sm}^d/\mathbb{F}_q, Zar) \to (\mathbf{Sch}^d/\mathbb{F}_q, eh)$ is continuous, i.e. if $\mathcal{F}$ is an eh-sheaf on $\mathbf{Sch}^d/\mathbb{F}_q$ then its restriction to $\mathbf{Sm}^d/\mathbb{F}_q$ is a Zariski sheaf. In other words, the induced eh-topology on $\mathbf{Sm}^d/\mathbb{F}_q$ is stronger than the Zariski topology; hence the second morphism is well defined.

Let $u : \mathbf{Sm}^d/\mathbb{F}_q \to \mathbf{Sch}^d/\mathbb{F}_q$ be the inclusion functor, and let $f : \mathrm{Sh}_{eh}(\mathbf{Sch}^d/\mathbb{F}_q) \xrightarrow{\sim} \mathrm{Sh}_{eh}(\mathbf{Sm}^d/\mathbb{F}_q)$ be the induced equivalence. We have a commutative square (see [3] III.1.3)

$$\begin{tikzcd} \mathrm{Sh}_{eh}(\mathbf{Sch}^d/\mathbb{F}_q) & \mathrm{Sh}_{eh}(\mathbf{Sm}^d/\mathbb{F}_q) \arrow[l, "f^{-1}"'] \\ \mathrm{PSh}(\mathbf{Sch}^d/\mathbb{F}_q) \arrow[u, "a"] & \mathrm{PSh}(\mathbf{Sm}^d/\mathbb{F}_q) \arrow[l, "u_!"'] \arrow[u, "a^{\mathrm{sm}}"'] \end{tikzcd}$$

where the vertical arrows are the associated sheaf functors. Let $\mathcal{F} \in \mathrm{PSh}(\mathbf{Sch}^d/\mathbb{F}_q)$ be a presheaf of sets and let u^* be the right adjoint of $u_!$. The adjunction morphism $u_!u^*\mathcal{F} \to \mathcal{F}$ is "bicouvrant" (see [3] III.4.1.1) hence $a(u_!u^*\mathcal{F}) \xrightarrow{\sim} a(\mathcal{F})$ is an isomorphism (see [3] II.5.3). Since the square above is commutative, we obtain

$$f^{-1} \circ a^{\mathrm{sm}} \circ u^*(\mathcal{F}) \simeq a \circ u_! \circ u^*(\mathcal{F}) \simeq a(\mathcal{F}).$$

So we have an isomorphism of left exact functors $f^{-1} \circ a^{\mathrm{sm}} \circ u^* \simeq a$, hence a similar isomorphism of functors between the categories of ring objects. Let $\mathcal{R}$ (resp. $\mathcal{O}$) be the presheaf of rings on $\mathbf{Sch}^d/\mathbb{F}_q$ (resp. on $\mathbf{Sm}^d/\mathbb{F}_q$) mapping X to $\mathcal{O}_X(X)$. By definition we have $\mathcal{O} = u^*\mathcal{R}$, $\mathcal{O}^{eh} = a(\mathcal{R})$ and $g^{-1}\mathcal{O} = a^{\mathrm{sm}}(\mathcal{O})$, where $g : \mathrm{Sh}_{eh}(\mathbf{Sm}^d/\mathbb{F}_q) \to \mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q)$ is the morphism of topoi defined above. We obtain

$$\rho^{-1}(\mathcal{O}) \simeq f^{-1} \circ g^{-1}(\mathcal{O}) \simeq f^{-1} \circ a^{\mathrm{sm}}(\mathcal{O}) \simeq f^{-1} \circ a^{\mathrm{sm}} \circ u^*(\mathcal{R}) \simeq a(\mathcal{R}) =: \mathcal{O}^{eh}.$$

□

We may therefore promote ρ into a morphism of ringed topoi

$$\rho : (\mathrm{Sh}_{eh}(\mathbf{Sch}^d/\mathbb{F}_q), \mathcal{O}^{eh}) \longrightarrow (\mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q), \mathcal{O}).$$

For any $X \in \mathbf{Sm}^d/\mathbb{F}_q$, we shall also consider the morphism of ringed topoi obtained by localisation over X:

$$\rho_{/X} : (\mathrm{Sh}_{eh}(\mathbf{Sch}^d/\mathbb{F}_q), \mathcal{O}^{eh})/yX \longrightarrow (\mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q), \mathcal{O})/X.$$

We denote by $\mathbb{Z}$ the constant sheaf on either $\mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q)$ or $\mathrm{Sh}_{eh}(\mathbf{Sch}^d/\mathbb{F}_q)$, and we apply the constructions of Section 2.1 to the unique morphism of rings $\mathbb{Z} \to \mathcal{O}^{eh}$ (respectively $\mathbb{Z} \to \mathcal{O}$) in the topos $\mathrm{Sh}_{eh}(\mathbf{Sch}^d/\mathbb{F}_q)$ (respectively in the topos $\mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q)$); see (3) and its proof.

Lemma 3.2. *Assume $R(d)$. We have*

$$L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n \simeq \rho^{-1}\left(L\Omega^*_{\mathcal{O}/\mathbb{Z}}/F^n\right) \tag{6}$$

and the complex of abelian sheaves $L\Omega^*_{\mathcal{O}/\mathbb{Z}}/F^n$ on $\mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q)$ is given by the complexes of abelian sheaves on X_{Zar}

$$\mathrm{res}_X\left(L\Omega^*_{\mathcal{O}/\mathbb{Z}}/F^n\right) = L\Omega^*_{X/\mathbb{Z}}/F^n := \mathrm{Tot}(\Omega^{<n}_{P_{\mathbb{Z}}(\mathcal{O}_X)/\mathbb{Z}}) \tag{7}$$

and obvious transition maps. Similarly, we have an isomorphism of complexes of $\mathcal{O}^{eh}$-modules

$$L\Lambda^i_{\mathcal{O}^{eh}}L_{\mathcal{O}^{eh}/\mathbb{Z}} \simeq \rho^{-1}\left(L\Lambda^i_{\mathcal{O}}L_{\mathcal{O}/\mathbb{Z}}\right) \tag{8}$$

and the complex of $\mathcal{O}$-modules $L\Lambda^i_{\mathcal{O}}L_{\mathcal{O}/\mathbb{Z}}$ is given by the complexes of $\mathcal{O}_X$-modules

$$\mathrm{res}_X\left(L\Lambda^i_{\mathcal{O}}L_{\mathcal{O}/\mathbb{Z}}\right) = L\Lambda^i_{\mathcal{O}_X}L_{X/\mathbb{Z}} := \mathrm{Tot}(\Omega^i_{P_{\mathbb{Z}}(\mathcal{O}_X)/\mathbb{Z}} \otimes_{P_{\mathbb{Z}}(\mathcal{O}_X)} \mathcal{O}_X) \tag{9}$$

and obvious transition maps. Finally, we have an isomorphism of $\mathcal{O}^{eh}$-modules

$$\Omega^i_{\mathcal{O}^{eh}/\mathbb{Z}} \simeq \rho^{-1}\Omega^i_{\mathcal{O}/\mathbb{Z}} \tag{10}$$

and the $\mathcal{O}$-module $\Omega^i_{\mathcal{O}/\mathbb{Z}}$ is given by the $\mathcal{O}_X$-modules

$$\mathrm{res}_X\left(\Omega^i_{\mathcal{O}/\mathbb{Z}}\right) = \Omega^i_{X/\mathbb{Z}} \tag{11}$$

and obvious transition maps.

Proof. The complex $L\Omega^*_{X/\mathbb{Z}}/F^n := \mathrm{Tot}(\Omega^{<n}_{P_{\mathbb{Z}}(\mathcal{O}_X)/\mathbb{Z}})$ is functorial on the nose in $X \in \mathbf{Sm}^d/\mathbb{F}_q$. Indeed, given a map $f : Y \to X$, there is a canonical morphism of complexes of abelian sheaves

$$f^{-1}L\Omega^*_{X/\mathbb{Z}}/F^n \simeq L\Omega^*_{f^{-1}\mathcal{O}_X/\mathbb{Z}}/F^n \to L\Omega^*_{Y/\mathbb{Z}}/F^n, \tag{12}$$

where the first map is supplied by Lemma 2.1 and the second map is induced by the structural morphism $f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$. The map $f^{-1}L\Omega^*_{X/\mathbb{Z}}/F^n \to L\Omega^*_{Y/\mathbb{Z}}/F^n$ is an isomorphism of complexes of abelian sheaves if $f : Y \to X$ is an open immersion. Similarly, the map f induces a morphism of complexes of $\mathcal{O}_Y$-modules

$$f^*L\Lambda^i_{\mathcal{O}_X}L_{X/\mathbb{Z}} \simeq L\Lambda^i_{f^{-1}\mathcal{O}_X}L_{f^{-1}\mathcal{O}_X/\mathbb{Z}} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y \to L\Lambda^i_{\mathcal{O}_Y}L_{\mathcal{O}_Y/\mathbb{Z}} \tag{13}$$

which is an isomorphism of complexes if f is an open immersion. We apply Lemma 2.1 to the morphism of topoi

$$s_X : X_{Zar} \longrightarrow \mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q)$$

and we observe that the transition maps

$$f^{-1}\mathrm{res}_X(L\Omega^*_{\mathcal{O}/\mathbb{Z}}/F^n) \to \mathrm{res}_Y(L\Omega^*_{\mathcal{O}/\mathbb{Z}}/F^n)$$

and

$$f^*\mathrm{res}_X(L\Lambda^i_{\mathcal{O}}L_{\mathcal{O}/\mathbb{Z}}/F^n) \to \mathrm{res}_Y(L\Lambda^i_{\mathcal{O}}L_{\mathcal{O}/\mathbb{Z}}/F^n)$$

can be identified with (12) and (13) respectively. This yields (7) and (9). We obtain (6) and (8) by applying Lemma 2.1 to the morphism

$$\rho : \mathrm{Sh}_{eh}(\mathbf{Sch}^d/\mathbb{F}_q) \longrightarrow \mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q)$$

since we have $\mathcal{O}^{eh} \simeq \rho^{-1}\mathcal{O}$ by (5). The proof of (10) and (11) is similar. $\square$

For a complex C of sheaves of modules on some topos, we denote by $\mathcal{H}^i(C)$ its i-th cohomology sheaf.

Lemma 3.3. *Let X be a smooth separated scheme of finite type over $\mathbb{F}_q$. Then there is a canonical isomorphism of sheaves of $\mathcal{O}_X$-modules*

$$\mathcal{H}^i(L\Lambda^n L_{X/\mathbb{Z}}[-n]) \simeq \Omega^{i\leq n}_{X/\mathbb{F}_q}$$

where $\Omega^{i\leq n}_{X/\mathbb{F}_q} := \Omega^i_{X/\mathbb{F}_q}$ for $0 \leq i \leq n$ and $\Omega^{i\leq n}_{X/\mathbb{F}_q} = 0$ otherwise. Moreover, for $f : Y \to X$ a morphism in $\mathbf{Sm}^d/\mathbb{F}_q$, the square of $\mathcal{O}_Y$-modules

$$\begin{tikzcd}
f^*\mathcal{H}^i(L\Lambda^n L_{X/\mathbb{Z}}[-n]) \arrow[r, "\sim"] \arrow[d] & f^*\Omega^{i\leq n}_{X/\mathbb{F}_q} \arrow[d] \\
\mathcal{H}^i(L\Lambda^n L_{Y/\mathbb{Z}}[-n]) \arrow[r, "\sim"] & \Omega^{i\leq n}_{Y/\mathbb{F}_q}
\end{tikzcd}$$

commutes, where the left vertical map is induced by (13) and the right vertical map is the evident one.

Proof. Let X be a scheme in $\mathbf{Sm}^d/\mathbb{F}_q$. We have an exact triangle in the derived category $\mathcal{D}(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules (see [6] for details):

$$\mathcal{O}_X[1] \to L_{X/\mathbb{Z}} \to \Omega^1_{X/\mathbb{F}_q} \stackrel{\omega_X}{\to} \mathcal{O}_X[2].$$

Let $U \subset X$ be an affine open subscheme. Then $\omega_U \in \mathrm{Ext}^2_{\mathcal{O}_U}(\Omega^1_{U/\mathbb{F}_q}, \mathcal{O}_U) = 0$ and there is a unique isomorphism

$$\alpha_U : L_{U/\mathbb{Z}} \xrightarrow{\sim} \mathcal{O}_U[1] \oplus \Omega^1_{U/\mathbb{F}_q}$$

in the derived category $\mathcal{D}(\mathcal{O}_U)$ of $\mathcal{O}_U$-modules, such that $\mathcal{H}^{-1}(\alpha_U) : \mathcal{H}^{-1}(L_{U/\mathbb{Z}}) \simeq \mathcal{O}_U$ and $\mathcal{H}^0(\alpha_U) : \mathcal{H}^0(L_{U/\mathbb{Z}}) \simeq \Omega^1_{X/\mathbb{F}_q}$ are the isomorphisms given by the triangle above. Indeed, the canonical map

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{O}_U)}(L_{U/\mathbb{Z}}, \mathcal{O}_U[1] \oplus \Omega^1_{U/\mathbb{F}_q})$$

$$\longrightarrow \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{H}^{-1}(L_{U/\mathbb{Z}}), \mathcal{O}_U) \oplus \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{H}^0(L_{U/\mathbb{Z}}), \Omega^1_{U/\mathbb{F}_q})$$

is an isomorphism, as it follows from the spectral sequence

$$\prod_{n\in\mathbb{Z}} \mathrm{Ext}^p(\mathcal{H}^n(L_{U/\mathbb{Z}}), \mathcal{H}^{n+q}(\mathcal{O}_U[1]\oplus\Omega^1_{U/\mathbb{F}_q})) \Rightarrow$$
$$\Rightarrow H^{p+q}(R\mathrm{Hom}(L_{U/\mathbb{Z}}, \mathcal{O}_U[1]\oplus\Omega^1_{U/\mathbb{F}_q}))$$

and from the fact higher Ext's vanish since U is affine and $\Omega^1_{X/\mathbb{F}_q}$ is locally free of finite rank. Then α_U is functorial in the open affine subscheme U, in the sense that, if $V \subseteq U$ is the inclusion of an open affine subscheme V, then $\alpha_U \mid V = \alpha_V$ by uniqueness of α_V. We obtain the following isomorphism in $\mathcal{D}(\mathcal{O}_U)$ (see [6] for details):

$$\begin{aligned} & L\Lambda^n L_{U/\mathbb{Z}} \\ \simeq\ & L\Lambda^n([\mathcal{O}_U \xrightarrow{0} \Omega^1_{U/\mathbb{F}_q}][1]) \\ \simeq\ & [\Gamma^n\mathcal{O}_U\otimes\Lambda^0\Omega^1_{U/\mathbb{F}_q} \xrightarrow{0} \Gamma^{n-1}\mathcal{O}_U\otimes\Lambda^1\Omega^1_{U/\mathbb{F}_q} \xrightarrow{0} \cdots \xrightarrow{0} \Gamma^0\mathcal{O}_U\otimes\Lambda^n\Omega^1_{U/\mathbb{F}_q}][n] \end{aligned}$$

where the differential maps are all trivial. This yields a canonical isomorphism of $\mathcal{O}_U$-modules

$$a_U : \mathcal{H}^i(L\Lambda^n L_{X/\mathbb{Z}}[-n]) \mid U \simeq \mathcal{H}^i(L\Lambda^n L_{U/\mathbb{Z}}[-n]) \simeq \Gamma^{n-i}\mathcal{O}_U \otimes_{\mathcal{O}_U} \Omega^i_{U/\mathbb{F}_q}$$

for any $i \in \mathbb{Z}$, where $\Gamma^{n-i}\mathcal{O}_U := 0$ for $n-i < 0$ and $\Omega^i_{U/\mathbb{F}_q} := 0$ for $i < 0$. Moreover, the isomorphisms a_U are compatible with the restriction maps given by inclusions of affine open subsets $V \subseteq U$, in the sense that $(a_U) \mid V = a_V$. Covering X by open affine subschemes U (recall that X is separated so that the intersection of two affine open subschemes is affine), the identifications a_U therefore give an isomorphism of sheaves of $\mathcal{O}_X$-modules

$$\mathcal{H}^i(L\Lambda^n L_{X/\mathbb{Z}}[-n]) \simeq \Gamma^{n-i}\mathcal{O}_X \otimes_{\mathcal{O}_X} \Omega^i_{X/\mathbb{F}_q}.$$

For $n-i \geq 0$, the $\mathcal{O}_X$-module $\Gamma^{n-i}\mathcal{O}_X$ is free of rank one with generator $\gamma_{n-i}(1)$ where $1 \in \mathcal{O}_X$ is the unit section and $\gamma_{n-i} : \mathcal{O}_X \to \Gamma^{n-i}\mathcal{O}_X$ the canonical map. So we obtain an isomorphism

$$\mathcal{H}^i(L\Lambda^n L_{X/\mathbb{Z}}[-n]) \simeq \Gamma^{n-i}\mathcal{O}_X \otimes_{\mathcal{O}_X} \Omega^i_{X/\mathbb{F}_q} \simeq \Omega^{i\leq n}_{X/\mathbb{F}_q}. \tag{14}$$

We now check that the isomorphism (14) is functorial in $X \in \mathbf{Sm}^d/\mathbb{F}_q$. Let Y and X be schemes in $\mathbf{Sm}^d/\mathbb{F}_q$ and let $f : Y \to X$ be an arbitrary map. There is a morphism of exact triangles (see [4] II.2.1.5)

$$\begin{array}{ccccccc} Lf^*\mathcal{O}_X[1] & \longrightarrow & Lf^*L_{X/\mathbb{Z}} & \longrightarrow & Lf^*\Omega^1_{X/\mathbb{F}_q} & \xrightarrow{Lf^*\omega_X} & Lf^*\mathcal{O}_X[2] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_Y[1] & \longrightarrow & L_{Y/\mathbb{Z}} & \longrightarrow & \Omega^1_{Y/\mathbb{F}_q} & \xrightarrow{\omega_Y} & \mathcal{O}_Y[2] \end{array}$$

Suppose first that X and Y are affine. Then $\omega_X = 0$ and $\omega_Y = 0$, and the square

$$\begin{tikzcd} Lf^*L_{X/\mathbb{Z}} \arrow[r, "\sim", "Lf^*\alpha_X"'] \arrow[d] & f^*\mathcal{O}_X[1] \oplus f^*\Omega^1_{X/\mathbb{F}_q} \arrow[d] \\ L_{Y/\mathbb{Z}} \arrow[r, "\sim", "\alpha_Y"'] & \mathcal{O}_Y[1] \oplus \Omega^1_{Y/\mathbb{F}_q} \end{tikzcd}$$

commutes in $\mathcal{D}(\mathcal{O}_Y)$, since a morphism $Lf^*L_{X/\mathbb{Z}} \to \mathcal{O}_Y[1] \oplus \Omega^1_{Y/\mathbb{F}_q}$ is determined by the morphisms it induces on cohomology (as for α_X above). Hence the bottom square in the following diagram

$$\begin{tikzcd} Lf^*L\Lambda^n L_{X/\mathbb{Z}} \arrow[r, "\sim"] \arrow[d] & f^*[\Gamma^n_{\mathcal{O}_X}\mathcal{O}_X \otimes \Lambda^0_{\mathcal{O}_X}\Omega^1_{X/\mathbb{F}_q} \xrightarrow{0} \cdots \xrightarrow{0} \Gamma^0_{\mathcal{O}_X}\mathcal{O}_X \otimes \Lambda^n_{\mathcal{O}_X}\Omega^1_{X/\mathbb{F}_q}][n] \arrow[d] \\ L\Lambda^n Lf^*L_{X/\mathbb{Z}} \arrow[r, "\sim"] \arrow[d] & [\Gamma^n_{\mathcal{O}_Y} f^*\mathcal{O}_X \otimes \Lambda^0_{\mathcal{O}_Y} f^*\Omega^1_{X/\mathbb{F}_q} \xrightarrow{0} \cdots \xrightarrow{0} \Gamma^0_{\mathcal{O}_Y} f^*\mathcal{O}_X \otimes \Lambda^n_{\mathcal{O}_Y} f^*\Omega^1_{X/\mathbb{F}_q}][n] \arrow[d] \\ L\Lambda^n L_{Y/\mathbb{Z}} \arrow[r, "\sim"] & [\Gamma^n\mathcal{O}_Y \otimes \Omega^0_{Y/\mathbb{F}_q} \xrightarrow{0} \cdots \xrightarrow{0} \Gamma^0\mathcal{O}_Y \otimes \Omega^n_{Y/\mathbb{F}_q}][n] \end{tikzcd}$$

commutes as well (see [4] I.4.3.1.3). Here the top left vertical map is induced by the derived version $Lf^*L\Lambda^n_{\mathcal{O}_X} \to L\Lambda^n_{\mathcal{O}_Y}Lf^*$ of the natural transformation $f^*\Lambda^n_{\mathcal{O}_X} \to \Lambda^n_{\mathcal{O}_Y}f^*$, and the top right vertical map is induced by $f^*\Lambda^i_{\mathcal{O}_X} \to \Lambda^i_{\mathcal{O}_Y}f^*$ and $f^*\Gamma^{n-i}_{\mathcal{O}_X} \to \Gamma^{n-i}_{\mathcal{O}_Y}f^*$. It follows that the upper square in the previous diagram commutes. Since the cohomology sheaves of $L\Lambda^n L_{X/\mathbb{Z}}$ are flat $\mathcal{O}_X$-modules, we have the isomorphism

$$f^*\mathcal{H}^i(L\Lambda^n L_{X/\mathbb{Z}}[-n]) \xrightarrow{\sim} \mathcal{H}^i(Lf^*L\Lambda^n L_{X/\mathbb{Z}}[-n]).$$

We obtain the following commutative square of $\mathcal{O}_Y$-modules

$$\begin{tikzcd} f^*\mathcal{H}^i(L\Lambda^n L_{X/\mathbb{Z}}[-n]) \arrow[r, "\sim"] \arrow[d] & f^*(\Gamma^{n-i}\mathcal{O}_X \otimes_{\mathcal{O}_X} \Omega^i_{X/\mathbb{F}_q}) \arrow[d] \\ \mathcal{H}^i(L\Lambda^n L_{Y/\mathbb{Z}}[-n]) \arrow[r, "\sim"] & \Gamma^{n-i}\mathcal{O}_Y \otimes_{\mathcal{O}_Y} \Omega^i_{Y/\mathbb{F}_q} \end{tikzcd}$$

where X and Y are affine schemes in $\mathbf{Sm}^d/\mathbb{F}_q$. Let $f : Y \to X$ be a map between arbitrary X, Y in $\mathbf{Sm}^d/\mathbb{F}_q$. Covering Y and X by affine open subschemes (compatibly with f) we see that the previous square commutes for arbitrary X and Y. The result follows because the identification of $\mathcal{O}_X$-modules $\Gamma^{n-i}_{\mathcal{O}_X}\mathcal{O}_X \simeq \mathcal{O}_X$ is functorial in X. Indeed, the map $f^*(\Gamma^{n-i}_{\mathcal{O}_X}\mathcal{O}_X) \to \Gamma^{n-i}_{\mathcal{O}_Y}\mathcal{O}_Y$ maps $\gamma_{n-i}(1)$ to itself. $\square$

REMARK 3.4. *An isomorphism of the form*

$$L\Lambda^n L_{X/\mathbb{Z}} \simeq [\mathcal{O}_X \xrightarrow{0} \Omega^1_{X/\mathbb{F}_q} \xrightarrow{0} \cdots \xrightarrow{0} \Omega^n_{X/\mathbb{F}_q}][n]$$

is false in general, e.g. take $n = 1$ and X such that $\alpha_X \neq 0$ (i.e. such that X has no lifting over $\mathbb{Z}/p^2\mathbb{Z}$).

REMARK 3.5. *In order to prove the main result of [6], one may use Lemma 3.3 above instead of ([6] Lemma 2).*

PROPOSITION 3.6. *Assume $R(d)$. There is a canonical isomorphism of sheaves of $\mathcal{O}^{eh}$-modules*

$$\mathcal{H}^i(L\Lambda^n L_{\mathcal{O}^{eh}/\mathbb{Z}}[-n]) \simeq \Omega^{i\leq n}_{\mathcal{O}^{eh}/\mathbb{F}_q}$$

where $\Omega^{i\leq n}_{\mathcal{O}^{eh}/\mathbb{F}_q} = \Omega^i_{\mathcal{O}^{eh}/\mathbb{F}_q}$ for $0 \leq i \leq n$ and $\Omega^{i\leq n}_{\mathcal{O}^{eh}/\mathbb{F}_q} = 0$ otherwise.

Proof. We first work in the ringed topos $(\mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q), \mathcal{O})$. By exactness of res_X, Lemma 3.2(9) and Lemma 3.3, we have

$$\begin{aligned} \mathrm{res}_X(\mathcal{H}^i(L\Lambda^n L_{\mathcal{O}/\mathbb{Z}}[-n])) &\simeq \mathcal{H}^i(\mathrm{res}_X(L\Lambda^n L_{\mathcal{O}/\mathbb{Z}}[-n])) \\ &\simeq \mathcal{H}^i(L\Lambda^n L_{X/\mathbb{Z}}[-n])) \\ &\simeq \Omega^{i\leq n}_{X/\mathbb{F}_q} \end{aligned}$$

for any X in $\mathbf{Sm}^d/\mathbb{F}_q$. Moreover, for a morphism $f: Y \to X$ in $\mathbf{Sm}^d/\mathbb{F}_q$, the transition map

$$\alpha_f : f^*\mathrm{res}_X(\mathcal{H}^i(L\Lambda^n L_{\mathcal{O}/\mathbb{Z}}[-n])) \longrightarrow \mathrm{res}_Y(\mathcal{H}^i(L\Lambda^n L_{\mathcal{O}/\mathbb{Z}}[-n]))$$

may be identified with the canonical map (see Lemma 3.2)

$$f^*\mathcal{H}^i(L\Lambda^n L_{X/\mathbb{Z}}[-n]) \longrightarrow \mathcal{H}^i(L\Lambda^n L_{Y/\mathbb{Z}}[-n])$$

which in turn may be identified with the canonical map

$$f^*\Omega^{i\leq n}_{X/\mathbb{F}_q} \longrightarrow \Omega^{i\leq n}_{Y/\mathbb{F}_q}$$

by Lemma 3.3. In view of (11), we obtain an isomorphism

$$\mathcal{H}^i(L\Lambda^n L_{\mathcal{O}/\mathbb{Z}}[-n]) \simeq \Omega^{i\leq n}_{\mathcal{O}/\mathbb{F}_q} \tag{15}$$

of $\mathcal{O}$-modules in the topos $\mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q)$. By Lemma 3.2 (8) and by exactness of ρ^{-1}, we have

$$\mathcal{H}^i(L\Lambda^n L_{\mathcal{O}^{eh}/\mathbb{Z}}) \simeq \mathcal{H}^i(\rho^{-1}L\Lambda^n L_{\mathcal{O}/\mathbb{Z}}) \simeq \rho^{-1}\mathcal{H}^i(L\Lambda^n L_{\mathcal{O}/\mathbb{Z}}). \tag{16}$$

By (16), (15) and (10), we obtain

$$\mathcal{H}^i(L\Lambda^n L_{\mathcal{O}^{eh}/\mathbb{Z}}[-n]) \simeq \rho^{-1}\mathcal{H}^i(L\Lambda^n L_{\mathcal{O}/\mathbb{Z}}[-n]) \simeq \rho^{-1}\Omega^{i\leq n}_{\mathcal{O}/\mathbb{F}_q} \simeq \Omega^{i\leq n}_{\mathcal{O}^{eh}/\mathbb{F}_q}.$$

□

Remark 3.7. *One may think of trying to prove Proposition 3.6 more directly using the exact triangle*

$$\mathcal{O}^{eh}[1] \to L_{\mathcal{O}^{eh}/\mathbb{Z}} \to \Omega^1_{\mathcal{O}^{eh}/\mathbb{F}_q} \xrightarrow{\omega^{eh}} \mathcal{O}^{eh}[2],$$

which is the image by ρ^{-1} of the exact triangle

$$\mathcal{O}[1] \to L_{\mathcal{O}/\mathbb{Z}} \to \Omega^1_{\mathcal{O}/\mathbb{F}_q} \xrightarrow{\omega} \mathcal{O}[2]$$

in the derived category of $\mathcal{O}$-modules on $\mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q)$. A direct computation of $L\Lambda^n L_{\mathcal{O}^{eh}/\mathbb{Z}}$ as in (14) would not work since the extension ω is non-trivial by Remark 3.4 and Lemma 3.2.

The following corollary follows immediately from Proposition 3.6.

Corollary 3.8. *If $R(d)$ holds then $L\Lambda^p_{\mathcal{O}^{eh}} L_{\mathcal{O}^{eh}/\mathbb{Z}}$ is concentrated in degrees $[-p, 0]$ and $L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n$ is concentrated in degrees $[0, n-1]$.*

Corollary 3.9. *Let X be a smooth projective scheme over $\mathbb{F}_q$ of dimension d and let $n \in \mathbb{Z}$ be an integer. If $R(d)$ holds then the canonical maps*

$$R\Gamma(X_{Zar}, L\Lambda^p_{\mathcal{O}_X} L_{X/\mathbb{Z}}) \to R\Gamma(X_{eh}, L\Lambda^p_{\mathcal{O}^{eh}} L_{\mathcal{O}^{eh}/\mathbb{Z}})$$

and

$$R\Gamma(X_{Zar}, L\Omega^*_{X/\mathbb{Z}}/F^n) \to R\Gamma(X_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n)$$

are quasi-isomorphisms.

Proof. The morphism of ringed topoi

$$(\mathrm{Sh}_{eh}(\mathbf{Sch}^d/\mathbb{F}_q), \mathcal{O}^{eh})/y(X) \xrightarrow{\rho/X} (\mathrm{Sh}_{Zar}(\mathbf{Sm}^d/\mathbb{F}_q), \mathcal{O})/X \xrightarrow{(4)} (X, \mathcal{O}_X). \quad (17)$$

induces a morphism of (derived Hodge to de Rham) spectral sequences from

$$E_1^{p,q} = H^q(X_{Zar}, L\Lambda^{p<n} L_{X/\mathbb{Z}}) \Longrightarrow H^{p+q}(X_{Zar}, L\Omega^*_{X/\mathbb{Z}}/F^n) \quad (18)$$

to

$${}'E_1^{p,q} = H^q(X_{eh}, L\Lambda^{p<n} L_{\mathcal{O}^{eh}/\mathbb{Z}}) \Longrightarrow H^{p+q}(X_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n). \quad (19)$$

Here the convergent spectral sequences (18) and (19) are obtained (using Corollary 3.8) as spectral sequences for the hypercohomology of filtered bounded below complexes. One is therefore reduced to showing that the maps

$$H^q(X_{Zar}, L\Lambda^p L_{X/\mathbb{Z}}) \to H^q(X_{eh}, L\Lambda^p L_{\mathcal{O}^{eh}/\mathbb{Z}})$$

are isomorphisms. By Lemma 3.3, Proposition 3.6 and Corollary 3.8, the morphism (17) induces a morphism of hypercohomology spectral sequences from

$$E_2^{i,j} = H^i(X_{Zar}, \Omega^{j\leq p}_{X/\mathbb{F}_q}) \Longrightarrow H^{i+j}(X_{Zar}, L\Lambda^p L_{X/\mathbb{Z}}[-p])$$

to

$$'E_2^{i,j} = H^i(X_{eh}, \Omega^{j\leq p}_{\mathcal{O}^{eh}/\mathbb{F}_q}) \Longrightarrow H^{i+j}(X_{eh}, L\Lambda^p L_{\mathcal{O}^{eh}/\mathbb{Z}}[-p]).$$

One is therefore reduced to showing that the map

$$H^i(X_{Zar}, \Omega^j_{X/\mathbb{F}_q}) \to H^i(X_{eh}, \Omega^j_{\mathcal{O}^{eh}/\mathbb{F}_q})$$

is an isomorphism for any i, j. Assuming $R(d)$, this follows from ([2] Theorem 4.7) since $\Omega^j_{\mathcal{O}^{eh}/\mathbb{F}_q} \simeq \rho^{-1}\Omega^j_{\mathcal{O}/\mathbb{F}_q}$. □

Recall from the introduction that one defines

$$\chi_c^{eh}(X/\mathbb{F}_q, \mathcal{O}, n) := \sum_{i\leq n, j\in\mathbb{Z}} (-1)^{i+j} \cdot (n-i) \cdot \dim_{\mathbb{F}_q} H^j_c(X_{eh}, \Omega^i_{\mathcal{O}^{eh}/\mathbb{F}_q}).$$

Corollary 3.10. *Let X be a separated scheme of finite type over $\mathbb{F}_q$ of dimension d and let $n \in \mathbb{Z}$ be an integer. If $R(d)$ holds then the complex $R\Gamma_c(X_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n)$ is bounded with finite cohomology groups, and we have*

$$\prod_{i\in\mathbb{Z}} \mid H^i_c(X_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n) \mid^{(-1)^i} \;=\; q^{\chi_c^{eh}(X/\mathbb{F}_q,\mathcal{O},n)}.$$

Proof. We consider the convergent spectral sequences

$$H^q_c(X_{eh}, L\Lambda^{p<n} L_{\mathcal{O}^{eh}/\mathbb{Z}}) \Longrightarrow H^{p+q}_c(X_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n) \tag{20}$$

and

$$H^i_c(X_{eh}, \Omega^{j\leq p}_{\mathcal{O}^{eh}/\mathbb{F}_q}) \Longrightarrow H^{i+j}_c(X_{eh}, L\Lambda^p L_{\mathcal{O}^{eh}/\mathbb{Z}}[-p]). \tag{21}$$

In view of Corollary 3.8 and the isomorphism (see [2] Remark before Lemma 3.5)

$$R\Gamma_c(X_{eh}, -) \simeq R\mathrm{Hom}(\mathbb{Z}^c_{eh}(X), -)$$

(20) and (21) may be obtained as spectral sequences for the hypercohomology of filtered bounded below complexes. The complex $R\Gamma_c(X_{eh}, \Omega^j_{\mathcal{O}^{eh}/\mathbb{F}_q}) \simeq R\Gamma_c(X_{eh}, \rho^{-1}\Omega^j_{\mathcal{O}/\mathbb{F}_q})$ is bounded with finite cohomology groups by ([2] Corollary 4.8). In view of (20) and (21), the complexes $R\Gamma_c(X_{eh}, L\Lambda^p L_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n)$ and $R\Gamma_c(X_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n)$ are also bounded with finite cohomology groups. By ([6] Lemma 1), the spectral sequences (20) and (21) give isomorphisms

$$\begin{aligned}
& \det{}_{\mathbb{Z}} R\Gamma_c(X_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n) && (22)\\
\xrightarrow{\sim}\; & \bigotimes_{p<n} \det{}^{(-1)^p}_{\mathbb{Z}} R\Gamma_c(X_{eh}, L\Lambda^p L_{\mathcal{O}^{eh}/\mathbb{Z}}) && (23)\\
\xrightarrow{\sim}\; & \bigotimes_{p<n} \det{}_{\mathbb{Z}} R\Gamma_c(X_{eh}, L\Lambda^p L_{\mathcal{O}^{eh}/\mathbb{Z}}[-p]) && (24)\\
\xrightarrow{\sim}\; & \bigotimes_{p<n} \left(\bigotimes_{i\leq p, j} \det{}^{(-1)^{i+j}}_{\mathbb{Z}} H^j_c(X_{eh}, \Omega^i_{\mathcal{O}^{eh}/\mathbb{F}_p}) \right) && (25)
\end{aligned}$$

such that the square of isomorphisms

$$\begin{tikzcd}
\left(\det_{\mathbb{Z}} R\Gamma_c(X_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n)\right)_{\mathbb{Q}} \arrow[r] \arrow[d, "\gamma"] & \left(\bigotimes_{p<n}\bigotimes_{i\leq p,j} \det_{\mathbb{Z}}^{(-1)^{i+j}} H^j_c(X_{eh}, \Omega^i_{\mathcal{O}^{eh}/\mathbb{F}_q})\right)_{\mathbb{Q}} \arrow[d, "\gamma'"] \\
\mathbb{Q} \arrow[r, "\mathrm{Id}"] & \mathbb{Q}
\end{tikzcd}$$

commutes, where the top horizontal map is induced by (25), and the vertical isomorphisms are the canonical trivializations (see Section 2.3). The result follows:

$$\begin{aligned}
& \mathbb{Z} \cdot \left(\prod_{i\in\mathbb{Z}} \mid H^i_c(X_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n) \mid^{(-1)^i} \right)^{-1} \\
= \quad & \gamma \left(\det_{\mathbb{Z}} R\Gamma_c(X_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n) \right) \\
= \quad & \gamma' \left(\bigotimes_{p<n} \bigotimes_{i\leq p,j} \det_{\mathbb{Z}}^{(-1)^{i+j}} H^j_c(X_{eh}, \Omega^i_{\mathcal{O}^{eh}/\mathbb{F}_q}) \right) \\
= \quad & \mathbb{Z} \cdot q^{-\chi_c^{eh}(X/\mathbb{F}_q, \mathcal{O}, n)}.
\end{aligned}$$

□

Recall from Section 2.3 the definitions of $\Delta(X/\mathbb{Z}, n)$ and λ_X.

Corollary 3.11. *Let X be a separated scheme of finite type over $\mathbb{F}_q$ of dimension d and let $n \in \mathbb{Z}$ be an integer. Assume that for any smooth projective variety Y of dimension $\leq d$ the Weil-étale cohomology groups $H^i_W(Y, \mathbb{Z}(n))$ are finitely generated for all i. If $R(d)$ holds, then one has*

$$\Delta(X/\mathbb{Z}, n) \quad = \quad \mathbb{Z} \cdot \lambda_X \left(\zeta^*(X, n)^{-1} \right).$$

Proof. All the schemes we consider in this proof are in $\mathbf{Sch}^d/\mathbb{F}_q$. For an open-closed decomposition $(U \xrightarrow{j} X \xleftarrow{i} Z)$, we have exact triangles

$$R\Gamma_c(U_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n) \to R\Gamma_c(X_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n) \to R\Gamma_c(Z_{eh}, L\Omega^*_{\mathcal{O}^{eh}/\mathbb{Z}}/F^n)$$

and

$$R\Gamma_{W,c}(U, \mathbb{Z}(n)) \to R\Gamma_{W,c}(X, \mathbb{Z}(n)) \to R\Gamma_{W,c}(Z, \mathbb{Z}(n)). \tag{26}$$

Moreover, the triangle (26) is compatible (in the obvious sense) with $\cup\theta$. This gives an isomorphism

$$\Delta(X/\mathbb{Z}, n) \simeq \Delta(U/\mathbb{Z}, n) \otimes_{\mathbb{Z}} \Delta(Z/\mathbb{Z}, n) \tag{27}$$

such that the square of isomorphisms

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\mathrm{Id}"] \arrow[d, "\lambda_X"] & \mathbb{R} \arrow[d, "\lambda_U \otimes \lambda_Z"] \\ \Delta(X/\mathbb{Z}, n)_{\mathbb{R}} \arrow[r, "(27)_{\mathbb{R}}"] & \Delta(U/\mathbb{Z}, n)_{\mathbb{R}} \otimes_{\mathbb{R}} \Delta(Z/\mathbb{Z}, n)_{\mathbb{R}} \end{tikzcd}$$

commutes. Similarly, one has

$$\zeta^*(X, n) = \zeta^*(U, n) \cdot \zeta^*(Z, n).$$

It follows that if the result is true for two out of the three schemes (X, U, Z) then it is true for the third. Moreover, the result is true for X smooth projective by [6], Corollary 3.9 and ([2] Theorem 4.3). It follows for arbitrary $X \in \mathbf{Sch}^d/\mathbb{F}_q$ by ([2] Lemma 2.7). □

Acknowlegments. I am grateful to Denis-Charles Cisinski and Matthias Flach for interesting comments. The author was supported by ANR-12-JS01-0007 and ANR-15-CE40-0002.

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