The Θ Function and the Weyl Law on Manifolds Without Conjugate Points

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Abstract. We prove that the usual Θ function on a Riemannian manifold without conjugate points is uniformly bounded from below. This extends a result of Green in two dimensions. We deduce that the Bérard remainder in the Weyl law is valid for a manifold without conjugate points, without any restriction on the dimension.

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1 Introduction

Let (M, g) be a compact Riemannian manifold of dimension n. The spectrum of its Laplacian is discrete. We denote the eigenvalues by $\mu_0 = 0 < \mu_1 \leq \dots$, and its counting function by

$$N(\lambda) := \#\{\mu_i \leq \lambda^2\}.$$

In 1977, Bérard proved the following:

Theorem 1 ([Bér77]). *Assume that $n = 2$ and M has no conjugate points, or that M has non-positive sectional curvature. Then, as λ tends to infinity,*

$$N(\lambda) = \frac{\mathrm{vol}(B^*M)}{(2\pi)^n}\lambda^n + \mathcal{O}\left(\frac{\lambda^{n-1}}{\log \lambda}\right). \tag{1}$$

In this note, we prove

Theorem 2. *It suffices to assume that M does not have conjugate points to obtain the above result in any dimension.*

This really *is* an improvement of theorem 1, as there exist manifolds without conjugate points whose curvature has no sign. One can find such examples in Gulliver [Gul75], or Ballmann-Brin-Burns [BBB87].

We will not enter into all the details of the original proof, as we will just make an observation on a crucial point in the arguments of Bérard. To obtain the theorem, Bérard studied the local behaviour of the wave trace via the Hadamard parametrix. The kernel $K(t,x,x')$ of the wave operator $\cos t\sqrt{-\Delta}$ on $\widetilde{M}$ – the universal cover of M – has an expansion of the form

$$K(t,x,x') = C_0 \sum_{k\geq 0} u_k(x,x')|t| \frac{(t^2 - d(x,x')^2)_+^{k-(n+1)/2}}{\Gamma(k-(n-1)/2)} \mod (C^\infty). \tag{2}$$

The coefficients u_k satisfy certain transport equations along the geodesic between x and x'. The expansion (2) is valid on the universal cover of M as soon as M has no conjugate points. A critical part of the proof of Bérard, which is the *only* spot where the negative curvature assumption is used, is the lemma:

Lemma 1 ([Bér77]). *Let (N,g) be the universal cover of a compact manifold without conjugate points. When $n = 2$ or if the curvature is non-positive, for all $k \geq 0$ and $l \geq 0$,*

$$\Delta^l_{x'} u_k(x,x') = \mathcal{O}(1) e^{\mathcal{O}(d(x,x'))}.$$

To prove theorem 2, it suffices to establish

Lemma 2. *The conclusion of lemma 1 holds with the sole assumption that (N,g) is a complete, simply connected Riemannian manifold without conjugate points and bounded geometry.*

To have bounded geometry means that the curvature tensor and all its covariant derivatives are bounded on N and that the injectivity radius is positively bounded from below. The latter is a given here since $\exp_x$ is a global diffeomorphism for any $x \in N$.

To understand the proof, we need to introduce the Θ function, announced in the title: for $x, x' \in N$

$$\Theta(x,x') = \det T_{\exp_x^{-1}(x')} \exp_x .$$

As $T_{\exp_x^{-1}(x')} \exp_x$ is a linear map between T_xN and $T_{x'}N$, this determinant is naturally computed taking as reference the volume form $d\,\mathrm{vol}_g$ at the points x and x'. We also define

$$\vartheta(x,x') = d(x,x')^{n-1}\Theta(x,x').$$

Now, we can give an explicit expression for the coefficients u_k (see [Bér77]):

$$u_0(x,x') = \frac{1}{\sqrt{\Theta}},$$
$$u_{k+1}(x,x') = \frac{1}{r^{k+1}\sqrt{\Theta}} \int_0^r s^k \sqrt{\Theta(x,x_s)}(-\Delta_{x'} u_k)(x,x_s) ds.$$

where $x' = \exp_x(ru)$ and $x_s = \exp_x(su)$. We can deduce that to complete the proof in the general case, it suffices to prove the most basic estimate, that is, $u_0 = \mathcal{O}(1)e^{\mathcal{O}(d(x,x'))}$. This fact was actually hinted at in the last remark of Bérard's paper.

The proof of lemma 2 will therefore be complete if we can prove this Riemannian geometry lemma:

Lemma 3. *Assume that (N,g) is a complete simply-connected manifold without conjugate points, and bounded sectional curvature. For all $\epsilon > 0$, there is a constant $C > 0$ so that $\vartheta(x,x') > C$ whenever $d(x,x') > \epsilon$.*

In the proof, this will be deduced from the lemma 4 on Jacobi equations, which gives some explicit estimates on the relation between the constants ϵ and C.

For surfaces, lemma 3 is due to Green (see lemma 2 in [Gre56]). While it may have been known for a while, we did not find any published statement, or proof, for the general case. In Eberlein [Ebe73], one can find a proof that for any (x,u), $\lim_{t\to\infty} \vartheta(x, \exp_x(tu)) = +\infty$, but the convergence is not uniform in (x,u) in higher dimension, so that this is not enough to deduce lemma 3. As a special case, Goto [Got78] proved the lemma for manifolds with no *focal points*.

Before we go on with the proof, we would like to make some observations.

1. In [HT15], Hassell and Tacy gave a uniform logarithmic improvement on the L^p norms of eigenfunctions, with the same assumptions on the manifold as in Bérard's theorem. According to their proof, the reason why they need non-positive curvature in dimension $n > 2$ is that they use lemma 1. Their result can thus be generalized to all closed manifolds without conjugate points. The same consideration applies to the article of Mroz and Strohmaier [MS16], whose results are thus extended to all complete compact manifolds without conjugate points; it is quite possible that this observation applies to other results in the litterature.

2. When investigating the Weyl law for some non-compact manifolds of finite volume with hyperbolic cusps (see [Bon15]), it was convenient to introduce a modified version of the Hadamard parametrix (2). This involved new coefficients $\tilde{u}_k$, which satisfy

$$\tilde{u}_0(x,x') = \sqrt{\frac{\sinh(r)^{n-1}}{\vartheta}}$$

$$\tilde{u}_{k+1}(x,x') =$$

$$\frac{1}{\sinh(r)}\int_0^r \left(\frac{\sinh(s)}{\sinh(r)}\right)^{k-\frac{n-1}{2}} \sqrt{\frac{\vartheta(x,x_s)}{\vartheta(x,x')}} \left(-\Delta_{x'} + k^2 - n + 1\right) \tilde{u}_k(x,x_s)ds,$$

still with $r = d(x,x')$. It was crucial to the proof therein that the same estimates as in lemma 1 hold also for the $\tilde{u}_k$. However, the proof of lemma 1 which can be found in the appendix of Bérard's article is sufficiently robust so that we can also reduce to the case of $\tilde{u}_0$, and this completes the proof of the results in [Bon15].

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2 Proof

The arguments we use are somewhat elementary, and they are inspired by the original proof of Green [Gre58], and some arguments from Eberlein [Ebe73]. However, for the convenience of the reader, our proof is (almost) self-contained.
There is a direct link between the Θ function and Jacobi fields. Let us fix for the moment a geodesic $\gamma(t)$, starting at x, of the form $\exp_x(tu)$. Then we choose a direct orthonormal basis in T_xN, whose last vector is u. Using parallel transport, this defines a family of parallel direct orthonormal frames along γ, and we can express Jacobi fields in those frames. They are found to satisfy the usual *matrix* equation:

$$X''(t) + \mathbb{K}(t)X(t) = 0 \tag{$*$}$$

where $\mathbb{K}(t)X(t) = R_{\gamma(t)}(X(t),\dot{\gamma}(t))\dot{\gamma}(t)$, R being the curvature tensor of N. In particular, $\mathbb{K}$ is a symmetric matrix, and $\mathbb{K}(t)\dot{\gamma} = 0$, so $\mathbb{K}(t)$ preserves the orthogonal of $\dot{\gamma}$. Hence we can decompose the Jacobi fields into a parallel part $c(t)\dot{\gamma}(t)$, and an orthogonal part. The parallel coefficient $c(t)$ is of the form $at + b$.
In these coordinates, the matrix for $T_{tu}\exp_x$ is $t^{-1}\mathbb{A}(t)$ where $\mathbb{A}(t)$ is the Jacobi matrix field such that $\mathbb{A}(0) = 0$ and $\mathbb{A}'(0) = \mathbb{1}$. If we decompose this into parallel and orthogonal fields, the parallel part is of course t, so we can abuse notations, and still denote by $\mathbb{A}(t)$ the orthogonal field. In what follows, we will only deal with orthogonal fields.
With the notations above,

$$\Theta(x,\exp_x(tu)) = \frac{det(\mathbb{A}(t))}{t^{n-1}}.$$

The condition that there are no conjugate points is equivalent to assuming that the field $\mathbb{A}(t)$ is invertible for $t \neq 0$, independently of the vector (x, u) (or equivalently, that for all x, $\exp_x$ is a global diffeomorphism). We also have

$$\vartheta(x, \exp_x(tu)) = \det \mathbb{A}(t).$$

More generally, we will consider the equation $(*)$ when $\mathbb{K}$ is a *bounded continuous family of symmetric matrices*. We will assume that $\mathbb{K}$ has *no conjugate points* in the sense that for any C^2 vector solution $v(t)$ to $(*)$, such that v vanishes for two different values of t, then v is identically zero. We will prove:

Lemma 4. *Under the assumptions above on $\mathbb{K}$, let $K_{max} = \sup \|\mathbb{K}\|$. Then for any $\epsilon > 0$, there is a $C > 0$ only depending on K_{max} and ϵ such that whenever $|t| > \epsilon$,*

$$\|\mathbb{A}(t)^{-1}\| \leq C.$$

In particular, if $K_{max} > 0$, when $t > \sqrt{1/(3K_{max})}$,

$$\|\mathbb{A}(t)^{-1}\|^2 \leq K_{max} \frac{24\sqrt{3}}{5} \coth \frac{1}{\sqrt{3}}.$$

We will use the Ricatti equation associated to $(*)$, that is

$$\mathbb{V}' + \mathbb{V}^2 + \mathbb{K} = 0. \qquad (**)$$

This is also a matrix-valued equation along $\gamma(t)$, and it is satisfied for $\mathbb{V}$'s of the form $\mathbb{B}'\mathbb{B}^{-1}$, where $\mathbb{B}$ is an invertible solution of $(*)$. The non-conjugacy assumption will imply the existence of solutions to $(**)$ on the interval $[0, +\infty)$, and this can be seen as the conceptual argument behind the proof.

The following lemma is fundamental for our proof. It can be found in Green for surfaces (see lemma 3 in [Gre58]), or in Eberlein in this level of generality (lemma 2.8 in [Ebe73]). It is actually a generalization of a result on Sturm-Liouville equations, known at least since E. Hopf. When we write an inequality between two matrices, they are assumed to be symmetric, and it means that the corresponding inequality holds between the associated quadratic forms.

Lemma 5. *Let $\mathbb{V}$ be a symmetric solution to the Ricatti equation $(**)$, defined for all $t > 0$, then $|\mathbb{V}| \leq k \coth kt$ as soon as $\mathbb{K} \geq -k^2 \mathbb{1}$ for all $t > 0$.*

Before going any further, let us make two remarks

1. It is useful to recall that if $\mathbb{B}$ and $\mathbb{C}$ are two solutions of $(*)$, their Wronskian is $\mathbb{W}(\mathbb{B}, \mathbb{C}) = \mathbb{B}^*\mathbb{C}' - \mathbb{B}'^*\mathbb{C}$ — here L^* is the transpose of L. It is constant. In particular, if $\mathbb{W}(\mathbb{B}, \mathbb{B}) = 0$ and $\mathbb{B}$ is invertible, the associated solution $\mathbb{B}'\mathbb{B}^{-1}$ of $(**)$ is symmetric.

2. Let $\mathbb{U} = \mathbb{A}'\mathbb{A}^{-1}$. One can check that it *is* a symmetric solution to $(**)$. Additionally, we find

$$\frac{d}{dt}\vartheta(x, \exp_x(tu)) = \vartheta(x, \exp_x(tu)) \operatorname{Tr} \mathbb{U}(t).$$

In particular, this implies the existence of a bound of the form $|\vartheta(t)|^{-1} = \mathcal{O}(t^{1-n})e^{\mathcal{O}(t)}$, where the constants are independent of x and u. This would probably be sufficient to obtain lemma 2, but we will nonetheless go on with the proof of lemma 3.

2.1 Green's method

Green's method gives us particularly useful solutions to $(*)$, that vanish only at infinity.
For $t \neq 0$, the field $\mathbb{D}_t$ is the unique solution to $(*)$ such that $\mathbb{D}_t(0) = \mathbb{1}$, and $\mathbb{D}_t(t) = 0$. The existence and uniqueness of such a field is assured by the non-conjugacy assumption. Let $\mathbb{D}_{\pm\infty}$ be the limit of the fields $\mathbb{D}_t$ as $t \to \pm\infty$. Let us recall the proof of existence of such limits given by Iturriaga [Itu02].
His first observation is that $(\mathbb{D}'_s(0) - \mathbb{D}'_t(0))$ is not singular, except at $s = t$. Indeed, let us assume that $(\mathbb{D}'_s(0) - \mathbb{D}'_t(0))v = 0$, with $s \neq t$. Then we get $\mathbb{D}_s(\tau)v = \mathbb{D}_t(\tau)v$ for all $\tau \in \mathbb{R}$. In particular, $\mathbb{D}_s(s)v = \mathbb{D}_s(t)v = 0$, and this implies that $\mathbb{D}_s(\tau)v = 0$ for all τ. Since $\mathbb{D}_s(0) = \mathbb{1}$, $v = 0$. On the other hand, $\mathbb{D}'_s(0) - \mathbb{D}'_t(0)$ depends continuously on s and t as long as they do not vanish, so the signature is constant in $\{t > s > 0\}$ and $\{t > 0 > s\}$. Now, we can see that

$$\mathbb{D}'_s(0) = -\frac{1}{s} + \mathcal{O}(s^2).$$

>From Iturriaga, we also learn that $\mathbb{D}'_s(0) - \mathbb{D}'_t(0)$ is always a symmetric matrix. Considering all the above, we deduce that when $0 < s < t$,

$$\mathbb{D}'_s(0) - \mathbb{D}'_t(0) < 0,$$

and when $s < 0 < t$,

$$\mathbb{D}'_s(0) - \mathbb{D}'_t(0) > 0.$$

In particular, since $\mathbb{D}'_t(0) - \mathbb{D}'_s(0)$ is increasing and bounded from above, it has to converge as $t \to +\infty$. It is a direct consequence that $\mathbb{D}_t(\tau)$ converges for all τ, to a solution $\mathbb{D}_{+\infty}(\tau)$ of $(*)$, and $\mathbb{D}'_t(0)$ has to converge to $\mathbb{D}'_{+\infty}(0)$. Usual arguments now give us that for $s, t > 0$,

$$\mathbb{D}_t(s) = \mathbb{A}(s) \int_s^t \mathbb{A}(\ell)^{-1}\mathbb{A}(\ell)^{-1*} d\ell.$$

In particular,

$$\mathbb{D}_{+\infty}(s) = \mathbb{A}(s) \int_s^{+\infty} \mathbb{A}(\ell)^{-1}\mathbb{A}(\ell)^{-1*} d\ell, \tag{3}$$

and $\mathbb{D}_{+\infty}(s) \to 0$ as $s \to +\infty$. We let

$$M(s) := \mathbb{D}'_{+\infty}(0) - \mathbb{D}'_s(0) = \int_s^{+\infty} \mathbb{A}(\ell)^{-1}\mathbb{A}(\ell)^{-1*} d\ell < \infty.$$

2.2 End of the proof

>From formula (3), we deduce that $\mathbb{D}_{+\infty}(s)$ is invertible for every $s \geq 0$. Hence the associated solution of $(**)$ $\mathbb{V} = \mathbb{D}'_{+\infty}\mathbb{D}_{+\infty}^{-1}$ is defined at least for all $s \geq 0$. It is also symmetric. Indeed, for any t, we have $\mathbb{W}(\mathbb{D}_t, \mathbb{D}_t) = 0$ by evaluation at t. Letting $t \to +\infty$, we get $\mathbb{W}(\mathbb{D}_{+\infty}, \mathbb{D}_{+\infty}) = 0$.

Now, we want to apply lemma 5 to solutions of $(**)$. An elementary computation gives

$$\mathbb{U}(t) - \mathbb{V}(t) = \mathbb{A}(t)^{-1*}M(t)^{-1}\mathbb{A}(t)^{-1}.$$

In particular,

$$\left|\mathbb{A}(t)^{-1*}M(t)^{-1}\mathbb{A}(t)^{-1}\right| \leq 2K_{max}^{1/2}\coth K_{max}^{1/2}t.$$

As $M(s)$ is symmetric, positive definite, $M^{-1}(s) \geq 1/\|M(s)\|$ — with the usual operator norm $\|\cdot\|$ — and

$$2\|M(t)\|K_{max}^{1/2}\coth K_{max}^{1/2}t \geq \mathbb{A}(t)^{-1*}\mathbb{A}(t)^{-1}.$$

As a consequence,

$$\|\mathbb{A}^{-1}(t)\|^2 \leq 2\|M(t)\|K_{max}^{1/2}\coth K_{max}^{1/2}t.$$

We finally get the result of this computation: for $t > s > 0$,

$$\|\mathbb{A}(t)^{-1}\|^2 \leq 2K_{max}^{1/2}\coth K_{max}^{1/2}t\|\mathbb{D}'_{-s}(0) - \mathbb{D}'_s(0)\|,$$

and according to Hadamard's inequality,

$$\vartheta^{-1}(t) \leq \left\{2K_{max}^{1/2}\coth K_{max}^{1/2}t\|\mathbb{D}'_{-s}(0) - \mathbb{D}'_s(0)\|_2\right\}^{\frac{n-1}{2}}.$$

Now, the last step of the proof is

Lemma 6. *Provided $s > 0$ is small enough, we can give a bound for $\|\mathbb{D}'_{-s}(0) - \mathbb{D}'_s(0)\|$ depending only on s and K_{max}.*

Observe that for the application we have in mind, K_{max} is bounded by the sup of the sectionnal curvature of (N, g).

Proof. Let $\mathbb{B}$ be the Jacobi matrix fields such that

$$\mathbb{B}(0) = \mathbb{1},\ \mathbb{B}'(0) = 0.$$

One can check that for $s \neq 0$, $\mathbb{D}'_s(0) = -\mathbb{A}(s)^{-1}\mathbb{B}(s)$. To prove the lemma, it suffices to bound both $\|A^{-1}\|$ and $\|B\|$ using only the fact that $\mathbb{K}$ is bounded with $K_{max} = \sup_t \|\mathbb{K}\|$.

Recall from the proof of the Cauchy-Lipschitz theorem that the Jacobi fields $\mathbb{A}$ and $\mathbb{B}$ are obtained, at least for small times, as fixed point of contraction mappings, respectively

$$T_A^\tau J(t) = t\cdot\mathbb{1} + \int_0^t (s-t)\mathbb{K}(s)J(s)ds, \text{ and } \quad T_B^\tau J(t) = \mathbb{1} + \int_0^t (s-t)\mathbb{K}(s)J(s)ds,$$

which are defined on C^0 matrix-valued functions on $t \in [-\tau, \tau]$, equipped with the norm $\|J\|_\tau = \sup_{|t|\leq\tau} \|J(t)\|$. These mappings have the same Lipschitz constant $\eta := \tau^2 K_{max}/2$, so we take $0 < \tau < \sqrt{2/K_{max}}$. From the Banach fixed point theorem, we know that $\mathbb{A} = \lim_n (T_A^\tau)^n(t \cdot \mathbb{1})$ and $\mathbb{B} = \lim_n (T_B^\tau)^n(\mathbb{1})$, and using the usual estimates,

$$\|\mathbb{A} - t \cdot \mathbb{1}\|_\tau \leq \frac{1}{1-\eta}\|T_A^\tau(t \cdot \mathbb{1}) - t \cdot \mathbb{1}\|_\tau, \text{ and } \quad \|\mathbb{B} - \mathbb{1}\|_\tau \leq \frac{1}{1-\eta}\|T_B^\tau(\mathbb{1}) - \mathbb{1}\|_\tau.$$

For τ small enough, we find that with a constant $C_{max} > 0$ only depending on K_{max},

$$\|\mathbb{A}(\tau) - \tau \cdot \mathbb{1}\| \leq C_{max}\tau^3, \text{ and } \quad \|\mathbb{B}(\tau) - \mathbb{1}\| \leq C_{max}\tau^2.$$

$\square$

REMARK 1. *To obtain the announced explicit bound, observe that* $C_{max}\tau^2 = \eta/(1-\eta)$, *so that when* $\tau < \sqrt{1/K_{max}}$,

$$\|\mathbb{D}'_\tau(0)\| \leq \frac{1}{\tau(1-2\eta)}.$$

We can optimize this with $\tau = 1/\sqrt{3K_{max}}$, *and it becomes* $6\sqrt{3K_{max}}/5$.

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