# Topological Conjugacy of Topological Markov Shifts and Cuntz–Krieger Algebras

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Received: June 23, 2016
Revised: February 8, 217

Communicated by Joachim Cuntz

Abstract.

For an irreducible non-permutation matrix $A$, the triplet $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ for the Cuntz-Krieger algebra $\mathcal{O}_A$, its canonical maximal abelian $C^*$-subalgebra $\mathcal{D}_A$, and its gauge action $\rho^A$ is called the Cuntz–Krieger triplet. We introduce a notion of strong Morita equivalence in the Cuntz–Krieger triplets, and prove that two Cuntz–Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent if and only if $A$ and $B$ are strong shift equivalent. We also show that the generalized gauge actions on the stabilized Cuntz–Krieger algebras are cocycle conjugate if the underlying matrices are strong shift equivalent. By clarifying K-theoretic behavior of the cocycle conjugacy, we investigate a relationship between cocycle conjugacy of the gauge actions on the stabilized Cuntz–Krieger algebras and topological conjugacy of the underlying topological Markov shifts.

2010 Mathematics Subject Classification: Primary 46L55; Secondary 46L35, 37B10.
Keywords and Phrases: Topological Markov shifts, topological conjugacy, strong shift equivalence, Cuntz–Krieger algebras, K-theory, gauge action.

[1]Supported by JSPS KAKENHI Grant Number 15K04896

## 1 Introduction and Preliminaries

Let $A = [A(i,j)]_{i,j=1}^N$ be an irreducible matrix with entries in $\{0,1\}$ with $1 < N \in \mathbb{N}$. We assume that $A$ is not any permutation matrix. In [7], J. Cuntz and W. Krieger have introduced a $C^*$-algebra $\mathcal{O}_A$ associated to the topological Markov shift $(X_A, \sigma_A)$. The $C^*$-algebra is called the Cuntz–Krieger algebra, which is a universal unique purely infinite simple $C^*$-algebra generated by partial isometries $S_1, \dots, S_N$ subject to the relations:

$$\sum_{j=1}^N S_j S_j^* = 1, \qquad S_i^* S_i = \sum_{j=1}^N A(i,j) S_j S_j^*, \quad i = 1, \dots, N. \tag{1.1}$$

For $t \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$, the correspondence $S_i \to e^{2\pi\sqrt{-1}t} S_i$, $i = 1, \dots, N$ gives rise to an automorphism of $\mathcal{O}_A$ denoted by $\rho_t^A$. The automorphisms $\rho_t^A, t \in \mathbb{T}$ yield an action of $\mathbb{T}$ on $\mathcal{O}_A$ called the gauge action. Cuntz and Krieger in [7] have shown that the algebra $\mathcal{O}_A$ has close relationships with the underlying dynamical system called topological Markov shift. Let us denote by $X_A$ the shift space

$$X_A = \{(x_n)_{n\in\mathbb{N}} \in \{1, \dots, N\}^{\mathbb{N}} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{N}\}. \tag{1.2}$$

Define the shift transformation $\sigma_A$ on $X_A$ by $\sigma_A((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}}$, which is a continuous surjection on $X_A$. The topological dynamical system $(X_A, \sigma_A)$ is called the one-sided topological Markov shift for matrix $A$. The two-sided topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$ is defined similarly with the shift space

$$\bar{X}_A = \{(x_n)_{n\in\mathbb{Z}} \in \{1, \dots, N\}^{\mathbb{Z}} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}\} \tag{1.3}$$

and the shift homeomorphism $\bar{\sigma}_A((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$ on $\bar{X}_A$.
Let us denote by $\mathcal{D}_A$ the $C^*$-subalgebra of $\mathcal{O}_A$ generated by the projections of the form: $S_{i_1} \cdots S_{i_n} S_{i_n}^* \cdots S_{i_1}^*, i_1, \dots, i_n = 1, \dots, N$. The subalgebra $\mathcal{D}_A$ is canonically isomorphic to the commutative $C^*$-algebra $C(X_A)$ of the complex valued continuous functions on $X_A$ by identifying the projection $S_{i_1} \cdots S_{i_n} S_{i_n}^* \cdots S_{i_1}^*$ with the characteristic function $\chi_{U_{i_1 \cdots i_n}} \in C(X_A)$ of the cylinder set $U_{i_1\cdots i_n}$ for the word $i_1 \cdots i_n$. Let us denote by $\mathcal{K}$ the $C^*$-algebra $\mathcal{K}(\ell^2(\mathbb{N}))$ of compact operators on a separable infinite dimensional Hilbert space $\ell^2(\mathbb{N})$ and by $\mathcal{C}$ its maximal abelian $C^*$-subalgebra of diagonal operators.
In [25], R. F. Williams proved that the topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate if and only if the matrices $A, B$ are strong shift equivalent. Two nonnegative matrices $A, B$ are said to be elementary equivalent if there exist nonnegative rectangular matrices $C, D$ such that $A = CD, B = DC$. We write it as $A \underset{C,D}{\approx} B$. If there exists a finite sequence of nonnegative matrices $A_0, A_1, \dots, A_n$ such that $A = A_0, B = A_n$ and $A_i$ is elementary equivalent to $A_{i+1}$ for $i = 0, 1, 2, \dots, n-1$, then $A$ and $B$ are said to be strong shift equivalent. Hence elementary equivalence generates topological conjugacy of two-sided topological Markov shifts.

Let $A$ be an irreducible non-permutation matrix. The triplet $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ for the Cuntz-Krieger algebra $\mathcal{O}_A$, its canonical maximal abelian $C^*$-subalgebra $\mathcal{D}_A$, and its gauge action $\rho^A$ is called the Cuntz–Krieger triplet for the matrix $A$. As pointed out in [11], two elementary equivalence matrices $A = CD, B = DC$ yield an $\mathcal{O}_A - \mathcal{O}_B$-imprimitivity bimodule via the Cuntz–Krieger algebra $\mathcal{O}_Z$ for the matrix $Z$ defined by $Z = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$.
In the first part of the paper, We will introduce a notion of strong Morita equivalence in the Cuntz–Krieger triplets, and prove the following theorem.

Theorem 1.1 (Corolary 2.19). *The Cuntz–Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent if and only if the matrices $A$ and $B$ are strong shift equivalent.*

It is well-known that two unital $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$ are strong Morita equivalent if and only if their stabilizations $\mathcal{A} \otimes \mathcal{K}$ and $\mathcal{B} \otimes \mathcal{K}$ are isomorphic by Brown–Green–Rieffel Theorem [3, Theorem 1.2] (cf. [2], [3], [4]). We will next study relationships between stabilized Cuntz–Krieger algebras with their gauge actions and strong shift equivalence for matrices. We must emphasize that Cuntz and Krieger in [7, Theorem 3.8] and Cuntz in [6, Theorem 2.3] have shown that the stabilized Cuntz–Krieger triplet $(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}, \rho^A \otimes \mathrm{id})$ is invariant under topological conjugacy of the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$. We will investigate stabilizations of generalized gauge actions from a view point of flow equivalence.
Let us denote by $C(X_A, \mathbb{Z})$ the set of $\mathbb{Z}$-valued continuous functions on $X_A$. For $f \in C(X_A, \mathbb{Z})$, define a one-parameter unitary group $U_t(f), t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ in $\mathcal{D}_A$ by

$$U_t(f) = \exp(2\pi\sqrt{-1}tf), \tag{1.4}$$

and an automorphism $\rho_t^{A,f}$ on $\mathcal{O}_A$ for each $t \in \mathbb{T}$ by

$$\rho_t^{A,f}(S_i) = U_t(f)S_i, \qquad i = 1, \dots, N. \tag{1.5}$$

For $f \equiv 1$, the action $\rho_t^{A,1}$ is the gauge action denoted by $\rho_t^A$. Suppose that $A = CD$ and $B = DC$ for some nonnegative rectangular matrices $C, D$. Then there exist homomorphisms $\varphi : C(X_A, \mathbb{Z}) \to C(X_B, \mathbb{Z})$ and $\psi : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z})$ such that

$$(\psi \circ \varphi)(f) = f \circ \sigma_A, \qquad (\varphi \circ \psi)(g) = g \circ \sigma_B \tag{1.6}$$

for $f \in C(X_A, \mathbb{Z})$ and $g \in C(X_B, \mathbb{Z})$. Let us denote by $(H^A, H_+^A)$ the ordered cohomology groups for the one-sided topological Markov shift $(X_A, \sigma_A)$ which has appeared in [17] by setting

$$H^A = C(X_A, \mathbb{Z})/\{\eta - \eta \circ \sigma_A \mid \eta \in C(X_A, \mathbb{Z})\}$$

and its positive cone

$$H_+^A = \{[\eta] \in H^A \mid \eta(x) \geq 0 \text{ for all } x \in X_A\}.$$

The ordered cohomology group $(\bar{H}^A, \bar{H}^A_+)$ for $(\bar{X}_A, \bar{\sigma}_A)$ has been considered by Y. T. Poon in [19]. The latter ordered group $(\bar{H}^A, \bar{H}^A_+)$ has been proved to be a complete invariant of flow equivalence of the two-sided topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$ by M. Boyle and D. Handelman in [1]. The two ordered groups $(\bar{H}^A, \bar{H}^A_+)$ and $(H^A, H^A_+)$ are actually isomorphic ([17, Lemma 3.1]).
In [15], the following result has been proved.

Theorem 1.2 ([15, Corollary 4.4]). *Suppose that $A$ and $B$ are strong shift equivalent. Then there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ satisfying $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$ and a homomorphism $\varphi : C(X_A, \mathbb{Z}) \to C(X_B, \mathbb{Z})$ of ordered groups which induces an isomorphism between $(H^A, H^A_+)$ and $(H^B, H^B_+)$ of ordered groups such that for each function $f \in C(X_A, \mathbb{Z})$ there exists a unitary one-cocycle $v^f_t \in \mathcal{U}(M(\mathcal{O}_A \otimes \mathcal{K}))$ relative to $\rho^{A,f} \otimes \mathrm{id}$ satisfying*

$$\Phi \circ \mathrm{Ad}(v^f_t) \circ (\rho^{A,f}_t \otimes \mathrm{id}) = (\rho^{B,\varphi(f)}_t \otimes \mathrm{id}) \circ \Phi \quad \textit{for } t \in \mathbb{T}.$$

In the second part of the present paper, we will study K-theoretic behavior of the above isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$. Let us denote by $\epsilon_A : K_0(\mathcal{O}_A) \to \mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N$ the isomorphism defined in [6, Proposition 3.1] satisfying $\epsilon_A([1_A]) = [(1, 1, \dots, 1)]$, where $1_A$ is the unit of $\mathcal{O}_A$. We will prove the following theorem.

Theorem 1.3 (Proposition 3.10 and Theorem 4.6). *Let $A$ and $B$ be elementary equivalent matrices, and choose matrices $C$ and $D$ satisfying $A = CD$ and $B = DC$. Then there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ satisfying $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$ and a unitary representation $t \in \mathbb{T} \to u^f_t \in M(\mathcal{D}_A \otimes \mathcal{C})$ for each $f \in C(X_A, \mathbb{Z})$ such that*

$$\Phi \circ \mathrm{Ad}(u^f_t) \circ (\rho^{A,f}_t \otimes \mathrm{id}) = (\rho^{B,\varphi(f)}_t \otimes \mathrm{id}) \circ \Phi \quad \textit{for } f \in C(X_A, \mathbb{Z}),\ t \in \mathbb{T}$$

*and the diagram*

$$\begin{array}{ccc} K_0(\mathcal{O}_A) & \xrightarrow{\Phi_*} & K_0(\mathcal{O}_B) \\ {\scriptstyle \epsilon_A}\downarrow & & \downarrow{\scriptstyle \epsilon_B} \\ \mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N & \xrightarrow{\Phi_{C^t}} & \mathbb{Z}^M/(\mathrm{id} - B^t)\mathbb{Z}^M \end{array}$$

*is commutative, where $\Phi_{C^t}$ is the isomorphism induced by multiplying by the matrix $C^t$.*

In the third part of the paper, we will study the converse of the above theorem for the gauge actions. We will introduce an invariant $K_0^{\mathrm{SSE}}(\mathcal{O}_A)$ which is a non-empty subset of $K_0(\mathcal{O}_A)$. The invariant $K_0^{\mathrm{SSE}}(\mathcal{O}_A)$ is realized as the subset of $\mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N$ consisting of the classes $[v]$ of vectors $v \in \mathbb{Z}^N$ such that $v = D^t_1 \cdots D^t_{n-1} D^t_n [1, 1, \dots, 1]^t$ for some strong shift equivalences $A \underset{C_1,D_1}{\approx} \cdots \underset{C_n,D_n}{\approx} D_n C_n$ (Proposition 5.7). We will then prove the following theorem.

Theorem 1.4 (Theorem 5.8). *Let $A, B$ be irreducible and non-permutation matrices. The following two assertions are equivalent.*

(i) *Two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate.*

(ii) *There exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ and a unitary representation $t \in \mathbb{T} \to v_t^A \in M(\mathcal{D}_A \otimes \mathcal{C})$ such that*

$$\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}, \qquad \Phi \circ \mathrm{Ad}(v_t^A) \circ (\rho_t^A \otimes \mathrm{id}) = (\rho_t^B \otimes \mathrm{id}) \circ \Phi \text{ for } t \in \mathbb{T},$$
$$\Phi_*(K_0^{\mathrm{SSE}}(\mathcal{O}_A)) = K_0^{\mathrm{SSE}}(\mathcal{O}_B).$$

The set $K_0^{\mathrm{SSE}}(\mathcal{O}_A)$ is always a non-empty subset of $K_0(\mathcal{O}_A)$. If in particular the condition $K_0^{\mathrm{SSE}}(\mathcal{O}_A) = K_0(\mathcal{O}_A)$ holds, the matrix $A$ is said to *have full units*. In this case, we have the following corollary.

Corollary 1.5 (Corollary 5.12). *Suppose that the matrices $A$ and $B$ have full units. Then the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate if and only if there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ of $C^*$-algebras and a unitary representation $t \in \mathbb{T} \to v_t^A \in M(\mathcal{D}_A \otimes \mathcal{C})$ such that*

$$\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}, \qquad \Phi \circ \mathrm{Ad}(v_t^A) \circ (\rho_t^A \otimes \mathrm{id}) = (\rho_t^B \otimes \mathrm{id}) \circ \Phi.$$

Throughout the paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{Z}_+$ the set of nonnegative integers, respectively. For the one-sided topological Markov shift $(X_A, \sigma_A)$, a word $\mu = (\mu_1, \dots, \mu_k)$ for $\mu_i \in \{1, \dots, N\}$ is said to be admissible for $X_A$ if $(\mu_1, \dots, \mu_k) = (x_1, \dots, x_k)$ for some element $(x_n)_{n \in \mathbb{N}} \in X_A$. The length of $\mu$ is denoted by $|\mu| = k$. We denote by $B_k(X_A)$ the set of all admissible words of length $k$. We similarly denote by $B_k(\bar{X}_A)$ the set of admissible words of length $k$, so that $B_k(\bar{X}_A) = B_k(X_A)$. The cylinder set $\{(x_n)_{n \in \mathbb{N}} \in X_A \mid x_1 = \mu_1, \dots, x_k = \mu_k\}$ for $\mu = (\mu_1, \dots, \mu_k) \in B_k(X_A)$ is denoted by $U_\mu$.

This paper is a revised version of arXiv:1604.02763v1, in which the given proofs of the main results were incorrect.

## 2 Strong Morita equivalence for Cuntz–Krieger triplets

There is a standard method to associate a Cuntz–Krieger algebra from a square matrix with entries in nonnegative integers as described in [7, Remark 2.16] (see also [23, Section 4]). Now we suppose that $A = [A(i,j)]_{i,j=1}^N$ is an $N \times N$ matrix with entries in nonnegative integers. Then the associated graph $G_A = (V_A, E_A)$ consists of the vertex set $V_A = \{v_1^A, \dots, v_N^A\}$ of $N$ vertices and the edge set $E_A = \{a_1, \dots, a_{N_A}\}$, where there are $A(i,j)$ edges from $v_i^A$ to $v_j^A$. Hence the total number of edges is $\sum_{i,j=1}^N A(i,j)$ denoted by $N_A$. For $a_i \in E_A$, denote by

$t(a_i), s(a_i)$ the terminal vertex of $a_i$, the source vertex of $a_i$, respectively. The graph $G_A$ has the $N_A \times N_A$ transition matrix $A^G = [A^G(i,j)]_{i,j=1}^{N_A}$ of edges defined by

$$A^G(i,j) = \begin{cases} 1 & \text{if } t(a_i) = s(a_j), \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

The Cuntz–Krieger algebra $\mathcal{O}_A$ for the matrix $A$ with entries in nonnegative integers is defined as the Cuntz–Krieger algebra $\mathcal{O}_{A^G}$ for the matrix $A^G$ which is the universal $C^*$-algebra generated by partial isometries $S_{a_i}$ indexed by edges $a_i, i = 1, \dots, N_A$ subject to the relations:

$$\sum_{j=1}^{N_A} S_{a_j} S_{a_j}^* = 1, \qquad S_{a_i}^* S_{a_i} = \sum_{j=1}^{N_A} A^G(i,j) S_{a_j} S_{a_j}^* \quad \text{ for } i = 1, \dots, N_A. \tag{2.2}$$

For a word $\mu = (\mu_1, \dots, \mu_k), \mu_i \in E_A$, we denote by $S_\mu$ the partial isometry $S_{\mu_1} \cdots S_{\mu_k}$.
As in the standard text books [9], [10] of symbolic dynamics, the two-sided topological Markov shift defined by a square matrix with entries in $\{0,1\}$ is naturally topologically conjugate to a topological Markov shift of the edge shift defined by the underlying directed graph. In what follows, we consider edge shifts and hence square matrices with entries in nonnegative integers (cf. [9], [10], [25], etc.). Such a matrix is simply called a nonnegative square matrix. For a nonnegative square matrix $A$, the two-sided shift space $\bar{X}_A$ is defined by the two-sided shift space $\bar{X}_{A^G}$ for the matrix $A^G$ which consists of the two-sided bi-infinite sequences of concatenated edges of the directed graph $G_A$.
Let $A$ and $B$ be elementary equivalent matrices, and choose matrices $C$ and $D$ satisfying $A = CD$ and $B = DC$. The sizes of the matrices $A$ and $B$ are denoted by $N$ and $M$ respectively, so that $C$ is an $N \times M$ matrix and $D$ is an $M \times N$ matrix, respectively. We set $Z = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$ as a block matrix, and we see

$$Z^2 = \begin{bmatrix} CD & 0 \\ 0 & DC \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

For the rectangular matrices $C$ and $D$, the vertex sets $V_C$ and $V_D$ are defined by the disjoint union $V_A \sqcup V_B$, and $C(i,j)$ directed edges are defined from the vertex $v_i^A$ to $v_j^B$, and $D(j,i)$ directed edges are defined from the vertex $v_j^B$ to $v_i^A$, respectively. The former forms a directed graph written $G_C = (V_C, E_C)$, and the latter forms a directed graph written $G_D = (V_D, E_D)$, Hene we have five directed graphs $G_A = (V_A, E_A), G_B = (V_B, E_B), G_C = (V_C, E_C), G_D = (V_D, E_D)$ and $G_Z = (V_Z, E_Z)$ associated to the nonnegative matrices $A, B, C, D$ and $Z$, respectively. In the identity

$$A(i,j) = \sum_{k=1}^{N_B} C(i,k) D(k,j) \qquad \text{for } i,j = 1, \dots, N_A,$$

the left hand side expresses the number of edges in $E_A$ starting with $v_i^A$ and ending with $v_j^A$, whereas the right hand side expresses the number of pairs of edges $E_C$ and $E_D$ starting with $v_i^A$ through some vertex $v_k^B$ and ending with $v_j^A$. Hence we may take a bijection, which is denoted by $\varphi_{A,CD}$, from $E_A$ to a subset of $E_C \times E_D$. The other identity $B = DC$ similarly admits us to take a bijection, which is denoted by $\varphi_{B,DC}$, from $E_B$ to a subset of $E_D \times E_C$. Let $S_c, S_d, c \in E_C, d \in E_D$ be the generating partial isometries of the Cuntz–Krieger algebra $\mathcal{O}_Z$ for the matrix $Z$, so that $\sum_{c\in E_C} S_c S_c^* + \sum_{d\in E_D} S_d S_d^* = 1$ and

$$S_c^* S_c = \sum_{d\in E_D} Z(c,d) S_d S_d^*, \qquad S_d^* S_d = \sum_{c\in E_C} Z(d,c) S_c S_c^*$$

for $c \in E_C, d \in E_D$. Since $S_c S_d \neq 0$ (resp. $S_d S_c \neq 0$) if and only if $\varphi_{A,CD}(a) = cd$ (resp. $\varphi_{B,DC}(b) = dc$) for a unique edge $a \in E_A$ (resp. $b \in E_B$), we may identify $cd$ (resp. $dc$) with $a$ (resp. $b$) through the map $\varphi_{A,CD}$ (resp. $\varphi_{B,DC}$). We may then write $S_{cd} = S_a$ (resp. $S_{dc} = S_b$) where $S_{cd}$ denotes $S_c S_d$ (resp. $S_{dc}$ denotes $S_d S_c$). We define two particular projections $P_C$ and $P_D$ in $\mathcal{D}_Z$ by $P_C = \sum_{c\in E_C} S_c S_c^*$ and $P_D = \sum_{d\in E_D} S_d S_d^*$ so that $P_C + P_D = 1$. It has been shown in [11] (cf. [15]) that

$$P_C \mathcal{O}_Z P_C = \mathcal{O}_A, \quad P_D \mathcal{O}_Z P_D = \mathcal{O}_B, \quad \mathcal{D}_Z P_C = \mathcal{D}_A, \quad \mathcal{D}_Z P_D = \mathcal{D}_B. \tag{2.3}$$

As in [11, Lemma 3.10], both $P_C$ and $P_D$ are full projections so that $P_C \mathcal{O}_Z P_D$ has a natural structure of $\mathcal{O}_A$–$\mathcal{O}_B$ imprimitivity bimodule that makes $\mathcal{O}_A$ and $\mathcal{O}_B$ strong Morita equivalent (cf. [16], [21], [22]).
Let $\rho^Z, \rho^A, \rho^B$ be the gauge actions of $\mathcal{T}$ on $\mathcal{O}_Z, \mathcal{O}_A, \mathcal{O}_B$, respectively. Since $S_c S_d$ (resp. $S_d S_c$) in $\mathcal{O}_Z$ is identified with $S_a$ in $\mathcal{O}_A$ (resp. $S_b$ in $\mathcal{O}_B$) if $\varphi_{A,CD}(a) = cd$ (resp. $\varphi_{B,DC}(b) = dc$, we have

$$\rho_t^Z|_{P_C \mathcal{O}_Z P_C} = \rho_{2t}^A \quad \text{on } \mathcal{O}_A, \qquad \rho_t^Z|_{P_D \mathcal{O}_Z P_D} = \rho_{2t}^B \quad \text{on } \mathcal{O}_B. \tag{2.4}$$

Let $A$ be an irreducible non-permutation matrix. The triplet $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ for the Cuntz-Krieger algebra $\mathcal{O}_A$, its canonical maximal abelian $C^*$-subalgebra $\mathcal{D}_A$, and its gauge action $\rho^A$ is called the Cuntz–Krieger triplet for the matrix $A$. In this section we will define the notion of strong Morita equivalence in Cuntz–Krieger triplets. We will then prove that the Cuntz–Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent if and only if the matrices $A$ and $B$ are strong shift equivalent. Let $A, B$ be irreducible non-permutation matrices.

Definition 2.1. The Cuntz–Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are said to be *strong Morita equivalent in* 1-*step* if there exist a Cuntz–Krieger triplet $(\mathcal{O}_Z, \mathcal{D}_Z, \rho^Z)$ for some nonnegative matrix $Z$ and projections $P_A, P_B \in \mathcal{D}_Z$ having the following properties:

(1) $P_A + P_B = 1$,

(2) $\mathcal{O}_Z$ contains both $\mathcal{O}_A$ and $\mathcal{O}_B$ as subalgebras, and
$P_A\mathcal{O}_Z P_A = \mathcal{O}_A$ and $P_B\mathcal{O}_Z P_B = \mathcal{O}_B$,

(3) $\mathcal{D}_Z P_A = \mathcal{D}_A$ and $\mathcal{D}_Z P_B = \mathcal{D}_B$,

(4) $\rho^Z_t|_{P_A\mathcal{O}_Z P_A} = \rho^A_{2t}$ on $\mathcal{O}_A$ and $\rho^Z_t|_{P_B\mathcal{O}_Z P_B} = \rho^B_{2t}$ on $\mathcal{O}_B$ for $t \in \mathbb{T}$.

In this case, we say that $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent in 1-step via $(\mathcal{O}_Z, \mathcal{D}_Z, \rho^Z)$. If two Cuntz–Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are connected through $n$-chains of strong Morita equivalences in 1-step, $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are said to be strong Morita equivalent in $n$-step, or simply, strong Morita equivalent.
The one-sided topological Markov shifts $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are said to be eventually conjugate if there exist a homeomorphism $h : X_A \to X_B$ and a nonnegative integer $K$ such that

$$\sigma_B^K(h(\sigma_A(x))) = \sigma_B^K(h(x)), \qquad x \in X_A,$$
$$\sigma_A^K(h^{-1}(\sigma_B(y))) = \sigma_A^K(h^{-1}(y)), \qquad y \in X_B.$$

It has been shown that there exists an isomorphism $\Phi : \mathcal{O}_A \longrightarrow \mathcal{O}_B$ satisfying $\Phi(\mathcal{D}_A) = \mathcal{D}_B$ and $\Phi \circ \rho^A_t = \rho^B_t \circ \Phi$, $t \in \mathbb{T}$ if and only if the one-sided topological Markov shifts $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are eventually conjugate ([15, Corollary 3.5]). The latter condition implies that their two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate by [14, Theorem 5.5] (cf. [14, Theorem 6.7]). Hence an isomorphic Cuntz–Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ yields a strong shift equivalence between the underlying matrices $A$ and $B$.

Proposition 2.2. *If $A$ and $B$ are elementary equivalent, then their Cuntz–Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent in 1-steps,*

*Proof.* Let $A$ and $B$ be elementary equivalent matrices, and choose matrices $C$ and $D$ satisfying $A = CD, B = DC$. Let $Z$ be the square matrix $Z = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$.
By the above discussions, there exist projections $P_C, P_D$ in $\mathcal{D}_Z$ satisfying $P_C + P_D = 1$ and (2.3) (2.4). $\square$

The main purpose of this section is to study the converse implication of Proposition 2.2.
We henceforth assume that $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent in 1-step via $(\mathcal{O}_Z, \mathcal{D}_Z, \rho^Z)$ for some matrix $Z$. We may take two projections $P_A, P_B$ in $\mathcal{D}_Z$ having the properties (1), (2), (3) and (4) in Definition 2.1. Let us denote by $G_Z = (V_Z, E_Z)$ the directed graph for the matrix $Z$. The Cuntz–Krieger algebra $\mathcal{O}_Z$ is then generated by partial isometries $S_\gamma, \gamma \in E_Z$ satisfying the relations:

$$\sum_{\eta \in E_Z} S_\eta S_\eta^* = 1, \qquad S_\gamma^* S_\gamma = \sum_{\eta \in E_Z} Z^G(\gamma, \eta) S_\eta S_\eta^* \quad \text{for } \gamma \in E_Z \tag{2.5}$$

where $Z^G(\gamma,\eta) = 1$ if $t(\gamma) = s(\eta)$, and 0 otherwise. We have the following lemmas.

Lemma 2.3. *Let $S_\gamma, \gamma \in E_Z$ be the generating partial isometries of $\mathcal{O}_Z$ satisfying* (2.5). *Then we have*

(i) $P_A S_\gamma P_A = P_B S_\gamma P_B = 0$.

(ii) $S_\gamma = P_A S_\gamma P_B + P_B S_\gamma P_A$.

(iii) $P_A S_\gamma = S_\gamma P_B$ *and* $P_B S_\gamma = S_\gamma P_A$.

*Proof.* By the equality $P_A + P_B = 1$, we have

$$S_\gamma = P_A S_\gamma P_A + P_A S_\gamma P_B + P_B S_\gamma P_A + P_B S_\gamma P_B.$$

Since $P_A S_\gamma P_A$ belongs to $P_A \mathcal{O}_Z P_A$ which is identified with $\mathcal{O}_A$, the condition (4) of Definition 2.1 gives rise to the equality

$$\rho^Z_t(P_A S_\gamma P_A) = \rho^A_{2t}(P_A S_\gamma P_A). \tag{2.6}$$

As $\rho^Z_t|_{\mathcal{D}_Z} = \mathrm{id}$ and $P_A, P_B \in \mathcal{D}_Z$, the left hand side for $t = \frac{1}{2}$ of (2.6) equals

$$P_A \rho^Z_{\frac{1}{2}}(S_\gamma) P_A = -P_A S_\gamma P_A.$$

As $\rho^A_1 = \mathrm{id}$, the right hand side for $t = \frac{1}{2}$ equals $P_A S_\gamma P_A$. Hence we have $P_A S_\gamma P_A = 0$ and similarly $P_B S_\gamma P_B = 0$. Therefore we know (i) and hence (ii). The assertion (iii) follows from (ii) since $P_A$ and $P_B$ are mutually orthogonal projections. □

Lemma 2.4.

$$\sum_{\gamma\in E_Z} S_\gamma P_A S^*_\gamma = P_B, \qquad \sum_{\gamma\in E_Z} S_\gamma P_B S^*_\gamma = P_A. \tag{2.7}$$

*Proof.* By Lemma 2.3, we know $S_\gamma P_A = P_B S_\gamma$ so that

$$\sum_{\gamma\in E_Z} S_\gamma P_A S^*_\gamma = \sum_{\gamma\in E_Z} P_B S_\gamma S^*_\gamma = P_B. \tag{2.8}$$

Similarly we see that $\sum_{\gamma\in E_Z} S_\gamma P_B S^*_\gamma = P_A$. □

We notice the identities in the following lemma which immediately come from Lemma 2.3 (iii).

Lemma 2.5. *For $\gamma_1, \gamma_2 \in E_Z$, we have the following identities.*

(i) $S_{\gamma_1} S_{\gamma_2} P_A = P_A S_{\gamma_1} S_{\gamma_2} \in \mathcal{O}_A$ *and* $S_{\gamma_1} S_{\gamma_2} P_B = P_B S_{\gamma_1} S_{\gamma_2} \in \mathcal{O}_B$.

(ii) $S_{\gamma_1} P_B S_{\gamma_2} = P_A S_{\gamma_1} P_B S_{\gamma_2} P_A \in \mathcal{O}_A$ *and* $S_{\gamma_1} P_A S_{\gamma_2} = P_B S_{\gamma_1} P_A S_{\gamma_2} P_B \in \mathcal{O}_B$.

LEMMA 2.6. *Let $\gamma_1, \gamma_2 \in E_Z$. Then $P_A S_{\gamma_1} \neq 0, P_B S_{\gamma_2} \neq 0$ and $Z^G(\gamma_1, \gamma_2) = 1$ if and only if $P_A S_{\gamma_1} S_{\gamma_2} \neq 0$.*

*Proof.* Since the identity

$$P_A S_{\gamma_1} S_{\gamma_2} = P_A S_{\gamma_1} P_B S_{\gamma_2} = S_{\gamma_1} S_{\gamma_2} P_A = Z(\gamma_1, \gamma_2) S_{\gamma_1} S_{\gamma_2} P_A \tag{2.9}$$

holds, the if part is obvious. It suffices to show the only if part. By the identity (2.9), we have

$$\begin{aligned}
(P_A S_{\gamma_1} S_{\gamma_2})^* (P_A S_{\gamma_1} S_{\gamma_2}) &= (S_{\gamma_1} S_{\gamma_2} P_A)^* S_{\gamma_1} S_{\gamma_2} P_A \\
&= P_A S_{\gamma_2}^* S_{\gamma_1}^* S_{\gamma_1} S_{\gamma_2} P_A \\
&= \sum_{\eta_1 \in E_Z} Z^G(\gamma_1, \eta_1) P_A S_{\gamma_2}^* S_{\eta_1} S_{\eta_1}^* S_{\gamma_2} P_A \\
&= Z^G(\gamma_1, \gamma_2) P_A S_{\gamma_2}^* S_{\gamma_2} P_A \\
&= Z^G(\gamma_1, \gamma_2) (P_B S_{\gamma_2})^* (P_B S_{\gamma_2}).
\end{aligned}$$

The above equalities ensure the only if part. □

LEMMA 2.7. *Let $\gamma_1, \gamma_2, \eta_1, \eta_2 \in E_Z$. Then $S_{\gamma_1} S_{\gamma_2} \neq 0, S_{\gamma_2} S_{\eta_1} \neq 0, P_A S_{\eta_1} S_{\eta_2} \neq 0$ if and only if $P_A S_{\gamma_1} S_{\gamma_2} S_{\eta_1} S_{\eta_2} \neq 0$.*

*Proof.* Since $P_A S_{\gamma_1} S_{\gamma_2} S_{\eta_1} S_{\eta_2} = S_{\gamma_1} S_{\gamma_2} P_A S_{\eta_1} S_{\eta_2}$, the if part is obvious. It suffices to show the only if part. We have

$$\begin{aligned}
& (P_A S_{\gamma_1} S_{\gamma_2} S_{\eta_1} S_{\eta_2})^* (P_A S_{\gamma_1} S_{\gamma_2} S_{\eta_1} S_{\eta_2}) \\
=& P_A S_{\eta_2}^* S_{\eta_1}^* S_{\gamma_2}^* S_{\gamma_1}^* S_{\gamma_1} S_{\gamma_2} S_{\eta_1} S_{\eta_2} P_A \\
=& \sum_{\zeta_1 \in E_Z} Z^G(\gamma_1, \zeta_1) P_A S_{\eta_2}^* S_{\eta_1}^* S_{\gamma_2}^* S_{\zeta_1} S_{\zeta_1}^* S_{\gamma_2} S_{\eta_1} S_{\eta_2} P_A \\
=& Z^G(\gamma_1, \gamma_2) \sum_{\zeta_2 \in E_Z} Z^G(\gamma_2, \zeta_2) P_A S_{\eta_2}^* S_{\eta_1}^* S_{\zeta_2} S_{\zeta_2}^* S_{\eta_1} S_{\eta_2} P_A \\
=& Z^G(\gamma_1, \gamma_2) Z^G(\gamma_2, \eta_1) \sum_{\zeta_3 \in E_Z} Z^G(\eta_1, \zeta_3) P_A S_{\eta_2}^* S_{\zeta_3} S_{\zeta_3}^* S_{\eta_2} P_A \\
=& Z^G(\gamma_1, \gamma_2) Z^G(\gamma_2, \eta_1) Z^G(\eta_1, \eta_2) P_A S_{\eta_2}^* S_{\eta_2} P_A.
\end{aligned}$$

The above equalities ensure the only if part. □

Now we are assuming that the Cuntz–Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent in 1-step via $(\mathcal{O}_Z, \mathcal{D}_Z, \rho^Z)$. Recall that $B_k(X_A)$ denotes the set of admissible words $\gamma_1 \cdots \gamma_k$ of $X_Z$ with length $k$. For $k = 2$, we see

$$B_2(X_A) = \{\gamma_1 \gamma_2 \mid Z^G(\gamma_1, \gamma_2) = 1\}.$$

Let us note that for $\gamma_1, \gamma_2 \in E_Z$, the word $\gamma_1 \gamma_2$ belongs to $B_2(X_A)$ if and only if $S_{\gamma_1} S_{\gamma_2} \neq 0$.

We introduce several directed graphs in this situation. Define edge sets $E_{\tilde{A}}, E_{\tilde{B}}, E_{\tilde{C}}, E_{\tilde{D}}$ by setting

$$\begin{aligned}
E_{\tilde{A}} &= \{(A, \gamma_1\gamma_2) \in \{A\} \times B_2(X_Z) \mid P_A S_{\gamma_1} S_{\gamma_2} \neq 0\},\\
E_{\tilde{B}} &= \{(B, \gamma_1\gamma_2) \in \{B\} \times B_2(X_Z) \mid P_B S_{\gamma_1} S_{\gamma_2} \neq 0\},\\
E_{\tilde{C}} &= \{(A, \gamma_1) \in \{A\} \times E_Z \mid P_A S_{\gamma_1} \neq 0\},\\
E_{\tilde{D}} &= \{(B, \gamma_1) \in \{B\} \times E_Z \mid P_B S_{\gamma_1} \neq 0\}
\end{aligned}$$

and vertex sets $V_{\tilde{A}s}, V_{\tilde{A}t}, V_{\tilde{B}s}, V_{\tilde{B}t}, V_{\tilde{C}s}, V_{\tilde{C}t}, V_{\tilde{D}s}, V_{\tilde{D}t}$ by setting

$$\begin{aligned}
V_{\tilde{A}s} &= \{(A, s(\gamma_1)) \in \{A\} \times V_Z \mid (A, \gamma_1\gamma_2) \in E_{\tilde{A}}\},\\
V_{\tilde{A}t} &= \{(A, t(\gamma_2)) \in \{A\} \times V_Z \mid (A, \gamma_1\gamma_2) \in E_{\tilde{A}}\},\\
V_{\tilde{B}s} &= \{(B, s(\gamma_1)) \in \{B\} \times V_Z \mid (B, \gamma_1\gamma_2) \in E_{\tilde{B}}\},\\
V_{\tilde{B}t} &= \{(B, t(\gamma_2)) \in \{B\} \times V_Z \mid (B, \gamma_1\gamma_2) \in E_{\tilde{B}}\},\\
V_{\tilde{C}s} &= \{(A, s(\gamma_1)) \in \{A\} \times V_Z \mid (A, \gamma_1) \in E_{\tilde{C}}\},\\
V_{\tilde{C}t} &= \{(B, t(\gamma_1)) \in \{A\} \times V_Z \mid (A, \gamma_1) \in E_{\tilde{C}}\},\\
V_{\tilde{D}s} &= \{(B, s(\gamma_1)) \in \{B\} \times V_Z \mid (B, \gamma_1) \in E_{\tilde{D}}\},\\
V_{\tilde{D}t} &= \{(A, t(\gamma_1)) \in \{B\} \times V_Z \mid (B, \gamma_1) \in E_{\tilde{D}}\}.
\end{aligned}$$

Lemma 2.8. *Keep the above notation. We have*

(i) $V_{\tilde{A}s} = V_{\tilde{A}t} = V_{\tilde{C}s} = V_{\tilde{D}t}$.

(ii) $V_{\tilde{B}s} = V_{\tilde{B}t} = V_{\tilde{D}s} = V_{\tilde{C}t}$.

*Proof.* (i) We will first show the equality $V_{\tilde{A}s} = V_{\tilde{A}t}$. Take an arbitrary vertex $(A, s(\gamma_1)) \in V_{\tilde{A}s}$ and $\gamma_2 \in E_Z$ with $P_A S_{\gamma_1} S_{\gamma_2} \neq 0$, so that $t(\gamma_1) = s(\gamma_2)$. We may find $\eta_1, \eta_2 \in E_Z$ such that $S_{\eta_1} S_{\eta_2} \neq 0$ and $t(\eta_2) = s(\gamma_1)$. By Lemma 2.7, we have $S_{\eta_1} S_{\eta_2} S_{\gamma_1} S_{\gamma_2} P_A \neq 0$. Since $S_{\eta_1} S_{\eta_2} S_{\gamma_1} S_{\gamma_2} P_A = P_A S_{\eta_1} S_{\eta_2} S_{\gamma_1} S_{\gamma_2}$, we have $P_A S_{\eta_1} S_{\eta_2} \neq 0$ so that $(A, t(\eta_2)) \in V_{\tilde{A}t}$ and hence $(A, s(\gamma_1) \in V_{\tilde{A}t}$. This shows that the inclusion relation $V_{\tilde{A}s} \subset V_{\tilde{A}t}$ holds. Similarly we obtain that $V_{\tilde{A}t} \subset V_{\tilde{A}s}$ so that $V_{\tilde{A}s} = V_{\tilde{A}t}$.
We will second show the equality $V_{\tilde{C}s} = V_{\tilde{D}t}$. Take an arbitrary vertex $(A, s(\gamma_1)) \in V_{\tilde{C}s}$. We see that $P_A S_{\gamma_1} \neq 0$ and hence $S_{\gamma_1} P_B \neq 0$. Now both matrices $A$ and $B$ are assumed to be irreducible and not any permutations, so that $Z$ and hence $Z^G$ are irreducible and not any permutations. This implies that $\sum_{\gamma' \in E_Z} S^*_{\gamma'} S_{\gamma'} \geq 1$. Hence we may find $\gamma_2 \in E_Z$ such that $S_{\gamma_2} S_{\gamma_1} P_B \neq 0$ so that $t(\gamma_2) = s(\gamma_1)$. Since $S_{\gamma_2} S_{\gamma_1} P_B = P_B S_{\gamma_2} S_{\gamma_1}$, we have $P_B S_{\gamma_2} \neq 0$. This implies that $(B, \gamma_2) \in E_{\tilde{D}}$ and $(A, t(\gamma_2)) \in V_{\tilde{D}t}$. As $t(\gamma_2) = s(\gamma_1)$, we obtain that $(A, s(\gamma_1)) \in V_{\tilde{D}t}$ so that $V_{\tilde{C}s} \subset V_{\tilde{D}t}$. We see $V_{\tilde{D}t} \subset V_{\tilde{C}s}$ similarly so that $V_{\tilde{C}s} = V_{\tilde{D}t}$.
We will finally show that $V_{\tilde{A}s} = V_{\tilde{C}s}$. Since the condition $P_A S_{\gamma_1} S_{\gamma_2} \neq 0$ implies $P_A S_{\gamma_1} \neq 0$, we have $V_{\tilde{A}s} \subset V_{\tilde{C}s}$. Conversely, for $(A, s(\gamma_1)) \in V_{\tilde{C}s}$, we have $P_A S_{\gamma_1} \neq 0$ so that $S_{\gamma_1} P_B \neq 0$. Since $P_B = \sum_{\gamma' \in E_Z} S_{\gamma'} P_A S^*_{\gamma'}$, we may find $\gamma_2 \in E_Z$ such that $S_{\gamma_1} S_{\gamma_2} P_A \neq 0$. Hence we see that $P_A S_{\gamma_1} S_{\gamma_2} \neq 0$ so that

$(A, s(\gamma_1)) \in V_{\tilde{A}s}$. This shows that $V_{\tilde{A}s} = V_{\tilde{C}s}$. Therefore (i) has been shown. (ii) is shown similarly. $\square$

Let us denote by $V_{\tilde{A}}$ and by $V_{\tilde{B}}$ the first four vertex sets and the second four vertex sets in Lemma 2.8, respectively. Namely we put

$$V_{\tilde{A}} := V_{\tilde{A}s} = V_{\tilde{A}t} = V_{\tilde{C}s} = V_{\tilde{D}t},$$
$$V_{\tilde{B}} := V_{\tilde{B}s} = V_{\tilde{B}t} = V_{\tilde{D}s} = V_{\tilde{C}t}.$$

For an edge $(A, \gamma_1\gamma_2) \in E_{\tilde{A}}$, define its source and terminal vertices by

$$s(A, \gamma_1\gamma_2) = (A, s(\gamma_1)) \in V_{\tilde{A}s}, \qquad t(A, \gamma_1\gamma_2) = (A, t(\gamma_2)) \in V_{\tilde{A}t}.$$

We then have a directed graph $(V_{\tilde{A}}, E_{\tilde{A}})$ denoted by $G_{\tilde{A}}$. We have a directed graph $G_{\tilde{B}} = (V_{\tilde{B}}, E_{\tilde{B}})$ similarly. From an edge $(A, \gamma_1) \in E_{\tilde{C}}$, define its source and terminal vertices by

$$s(A, \gamma_1) = (A, s(\gamma_1)) \in V_{\tilde{C}s}, \qquad t(A, \gamma_1) = (A, t(\gamma_1)) \in V_{\tilde{C}t}.$$

We have a directed graph $G_{\tilde{C}} = (V_{\tilde{A}} \xrightarrow{E_{\tilde{C}}} V_{\tilde{B}})$ and similarly $G_{\tilde{D}} = (V_{\tilde{B}} \xrightarrow{E_{\tilde{D}}} V_{\tilde{A}})$. Let $\tilde{A}$ be the vertex transition matrix $\tilde{A} : V_{\tilde{A}} \times V_{\tilde{A}} \longrightarrow \mathbb{Z}_+$ of the directed graph $G_{\tilde{A}}$ which is defined by

$$\tilde{A}((A, u), (A, v)) = |\{(A, \gamma_1\gamma_2) \in E_{\tilde{A}} \mid s(\gamma_1) = u, t(\gamma_2) = v\}|$$

for $(A, u), (A, v) \in V_{\tilde{A}}$. The edge transition matrix $\tilde{A}^G : E_{\tilde{A}} \times E_{\tilde{A}} \longrightarrow \{0, 1\}$ of $G_{\tilde{A}}$ is defined by

$$\tilde{A}^G(\gamma_1\gamma_2, \eta_1\eta_2) = \begin{cases} 1 & \text{if } t(A, \gamma_1\gamma_2) = s(A, \eta_1\eta_2), \\ 0 & \text{otherwise} \end{cases}$$

for $(A, \gamma_1\gamma_2), (A, \eta_1\eta_2) \in E_{\tilde{A}}$. We similarly have the vertex transition matrices $\tilde{B}, \tilde{C}, \tilde{D}$ and the edge transition matrices $\tilde{B}^G, \tilde{C}^G, \tilde{D}^G$ of the directed graphs $G_{\tilde{B}}, G_{\tilde{C}}, G_{\tilde{D}}$, respectively.

Proposition 2.9. *The matrices $\tilde{A}$ and $\tilde{B}$ are elementary equivalent such that*

$$\tilde{A} = \tilde{C}\tilde{D} \quad \text{and} \quad \tilde{B} = \tilde{D}\tilde{C}.$$

*Hence the two-sided topological Markov shifts $(\bar{X}_{\tilde{A}}, \bar{\sigma}_{\tilde{A}})$ and $(\bar{X}_{\tilde{B}}, \bar{\sigma}_{\tilde{B}})$ are topologically conjugate.*

*Proof.* For $(A, \gamma_1\gamma_2)$ with $\gamma_1, \gamma_2 \in E_Z$, Lemma 2.6 ensures that $(A, \gamma_1) \in E_{\tilde{C}}, (B, \gamma_2) \in E_{\tilde{D}}, Z^G(\gamma_1, \gamma_2) = 1$ if and only if $(A, \gamma_1\gamma_2) \in E_{\tilde{A}}$. Since $t(A, \gamma_1) = s(B, \gamma_2)$ if and only if $Z^G(\gamma_1, \gamma_2) = 1$, we know that $\tilde{A} = \tilde{C}\tilde{D}$, and $\tilde{B} = \tilde{D}\tilde{C}$ similarly. $\square$

A directed graph $G = (V, E)$ with vertex set $V$ and edge set $E$ is said to be bipartite if $V$ and $E$ may be decomposed into disjoint unions $V = V_1 \sqcup V_2$ and $E = E_{12} \sqcup E_{21}$ such that

$$V_1 = \{s(\gamma) \in V \mid \gamma \in E_{12}\} = \{t(\gamma) \in V \mid \gamma \in E_{21}\},$$
$$V_2 = \{s(\gamma) \in V \mid \gamma \in E_{21}\} = \{t(\gamma) \in V \mid \gamma \in E_{12}\}.$$

Let $E_{\tilde{Z}} = E_{\tilde{C}} \cup E_{\tilde{D}}$ and $V_{\tilde{Z}} = V_{\tilde{A}} \cup V_{\tilde{B}}$. It is now obvious that the directed graph $G_{\tilde{Z}} = (V_{\tilde{Z}}, E_{\tilde{Z}})$ is bipartite. Let us denote by $\tilde{Z}$ and $\tilde{Z}^G$ the vertex transition matrix and the edge transition matrix of the directed graph $G_{\tilde{Z}}$, respectively. Since $G_{\tilde{Z}}$ is bipartite, by the above proposition, we have

$$\tilde{Z} = \begin{bmatrix} 0 & \tilde{C} \\ \tilde{D} & 0 \end{bmatrix}, \qquad \tilde{Z}^2 = \begin{bmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{bmatrix}.$$

We will study the relationship between the two matrices $\tilde{Z}$ and $Z$. For $\gamma \in E_Z$, denote by $S_{(A,\gamma)}, S_{(B,\gamma)}$ the partial isometries $P_A S_\gamma, P_B S_\gamma$, respectively, so that $S_\gamma = S_{(A,\gamma)} + S_{(B,\gamma)}$.

Lemma 2.10. *Let $\gamma_1, \gamma_2 \in E_Z$ satisfy $Z^G(\gamma_1, \gamma_2) = 1$.*

(i) $S_{(B,\gamma_2)} \neq 0$ *implies* $S_{(A,\gamma_1)} \neq 0$.

(ii) $S_{(A,\gamma_2)} \neq 0$ *implies* $S_{(B,\gamma_1)} \neq 0$.

*Proof.* (i) Since $S_{(A,\gamma_1)} S_{(B,\gamma_2)} = P_A S_{\gamma_1} P_B S_{\gamma_2} = S_{\gamma_1} S_{\gamma_2} P_A$, we have

$$\begin{aligned}(S_{(A,\gamma_1)} S_{(B,\gamma_2)})^* (S_{(A,\gamma_1)} S_{(B,\gamma_2)}) &= P_A S_{\gamma_2}^* S_{\gamma_1}^* S_{\gamma_1} S_{\gamma_2} P_A \\ &= \sum_{\eta_1 \in E_Z} Z^G(\gamma_1, \eta_1) P_A S_{\gamma_2}^* S_{\eta_1} S_{\eta_1}^* S_{\gamma_2} P_A \\ &= Z^G(\gamma_1, \gamma_2) S_{(B,\gamma_2)}^* S_{(B,\gamma_2)}.\end{aligned}$$

The above equality ensures the assertion. (ii) is shown similarly. □

Lemma 2.11. *Either of the following two situations occurs:*

(1) *Both $S_{(A,\gamma)}$ and $S_{(B,\gamma)}$ are not zeros for all $\gamma \in E_Z$. In this case we have $\tilde{C}^G = \tilde{D}^G = Z^G$ so that $\tilde{A} = \tilde{B}$ and $\tilde{Z} = \begin{bmatrix} 0 & Z \\ Z & 0 \end{bmatrix}$*

(2) *For each $\gamma \in E_Z$, either $S_{(A,\gamma)} = 0, S_{(B,\gamma)} \neq 0$ or $S_{(A,\gamma)} \neq 0, S_{(B,\gamma)} = 0$ holds. In this case we have $\tilde{Z} = Z$.*

*Proof.* Suppose that there exists $\gamma_0 \in E_Z$ such that both conditions $S_{(A,\gamma_0)} \neq 0$ and $S_{(B,\gamma_0)} \neq 0$ hold. Since the directed graph $G_Z = (V_Z, E_Z)$ is irreducible, for any edge $\gamma \in E_Z$, there exists a finite sequence of edges $\gamma_1, \dots, \gamma_n$ in $E_Z$ such that

$$Z^G(\gamma, \gamma_1) = Z^G(\gamma_1, \gamma_2) = \cdots = Z^G(\gamma_n, \gamma_0) = 1.$$

By the preceding lemma, any edge $\eta \in E_Z$ satisfying $Z^G(\eta, \gamma_0) = 1$ forces that $S_{(A,\eta)} \neq 0$ and $S_{(B,\eta)} \neq 0$. By using this argument repeatedly, we see that $S_{(A,\gamma)} \neq 0$ and $S_{(B,\gamma)} \neq 0$. Hence either of the following two cases occurs:

(1) Both $S_{(A,\gamma)}$ and $S_{(B,\gamma)}$ are not zeros for all $\gamma \in E_Z$.

(2) For each $\gamma \in E_Z$, either $S_{(A,\gamma)} = 0$ or $S_{(B,\gamma)} = 0$.

Case (1): We have the following equalities.

$$\begin{aligned}
S_\gamma^* S_\gamma =&(S_{(A,\gamma)}^* + S_{(B,\gamma)}^*)(S_{(A,\gamma)} + S_{(B,\gamma)})\\
=&S_{(A,\gamma)}^* S_{(A,\gamma)} + S_{(B,\gamma)}^* S_{(B,\gamma)}\\
=&\sum_{(B,\eta)\in E_{\tilde{D}}} \tilde{C}^G((A,\gamma),(B,\eta)) S_{(B,\eta)} S_{(B,\eta)}^*\\
&+ \sum_{(A,\eta)\in E_{\tilde{C}}} \tilde{D}^G((B,\gamma),(A,\eta)) S_{(A,\eta)} S_{(A,\eta)}^*.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
S_\gamma^* S_\gamma =& \sum_{\eta\in E_Z} Z^G(\gamma,\eta) S_\eta S_\eta^*\\
=& \sum_{\eta\in E_Z} Z^G(\gamma,\eta)(P_B S_\eta S_\eta^* P_B + P_A S_\eta S_\eta^* P_A)\\
=& \sum_{\eta\in E_Z} Z^G(\gamma,\eta) S_{(B,\eta)} S_{(B,\eta)}^* + \sum_{\eta\in E_Z} Z^G(\gamma,\eta) S_{(A,\eta)} S_{(A,\eta)}^*.
\end{aligned}$$

Since both $S_{(A,\gamma)} \neq 0$ and $S_{(B,\gamma)} \neq 0$ for all $\gamma \in E_Z$, we have

$$\tilde{C}^G((A,\gamma),(B,\eta)) = Z^G(\gamma,\eta), \qquad \tilde{D}^G((B,\gamma),(A,\eta)) = Z^G(\gamma,\eta)$$

for all $\gamma, \eta \in E_Z$. Hence we have $\tilde{C}^G = \tilde{D}^G = Z^G$ so that $\tilde{A}^G = \tilde{B}^G$ and hence $\tilde{A} = \tilde{B}$. As $\tilde{Z} = \begin{bmatrix} 0 & \tilde{C} \\ \tilde{D} & 0 \end{bmatrix}$, we have $\tilde{Z}^G = \begin{bmatrix} 0 & Z^G \\ Z^G & 0 \end{bmatrix}$ and hence $\tilde{Z} = \begin{bmatrix} 0 & Z \\ Z & 0 \end{bmatrix}$.
Case (2): Suppose that for each $\gamma \in E_Z$, either $S_{(A,\gamma)} \neq 0$ or $S_{(B,\gamma)} \neq 0$ occurs. Since the identity

$$S_\gamma^* S_\gamma = S_{(A,\gamma)}^* S_{(A,\gamma)} + S_{(B,\gamma)}^* S_{(B,\gamma)}$$

always holds, the situation $S_{(A,\gamma)} \neq 0$ or $S_{(B,\gamma)} \neq 0$ occurs. Hence in this case we see that for each $\gamma \in E_Z$, either $S_{(A,\gamma)} = 0, S_{(B,\gamma)} \neq 0$ or $S_{(A,\gamma)} \neq 0, S_{(B,\gamma)} = 0$ occurs. This implies that the edge set $E_Z$ is a disjoint union $E_Z = E_{\tilde{C}} \sqcup E_{\tilde{D}}$. As $S_{(A,\gamma_1)} S_{(A,\gamma_2)} = 0, S_{(B,\gamma_1)} S_{(B,\gamma_2)} = 0$ for all $\gamma_1, \gamma_2 \in E_Z$, we have $Z = \begin{bmatrix} 0 & \tilde{C} \\ \tilde{D} & 0 \end{bmatrix}$ so that $\tilde{Z} = Z$. □

We thus see the following lemma and proposition.

Lemma 2.12. *We have a natural identification between the Cuntz–Krieger triplets* $(\mathcal{O}_{\tilde{Z}}, \mathcal{D}_{\tilde{Z}}, \rho^{\tilde{Z}})$ *and* $(\mathcal{O}_Z, \mathcal{D}_Z, \rho^Z)$.

*Proof.* For $\gamma \in E_Z$, we have that $S_\gamma = P_A S_\gamma + P_B S_\gamma$. If $P_A S_\gamma \neq 0$, then $(A, \gamma) \in E_{\tilde{C}}$. If $P_B S_\gamma \neq 0$, then $(B, \gamma) \in E_{\tilde{D}}$. Hence $S_\gamma$ belongs to the $C^*$-algebra $C^*(S_{(A,\gamma)}, S_{(B,\gamma')} \mid (A,\gamma) \in E_{\tilde{C}}, (B,\gamma') \in E_{\tilde{D}})$ generated by $S_{(A,\gamma)}, S_{(B,\gamma')}$ with $(A,\gamma) \in E_{\tilde{C}}$, $(B,\gamma') \in E_{\tilde{D}}$. Hence we have

$$C^*(S_{(A,\gamma)}, S_{(B,\gamma')} \mid (A,\gamma) \in E_{\tilde{C}}, (B,\gamma') \in E_{\tilde{D}}) = \mathcal{O}_Z.$$

Since $E_{\tilde{Z}} = E_{\tilde{C}} \cup E_{\tilde{D}}$ and $V_{\tilde{Z}} = V_{\tilde{C}} \cup V_{\tilde{D}}$, the algebra $C^*(S_{(A,\gamma)}, S_{(B,\gamma')} \mid (A,\gamma) \in E_{\tilde{C}}, (B,\gamma') \in E_{\tilde{D}})$ is nothing but $\mathcal{O}_{\tilde{Z}}$, so that $\mathcal{O}_{\tilde{Z}}$ is identified with $\mathcal{O}_Z$ through the correspondence between $S_{(A,\gamma)} + S_{(B,\gamma)} \in \mathcal{O}_{\tilde{Z}}$ and $S_\gamma \in \mathcal{O}_Z$. We then have

$$\begin{aligned} S_\gamma S_\gamma^* =& (S_{(A,\gamma)} + S_{(B,\gamma)})(S_{(A,\gamma)} + S_{(B,\gamma)})^* \\ =& (P_A S_\gamma + P_B S_\gamma)(P_A S_\gamma + P_B S_\gamma)^* \\ =& P_A S_\gamma S_\gamma^* P_A + P_B S_\gamma S_\gamma^* P_A + P_A S_\gamma S_\gamma^* P_B + P_B S_\gamma S_\gamma^* P_B \\ =& P_A S_\gamma S_\gamma^* P_A + P_B S_\gamma S_\gamma^* P_B. \end{aligned}$$

Similarly, by a routine calculation, we have the equalities

$$\begin{aligned} S_{\gamma_1} S_{\gamma_2} \cdots S_{\gamma_n} S_{\gamma_n}^* \cdots S_{\gamma_2}^* S_{\gamma_1}^* =& P_A S_{\gamma_1} P_B S_{\gamma_2} \cdots S_{\gamma_n} S_{\gamma_n}^* \cdots S_{\gamma_2}^* P_B S_{\gamma_1}^* P_A \\ &+ P_B S_{\gamma_1} P_A S_{\gamma_2} \cdots S_{\gamma_n} S_{\gamma_n}^* \cdots S_{\gamma_2}^* P_A S_{\gamma_1}^* P_B \end{aligned}$$

and

$$\begin{aligned} P_A S_{\gamma_1} S_{\gamma_2} \cdots S_{\gamma_n} S_{\gamma_n}^* \cdots S_{\gamma_2}^* S_{\gamma_1}^* P_A =& P_A S_{\gamma_1} P_B S_{\gamma_2} \cdots S_{\gamma_n} S_{\gamma_n}^* \cdots S_{\gamma_2}^* P_B S_{\gamma_1}^* P_A, \\ P_B S_{\gamma_1} S_{\gamma_2} \cdots S_{\gamma_n} S_{\gamma_n}^* \cdots S_{\gamma_2}^* S_{\gamma_1}^* P_B =& P_B S_{\gamma_1} P_A S_{\gamma_2} \cdots S_{\gamma_n} S_{\gamma_n}^* \cdots S_{\gamma_2}^* P_A S_{\gamma_1}^* P_B. \end{aligned}$$

These equalities give us a natural identification between $\mathcal{D}_{\tilde{Z}}$ and $\mathcal{D}_Z$.
For $t \in \mathbb{T}$, we have

$$\begin{aligned} \rho_t^Z(S_\gamma) =& \rho_t^Z(P_A S_\gamma + P_B S_\gamma) \\ =& P_A \rho_t^Z(S_\gamma) + P_B \rho_t^Z(S_\gamma) \\ =& \exp(2\pi\sqrt{-1}t) P_A S_\gamma + \exp(2\pi\sqrt{-1}t) P_B S_\gamma \\ =& \rho_t^{\tilde{Z}}(S_{(A,\gamma)}) + \rho_t^{\tilde{Z}}(S_{(B,\gamma)}) \\ =& \rho_t^{\tilde{Z}}(S_{(A,\gamma)} + S_{(B,\gamma)}). \end{aligned}$$

Therefore the Cuntz–Krieger triplets $(\mathcal{O}_{\tilde{Z}}, \mathcal{D}_{\tilde{Z}}, \rho^{\tilde{Z}})$ and $(\mathcal{O}_Z, \mathcal{D}_Z, \rho^Z)$ are naturally identified with each other. ∎

Proposition 2.13. $\tilde{Z} = Z$.

*Proof.* By Lemma 2.11, we know that the either of the following two cases occurs:

$$(1)\quad \tilde{Z} = \begin{bmatrix} 0 & Z \\ Z & 0 \end{bmatrix}, \qquad\qquad (2)\quad \tilde{Z} = Z.$$

We assume the first case $\tilde{Z} = \begin{bmatrix} 0 & Z \\ Z & 0 \end{bmatrix}$. Let $I_Z$ denote the identity matrix whose size is the same as that of $Z$. By the unitary $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I_Z & I_Z \\ I_Z & -I_Z \end{bmatrix}$, we have $U\tilde{Z}U^* = \begin{bmatrix} Z & 0 \\ 0 & -Z \end{bmatrix}$, so that $\mathrm{Sp}^\times(\tilde{Z}) = \mathrm{Sp}^\times(Z) \cup (-\mathrm{Sp}^\times(Z))$, where $\mathrm{Sp}^\times(Z)$ denotes the set of non zero spectra of $Z$. By Lemma 2.12, the Cuntz–Krieger triplets $(\mathcal{O}_{\tilde{Z}}, \mathcal{D}_{\tilde{Z}}, \rho^{\tilde{Z}})$ and $(\mathcal{O}_Z, \mathcal{D}_Z, \rho^Z)$ are isomorphic. Hence, as we noted in the paragraph before Proposition 2.2, the two-sided topological Markov shifts $(\bar{X}_{\tilde{Z}}, \bar{\sigma}_{\tilde{Z}})$ and $(\bar{X}_Z, \bar{\sigma}_Z)$ become topologically conjugate, so that $\mathrm{Sp}^\times(\tilde{Z}) = \mathrm{Sp}^\times(Z)$ by a general theory of symbolic dynamics (cf. [10]). This is a contradiction, and the case (1) does not occur. $\square$

We will next study the bipartite graph $G_{\tilde{A}}$ from the $C^*$-algebraic view point. For $(A, \gamma_1\gamma_2) \in E_{\tilde{A}}$, define the partial isometry

$$S_{(A,\gamma_1\gamma_2)} = P_A S_{\gamma_1} S_{\gamma_2}.$$

LEMMA 2.14. *The $C^*$-subalgebra $C^*(S_{(A,\gamma_1\gamma_2)}; (A, \gamma_1\gamma_2) \in E_{\tilde{A}})$ of $\mathcal{O}_Z$ is isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_{\tilde{A}}$ for the matrix $\tilde{A}$.*

*Proof.* We first notice the identity

$$\sum_{(A,\gamma_1\gamma_2)\in E_{\tilde{A}}} S_{(A,\gamma_1\gamma_2)} S^*_{(A,\gamma_1\gamma_2)} = \sum_{\gamma_1,\gamma_2 \in E_Z} P_A S_{\gamma_1} S_{\gamma_2} S^*_{\gamma_2} S^*_{\gamma_1} P_A = P_A$$

holds. We also have

$$\begin{aligned}
& S^*_{(A,\gamma_1\gamma_2)} S_{(A,\gamma_1\gamma_2)} \\
=& P_A S^*_{\gamma_2} S^*_{\gamma_1} S_{\gamma_1} S_{\gamma_2} P_A \\
=& \sum_{\zeta_1 \in E_Z} Z^G(\gamma_1, \zeta_1) P_A S^*_{\gamma_2} S_{\zeta_1} S^*_{\zeta_1} S_{\gamma_2} P_A \\
=& \sum_{\eta_1 \in E_Z} Z^G(\gamma_1, \gamma_2) Z^G(\gamma_2, \eta_1) P_A S_{\eta_1} S^*_{\eta_1} P_A \\
=& \sum_{\eta_1,\eta_2 \in E_Z} Z^G(\gamma_1, \gamma_2) Z^G(\gamma_2, \eta_1) Z^G(\eta_1, \eta_2) P_A S_{\eta_1} S_{\eta_2} S^*_{\eta_2} S^*_{\eta_1} P_A.
\end{aligned}$$

For $(A, \gamma_1\gamma_2), (A, \eta_1\eta_2) \in E_{\tilde{A}}$, the condition $t(A, \gamma_1\gamma_2) = s(A, \eta_1\eta_2)$ holds if and only if $Z^G(\gamma_2, \eta_1) = 1$. Hence we know

$$Z^G(\gamma_1, \gamma_2) Z^G(\gamma_2, \eta_1) Z^G(\eta_1, \eta_2) = \tilde{A}^G(\gamma_1\gamma_2, \eta_1\eta_2).$$

By the above equalities, we have

$$S^*_{(A,\gamma_1\gamma_2)}S_{(A,\gamma_1\gamma_2)} = \sum_{(A,\eta_1\eta_2)\in E_{\tilde{A}}} \tilde{A}^G(\gamma_1\gamma_2,\eta_1\eta_2)S_{(A,\eta_1\eta_2)}S^*_{(A,\eta_1\eta_2)},$$

thus proving that the $C^*$-subalgebra $C^*(S_{(A,\gamma_1\gamma_2)};(A,\gamma_1\gamma_2)\in E_{\tilde{A}})$ of $\mathcal{O}_Z$ is isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_{\tilde{A}}$ for the matrix $\tilde{A}$. ☐

Lemma 2.15. *The $C^*$-subalgebra $C^*(S_{(A,\gamma_1\gamma_2)};(A,\gamma_1\gamma_2)\in E_{\tilde{A}})$ of $\mathcal{O}_Z$ is nothing but $P_A\mathcal{O}_ZP_A$. Hence the Cuntz–Krieger algebra $\mathcal{O}_{\tilde{A}}$ is isomorphic to $\mathcal{O}_A$.*

*Proof.* Since $S_{(A,\gamma_1\gamma_2)} = P_AS_{\gamma_1}S_{\gamma_2}P_A$ for $(A,\gamma_1\gamma_2)\in E_{\tilde{A}}$, we have $C^*(S_{(A,\gamma_1\gamma_2)};(A,\gamma_1\gamma_2)\in E_{\tilde{A}})\subset P_A\mathcal{O}_ZP_A$. We will show the converse inclusion relation. Take an arbitrary fixed $X\in\mathcal{O}_Z$ with $P_AXP_A\neq 0$. Let $\mathcal{P}_Z$ be the dense $*$-subalgebra of $\mathcal{O}_Z$ algebraically generated by $S_\gamma,\gamma\in E_Z$. We may find $X_n\in\mathcal{P}_Z$ such that $\|X-X_n\|\to 0$. Since $\|P_AXP_A-P_AX_nP_A\|\le\|X-X_n\|\to 0$, it suffices to show that $P_AX_nP_A$ belongs to $C^*(S_{(A,\gamma_1\gamma_2)};(A,\gamma_1\gamma_2)\in E_{\tilde{A}})$. By [7, Lemma 2.2], any element of the subalgebra $\mathcal{P}_Z$ is a finite linear combination of elements of the form $S_\mu S_iS_i^*S_\nu^*$ for some admissible words $\mu=(\mu_1,\dots,\mu_m),\nu=(\nu_1,\dots,\nu_n)$ in $X_Z$. Assume that $P_AS_\mu S_iS_i^*S_\nu^*P_A\neq 0$. Since $P_AS_j=S_jP_B$, we have

$$P_AS_\mu = P_AS_{\mu_1}\cdots S_{\mu_m} = \begin{cases} S_{\mu_1}\cdots S_{\mu_m}P_A & \text{if } m \text{ is even,}\\ S_{\mu_1}\cdots S_{\mu_m}P_B & \text{if } m \text{ is odd.}\end{cases} \tag{2.10}$$

The assumption $P_AS_\mu S_iS_i^*S_\nu^*P_A\neq 0$ forces the numbers $m,n$ to be both even, or both odd.
Case 1: $m,n$ are both even.
We have

$$\begin{aligned}
&P_AS_\mu S_iS_i^*S_\nu^*P_A\\
=&P_AS_{\mu_1}S_{\mu_2}P_AS_{\mu_3}S_{\mu_4}P_A\cdots P_AS_{\mu_{m-1}}S_{\mu_m}P_AS_iS_i^*P_A\\
&\cdot S^*_{\nu_n}S^*_{\nu_{n-1}}P_A\cdots S^*_{\nu_4}S^*_{\nu_3}P_AS^*_{\nu_2}S^*_{\nu_1}P_A\\
=&S_{(A,\mu_1\mu_2)}S_{(A,\mu_3\mu_4)}\cdots S_{(A,\mu_{m-1}\mu_m)}P_AS_iS_i^*P_AS^*_{(A,\nu_{n-1}\nu_n)}\cdots S^*_{(A,\nu_3\nu_4)}S^*_{(A,\nu_1\nu_2)}.
\end{aligned}$$

Now we have

$$P_AS_iS_i^*P_A = \sum_{j\in E_Z}P_AS_iS_jS_j^*S_i^*P_A = \sum_{j\in E_Z}S_{(A,ij)}S^*_{(A,ij)}$$

so that $P_AS_\mu S_iS_i^*S_\nu^*P_A$ is a finite linear combination of products of the elements $S_{(A,\gamma_1\gamma_2)},S^*_{(A,\gamma_1\gamma_2)}$ for $(A,\gamma_1\gamma_2)\in E_{\tilde{A}}$ and hence it belongs to $C^*(S_{(A,\gamma_1\gamma_2)};(A,\gamma_1\gamma_2)\in E_{\tilde{A}})$.
Case 2: $m,n$ are both odd.

Similarly to Case 1, we have

$$\begin{aligned}&P_A S_\mu S_i S_i^* S_\nu^* P_A\\ =&S_{(A,\mu_1\mu_2)}\cdots S_{(A,\mu_{m-2}\mu_{m-1})}S_{(A,\mu_m i)}S^*_{(A,\nu_n i)}S^*_{(A,\nu_{n-2}\nu_{n-1})}\cdots S^*_{(A,\nu_1\nu_2)}\end{aligned}$$

so that $P_A S_\mu S_i S_i^* S_\nu^* P_A$ belongs to $C^*(S_{(A,\gamma_1\gamma_2)}; (A,\gamma_1\gamma_2)\in E_{\tilde{A}})$. ☐

PROPOSITION 2.16. *The Cuntz–Krieger triplet $(\mathcal{O}_{\tilde{A}}, \mathcal{D}_{\tilde{A}}, \rho^{\tilde{A}})$ for the matrix $\tilde{A}$ is isomorphic to $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$.*

*Proof.* By Lemma 2.14 and Lemma 2.15, we know that

$$\mathcal{O}_{\tilde{A}} = C^*(S_{(A,\gamma_1\gamma_2)}; (A,\gamma_1\gamma_2)\in E_{\tilde{A}}) = P_A\mathcal{O}_Z P_A = \mathcal{O}_A. \tag{2.11}$$

Under the identification between $C^*(S_{(A,\gamma_1\gamma_2)}; (A,\gamma_1\gamma_2)\in E_{\tilde{A}})$ and $P_A\mathcal{O}_Z P_A$ in Lemma 2.15, the $C^*$-subalgebra

$$\begin{aligned}C^*(S_{(A,\gamma_1\gamma_2)}\cdots S_{(A,\gamma_{n-1}\gamma_n)}S^*_{(A,\gamma_{n-1}\gamma_n)}\cdots S^*_{(A,\gamma_1\gamma_2)};\\ (A,\gamma_1\gamma_2),\dots,(A,\gamma_{n-1}\gamma_n)\in E_{\tilde{A}})\end{aligned}$$

of $C^*(S_{(A,\gamma_1\gamma_2)}; (A,\gamma_1\gamma_2)\in E_{\tilde{A}})$ generated by the projections

$$S_{(A,\gamma_1\gamma_2)}\cdots S_{(A,\gamma_{n-1}\gamma_n)}S^*_{(A,\gamma_{n-1}\gamma_n)}\cdots S^*_{(A,\gamma_1\gamma_2)}$$

for $(A,\gamma_1\gamma_2),\dots,(A,\gamma_{n-1}\gamma_n)\in E_{\tilde{A}}$ is naturally identified with the $C^*$-subalgebra $P_A\mathcal{D}_Z P_A$ of $\mathcal{D}_Z$, so that $\mathcal{D}_{\tilde{A}} = \mathcal{D}_A$. By regarding the generating partial isometry $S_{(A,\gamma_1\gamma_2)}$ for $(A,\gamma_1\gamma_2)\in E_{\tilde{A}}$ as an element of $P_A\mathcal{O}_Z P_A = \mathcal{O}_A$, we have

$$\begin{aligned}\rho^{\tilde{A}}_{2t}(S_{(A,\gamma_1\gamma_2)}) =& e^{2\pi\sqrt{-1}2t}S_{(A,\gamma_1\gamma_2)}\\ =&P_A e^{2\pi\sqrt{-1}t}S_{\gamma_1}e^{2\pi\sqrt{-1}t}S_{\gamma_2}\\ =&P_A\rho^Z_t(S_{\gamma_1})\rho^Z_t(S_{\gamma_2})\\ =&\rho^Z_t(P_A S_{\gamma_1}S_{\gamma_2}).\end{aligned}$$

Since $P_A S_{\gamma_1}S_{\gamma_2}\in P_A\mathcal{O}_Z P_A = \mathcal{O}_A$ and $\rho^Z_t|_{P_A\mathcal{O}_Z P_A} = \rho^A_{2t}$ on $\mathcal{O}_A$, we have

$$\rho^Z_t(P_A S_{\gamma_1}S_{\gamma_2}) = \rho^A_{2t}(P_A S_{\gamma_1}S_{\gamma_2}) = \rho^A_{2t}(S_{(A,\gamma_1\gamma_2)})$$

so that $\rho^{\tilde{A}}_{2t} = \rho^A_{2t}$ for all $t\in\mathbb{T}$ and hence $\rho^{\tilde{A}} = \rho^A$. ☐

We thus have

PROPOSITION 2.17. *Suppose that the Cuntz–Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent in 1-step. Then the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate.*

*Proof.* Assume that the Cuntz–Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent in 1-step. By Proposition 2.9, the matrices $\tilde{A}, \tilde{B}$ are elementary equivalent so that their two-sided topological Markov shifts $(\bar{X}_{\tilde{A}}, \bar{\sigma}_{\tilde{A}})$ and $(\bar{X}_{\tilde{B}}, \bar{\sigma}_{\tilde{B}})$ are topologically conjugate. Proposition 2.16 with [15, Corollary 3.5] ensures that the ons-sided topological Markov shifts $(X_{\tilde{A}}, \sigma_{\tilde{A}})$ and $(X_A, \sigma_A)$ are eventually conjugate and hence strongly continuous orbit equivalent in the sense of [15]. Since the latter property yields topological conjugacy of their two-sided topological Markov shifts, the two-sided topological Markov shifts $(\bar{X}_{\tilde{A}}, \bar{\sigma}_{\tilde{A}})$ and $(\bar{X}_A, \bar{\sigma}_A)$ are topologically conjugate. Similarly we know that the two-sided topological Markov shifts $(\bar{X}_{\tilde{B}}, \bar{\sigma}_{\tilde{B}})$ and $(\bar{X}_A, \bar{\sigma}_B)$ are topologically conjugate. Therefore we get the assertion. □

Now we reach one of the main results of the paper.

THEOREM 2.18. *Let $A, B$ be irreducible non-permutation matrices. The Cuntz–Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent if and only if their two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate.*

*Proof.* The if part comes from Proposition 2.2. The only if part follows from Proposition 2.17. □

By the Williams's fundamental theorem on topological Markov shifts which states that two irreducible matrices $A$ and $B$ are strong shift equivalent if and only if their two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate ([25]), we obtain the following corollary.

COROLLARY 2.19. *Let $A, B$ be irreducible non-permutation matrices. The Cuntz–Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent if and only if the matrices $A$ and $B$ are strong shift equivalent.*

## 3 STRONG SHIFT EQUIVALENCE AND CIRCLE ACTIONS ON $\mathcal{O}_A$

It is well-known that two unital $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$ are strong Morita equivalent if and only if their stabilizations $\mathcal{A} \otimes \mathcal{K}$ and $\mathcal{B} \otimes \mathcal{K}$ are isomorphic by Brown–Green–Rieffel Theorem [3, Theorem 1.2] (cf. [2], [3], [4], [20]). We will next study relationships between stabilized Cuntz–Krieger algebras with their gauge actions and strong shift equivalence matrices. We will investigate stabilizations of generalized gauge actions from a view point of flow equivalence.
Recall that for a function $f \in C(X_A, \mathbb{Z})$ and $t \in \mathbb{T}$, an automorphism $\rho_t^{A,f} \in \mathrm{Aut}(\mathcal{O}_A)$ is defined by $\rho_t^{A,f}(S_i) = U_t(f)S_i, i = 1, \dots, N, t \in \mathbb{T}$ for the unitary $U_t(f) = \exp(2\pi\sqrt{-1}tf) \in \mathcal{D}_A$ as in (1.5). It is easy to see that the automorphisms $\rho_t^{A,f}, t \in \mathbb{T}$ yield an action of $\mathbb{T}$ to $\mathcal{O}_A$ such that $\rho_t^{A,f}(a) = a$ for all $a \in \mathcal{D}_A$. For $f \in C(X_A, \mathbb{Z})$ and $n \in \mathbb{Z}_+$, let us denote by $f^n$ the function $f^n(x) = \sum_{i=0}^{n-1} f(\sigma_A^i(x)), x \in X_A$. We know that the identity

$$\rho_t^{A,f}(S_\mu) = U_t(f^n)S_\mu \tag{3.1}$$

for $f \in C(X_A, \mathbb{Z}), \mu = (\mu_1, \dots, \mu_n) \in B_n(X_A), t \in \mathbb{T}$ holds (cf. [15, Lemma 3.1]).

For a $C^*$-algebra $\mathcal{A}$ without unit, let $M(\mathcal{A})$ stand for its multiplier $C^*$-algebra defined by

$$M(\mathcal{A}) = \{a \in \mathcal{A}^{**} \mid a\mathcal{A} \subset \mathcal{A},\, \mathcal{A}a \subset \mathcal{A}\}$$

where $\mathcal{A}^{**}$ denotes the second dual $(\mathcal{A}^*)^*$ of the $C^*$-algebra $\mathcal{A}$. An action $\alpha$ of $\mathbb{T}$ to $\mathcal{A}$ extends to $M(\mathcal{A})$ and is still denoted by $\alpha$. For an action $\alpha$ of $\mathbb{T}$ to $\mathcal{A}$, a unitary one-cocycle $u_t, t \in \mathbb{T}$ relative to $\alpha$ is a strongly continuous map $t \in \mathbb{T} \to u_t \in \mathcal{U}(M(\mathcal{A}))$ to the unitary group $\mathcal{U}(M(\mathcal{A}))$ satisfying $u_{t+s} = u_s \alpha_s(u_t), s, t \in \mathbb{T}$. The following proposition has been proved in [15].

PROPOSITION 3.1 ([15, Proposition 4.3]). *Let $A$ and $B$ be elementary equivalent matrices, and choose matrices $C$ and $D$ satisfying $A = CD$ and $B = DC$. Then there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ satisfying $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$ and a homomorphism $\varphi : C(X_A, \mathbb{Z}) \to C(X_B, \mathbb{Z})$ of ordered groups such that for each function $f \in C(X_A, \mathbb{Z})$ there exists a unitary one-cocycle $u_t^f \in \mathcal{U}(M(\mathcal{O}_A \otimes \mathcal{K}))$ relative to $\rho^{A,f} \otimes \mathrm{id}$ such that*

$$\Phi \circ \mathrm{Ad}(u_t^f) \circ (\rho_t^{A,f} \otimes \mathrm{id}) = (\rho_t^{B,\varphi(f)} \otimes \mathrm{id}) \circ \Phi \quad \textit{for } t \in \mathbb{T}. \tag{3.2}$$

In this section, we will first review the proof in [15] of the above proposition to investigate the K-theoretic behavior of the above isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$. The proof of the above proposition is based on the the proof of [11, Proposition 4.1], in which Morita equivalence of $C^*$-algebras has been used (cf. [2], [3], [4], [5], [8], [12], [16], [18], [24]).

Let $A$ and $B$ be elementary equivalent matrices, and choose matrices $C$ and $D$ satisfying $A = CD$ and $B = DC$. As in the previous section, the equality $A(i,j) = \sum_{k=1}^{N_B} C(i,k)D(k,j)$ for $i, j = 1, \dots, N_A$ forces that the cardinal numbers of the two sets $\{a \in E_A \mid s(a) = v_i^A, t(a) = v_j^A\}$ and $\{(c,d) \in E_C \times E_D \mid s(c) = v_i^A,\, t(c) = s(d),\, t(d) = v_j^A\}$ coincide. Hence we may take a bijection from $E_A$ to the above subset of $E_C \times E_D$. We fix it and write it as $\varphi_{A,CD}$. By the other equality $B = DC$, one may take a bijection written $\varphi_{B,DC}$ from $E_B$ to a subset of $E_D \times E_C$ similarly. We set $Z = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$ as a block matrix, and use the same notation as in the previous sections.

For an arbitrary fixed function $f \in C(X_A, \mathbb{Z})$, we may regard it as an element of $\mathcal{D}_A$ and hence of $\mathcal{D}_Z$ by identifying it with $f \oplus 0$ in $\mathcal{D}_A \oplus \mathcal{D}_B = \mathcal{D}_Z$. As

$$\exp(2\pi\sqrt{-1}t(f \oplus 0)) = \exp(2\pi\sqrt{-1}tf) \oplus P_D \in \mathcal{U}(\mathcal{D}_Z),$$

the automorphism $\rho_t^{Z,f\oplus 0}$ of $\mathcal{O}_Z$ for $t \in \mathbb{T}$ defined by (1.5) satisfies

$$\rho_t^{Z,f\oplus 0}(S_c) = \exp(2\pi\sqrt{-1}tf)S_c \quad \text{for } c \in E_C, \tag{3.3}$$

$$\rho_t^{Z,f\oplus 0}(S_d) = S_d \quad \text{for } d \in E_D. \tag{3.4}$$

Fix $c \in E_C$ and $d \in E_D$ such that $t(c) = s(d)$, and let $a \in E_A$ be the unique edge satisfying $\varphi_{A,CD}(a) = cd$. Let $b \in E_B$ be the unique edge in $E_B$ satisfying $\varphi_{B,DC}(b) = dc$, in a similar way. The equalities (3.3), (3.4) imply

$$\rho_t^{Z,f\oplus 0}(S_cS_d) = \exp(2\pi\sqrt{-1}tf)S_cS_d = \rho_t^{A,f}(S_a),$$
$$\rho_t^{Z,f\oplus 0}(S_dS_c) = S_d\exp(2\pi\sqrt{-1}tf)S_c = S_d\exp(2\pi\sqrt{-1}tf)S_d^*S_b.$$

We set $\varphi(f) = \sum_{d\in E_D} S_d f S_d^* \in \mathcal{D}_Z$. As $P_D\varphi(f)P_D = \varphi(f)$, we see that $\varphi(f) \in \mathcal{D}_B$ and hence $\varphi(f) \in C(X_B, \mathbb{Z})$ satisfies

$$\sum_{d\in E_D} S_d\exp(2\pi\sqrt{-1}tf)S_d^* = \exp(2\pi\sqrt{-1}t\varphi(f)) \in \mathcal{U}(\mathcal{D}_B).$$

We similarly set $\psi(g) = \sum_{c\in E_C} S_c g S_c^* \in C(X_A, \mathbb{Z})$ for $g \in C(X_B, \mathbb{Z})$. We thus see the following lemma.

Lemma 3.2 ([15, Lemma 4.1]). *For $f \in C(X_A, \mathbb{Z}), g \in C(X_B, \mathbb{Z})$ and $t \in \mathbb{T}$, we have*

$$\rho_t^{Z,f\oplus 0}(S_cS_d) = \rho_t^{A,f}(S_a), \qquad \rho_t^{Z,f\oplus 0}(S_dS_c) = \rho_t^{B,\varphi(f)}(S_b), \tag{3.5}$$
$$\rho_t^{Z,0\oplus g}(S_dS_c) = \rho_t^{B,g}(S_b), \qquad \rho_t^{Z,0\oplus g}(S_cS_d) = \rho_t^{A,\psi(g)}(S_a) \tag{3.6}$$

*where $a \in E_A, b \in E_B$ and $c \in E_C, d \in E_D$ satisfy $\varphi_{A,CD}(a) = cd$ and $\varphi_{B,DC}(b) = dc$, respectively.*

We note that the homomorphisms $\varphi : C(X_A, \mathbb{Z}) \to C(X_B, \mathbb{Z})$ and $\psi : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z})$ satisfy the equalities

$$(\psi \circ \varphi)(f) = f \circ \sigma_A, \qquad (\varphi \circ \psi)(g) = g \circ \sigma_B$$

for $f \in C(X_A, \mathbb{Z})$ and $g \in C(X_B, \mathbb{Z})$ ([15, Lemma 4.2]).
By [11, Proposition 4.1], one may find partial isometries $v_A, v_B \in M(\mathcal{O}_Z \otimes \mathcal{K})$ such that

$$v_A^*v_A = v_B^*v_B = 1 \otimes 1, \qquad v_Av_A^* = P_C \otimes 1, \qquad v_Bv_B^* = P_D \otimes 1. \tag{3.7}$$

Since

$$\mathrm{Ad}(v_A^*) : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_Z \otimes \mathcal{K} \quad \text{and} \quad \mathrm{Ad}(v_B^*) : \mathcal{O}_B \otimes \mathcal{K} \to \mathcal{O}_Z \otimes \mathcal{K} \tag{3.8}$$

are isomorphisms satisfying

$$\mathrm{Ad}(v_A^*)(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_Z \otimes \mathcal{C} \quad \text{and} \quad \mathrm{Ad}(v_B^*)(\mathcal{D}_B \otimes \mathcal{C}) = \mathcal{D}_Z \otimes \mathcal{C}.$$

By putting

$$w = v_Bv_A^* \in M(\mathcal{O}_Z \otimes \mathcal{K}), \tag{3.9}$$
$$\Phi = \mathrm{Ad}(w) : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}, \tag{3.10}$$
$$u_t^{A,f} = w^*(\rho_t^{Z,f\oplus 0} \otimes \mathrm{id})(w) \quad \text{for } f \in C(X_A, \mathbb{Z}), \tag{3.11}$$
$$u_t^{B,g} = w(\rho_t^{Z,0\oplus g} \otimes \mathrm{id})(w^*) \quad \text{for } g \in C(X_B, \mathbb{Z}), \tag{3.12}$$

they satisfy $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$ and the equalities

$$\Phi \circ \mathrm{Ad}(u_t^{A,f}) \circ (\rho_t^{A,f} \otimes \mathrm{id}) = (\rho_t^{B,\varphi(f)} \otimes \mathrm{id}) \circ \Phi \quad \text{for } f \in C(X_A, \mathbb{Z}),$$
$$\Phi \circ (\rho_t^{A,\psi(g)} \otimes \mathrm{id}) = \mathrm{Ad}(u_t^{B,g}) \circ (\rho_t^{B,g} \otimes \mathrm{id}) \circ \Phi \quad \text{for } g \in C(X_B, \mathbb{Z}).$$

The above discussion is a sketch of the proof of Proposition 3.1 given in [15]. In what follows, we will reconstruct partial isometries $v_A, v_B$ satisfying (3.7) to investigate the K-theoretic behavior of the map $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ in the following section. The idea of the reconstruction is due to the proof of [2, Lemma 2.5] (cf. [11, Proposition 4.1]).

We are assuming that $A = CD, B = DC$. Keep the notation as in the preceding section. Put $E_C = \{c_1, \dots, c_{N_C}\}$ and $E_D = \{d_1, \dots, d_{N_D}\}$ for the matrices $C$ and $D$ respectively. For $k = 1, \dots, N_D$, take $c(k) \in E_C$ such that $c(k)d_k \in B_2(X_Z)$ so that we have

$$S_{c(k)}^* S_{c(k)} \geq S_{d_k} S_{d_k}^*.$$

Similarly for $l = 1, \dots, N_C$, take $d(l) \in E_D$ such that $d(l)c_l \in B_2(X_Z)$ so that we have

$$S_{d(l)}^* S_{d(l)} \geq S_{c_l} S_{c_l}^*.$$

Put

$$U_0 = P_C, \qquad U_k = S_{c(k)} S_{d_k} S_{d_k}^* \quad \text{for } k = 1, \dots, N_D, \tag{3.13}$$
$$T_0 = P_D, \qquad T_l = S_{d(l)} S_{c_l} S_{c_l}^* \quad \text{for } l = 1, \dots, N_C. \tag{3.14}$$

We then have

$$\sum_{k=1}^{N_D} U_k^* U_k = \sum_{k=1}^{N_D} S_{d_k} S_{d_k}^* S_{c(k)}^* S_{c(k)} S_{d_k} S_{d_k}^* = \sum_{k=1}^{N_D} S_{d_k} S_{d_k}^* = P_D,$$
$$\sum_{k=1}^{N_C} T_l^* T_l = \sum_{l=1}^{N_C} S_{c_l} S_{c_l}^* S_{d(l)}^* S_{d(l)} S_{c_l} S_{c_l}^* = \sum_{l=1}^{N_C} S_{c_l} S_{c_l}^* = P_C.$$

We decompose the set $\mathbb{N}$ of natural numbers into disjoint infinite subsets $\mathbb{N} = \cup_{j=1}^{\infty} \mathbb{N}_j$, and decompose $\mathbb{N}_j$ for each $j$ once again into disjoint infinite sets $\mathbb{N}_j = \cup_{k=0}^{\infty} \mathbb{N}_{j_k}$. Let $\{e_{i,j}\}_{i,j \in \mathbb{N}}$ be a set of matrix units which generate the algebra $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$. Put the projections $f_j = \sum_{i \in \mathbb{N}_j} e_{i,i}$ and $f_{j_k} = \sum_{i \in \mathbb{N}_{j_k}} e_{i,i}$, both of which converge in the strong operator topology on $\ell^2(\mathbb{N})$. Take a partial isometry $s_{j_k,j}$ such that $s_{j_k,j}^* s_{j_k,j} = f_j, s_{j_k,j} s_{j_k,j}^* = f_{j_k}$ and put $s_{j,j_k} = s_{j_k,j}^*$. We set for $n = 1, 2, \dots,$

$$u_n = \sum_{k=1}^{N_D} U_k \otimes s_{n_k,n}, \qquad w_n = P_C \otimes s_{n_0,n} + u_n,$$
$$t_n = \sum_{l=1}^{N_C} T_l \otimes s_{n_l,n}, \qquad z_n = P_D \otimes s_{n_0,n} + t_n.$$

Lemma 3.3. *Keep the above notations.*

(i) $w_n^* w_n = 1 \otimes f_n$ *and* $w_n w_n^* \leq P_C \otimes f_n$.

(ii) $z_n^* z_n = 1 \otimes f_n$ *and* $z_n z_n^* \leq P_D \otimes f_n$.

*Proof.* (i) Since $u_n^* u_n = P_D \otimes f_n$, we have

$$w_n^* w_n = P_C \otimes f_n + u_n^* u_n = P_C \otimes f_n + P_D \otimes f_n = 1 \otimes f_n.$$

On the other hand, we know that $u_n(P_C \otimes s_{n,n_0}) = (P_C \otimes s_{n,n_0})u_n^* = 0$ so that we have

$$w_n w_n^* = P_C \otimes f_{n_0} + u_n u_n^* = P_C \otimes f_{n_0} + \sum_{k=1}^{N_D} S_{c(k)} S_{d_k} S_{d_k}^* S_{c(k)}^* \otimes f_{n_k}.$$

As $f_{n_0}, f_{n_k} \leq f_n$, we have

$$w_n w_n^* \leq P_C \otimes f_n.$$

(ii) is shown similarly. ☐

We will reconstruct and study the isometry $v_A$ in (3.7). Let $f_{n,m}$ be a partial isometry satisfying $f_{n,m}^* f_{n,m} = f_m$, $f_{n,m} f_{n,m}^* = f_n$. We put

$$\begin{aligned} v_1 &= w_1 = P_C \otimes s_{1_0,1} + u_1, \\ v_{2n} &= (P_C \otimes f_n - v_{2n-1} v_{2n-1}^*)(P_C \otimes f_{n,n+1}) \quad \text{for } 1 \leq n \in \mathbb{N}, \\ v_{2n-1} &= w_n(1 \otimes f_n - v_{2n-2}^* v_{2n-2}) \quad \text{for } 2 \leq n \in \mathbb{N}. \end{aligned}$$

Lemma 3.4. *Keep the above notation.*

(i) $v_{2n-2}^* v_{2n-2} + v_{2n-1}^* v_{2n-1} = 1 \otimes f_n$.

(ii) $v_{2n-1} v_{2n-1}^* + v_{2n} v_{2n}^* = P_C \otimes f_n$.

*Proof.* (i) As $w_n^* w_n = 1 \otimes f_n$, we have

$$\begin{aligned} & v_{2n-2}^* v_{2n-2} + v_{2n-1}^* v_{2n-1} \\ =& v_{2n-2}^* v_{2n-2} + (1 \otimes f_n - v_{2n-2}^* v_{2n-2}) w_n^* w_n (1 \otimes f_n - v_{2n-2}^* v_{2n-2}) \\ =& v_{2n-2}^* v_{2n-2} + 1 \otimes f_n - v_{2n-2}^* v_{2n-2} \\ =& 1 \otimes f_n. \end{aligned}$$

(ii) We have

$$\begin{aligned} & v_{2n-1} v_{2n-1}^* + v_{2n} v_{2n}^* \\ =& v_{2n-1} v_{2n-1}^* + (P_C \otimes f_n - v_{2n-1} v_{2n-1}^*)(P_C \otimes f_n)(P_C \otimes f_n - v_{2n-1} v_{2n-1}^*) \\ =& v_{2n-1} v_{2n-1}^* + P_C \otimes f_n - v_{2n-1} v_{2n-1}^* \\ =& P_C \otimes f_n. \end{aligned}$$

☐

By the above lemma, one may see that the summations $\sum_{n=1}^{\infty} v_{2n-2}$ and $\sum_{n=1}^{\infty} v_{2n-1}$ converge in $M(\mathcal{O}_Z \otimes \mathcal{K})$ to certain partial isometries written $v_{ev}$ and $v_{od}$ respectively in the strict topology of the multiplier algebra of $\mathcal{O}_Z \otimes \mathcal{K}$. Similarly we obtain a partial isometry $v_A = \sum_{n=1}^{\infty} v_n$ in $M(\mathcal{O}_Z \otimes \mathcal{K})$ in the strict topology. Therefore we have the next lemma.

Lemma 3.5. *The partial isometries $v_{ev}, v_{od}$ and $v_A$ defined above satisfy the following relations:*

(i) $v_A = v_{od} + v_{ev}$.

(ii) $v_{od}^* v_{od} + v_{ev}^* v_{ev} = 1 \otimes 1$.

(iii) $v_{od} v_{od}^* + v_{ev} v_{ev}^* = P_C \otimes 1$.

(iv) $v_A^* v_A = 1 \otimes 1$ *and* $v_A v_A^* = P_C \otimes 1$.

We put

$$q_{od}^C = \sum_{n=1}^{\infty} v_{2n-1}(P_C \otimes 1) v_{2n-1}^*, \qquad q_{od}^D = \sum_{n=1}^{\infty} v_{2n-1}(P_D \otimes 1) v_{2n-1}^*$$

so that

$$q_{od}^C + q_{od}^D = v_{od} v_{od}^* \quad \text{and hence} \quad q_{od}^C + q_{od}^D + v_{ev} v_{ev}^* = P_C \otimes 1.$$

We will show the following lemma.

Lemma 3.6. $v_A(\rho_t^{Z,f\oplus 0} \otimes \mathrm{id})(v_A^*) = q_{od}^C + (U_t(-f) \otimes 1) q_{od}^D + v_{ev} v_{ev}^*$.

*Proof.* We notice that $\rho_t^{Z,f\oplus 0}(S_c) = U_t(f) S_c$ for $c \in E_C$ and $\rho_t^{Z,f\oplus 0}(S_d) = S_d$ for $d \in E_D$. As $v_{2n-1} v_{2n-1}^* \in D_Z \otimes \mathcal{C}$, we have $(\rho_t^{Z,f\oplus 0} \otimes \mathrm{id})(v_{2n-1} v_{2n-1}^*) = v_{2n-1} v_{2n-1}^*$ and hence $(\rho_t^{Z,f\oplus 0} \otimes \mathrm{id})(v_{ev}) = v_{ev}$. We then have

$$\begin{aligned} v_A(\rho_t^{Z,f\oplus 0} \otimes \mathrm{id})(v_A^*) &= v_{od}(\rho_t^{Z,f\oplus 0} \otimes \mathrm{id})(v_{od}^*) + v_{ev}(\rho_t^{Z,f\oplus 0} \otimes \mathrm{id})(v_{ev}^*) \\ &= \sum_{n=1}^{\infty} v_{2n-1}(\rho_t^{Z,f\oplus 0} \otimes \mathrm{id})(v_{2n-1}^*) + v_{ev} v_{ev}^*. \end{aligned}$$

Since

$$v_1(P_C \otimes 1) = P_C \otimes s_{1_0,1} \quad \text{and} \quad v_1(P_D \otimes 1) = \sum_{k=1}^{N_D} S_{c(k)} S_{d_k} S_{d_k}^* \otimes s_{1_k,1},$$

we have

$$\begin{aligned}(\rho_t^{Z,f\oplus 0}\otimes \mathrm{id})(v_1^*) &= (P_C\otimes 1)v_1^* + (\rho_t^{Z,f\oplus 0}\otimes \mathrm{id})((P_D\otimes 1)v_1^*)\\ &= (P_C\otimes 1)v_1^* + \sum_{k=1}^{N_D} S_{d_k}S_{d_k}^*\rho_t^{Z,f\oplus 0}(S_{c(k)}^*)\otimes s_{1_k,1}^*\\ &= (P_C\otimes 1)v_1^* + \sum_{k=1}^{N_D} S_{d_k}S_{d_k}^*S_{c(k)}^*U_t(-f)\otimes s_{1_k,1}^*\\ &= (P_C\otimes 1)v_1^* + (P_D\otimes 1)v_1^*(U_t(-f)\otimes 1),\end{aligned}$$

so that

$$\begin{aligned}v_1(\rho_t^{Z,f\oplus 0}\otimes \mathrm{id})(v_1^*) &= v_1(P_C\otimes 1)v_1^* + v_1(P_D\otimes 1)v_1^*(U_t(-f)\otimes 1)\\ &= v_1(P_C\otimes 1)v_1^* + (U_t(-f)\otimes 1)v_1(P_D\otimes 1)v_1^*.\end{aligned}$$

For $2\le n\in\mathbb{N}$, we have

$$\begin{aligned}v_{2n-1}(P_C\otimes 1) &= (P_C\otimes s_{n_0,n})(1\otimes f_n - v_{2n-2}^*v_{2n-2}),\\ v_{2n-1}(P_D\otimes 1) &= \sum_{k=1}^{N_D}(S_{c(k)}S_{d_k}S_{d_k}^*\otimes s_{n_k,n})(1\otimes f_n - v_{2n-2}^*v_{2n-2}),\end{aligned}$$

and hence

$$\begin{aligned}&(\rho_t^{Z,f\oplus 0}\otimes \mathrm{id})((P_D\otimes 1)v_{2n-1}^*)\\ =&(1\otimes f_n - v_{2n-2}^*v_{2n-2})\sum_{k=1}^{N_D} S_{d_k}S_{d_k}^*\rho_t^{Z,f\oplus 0}(S_{c(k)}^*)\otimes s_{n_k,n}^*\\ =&(1\otimes f_n - v_{2n-2}^*v_{2n-2})\sum_{k=1}^{N_D} S_{d_k}S_{d_k}^*S_{c(k)}^*U_t(-f)\otimes s_{n_k,n}^*\\ =&(P_D\otimes 1)v_{2n-1}^*(U_t(-f)\otimes 1)\end{aligned}$$

so that

$$\begin{aligned}&v_{2n-1}(\rho_t^{Z,f\oplus 0}\otimes \mathrm{id})(v_{2n-1}^*)\\ =&v_{2n-1}(P_C\otimes 1)v_{2n-1}^* + v_{2n-1}(P_D\otimes 1)v_{2n-1}^*(U_t(-f)\otimes 1)\\ =&v_{2n-1}(P_C\otimes 1)v_{2n-1}^* + (U_t(-f)\otimes 1)v_{2n-1}(P_D\otimes 1)v_{2n-1}^*.\end{aligned}$$

Therefore we have

$$v_{od}(\rho_t^{Z,f\oplus 0}\otimes \mathrm{id})(v_{od}^*) = q_{od}^C + (U_t(-f)\otimes 1)q_{od}^D$$

and hence

$$v_A(\rho_t^{Z,f\oplus 0}\otimes \mathrm{id})(v_A^*) = q_{od}^C + (U_t(-f)\otimes 1)q_{od}^D + v_{ev}v_{ev}^*.$$

□

By using $t_n, z_n$ instead of $u_n, w_n$ respectively, we similarly obtain a partial isometry $v_B$ in $M(\mathcal{O}_Z \otimes \mathcal{K})$ in the strict topology. We then have the following lemmas.

Lemma 3.7.

(i) *The partial isometry* $v_A(\rho_t^{Z,f\oplus 0} \otimes \mathrm{id})(v_A^*)$ *for* $f \in C(X_A, \mathbb{Z}), t \in \mathbb{T}$ *belongs to* $M(\mathcal{D}_A \otimes \mathcal{C})$ *and satisfies*

$$v_A(\rho_t^{Z,(f_1+f_2)\oplus 0} \otimes \mathrm{id})(v_A^*) = v_A(\rho_t^{Z,f_1\oplus 0} \otimes \mathrm{id})(v_A^*) v_A(\rho_t^{Z,f_2\oplus 0} \otimes \mathrm{id})(v_A^*)$$

*for* $f_1, f_2 \in C(X_A, \mathbb{Z}), t \in \mathbb{T}$.

(ii) *The partial isometry* $v_B(\rho_t^{Z,0\oplus g} \otimes \mathrm{id})(v_B^*)$ *for* $g \in C(X_B, \mathbb{Z}), t \in \mathbb{T}$ *belongs to* $M(\mathcal{D}_B \otimes \mathcal{C})$ *and satisfies*

$$v_B(\rho_t^{Z,0\oplus(g_1+g_2)} \otimes \mathrm{id})(v_B^*) = v_B(\rho_t^{Z,0\oplus g_1} \otimes \mathrm{id})(v_B^*) v_B(\rho_t^{Z,0\oplus g_2} \otimes \mathrm{id})(v_B^*)$$

*for* $g_1, g_2 \in C(X_B, \mathbb{Z}), t \in \mathbb{T}$.

*Proof.* (i) Since the projections $q_{od}^C, q_{od}^D, v_{ev}v_{ev}^*$ all belong to the multiplier algebra $M(\mathcal{D}_A \otimes \mathcal{C})$ of $\mathcal{D}_A \otimes \mathcal{C}$, the preceding lemma ensures that the partial isometry $v_A(\rho^{Z,f\oplus 0} \otimes \mathrm{id})(v_A^*)$ belongs to $M(\mathcal{D}_A \otimes \mathcal{C})$. As $U_t(f_1+f_2) = U_t(f_1)U_t(f_2)$, the desired equality follows.
(ii) is shown similarly. ∎

Lemma 3.8.

(i) $(\rho_t^{Z,0\oplus g} \otimes \mathrm{id})(v_A) = v_A$ *for* $g \in C(X_B, \mathbb{Z}), t \in \mathbb{T}$.

(ii) $(\rho_t^{Z,f\oplus 0} \otimes \mathrm{id})(v_B) = v_B$ *for* $f \in C(X_A, \mathbb{Z}), t \in \mathbb{T}$.

*Proof.* (i) Since $\rho_t^{Z,0\oplus g}(S_c) = S_c, \rho_t^{Z,0\oplus g}(S_d) = e^{2\pi\sqrt{-1}tg} S_d$, we have

$$\begin{aligned}\rho_t^{Z,0\oplus g}(U_k) =& \rho_t^{Z,0\oplus g}(S_{c(k)} S_{d_k} S_{d_k}^*) \\ =& S_{c(k)} e^{2\pi\sqrt{-1}tg} S_{d_k} S_{d_k}^* e^{-2\pi\sqrt{-1}tg} \\ =& S_{c(k)} S_{d_k} S_{d_k}^* = U_k.\end{aligned}$$

Hence $(\rho_t^{Z,0\oplus g} \otimes \mathrm{id})(u_n) = u_n$ so that $(\rho_t^{Z,0\oplus g} \otimes \mathrm{id})(w_n) = w_n$. We then have

$$(\rho_t^{Z,0\oplus g} \otimes \mathrm{id})(v_1) = (\rho_t^{Z,0\oplus g} \otimes \mathrm{id})(P_C \otimes s_{1_0,1} + u_1) = P_C \otimes s_{1_0,1} + u_1 = v_1.$$

Since $v_{2n-1}v_{2n-1}^*, v_{2n-2}^* v_{2n-2} \in \mathcal{D}_Z \otimes \mathcal{C}$ and the restriction of $\rho_t^{Z,0\oplus g} \otimes \mathrm{id}$ to $\mathcal{D}_Z \otimes \mathcal{C}$ is the identity, we easily know that

$$(\rho_t^{Z,0\oplus g} \otimes \mathrm{id})(v_{2n}) = v_{2n}, \qquad (\rho_t^{Z,0\oplus g} \otimes \mathrm{id})(v_{2n-1}) = v_{2n-1} \quad \text{for } n \in \mathbb{N}.$$

We thus have $(\rho_t^{Z,0\oplus g} \otimes \mathrm{id})(v_n) = v_n$ for all $n \in \mathbb{N}$ and hence $(\rho_t^{Z,0\oplus g} \otimes \mathrm{id})(v_A) = v_A$.
(ii) is shown similarly. ∎

We put

$$\begin{aligned} w &= v_B v_A^* \in M(\mathcal{O}_Z \otimes \mathcal{K}), \\ u_t^{A,f} &= w^*(\rho_t^{Z,f\oplus 0} \otimes \mathrm{id})(w) \quad \text{for } f \in C(X_A, \mathbb{Z}), \\ u_t^{B,g} &= w(\rho_t^{Z,0\oplus g} \otimes \mathrm{id})(w^*) \quad \text{for } g \in C(X_B, \mathbb{Z}). \end{aligned}$$

By Lemma 3.8, we have

$$u_t^{A,f} = v_A v_B^*(\rho_t^{Z,f\oplus 0} \otimes \mathrm{id})(v_B)(\rho_t^{Z,f\oplus 0} \otimes \mathrm{id})(v_A^*) = v_A(\rho_t^{Z,f\oplus 0} \otimes \mathrm{id})(v_A^*) \tag{3.15}$$

and similarly $u_t^{B,g} = v_B(\rho_t^{Z,0\oplus g} \otimes \mathrm{id})(v_B^*)$.

Lemma 3.9.

(i) *For each $f \in C(X_A, \mathbb{Z})$, the unitaries $u_t^{A,f}, t \in \mathbb{T}$ give rise to a unitary representation of $\mathbb{T}$ in $M(\mathcal{D}_A \otimes \mathcal{C})$ which satisfies $u_t^{A,f_1+f_2} = u_t^{A,f_1} u_t^{A,f_2}$ for $f_1, f_2 \in C(X_A, \mathbb{Z})$.*

(ii) *For each $g \in C(X_B, \mathbb{Z})$, the unitaries $u_t^{B,g}, t \in \mathbb{T}$ give rise to a unitary representation of $\mathbb{T}$ in $M(\mathcal{D}_B \otimes \mathcal{C})$ which satisfies $u_t^{B,g_1+g_2} = u_t^{B,g_1} u_t^{B,g_2}$ for $g_1, g_2 \in C(X_B, \mathbb{Z})$.*

*Proof.* (i) By Lemma 3.6 and (3.15), we have

$$\begin{aligned} u_t^{A,f} u_s^{A,f} =& v_A(\rho_t^{Z,f\oplus 0} \otimes \mathrm{id})(v_A^*) v_A(\rho_s^{Z,f\oplus 0} \otimes \mathrm{id})(v_A^*) \\ =& (q_{od}^C + (U_t(-f) \otimes 1) q_{od}^D + v_{ev} v_{ev}^*)(q_{od}^C + (U_s(-f) \otimes 1) q_{od}^D + v_{ev} v_{ev}^*) \\ =& q_{od}^C + (U_{t+s}(-f) \otimes 1) q_{od}^D + v_{ev} v_{ev}^* = u_{t+s}^{A,f}. \end{aligned}$$

The equality $u_t^{A,f_1+f_2} = u_t^{A,f_1} u_t^{A,f_2}$ immediately follows from Lemma 3.7. (ii) is shown similarly. ☐

We thus have

Proposition 3.10. *Let $A, B$ be nonnegative irreducible and non-permutation matrices. Suppose that they are elementary equivalent, and choose matrices $C$ and $D$ satisfying $A = CD$, $B = DC$. Then there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ satisfying $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$, and unitary representations $t \in \mathbb{T} \to u_t^{A,f} \in M(\mathcal{D}_A \otimes \mathcal{C})$ for each $f \in C(X_A, \mathbb{Z})$ and $t \in \mathbb{T} \to u_t^{B,g} \in M(\mathcal{D}_B \otimes \mathcal{C})$ for each $g \in C(X_B, \mathbb{Z})$ such that*

$$\begin{aligned} \Phi \circ \mathrm{Ad}(u_t^{A,f}) \circ (\rho_t^{A,f} \otimes \mathrm{id}) &= (\rho_t^{B,\varphi(f)} \otimes \mathrm{id}) \circ \Phi \quad \textit{for } f \in C(X_A, \mathbb{Z}), \\ \Phi \circ (\rho_t^{A,\psi(g)} \otimes \mathrm{id}) &= \mathrm{Ad}(u_t^{B,g}) \circ (\rho_t^{B,g} \otimes \mathrm{id}) \circ \Phi \quad \textit{for } g \in C(X_B, \mathbb{Z}). \end{aligned}$$

*Proof.* As in the proof of [15, Proposition 4.3], the map $\Phi = \mathrm{Ad}(w)$ where $w = v_B v_A^*$ gives rise to an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ such that $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$ and

$$\Phi \circ \mathrm{Ad}(u_t^{A,f}) \circ (\rho_t^{A,f} \otimes \mathrm{id}) = (\rho_t^{B,\varphi(f)} \otimes \mathrm{id}) \circ \Phi.$$

The other equality is shown similarly. ☐

Since both homomorphisms $\varphi : C(X_A, \mathbb{Z}) \to C(X_B, \mathbb{Z})$ and $\psi : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z})$ satisfy $\varphi(1) = 1, \psi(1) = 1$, we have the following corollary.

COROLLARY 3.11 (cf. [7, Theorem 3.8], [6, Theorem 2.3]). *Let $A, B$ be irreducible non-permutation matrices. Suppose that the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate. Then there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ of $C^*$-algebras satisfying $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$, and unitary representations $t \in \mathbb{T} \to v_t^A \in M(\mathcal{D}_A \otimes \mathcal{C})$ and $t \in \mathbb{T} \to v_t^B \in M(\mathcal{D}_B \otimes \mathcal{C})$ such that*

$$\Phi \circ \mathrm{Ad}(v_t^A) \circ (\rho_t^A \otimes \mathrm{id}) = (\rho_t^B \otimes \mathrm{id}) \circ \Phi,$$
$$\Phi \circ (\rho_t^A \otimes \mathrm{id}) = \mathrm{Ad}(v_t^B) \circ (\rho_t^B \otimes \mathrm{id}) \circ \Phi$$

*where $\rho_t^A$ and $\rho_t^B$ are the gauge actions on $\mathcal{O}_A$ and $\mathcal{O}_B$, respectively.*

REMARK 3.12. We must emphasize that Cuntz–Krieger in [7, Theorem 3.8] and Cuntz in [6, Theorem 2.3] have shown that the stabilized Cuntz–Krieger triplet $(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}, \rho^A \otimes \mathrm{id})$ is invariant under topological conjugacy of the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$. Hence the above corollary is weaker than their result.

Before ending this section, we will introduce a notion of strong Morita equivalence in the stabilized Cuntz–Krieger triplets. The triplet $(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}, \rho^A \otimes \mathrm{id})$ is called the stabilized Cuntz–Krieger triplet. Two stabilized Cuntz–Krieger triplets $(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}, \rho^A \otimes \mathrm{id})$ and $(\mathcal{O}_B \otimes \mathcal{K}, \mathcal{D}_B \otimes \mathcal{C}, \rho^B \otimes \mathrm{id})$ are said to be *strong Morita equivalent in* 1*-step* if there exist a stabilized Cuntz–Krieger triplet $(\mathcal{O}_Z \otimes \mathcal{K}, \mathcal{D}_Z \otimes \mathcal{C}, \rho^Z \otimes \mathrm{id})$ and isomorphisms of $C^*$-algebras

$$\Phi_A : \mathcal{O}_Z \otimes \mathcal{K} \longrightarrow \mathcal{O}_A \otimes \mathcal{K}, \qquad \Phi_B : \mathcal{O}_Z \otimes \mathcal{K} \longrightarrow \mathcal{O}_B \otimes \mathcal{K}$$

satisfying

$$\Phi_A(\mathcal{D}_Z \otimes \mathcal{C}) = \mathcal{D}_A \otimes \mathcal{C}, \qquad \Phi_B(\mathcal{D}_Z \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C},$$
$$\begin{aligned}\rho_t^Z \otimes \mathrm{id} =& (\Phi_B^{-1} \circ \rho_t^B \otimes \mathrm{id} \circ \Phi_B) \circ (\Phi_A^{-1} \circ \rho_t^A \otimes \mathrm{id} \circ \Phi_A) \\ =& (\Phi_A^{-1} \circ \rho_t^A \otimes \mathrm{id} \circ \Phi_A) \circ (\Phi_B^{-1} \circ \rho_t^B \otimes \mathrm{id} \circ \Phi_B).\end{aligned}$$

If two stabilized Cuntz–Krieger triplets $(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}, \rho^A \otimes \mathrm{id})$ and $(\mathcal{O}_B \otimes \mathcal{K}, \mathcal{D}_B \otimes \mathcal{C}, \rho^B \otimes \mathrm{id})$ are connected by $n$-chains of strong Morita equivalences in 1-step, they are said to be strong Morita equivalent in $n$-step, or simply strong Morita equivalent.

PROPOSITION 3.13. *Let $A$ and $B$ be irreducible and not any permutation matrices. Suppose that $A, B$ are elementary equivalent. Then the stabilized Cuntz–Krieger triplets $(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}, \rho^A \otimes \mathrm{id})$ and $(\mathcal{O}_B \otimes \mathcal{K}, \mathcal{D}_B \otimes \mathcal{C}, \rho^B \otimes \mathrm{id})$ are strong Morita equivalent in* 1*-step.*

*Proof.* Let $Z = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$. Take isometries $v_A, v_B \in M(\mathcal{O}_Z \otimes \mathcal{K})$ satisfying (3.7). By Lemma 3.8, the following identities hold

$$(\rho_t^{Z,0\oplus 1} \otimes \mathrm{id})(v_A) = v_A, \qquad (\rho_t^{Z,1\oplus 0} \otimes \mathrm{id})(v_B) = v_B.$$

Define $\Phi_A = \mathrm{Ad}(v_A), \Phi_B = \mathrm{Ad}(v_B)$. As in (3.8), they give rise to isomorphisms

$$\Phi_A : \mathcal{O}_Z \otimes \mathcal{K} \longrightarrow \mathcal{O}_A \otimes \mathcal{K}, \qquad \Phi_B : \mathcal{O}_Z \otimes \mathcal{K} \longrightarrow \mathcal{O}_B \otimes \mathcal{K}$$

satisfying

$$\Phi_A(\mathcal{D}_Z \otimes \mathcal{C}) = \mathcal{D}_A \otimes \mathcal{C}, \qquad \Phi_B(\mathcal{D}_Z \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}.$$

Since we see

$$\begin{aligned} \rho_t^{Z,0\oplus 1}(S_c) &= S_c, \qquad \rho_t^{Z,0\oplus 1}(S_d) = e^{2\pi\sqrt{-1}t} S_d, \\ \rho_t^{Z,1\oplus 0}(S_c) &= e^{2\pi\sqrt{-1}t} S_c, \qquad \rho_t^{Z,1\oplus 0}(S_d) = S_d \end{aligned}$$

for $c \in C, d \in D$, we have for $x \otimes K \in \mathcal{O}_Z \otimes \mathcal{K}$

$$\begin{aligned} ((\rho_t^A \otimes \mathrm{id}) \circ \Phi_A)(x \otimes K) =& (\rho_t^{Z,0\oplus 1} \otimes \mathrm{id})(v_A(x \otimes K)v_A^*) \\ =& v_A(\rho_t^{Z,0\oplus 1} \otimes \mathrm{id})(x \otimes K)v_A^* \\ =& \Phi_A \circ (\rho_t^{Z,0\oplus 1} \otimes \mathrm{id})(x \otimes K). \end{aligned}$$

Hence we have $(\rho_t^A \otimes \mathrm{id}) \circ \Phi_A = \Phi_A \circ (\rho_t^{Z,0\oplus 1} \otimes \mathrm{id})$ and similarly $(\rho_t^B \otimes \mathrm{id}) \circ \Phi_B = \Phi_B \circ (\rho_t^{Z,1\oplus 0} \otimes \mathrm{id})$. Since

$$\rho_t^Z \otimes \mathrm{id} = (\rho_t^{Z,1\oplus 0} \otimes \mathrm{id}) \circ (\rho_t^{Z,0\oplus 1} \otimes \mathrm{id}) = (\rho_t^{Z,0\oplus 1} \otimes \mathrm{id}) \circ (\rho_t^{Z,1\oplus 0} \otimes \mathrm{id}),$$

we know the assertion. ☐

Therefore we have the following corollary.

Corollary 3.14. *If $A, B$ are strong shift equivalent, then the stabilized Cuntz–Krieger triplets $(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}, \rho^A \otimes \mathrm{id})$ and $(\mathcal{O}_B \otimes \mathcal{K}, \mathcal{D}_B \otimes \mathcal{C}, \rho^B \otimes \mathrm{id})$ are strong Morita equivalent.*

## 4 Behavior on K-theory

In this section we will study the behavior of the isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ in Proposition 3.10 on their K-groups $\Phi_* : K_0(\mathcal{O}_A) \to K_0(\mathcal{O}_B)$ under the condition $A = CD, B = DC$.

Recall that $A = [A(i,j)]_{i,j=1}^N$ is an $N \times N$ matrix with entries in nonnegative integers. Then the associated graph $G_A = (V_A, E_A)$ consists of the vertex set $V_A = \{v_1^A, \dots, v_N^A\}$ of $N$ vertices and edge set $E_A = \{a_1, \dots, a_{N_A}\}$, where there are $A(i,j)$ edges from $v_i^A$ to $v_j^A$. Denote by $t(a_i), s(a_i)$ the terminal vertex of $a_i$, the source vertex of $a_i$, respectively. The graph $G_A$ has the $N_A \times N_A$ transition matrix $A^G = [A^G(i,j)]_{i,j=1}^{N_A}$ of edges defined by (2.1). The Cuntz–Krieger algebra $\mathcal{O}_A$ is defined as the Cuntz–Krieger algebra $\mathcal{O}_{A^G}$ for the matrix $A^G$ which is the universal $C^*$-algebra generated by partial isometries $S_{a_i}, i = 1, \dots, N_A$ subject to the relations (2.2). We similarly consider the $N_B \times N_B$ matrix $B^G$ with entries in $\{0,1\}$ for the graph $G_B = (V_B, E_B)$ of the matrix $B$ with vertex set $V_B = \{v_1^B, \dots, v_M^B\}$ and edge set $E_B = \{b_1, \dots, b_{N_B}\}$, so that we have the other Cuntz–Krieger algebra $\mathcal{O}_{B^G}$ for the matrix $B^G$ which is denoted by $\mathcal{O}_B$.

Now we are assuming that $A = CD$ and $B = DC$ for some nonnegative rectangular matrices $C$ and $D$. Both $A$ and $B$ are also assumed to be irreducible and not any permutations. Since $A = CD$, the edge set $E_A$ is regarded as a subset of the product $E_C \times E_D$ of those of $E_C$ and $E_D$. As in Section 2, we may take a bijection $\varphi_{A,CD}$ from $E_A$ to a subset of $E_C \times E_D$. For any $a_i \in E_A$, there uniquely exist $c(a_i) \in E_C$ and $d(a_i) \in E_D$ such that $\varphi_{A,CD}(a_i) = c(a_i)d(a_i)$. We write it simply as $a_i = c(a_i)d(a_i)$. Similarly, for any edge $b_l \in E_B$, there uniquely exist $d(b_l) \in E_D$ and $c(b_l) \in E_C$ such that $\varphi_{B,DC}(b_l) = d(b_l)c(b_l)$, simply written $b_l = d(b_l)c(b_l)$. We define the $N_A \times N_B$ matrix $\hat{D} = [\hat{D}(i,l)]_{i=1,\dots,N_A}^{l=1,\dots,N_B}$ by

$$\hat{D}(i,l) = \begin{cases} 1 & \text{if } d(a_i) = d(b_l), \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Lemma 4.1. *The matrix $\hat{D}^t : \mathbb{Z}^{N_A} \to \mathbb{Z}^{N_B}$ induces a homomorphism from $\mathbb{Z}^{N_A}/(\mathrm{id} - (A^G)^t)\mathbb{Z}^{N_A}$ to $\mathbb{Z}^{N_B}/(\mathrm{id} - (B^G)^t)\mathbb{Z}^{N_B}$ as abelian groups.*

*Proof.* For $i = 1, \dots, N_A$ and $l = 1, \dots, N_B$, we know that both

$$[A^G\hat{D}](i,l) = \sum_{j=1}^{N_A} A^G(i,j)\hat{D}(j,l) \quad \text{and} \quad [\hat{D}B^G](i,l) = \sum_{k=1}^{N_B} \hat{D}(i,k)B^G(k,l)$$

are the cardinal number of the set $\{c \in E_C \mid d(a_i)cd(b_l) \in B_3(X_Z)\}$. Hence we have $A^G\hat{D} = \hat{D}B^G$. We then have that $\hat{D}^t(\mathrm{id} - (A^G)^t)\mathbb{Z}^{N_A} \subset (\mathrm{id} - (B^G)^t)\mathbb{Z}^{N_B}$ so that $\hat{D}^t$ induces a desired homomorphism. $\square$

We denote by $\Phi_{\hat{D}^t}$ the above homomorphism from $\mathbb{Z}^{N_A}/(\mathrm{id} - (A^G)^t)\mathbb{Z}^{N_A}$ to $\mathbb{Z}^{N_B}/(\mathrm{id} - (B^G)^t)\mathbb{Z}^{N_B}$ induced by $\hat{D}^t$.

Let us denote by $[e_i^{N_A}]$ the class of the vector $e_i^{N_A} = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0) \in \mathbb{Z}^{N_A}$ in $\mathbb{Z}^{N_A}/(\mathrm{id} - (A^G)^t)\mathbb{Z}^{N_A}$. It was shown in [6] that the correspondence $\epsilon_{A^G} : K_0(\mathcal{O}_{A^G}) \to \mathbb{Z}^{N_A}/(\mathrm{id} - (A^G)^t)\mathbb{Z}^{N_A}$ defined by $\epsilon_{A^G}([S_{a_i}S_{a_i}^*]) = [e_i^{N_A}]$ yields an isomorphism of abelian groups. We then have

Proposition 4.2. *Suppose that $A = CD, B = DC$. Let $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ be the isomorphism constructed in the proof of Proposition 3.10 such that $\Phi =$*

$\mathrm{Ad}(w)$ *with* $w = v_B v_A^*$ *for the isometry* $v_A$ *as well as* $v_B$ *defined before Lemma 3.5. Then the diagram*

$$\begin{array}{ccc} K_0(\mathcal{O}_{A^G}) & \xrightarrow{\Phi_*} & K_0(\mathcal{O}_{B^G}) \\ {\scriptstyle \epsilon_{A^G}}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon_{B^G}} \\ \mathbb{Z}^{N_A}/(\mathrm{id} - (A^G)^t)\mathbb{Z}^{N_A} & \xrightarrow{\Phi_{\hat{D}^t}} & \mathbb{Z}^{N_B}/(\mathrm{id} - (B^G)^t)\mathbb{Z}^{N_B} \end{array}$$

*is commutative.*

*Proof.* We note that $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$ has a countable basis and $\mathbb{N}$ is decomposed such as $\mathbb{N} = \cup_{j=1}^{\infty} \mathbb{N}_j$ where $\mathbb{N}_j$ is also a disjoint infinite set such as $\mathbb{N}_j = \cup_{k=0}^{\infty} \mathbb{N}_{j_k}$ with disjoint infinite sets $\mathbb{N}_{j_k}$ for every $k = 0, 1, 2, \dots$. We write $\mathbb{N}_{j_k}$ as $\mathbb{N}_{j_k} = \{j_k(0), j_k(1), j_k(2), \dots\}$. In particular for $j = 1, k = 0$, we denote by $\bar{n} = 1_0(n)$ for $n = 0, 1, 2, \dots$ so that $\mathbb{N}_{1_0} = \{\bar{0}, \bar{1}, \bar{2}, \dots\}$. Let $p_{\bar{n}}, n = 0, 1, 2, \dots$ be the sequence of projections of rank one in $\mathcal{K}$ such that $\sum_{n=0}^{\infty} p_{\bar{n}} = f_{1_0}$. By [6, Proposition 3.1], the group $K_0(\mathcal{O}_{A^G})$ is generated by the projections of the form

$$S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}, \qquad i = 1, \dots, N_A.$$

Denote by $1_A$ the unit of $\mathcal{O}_{A^G}$ so that $[1_A] = \sum_{i=1}^{N_A} [S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}]$ in $K_0(\mathcal{O}_{A^G})$. Let $\Phi = \mathrm{Ad}(w) : \mathcal{O}_{A^G} \otimes \mathcal{K} \to \mathcal{O}_{B^G} \otimes \mathcal{K}$ be the isomorphism constructed in the proof of Proposition 3.10. Hence $\Phi_* : K_0(\mathcal{O}_{A^G}) \to K_0(\mathcal{O}_{B^G})$ satisfies $\Phi_*([S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}]) = [w(S_{a_i} S_{a_i}^* \otimes p_{\bar{0}})^* w^*]$. To complete the proof of the proposition, we provide the following two lemmas. $\square$

Let $l(i)$ be the number $l = 1, \dots, N_C$ satisfying $c_l = c(a_i)$ so that $d(l(i)) \in E_D$ satisfies $T_{l(i)} = S_{d(l(i))} S_{c(a_i)} S_{c(a_i)}^*$ in (3.14). We put $s_{1_{l(i)}, 1_0} = s_{1_{l(i)}, 1} s_{1, 1_0}$ and $s_{1_0, 1_{l(i)}} = s_{1_{l(i)}, 1_0}^*$.

Lemma 4.3. *Keep the above notation.*

(i) $w(S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}) w^* = v_B (S_{a_i} S_{a_i}^* \otimes s_{1, 1_0} p_{\bar{0}} s_{1_0, 1}) v_B^*$.

(ii) $v_B (S_{a_i} S_{a_i}^* \otimes s_{1, 1_0} p_{\bar{0}} s_{1_0, 1}) v_B^* = S_{d(l(i))} S_{c(a_i)} S_{d(a_i)} S_{d(a_i)}^* S_{c(a_i)}^* S_{d(l(i))}^* \otimes s_{1_{l(i)}, 1_0} p_{\bar{0}} s_{1_0, 1_{l(i)}}$.

*Proof.* (i) The unitary $w$ is given by $w = v_B v_A^*$. We know $v_A = \sum_{n=1}^{\infty} v_n$ and $v_1 = P_C \otimes s_{1_0, 1} + \sum_{k=1}^{N_D} U_k \otimes s_{1_k, 1}$. As $p_{\bar{0}} s_{1_k, 1} = 0$ for $k = 1, \dots, N_D$, we have

$$\begin{aligned} v_A^* (S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}) v_A &= v_1^* (S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}) v_1 \\ &= (P_C \otimes s_{1_0, 1})^* (S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}) (P_C \otimes s_{1_0, 1}) \\ &= S_{a_i} S_{a_i}^* \otimes s_{1, 1_0} p_{\bar{0}} s_{1_0, 1}. \end{aligned}$$

(ii) For $c_l \in E_C = \{c_1, \dots, c_{N_C}\}$ and $a_i \in E_A$, we note that $S_{c_l}^* S_{a_i} = S_{c_l}^* S_{c(a_i)} S_{d(a_i)}$ if $c_l = c(a_i)$, otherwise zero. Hence we have

$$
\begin{aligned}
& v_B (S_{a_i} S_{a_i}^* \otimes s_{1,1_0} p_{\bar{0}} s_{1_0,1}) v_B^* \\
=& \left( \sum_{l=1}^{N_C} T_l \otimes s_{1_l,1} \right) \left( S_{a_i} S_{a_i}^* \otimes s_{1,1_0} p_{\bar{0}} s_{1_0,1} \right) \left( \sum_{l'=1}^{N_C} T_{l'} \otimes s_{1_{l'},1} \right)^* \\
=& \sum_{l=1}^{N_C} S_{d(l)} S_{c_l} S_{c_l}^* S_{a_i} S_{a_i}^* S_{c_l} S_{c_l}^* S_{d(l)}^* \otimes s_{1_l,1} s_{1,1_0} p_{\bar{0}} s_{1_0,1} s_{1_l,1}^* \\
=& S_{d(l(i))} S_{c(a_i)} S_{d(a_i)} S_{d(a_i)}^* S_{c(a_i)}^* S_{d(l(i))}^* \otimes s_{1_{l(i)},1_0} p_{\bar{0}} s_{1_0,1_{l(i)}}.
\end{aligned}
$$

∎

Lemma 4.4. $S_{d(a_i)} S_{d(a_i)}^* = \sum_{l=1}^{N_B} \hat{D}(i,l) S_{b_l} S_{b_l}^*$.

*Proof.* In the algebra $\mathcal{O}_{B^G}$, we have $\sum_{l=1}^{N_B} S_{b_l} S_{b_l}^* = 1$. As $b_l = d(b_l)c(b_l)$, it implies that $\sum_{l=1}^{N_B} S_{d(b_l)} S_{c(b_l)} S_{c(b_l)}^* S_{d(b_l)}^* = P_D$ in $\mathcal{O}_Z$. By multiplying $S_{d(a_i)} S_{d(a_i)}^*$ to the equality we have

$$
\sum_{l=1}^{N_B} S_{d(a_i)} S_{d(a_i)}^* S_{d(b_l)} S_{c(b_l)} S_{c(b_l)}^* S_{d(b_l)}^* S_{d(a_i)} S_{d(a_i)}^* = S_{d(a_i)} S_{d(a_i)}^*.
$$

Since

$$
S_{d(a_i)} S_{d(a_i)}^* S_{d(b_l)} = \hat{D}(i,l) S_{d(b_l)},
$$

we have

$$
\sum_{l=1}^{N_B} \hat{D}(i,l) S_{d(b_l)} S_{c(b_l)} S_{c(b_l)}^* S_{d(b_l)}^* = S_{d(a_i)} S_{d(a_i)}^*.
$$

As $S_{b_l} = S_{d(b_l)} S_{c(b_l)}$, we get the desired equality. ∎

*Proof of Proposition 4.2*:
By using Lemma 4.3, we have the equalities in $K_0(\mathcal{O}_{B^G})$:

$$
\Phi_*([S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}]) = [S_{d(l(i))} S_{c(a_i)} S_{d(a_i)} S_{d(a_i)}^* S_{c(a_i)}^* S_{d(l(i))}^* \otimes s_{1_{l(i)},1_0} p_{\bar{0}} s_{1_0,1_{l(i)}}].
$$

Since

$$
\begin{aligned}
& [S_{d(l(i))} S_{c(a_i)} S_{d(a_i)} S_{d(a_i)}^* S_{c(a_i)}^* S_{d(l(i))}^* \otimes s_{1_{l(i)},1_0} p_{\bar{0}} s_{1_0,1_{l(i)}}] \\
=& [S_{d(a_i)} S_{d(a_i)}^* \otimes f_{1_0} p_{\bar{0}} f_{1_0}] \quad \text{in} \quad K_0(\mathcal{O}_{B^G}),
\end{aligned}
$$

and $f_{1_0} p_{\bar{0}} f_{1_0} = p_{\bar{0}}$, we have

$$
\Phi_*([S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}]) = [S_{d(a_i)} S_{d(a_i)}^* \otimes p_{\bar{0}}].
$$

As $\epsilon_{A^G}([S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}]) = [e_i^{N_A}]$ and $\epsilon_{B^G}([S_{b_l} S_{b_l}^* \otimes p_{\bar{0}}]) = [e_l^{N_B}]$, By using Lemma 4.4, we complete the proof of Proposition 4.2. ∎

Let $S_A$ and $R_A$ be the $N_A \times N$ matrix and $N \times N_A$ matrix defined by

$$S_A(i,j) = \begin{cases} 1 & \text{if } t(a_i) = v_j^A, \\ 0 & \text{otherwise,} \end{cases} \qquad R_A(j,i) = \begin{cases} 1 & \text{if } v_j^A = s(a_i), \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, \dots, N_A$ and $j = 1, \dots, N$, respectively. We then have $A = R_A S_A$ and $A^G = S_A R_A$. We similarly have the matrices $S_B, R_B$ for the other matrix $B$ such that $B = R_B S_B$ and $B^G = S_B R_B$. The matrix $S_A^t : \mathbb{Z}^{N_A} \to \mathbb{Z}^N$ induces a homomorphism $\mathbb{Z}^{N_A}/(\mathrm{id} - (A^G)^t)\mathbb{Z}^{N_A} \to \mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N$ of abelian groups which is actually an isomorphism since its inverse is given by a homomorphism induced by $R_A^t$. The above isomorphism is denoted by $\Phi_{S_A^t}$. We have an isomorphism $\Phi_{S_B^t} : \mathbb{Z}^{N_B}/(\mathrm{id} - (B^G)^t)\mathbb{Z}^{N_B} \to \mathbb{Z}^M/(\mathrm{id} - B^t)\mathbb{Z}^M$ in a similar way.

Now we are assuming that $A = CD, B = DC$ so that $AC = CB$ and hence $C^t A^t = B^t C^t$. The matrix $C^t : \mathbb{Z}^N \to \mathbb{Z}^M$ induces a homomorphism from $\mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N$ to $\mathbb{Z}^M/(\mathrm{id} - B^t)\mathbb{Z}^M$ as abelian groups, which is denoted by $\Phi_{C^t}$. It is actually an isomorphism with $\Phi_{D^t}$ as its inverse. We notice the following lemma. The second assertion (ii) is pointed out by Hiroki Matui. The author thanks him for his advice.

Lemma 4.5. (i) *The diagram*

$$\begin{array}{ccc} \mathbb{Z}^{N_A}/(\mathrm{id} - (A^G)^t)\mathbb{Z}^{N_A} & \xrightarrow{\Phi_{\hat{D}^t}} & \mathbb{Z}^{N_B}/(\mathrm{id} - (B^G)^t)\mathbb{Z}^{N_B} \\ {\scriptstyle \Phi_{S_A^t}}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_{S_B^t}} \\ \mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N & \xrightarrow{\Phi_{C^t}} & \mathbb{Z}^M/(\mathrm{id} - B^t)\mathbb{Z}^M \end{array}$$

*is commutative.*

(ii) $\Phi_{S_A^t}([(1,1,\dots,1)]) = [(1,1,\dots,1)]$.

*Proof.* (i) Since $\Phi_{\hat{D}}$ is induced by the matrix $\hat{D}^t$, it suffices to prove the equality $\hat{D} S_B = S_A C$. Let $(i,j)$ be $i = 1, \dots, N_A$ and $j = 1, \dots, M$ so that $a_i \in E_A$ and $v_j^B \in V_B$. Let $k$ be such that $t(a_i) = v_k^A$. Hence we have

$$[S_A C](i,j) = \sum_{n=1}^{N} S_A(i,n) C(n,j) = C(k,j)$$

which is the number of edges of $E_C$ leaving $v_k^A$ and terminating at $v_j^B$. On the other hand,

$$[\hat{D} S_B](i,j) = \sum_{l=1}^{M_B} \hat{D}(i,l) S_B(l,j).$$

It is easy to see that the above number is also $C(k,j)$.

(ii) Since $A = R_A S_A$, for each $k = 1, \dots, N_A$ with $a_k \in E_A$ there exists a unique $i = 1, \dots, N$ such that $s(a_k) = v_i^A$. Hence $\sum_{i=1}^N R_A(i,k) = 1$ so that we have for each $j = 1, \dots, N$

$$\begin{aligned}\sum_{i=1}^N A^t(j,i) &= \sum_{i=1}^N \sum_{k=1}^{N_A} R_A(i,k) S_A(k,j) \\ &= \sum_{k=1}^{N_A} \left( \sum_{i=1}^N R_A(i,k) \right) S_A(k,j) \\ &= \sum_{k=1}^{N_A} S_A^t(j,k).\end{aligned}$$

We then see

$$\begin{aligned}\Phi_{S_A^t}([(1,1,\dots,1)]) &= [(\sum_{k=1}^{N_A} S_A(k,1), \sum_{k=1}^{N_A} S_A(k,2), \dots, \sum_{k=1}^{N_A} S_A(k,N))] \\ &= [(\sum_{i=1}^N A^t(1,i), \sum_{i=1}^N A^t(2,i), \dots, \sum_{i=1}^N A^t(N,i))] \\ &= [(1,1,\dots,1)] \quad \text{in } \mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N.\end{aligned}$$

□

Put

$$\epsilon_A = \Phi_{S_A^t} \circ \epsilon_{A^G} : K_0(\mathcal{O}_A) \to \mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N, \tag{4.2}$$

which is an isomorphism of groups such that $\epsilon_A([1_A]) = [(1,1,\dots,1)]$. We thus reach the following theorem:

Theorem 4.6. *Suppose that two nonnegative irreducible matrices $A, B$ satisfy $A = CD, B = DC$ for some nonnegative rectangular matrices $C, D$. Let $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ be the isomorphism constructed in the proof of Proposition 3.10 such that $\Phi = \mathrm{Ad}(w)$ with $w = v_B v_A^*$ for the isometry $v_A$ as well as $v_B$ defined before Lemma 3.5. Then the diagram*

$$\begin{array}{ccc} K_0(\mathcal{O}_A) & \xrightarrow{\Phi_*} & K_0(\mathcal{O}_B) \\ {\scriptstyle \epsilon_A}\downarrow & & \downarrow{\scriptstyle \epsilon_B} \\ \mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N & \xrightarrow{\Phi_{C^t}} & \mathbb{Z}^M/(\mathrm{id} - B^t)\mathbb{Z}^M \end{array}$$

*is commutative, where all maps are isomorphisms of abelian groups.*

We write $A \underset{C,D}{\approx} B$ if $A = CD$, $B = DC$. Recall that $A, B$ are said to be strong shift equivalent in $n$-step if there exist a finite sequence of square matrices

$A_1, \dots, A_{n-1}$ and two finite sequences of rectangular matrices $C_1, \dots, C_n$ and $D_1, \dots, D_n$ such that

$$A = A_0 \underset{C_1,D_1}{\approx} A_1, \quad A_1 \underset{C_2,D_2}{\approx} A_2, \quad \dots, \quad A_{n-1} \underset{C_n,D_n}{\approx} A_n = B.$$

This situation is written

$$A \underset{C_1,D_1}{\approx} \cdots \underset{C_n,D_n}{\approx} B. \tag{4.3}$$

R. F. Williams proved that two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate if and only if $A$ and $B$ are strong shift equivalent in $n$-step for some $n$ ([25]). Hence we have the following corollary.

Corollary 4.7. *Suppose that two matrices $A, B$ are strong shift equivalent in $n$-step for some two sequences of rectangular matrices $C_1, \dots, C_n$ and $D_1, \dots, D_n$ as in* (4.3)*. Then there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ of $C^*$-algebras and a unitary representation $t \in \mathbb{T} \to v_t^A \in M(\mathcal{D}_A \otimes \mathcal{C})$ such that*

$$\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}, \qquad \Phi \circ \mathrm{Ad}(v_t^A) \circ (\rho_t^A \otimes \mathrm{id}) = (\rho_t^B \otimes \mathrm{id}) \circ \Phi,$$

*and the following diagram is commutative*

$$\begin{array}{ccc}
K_0(\mathcal{O}_A) & \xrightarrow{\Phi_*} & K_0(\mathcal{O}_B) \\
{\scriptstyle \epsilon_A}\downarrow & & \downarrow{\scriptstyle \epsilon_B} \\
\mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N & \xrightarrow{\Phi_{(C_1C_2\cdots C_n)^t}} & \mathbb{Z}^M/(\mathrm{id} - B^t)\mathbb{Z}^M.
\end{array}$$

We note that the inverse of
$\Phi_{(C_1C_2\cdots C_n)^t} : \mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N \to \mathbb{Z}^M/(\mathrm{id} - B^t)\mathbb{Z}^M$ is given by
$\Phi_{(D_n\cdots D_2D_1)^t} : \mathbb{Z}^M/(\mathrm{id} - B^t)\mathbb{Z}^M \to \mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N$.

## 5 Converse and Invariant

In this section, we will study the converse of Corollary 3.11 by using Corollary 4.7. We fix a projection $p_1$ of rank one in $\mathcal{K}$.

Proposition 5.1. *The following assertions are equivalent.*

(i) *There exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ of $C^*$-algebras and a unitary one-cocycle $u_t \in M(\mathcal{O}_B \otimes \mathcal{K}), t \in \mathbb{T}$ relative to $\rho_t^B \otimes \mathrm{id}$ such that*

$$\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}, \quad \Phi \circ (\rho_t^A \otimes \mathrm{id}) = \mathrm{Ad}(u_t) \circ (\rho_t^B \otimes \mathrm{id}) \circ \Phi, \tag{5.1}$$

$$\Phi_*([1_A \otimes p_1]) = [1_B \otimes p_1] \text{ in } K_0(\mathcal{O}_B). \tag{5.2}$$

(ii) *There exist an isomorphism $\varphi : \mathcal{O}_A \to \mathcal{O}_B$ and a unitary one-cocycle $v_t \in U(\mathcal{O}_B), t \in \mathbb{T}$ relative to $\rho_t^B$ on $\mathcal{O}_B$ such that*

$$\varphi(\mathcal{D}_A) = \mathcal{D}_B \quad \text{and} \quad \varphi \circ \rho_t^A = \mathrm{Ad}(v_t) \circ \rho_t^B \circ \varphi, \qquad t \in \mathbb{T}. \tag{5.3}$$

*Proof.* The implication (ii) $\Longrightarrow$ (i) is obvious by putting $\Phi = \varphi \otimes \mathrm{id}$ and $u_t = v_t \otimes 1$. We will show the implication (i) $\Longrightarrow$ (ii) in the following way. By [13, Proposition 3.13], the condition $\Phi_*([1_A \otimes p_1]) = [1_B \otimes p_1]$ in $K_0(\mathcal{O}_B)$ ensures that there exists a partial isometry $V \in \mathcal{O}_B \otimes \mathcal{K}$ satisfying the following conditions:

$$V(\mathcal{D}_B \otimes \mathcal{C})V^* \subset \mathcal{D}_B \otimes \mathcal{C}, \qquad V^*(\mathcal{D}_B \otimes \mathcal{C})V \subset \mathcal{D}_B \otimes \mathcal{C},$$
$$VV^* = 1_B \otimes p_1, \quad V^*V = \Phi(1_A \otimes p_1).$$

Put $\Psi = \mathrm{Ad}(V) \circ \Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$. It is straightforward to see that

$$\Psi(\mathcal{O}_A \otimes \mathbb{C}p_1) = \mathcal{O}_B \otimes \mathbb{C}p_1, \qquad \Psi(\mathcal{D}_A \otimes \mathbb{C}p_1) = \mathcal{D}_B \otimes \mathbb{C}p_1,$$
$$\Psi(1_A \otimes p_1) = 1_B \otimes p_1.$$

It is clear that $\Psi_* = \Phi_* : K_0(\mathcal{O}_A) \to K_0(\mathcal{O}_B)$. We identify $\mathcal{O}_B \otimes \mathbb{C}p_1$ with $\mathcal{O}_B$. Put the partial isometry $v_t = Vu_t(\rho_t^B \otimes \mathrm{id})(V^*) \in \mathcal{O}_B \otimes \mathcal{K}$. Since $v_t = (1_B \otimes p_1)v_t(1_B \otimes p_1)$, by this identification, $v_t$ belongs to $\mathcal{O}_B$. Define $\varphi : \mathcal{O}_A \to \mathcal{O}_B$ by setting $\varphi(a) = \Psi(a \otimes p_1)$ for $a \in \mathcal{O}_A$. It then follows that

$$\begin{aligned}
\varphi(\rho_t^A(a)) \otimes p_1 =& V\Phi(\rho_t^A(a) \otimes p_1)V^* \\
=& V(\mathrm{Ad}(u_t) \circ (\rho_t^B \otimes \mathrm{id}) \circ \Phi)(a \otimes p_1)V^* \\
=& Vu_t(\rho_t^B \otimes \mathrm{id})(V^*)(\rho_t^B \otimes \mathrm{id})\Phi(V(a \otimes p_1)V^*)(\rho_t^B \otimes \mathrm{id})(V)u_t^*V^* \\
=& v_t((\rho_t^B \otimes \mathrm{id}) \circ \Psi)(a \otimes p_1)v_t^* \\
=& (\mathrm{Ad}(v_t) \circ (\rho_t^B \circ \varphi)(a)) \otimes p_1
\end{aligned}$$

so that we have $\varphi(\rho_t^A(a)) = (\mathrm{Ad}(v_t) \circ \rho_t^B \circ \varphi)(a)$. Since we have

$$\begin{aligned}
(\rho_t^B \otimes \mathrm{id})(\Phi(1_A \otimes p_1)) =& (\mathrm{Ad}(u_t^*) \circ \Phi \circ (\rho_t^A \otimes \mathrm{id}))(1_A \otimes p_1) \\
=& u_t^*\Phi(1_A \otimes p_1)u_t = u_t^*V^*Vu_t,
\end{aligned}$$

we have

$$\begin{aligned}
v_t\rho_t^B(v_s) =& Vu_t(\rho_t^B \otimes \mathrm{id})(V^*)(\rho_t^B \otimes \mathrm{id})(Vu_s(\rho_s^B \otimes \mathrm{id})(V^*)) \\
=& Vu_t(\rho_t^B \otimes \mathrm{id})(V^*V)(\rho_t^B \otimes \mathrm{id})(u_s)(\rho_t^B \circ \rho_s^B \otimes \mathrm{id})(V^*) \\
=& Vu_t(\rho_t^B \otimes \mathrm{id})(\Phi(1_A \otimes p_1))(\rho_t^B \otimes \mathrm{id})(u_s)(\rho_{t+s}^B \otimes \mathrm{id})(V^*) \\
=& Vu_tu_t^*V^*Vu_t(\rho_t^B \otimes \mathrm{id})(u_s)(\rho_{t+s}^B \otimes \mathrm{id})(V^*) \\
=& Vu_t(\rho_t^B \otimes \mathrm{id})(u_s)(\rho_{t+s}^B \otimes \mathrm{id})(V^*) \\
=& Vu_{t+s}(\rho_{t+s}^B \otimes \mathrm{id})(V^*) \\
=& v_{t+s}.
\end{aligned}$$

Hence $v_t, t \in \mathbb{T}$ is a unitary one-cocycle relative to $\rho^B$. $\square$

REMARK 5.2. Let $v_t$ in $\mathcal{O}_B$ be a unitary one-cocycle relative to $\rho^B_t$ satisfying (5.3). For $a \in \mathcal{D}_A$, we see that $\varphi(\rho^A_t(a)) = \mathrm{Ad}(v_t)(\rho^B_t(\varphi(a)))$. As $\rho^A_t(a) = a$ and $\varphi(a)$ belongs to $\mathcal{D}_B$ so that we have $\varphi(a) = \mathrm{Ad}(v_t)(\varphi(a))$. Hence $v_t$ commutes with any element of $\mathcal{D}_B$. This implies that $v_t$ belongs to $\mathcal{D}_B$ and hence it is fixed by the action $\rho^B$. Therefore a unitary one-cocycle $v_t$ in $\mathcal{O}_B$ relative to $\rho^B_t$ satisfying (5.3) automatically belongs to $\mathcal{D}_B$ and yields a unitary representation $t \in \mathbb{T} \to v_t \in \mathcal{D}_B$. Since the unitary $u_t$ in (5.1) is given by $u_t = v_t \otimes 1$ from the unitary $v_t$ satisfying (5.3), the unitary one-cocycle $u_t$ in the statement (i) of the above proposition can be taken as a unitary representation $t \in \mathbb{T} \to u_t \in M(\mathcal{D}_B \otimes \mathcal{C})$ which is fixed by the action $\rho^B_t \otimes \mathrm{id}$.

COROLLARY 5.3. *If there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ of $C^*$-algebras and a unitary one-cocycle $u_t$ in $M(\mathcal{O}_B \otimes \mathcal{K})$ relative to $\rho^B_t \otimes \mathrm{id}$ such that*

$$\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}, \qquad \Phi \circ (\rho^A_t \otimes \mathrm{id}) = \mathrm{Ad}(u_t) \circ (\rho^B_t \otimes \mathrm{id}) \circ \Phi,$$
$$\Phi_*([1_A \otimes p_1]) = [1_B \otimes p_1] \text{ in } K_0(\mathcal{O}_B),$$

*then two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate.*

*Proof.* Suppose that there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ and a unitary one-cocycle $u_t$ in $M(\mathcal{O}_B \otimes \mathcal{K})$ relative to $\rho^B_t \otimes \mathrm{id}$ satisfying the above equalities. Proposition 5.1 tells us that there exist an isomorphism $\varphi : \mathcal{O}_A \to \mathcal{O}_B$ and a unitary one-cocycle $v_t \in U(\mathcal{O}_B), t \in \mathbb{T}$ relative to $\rho^B_t$ on $\mathcal{O}_B$ satisfying (5.3). Hence the one-sided topological Markov shifts $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are strongly continuous orbit equivalent by [14, Theorem 6.7]. It also implies topological conjugacy of their two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ by [14, Theorem 5.5]. □

DEFINITION 5.4. An isomorphism $\xi : \mathcal{O}_B \otimes \mathcal{K} \to \mathcal{O}_A \otimes \mathcal{K}$ of $C^*$-algebras is said to be *induced from strong shift equivalence* if there exist a strong shift equivalence $A \underset{C_1,D_1}{\approx} \cdots \underset{C_n,D_n}{\approx} B$ and a unitary one-cocycle $u_t$ in $M(\mathcal{O}_A \otimes \mathcal{K})$ relative to $\rho^A_t \otimes \mathrm{id}$ such that

$$\xi(\mathcal{D}_B \otimes \mathcal{C}) = \mathcal{D}_A \otimes \mathcal{C}, \qquad \xi \circ (\rho^B_t \otimes \mathrm{id}) = \mathrm{Ad}(u_t) \circ (\rho^A_t \otimes \mathrm{id}) \circ \xi,$$
$$\xi_* = \epsilon_A^{-1} \circ \Phi_{(D_n \cdots D_2 D_1)^t} \circ \epsilon_B : K_0(\mathcal{O}_B) \to K_0(\mathcal{O}_A).$$

In this case, we say that $\xi : \mathcal{O}_B \otimes \mathcal{K} \to \mathcal{O}_A \otimes \mathcal{K}$ is induced from strong shift equivalence $A \underset{C_1,D_1}{\approx} \cdots \underset{C_n,D_n}{\approx} B$.

We will define the *strong shift equivalence invariant subset* $\mathrm{K}_0^{\mathrm{SSE}}(\mathcal{O}_A)$ of $K_0(\mathcal{O}_A)$ as follows.

DEFINITION 5.5.

$$\begin{aligned}\mathrm{K}_0^{\mathrm{SSE}}(\mathcal{O}_A) = \{[p] \in K_0(\mathcal{O}_A) \mid\ & \text{there exist a square matrix } B \text{ and} \\ & \text{an isomorphism } \xi : \mathcal{O}_B \otimes \mathcal{K} \to \mathcal{O}_A \otimes \mathcal{K} \text{ induced from} \\ & \text{strong shift equivalence such that } \xi_*([1_B]) = [p] \text{ in } K_0(\mathcal{O}_A)\}.\end{aligned}$$

We note that the class $[1_A]$ in $K_0(\mathcal{O}_A)$ of the unit $1_A$ of $\mathcal{O}_A$ always belongs to the set $\mathrm{K}_0^{\mathrm{SSE}}(\mathcal{O}_A)$, because we may take $B = A$ and $\xi = \mathrm{id}$.

PROPOSITION 5.6. *Suppose that there exists a topological conjugacy between* $(\bar{X}_A, \bar{\sigma}_A)$ *and* $(\bar{X}_B, \bar{\sigma}_B)$. *Then there exists an isomorphism* $\eta : K_0(\mathcal{O}_A) \to K_0(\mathcal{O}_B)$ *satisfying* $\eta(\mathrm{K}_0^{SSE}(\mathcal{O}_A)) = \mathrm{K}_0^{SSE}(\mathcal{O}_B)$. *Hence the pair* $(K_0(\mathcal{O}_A), \mathrm{K}_0^{SSE}(\mathcal{O}_A))$ *is an invariant under topological conjugacy of two-sided topological Markov shifts.*

*Proof.* Suppose that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate so that $A \underset{C_1,D_1}{\approx} \cdots \underset{C_n,D_n}{\approx} B$ for some nonnegative rectangular matrices $C_1, D_1, \dots, C_n, D_n$. By Corollary 4.7, the strong shift equivalence induces an isomorphism $\xi_{BA} : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ and a unitary one-cocycle $u_t$ in $M(\mathcal{O}_B \otimes \mathcal{K})$ relative to $\rho_t^B \otimes \mathrm{id}$ such that

$$\xi_{BA}(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}, \qquad \xi_{BA} \circ (\rho_t^A \otimes \mathrm{id}) = \mathrm{Ad}(u_t) \circ (\rho_t^B \otimes \mathrm{id}) \circ \xi_{BA},$$
$$\xi_{BA*} = \Phi_{(C_1 \cdots C_n)^t} : K_0(\mathcal{O}_A) \to K_0(\mathcal{O}_B).$$

Put $\eta = \xi_{BA*} : K_0(\mathcal{O}_A) \to K_0(\mathcal{O}_B)$. Take an element $[p] \in \mathrm{K}_0^{\mathrm{SSE}}(\mathcal{O}_A)$. There exist a square nonnegative matrix $A'$ and an isomorphism $\xi_{AA'} : \mathcal{O}_{A'} \otimes \mathcal{K} \to \mathcal{O}_A \otimes \mathcal{K}$ of $C^*$-algebras induced from strong shift equivalence $A' \underset{C'_1,D'_1}{\approx} \cdots \underset{C'_{n'},D'_{n'}}{\approx} A$ such that $\xi_{AA'*}([1_{A'}]) = [p]$ in $K_0(\mathcal{O}_A)$. Then the isomorphism $\xi_{BA} \circ \xi_{AA'} : \mathcal{O}_{A'} \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ is induced from strong shift equivalence

$$A' \underset{C'_1,D'_1}{\approx} \cdots \underset{C'_{n'},D'_{n'}}{\approx} A \underset{C_1,D_1}{\approx} \cdots \underset{C_n,D_n}{\approx} B$$

such that $\eta([p]) = (\xi_{BA} \circ \xi_{AA'})_*([1_{A'}])$ in $K_0(\mathcal{O}_B)$ so that $\eta([p]) \in \mathrm{K}_0^{\mathrm{SSE}}(\mathcal{O}_B)$. $\square$

Suppose that two matrices $A, B$ are strong shift equivalent in $n$-step such as (4.3). The matrix $B$ in (4.3) is given by $B = D_n C_n$ so that (4.3) is written as

$$A \underset{C_1,D_1}{\approx} \cdots \underset{C_n,D_n}{\approx} D_n C_n. \tag{5.4}$$

We set the following sequence $\mathrm{SSE}_n(A), n = 1, 2, \dots$ of subsets of the group $\mathbb{Z}^N$

$$\begin{aligned}&\mathrm{SSE}_n(A)\\ =&\{v \in \mathbb{Z}^N \mid v = D_1^t \cdots D_{n-1}^t D_n^t [1, 1, \dots, 1]^t,\ A \underset{C_1,D_1}{\approx} \cdots \underset{C_n,D_n}{\approx} D_n C_n\},\end{aligned}$$

where $[1, 1, \dots, 1]^t$ denotes (the row size of $D_n$) $\times 1$ matrix whose entries are all 1's. We define the sequence $\mathrm{K}_{\mathrm{alg},n}^{\mathrm{SSE}}(A), n = 1, 2, \dots$ of subsets of the group $\mathbb{Z}^N / (\mathrm{id} - A^t)\mathbb{Z}^N$ by

$$\mathrm{K}_{\mathrm{alg},n}^{\mathrm{SSE}}(A) = \{[v] \in \mathbb{Z}^N / (\mathrm{id} - A^t)\mathbb{Z}^N \mid v \in \mathrm{SSE}_n(A)\}, \quad n = 1, 2, \dots.$$

We then define the subset $\mathrm{K}_{\mathrm{alg}}^{\mathrm{SSE}}(A)$ of $\mathbb{Z}^N/(\mathrm{id}-A^t)\mathbb{Z}^N$ by

$$\mathrm{K}_{\mathrm{alg}}^{\mathrm{SSE}}(A) = \cup_{n=1}^{\infty} \mathrm{K}_{\mathrm{alg},n}^{\mathrm{SSE}}(A).$$

By Corollary 4.7, we have the following proposition

Proposition 5.7. *Let $\epsilon_A : K_0(\mathcal{O}_A) \to \mathbb{Z}^N/(\mathrm{id}-A^t)\mathbb{Z}^N$ be the isomorphism defined in* (4.2). *Then we have*

$$\epsilon_A(K_0^{SSE}(\mathcal{O}_A)) = \mathrm{K}_{\mathrm{alg}}^{SSE}(A).$$

*Proof.* For $[p] \in K_0^{\mathrm{SSE}}(\mathcal{O}_A)$, there exist a nonnegative square matrix $B$ with a strong shift equivalence $A \underset{C_1,D_1}{\approx} \cdots \underset{C_n,D_n}{\approx} B$, an isomorphism $\xi : \mathcal{O}_B \otimes \mathcal{K} \to \mathcal{O}_A \otimes \mathcal{K}$ of $C^*$-algebras and a unitary one-cocycle $u_t, t \in \mathbb{T}$ relative to $\rho^A \otimes \mathrm{id}$ such that

$$\xi(\mathcal{D}_B \otimes \mathcal{C}) = \mathcal{D}_A \otimes \mathcal{C}, \qquad \xi \circ (\rho_t^B \otimes \mathrm{id}) = \mathrm{Ad}(u_t) \circ (\rho_t^A \otimes \mathrm{id}) \circ \xi, \tag{5.5}$$

$$\xi_* = \epsilon_A^{-1} \circ \Phi_{(D_n \cdots D_2 D_1)^t} \circ \epsilon_B : K_0(\mathcal{O}_B) \to K_0(\mathcal{O}_A) \quad \text{and} \quad \xi_*([1_B]) = [p]. \tag{5.6}$$

Since $\epsilon_B([1_B]) = [[1,1,\dots,1]^t]$ in $\mathbb{Z}^M/(\mathrm{id}-B^t)\mathbb{Z}^M$, we have

$$\epsilon_A([p]) = \epsilon_A \circ \xi_*([1_B]) \tag{5.7}$$

$$= \Phi_{(D_n \cdots D_2 D_1)^t} \circ \epsilon_B([1_B]) = \Phi_{(D_n \cdots D_2 D_1)^t}([1,1,\dots,1]^t) \tag{5.8}$$

so that $\epsilon_A([p]) \in \mathrm{K}_{\mathrm{alg}}^{\mathrm{SSE}}(A)$ and hence $\epsilon_A(K_0^{\mathrm{SSE}}(\mathcal{O}_A)) \subset \mathrm{K}_{\mathrm{alg}}^{\mathrm{SSE}}(A)$.
Conversely, take an arbitrary element $[v] \in \mathrm{K}_{\mathrm{alg}}^{\mathrm{SSE}}(A)$. We may find a strong shift equivalence $A \underset{C_1,D_1}{\approx} \cdots \underset{C_n,D_n}{\approx} D_n C_n$ such that $v = (D_n \cdots D_2 D_1)^t [1,1,\dots,1]^t$. Put $B = D_n C_n$. By Corollary 4.7, there exist an isomorphism $\xi : \mathcal{O}_B \otimes \mathcal{K} \to \mathcal{O}_A \otimes \mathcal{K}$ of $C^*$-algebras and a unitary one-cocycle $u_t, t \in \mathbb{T}$ relative to $\rho^A \otimes \mathrm{id}$ satisfying (5.5) and $\xi_* = \epsilon_A^{-1} \circ \Phi_{(D_n \cdots D_2 D_1)^t} \circ \epsilon_B : K_0(\mathcal{O}_B) \to K_0(\mathcal{O}_A)$. Put $[p] = \xi_*([1_B])$ which belongs to $K_0^{\mathrm{SSE}}(\mathcal{O}_A)$. By the same equalities as (5.7), (5.8), we get $\epsilon_A([p]) = \Phi_{(D_n \cdots D_2 D_1)^t}([1,1,\dots,1]^t)$ which is the class of $[v]$. This shows that $\epsilon_A(K_0^{\mathrm{SSE}}(\mathcal{O}_A)) \supset \mathrm{K}_{\mathrm{alg}}^{\mathrm{SSE}}(A)$. ∎

Theorem 5.8. *Let $A, B$ be nonnegative irreducible and non-permutation matrices. The following two assertions are equivalent.*

(i) *Two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate.*

(ii) *There exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ of $C^*$-algebras and a unitary one-cocycle $u_t$ in $M(\mathcal{O}_B \otimes \mathcal{K})$ relative to $\rho_t^B \otimes \mathrm{id}$ such that*

$$\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}, \qquad \Phi \circ (\rho_t^A \otimes \mathrm{id}) = \mathrm{Ad}(u_t) \circ (\rho_t^B \otimes \mathrm{id}) \circ \Phi,$$
$$\Phi_*(K_0^{SSE}(\mathcal{O}_A)) = K_0^{SSE}(\mathcal{O}_B) \text{ in } K_0(\mathcal{O}_B).$$

*Proof.* (i) $\Longrightarrow$ (ii): The assertion follows from Corollary 3.11 and Proposition 5.6.
(ii) $\Longrightarrow$ (i): Suppose that there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ of $C^*$-algebras and a unitary one-cocycle $u_t$ in $M(\mathcal{O}_B \otimes \mathcal{K})$ relative to $\rho^B_t \otimes \mathrm{id}$ satisfying the conditions of (ii). Take a projection $p_1$ of rank one in $\mathcal{K}$. Put the projection $p = \Phi(1_A \otimes p_1) \in \mathcal{O}_B \otimes \mathcal{K}$. As $[1_A] \in K_0^{\mathrm{SSE}}(\mathcal{O}_A)$ and $\Phi_*(K_0^{\mathrm{SSE}}(\mathcal{O}_A)) = K_0^{\mathrm{SSE}}(\mathcal{O}_B)$, the class $[p] = \Phi_*([1_A])$ of $p$ in $K_0(\mathcal{O}_B)$ belongs to $K_0^{\mathrm{SSE}}(\mathcal{O}_B)$. One may take a nonnegative square matrix $B'$ and an isomorphism $\gamma : \mathcal{O}_B \otimes \mathcal{K} \to \mathcal{O}_{B'} \otimes \mathcal{K}$ with a unitary one-cocycle $u'_t$ in $M(\mathcal{O}_{B'} \otimes \mathcal{K})$ relative to $\rho^{B'}_t \otimes \mathrm{id}$ induced from strong shift equivalence $B \underset{C_1,D_1}{\approx} \cdots \underset{C_n,D_n}{\approx} B'$ satisfying

$$\gamma(\mathcal{D}_B \otimes \mathcal{C}) = \mathcal{D}_{B'} \otimes \mathbb{C}, \qquad \gamma \circ (\rho^B_t \otimes \mathrm{id}) = \mathrm{Ad}(u'_t) \circ (\rho^{B'}_t \otimes \mathrm{id}) \circ \gamma,$$
$$\gamma_*([p]) = [1_{B'}] \quad \text{ in } K_0(\mathcal{O}_{B'}).$$

Then the isomorphism $\gamma \circ \Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_{B'} \otimes \mathcal{K}$ satisfies the conditions

$$(\gamma \circ \Phi)(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_{B'} \otimes \mathbb{C},$$
$$(\gamma \circ \Phi) \circ (\rho^A_t \otimes \mathrm{id}) = \mathrm{Ad}(\gamma(u_t)u'_t) \circ (\rho^{B'}_t \otimes \mathrm{id}) \circ (\gamma \circ \Phi),$$
$$(\gamma \circ \Phi)_*([1_A]) = [1_{B'}] \quad \text{ in } K_0(\mathcal{O}_{B'}).$$

By Corollary 5.3, the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_{B'}, \bar{\sigma}_{B'})$ are topologically conjugate. Since $(\bar{X}_B, \bar{\sigma}_B)$ and $(\bar{X}_{B'}, \bar{\sigma}_{B'})$ are topologically conjugate, so are $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$. ☐

REMARK 5.9. The unitary one-cocycle $u_t$ in $M(\mathcal{O}_B \otimes \mathcal{K})$ in (ii) of the above theorem can be taken as a unitary representation $t \in \mathbb{T} \to u_t \in M(\mathcal{O}_B \otimes \mathcal{K})$ by Corollary 3.11.

DEFINITION 5.10. A nonnegative square matrix $A = [A(i,j)]_{i,j=1}^N$ is said to have *full strong shift equivalent units in* $K_0$*-group* if $\mathrm{K}^{\mathrm{SSE}}_{\mathrm{alg}}(A) = \mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N$. We simply call it that $A$ *has full units.*

By Proposition 5.7, $A$ has full units if and only if $K_0^{\mathrm{SSE}}(\mathcal{O}_A) = K_0(\mathcal{O}_A)$. Since the subset $K_0^{\mathrm{SSE}}(\mathcal{O}_A) \subset K_0(\mathcal{O}_A)$ is invariant under topological conjugacy of two-sided topological Markov shifts by Proposition 5.6, we have

PROPOSITION 5.11. *Suppose that two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate. Then $A$ has full units if and only if $B$ has full units.*

As a consequence of Theorem 5.8, we have the following corollary.

COROLLARY 5.12. *Suppose that both $A$ and $B$ have full units. Then the following two assertions are equivalent.*

(i) *Two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate.*

(ii) *There exist an isomorphism* $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ *of* $C^*$*-algebras and a unitary one-cocycle* $u_t$ *in* $M(\mathcal{O}_B \otimes \mathcal{K})$ *relative to* $\rho_t^B \otimes \mathrm{id}$ *such that*

$$\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}, \qquad \Phi \circ (\rho_t^A \otimes \mathrm{id}) = \mathrm{Ad}(u_t) \circ (\rho_t^B \otimes \mathrm{id}) \circ \Phi.$$

Example 5.13.
1. *If* $K_0(\mathcal{O}_A) = 0$*, then* $A$ *has full units.*
2. *Let* $A$ *be the* $1 \times 1$ *matrix* $[N]$ *whose entry is* $N$ *with* $1 < N \in \mathbb{N}$*. Then the matrix* $A$ *has full units. For any* $0 \le k \le N-1$*, let* $C$ *be the* $1 \times (k+1)$ *matrix* $[1, \dots, 1, N-k]$ *and* $D$ *the* $(k+1) \times 1$*matrix* $(1, 1, \dots, 1)^t$*. Then* $A = CD$ *and* $D^t[1, \dots, 1]^t = k+1$*. Hence* $[k+1] \in \mathbb{Z}/(1-N)\mathbb{Z}$ *so that* $\mathrm{K}_{alg}^{SSE}(A) = \mathbb{Z}/(1-N)\mathbb{Z} = K_0(\mathcal{O}_A)$*.*

There is no known example of irreducible, non permutation matrix $A$ such that $A$ does not have full units.

*Acknowledgments:* The author is very grateful to the referee for his careful reading, valuable advices and useful suggestions in the presentation of the paper. This work was done during staying at the Institut Mittag-Leffler in the research program *Classification of operator algebras: complexity, rigidity and dynamics*, January–April 2016. The author is grateful to the Institut Mittag-Leffler for its wonderful research environment, hospitality and support. The author also thanks Hiroki Matui for his advices in Lemma 4.5 and suggestions.

Remark 5.14. After submitting the paper, there were several progress in the following papers related to this paper:
1. T. M. Carlsen and J. Rout, *Diagonal-preserving gauge invariant isomorphisms of graph* $C^*$*-algebras*, preprint, arXiv: 1610.00692 [mathOA].
2. K. Matsumoto, *State splitting, strong shift equivalence and stable isomorphism of Cuntz–Krieger algebras*, preprint, arXiv: 1611.06627 [mathOA].
In the paper 1, the converse implication of [7, Theorem 3.8] was proved, In the paper 2, strong shift equivalence class of the matrix $A$ was described in terms of $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_{A^t}, \mathcal{D}_{A^t}, \rho^{A^t})$.

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