Potentially Crystalline Lifts of Certain Prescribed Types

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Abstract. We prove several results concerning the existence of potentially crystalline lifts of prescribed Hodge–Tate weights and inertial types of a given representation $\overline{r}: G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, where $K/\mathbb{Q}_p$ is a finite extension. Some of these results are proved by purely local methods, and are expected to be useful in the application of automorphy lifting theorems. The proofs of the other results are global, making use of automorphy lifting theorems.

1. Introduction

Let p be a prime, let $K/\mathbb{Q}_p$ be a finite extension, and let $\overline{r}: G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be a continuous representation. For many reasons, it is a natural and important question to study the lifts of $\overline{r}$ to de Rham representations $r: G_K \to \mathrm{GL}_n(\overline{\mathbb{Z}}_p)$; for example, the de Rham lifts of fixed Hodge and inertial types are parameterised by a universal (framed) deformation ring thanks to [Kis08], and the study of these deformation rings is an important step in proving automorphy lifting theorems, going back to Wiles' proof of Fermat's Last Theorem, which made use of Ramakrishna's work on flat deformations [Ram02].

It is therefore slightly vexing that (as far as we are aware) it is currently an open problem to prove that for a general choice of $\overline{r}$, a single such lift r exists (equivalently, to show for each $\overline{r}$ that at least one of Kisin's deformation rings is nonzero). Some results in this direction can be found in the Ph.D. thesis of Alain Muller [Mul13]. This note sheds little further light on this question, but rather investigates the question of congruences between de Rham

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representations of different Hodge and inertial types; that is, in many of our results we suppose the existence of a single lift, and see what other lifts (of differing Hodge and inertial types) we can produce from this. The existence of congruences between representations of differing such types is conjecturally governed by the (generalised) Breuil–Mézard conjecture (at least for regular Hodge types; see [EG14]). This conjecture is almost completely open beyond the case of $\mathrm{GL}_2/\mathbb{Q}_p$, so it is of interest to prove unconditional results.

We prove several such results in this paper, by a variety of different methods. Some of our results make use of the notion of a *potentially diagonalisable* Galois representation, which was introduced in [BLGGT14], and is very important in automorphy lifting theorems. It is expected ([EG14, Conj. A.3]) that every $\overline{r}$ admits a potentially diagonalisable lift of regular weight[2], but this is at present known only if $n \leq 3$ or $\overline{r}$ is semisimple; see for example [CEG$^+$16, Lem. 2.2], and the proof of [Mul13, Prop. 2.5.7] for the case $n = 3$. It seems plausible that these arguments could be extended to cover other small dimensions, but the case of general n seems to be surprisingly difficult.

We recall that an n-dimensional de Rham representation of G_K is said to have Hodge type 0 if for any continuous embedding $K \hookrightarrow \overline{\mathbb{Q}}_p$ the corresponding Hodge–Tate weights are $0, 1, \ldots, n-1$; while if $K/\mathbb{Q}_p$ is unramified, a crystalline representation of G_K is said to be Fontaine–Laffaille if for each continuous embedding $K \hookrightarrow \overline{\mathbb{Q}}_p$ the corresponding Hodge–Tate weights are all contained in an interval of the form $[i, i + p - 2]$. We remark that we will normalise Hodge–Tate weights so that the cyclotomic character ε has Hodge–Tate weight -1.

Our first result is the following theorem, which will be used in forthcoming work of Arias de Reyna and Dieulefait.

Theorem A. (Cor. 2.3.4) *Suppose that $K/\mathbb{Q}_p$ is unramified, and fix an integer $n \geq 1$. Then there is a finite extension K'/K, depending only on n and K, with the following property: if $\overline{r} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ has a Fontaine–Laffaille lift, then it also has a potentially diagonalisable lift $r : G_K \to \mathrm{GL}_n(\overline{\mathbb{Z}}_p)$ of Hodge type 0 with the property that $r|_{G_{K'}}$ is crystalline.*

In fact this is a special case of a result (Cor. 2.1.11) that holds for a more general class of representations $\overline{r}$ that we call *peu ramifiée*, and with no assumption that the finite extension $K/\mathbb{Q}_p$ is unramified. We expect that this result should even be true without the assumption that $\overline{r}$ is peu ramifiée, but we do not know how to prove this; indeed, as mentioned above, we do not know how to produce a single de Rham lift in general!

To explain why this result is reasonable, and to give some indication of the proof, we focus on the case that $K = \mathbb{Q}_p$ and $n = 2$. Assume for simplicity in the following discussion that $p > 2$. One way to see that we should expect the result to be true (at least if we remove "potentially diagonalisable" from the statement) is that it is then the local Galois analogue of the well-known

[2]Recall that a de Rham representation of G_K is said to have *regular* weight if for any continuous embedding $K \hookrightarrow \overline{\mathbb{Q}}_p$, the corresponding Hodge–Tate weights are all distinct.

statement that every modular eigenform of level prime to p is congruent to one of weight 2 and bounded level at p. Indeed, via the mechanism of modularity lifting theorems and potential modularity, it is possible to turn this analogy into a proof. (See Theorem C below. Since all potentially Barsotti–Tate representations are known to be potentially diagonalisable, this literally proves Theorem A in this case, but this deduction cannot be made if $n > 2$.)

Since these global methods are (at least at present) unable to handle the case $n > 2$, a local approach is needed, which we again motivate via the case $K = \mathbb{Q}_p$ and $n = 2$. The possible $\overline{r} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ are well-understood; they are either irreducible representations, in which case they are induced from characters of the unramified quadratic extension $\mathbb{Q}_{p^2}$ of $\mathbb{Q}_p$, or they are reducible, and are extensions of unramified twists of powers of the mod p cyclotomic character ω. In the first case, the representations are induced from characters of $G_{\mathbb{Q}_{p^2}}$ which become unramified after restriction to any totally ramified extension of degree $p^2 - 1$, and it is straightforward to produce the required lifts by considering inductions of potentially crystalline characters of $G_{\mathbb{Q}_{p^2}}$ which become crystalline over such an extension; see Lemma 2.1.12. Such representations are automatically potentially diagonalisable, as after restriction to some finite extension they are even a direct sum of crystalline characters.

This leaves the case that $\overline{r}$ is reducible. After twisting, we may assume that $\overline{r}$ is an extension of an unramified twist of ω^{-i} by the trivial character, for some $0 \le i \le p-2$. Then the natural way to lift to characteristic zero and Hodge type 0 is to try to lift to an extension of an unramified twist of $\varepsilon^{-1}\widetilde{\omega}^{1-i}$ by the trivial character, where $\widetilde{\omega}$ is the Teichmüller lift of ω; this is promising because any such extension is at least potentially semistable, and becomes semistable over $\mathbb{Q}_p(\zeta_p)$ (which is in particular independent of the specific reducible $\overline{r}$ under consideration), and if it is potentially crystalline, then it is also potentially diagonalisable (as it is known that any successive extension of characters which is potentially crystalline is also potentially diagonalisable).

The problem of producing such lifts is one of Galois cohomology, and Tate's duality theorems show that when $i \neq 1$ there is no obstruction to lifting.[3] It is also easy to check that in this case the lifts are automatically potentially crystalline. However, when $i = 1$ the situation is more complicated. Then one can check that très ramifiée extensions of ω^{-1} by the trivial character do not lift to extensions of a non-trivial unramified twist of ε^{-1} by the trivial character, but only lift to semistable non-crystalline extensions of ε^{-1} by the trivial character. However, this is the only obstruction to carrying out the strategy in this case; and in fact, since très ramifiée representations do not have Fontaine–Laffaille lifts, the result also follows in the case $i = 1$.

We prove Theorem A by a generalisation of this strategy: we write $\overline{r}$ as an extension of irreducible representations, lift the irreducible representations as

[3] This is true even for the ramified self-extensions of the trivial character in the case $i = 0$, which are not Fontaine–Laffaille, although they are peu ramifiée in the sense of this paper (Definition 2.1.3).

inductions of crystalline characters, and then lift the extension classes. However, the issues that arose in the previous paragraph in the case $i = 1$ are more complicated in general. To address this, we make use of the following observation: in the case considered in the previous paragraphs (that is, $K = \mathbb{Q}_p$, $n = 2$, and $\overline{r}$ has a trivial subrepresentation), if $\overline{r}$ is not très ramifiée then it admits "many" reducible crystalline lifts; indeed, it can be lifted as an extension by the trivial character of *any* unramified twist of ε^{-i} that lifts the corresponding character mod p.

This freedom to twist by unramified characters is in marked contrast to the behaviour in the très ramifiée case, and can be exploited in the Galois cohomology calculations used to produce the potentially crystalline lifts of Hodge type 0. Motivated by these observations, we introduce a generalisation (Definition 2.1.3) of the classical notion of peu ramifiée representations, and we prove by direct Galois cohomology arguments that the peu ramifiée condition allows great flexibility in the production of lifts to varying reducible representations (see Theorem 2.1.8 and Corollary 2.1.11).

Conversely, every representation that admits enough lifts of the sort promised by Theorem 2.1.8 must in fact be peu ramifiée (see Proposition 2.2.4 for a precise statement); such a representation is said to admit "highly twisted lifts." We show that representations that admit Fontaine–Laffaille lifts also admit highly twisted lifts (Proposition 2.3.1), and so deduce that Corollary 2.1.11 applies whenever the residual representation is Fontaine–Laffaille. Theorem A follows.

Using roughly the same purely local methods, we additionally prove the following.

Theorem B. (Cor. 2.1.13) *Suppose that* $\overline{r} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ *is peu ramifiée. Then* $\overline{r}$ *has a crystalline lift of some Serre weight* (*in the sense of Section 1.2.4*).

In contrast to these relatively concrete local arguments, in Section 3 we use global methods, and in particular the potential automorphy machinery of [BLGGT14]. Our first result is the following, which takes as input a potentially crystalline lift that could have highly ramified inertial type, or highly spread out Hodge–Tate weights, and produces a crystalline lift of small Hodge–Tate weights.

Theorem C. (Thm. 3.1.2) *Suppose that* $p \nmid 2n$, *and that* $\overline{r} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ *has a potentially diagonalisable lift of some regular weight. Then the following hold.*

(1) *There exists a finite extension* K'/K (*depending only on* n *and* K, *and not on* $\overline{r}$) *such that* $\overline{r}$ *has a lift* $r : G_K \to \mathrm{GL}_n(\overline{\mathbb{Z}}_p)$ *of Hodge type* 0 *that becomes crystalline over* K'.
(2) $\overline{r}$ *has a crystalline lift of some Serre weight.*

The first part of this result should be contrasted with Theorem A above, while the second part should be contrasted with Theorem B. For instance, we remark that it follows from Theorem A (or more precisely, from its more general statement for peu ramifiée representations) that every peu ramifiée representation $\overline{r}$ admits a potentially diagonalisable lift of some regular weight, whereas this latter condition on $\overline{r}$ is an input to Theorem C.

If $K/\mathbb{Q}_p$ is unramified and $\overline{r}$ admits a lift of extended FL weight (see Section 1.2.4 for this terminology), we also show the following "weak Breuil–Mézard result".

Theorem D. (Thm. 3.1.5) *Suppose that $p \neq n$, that $K/\mathbb{Q}_p$ is unramified, and that $\overline{r} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ has a crystalline lift of some extended FL weight F. If F is a Jordan–Hölder factor of $\overline{\sigma}(\lambda, \tau)$ for some λ, τ, then $\overline{r}$ has a potentially crystalline lift of type (λ, τ).*

Since there is no restriction on λ or τ, this result seems to be well beyond anything that can currently be proved directly using integral p-adic Hodge theory.

If we knew that all potentially crystalline lifts were potentially diagonalisable, then the special case of Theorem A in which the given Fontaine–Laffaille lift is regular would be an easy consequence of part (1) of Theorem C (note that the existence of a regular Fontaine–Laffaille lift implies that $p > n$). However, we do not know how to prove that general potentially crystalline representations are potentially diagonalisable (and we do not have any strong evidence that it should be true).

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1.2. Notation and conventions. Fix a prime p, and let $K/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_K$. Write G_K for the absolute Galois group of K, I_K for the inertia subgroup of G_K, and $\mathrm{Frob}_K \in G_K$ for a choice of geometric Frobenius. All representations of G_K are assumed without further comment to be continuous. Write v_K for the p-adic valuation on K taking the value 1 on a uniformiser of K, as well as for the unique extension of this valuation to any algebraic extension of K.

1.2.1. *Inertial types.* An *inertial type* is a representation $\tau : I_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ with open kernel which extends to the Weil group W_K. We say that a de Rham representation $r : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ has inertial type τ if the restriction to I_K of the Weil–Deligne representation $\mathrm{WD}(r)$ associated to r is equivalent to τ. Given an inertial type τ, there is a (not necessarily unique) finite-dimensional smooth irreducible $\overline{\mathbb{Q}}_p$-representation $\sigma(\tau)$ of $\mathrm{GL}_n(\mathcal{O}_K)$ associated to τ by the "inertial local Langlands correspondence", which we normalise as in [EG14, Conj. 4.1.3]. (Note that there is an unfortunate difference in conventions between this and that of [EG14, Thm. 4.1.5], but it is this normalisation that is used in the remainder of [EG14].) We can and do suppose that $\sigma(\tau)$ is defined over $\overline{\mathbb{Z}}_p$.

1.2.2. *Hodge–Tate weights and Hodge types.* If W is a de Rham representation of G_K over $\overline{\mathbb{Q}}_p$, and $\kappa : K \hookrightarrow \overline{\mathbb{Q}}_p$, then we will write $\mathrm{HT}_\kappa(W)$ for the multiset of Hodge–Tate weights of W with respect to κ. By definition, the multiset $\mathrm{HT}_\kappa(W)$ contains i with multiplicity $\dim_{\overline{\mathbb{Q}}_p}(W \otimes_{\kappa,K} \widehat{\overline{K}}(i))^{G_K}$. Thus for example if ε denotes the p-adic cyclotomic character of G_K, then $\mathrm{HT}_\kappa(\varepsilon) = \{-1\}$ for all κ.

We say that W has *regular* Hodge–Tate weights if for each κ, the elements of $\mathrm{HT}_\kappa(W)$ are pairwise distinct. Let $\mathbb{Z}^n_+$ denote the set of tuples $(\lambda_1, \dots, \lambda_n)$ of integers with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then if W has regular Hodge–Tate weights, there is a unique $\lambda = (\lambda_{\kappa,i}) \in (\mathbb{Z}^n_+)^{\mathrm{Hom}_{\mathbb{Q}_p}(K,\overline{\mathbb{Q}}_p)}$ such that for each $\kappa : K \hookrightarrow \overline{\mathbb{Q}}_p$,

$$\mathrm{HT}_\kappa(W) = \{\lambda_{\kappa,1} + n - 1, \lambda_{\kappa,2} + n - 2, \dots, \lambda_{\kappa,n}\},$$

and we say that W is regular of *Hodge type* λ.

1.2.3. *Representations of* GL_n *and Serre weights.* For any $\lambda \in \mathbb{Z}^n_+$, view λ as a dominant weight (with respect to the upper triangular Borel subgroup) of the algebraic group GL_n in the usual way, and let M'_λ be the algebraic $\mathcal{O}_K$-representation of GL_n given by

$$M'_\lambda := \mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(w_0\lambda)_{/\mathcal{O}_K}$$

where B_n is the Borel subgroup of upper-triangular matrices of GL_n, and w_0 is the longest element of the Weyl group (see [Jan03] for more details of these notions, and note that M'_λ has highest weight λ). Write M_λ for the $\mathcal{O}_K$-representation of $\mathrm{GL}_n(\mathcal{O}_K)$ obtained by evaluating M'_λ on $\mathcal{O}_K$. For any $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}_{\mathbb{Q}_p}(K,\overline{\mathbb{Q}}_p)}$ we write L_λ for the $\overline{\mathbb{Z}}_p$-representation of $\mathrm{GL}_n(\mathcal{O}_K)$ defined by

$$L_\lambda := \otimes_{\kappa:K\hookrightarrow\overline{\mathbb{Q}}_p} M_{\lambda_\kappa} \otimes_{\mathcal{O}_K,\kappa} \overline{\mathbb{Z}}_p.$$

Let k be the residue field of K. We call isomorphism classes of irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_n(k)$ *Serre weights*; they can be parameterised as follows. We say that an element (a_i) of $\mathbb{Z}^n_+$ is *p-restricted* if $p - 1 \geq a_i - a_{i+1}$ for all $1 \leq i \leq n-1$, and we write $X_1^{(n)}$ for the set of p-restricted elements. Given any $a \in X_1^{(n)}$, we define the k-representation P_a of $\mathrm{GL}_n(k)$ to be the representation obtained by evaluating $\mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(w_0 a)_{/k}$ on k, and let N_a be the irreducible sub-k-representation of P_a generated by the highest weight vector (that this is indeed irreducible follows from the analogous result for the algebraic group GL_n, cf. II.2.2–II.2.6 in [Jan03], and the appendix to [Her09]).

If $a = (a_{\overline{\kappa},i}) \in (X_1^{(n)})^{\mathrm{Hom}(k,\overline{\mathbb{F}}_p)}$, write $a_{\overline{\kappa}}$ for the component of a indexed by $\overline{\kappa} \in \mathrm{Hom}(k, \overline{\mathbb{F}}_p)$. If $a \in (X_1^{(n)})^{\mathrm{Hom}(k,\overline{\mathbb{F}}_p)}$ then we define an irreducible $\overline{\mathbb{F}}_p$-representation F_a of $\mathrm{GL}_n(k)$ by

$$F_a := \otimes_{\overline{\kappa}\in\mathrm{Hom}(k,\overline{\mathbb{F}}_p)} N_{a_{\overline{\kappa}}} \otimes_{k,\overline{\kappa}} \overline{\mathbb{F}}_p.$$

The representations F_a are irreducible, and every Serre weight is (isomorphic to one) of the form F_a for some a. The choice of a is not unique: one has $F_a \cong F_{a'}$ if and only if there exist integers $x_{\overline{\kappa}}$ such that $a_{\overline{\kappa},i} - a'_{\overline{\kappa},i} = x_{\overline{\kappa}}$ for all

$\overline{\kappa}, i$ and, for any labeling $\overline{\kappa}_j$ of the elements of $\mathrm{Hom}(k, \overline{\mathbb{F}}_p)$ such that $\overline{\kappa}_j^p = \overline{\kappa}_{j+1}$ we have $\sum_{j=0}^{f-1} p^j x_{\overline{\kappa}_j} \equiv 0 \pmod{p^f - 1}$, where $f = [k : \mathbb{F}_p]$. In this case we write $a \sim a'$.

We remark that if $K/\mathbb{Q}_p$ is unramified and $a \in (X_1^{(n)})^{\mathrm{Hom}(k,\overline{\mathbb{F}}_p)}$ satisfies $a_{\overline{\kappa},1} - a_{\overline{\kappa},n} \leq p - (n-1)$ for each $\overline{\kappa}$, then $L_a \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p \cong F_a$ as representations of $\mathrm{GL}_n(\mathcal{O}_K)$. The reason is that $P_b = N_b$ whenever $b \in \mathbb{Z}_+^n$ satisfies $b_1 - b_n \leq p - (n-1)$ (cf. [Jan03, II.5.6]).

1.2.4. *Potentially crystalline representations.* An element $\lambda \in (\mathbb{Z}_+^n)^{\mathrm{Hom}_{\mathbb{Q}_p}(K,\overline{\mathbb{Q}}_p)}$ is said to be a *lift* of an element $a \in (X_1^{(n)})^{\mathrm{Hom}(k,\overline{\mathbb{F}}_p)}$ if for each $\overline{\kappa} \in \mathrm{Hom}(k, \overline{\mathbb{F}}_p)$ there exists $\kappa_{\overline{\kappa}} \in \mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p)$ lifting $\overline{\kappa}$ such that $\lambda_{\kappa_{\overline{\kappa}}} = a_{\overline{\kappa}}$, and $\lambda_{\kappa'} = 0$ for all other $\kappa' \neq \kappa_{\overline{\kappa}}$ in $\mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p)$ lifting $\overline{\kappa}$. If λ is a lift of a, then F_a is a Jordan–Hölder factor of $L_\lambda \otimes \overline{\mathbb{F}}_p$.

Given a pair (λ, τ), we say that a potentially crystalline representation W of G_K over $\overline{\mathbb{Q}}_p$ has type (λ, τ) if it is regular of Hodge type λ, and has inertial type τ. Write $\sigma(\lambda, \tau)$ for $L_\lambda \otimes_{\overline{\mathbb{Z}}_p} \sigma(\tau)$, a $\overline{\mathbb{Z}}_p$-representation of $\mathrm{GL}_n(\mathcal{O}_K)$, and write $\overline{\sigma}(\lambda, \tau)$ for the semisimplification of $\sigma(\lambda, \tau) \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p$. Then the action of $\mathrm{GL}_n(\mathcal{O}_K)$ on $\overline{\sigma}(\lambda, \tau)$ factors through $\mathrm{GL}_n(k)$, so that the Jordan–Hölder factors of $\overline{\sigma}(\lambda, \tau)$ are Serre weights.

If $\overline{r} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ has a crystalline lift W of type $(\lambda, 1)$ (that is, W is crystalline of Hodge type λ), and λ is a lift of some $a \in (X_1^{(n)})^{\mathrm{Hom}(k,\overline{\mathbb{F}}_p)}$, then we say that $\overline{r}$ has a *crystalline lift of Serre weight* F_a. This terminology is sensible because the existence of a crystalline lift of Hodge type λ for some lift λ of a does not depend on the choice of the element a in its equivalence class under the equivalence relation $\sim$ (cf. [GHS15, Lem. 7.1.1]).

If furthermore $K/\mathbb{Q}_p$ is unramified, and $a_{\overline{\kappa},1} - a_{\overline{\kappa},n} \leq p - 1 - n$ for all $\overline{\kappa}$, then we say that a (or F_a) is an *FL weight*, and that $\overline{r}$ *has a crystalline lift of FL weight* F_a. If instead $a_{\overline{\kappa},1} - a_{\overline{\kappa},n} \leq p - n$ for all $\overline{\kappa}$, then we say that a (or F_a) is an *extended FL weight*, and that $\overline{r}$ has a *crystalline lift of extended FL weight* F_a.

1.2.5. *Potential diagonalisability.* Following [BLGGT14], we say that a potentially crystalline representation $r : G_K \to \mathrm{GL}_n(\overline{\mathbb{Z}}_p)$ with distinct Hodge–Tate weights is *potentially diagonalisable* if for some finite extension K'/K, $r|_{G_{K'}}$ is crystalline, and the corresponding $\overline{\mathbb{Q}}_p$ point of the corresponding crystalline deformation ring lies on the same irreducible component as some direct sum of crystalline characters. (For example, it follows from the main theorem of [GL14] that any crystalline representation of extended FL weight is potentially diagonalisable.)

2. Local existence of lifts in the residually Fontaine–Laffaille case

2.1. Peu ramifiée representations. Recall that for any discrete G_K-module X, the space $H^1_{\mathrm{ur}}(G_K, X)$ of unramified classes in $H^1(G_K, X)$ is the kernel of the restriction map $H^1(G_K, X) \to H^1(I_K, X)$; by the inflation-restriction sequence, this is the same as the image of the inflation map $H^1(G_K/I_K, X^{I_K}) \hookrightarrow H^1(G_K, X)$. We will make use of the following well-known fact.

Lemma 2.1.1. *Suppose that X is a discrete G_K-module that is moreover a finite-dimensional vector space over a field $\mathbb{F}$. Then*

$$\dim_{\mathbb{F}} H^1_{\mathrm{ur}}(G_K, X) = \dim_{\mathbb{F}} H^0(G_K, X).$$

Proof. We have

$$\dim_{\mathbb{F}} H^1(G_K/I_K, X^{I_K}) = \dim_{\mathbb{F}} H^0(G_K/I_K, X^{I_K}) = \dim_{\mathbb{F}} H^0(G_K, X),$$

the first equality coming from the fact that $H^i(G_K/I_K, X^{I_K})$ for $i = 0, 1$ are, respectively, the invariants and co-invariants of X^{I_K} under $\mathrm{Frob}_K - 1$. $\square$

Definition 2.1.2. Suppose that $K/\mathbb{Q}_p$ is a finite extension and $\mathbb{F}$ is a field of characteristic p. Consider a representation $\overline{r} : G_K \to \mathrm{GL}_n(\mathbb{F})$, let $\overline{V}$ be the underlying $\mathbb{F}[G_K]$-module of $\overline{r}$, and suppose that $0 = \overline{U}_0 \subset \overline{U}_1 \subset \cdots \subset \overline{U}_\ell = \overline{V}$ is an increasing filtration on $\overline{V}$ by $\mathbb{F}[G_K]$-submodules. Write $\overline{V}_i := \overline{U}_i/\overline{U}_{i-1}$. We say that $\overline{r}$ is *peu ramifiée with respect to the filtration* $\{\overline{U}_i\}$ if for all $1 \le i \le \ell$ the class in $H^1(G_K, \mathrm{Hom}_{\mathbb{F}}(\overline{V}_i, \overline{U}_{i-1}))$ defined by $\overline{U}_i$ (regarded as an extension of $\overline{V}_i$ by $\overline{U}_{i-1}$) is annihilated under Tate local duality by $H^1_{\mathrm{ur}}(G_K, \mathrm{Hom}_{\mathbb{F}}(\overline{U}_{i-1}, \overline{V}_i(1)))$.

Since group cohomology is compatible with base change for field extensions, so is Definition 2.1.2: that is, if $\mathbb{F}'/\mathbb{F}$ is any field extension, then $\overline{r}$ is peu ramifiée with respect to some filtration $\{\overline{U}_i\}$ if and only if $\overline{r} \otimes_{\mathbb{F}} \mathbb{F}'$ is peu ramifiée with respect to the filtration $\{\overline{U}_i \otimes_{\mathbb{F}} \mathbb{F}'\}$.
Definition 2.1.2 is most interesting in the case where the filtration $\{\overline{U}_i\}$ is *saturated*, i.e., where the graded pieces $\overline{V}_i$ are irreducible. (For instance, any $\overline{r}$ will trivially be peu ramifiée with respect to the one-step filtration $0 = \overline{U}_0 \subset \overline{U}_1 = \overline{V}$.) This motivates the following further definition.

Definition 2.1.3. We say that $\overline{r}$ is *peu ramifiée* if there exists a saturated filtration $\{\overline{U}_i\}$ with respect to which $\overline{r}$ is peu ramifiée as in Definition 2.1.2.

Examples 2.1.4.

(1) If $n = 2$ and $\overline{r} \cong \begin{pmatrix} \chi\omega & * \\ 0 & \chi \end{pmatrix}$ for some character χ, then Definition 2.1.3 coincides with the usual definition of peu ramifiée. (Recall that ω denotes the mod p cyclotomic character.) Indeed, the duality pairing $H^1(G_K, \mathbb{F}_p(1)) \times H^1(G_K, \mathbb{F}_p) \to \mathbb{Q}_p/\mathbb{Z}_p$ can be identified (via the Kummer and Artin maps) with the evaluation map

$$K^\times/(K^\times)^p \times \mathrm{Hom}(K^\times, \mathbb{F}_p) \to \mathbb{F}_p \hookrightarrow \mathbb{Q}_p/\mathbb{Z}_p,$$

from which it is immediate that the classes in $H^1(G_K, \mathbb{F}_p(1))$ that are annihilated by $H^1_{\mathrm{ur}}(G_K, \mathbb{F}_p)$ are precisely those which are identified with $\mathcal{O}_K^\times/(\mathcal{O}_K^\times)^p$ by the Kummer map.

(2) If $\overline{r}$ is semisimple then trivially $\overline{r}$ is peu ramifiée.

(3) If there are no nontrivial G_K-maps $\overline{U}_{i-1} \to \overline{V}_i(1)$ for any i (e.g. if one has $\overline{V}_j \not\cong \overline{V}_i(1)$ for all $j < i$) then $\overline{r}$ is necessarily peu ramifiée because by Lemma 2.1.1 we have $H^1_{\mathrm{ur}}(G_K, \mathrm{Hom}_{\mathbb{F}}(\overline{U}_{i-1}, \overline{V}_i(1))) = 0$.

(4) Suppose $K/\mathbb{Q}_p$ is unramified. We will prove in Section 2.3 that Fontaine–Laffaille representations are peu ramifiée, so that all of the main results in this section will apply to Fontaine–Laffaille representations.

Example 2.1.5. If $\overline{r}$ is peu ramifiée, it is natural to ask whether $\overline{r}$ is peu ramifiée with respect to every (saturated) filtration on $\overline{r}$. This is not the case. Suppose, for instance, that K does not contain the p-th roots of unity (so $\omega \neq 1$) and

$$\overline{r} \cong \begin{pmatrix} \omega & *_1 & *_2 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

where the class of the cocycle $*_1$ is nontrivial and peu ramifiée, and the cocycle $*_2$ is très ramifiée. For the filtration on $\overline{r}$ in which $\overline{U}_i$ is the span of the first i vectors giving rise to the above matrix representation (so that the action of $G_{\mathbb{Q}_p}$ on $\overline{U}_i$ is given by the upper-left $i \times i$ block), the representation $\overline{r}$ is peu ramifiée. This is clear at the first two steps in the filtration, and for the third step one notes (as in Example 2.1.4(3)) that there are no nontrivial maps $\overline{U}_2 \to \overline{V}_3(1)$.

On the other hand, if one defines a new filtration on $\overline{r}$ by replacing $\overline{U}_2$ with the span of the first and third basis vectors giving rise to the above matrix representation, then $\overline{r}$ is not peu ramifiée with respect to the new filtration, because the new $\overline{U}_2$ is très ramifiée.

Remark 2.1.6. One consequence of the preceding example is that the collection of peu ramifiée representations is not closed under taking arbitrary subquotients. On the other hand, if $\overline{r}$ is peu ramifiée with respect to the filtration $\{\overline{U}_i\}$, then for any $a \leq b$ it is not difficult to check that $\overline{U}_b/\overline{U}_a$ is peu ramifiée with respect to the induced filtration $\{\overline{U}_{a+i}/\overline{U}_a\}_{0 \leq i \leq b-a}$.

Using the preceding example one can similarly see that the collection of peu ramifiée representations is not closed under contragredients.

Remark 2.1.7. In some sense we are making an arbitrary choice by demanding that we first lift $\overline{U}_1$, then to $\overline{U}_2$, then to $\overline{U}_3$, and so forth. One could equally well lift in other orders, and as Example 2.1.5 shows, this can make a difference. However, since Definition 2.1.2 will suffice for our purposes, we do not elaborate further on this point.

We say that a $\overline{\mathbb{Z}}_p$-*lift* of an $\overline{\mathbb{F}}_p[G_K]$-module $\overline{V}$ is a $\overline{\mathbb{Z}}_p[G_K]$-module V that is free as a $\overline{\mathbb{Z}}_p$-module, together with a $\overline{\mathbb{F}}_p[G_K]$-isomorphism $V \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p \cong \overline{V}$. We have introduced the notion of a peu ramifiée representation (Definition 2.1.2) in order to prove the following result, to the effect that peu ramifiée representations have many $\overline{\mathbb{Z}}_p$-lifts.

Theorem 2.1.8. *Suppose that $K/\mathbb{Q}_p$ is a finite extension. Consider a representation $\overline{r} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ that is peu ramifiée with respect to the increasing filtration $\{\overline{U}_i\}$, so that $\overline{r}$ may be written as*

$$\overline{r} = \begin{pmatrix} \overline{V}_1 & \dots & * \\ & \ddots & \vdots \\ & & \overline{V}_\ell \end{pmatrix},$$

where the $\overline{V}_i := \overline{U}_i/\overline{U}_{i-1}$ are the graded pieces of the filtration.
For each i, suppose that we are given a $\overline{\mathbb{Z}}_p$-representation V_i of G_K lifting $\overline{V}_i$. Then there exist unramified characters $\psi_1, \dots, \psi_\ell$ with trivial reduction such that $\overline{r}$ may be lifted to a representation r of the form

$$r = \begin{pmatrix} V_1 \otimes \psi_1 & \dots & * \\ & \ddots & \vdots \\ & & V_\ell \otimes \psi_\ell \end{pmatrix}.$$

More precisely, r is equipped with an increasing filtration $\{U_i\}$ by $\overline{\mathbb{Z}}_p$-direct summands such that $U_i/U_{i-1} \cong V_i \otimes \psi_i$ and $r \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p \cong \overline{r}$ induces $U_i \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p \cong \overline{U}_i$, for each $1 \le i \le \ell$.
In fact, there are infinitely many choices of characters $(\psi_1, \dots, \psi_\ell)$ for which this is true, in the strong sense that for any $1 \le i \le \ell$, if $(\psi_1, \dots, \psi_{i-1})$ can be extended to an ℓ-tuple of characters for which such a lift exists, then there are infinitely many choices of ψ_i such that $(\psi_1, \dots, \psi_i)$ can also be extended to such an ℓ-tuple.

Proof. We proceed by induction on ℓ, the case $\ell = 1$ being trivial. From the induction hypothesis, we can find $\psi_1, \dots, \psi_{\ell-1}$ so that $\overline{U} := \overline{U}_{\ell-1}$ can be lifted to some

$$U := \begin{pmatrix} V_1 \otimes \psi_1 & \dots & * \\ & \ddots & \vdots \\ & & V_{\ell-1} \otimes \psi_{\ell-1} \end{pmatrix}.$$

as in the statement of the theorem. It suffices to prove that for each such choice of $\psi_1, \dots, \psi_{\ell-1}$, there exist infinitely many choices of ψ_ℓ for which $\overline{r}$ lifts to an extension of $V_\ell \otimes \psi_\ell$ by U as in the statement of the theorem.
Choose the field $E/\mathbb{Q}_p$ large enough so that U and V_ℓ are realisable over $\mathcal{O}_E$, and so that $\overline{r}$ is realisable over the residue field of E. Suppose that F/E is a finite extension with ramification degree $e(F/E) > (\dim \overline{V}_\ell)(\dim \overline{U})$, write $\mathcal{O}$ for the integers of F and $\mathbb{F}$ for its residue field, and let $\psi : G_K \to \mathcal{O}^\times$ be an unramified character such that $0 < v_E(\psi(\mathrm{Frob}_K) - 1) < 1/(\dim \overline{V}_\ell)(\dim \overline{U})$. In the remainder of this argument, when we write U and V_ℓ we will mean

their (chosen) realisations over $\mathcal{O}$, and similarly $\overline{U}$ and $\overline{V}_\ell$ will mean their realisations over $\mathbb{F}$ obtained by reduction from U and V_ℓ.

Extensions of $V_\ell \otimes \psi$ by U correspond to elements of $H^1(G_K, \mathrm{Hom}_{\mathcal{O}}(V_\ell \otimes \psi, U))$, while $\overline{r}$ corresponds to an element c of $\mathrm{Ext}^1_{\mathbb{F}[G_K]}(\overline{V}_\ell, \overline{U})$, which we identify with $H^1(G_K, \mathrm{Hom}_{\mathbb{F}}(\overline{V}_\ell, \overline{U}))$. By hypothesis (together with the remark about base change immediately following Definition 2.1.2) the class c is annihilated by $H^1_{\mathrm{ur}}(G_K, \mathrm{Hom}_{\mathbb{F}}(\overline{U}, \overline{V}_\ell(1)))$ under Tate local duality. Taking the cohomology of the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}}(V_\ell \otimes_{\mathcal{O}} \psi, U) \xrightarrow{\varpi} \mathrm{Hom}_{\mathcal{O}}(V_\ell \otimes_{\mathcal{O}} \psi, U) \to \mathrm{Hom}_{\mathbb{F}}(\overline{V}_\ell, \overline{U}),$$

we have in particular an exact sequence

$$H^1(G_K, \mathrm{Hom}_{\mathcal{O}}(V_\ell \otimes_{\mathcal{O}} \psi, U)) \to H^1(G_K, \mathrm{Hom}_{\mathbb{F}}(\overline{V}_\ell, \overline{U})) \xrightarrow{\delta} \\ \xrightarrow{\delta} H^2(G_K, \mathrm{Hom}_{\mathcal{O}}(V_\ell \otimes_{\mathcal{O}} \psi, U)),$$

so it is enough to check that that $c \in \ker(\delta)$ except for finitely many choices of ψ.

From Tate duality, we have the dual map

$$H^0(G_K, \mathrm{Hom}_{\mathcal{O}}(U, V_\ell(1) \otimes_{\mathcal{O}} \psi) \otimes F/\mathcal{O}) \quad \xrightarrow{\delta^\vee} \quad H^1(G_K, \mathrm{Hom}_{\mathbb{F}}(\overline{U}, \overline{V}_\ell(1)).$$

As $\ker(\delta)^\perp = \mathrm{im}(\delta^\vee)$, it is enough to show that $\mathrm{im}(\delta^\vee)$ is contained in $H^1_{\mathrm{ur}}(G_K, \mathrm{Hom}_{\mathbb{F}}(\overline{U}, \overline{V}_\ell(1)))$ except, again, for possibly finitely many choices of ψ. Letting $X = \mathrm{Hom}_{\mathcal{O}}(U, V_\ell(1))$, we first claim that $(X \otimes_{\mathcal{O}} \mathcal{O}(\psi))^{G_K} = 0$ for all but finitely many choices of ψ. Indeed, if

$$(X \otimes_{\mathcal{O}} \mathcal{O}(\psi))^{G_K} = \mathrm{Hom}_{\mathcal{O}[G_K]}(U, V_\ell(1) \otimes_{\mathcal{O}} \psi) \neq 0$$

then we must have $W \cong Z(1) \otimes_{\mathcal{O}} \psi$ for some Jordan–Hölder factor W of U and Z of V_ℓ. This can happen for only finitely many choices of ψ (by determinant considerations applied to each of the finitely many pairs W, Z). Now we are done by the following proposition. $\square$

PROPOSITION 2.1.9. *Let $F/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$. Let X be an $\mathcal{O}[G_K]$-module that is free of finite rank as an $\mathcal{O}$-module. Suppose that there is a field lying E lying between F and $\mathbb{Q}_p$ such that X is realisable over $\mathcal{O}_E$ and with ramification index $e(F/E) > \mathrm{rank}_{\mathcal{O}}(X)$. Let $\psi : G_K \to \mathcal{O}^\times$ be an unramified character such that $0 < v_E(\psi(\mathrm{Frob}_K) - 1) < 1/\mathrm{rank}_{\mathcal{O}}(X)$.*

Assume further that $(X \otimes_{\mathcal{O}} \mathcal{O}(\psi))^{G_K} = 0$. Then the image of

$$\delta^\vee : H^0(G_K, (X \otimes_{\mathcal{O}} \mathcal{O}(\psi)) \otimes_{\mathcal{O}} F/\mathcal{O}) \to H^1(G_K, X \otimes_{\mathcal{O}} \mathbb{F})$$

is equal to the subspace of unramified classes, and in particular depends only on $X \otimes_{\mathcal{O}} \mathbb{F}$, and not on X, F, or ψ.

Proof. The statement is unchanged upon replacing E with the maximal unramified extension E^{ur} of E contained in F. We are therefore reduced to the case where F/E is totally ramified (so that in particular $\mathbb{F}$ is also the residue field of E).

Let $X_{\mathcal{O}_E}$ be a realisation of X over $\mathcal{O}_E$. Write $\overline{X} = X_{\mathcal{O}_E} \otimes_{\mathcal{O}_E} \mathbb{F} = X \otimes_{\mathcal{O}} \mathbb{F}$ and $X_\psi = X_{\mathcal{O}_E} \otimes_{\mathcal{O}_E} \mathcal{O}(\psi)$. The inclusion $\iota : X_{\mathcal{O}_E} \hookrightarrow X_\psi$ sending $x \mapsto x \otimes 1$ is a map of $\mathcal{O}_E$-modules inducing an isomorphism of $\mathbb{F}[G_K]$-modules $\overline{X} \cong X_\psi \otimes_{\mathcal{O}} \mathbb{F}$. Moreover for any $g \in G_K$ and $x \in X_{\mathcal{O}_E}$ we have $g \cdot \iota(x) = \psi(g)(\iota(g \cdot x))$, so that the map ι is at least I_K-linear.

Define $\alpha = \psi(\mathrm{Frob}_K)^{-1} - 1$ and write $N = \mathrm{Frob}_K - 1$, which acts on $\overline{X}^{I_K}$ with kernel $\ker(N) = \overline{X}^{G_K}$. We have an isomorphism

$$H^1(G_K/I_K, \overline{X}^{I_K}) \cong \overline{X}^{I_K}/N\overline{X}^{I_K}$$

induced by evaluation at Frob_K. Note that any class in this quotient space has a representative in $\cup_{i=0}^{\infty} \ker(N^i)$, as can be seen for example by writing $\overline{X}^{I_K} = \overline{Y} \oplus \overline{Z}$ with N nilpotent on $\overline{Y}$ and invertible on $\overline{Z}$. Hence to see that the image of $\delta^\vee$ contains all unramified classes, it suffices to exhibit for $\overline{f} \in \cup_{i=0}^{\infty} \ker(N^i)$ an element $e_{\overline{f}} \in (X_\psi \otimes_{\mathcal{O}} F/\mathcal{O})^{G_K}$ such that $\delta^\vee(e_{\overline{f}}) = [\overline{f}]$ in $H^1(G_K/I_K, \overline{X}^{I_K})$.

Suppose then that $\overline{f} \in \cup_{i=0}^{\infty} \ker(N^i)$ is nonzero. Let $i \geq 0$ be the largest integer such that $N^i\overline{f} \neq 0$, and let $\overline{f}_i := \overline{f}$. For each $0 \leq j \leq i$ let $f_j \in X_{\mathcal{O}_E}$ be a lift of $N^{i-j}\overline{f}_i$, and define

$$f^* = \sum_{j=0}^{i} \alpha^j \cdot \iota(f_j) \in X_\psi.$$

Since $\overline{f}_j \in \overline{X}^{I_K}$, it follows that for $g \in I_K$ we have $g(f_j) \equiv f_j \pmod{\varpi_E X_{\mathcal{O}_E}}$ with $\varpi_E \in \mathcal{O}_E$ a uniformiser, and so also $g(f^*) \equiv f^* \pmod{\varpi_E X_\psi}$.

Now let us compute $(\mathrm{Frob}_K - 1)(f^*)$. Noting that $(\mathrm{Frob}_K - 1)f_j \equiv f_{j-1} \pmod{\varpi_E X_{\mathcal{O}_E}}$, with $f_{-1} := 0$, and recalling that $(1+\alpha)(\mathrm{Frob}_K \cdot \iota(x)) = \iota(\mathrm{Frob}_K \cdot x)$, we have

$$\begin{aligned}
(1+\alpha)(\mathrm{Frob}_K(f^*) - f^*) &= \sum_{j=0}^{i} \alpha^j \iota(\mathrm{Frob}_K(f_j)) - (1+\alpha)\sum_{j=0}^{i} \alpha^j \iota(f_j) \\
&= \sum_{j=0}^{i} \alpha^j \iota((\mathrm{Frob}_K - 1)f_j) - \sum_{j=0}^{i} \alpha^{j+1}\iota(f_j) \\
&\equiv \sum_{j=0}^{i} \alpha^j \iota(f_{j-1}) - \sum_{j=0}^{i} \alpha^{j+1}\iota(f_j) \pmod{\varpi_E X_\psi} \\
&\equiv -\alpha^{i+1}\iota(f_i) \pmod{\varpi_E X_\psi}.
\end{aligned}$$

Note that $N^{i+1}\overline{f}_i = 0$ and $N^i\overline{f}_i \neq 0$, so that $i+1 \leq \dim_{\mathbb{F}} \overline{X}^{I_K} \leq \mathrm{rank}_{\mathcal{O}} X$. Therefore $v_E(\alpha^{i+1}) < 1$, and we deduce that $g(f^*) \equiv f^* \pmod{\alpha^{i+1} X_\psi}$ for all $g \in G_K$, or in other words $f^* \otimes \alpha^{-i-1} \in (X_\psi \otimes F/\mathcal{O})^{G_K}$.

Furthermore, if $c_{\overline{f}} := \delta^\vee(f^* \otimes \alpha^{-i-1}) \in H^1(G_K, \overline{X})$, then $c_{\overline{f}}(g)$ is by definition the image in $\overline{X}$ of $\alpha^{-i-1}(g(f^*) - f^*)$. So on the one hand $c_{\overline{f}}$ is unramified (because $v_E(\alpha^{i+1}) < v_E(\varpi_E) = 1$), while on the other hand

$c_{\overline{f}}(\mathrm{Frob}_K) = -\overline{f}_i$. Thus we can take $e_{\overline{f}} := -f^* \otimes \alpha^{-i-1}$, and we have shown that $H^1(G_K/I_K, \overline{X}^{I_K}) \subset \operatorname{im}\delta^\vee$.
On the other hand, since $X_\psi^{G_K}$ is assumed to be trivial, we have that $(X_\psi \otimes F/\mathcal{O})^{G_K}$ is of finite length; so if ϖ_F is a uniformiser of F, then

$$\begin{aligned}\dim(\operatorname{im}\delta^\vee) &= \dim((X_\psi \otimes F/\mathcal{O})^{G_K}/\varpi_F)\\ &= \dim((X_\psi \otimes F/\mathcal{O})^{G_K}[\varpi_F]) = \dim \overline{X}^{G_K} = \dim(\ker N).\end{aligned}$$

On the other hand $\dim(\ker N) = \dim(\operatorname{coker} N) = \dim H^1(G_K/I_K, \overline{X}^{I_K})$, and the result follows. □

Theorem 2.1.8 implies the following result on the existence of certain potentially crystalline Galois representations.

Proposition 2.1.10. *Suppose that $K/\mathbb{Q}_p$ is a finite extension. Consider a representation $\overline{r} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ that is peu ramifiée with respect to the increasing filtration $\{\overline{U}_i\}$, so that $\overline{r}$ may be written as*

$$\overline{r} = \begin{pmatrix} \overline{V}_1 & \dots & * \\ & \ddots & \vdots \\ & & \overline{V}_\ell \end{pmatrix},$$

where the $\overline{V}_i := \overline{U}_i/\overline{U}_{i-1}$ are the graded pieces of the filtration.
For each i, suppose that we are given a $\overline{\mathbb{Z}}_p$-representation V_i of G_K lifting $\overline{V}_i$ such that:

- *each V_i is potentially crystalline, and*
- *for each $1 \le i < \ell$ and each $\kappa : K \hookrightarrow \overline{\mathbb{Q}}_p$, every element of $\mathrm{HT}_\kappa(V_{i+1})$ is strictly greater than every element of $\mathrm{HT}_\kappa(V_i)$.*

Then $\overline{r}$ may be lifted to a potentially crystalline representation r of the form

$$r = \begin{pmatrix} V_1 \otimes \psi_1 & \dots & * \\ & \ddots & \vdots \\ & & V_\ell \otimes \psi_\ell \end{pmatrix},$$

where each ψ_i is an unramified character with trivial reduction, and if K'/K is a finite extension such that each $V_i|_{G_{K'}}$ is crystalline, then $r|_{G_{K'}}$ is also crystalline.
In fact, there are infinitely many choices of characters $(\psi_1, \dots, \psi_\ell)$ for which this is true, in the strong sense that for any $1 \le i \le \ell$, if $(\psi_1, \dots, \psi_{i-1})$ can be extended to an ℓ-tuple of characters for which such a lift exists, then there are infinitely many choices of ψ_i such that $(\psi_1, \dots, \psi_i)$ can also be extended to such an ℓ-tuple.

Proof. This follows from Theorem 2.1.8 along with standard facts about extensions of de Rham representations. Indeed, by [Nek93, Prop. 1.28(2)] and our assumption on the Hodge–Tate weights of the V_i, the representation $r|_{G_{K'}}$ is semistable for any r as in Theorem 2.1.8 and any K' as above. Then by repeated application of the third part of [Nek93, Prop. 1.24(2)], as well as [Nek93, Prop. 1.26], this semistable representation is guaranteed to be crystalline as long as there is no $G_{K'}$-equivariant surjection $(V_j \otimes V_i^*)(\psi_j \psi_i^{-1}) \twoheadrightarrow \varepsilon$ for any $j < i$. Once $\psi_1, \ldots, \psi_{i-1}$ have been determined, this can be arranged by avoiding finitely many possibilities for ψ_i. $\square$

We give two sample applications of Proposition 2.1.10. The following Corollary will be used in forthcoming work of Arias de Reyna and Dieulefait (in the special case where $\overline{r}$ is Fontaine–Laffaille and the Hodge type λ is 0).

Corollary 2.1.11. *Fix an integer $n \geq 1$. Then there is a finite extension K'/K, depending only on n, with the following property: if $\overline{r} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is peu ramifiée and $\lambda = (\lambda_{\kappa,i}) \in (\mathbb{Z}^n_+)^{\mathrm{Hom}_{\mathbb{Q}_p}(K,\overline{\mathbb{Q}}_p)}$, then $\overline{r}$ has a potentially diagonalisable lift $r : G_K \to \mathrm{GL}_n(\overline{\mathbb{Z}}_p)$ that is regular of Hodge type λ, with the property that $r|_{G_{K'}}$ is crystalline.*

Proof. Write $\overline{r}$ as in Proposition 2.1.10 with $\overline{V}_i$ irreducible for all i, and set $d_i = \dim_{\overline{\mathbb{F}}_p} \overline{U}_i$. By Proposition 2.1.10 and [BLGGT14, Lem. 1.4.3], it is enough to show that there is a finite extension K'/K depending only on n, with the property that we may lift each $\overline{V}_i$ to a potentially crystalline representation V_i, such that for all i, κ the set $\mathrm{HT}_\kappa(V_i)$ is equal to $\{\lambda_{\kappa,n-j} + j : j \in [d_{i-1}, d_i - 1]\}$, with the additional property that $V_i|_{G_{K'}}$ is isomorphic to a direct sum of crystalline characters. This is immediate from Lemma 2.1.12 below. $\square$

Lemma 2.1.12. *Let $d \geq 1$ be an integer. Let K_d be the unramified extension of K of degree d, and define L to be any totally ramified extension of K_d of degree $|k_d^\times|$, where k_d is the residue field of K_d. Let $\overline{r} : G_K \to \mathrm{GL}_d(\overline{\mathbb{F}}_p)$ be an irreducible representation. Then for any collection of multisets of d integers $\{h_{\kappa,1}, \ldots, h_{\kappa,d}\}$, one for each continuous embedding $\kappa : K \hookrightarrow \overline{\mathbb{Q}}_p$, there is a lift of $\overline{r}$ to a representation $r : G_K \to \mathrm{GL}_d(\overline{\mathbb{Z}}_p)$, such that $r|_{G_L}$ is isomorphic to a direct sum of crystalline characters, and for each κ we have $\mathrm{HT}_\kappa(r) = \{h_{\kappa,1}, \ldots, h_{\kappa,d}\}$.*

Proof. Since $\overline{r}$ is irreducible, we can write $\overline{r} \cong \mathrm{Ind}_{G_{K_d}}^{G_K} \overline{\psi}$, and $\overline{\psi} : G_{K_d} \to \overline{\mathbb{F}}_p^\times$ is a character. Choose a crystalline character $\chi : G_{K_d} \to \overline{\mathbb{Q}}_p^\times$ with the property that for each continuous embedding $\kappa : K \hookrightarrow \overline{\mathbb{Q}}_p$ we have

$$\bigcup_{\tilde{\kappa}|_K = \kappa} \mathrm{HT}_{\tilde{\kappa}}(\chi) = \{h_{\kappa,1}, \ldots, h_{\kappa,d}\},$$

where the union is taken as multisets. (That such a character exists is well-known; see e.g. [Ser79, §2.3, Cor. 2].) If we let $\theta : G_{K_d} \to \overline{\mathbb{Z}}_p^\times$ be the Teichmüller lift of $\overline{\psi}\overline{\chi}^{-1}$, then we may take $r := \mathrm{Ind}_{G_{K_d}}^{G_K} \chi\theta$, which has the correct Hodge–Tate weights by [GHS15, Cor. 7.1.3]. (Note that ${}^g\theta|_{G_L}$ is unramified for any $g \in G_K$.) $\square$

As a second application of Proposition 2.1.10, we show that each peu ramifié representation has a crystalline lift of some Serre weight.

Corollary 2.1.13. *Suppose that $K/\mathbb{Q}_p$ is a finite extension, and that $\overline{r} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is peu ramifiée. Then $\overline{r}$ has a crystalline lift of some Serre weight.*

Proof. When $\overline{r}$ is irreducible, this is straightforward from [GHS15, Thm B.1.1]. (One only has to note that when $\overline{r}$ is irreducible, an obvious lift of $\overline{r}$ in the terminology of [GHS15, §7] is always an unramified twist of a true lift of $\overline{r}$.)
In the general case, suppose that $\overline{r}$ is peu ramifiée with respect to the filtration $\{\overline{U}_i\}$, and as usual set $\overline{V}_i := \overline{U}_i/\overline{U}_{i-1}$. By the previous paragraph, for each $\overline{V}_i$ we are able to choose a crystalline lift V_i of some Serre weight. By an argument as in the fourth paragraph of the proof of [GHS15, Thm B.1.1] it is possible to arrange that every element of $\mathrm{HT}_\kappa(V_{i+1})$ is strictly greater than every element of $\mathrm{HT}_\kappa(V_i)$, and that $\oplus_i V_i$ is a crystalline lift of $\oplus_i \overline{V}_i$ of some Serre weight. (This is just a matter of replacing each V_i with a twist by a suitably-chosen crystalline character of trivial reduction.) Now the Corollary follows directly from Proposition 2.1.10 (with $K' = K$). $\square$

2.2. Highly twisted lifts. In this section we give a criterion (Proposition 2.2.4) for checking that a representation is peu ramifiée, which we will apply to show in Section 2.3 that Fontaine–Laffaille representations are peu ramifiée.

Definition 2.2.1. Suppose that $K/\mathbb{Q}_p$ is a finite extension. Consider a representation $\overline{r} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, let $\overline{V}$ be the underlying $\overline{\mathbb{F}}_p[G_K]$-module of $\overline{r}$, and suppose that $0 = \overline{U}_0 \subset \overline{U}_1 \subset \cdots \subset \overline{U}_\ell = \overline{V}$ is an increasing filtration on $\overline{V}$ by $\overline{\mathbb{F}}_p[G_K]$-submodules. Denote $\overline{V}_i := \overline{U}_i/\overline{U}_{i-1}$ for $1 \le i \le \ell$, the graded pieces of the filtration.
We say that $\overline{r}$ *admits highly twisted lifts with respect to the filtration* $\{\overline{U}_i\}$ if there exist $\overline{\mathbb{Z}}_p$-lifts V_i of the $\overline{V}_i$, and a family of $\overline{\mathbb{Z}}_p$-lifts $V(\psi_1, \ldots, \psi_\ell)$ of $\overline{V}$ indexed by a nonempty set Ψ of ℓ-tuples of unramified characters $\psi_i : G_K \to \overline{\mathbb{Z}}_p^\times$ with trivial reduction modulo $\mathfrak{m}_{\overline{\mathbb{Z}}_p}$, having the following additional properties:

- Each $V(\psi_1, \ldots, \psi_\ell)$ is equipped with an increasing filtration $\{U(\psi_1, \ldots, \psi_\ell)_i\}$ by $\overline{\mathbb{Z}}_p[G_K]$-submodules that are $\overline{\mathbb{Z}}_p$-direct summands.
- We have $U(\psi_1, \ldots, \psi_\ell)_i/U(\psi_1, \ldots, \psi_\ell)_{i-1} \cong V_i \otimes \psi_i$ for each $1 \le i \le \ell$.
- The isomorphism $V(\psi_1, \ldots, \psi_\ell) \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p \cong \overline{V}$ induces isomorphisms $U(\psi_1, \ldots, \psi_\ell)_i \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p \cong \overline{U}_i$ for each $1 \le i \le \ell$.
- $U(\psi_1, \ldots, \psi_\ell)_i$ depends up to isomorphism only on $\psi_1, \ldots, \psi_i$ (that is, it does not depend on $\psi_{i+1}, \ldots, \psi_\ell$).
- For each $(\psi_1, \ldots, \psi_i)$ that extends to an element of Ψ and for each $\epsilon > 0$, there exists ψ_{i+1} such that $(\psi_1, \ldots, \psi_{i+1})$ extends to an element of Ψ, with the further property that $0 < v_{\mathbb{Q}_p}(\psi_{i+1}(\mathrm{Frob}_K) - 1) < \epsilon$.

If moreover the set Ψ can be taken to be the set of all ℓ-tuples of unramified characters $\psi_i : G_K \to \overline{\mathbb{Z}}_p^\times$ with trivial reduction modulo $\mathfrak{m}_{\overline{\mathbb{Z}}_p}$, we say that $\overline{r}$ admits universally twisted lifts with respect to the filtration $\{\overline{U}_i\}$.

As with Definition 2.1.2, the preceding definition is most interesting in the case where the filtration $\{\overline{U}_i\}$ is saturated, and so we make the following further definition.

Definition 2.2.2. We say that $\overline{r}$ *admits highly* (*resp. universally*) *twisted lifts* if it admits highly (resp. universally) twisted lifts as in Definition 2.2.1 with respect to some saturated filtration.

Remark 2.2.3. It is natural to ask whether, if $\overline{r}$ admits highly (resp. universally) twisted lifts with respect to some saturated filtration as in Definition 2.2.2, it admits highly (resp. universally) twisted lifts with respect to any such filtration. Proposition 2.2.4 below, in combination with Example 2.1.5, gives a negative answer to this question in the highly twisted case.
In fact, Example 2.1.5 also shows that the above question has a negative answer in the universally twisted case. Suppose for simplicity that $K/\mathbb{Q}_p$ is unramified and that $p > 2$. Then $\overline{r}$ in Example 2.1.5 admits universally twisted lifts for the first filtration considered there. To see this, we first note that the first block $\overline{U}_2 = \begin{pmatrix} \omega & *_1 \\ & 1 \end{pmatrix}$ admits universally twisted lifts for $V_1 = \varepsilon$ and $V_2 = 1$ by Proposition 2.3.1 below, because $\overline{U}_2$ is Fontaine–Laffaille. Since there is no nontrivial map $\overline{U}_2 \to \overline{V}_3(1)$, one easily checks that $\overline{r}$ admits universally twisted lifts for this filtration. However, $\overline{r}$ does not admit universally twisted lifts for the second filtration considered in Example 2.1.5. This is because the first block $\overline{U}'_2 = \begin{pmatrix} \omega & *_2 \\ & 1 \end{pmatrix}$ does not admit universally twisted lifts (e.g. by Proposition 2.2.4).

Proposition 2.2.4. *Let $K/\mathbb{Q}_p$ be a finite extension, and let $\{\overline{U}_i\}$ be an increasing filtration on the representation $\overline{r} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$. Then $\overline{r}$ is peu ramifiée with respect to $\{\overline{U}_i\}$ if and only if it admits highly twisted lifts with respect to $\{\overline{U}_i\}$.*

Proof. An inspection of the proof of Theorem 2.1.8 already gives the "only if" implication (for *any* choice of V_i's lifting $\overline{V}_i$).
For the other direction, we assume that $\overline{r}$ admits highly twisted lifts with respect to the filtration $\{\overline{U}_i\}$ and some $\overline{\mathbb{Z}}_p$-lifts V_i of the $\overline{V}_i$. We proceed by induction on ℓ, the length of the filtration. By the induction hypothesis we may assume that for all $i < \ell$ the class in $H^1(G_K, \mathrm{Hom}_{\mathbb{F}}(\overline{V}_i, \overline{U}_{i-1}))$ defined by $\overline{U}_i$ is annihilated under Tate local duality by $H^1_{\mathrm{ur}}(G_K, \mathrm{Hom}_{\mathbb{F}}(\overline{U}_{i-1}, \overline{V}_i(1)))$, and it remains to prove this for $i = \ell$. Choose any $(\psi_1, \ldots, \psi_{\ell-1})$ that extends to an element of the set Ψ (as in Definition 2.2.1 for $\overline{r}$), and let $U := U(\psi_1, \ldots, \psi_\ell)_{\ell-1}$ (where ψ_ℓ is any character such that $(\psi_1, \ldots, \psi_\ell) \in \Psi$); note that this is independent of ψ_ℓ.

Let $\mathcal{S}$ be the set of characters $\psi : G_K \to \overline{\mathbb{Z}}_p^\times$ such that $(\mathrm{Hom}_{\overline{\mathbb{Z}}_p}(U, V_\ell(1)) \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Z}}_p(\psi))^{G_K} \neq 0$. As in the proof of Theorem 2.1.8 we see that $\mathcal{S}$ is finite. Let $E/\mathbb{Q}_p$ be a finite extension such that U and V_ℓ are realisable over $\mathcal{O}_E$. It follows from the highly twisted lift condition on $\overline{r}$ that there exists ψ_ℓ having the following properties:

(i) $(\psi_1, \dots, \psi_\ell) \in \Psi$,
(ii) $\psi_\ell \notin \mathcal{S}$, and
(iii) $0 < v_E(\psi_\ell(\mathrm{Frob}_K) - 1) < 1/(\dim \overline{U}_{\ell-1})(\dim \overline{V}_\ell)$.

Let F/E be a finite extension over which ψ_ℓ and $V(\psi_1, \dots, \psi_\ell)$ are both realisable. Write $\mathcal{O}$ for the ring of integers of F, and $\mathbb{F}$ for its residue field. For the remainder of this proof, when we write U, V_ℓ, ψ_ℓ we will mean their chosen realisations over F, and similarly for $\overline{U}$, $\overline{V}_\ell$ over $\mathbb{F}$ (obtained by reduction).
Set $X = \mathrm{Hom}_{\mathcal{O}}(U, V_\ell(1))$. As in the proof of Theorem 2.1.8, write δ for the connection map

$$H^1(G_K, \mathrm{Hom}_{\mathbb{F}}(\overline{V}_\ell, \overline{U})) \xrightarrow{\delta} H^2(G_K, \mathrm{Hom}_{\mathcal{O}}(V_\ell \otimes_{\mathcal{O}} \psi_\ell, U)).$$

The existence of the lift $V(\psi_1, \dots, \psi_\ell)$ (i.e. the property (i) of ψ_ℓ) shows that the class $c \in H^1(G_K, \mathrm{Hom}_{\mathbb{F}}(\overline{V}_\ell, \overline{U}))$ defining $\overline{r}$ lies in $\ker(\delta)$. On the other hand, the properties (ii) and (iii) of ψ_ℓ mean that Proposition 2.1.9 applies (with ψ_ℓ playing the role of ψ) to show that the dual map $\delta^\vee$ has image $H^1_{\mathrm{ur}}(G_K, X \otimes_{\mathcal{O}} \mathbb{F})$. Since $c \in \ker(\delta)$ it is annihilated under Tate local duality by this image, and we deduce that $\overline{r}$ is peu ramifiée. □

COROLLARY 2.2.5. *Suppose that $\overline{r}$ admits highly twisted lifts with respect to the filtration $\{\overline{U}_i\}$. Then $\overline{r}$ satisfies the definition of admitting highly twisted lifts with respect to the filtration $\{\overline{U}_i\}$ for* any *lifts V_i of the $\overline{V}_i$.*

Proof. This is immediate from Proposition 2.2.4 along with the first sentence of its proof. □

Remark 2.2.6. The above corollary fails if we replace 'highly twisted' with 'universally twisted'. For instance, consider Example 2.1.4(1) with $K/\mathbb{Q}_p$ unramified, the extension class $*$ peu ramifiée, and $\chi = 1$. It admits universally twisted lifts if we set $V_1 = \varepsilon$ and $V_2 = 1$. (This will follow from Proposition 2.3.1 below.) But it does not admit universally twisted lifts for $V_1 = \varepsilon^p$ and $V_2 = 1$.

Remark 2.2.7. We do not know whether there exist representations that admit highly twisted lifts but not universally twisted lifts.

2.3. FONTAINE–LAFFAILLE REPRESENTATIONS. In this section we will prove that representations which admit a Fontaine–Laffaille lift also admit universally twisted lifts, and so by Proposition 2.2.4 are peu ramifiée. We begin by recalling the formulation of unipotent Fontaine–Laffaille theory in [DFG04, §1.1.2]. Throughout this section let $K/\mathbb{Q}_p$ be a finite unramified extension with integer ring $\mathcal{O}_K$, and write Frob_p for the absolute geometric Frobenius on K.

Let $\mathcal{O}$ be the ring of integers in E, a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}$. We assume that E is sufficiently large as to contain the image of some (hence any) continuous embedding of K into an algebraic closure of E. Fix an integer $0 \le h \le p-1$, and let $\mathcal{MF}^h_{\mathcal{O}}$ denote the category of finitely generated $\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{O}$-modules M together with

- a decreasing filtration $\mathrm{Fil}^s M$ by $\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{O}$-submodules which are $\mathcal{O}_K$-direct summands with $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{h+1} M = \{0\}$;
- and $\mathrm{Frob}_p^{-1} \otimes 1$-linear maps $\Phi^s : \mathrm{Fil}^s M \to M$ with $\Phi^s|_{\mathrm{Fil}^{s+1} M} = p\Phi^{s+1}$ and $\sum_s \Phi^s(\mathrm{Fil}^s M) = M$.

We say that an object M of $\mathcal{MF}^{p-1}_{\mathcal{O}}$ is *étale* if $\mathrm{Fil}^{p-1} M = M$, and define $\mathcal{MF}^{p-1,u}_{\mathcal{O}}$ to be the full subcategory of $\mathcal{MF}^{p-1}_{\mathcal{O}}$ consisting of objects with no nonzero étale quotients. Such objects are said to be *unipotent*. Note that $\mathcal{MF}^{p-2}_{\mathcal{O}}$ is a subcategory of $\mathcal{MF}^{p-1,u}_{\mathcal{O}}$.

In the following paragraphs, let $\mathcal{MF}_{\mathcal{O}}$ denote either $\mathcal{MF}^h_{\mathcal{O}}$ (for $0 \le h \le p-2$) or $\mathcal{MF}^{p-1,u}_{\mathcal{O}}$ (for $h = p-1$). Let $\mathrm{Rep}_{\mathcal{O}}(G_K)$ denote the category of finitely generated $\mathcal{O}$-modules with a continuous G_K-action. There is an exact, fully faithful, covariant functor of $\mathcal{O}$-linear categories $T_K : \mathcal{MF}_{\mathcal{O}} \to \mathrm{Rep}_{\mathcal{O}}(G_K)$. This is the functor denoted $\mathbb{V}$ in [DFG04, §1.1.2]. The essential image of T_K is closed under taking subquotients. If M is an object of $\mathcal{MF}_{\mathcal{O}}$, then the length of M as an $\mathcal{O}$-module is $[K : \mathbb{Q}_p]$ times the length of $T_K(M)$ as an $\mathcal{O}$-module. Let $\mathcal{MF}_{\mathbb{F}}$ denote the full subcategory of $\mathcal{MF}_{\mathcal{O}}$ consisting of objects killed by the maximal ideal of $\mathcal{O}$ and let $\mathrm{Rep}_{\mathbb{F}}(G_K)$ denote the category of finite $\mathbb{F}$-modules with a continuous G_K-action. Then T_K restricts to a functor $\mathcal{MF}_{\mathbb{F}} \to \mathrm{Rep}_{\mathbb{F}}(G_K)$. If M is an object of $\mathcal{MF}_{\mathbb{F}}$ and κ is a continuous embedding $K \hookrightarrow \overline{\mathbb{Q}}_p$, we let $\mathrm{FL}_\kappa(M)$ denote the multiset of integers i such that $\mathrm{gr}^i M \otimes_{\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{O}, \kappa \otimes 1} \mathcal{O} \neq \{0\}$ and i is counted with multiplicity equal to the $\mathbb{F}$-dimension of this space. If M is a p-torsion free object of $\mathcal{MF}_{\mathcal{O}}$ then $T_K(M) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline and for every continuous embedding $\kappa : K \hookrightarrow \overline{\mathbb{Q}}_p$ we have

$$\mathrm{HT}_\kappa(T_K(M) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = \mathrm{FL}_\kappa(M \otimes_{\mathcal{O}} \mathbb{F}).$$

Moreover, if Λ is a G_K-invariant lattice in a crystalline representation V of G_K with all its Hodge–Tate numbers in the range $[0, h]$, having (when $h = p-1$) no nontrivial quotient isomorphic to a twist of an unramified representation by $\varepsilon^{-(p-1)}$, then Λ is in the essential image of T_K. If some twist of $\overline{r} : G_K \to \mathrm{GL}_n(\mathbb{F})$ lies in the essential image of T_K on $\mathcal{MF}^{p-2}_{\mathcal{O}}$, we say that $\overline{r}$ *admits a Fontaine–Laffaille lift*, while if some twist of $\overline{r}$ lies in the essential image of T_K on $\mathcal{MF}^{p-1,u}_{\mathcal{O}}$ we say that it *admits a unipotent extended Fontaine–Laffaille lift*.

The proof of the following result is essentially the same as that of [BLGGT14, Lem. 1.4.2]. (We remark that [BLGGT14, §1.4] uses the formulation of Fontaine–Laffaille theory as [CHT08, §2.4.1], which in fact is equivalent to that of [DFG04, §1.1.2] (at least on $\mathcal{MF}^{p-2}_{\mathcal{O}}$), although this equivalence is not needed for the following argument.)

Proposition 2.3.1. *Let $K/\mathbb{Q}_p$ be unramified. Consider a representation $\overline{r} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ with an increasing filtration $\{\overline{U}_i\}$ such that $\overline{U}_0 = 0$ and $\overline{U}_\ell = \overline{r}$, so that $\overline{r}$ may be written as*

$$\overline{r} = \begin{pmatrix} \overline{V}_1 & \cdots & * \\ & \ddots & \vdots \\ & & \overline{V}_\ell \end{pmatrix},$$

where the $\overline{V}_i = \overline{U}_i/\overline{U}_{i-1}$ are the graded pieces of the filtration.
Suppose that $\overline{r}$ admits a Fontaine–Laffaille (resp. unipotent extended Fontaine–Laffaille) lift. Then $\overline{r}$ admits universally twisted lifts with respect to the filtration $\{\overline{U}_i\}$; indeed, it admits universally twisted Fontaine–Laffaille (resp. unipotent extended Fontaine–Laffaille) lifts. In either case $\overline{r}$ is peu ramifiée.

Remark 2.3.2. By duality, the same result holds when $\overline{r}$ admits a *nilpotent* extended Fontaine–Laffaille lift, i.e., if some twist of $\overline{r}$ lies in the essential image of T_K on $\mathcal{MF}_{\mathcal{O}}^{p-1,n}$, the full subcategory of $\mathcal{MF}_{\mathcal{O}}^{p-1}$ whose objects admit no nonzero subobject M with $\mathrm{Fil}^1 M = 0$. We refer the reader to [GL14] for a further discussion of nilpotent Fontaine–Laffaille theory.
Similar arguments (which we omit to keep the paper at a reasonable length) can be used to show that the same result holds when $\overline{r}$ is finite flat (for arbitrary $K/\mathbb{Q}_p$; in this case the argument uses Kisin modules).

Proof of Proposition 2.3.1. Since the truth of this proposition for $\overline{r}$ evidently implies its truth for any twist of $\overline{r}$ (using the fact that every character of G_K admits a crystalline lift), we reduce to the case that $\overline{r}$ lies in the essential image of T_K on $\mathcal{MF}_{\mathcal{O}}^{p-2}$ (or on $\mathcal{MF}_{\mathcal{O}}^{p-1,u}$, in the unipotent extended case).
The case that each $\overline{V}_i$ is one-dimensional is essentially found in [BLGGT14, Lem. 1.4.2] and, as previously remarked, we will follow the proof of that result closely. We can and do suppose that $\overline{r}$ is defined over some finite field $\mathbb{F}$, and we fix a finite extension E of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$.
Let $\overline{V}$ be the underlying $\mathbb{F}$-vector space of $\overline{r}$, and let $\overline{M}$ denote the object of $\mathcal{MF}_{\mathbb{F}}$ corresponding to $\overline{V}$, which exists by our assumption that $\overline{r}$ has a (possibly unipotent extended) Fontaine–Laffaille lift. Then we have a filtration

$$\overline{M} = \overline{M}_\ell \supset \overline{M}_{\ell-1} \supset \cdots \supset \overline{M}_1 \supset \overline{M}_0 = (0)$$

by $\mathcal{MF}_{\mathbb{F}}$-subobjects such that $\overline{M}_i$ corresponds to $\overline{U}_i$ and so $\overline{M}_i/\overline{M}_{i-1}$ corresponds to $\overline{V}_i$. Then we claim that we can find an object M of $\mathcal{MF}_{\mathcal{O}}$ which is p-torsion free together with a filtration by $\mathcal{MF}_{\mathcal{O}}$-subobjects

$$M = M_\ell \supset M_{\ell-1} \supset \cdots \supset M_1 \supset M_0 = (0)$$

and an isomorphism

$$M \otimes_{\mathcal{O}} \mathbb{F} \cong \overline{M}$$

under which $M_i \otimes_{\mathcal{O}} \mathbb{F}$ maps isomorphically to $\overline{M}_i$ for all i.
Write $d_i := \dim \overline{V}_i$. We note first that $\overline{M}$ has an $\mathbb{F}$-basis $\overline{e}_{i,\kappa}$ for $i = 1, \ldots, n$ and $\kappa \in \mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p)$ such that

- the residue field k of K acts on $\overline{e}_{i,\kappa}$ via κ;
- $\overline{M}_j$ is spanned over $\mathbb{F}$ by the $\overline{e}_{i,\kappa}$ for $i \leq d_1 + \cdots + d_j$;
- and for each j, s there is a subset $\Omega_{j,s} \subset \{1, \ldots, n\} \times \mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p)$ such that $\overline{M}_j \cap \mathrm{Fil}^s \overline{M}$ is spanned over $\mathbb{F}$ by the $\overline{e}_{i,\kappa}$ for $(i,\kappa) \in \Omega_{j,s}$.

(Such a basis is easily constructed recursively in j. The case $j = 1$ is trivial, and it is straightforward to extend a basis of this kind for $\overline{M}_{j-1}$ to one for $\overline{M}_j$.)
We put $\Omega_s := \Omega_{\ell,s}$.
Then we define M to be the free $\mathcal{O}$-module with basis $e_{i,\kappa}$ for $i = 1, \ldots, n$ and $\kappa \in \mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p)$.

- We let $\mathcal{O}_K$ act on $e_{i,\kappa}$ via κ;
- we define M_j to be the $\mathcal{O}$-submodule generated by the $e_{i,\kappa}$ with $i \leq d_1 + \cdots + d_j$;
- and we define $\mathrm{Fil}^s M$ to be the $\mathcal{O}$-submodule spanned by the $e_{i,\kappa}$ for $(i,\kappa) \in \Omega_s$.

We define $\Phi^s : \mathrm{Fil}^s M \to M$ by reverse induction on s. If we have defined Φ^{s+1} we define Φ^s as follows:

- If $(i,\kappa) \in \Omega_{s+1}$ then $\Phi^s e_{i,\kappa} = p\Phi^{s+1} e_{i,\kappa}$.
- If $(i,\kappa) \in \Omega_s - \Omega_{s+1}$ then $\Phi^s e_{i,\kappa}$ is chosen to be any lift of $\overline{\Phi}^s \overline{e}_{i,\kappa}$ in $\sum_{i' \leq d_1 + \cdots + d_j} \mathcal{O} \cdot e_{i',\kappa \circ \mathrm{Frob}_p}$, where j is minimal such that $i \leq d_1 + \cdots + d_j$.

It follows from Nakayama's lemma that M is an object of $\mathcal{MF}^h_{\mathcal{O}}$. When $h = p-1$, suppose that $M \to M'$ is a nontrivial étale quotient of M. We can without loss of generality replace M' with $M' \otimes_{\mathcal{O}} \mathbb{F}$; but then the map $M \to M'$ would factor through $\overline{M}$, contradicting the assumption that $\overline{M}$ is an object of $\mathcal{MF}_{\mathcal{O}}$ (and not just $\mathcal{MF}^{p-1}_{\mathcal{O}}$). It follows that M is also an object of $\mathcal{MF}_{\mathcal{O}}$. In the same way we see that $\{M_i\}$ is an increasing filtration of subobjects of M in $\mathcal{MF}_{\mathcal{O}}$.
It is immediate that M verifies the desired property that $M_i \otimes_{\mathcal{O}} \mathbb{F}$ maps isomorphically to $\overline{M}_i$ under the isomorphism $M \otimes_{\mathcal{O}} \mathbb{F} \cong \overline{M}$.
Set $V_i := T_K(M_i/M_{i-1}) \otimes_{\mathcal{O}} \overline{\mathbb{Z}}_p$. We claim that for this choice of V_i, the conditions of Definition 2.2.1 are satisfied. Since Fontaine–Laffaille theory is compatible in an obvious fashion with extension of scalars from E to a finite extension of E, we can and do suppose that the characters ψ_i are valued in $\mathcal{O}^\times$. Then the objects of $\mathcal{MF}_{\mathcal{O}}$ corresponding to the desired lifts $V(\psi_1, \ldots, \psi_\ell)$ are obtained from M by rescaling the maps Φ^s. More precisely, if we let $M(\psi_1, \ldots, \psi_\ell)$ be defined from M by rescaling $\Phi^s e_{i,\kappa}$ by $\psi_j(\mathrm{Frob}_K)$ for $(i,\kappa) \in \Omega_{j,s} \setminus \Omega_{j-1,s}$, then one can take $V(\psi_1, \ldots, \psi_\ell) = T_K(M(\psi_1, \ldots, \psi_\ell)) \otimes_{\mathcal{O}} \overline{\mathbb{Z}}_p$. (To establish the second bullet point in Definition 2.2.1, note from [DFG04, p. 670] that T_K is compatible with tensor products, and use (1) of *loc. cit.* to compute the Fontaine–Laffaille module corresponding to each ψ_i.) $\square$

Corollary 2.3.3. *Suppose that $\overline{r}$ admits a (possibly unipotent extended) Fontaine–Laffaille lift. Then the conclusions of Theorem 2.1.8 and Proposition 2.1.10 hold for $\overline{r}$ with respect to any separated, exhaustive increasing filtration $\{\overline{U}_i\}$ on $\overline{r}$.*

Corollary 2.3.4. *Suppose that $\overline{r}$ admits a (possibly unipotent extended) Fontaine–Laffaille lift. Then the conclusion of Corollary 2.1.11 holds for $\overline{r}$.*

The following result will be used in [GHS15].

Corollary 2.3.5. *Suppose that $\overline{r} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ admits a (possibly unipotent extended) Fontaine–Laffaille lift. Suppose also that*

$$\overline{r} = \begin{pmatrix} \overline{\chi}_1 & \cdots & * \\ & \ddots & \vdots \\ & & \overline{\chi}_n \end{pmatrix}.$$

Suppose that $h_1 > \cdots > h_n$ are integers such that $\overline{\chi}_i|_{I_{\mathbb{Q}_p}} = \omega^{h_i}$. Then $\overline{r}$ has a crystalline lift of the form

$$r = \begin{pmatrix} \chi_1 & \cdots & * \\ & \ddots & \vdots \\ & & \chi_n \end{pmatrix},$$

where $\chi_i|_{I_{\mathbb{Q}_p}} = \varepsilon^{h_i}$.

Proof. This is immediate from Corollary 2.3.3, taking the V_i to be appropriate unramified twists of ε^{h_i}. □

3. de Rham lifts by global methods

3.1. Potential automorphy and globalisation. In this section, we make use of (global) potential automorphy techniques to produce potentially crystalline lifts. Ultimately, these results rely on those of [BLGGT14], but the actual global results we need are those of [EG14, App. A].

The key idea is as follows: by the methods of [BLGGT14] and [Cal12], we can often realise $\overline{r} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ as the restriction to a decomposition group of $\overline{\rho}$, the reduction mod p of the p-adic Galois representation associated to an automorphic representation on some unitary group. Then the existence of congruences between automorphic representations of different weights and types produces lifts of $\overline{r}$ of the corresponding Hodge and inertial types.

To keep this paper from becoming longer than necessary, and to avoid obscuring the relatively simple arguments that we need to make, we will not recall the precise definitions of the spaces of automorphic forms that we work with; the details may be found in [EG14] (and the papers referenced therein). Suppose from now until the end of Lemma 3.1.1 that:

- $p \nmid 2n$, and
- $\overline{r}$ has a potentially diagonalisable lift of some type $(\lambda_{\overline{r}}, \tau_{\overline{r}})$.

Then in particular Conjecture A.3 of [EG14] holds for $\overline{r}$, so that by [EG14, Cor. A.7], there is a CM field F with maximal totally real field F^+, and an irreducible representation $\overline{\rho} : G_{F^+} \to \mathcal{G}_n(\overline{\mathbb{F}}_p)$ (where $\mathcal{G}_n$ is the algebraic group defined in [CHT08, §2.1]) which is automorphic in the sense of [EG14, Def. 5.3.1], and which globalises $\overline{r}$ in the sense that for each place $v \mid p$ of F^+ we have that v splits in F and that there is a place $\tilde{v}$ of F lying over v such that $F_{\tilde{v}} \cong K$ and $\overline{\rho}|_{G_{F_{\tilde{v}}}} \cong \overline{r}$. The above data will remain fixed throughout this section.

Suppose that for each place $v \mid p$ of F^+ we fix a representation of $\mathrm{GL}_n(\mathcal{O}_K)$ on a finite $\overline{\mathbb{Z}}_p$-module W_v. Via the isomorphisms $\iota_{\tilde{v}}$ of [EG14, §5.2], we can regard $W := \otimes_{\overline{\mathbb{Z}}_p, v|p} W_v$ as a representation of $G(\mathcal{O}_{F^+,p})$, where G is a certain unitary group. Then there is a space of algebraic modular forms $S(U, W)$, as in [EG14, §5.2]. (In fact, [EG14] works with coefficients in the ring of integers of some finite extension of $\mathbb{Q}_p$, rather than with $\overline{\mathbb{Z}}_p$-coefficients, but this makes no difference for the arguments we are making here.)

In particular, for any $(\lambda_v, \tau_v)_{v|p}$ a space of automorphic forms $S_{\lambda,\tau}(U, \overline{\mathbb{Z}}_p)$ is defined in [EG14, §5.2] for certain sufficiently small compact open subgroups $U \subset G(\mathbb{A}^{\infty}_{F^+})$ which are hyperspecial at p, corresponding to taking each W_v to be $\sigma(\lambda_v, \tau_v)$. Examining the proof of [EG14, Cor. A.7], we see that in fact we have $S_{\lambda_{\overline{r}}, \tau_{\overline{r}}}(U, \overline{\mathbb{Z}}_p)_{\mathfrak{m}} \neq 0$, where $\mathfrak{m}$ is as in [EG14, Def. 5.3.1], and in a mild abuse of notation we write $(\lambda_{\overline{r},v}, \tau_{\overline{r},v}) = (\lambda_{\overline{r}}, \tau_{\overline{r}})$ for all $v \mid p$. (This says that there is an automorphic representation π of weight $\lambda_{\overline{r}}$ and type $\tau_{\overline{r}}$, whose associated p-adic Galois representation ρ_π lifts $\overline{\rho}$; this representation ρ_π is the representation ρ constructed in [EG14, Lem. A.5].)

Lemma 3.1.1. *Keep the notation and assumptions of the preceding discussion.*
(1) *If for some choice of $(\lambda_v, \tau_v)_{v|p}$ we have $S_{\lambda,\tau}(U, \overline{\mathbb{Z}}_p)_{\mathfrak{m}} \neq 0$, then for each $v \mid p$, $\overline{r}$ has a potentially crystalline lift of type (λ_v, τ_v).*
(2) *$S_{\lambda,\tau}(U, \overline{\mathbb{Z}}_p)_{\mathfrak{m}} \neq 0$ if and only if there are Serre weights F_v of $\mathrm{GL}_n(k)$ such that*

- *$S(U, \otimes_{\overline{\mathbb{F}}_p, v|p} F_v)_{\mathfrak{m}} \neq 0$, and*
- *for all $v \mid p$, F_v is a Jordan–Hölder factor of $\overline{\sigma}(\lambda_v, \tau_v)$.*

Proof. (1) is immediate from [EG14, Prop. 5.3.2]. (We remind the reader that $S_{\lambda,\tau}(U, \overline{\mathbb{Z}}_p)$ is torsion-free.) In the case that τ is trivial, (2) is [BLGG15, Lem. 2.1.6], and the proof goes over unchanged to the general case. □

Theorem 3.1.2. *Suppose that $p \nmid 2n$, and that $\overline{r}$ has a potentially diagonalisable lift of some regular weight. Then the following hold.*

(1) *There exists a finite extension K'/K (depending only on n and K, and not on $\overline{r}$) such that $\overline{r}$ has a lift $r : G_K \to \mathrm{GL}_n(\overline{\mathbb{Z}}_p)$ of Hodge type 0 that becomes crystalline over K'.*
(2) *The representation $\overline{r}$ has a crystalline lift of some Serre weight.*

Proof. We begin with the proof of (2), since the argument is much shorter. Let r be the given potentially diagonalisable lift, and as above, write $(\lambda_{\overline{r}}, \tau_{\overline{r}})$

for the type of r. By Lemma 3.1.1(2), there are Jordan–Hölder factors F_{a_v} of $\overline{\sigma}(\lambda_{\overline{r}}, \tau_{\overline{r}})$ (possibly varying with v) such that $S(U, \otimes_{\overline{\mathbb{F}}_p, v|p} F_{a_v})_{\mathfrak{m}} \neq 0$. Let λ_v be a lift of a_v for each v, and let τ_v be trivial for each v. Applying Lemma 3.1.1(2) again, we see that $S_{\lambda,1}(U, \overline{\mathbb{Z}}_p)_{\mathfrak{m}} \neq 0$. By Lemma 3.1.1(1), $\overline{r}$ has a crystalline lift of Hodge type λ_v for each $v \mid p$; any such lift will do.

Turn now to (1). As in the previous part we get $S(U, V)_{\mathfrak{m}} \neq 0$ for some irreducible representation $V = \otimes_{v|p} V_v$ of $G(\mathcal{O}_{F^+} \otimes \mathbb{Z}_p)$ over $\overline{\mathbb{F}}_p$. Let $T \subset B \subset \mathrm{GL}_n$ denote the subgroups of diagonal and upper-triangular matrices, as algebraic groups over $\mathbb{Z}$. Consider V_v as a representation of $\mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}}}) =: K_v$ via $\iota_{\tilde{v}}$. Let $I_v \subset K_v$ denote the preimage of $B(k_{\tilde{v}}) \subset \mathrm{GL}_n(k_{\tilde{v}})$. Then we can choose a character $\overline{\chi}_v : I_v \to \overline{\mathbb{F}}_p^{\times}$ such that $V_v|_{I_v} \twoheadrightarrow \overline{\chi}_v$.

Let $q := \#k$. We claim that for any $s \geq 1$ such that $q^{s-1} \geq n$, we can find a (smooth) lift $\chi_v = \chi_{1,v} \otimes \cdots \otimes \chi_{n,v} : T(\mathcal{O}_{F_{\tilde{v}}}) \to \overline{\mathbb{Z}}_p^{\times}$ of $\overline{\chi}_v|_{T(\mathcal{O}_{F_{\tilde{v}}})} = \overline{\chi}_{1,v} \otimes \cdots \otimes \overline{\chi}_{n,v}$ such that the $\{\chi_{i,v}\}_{i=1}^n$ are pairwise distinct and $\chi_{i,v}|_{1+\varpi_{\tilde{v}}^s \mathcal{O}_{F_{\tilde{v}}}} = 1$ for all i. Indeed, recalling that $F_{\tilde{v}} \cong K$, write $\mathcal{O}_{F_{\tilde{v}}}^{\times}/(1 + \varpi_{\tilde{v}}^s \mathcal{O}_{F_{\tilde{v}}}) \cong k^{\times} \times H$ (via the Teichmüller splitting), where H is abelian of order q^{s-1}. Then each $\overline{\chi}_{i,v}|_{k^{\times}}$ lifts uniquely to $\overline{\mathbb{Z}}_p^{\times}$, whereas $\overline{\chi}_{i,v}|_H = 1$ and can be lifted arbitrarily to $\overline{\mathbb{Z}}_p^{\times}$. Hence it is enough to note that $\# \mathrm{Hom}(H, \overline{\mathbb{Z}}_p^{\times}) = \#H = q^{s-1} \geq n$, and this proves the claim. For the rest of the proof, we fix such a choice of s and χ_v.

Now, [Roc98, §3] (applied with a standard Chevalley basis such that $U_{\alpha,0} = U_\alpha \cap \mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}}})$ for all roots α) provides a pair $(J_{\chi_v}, \rho_{\chi_v})$ consisting of a compact open subgroup $J_{\chi_v} \subset I_v$ that contains $T(\mathcal{O}_{F_{\tilde{v}}})$ and a smooth character $\rho_{\chi_v} : J_{\chi_v} \to \overline{\mathbb{Z}}_p^{\times}$ such that $\rho_{\chi_v}|_{T(\mathcal{O}_{F_{\tilde{v}}})} = \chi_v$. By construction, $\overline{\rho}_{\chi_v}$ is the restriction of $\overline{\chi}_v$ to J_{χ_v}, so by Frobenius reciprocity we get a K_v-equivariant map $V_v \hookrightarrow \mathrm{Ind}_{J_{\chi_v}}^{K_v}(\overline{\rho}_{\chi_v})$.

In particular, $S\left(U, \otimes_{v|p}\left(\mathrm{Ind}_{J_{\chi_v}}^{K_v} \rho_{\chi_v}\right)\right)_{\mathfrak{m}} \neq 0$. Using Deligne–Serre lifting we get an automorphic representation π of $G(\mathbb{A}_{F^+})$ with associated Galois representation ρ_π lifting $\overline{\rho}|_{G_F} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ such that (i) $\pi_\infty \cong \mathbf{1}$ and (ii) $\mathrm{Hom}_{K_v}(\mathrm{Ind}_{J_{\chi_v}}^{K_v} \rho_{\chi_v}^{-1}, \pi_v) \neq 0$ (again via $\iota_{\tilde{v}}$) for any $v \mid p$. Applying [Roc98, Thm 7.7] (noting there are no restrictions on p, cf. [CHT08, Lem. 3.1.6]), we deduce that π_v is a subquotient of $\mathrm{Ind}_{B(F_{\tilde{v}})}^{G(F_{\tilde{v}})}(\tilde{\chi}_v^{-1})$ for some $\tilde{\chi}_v : T(F_{\tilde{v}}) \to \overline{\mathbb{Q}}_p^{\times}$ extending χ_v. (Note that $J_{\chi_v} = J_{\chi_v^{-1}}$.) Since the characters $\{\chi_{i,v}\}_{i=1}^n$ are pairwise distinct, the Bernstein–Zelevinsky irreducibility criterion implies that $\pi_v \cong \mathrm{Ind}_{B(F_{\tilde{v}})}^{G(F_{\tilde{v}})}(\tilde{\chi}_v^{-1})$.

It follows that $\mathrm{rec}(\pi_v)$ has $N = 0$ and on inertia is of the form $\chi_{1,v}^{-1} \oplus \cdots \oplus \chi_{n,v}^{-1}$ via the local Artin map, where rec denotes the local Langlands correspondence, normalised as in [EG14] (i.e., as in [HT01]). Using Lemma 3.1.4 below there exists a finite extension K'/K depending only on n and K such that $\mathrm{rec}(\pi_v)|_{I_{K'}}$ is trivial. Applying local-global compatibility at p to ρ_π, we deduce that for any $v \mid p$, the representation $\rho_\pi|_{G_{F_{\tilde{v}}}}$ provides a desired lift of $\overline{\rho}|_{G_{F_{\tilde{v}}}} \cong \overline{r}$. □

Remark 3.1.3. The argument in the proof above shows that if $\chi_{i,v} : \mathcal{O}_{F_v}^\times \to \overline{\mathbb{Q}}_p^\times$ are pairwise distinct smooth characters of $\mathcal{O}_{F_v}^\times$ (or equivalently of I_{F_v}), then $\mathrm{Ind}_{J_{\chi_v}}^{K_v} \rho_{\chi_v}$ is a K_v-type corresponding to $\oplus_{i=1}^n \chi_{i,v}$ under the inertial Langlands correspondence, i.e. [EG14, Conj. 4.1.3] holds with $\sigma(\oplus_{i=1}^n \chi_{i,v}) \cong \mathrm{Ind}_{J_{\chi_v}}^{K_v} \rho_{\chi_v}^{-1}$.

Lemma 3.1.4. *Suppose $K/\mathbb{Q}_p$ is a finite extension and $s \geq 1$. Then there exists a finite extension L/K such that any smooth character $\chi : W_K \to \mathbb{C}^\times$ that is trivial on the ramification subgroup G_K^s satisfies $\chi|_{I_L} = 1$.*

Proof. By local class field theory there exists a finite extension M_s/K^{nr} that is independent of χ such that $\chi|_{G_{M_s}} = 1$. (We can take M_s/K abelian such that K^{ab}/M has Galois group $1 + \varpi_K^s \mathcal{O}_K$, with ϖ_K a uniformiser of K.) Then we choose L/K finite such that M_s is contained in $L \cdot K^{\mathrm{nr}} = L^{\mathrm{nr}}$. This implies $\chi|_{I_L} = 1$. In fact, this argument shows that we can take L/K of degree $q^{s-1}(q-1)$, where $q = \#k$. □

Our final result may be viewed as a "weak Breuil–Mézard"-type statement.

Theorem 3.1.5. *Suppose that $p \neq n$, that $K/\mathbb{Q}_p$ is unramified, and that $\overline{r}$ has a crystalline lift of weight F for some extended FL weight F. If F is a Jordan–Hölder factor of $\overline{\sigma}(\lambda, \tau)$ for some λ, τ, then $\overline{r}$ has a potentially crystalline lift of type (λ, τ).*

Proof. Choose $a \in (X_1^{(n)})^{\mathrm{Hom}(k, \overline{\mathbb{F}}_p)}$ such that $F \cong F_a$. The conditions that $p \neq n$ and $\overline{r}$ has a crystalline lift of weight F_a with a an extended FL weight imply that $p > n$; so either $p \nmid 2n$, as we have assumed throughout this section, or else $p = 2$ and $n = 1$.

First suppose that $p \nmid 2n$. By the main result of [GL14] any crystalline representation of extended FL weight is potentially diagonalisable. Let λ' be the lift of a (uniquely defined, as $K/\mathbb{Q}_p$ is unramified). Since $a_{\overline{\kappa},1} - a_{\overline{\kappa},n} \leq p-(n-1)$ for each $\overline{\kappa}$, $F_a = L_a \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p$ (see §1.2.3). By hypothesis, we can apply the constructions from the paragraphs preceding Lemma 3.1.1 with $\lambda_{\overline{r}} = \lambda'$ and $\tau_{\overline{r}} = 1$ to deduce that $S_{\lambda',1}(U, \overline{\mathbb{Z}}_p)_{\mathfrak{m}} \neq 0$. By Lemma 3.1.1(2), $S(U, \otimes_{\overline{\mathbb{F}}_p, v|p} F_a)_{\mathfrak{m}} \neq 0$. Applying Lemma 3.1.1(2) with $(\lambda_v, \tau_v) = (\lambda, \tau)$ for each v, we see that $S_{\lambda,\tau}(U, \overline{\mathbb{Z}}_p)_{\mathfrak{m}} \neq 0$, and the result follows from Lemma 3.1.1(1).

On the other hand, the case $n = 1$ is an easy consequence of local class field theory: $\sigma(\tau)^\vee$ is obtained from τ by local class field theory, so that the locally algebraic characters of $K^\times$ extending $\sigma(\lambda, \tau)$ correspond to de Rham characters of type (λ, τ), while $\overline{r}|_{I_K}$ corresponds to F_a. □

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