The Mumford–Tate Conjecture for the Product of an Abelian Surface and a K3 Surface

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Abstract. The Mumford–Tate conjecture is a precise way of saying that the Hodge structure on singular cohomology conveys the same information as the Galois representation on ℓ-adic étale cohomology, for an algebraic variety over a finitely generated field of characteristic 0. This paper presents a proof of the Mumford–Tate conjecture in degree 2 for the product of an abelian surface and a K3 surface.

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1 Introduction

The main result of this paper is the following theorem. The next paragraph recalls the Mumford–Tate conjecture; and §1.5 gives an outline of the proof.

1.1 Theorem. — *Let K be a finitely generated subfield of $\mathbb{C}$. If A is an abelian surface over K and X is a K3 surface over K, then the Mumford–Tate conjecture is true for* $\mathrm{H}^2(A \times X)(1)$.

1.2 The Mumford–Tate conjecture. — Let K be a finitely generated field of characteristic 0; and let $K \hookrightarrow \mathbb{C}$ be an embedding of K into the complex numbers. Let $\bar{K}$ be the algebraic closure of K in $\mathbb{C}$. Let X/K be a smooth projective variety. One may attach several cohomology groups to X. For the purpose of this article we are interested in two cohomology theories: Betti cohomology and ℓ-adic étale cohomology (for a prime number ℓ). We will write $\mathrm{H}^w_{\mathrm{B}}(X)$ for the $\mathbb{Q}$-Hodge structure $\mathrm{H}^w_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Q})$ in weight w. Similarly, we write $\mathrm{H}^w_{\ell}(X)$ for the $\mathrm{Gal}(\bar{K}/K)$-representation $\mathrm{H}^W_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$.

The Mumford–Tate conjecture was first posed for abelian varieties, by Mumford in [22], and later extended to general algebraic varieties. The conjecture is a precise way of saying that the cohomology groups $\mathrm{H}^w_{\mathrm{B}}(X)$ and $\mathrm{H}^w_\ell(X)$ contain the same information about X. To make this precise, let $\mathrm{G}_{\mathrm{B}}(\mathrm{H}^w_{\mathrm{B}}(X))$ be the Mumford–Tate group of the Hodge structure $\mathrm{H}^w_{\mathrm{B}}(X)$, and let $\mathrm{G}^\circ_\ell(\mathrm{H}^w_\ell(X))$ be the connected component of the Zariski closure of $\mathrm{Gal}(\bar{K}/K)$ in $\mathrm{GL}(\mathrm{H}^w_\ell(X))$. The comparison theorem by Artin, comparing singular cohomology with étale cohomology, canonically identifies $\mathrm{GL}(\mathrm{H}^w_{\mathrm{B}}(X)) \otimes \mathbb{Q}_\ell$ with $\mathrm{GL}(\mathrm{H}^w_\ell(X))$. The Mumford–Tate conjecture (for the prime ℓ, and the embedding $K \hookrightarrow \mathbb{C}$) states that under this identification

$$\mathrm{G}_{\mathrm{B}}(\mathrm{H}^w_{\mathrm{B}}(X)) \otimes \mathbb{Q}_\ell \cong \mathrm{G}^\circ_\ell(\mathrm{H}^w_\ell(X)).$$

1.3 Notation and terminology. — Like above, let K be a finitely generated field of characteristic 0; and fix an embedding $K \hookrightarrow \mathbb{C}$. In this article we use the language of motives in the sense of André, [2]. To be precise, our category of base pieces is the category of smooth projective varieties over K, and our reference cohomology is Betti cohomology, $\mathrm{H}_{\mathrm{B}}(\_)$. We stress that $\mathrm{H}_{\mathrm{B}}(\_)$ depends on the chosen embedding $K \hookrightarrow \mathbb{C}$, but the resulting category of motives does not depend on $K \hookrightarrow \mathbb{C}$, see proposition 2.3 of [2]. We write $\mathrm{H}^w(X)$ for the motive of weight w associated with a smooth projective variety X/K.
The Mumford–Tate conjecture naturally generalises to motives, as follows: Let M be a motive. We will write $\mathrm{H}_{\mathrm{B}}(M)$ for its Hodge realisation; $\mathrm{H}_\ell(M)$ for its ℓ-adic realisation; $\mathrm{G}_{\mathrm{B}}(M)$ for its Mumford–Tate group (*i.e.,* the Mumford–Tate group of $\mathrm{H}_{\mathrm{B}}(M)$); and $\mathrm{G}^\circ_\ell(M)$ for $\mathrm{G}^\circ_\ell(\mathrm{H}_\ell(M))$. We will use the notation $\mathrm{MTC}_\ell(M)$ for the conjectural statement

$$\mathrm{G}_{\mathrm{B}}(M) \otimes \mathbb{Q}_\ell \cong \mathrm{G}^\circ_\ell(M),$$

and $\mathrm{MTC}(M)$ for the assertion $\forall \ell : \mathrm{MTC}_\ell(M)$.
The following remark allows us to take finitely generated extensions of the base field, whenever needed. If $K \subset L$ is an extension of finitely generated subfields of $\mathbb{C}$, and if M is a motive over K, then $\mathrm{MTC}(M) \iff \mathrm{MTC}(M_L)$; see proposition 1.3 of [19].
In this paper, we never use specific properties of the chosen embedding $K \hookrightarrow \mathbb{C}$, and all statements are valid for every such embedding. In particular, we will speak about subfields of $\mathbb{C}$, where the embedding is implicit.
In this paper, we will use compatible systems of ℓ-adic representations. For more information we refer to the book [27] by Serre, the letters of Serre to Ribet (see [28]), or the work of Larsen and Pink [16, 17].
Throughout this paper, A is an abelian variety, over some base field. (Outside section 5, it is even an abelian surface.) Assume A is absolutely simple; and choose a polarisation on A. Let $(D, \dagger)$ denote its endomorphism ring $\mathrm{End}^0(A)$ together with the Rosati involution associated with the polarisation. The simple algebra D together with the positive involution $\dagger$ has a certain type in the

Albert classification that does not depend on the chosen polarisation (see theorem 2 on page 201 of [23]). We say that A is of type X if $(D, \dagger)$ is of type X, where X runs through $\{\text{I}, \dots, \text{IV}\}$. If E denotes the center of D, with degree $e = [E : \mathbb{Q}]$, then we also say that A is of type X(e).
Whenever we speak of simple groups or simple Lie algebras, we mean *non-commutative* simple groups, and *non-abelian* simple Lie-algebras.
Let T be a type of Dynkin diagram (*e.g.,* A_n, B_n, C_n or D_n). Let $\mathfrak{g}$ be a semisimple Lie algebra over K. We say that T does not occur in the Lie type of $\mathfrak{g}$, if the Dynkin diagram of $\mathfrak{g}_{\bar{K}}$ does not have a component of type T. For a semisimple group G over K, we say that T does not occur in the Lie type of G, if T does not occur in the Lie type of $\mathrm{Lie}(G)$.

1.4 — Let K be a finitely generated subfield of $\mathbb{C}$. Let A/K be an abelian surface, and let X/K be a K3 surface. Since $\mathrm{H}^1(X) = 0$, Künneth's theorem gives $\mathrm{H}^2(A \times X) \cong \mathrm{H}^2(A) \oplus \mathrm{H}^2(X)$. Recall that the Mumford–Tate conjecture for A is known in degree 1, and hence in all degrees. (This is classical, but see corollary 4.4 of [18] for a reference.) The Mumford–Tate conjecture for X (in degree 2) is true as well, by [30, 31, 1]. Still, it is not a formal consequence that the Mumford–Tate conjecture for $A \times X$ is true in degree 2: for two motives M_1 and M_2, it is a formal fact that $\mathrm{G_B}(M_1 \oplus M_2) \hookrightarrow \mathrm{G_B}(M_1) \times \mathrm{G_B}(M_2)$, with surjective projections onto $\mathrm{G_B}(M_1)$ and $\mathrm{G_B}(M_2)$. However, this inclusion need not be surjective. For example, if $M_1 = M_2$, then map in question is the diagonal inclusion. Similar remarks hold for $\mathrm{G}_\ell^\circ(\_)$.

1.5 Outline of the proof. — Roughly speaking, the proof of theorem 1.1 as presented in this paper, is as follows:

1. We replace $\mathrm{H}^2(A)(1)$ and $\mathrm{H}^2(X)(1)$ by their transcendental parts M_A and M_X. It suffices to prove $\mathrm{MTC}(M_A \oplus M_X)$, see lemma 7.2.
2. It suffices to show that $\mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}} \cong \mathrm{G}_\ell^\circ(M_A)^{\mathrm{der}} \times \mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}}$, see lemma 7.3.

Write $\mathfrak{g}_A$ for $\mathrm{Lie}(\mathrm{G_B}(M_A)^{\mathrm{der}})$. There exists a number field F_A acting on $\mathfrak{g}_A$, such that $\mathfrak{g}_A$ viewed as F_A-Lie algebra is a direct sum of absolutely simple factors. Analogously we find $\mathfrak{g}_X$ and F_X; see §6.4. We explicitly compute $\mathfrak{g}_A$, $\mathfrak{g}_X$, F_A, and F_X in remark 6.3 and remark 6.6.

3. Using Čebotarëv's density theorem, we show that (2.) can be applied if $F_A \ncong F_X$; see lemma 7.4.
4. By Goursat's lemma, (2.) can also be applied if $\mathfrak{g}_A \otimes \mathbb{C}$ and $\mathfrak{g}_X \otimes \mathbb{C}$ do not have a common simple factor; see lemma 7.6.
5. The previous two items cover almost all cases. The remaining cases (listed in §7.5) fall into two subcases.
   (a) If $\dim(M_X) \leq 5$, we show that we may replace X by its Kuga–Satake abelian variety B. We then prove $\mathrm{MTC}(A \times B)$ using techniques of Lombardo, developed in [18]; see lemma 7.7. Note: In this case the condition in (2.) might not hold (for example if X is the Kummer variety associated with A).
   (b) In the final case $F_A \cong F_X$ is totally real quadratic. We show that

if the condition in (2.) does not hold, then there is a prime number ℓ for which $\mathrm{H}_\ell(M_X)$ is not an irreducible Galois representation. However, by assembling results from [4], [6], and [34], we see that there exist places for which the characteristic polynomial of Frobenius acting on $\mathrm{H}_\ell(M_X)$ is an irreducible polynomial. This leads to a contradiction, and thus (2.) can be applied in this final case; see lemma 7.9.

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2 GALOIS THEORETICAL PRELIMINARIES

Recall the following elementary result from finite group theory.

2.1 LEMMA. — *Let T be a set with a transitive action by a finite group G. Fix $n \in \mathbb{Z}_{\geq 0}$, and let $C \subset G$ be the set of elements $g \in G$ that fix at least n points of T. If $n \cdot |C| \geq |G|$, then $|T| = n$.*
Proof. By computing the cardinality of $\{(g,t) \in G \times T \mid gt = t\}$ in two distinct ways, one gets the formula $|G| \cdot |G \backslash T| = \sum_{g \in G} |T^g|$. From this we derive,

$$1 = |G \backslash T| = \frac{1}{|G|} \sum_{g \in G} |T^g| \geq \frac{n \cdot |C|}{|G|} \geq 1.$$

Hence $n \cdot |C| = |G|$ and all elements in C have exactly n fixed points. In particular the identity element has n fixed points, which implies $|T| = n$. □
Note that, in the setting of the previous lemma, if the action of G on T is faithful, then $|G| = n$, and T is principal homogeneous under G.

2.2 LEMMA. — *Let F_1 be a Galois extension of $\mathbb{Q}$. Let F_2 be a number field such that for all prime numbers ℓ, the product of local fields $F_1 \otimes \mathbb{Q}_\ell$ is a factor of $F_2 \otimes \mathbb{Q}_\ell$. Then $F_1 \cong F_2$.*

[1] A preliminary version of lemma 2.2 arose from a question on MathOverflow titled "How simple does a $\mathbb{Q}$-simple group remain after base change to $\mathbb{Q}_\ell$?" (`http://mathoverflow.net/q/214603/78087`). The answers also inspired lemma 2.1.

Proof. Let L be a Galois closure of F_2, and let G be the Galois group $\mathrm{Gal}(L/\mathbb{Q})$, which acts naturally on the set of field embeddings $\Sigma = \mathrm{Hom}(F_2, L)$. Let n be the degree of F_1, and let C be the set $\{g \in G \mid |\Sigma^g| \geq n\}$ of elements in G that have at least n fixed points in Σ.
By Čebotarëv's density theorem (Satz VII.13.4 of [24]), the set of primes that split completely in $F_1/\mathbb{Q}$ has density $1/n$. Therefore, the set of primes ℓ for which $F_2 \otimes \mathbb{Q}_\ell$ has a semisimple factor isomorphic to $(\mathbb{Q}_\ell)^n$ must have density $\geq 1/n$. In other words, $n \cdot |C| \geq |G|$. By lemma 2.1, this implies $|\Sigma| = n$, and since G acts faithfully on Σ, we find that $F_2/\mathbb{Q}$ is Galois of degree n. Because Galois extensions of number fields can be recovered from their set of splitting primes (Satz VII.13.9 of [24]), we conclude that $F_1 \cong F_2$. □

2.3 Lemma. — *Let F_1 be a quadratic extension of $\mathbb{Q}$. Let F_2 be a number field of degree ≤ 5 over $\mathbb{Q}$. If for all prime numbers ℓ, the products of local fields $F_1 \otimes \mathbb{Q}_\ell$ and $F_2 \otimes \mathbb{Q}_\ell$ have an isomorphic factor, then $F_1 \cong F_2$.*
Proof. Let L be a Galois closure of F_2, and let G be the Galois group $\mathrm{Gal}(L/\mathbb{Q})$, which acts naturally on the set of field embeddings $\Sigma = \mathrm{Hom}(F_2, L)$. Observe that G acts transitively on Σ, and we identify G with its image in $\mathfrak{S}(\Sigma)$. Write n for the degree of F_2 over $\mathbb{Q}$, which also equals $|\Sigma|$. The order of G is divisible by n. Hence, if n is prime, then G must contain an n-cycle.
Suppose that G contains an n-cycle. By Čebotarëv's density theorem there must be a prime number ℓ that is inert in $F_2/\mathbb{Q}$. By our assumption $F_2 \otimes \mathbb{Q}_\ell$ also contains a factor of at most degree 2 over $\mathbb{Q}_\ell$. This shows that $n = 2$.
If $n = 4$, then G does not contain an n-cycle if and only if it is isomorphic to V_4 or A_4. If $G \cong V_4$, then only the identity element has fixed points, and by Čebotarëv's density theorem this means that the set of primes ℓ for which $F_2 \otimes \mathbb{Q}_\ell$ has a factor $\mathbb{Q}_\ell$ has density $1/4$, whereas the set of primes splitting in $F_1/\mathbb{Q}$ has density $1/2$. On the other hand, if $G \cong A_4$, then only 3 of the 12 elements have a 2-cycle in the cycle decomposition, and by Čebotarëv's density theorem this means that the set of primes ℓ for which $F_2 \otimes \mathbb{Q}_\ell$ has a factor isomorphic to a quadratic extension of $\mathbb{Q}_\ell$ has density $1/4$, whereas the set of primes inert in $F_1/\mathbb{Q}$ has density $1/2$. This gives a contradiction. We conclude that n must be 2; and therefore $F_1 \cong F_2$, by lemma 2.2. □

3 A result on semisimple groups over number fields

Throughout this section K is a field of characteristic 0.

3.1 Remark. — Let $\mathfrak{h} \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be Lie algebras over K such that $\mathfrak{h}$ projects surjectively onto $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Assume that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are finite-dimensional and semisimple. Then there exist semisimple Lie algebras $\mathfrak{s}_1$, $\mathfrak{t}$, and $\mathfrak{s}_2$ such that $\mathfrak{g}_1 \cong \mathfrak{s}_1 \oplus \mathfrak{t}$, $\mathfrak{g}_2 \cong \mathfrak{t} \oplus \mathfrak{s}_2$, and $\mathfrak{h} \cong \mathfrak{s}_1 \oplus \mathfrak{t} \oplus \mathfrak{s}_2$. (To see this, recall that a finite-dimensional semisimple Lie algebra is the sum of its simple ideals.)

3.2 Lemma. — *Let $K \subset F$ be a finite field extension, and let G/F be an algebraic group. If $\mathrm{Lie}(G)_K$ denotes the Lie algebra $\mathrm{Lie}(G)$ viewed as K-algebra, then $\mathrm{Lie}(\mathrm{Res}_{F/K} G)$ is naturally isomorphic to $\mathrm{Lie}(G)_K$.*

Proof. The Lie algebra $\mathrm{Lie}(G)$ is the kernel of the natural map $G(F[\varepsilon]) \to G(F)$; where the Lie bracket may be described as follows: Consider the map

$$\begin{aligned} G(F[\varepsilon_1]) \times G(F[\varepsilon_2]) &\longrightarrow G(F[\varepsilon_1, \varepsilon_2]) \\ (X_1, X_2) &\longmapsto X_1 X_2 X_1^{-1} X_2^{-1} \end{aligned}$$

If $(X_1, X_2) \in \mathrm{Lie}(G) \times \mathrm{Lie}(G)$, then a calculation shows that its image lands in $G(F[\varepsilon_1 \cdot \varepsilon_2]) \cong G(F[\varepsilon])$. What is more, it lands in $\mathrm{Lie}(G) \subset G(F[\varepsilon])$. We denote this image with $[X_1, X_2]$, the Lie bracket of X_1 and X_2.
Let $G_{F/K}$ denote $\mathrm{Res}_{F/K}\, G$. The following diagram shows that $\mathrm{Lie}(G_{F/K})$ is canonically identified with $\mathrm{Lie}(G)_K$ as K-vector space, and that the Lie bracket is preserved.

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Lie}(G_{F/K}) \arrow[r] & G_{F/K}(K[\varepsilon]) \arrow[r] \arrow[d, "\wr"] & G_{F/K}(K) \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & \mathrm{Lie}(G) \arrow[r] & G(F[\varepsilon]) \arrow[r] & G(F) \arrow[r] & 0 \end{tikzcd}$$ □

3.3 — Let F/K be a finite field extension. Let $\mathfrak{g}$ be a Lie algebra over F, and write $(\mathfrak{g})_K$ for the Lie algebra $\mathfrak{g}$ viewed as K-algebra. If $\mathfrak{g}$ is an F-simple Lie algebra, then $(\mathfrak{g})_K$ is K-simple: indeed, if $(\mathfrak{g})_K \cong \mathfrak{h} \oplus \mathfrak{h}'$, and $X \in \mathfrak{h}$, then $[fX, X'] = 0$ for all $f \in F$ and $X' \in \mathfrak{h}'$. This implies $fX \in \mathfrak{h}$, hence $\mathfrak{h} = \mathfrak{g}$.

3.4 Lemma. — *For $i = 1, 2$, let $K \subset F_i$ be a finite field extension, and let $\mathfrak{g}_i/F_i$ be a product of absolutely simple Lie algebras (cf. our conventions in §1.3). If $(\mathfrak{g}_1)_K$ and $(\mathfrak{g}_2)_K$ have an isomorphic factor, then $F_1 \cong_K F_2$.*
Proof. By the remark before this lemma, the K-simple factors of $(\mathfrak{g}_i)_K$ are all of the form $(\mathfrak{t}_i)_K$, where $\mathfrak{t}_i$ is an F_i-simple factor of $\mathfrak{g}_i$. So if $(\mathfrak{g}_1)_K$ and $(\mathfrak{g}_2)_K$ have an isomorphic factor, then there exist F_i-simple factors $\mathfrak{t}_i$ of $\mathfrak{g}_i$ for which there exists an isomorphism $f\colon (\mathfrak{t}_1)_K \to (\mathfrak{t}_2)_K$. Let $\bar{K}$ be an algebraic closure of K. Observe that

$$(\mathfrak{t}_i)_K \otimes_K \bar{K} \cong \bigoplus_{\sigma \in \mathrm{Hom}_K(F_i, \bar{K})} \mathfrak{t}_i \otimes_{F_i, \sigma} \bar{K},$$

and note that $\mathrm{Gal}(\bar{K}/K)$ acts transitively on $\mathrm{Hom}_K(F_i, \bar{K})$. By assumption, the $\mathfrak{t}_i$ are absolutely simple, hence the $\mathfrak{t}_i \otimes_{F_i,\sigma} \bar{K}$ are precisely the simple ideals of $(\mathfrak{t}_i)_K \otimes_K \bar{K}$. Thus the isomorphism f gives a $\mathrm{Gal}(\bar{K}/K)$-equivariant bijection between the simple ideals of $(\mathfrak{t}_1)_K \otimes_K \bar{K}$ and $(\mathfrak{t}_2)_K \otimes_K \bar{K}$; and therefore between $\mathrm{Hom}_K(F_1, \bar{K})$ and $\mathrm{Hom}_K(F_2, \bar{K})$ as $\mathrm{Gal}(\bar{K}/K)$-sets. This proves the result. □

3.5 Lemma. — *For $i = 1, 2$, let F_i be a number field, and let G_i/F_i be an almost direct product of connected absolutely simple F_i-groups. Let ℓ be a prime number, and let $\iota_\ell\colon G \hookrightarrow (\mathrm{Res}_{F_1/\mathbb{Q}}\, G_1)_{\mathbb{Q}_\ell} \times (\mathrm{Res}_{F_2/\mathbb{Q}}\, G_2)_{\mathbb{Q}_\ell}$ be a subgroup over $\mathbb{Q}_\ell$, with surjective projections onto both factors. If ι_ℓ is not an isomorphism, then $F_1 \otimes \mathbb{Q}_\ell$ and $F_2 \otimes \mathbb{Q}_\ell$ have an isomorphic simple factor.*

Proof. Note that $(\mathrm{Res}_{F_i/\mathbb{Q}} G_i) \otimes \mathbb{Q}_\ell \cong \prod_{\lambda|\ell} \mathrm{Res}_{F_{i,\lambda}/\mathbb{Q}_\ell}(G_i \otimes_{F_i} F_\lambda)$. By remark 3.1 there exist places λ_i of F_i over ℓ such that $\mathrm{Lie}(\mathrm{Res}_{F_{1,\lambda_1}/\mathbb{Q}_\ell}(G_1 \otimes_{F_1} F_{1,\lambda_1}))$ and $\mathrm{Lie}(\mathrm{Res}_{F_{2,\lambda_2}/\mathbb{Q}_\ell}(G_2 \otimes_{F_2} F_{2,\lambda_2}))$ have an isomorphic factor. By lemmas 3.2 and 3.4, this implies that $F_{1,\lambda_1} \cong_{\mathbb{Q}_\ell} F_{2,\lambda_2}$, which proves the lemma. □

4 Several results on abelian motives

4.1 Definition. — A motive M over a field $K \subset \mathbb{C}$ is *abelian* (or an *abelian motive*) if it satisfies one of the following three equivalent conditions:

1. M is isomorphic to an object in the Tannakian subcategory generated by *all* abelian varieties over K.
2. There exists an abelian variety A over K such that M is isomorphic to an object in the Tannakian subcategory $\langle \mathrm{H}(A)\rangle^{\otimes}$ generated by $\mathrm{H}(A)$. (Note that $\langle \mathrm{H}(A)\rangle^{\otimes} = \langle \mathrm{H}^1(A)\rangle^{\otimes}$.)
3. There is an isomorphism $M \cong \bigoplus_i M_i$, and for each M_i there exists an abelian variety A_i such that M_i is a subobject of $\mathrm{H}(A_i)(n_i)$.

4.2 Theorem. — *The Hodge realisation functor* $\mathrm{H}_\mathrm{B}(\_)$ *is fully faithful on the subcategory of abelian motives over* $\mathbb{C}$*. As a consequence, if K is a finitely generated subfield of* $\mathbb{C}$*, and if M is an abelian motive over K, then the natural inclusion* $\mathrm{G}_\mathrm{B}(M) \subset \mathrm{G}(M)^\circ$ *is an isomorphism, and* $\mathrm{G}_\ell^\circ(M) \subset \mathrm{G}_\mathrm{B}(M) \otimes \mathbb{Q}_\ell$.
Proof. This is an immediate consequence of théorème 0.6.2 of [2]. □

4.3 Remark. — All the motives in this article are abelian motives. For the motives coming from abelian varieties, this is obvious. The motive $\mathrm{H}^2(X)$, where X is a K3 surface, is also an abelian motive, by théorème 0.6.3 of [2].

4.4 Proposition. — *The Mumford–Tate conjecture on centres is true for abelian motives. In other words, let M be an abelian motive. Let* $Z_\mathrm{B}(M)$ *be the centre of the Mumford–Tate group* $\mathrm{G}_\mathrm{B}(M)$*, and let* $Z_\ell(M)$ *be the centre of* $\mathrm{G}_\ell^\circ(M)$*. Then* $Z_\ell(M) \cong Z_\mathrm{B}(M) \otimes \mathbb{Q}_\ell$.
Proof. The result is true for abelian varieties (see theorem 1.3.1 of [33] or corollary 2.11 of [32]).
By definition of abelian motive, there is an abelian variety A such that M is contained in the Tannakian subcategory of motives generated by $\mathrm{H}(A)$. This yields a surjection $\mathrm{G}_\mathrm{B}(A) \twoheadrightarrow \mathrm{G}_\mathrm{B}(M)$. Since $\mathrm{G}_\mathrm{B}(A)$ is reductive, $Z_\mathrm{B}(M)$ is the image of $Z_\mathrm{B}(A)$ under this map. The same is true on the ℓ-adic side. (Note that $\mathrm{G}_\ell^\circ(A)$ is reductive, by Satz 3 in §5 of [10].) Thus we obtain a commutative diagram with solid arrows

$$\begin{tikzcd}
Z_\ell(A) \arrow[r, two heads] \arrow[d, "\wr"] & Z_\ell(M) \arrow[r, hook] \arrow[d, dotted, "\wr"] & \mathrm{G}_\ell^\circ(M) \arrow[d, hook] \\
Z_\mathrm{B}(A) \otimes \mathbb{Q}_\ell \arrow[r, two heads] & Z_\mathrm{B}(M) \otimes \mathbb{Q}_\ell \arrow[r, hook] & \mathrm{G}_\mathrm{B}(M) \otimes \mathbb{Q}_\ell
\end{tikzcd}$$

which shows that the dotted arrow exists and is an isomorphism. (The vertical arrow on the right exists and is an inclusion, by theorem 4.2.) □

4.5 Lemma. — *Let M be an abelian motive. Assume that the ℓ-adic realisations of M form a compatible system of ℓ-adic representations. If there is one prime ℓ for which the absolute rank of $\mathrm{G}_\ell^\circ(M)$ is equal to the absolute rank of $\mathrm{G}_\mathrm{B}(M)$, then the Mumford–Tate conjecture is true for M.*

Proof. The absolute rank of $\mathrm{G}_\ell^\circ(M)$ does not depend on ℓ (see proposition 6.12 of [16] or Serre's letter to Ribet). Since M is an abelian motive, we have $\mathrm{G}_\ell^\circ(M) \subset \mathrm{G}_\mathrm{B}(M) \otimes \mathbb{Q}_\ell$ (see theorem 4.2). The Mumford–Tate conjecture follows from Borel–de Siebenthal theory[2]: since $\mathrm{G}_\ell^\circ(M) \subset \mathrm{G}_\mathrm{B}(M) \otimes \mathbb{Q}_\ell$ has maximal rank, it is equal to the connected component of the centraliser of its centre. By proposition 4.4, we know that the centre of $\mathrm{G}_\ell^\circ(M)$ equals the centre of $\mathrm{G}_\mathrm{B}(M) \otimes \mathbb{Q}_\ell$. Hence $\mathrm{G}_\ell^\circ(M) \cong \mathrm{G}_\mathrm{B}(M) \otimes \mathbb{Q}_\ell$. □

4.6 — Let K be a finitely generated subfield of $\mathbb{C}$. A pair (A, X), consisting of an abelian surface A and a K3 surface X over K, is said to satisfy condition 4.6 for ℓ if

$$\mathrm{G}_\ell^\circ\big(\mathrm{H}^2(A \times X)(1)\big)^{\mathrm{der}} \hookrightarrow \mathrm{G}_\ell^\circ\big(\mathrm{H}^2(A)(1)\big)^{\mathrm{der}} \times \mathrm{G}_\ell^\circ\big(\mathrm{H}^2(X)(1)\big)^{\mathrm{der}}$$

is an isomorphism.

4.7 — Let ℓ be a prime number. Let G_1 and G_2 be connected reductive groups over $\mathbb{Q}_\ell$. By a *(G_1, G_2)-tuple* over K we shall mean a pair (A, X), where A is an abelian surface over K, and X is a K3 surface over K such that $\mathrm{G}_\ell^\circ(\mathrm{H}^2(A)(1)) \cong G_1$ and $\mathrm{G}_\ell^\circ(\mathrm{H}^2(X)(1)) \cong G_2$. We will show in section 7 that there exist groups G_1 and G_2 that satisfy the hypothesis of the following lemma, namely that condition 4.6 for ℓ is satisfied for all (G_1, G_2)-tuples over number fields.

4.8 Lemma. — *Let ℓ be a prime number. Let G_1 and G_2 be connected reductive groups over $\mathbb{Q}_\ell$. If for all number fields K, all (G_1, G_2)-tuples (A, X) over K satisfy condition 4.6 for ℓ, then for all finitely generated subfields L of $\mathbb{C}$, all (G_1, G_2)-tuples (A, X) over L satisfy condition 4.6 for ℓ.*

Proof. Let (A, X) be a (G_1, G_2)-tuple over a finitely generated field L. Then there exists a (not necessarily proper) integral scheme $S/\mathbb{Q}$, with generic point η, an abelian scheme $\mathcal{A}/S$, and a K3 surface $\mathcal{X}/S$, such that L is isomorphic to the function field of S, $\mathcal{A}_\eta \cong A$, and $\mathcal{X}_\eta \cong X$.

By a result of Serre (using Hilbert's irreducibility theorem, see Serre's letters to Ribet [28], or section 10.6 of [26]), there exists a closed point $c \in S$ such that $\mathrm{G}_\ell^\circ(\mathrm{H}_\ell^2(\mathcal{A}_c \times \mathcal{X}_c)(1)) \cong \mathrm{G}_\ell^\circ(\mathrm{H}_\ell^2(A \times X)(1))$. Since $\mathrm{G}_\ell^\circ(\mathrm{H}_\ell^2(A \times X)(1))$ surjects onto $\mathrm{G}_\ell^\circ(\mathrm{H}_\ell^2(A)(1))$, we find that $\mathrm{G}_\ell^\circ(\mathrm{H}_\ell^2(\mathcal{A}_c)(1)) \cong \mathrm{G}_\ell^\circ(\mathrm{H}_\ell^2(A)(1))$, and similar for X. The following diagram shows that shows that (A, X) satisfies condition 4.6 for ℓ

[2] The original article [5], by Borel and de Siebenthal deals with the case of compact Lie groups. See `http://www.math.ens.fr/~gille/prenotes/bds.pdf` for a proof in the case of reductive algebraic groups.

if $(\mathcal{A}_c, \mathcal{X}_c)$ satisfies it.

$$\begin{array}{ccc}
\mathrm{G}_\ell^\circ(\mathrm{H}^2(\mathcal{A}_c \times \mathcal{X}_c)(1))^{\mathrm{der}} & \hookrightarrow & \mathrm{G}_\ell^\circ(\mathrm{H}^2(\mathcal{A}_c)(1))^{\mathrm{der}} \times \mathrm{G}_\ell^\circ(\mathrm{H}^2(\mathcal{X}_c)(1))^{\mathrm{der}} \\
\downarrow \wr & & \downarrow \wr \\
\mathrm{G}_\ell^\circ(\mathrm{H}^2(A \times X)(1))^{\mathrm{der}} & \hookrightarrow & \mathrm{G}_\ell^\circ(\mathrm{H}^2(A)(1))^{\mathrm{der}} \times \mathrm{G}_\ell^\circ(\mathrm{H}^2(X)(1))^{\mathrm{der}}
\end{array} \qquad \square$$

5 Some remarks on the Mumford–Tate conjecture for abelian varieties

5.1 — For the convenience of the reader, we copy some results from [18]. Before we do that, let us recall the notion of the *Hodge group*, $\mathrm{Hdg}_{\mathrm{B}}(A)$, of an abelian variety. Let A be an abelian variety over a finitely generated field $K \subset \mathbb{C}$. By definition, the Mumford–Tate group of an abelian variety is $\mathrm{G}_{\mathrm{B}}(A) = \mathrm{G}_{\mathrm{B}}(\mathrm{H}^1_{\mathrm{B}}(A)) \subset \mathrm{GL}(\mathrm{H}^1_{\mathrm{B}}(A))$, and we put

$$\mathrm{Hdg}_{\mathrm{B}}(A) = (\mathrm{G}_{\mathrm{B}}(A) \cap \mathrm{SL}(\mathrm{H}^1_{\mathrm{B}}(A)))^\circ \quad \text{and} \quad \mathrm{Hdg}_\ell(A) = (\mathrm{G}_\ell(A) \cap \mathrm{SL}(\mathrm{H}^1_\ell(A)))^\circ.$$

The equivalence

$$\mathrm{MTC}_\ell(A) \quad \Longleftrightarrow \quad \mathrm{Hdg}_{\mathrm{B}}(A) \otimes \mathbb{Q}_\ell \cong \mathrm{Hdg}_\ell(A).$$

is a consequence of proposition 4.4.

5.2 Definition (1.1 in [18]). — Let A be an absolutely simple abelian variety of dimension g over K. The endomorphism ring $D = \mathrm{End}^0(A)$ is a division algebra. Write E for the centre of D. The ring E is a field, either TR (totally real) or CM. Write e for $[E : \mathbb{Q}]$. The degree of D over E is a perfect square d^2. The *relative dimension* of A is

$$\mathrm{reldim}(A) = \begin{cases} \frac{g}{de}, & \text{if } A \text{ is of type I, II, or III,} \\ \frac{2g}{de}, & \text{if } A \text{ is of type IV.} \end{cases}$$

Note that $d = 1$ if A is of type I, and $d = 2$ if A is of type II or III.
In definition 2.22 of [18], Lombardo defines when an abelian variety is of *general Lefschetz type*. This definition is a bit unwieldy, and its details do not matter too much for our purposes. What matters are the following results, that prove that certain abelian varieties are of general Lefschetz type, and that show why this notion is relevant for us.

5.3 Lemma. — *Let A be an absolutely simple abelian variety over a finitely generated subfield of $\mathbb{C}$. Assume that A is of type* I *or* II. *If* $\mathrm{reldim}(A)$ *is odd, or equal to* 2*, then A is of general Lefschetz type.*
Proof. If $\mathrm{reldim}(A)$ is odd, then this follows from theorems 6.9 and 7.12 of [3]. Lombardo notes (remark 2.25 in [18]) that the proof of [3] also works if $\mathrm{reldim}(A) = 2$, and also refers to theorem 8.5 of [7] for a proof of that fact. $\square$

5.4 Lemma. — *Let K be a finitely generated subfield of $\mathbb{C}$. Let A_1 and A_2 be two abelian varieties over K that are isogenous to products of abelian varieties of general Lefschetz type. If D_4 does not occur in the Lie type of $\mathrm{Hdg}_\ell(A_1)$ and $\mathrm{Hdg}_\ell(A_2)$, then either*

$$\mathrm{Hom}_K(A_1, A_2) \neq 0, \quad \textit{or} \quad \mathrm{Hdg}_\ell(A_1 \times A_2) \cong \mathrm{Hdg}_\ell(A_1) \times \mathrm{Hdg}_\ell(A_2).$$

Proof. This is remark 4.3 of [18], where Lombardo observes that, under the assumption of the lemma, theorem 4.1 of [18] can be applied to products of abelian varieties of general Lefschetz type. □

5.5 Lemma. — *Let A be an abelian variety over a finitely generated field $K \subset \mathbb{C}$. Let $L \subset \mathbb{C}$ be a finite extension of K for which A_L is isogenous over L to a product of absolutely simple abelian varieties $\prod A_i^{k_i}$. Assume that for all i the following conditions are satisfied:*

(a) either A_i is of general Lefschetz type or A_i is of CM *type;*
(b) the Lie type of $\mathrm{Hdg}_\ell(A_i)$ does not contain D_4;
(c) the Mumford–Tate conjecture is true for A_i.

Under these conditions the Mumford–Tate conjecture is true for A.

Proof. Recall that $\mathrm{MTC}(A) \iff \mathrm{MTC}(A_L)$. Note that $\mathrm{MTC}(A_L)$ is equivalent to $\mathrm{MTC}(\prod A_i)$, since $\mathrm{H}^1(A_L)$ and $\mathrm{H}^1(\prod A_i) = \bigoplus \mathrm{H}^1(A_i)$ generate the same Tannakian subcategory of motives. Observe that

$$\mathrm{Hdg}_\ell(\prod A_i) \subset \mathrm{Hdg}_\mathrm{B}(\prod A_i) \otimes \mathbb{Q}_\ell \subset \prod \mathrm{Hdg}_\mathrm{B}(A_i) \otimes \mathbb{Q}_\ell = \prod \mathrm{Hdg}_\ell(A_i),$$

where the first inclusion is Deligne's "Hodge = absolute Hodge" theorem (see proposition 2.9 and theorem 2.11 of [9], or see theorem 4.2); the second inclusion follows from the fact that the Hodge group of a product is a subgroup of the product of the Hodge groups (with surjective projections); and the last equality is condition (c).

If we ignore the factors that are CM, then an inductive application of the previous lemma yields $\mathrm{Hdg}_\ell(A) = \prod \mathrm{Hdg}_\ell(A_i)$. If we do not ignore the factors that are CM, then we actually get $\mathrm{Hdg}_\ell(A)^{\mathrm{der}} = \prod \mathrm{Hdg}_\ell(A_i)^{\mathrm{der}}$. Together with proposition 4.4, this proves $\mathrm{Hdg}_\ell(A) = \mathrm{Hdg}_\mathrm{B}(A) \otimes \mathbb{Q}_\ell$. □

As an illustrative application of this result, Lombardo observes in corollary 4.5 of [18] that the Mumford–Tate conjecture is true for arbitrary products of elliptic curves and abelian surfaces.

6 Hodge theory of K3 surfaces and abelian surfaces

6.1 — In this section we recall some results of Zarhin that describe all possible Mumford–Tate groups of Hodge structures of K3 type, *i.e.,* Hodge structures of weight 0 with Hodge numbers of the form $(1, n, 1)$.

The canonical example of a Hodge structure of K3 type is the Tate twist of the cohomology in degree 2 of a complex K3 surface X. Namely the Hodge structure $\mathrm{H}^2_{\mathrm{B}}(X)(1)$ has Hodge numbers $(1,20,1)$. Another example is provided by abelian surfaces, which is the content of remark 6.6 below.

6.2 Theorem. — *Let (V,ψ) be a polarised irreducible Hodge structure of K3 type.*

1. *The endomorphism algebra E of V is a field.*
2. *The field E is* TR *(totally real) or* CM.
3. *If E is* TR, *then $\dim_E(V) \geq 3$.*
4. *Let E_0 be the maximal totally real subfield of E. Let $\tilde{\psi}$ be the unique E-bilinear (resp. hermitian) form such that $\psi = \mathrm{tr}_{E/\mathbb{Q}} \circ \tilde{\psi}$ if E is* TR *(resp.* CM*). The Mumford–Tate group of V is*

$$\mathrm{G}_{\mathrm{B}}(V) \cong \begin{cases} \mathrm{Res}_{E/\mathbb{Q}} \mathrm{SO}(\tilde{\psi}), & \text{if } E \text{ is TR;} \\ \mathrm{Res}_{E_0/\mathbb{Q}} \mathrm{U}(\tilde{\psi}), & \text{if } E \text{ is CM.} \end{cases}$$

(Here $\mathrm{U}(\tilde{\psi})$ is the unitary group over E_0 associated with the hermitian form $\tilde{\psi}$.)

Proof. The first (resp. second) claim is theorem 1.6.a (resp. theorem 1.5) of [35]; the third claim is observed by Van Geemen, in lemma 3.2 of [13]; and the final claim is a combination of theorems 2.2 and 2.3 of [35]. (We note that [35] deals with Hodge groups, but because our Hodge structure has weight 0, the Mumford–Tate group and the Hodge group coincide.) □

6.3 Remark. — Let V, E and $\tilde{\psi}$ be as in theorem 6.2. If E is CM, and $\dim_E(V) = 1$, then $\mathrm{U}(\tilde{\psi})^{\mathrm{der}} = \mathrm{SU}(\tilde{\psi}) = 1$; while if $\dim_E(V) > 1$, then $\mathrm{U}(\tilde{\psi})^{\mathrm{der}} = \mathrm{SU}(\tilde{\psi})$ is absolutely simple over E_0. If E is TR and $\dim_E(V) \neq 4$, then $\mathrm{SO}(\tilde{\psi})$ is absolutely simple over E. Assume E is TR and $\dim_E(V) = 4$. In this case $\mathrm{SO}(\tilde{\psi})$ is not absolutely simple over E: it has Lie type $D_2 = A_1 \oplus A_1$. In this remark we will take a close look at this special case, because a good understanding of it will play a crucial rôle in the proof of lemma 7.9.

Geometrically we find $\mathrm{SO}(\tilde{\psi})_{\bar{E}} \cong (\mathrm{SL}_{2,\bar{E}} \times \mathrm{SL}_{2,\bar{E}})/\langle(-1,-1)\rangle$. We distinguish the following two cases:

1. $\mathrm{SO}(\tilde{\psi})$ is not simple over E. The fact most relevant to us is the existence of a quaternion algebra D/E such that $\mathrm{SO}(\tilde{\psi}) \cong (N \times N^{\mathrm{op}})/\langle(-1,-1)\rangle$ where N is the group over E of elements in $D^\star$ that have norm 1, and likewise for $N^{\mathrm{op}} \subset (D^{\mathrm{op}})^\star$. One can read more about the details of this claim in section 8.1 of [19]. This situation is also described in section 26.B of [15], where the quaternion algebra is replaced by $D \times D$ viewed as quaternion algebra over $E \times E$. This might be slightly more natural, but it requires bookkeeping of étale algebras which makes the proof in section 7 more difficult than necessary.
2. $\mathrm{SO}(\tilde{\psi})$ is simple over E. This means that the action of $\mathrm{Gal}(\bar{E}/E)$ on $\mathrm{SO}(\tilde{\psi})_{\bar{E}}$ interchanges the two factors $\mathrm{SL}_{2,\bar{E}}$. The stabilisers of these factors are subgroups of index 2 that coincide. This subgroup fixes a quadratic extension F/E. From our description of the geometric situation, together with the description of the stabilisers, we see that

$\mathrm{Spin}(\tilde{\psi}) = \mathrm{Res}_{F/E}\mathscr{G}$ is a $(2:1)$-cover of $\mathrm{SO}(\tilde{\psi})$, where $\mathscr{G}$ is an absolutely simple, simply connected group of Lie type A_1 over F.

What we have gained is that in all cases we have a description (up to isogeny) of $\mathrm{G_B}(V)^{\mathrm{der}}$ as Weil restriction of a group that is an *almost direct product* of groups that are *absolutely simple*. This allows us to apply lemma 3.4, which will play an important rôle in section 7.

6.4 Notation and terminology. — Let V, E and $\tilde{\psi}$ be as in theorem 6.2. To harmonise the proof in section 7, we unify notation as follows:

$$F = \begin{cases} E_0 & \text{if } E \text{ is CM,} \\ E & \text{if } E \text{ is TR and } \dim_E(V) \neq 4, \\ E & \text{if } E \text{ is TR, } \dim_E(V) = 4, \text{ and we are in case 6.3.1,} \\ F & \text{if } E \text{ is TR, } \dim_E(V) = 4, \text{ and we are in case 6.3.2.} \end{cases}$$

Similarly

$$\mathscr{G} = \begin{cases} \mathrm{U}(\tilde{\psi}) & \text{if } E \text{ is CM,} \\ \mathrm{SO}(\tilde{\psi}) & \text{if } E \text{ is TR and } \dim_E(V) \neq 4, \\ \mathrm{SO}(\tilde{\psi}) & \text{if } E \text{ is TR, } \dim_E(V) = 4, \text{ and we are in case 6.3.1,} \\ \mathscr{G} & \text{if } E \text{ is TR, } \dim_E(V) = 4, \text{ and we are in case 6.3.2.} \end{cases}$$

We stress that $\mathscr{G}^{\mathrm{der}}$ is an almost direct product of absolutely simple groups over F. In section 7, most of the time it is enough to know that $\mathrm{G_B}(V)$ is isogenous to $\mathrm{Res}_{F/\mathbb{Q}}\mathscr{G}$. When we need more detailed information, it is precisely the case that E is TR and $\dim_E(V) = 4$. For this case we gave a description of $\mathscr{G}$ in the previous remark.

6.5 — Let V, E and $\tilde{\psi}$ be as in theorem 6.2. Write n for $\dim_E(V)$. If E is TR, then we say that the group $\mathrm{SO}(\tilde{\psi})$ over E is a group of type $\mathrm{SO}_{n,E}$. We also say that $\mathrm{G_B}(V)$ is of type $\mathrm{Res}_{E/\mathbb{Q}}\mathrm{SO}_{n,E}$. Similarly, if E is CM, with maximal totally real subfield E_0, then we say that the group $\mathrm{U}(\tilde{\psi})$ over E_0 is a group of type U_{n,E_0}, and that $\mathrm{G_B}(V)$ is of type $\mathrm{Res}_{E_0/\mathbb{Q}}\mathrm{U}_{n,E_0}$.

6.6 Remark. — Let A be an abelian surface over $\mathbb{C}$. Recall that $\mathrm{H}^2_{\mathrm{B}}(A)(1)$ has dimension 6. Let H be the transcendental part of $\mathrm{H}^2_{\mathrm{B}}(A)(1)$ and let ρ denote the Picard number of A, so that $\dim_{\mathbb{Q}}(H) + \rho = 6$. Observe that H is an irreducible Hodge structure of K3 type. In this remark we explicitly calculate what Zarhin's classification (theorem 6.2) means for H. If A is simple, then the Albert classification of endomorphism algebras of abelian varieties states that $\mathrm{End}(A) \otimes \mathbb{Q}$ can be one of the following:

1. The field of rational numbers, $\mathbb{Q}$. In this case $\rho = 1$ and $\mathrm{G_B}(H)$ is of type $\mathrm{SO}_{5,\mathbb{Q}}$.
2. A real quadratic extension $F/\mathbb{Q}$. In this case $\rho = 2$ and $\mathrm{G_B}(H)$ is of type $\mathrm{SO}_{4,\mathbb{Q}}$. By exemple 3.2.2(a) of [11], we see that $\mathrm{Nm}_{F/\mathbb{Q}}(\mathrm{H}^1(A)) \hookrightarrow \bigwedge^2 \mathrm{H}^1(A) \cong \mathrm{H}^2(A)$, where $\mathrm{Nm}(\_)$ is the norm functor studied in [11].

This norm map identifies $\mathrm{Nm}_{F/\mathbb{Q}}(\mathrm{H}^1(A))(1)$ with the transcendental part H. Observe that consequently the Hodge group $\mathrm{Hdg}_{\mathrm{B}}(\mathrm{H}^1(A)) = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{SL}_{2,F}$ is a $(2:1)$-cover of $\mathrm{G}_{\mathrm{B}}(H)$.

3. An indefinite quaternion algebra $D/\mathbb{Q}$. (This means $D \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$.) In this case $\rho = 3$ and $\mathrm{G}_{\mathrm{B}}(H)$ is of type $\mathrm{SO}_{3,\mathbb{Q}}$.
4. A CM field $E/\mathbb{Q}$ of degree 4. In this case $\rho = 2$ and $\mathrm{G}_{\mathrm{B}}(H)$ is of type $\mathrm{Res}_{E_0/\mathbb{Q}} \mathrm{U}_{1,E_0}$.

(Note that the endomorphism algebra of A cannot be an imaginary quadratic field, by theorem 5 of [29].) If A is isogenous to the product of two elliptic curves $Y_1 \times Y_2$, then there are the following options:

5. The elliptic curves are not isogenous, and neither of them is of CM type, in which case $\rho = 2$ and $\mathrm{G}_{\mathrm{B}}(H)$ is of type $\mathrm{SO}_{4,\mathbb{Q}}$. Indeed, $\mathrm{Hdg}_{\mathrm{B}}(Y_1)$ and $\mathrm{Hdg}_{\mathrm{B}}(Y_2)$ are isomorphic to $\mathrm{SL}_{2,\mathbb{Q}}$. Note that $H = \mathrm{H}^2_{\mathrm{B}}(A)(1)^{\mathrm{tra}}$ is isomorphic to the exterior tensor product $\big(\mathrm{H}^1_{\mathrm{B}}(Y_1) \boxtimes \mathrm{H}^1_{\mathrm{B}}(Y_2)\big)(1)$. We find that $\mathrm{G}_{\mathrm{B}}(H)$ is the image of the canonical map $\mathrm{SL}_{2,\mathbb{Q}} \times \mathrm{SL}_{2,\mathbb{Q}} \to \mathrm{GL}(H)$. The kernel of this map is $\langle(-1,-1)\rangle$.
6. The elliptic curves are not isogenous, one has endomorphism algebra $\mathbb{Q}$, and the other has CM by an imaginary quadratic extension $E/\mathbb{Q}$. In this case $\rho = 2$ and $\mathrm{G}_{\mathrm{B}}(H)$ is of type $\mathrm{U}_{2,\mathbb{Q}}$.
7. The elliptic curves are not isogenous, and Y_i (for $i = 1, 2$) has CM by an imaginary quadratic extension $E_i/\mathbb{Q}$. Observe that $E_1 \ncong E_2$, since Y_1 and Y_2 are not isogenous. Let $E/\mathbb{Q}$ be the compositum of E_1 and E_2, which is a CM field of degree 4 over $\mathbb{Q}$. In this case $\rho = 2$ and $\mathrm{G}_{\mathrm{B}}(H)$ is of type $\mathrm{Res}_{E_0/\mathbb{Q}} \mathrm{U}_{1,E_0}$.
8. The elliptic curves are isogenous and have trivial endomorphism algebra. In this case $\rho = 3$ and $\mathrm{G}_{\mathrm{B}}(H)$ is of type $\mathrm{SO}_{3,\mathbb{Q}}$.
9. The elliptic curves are isogenous and have CM by an imaginary quadratic extension $E/\mathbb{Q}$. In this case $\rho = 4$ and $\mathrm{G}_{\mathrm{B}}(H)$ is of type $\mathrm{U}_{1,\mathbb{Q}}$.

7 Main theorem: the Mumford–Tate conjecture for the product of an abelian surface and a K3 surface

7.1 — Let K be a finitely generated subfield of $\mathbb{C}$. Let A be an abelian surface over K, and let M_A denote the transcendental part of the motive $\mathrm{H}^2(A)(1)$. (The Hodge structure H in remark 6.6 is the Betti realisation $\mathrm{H}_{\mathrm{B}}(M_A)$ of M_A.) Let X be a K3 surface over K, and let M_X denote the transcendental part of the motive $\mathrm{H}^2(X)(1)$. Let E_A (resp. E_X) be the endomorphism algebra of M_A (resp. M_X).

Recall from §6.4 that we associated a field F and a group $\mathscr{G}$ with every Hodge structure V of K3 type. The important properties of F and G are that

» $\mathscr{G}^{\mathrm{der}}$ is an almost direct product of absolutely simple groups over F; and
» $\mathrm{Res}_{F/\mathbb{Q}} \mathscr{G}$ is isogenous to $\mathrm{G}_{\mathrm{B}}(V)$.

Let F_A and $\mathscr{G}_A$ be the field and group associated with $\mathrm{H}_{\mathrm{B}}(M_A)$ as in §6.4. Similarly, let F_X and $\mathscr{G}_X$ be the field and group associated with $\mathrm{H}_{\mathrm{B}}(M_X)$.

Concretely, for F_A this means that

$$F_A \cong \begin{cases} \mathrm{End}(A) \otimes \mathbb{Q} & \text{in case 6.6.2 (so } F_A \text{ is TR of degree 2)} \\ E_{A,0} & \text{in cases 6.6.4 and 6.6.7 (so } F_A \text{ is TR of degree 2)} \\ \mathbb{Q} & \text{otherwise.} \end{cases}$$

We summarise the notation for easy review during later parts of this section:

K	finitely generated subfield of $\mathbb{C}$
A	abelian surface over K
M_A	transcendental part of the motive $\mathrm{H}^2(A)(1)$
F_A	field associated with the Hodge structure $\mathrm{H_B}(M_A)$, as in §6.4
$\mathscr{G}_A$	group over F_A such that $\mathrm{Res}_{F_A/\mathbb{Q}}\mathscr{G}_A$ is isogenous to $\mathrm{G_B}(M_A)$, as in §6.4
X	K3 surface over K
M_X	transcendental part of the motive $\mathrm{H}^2(X)(1)$
F_X	field associated with the Hodge structure $\mathrm{H_B}(M_X)$, as in §6.4
$\mathscr{G}_X$	group over F_X such that $\mathrm{Res}_{F_X/\mathbb{Q}}\mathscr{G}_X$ is isogenous to $\mathrm{G_B}(M_X)$, as in §6.4
E_X	the endomorphism algebra of M_X

The proof of the main theorem (1.1) will take the remainder of this article. There are four main parts going into the proof, which are lemmas 7.4, 7.6, 7.7 and 7.9. The lemmas 7.2, 7.3 and 7.8 and corollary 7.10 are small reductions and intermediate results. Together lemmas 7.4, 7.6 and 7.7 deal with almost all combinations of abelian surfaces and K3 surfaces. Lemma 7.9 is rather technical, and is the only place in the proof where we use that M_X really is a motive coming from a K3 surface.

7.2 LEMMA. —
- » *The Mumford–Tate conjecture for* $\mathrm{H}^2(A\times X)(1)$ *is equivalent to* $\mathrm{MTC}(M_A \oplus M_X)$.
- » *The ℓ-adic realisations of $M_A \oplus M_X$ form a compatible system of ℓ-adic representations.*

Proof. The first claim is because $\mathrm{H}^2(A\times X)(1)$ and $M_A \oplus M_X$ generate the same Tannakian subcategory of motives. By théorème 1.6 of [8], the $\mathrm{H}^2_\ell(A\times X)(1)$ form a compatible system of ℓ-adic representations and we only remove Tate classes to obtain $\mathrm{H}_\ell(M_A \oplus M_X)$; hence the ℓ-adic realisations of $M_A \oplus M_X$ also form a compatible system of ℓ-adic representations. □

7.3 LEMMA. — *If for some prime ℓ, the natural morphism*

$$\iota_\ell \colon \mathrm{G}^\circ_\ell(M_A \oplus M_X)^{\mathrm{der}} \hookrightarrow \mathrm{G}^\circ_\ell(M_A)^{\mathrm{der}} \times \mathrm{G}^\circ_\ell(M_X)^{\mathrm{der}}$$

is an isomorphism (that is, if condition 4.6 holds), then the Mumford–Tate conjecture for $M_A \oplus M_X$ is true.

Proof. The absolute rank of $\mathrm{G_B}(M_A \oplus M_X)$ is bounded from above by the sum of the absolute ranks of $\mathrm{G_B}(M_A)^{\mathrm{der}}$, $\mathrm{G_B}(M_X)^{\mathrm{der}}$, and the centre of $\mathrm{G_B}(M_A \oplus M_X)$. By proposition 4.4, we know that the Mumford–Tate conjecture for $M_A \oplus M_X$ is true on the centres of $\mathrm{G_B}(M_A \oplus M_X) \otimes \mathbb{Q}_\ell$ and $\mathrm{G}^\circ_\ell(M_A \oplus M_X)$. Hence if $\iota_\ell \colon \mathrm{G}^\circ_\ell(M_A \oplus M_X)^{\mathrm{der}} \hookrightarrow \mathrm{G}^\circ_\ell(M_A)^{\mathrm{der}} \times \mathrm{G}^\circ_\ell(M_X)^{\mathrm{der}}$ is an isomorphism, then the absolute rank of $\mathrm{G}_\ell^\circ(M_A \oplus M_X)$ is bounded from below by the sum of the absolute ranks of $\mathrm{G}_\mathrm{B}(M_A)^\mathrm{der}$, $\mathrm{G}_\mathrm{B}(M_X)^\mathrm{der}$, and the centre of $\mathrm{G}_\mathrm{B}(M_A \oplus M_X)$. The result follows from lemma 4.5. □

7.4 Lemma. — *If $F_A \ncong F_X$, then* $\mathrm{MTC}(M_A \oplus M_X)$ *is true.*

Proof. By lemma 7.3 we are done if there is some prime number ℓ, for which $\iota_\ell \colon \mathrm{G}_\ell^\circ(M_A \oplus M_X)^\mathrm{der} \hookrightarrow \mathrm{G}_\ell^\circ(M_A)^\mathrm{der} \times \mathrm{G}_\ell^\circ(M_X)^\mathrm{der}$ is an isomorphism. Hence assume that for all ℓ, the morphism ι_ℓ is not an isomorphism. This will imply that $F_A \cong F_X$.

By lemma 3.5, we see that $F_{A,\ell} = F_A \otimes \mathbb{Q}_\ell$ and $F_{X,\ell} = F_X \otimes \mathbb{Q}_\ell$ have an isomorphic factor. If F_A is isomorphic to $\mathbb{Q}$, then $F_{X,\ell}$ has a factor $\mathbb{Q}_\ell$ for each ℓ, and we win by lemma 2.2.

Next suppose that $F_A \ncong \mathbb{Q}$, in which case F_A is a real quadratic extension of $\mathbb{Q}$. If $\mathscr{G}_X^\mathrm{der}$ is not an absolutely simple group, then it is of type SO_{4,E_X}. In particular $\dim_{E_X}(M_X) = 4$ and $F_X \cong E_X$. Since $\dim_\mathbb{Q}(M_X) \le 22$ we find $[F_X : \mathbb{Q}] \le 5$, and we conclude by lemma 2.3.

Finally, suppose that $\mathscr{G}_X^\mathrm{der}$ is an absolutely simple group over F_X. We want to apply lemma 2.2, so we need to show that $F_{A,\ell}$ is a factor of $F_{X,\ell}$, for all prime numbers ℓ. For the primes that are inert in $F_{A,\ell}$ this is obvious. We are thus left to show that $F_{X,\ell}$ has at least two factors $\mathbb{Q}_\ell$ for every prime ℓ that splits in F_A.

Note that $\mathscr{G}_A^\mathrm{der}$ is an absolutely simple group over F_A of Lie type A_1. Using remark 3.1 we find, for each prime ℓ, semisimple Lie algebras $\mathfrak{s}_{A,\ell}$, $\mathfrak{t}_\ell$ and $\mathfrak{s}_{X,\ell}$ such that

$$\begin{aligned}
\mathrm{Lie}(\mathscr{G}_A^\mathrm{der}) \cong \mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A)^\mathrm{der}) &\cong \mathfrak{s}_{A,\ell} \oplus \mathfrak{t}_\ell \\
\mathrm{Lie}(\mathscr{G}_X^\mathrm{der}) \cong \mathrm{Lie}(\mathrm{G}_\ell^\circ(M_X)^\mathrm{der}) &\cong \qquad\quad \mathfrak{t}_\ell \oplus \mathfrak{s}_{X,\ell} \\
\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A \oplus M_X)^\mathrm{der}) &\cong \mathfrak{s}_{A,\ell} \oplus \mathfrak{t}_\ell \oplus \mathfrak{s}_{X,\ell}.
\end{aligned}$$

The absolute ranks of $\mathrm{G}_\ell^\circ(M_A)^\mathrm{der}$, $\mathrm{G}_\ell^\circ(M_X)^\mathrm{der}$, and $\mathrm{G}_\ell^\circ(M_A \oplus M_X)^\mathrm{der}$ do not depend on ℓ, by proposition 4.4 and remark 6.13 of [16] (or the letters of Serre to Ribet in [28]).[3] Since the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

is invertible, we find that the absolute ranks of the Lie algebras $\mathfrak{s}_{A,\ell}$, $\mathfrak{t}_\ell$, and $\mathfrak{s}_{X,\ell}$ do not depend on ℓ,

If ℓ is a prime that is inert in F_A, then $\mathscr{G}_A^\mathrm{der} \otimes_{F_A} F_{A,\ell}$ is an absolutely simple group. Since $\mathfrak{t}_\ell \ne 0$, we conclude that $\mathfrak{s}_{A,\ell} = 0$. By the independence of the absolute ranks, $\mathfrak{s}_{A,\ell} = 0$ for all primes ℓ. Consequently, if ℓ is a prime that splits in F_A, then $\mathfrak{t}_\ell$ has two simple factors that are absolutely simple Lie algebras

[3] More generally, Hui proved that for every semisimple system of compatible representations the semisimple rank does not depend on ℓ, see theorem 3.19 of [14].

over $\mathbb{Q}_\ell$ of Lie type A_1. Since $\mathscr{G}_X^{\mathrm{der}}$ is an absolutely simple group over F_X, we conclude that $F_{X,\ell}$ has at least two factors $\mathbb{Q}_\ell$, for every prime ℓ that splits in F_A. □

7.5 — From now on, we assume that $F_A \cong F_X$, which we will simply denote with F. We single out the following cases, and prove the Mumford–Tate conjecture for $M_A \oplus M_X$ for all other cases in the next lemma.

1. $\mathrm{G_B}(M_A)$ and $\mathrm{G_B}(M_X)$ are both of type $\mathrm{SO}_{5,\mathbb{Q}}$;
2. $\mathrm{G_B}(M_A)$ is of type $\mathrm{SO}_{3,\mathbb{Q}}$, or $\mathrm{SO}_{4,\mathbb{Q}}$, or $\mathrm{U}_{2,\mathbb{Q}}$, and the type of $\mathrm{G_B}(M_X)$ is also one of these types;
3. F is a real quadratic extension of $\mathbb{Q}$, A is an absolutely simple abelian surface with endomorphisms by F (so $\mathscr{G}_A \cong \mathrm{SL}_{2,F}$), and
    1. $\mathrm{G_B}(M_X)$ is of type $\mathrm{SO}_{3,F}$ or $\mathrm{U}_{2,F}$; or
    2. $\mathrm{G_B}(M_X)$ is non-simple of type $\mathrm{SO}_{4,F}$ as in case 6.3.1 of remark 6.3.

Note that in the third case we did not forget case 6.3.2, since that is covered in case 2. We point out that in the first two cases $\dim(M_X) \leq 5$, which can be deduced from theorem 6.2.

7.6 Lemma. — *If M_A and M_X do not fall into one of the cases listed in §7.5, then the Mumford–Tate conjecture for $M_A \oplus M_X$ is true.*

Proof. By lemma 7.3 we are done if there is some prime number ℓ, for which $\iota_\ell \colon \mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}} \hookrightarrow \mathrm{G}_\ell^\circ(M_A)^{\mathrm{der}} \times \mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}}$ is an isomorphism.

Recall that $\mathbb{C} \cong \overline{\mathbb{Q}_\ell}$, as fields. If the Dynkin diagram of $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A)^{\mathrm{der}})_\mathbb{C}$ and the Dynkin diagram of $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}})_\mathbb{C}$ have no common components, then ι_ℓ must be an isomorphism, and we win. Recall that $\mathrm{MTC}(M_A)$ and $\mathrm{MTC}(M_X)$ are known. Thus ι_ℓ is an isomorphism if the Dynkin diagram of $\mathrm{Lie}(\mathrm{G_B}(M_A)^{\mathrm{der}})_\mathbb{C}$ and the Dynkin diagram of $\mathrm{Lie}(\mathrm{G_B}(M_X)^{\mathrm{der}})_\mathbb{C}$ have no common components. By inspection of theorem 6.2 and remark 6.6, we see that this holds, except for the cases listed in §7.5. □

7.7 Lemma. — *The Mumford–Tate conjecture for $M_A \oplus M_X$ is true if $\dim(M_X) \leq 5$. In particular, the Mumford–Tate conjecture is true for the first two cases listed in §7.5.*

Proof. If $\dim(M_X) = 2$, then $\mathrm{G_B}(M_X)$ is commutative, and we are done by lemma 7.3. Let B be the Kuga–Satake variety associated with $\mathrm{H_B}(M_X)$. This is a complex abelian variety of dimension $2^{\dim(M_X)-2}$. Up to a finitely generated extension of K, we may assume that B is defined over K. (In fact, B *is* defined over K, by work of Rizov, [25].) We may and do allow ourselves a finite extension of K, to assure that B is isogenous to a product of absolutely simple abelian varieties over K. By proposition 6.4.3 of [1] we deduce that M_X is a submotive of $\mathrm{End}(\mathrm{H}^1(B))$. (Alternatively, see proposition 6.3.3 of [12] for a direct argument that $\mathrm{H_B}(M_X)$ is a sub-$\mathbb{Q}$-Hodge structure of $\mathrm{End}(\mathrm{H}^1_\mathrm{B}(B))$; and use that M_X is an abelian motive together with theorem 4.2.) Consequently, $\mathrm{MTC}(A \times B)$ implies $\mathrm{MTC}(M_A \oplus M_X)$, for if the Mumford–Tate conjecture holds for a motive M, then it holds for all motives in the Tannakian subcategory $\langle M \rangle^\otimes$ generated by M.

Recall that the even Clifford algebra $C^+(M_X) = C^+(\mathrm{H_B}(M_X))$ acts faithfully on B. Theorem 7.7 of [12] gives a description of $C^+(M_X)$; thus describing a subalgebra of $\mathrm{End}^0(B)$.

» If $\dim(M_X) = 3$, then $\dim(B) = 2$ and $C^+(M_X)$ is a quaternion algebra over $\mathbb{Q}$.

» If $\dim(M_X) = 4$, then $\dim(B) = 4$ and $C^+(M_X)$ is either a product $D \times D$, where D is a quaternion algebra over $\mathbb{Q}$; or $C^+(M_X)$ is a quaternion algebra over a totally real quadratic extension of $\mathbb{Q}$.

» If $\dim(M_X) = 5$, then $\dim(B) = 8$ and $C^+(M_X)$ is a matrix algebra $M_2(D)$, where D is a quaternion algebra over $\mathbb{Q}$.

We claim that $A \times B$ satisfies the conditions of lemma 5.5. First of all, observe that A satisfies those conditions, which can easily be seen by reviewing remark 6.6. We are done if we check that B satisfies the conditions as well.

» If $\dim(M_X) = 3$, then B is either a simple abelian surface, or isogenous to the square of an elliptic curve. In both cases, B satisfies the conditions of lemma 5.5.

» If $\dim(M_X) = 4$, and $C^+(M_X)$ is $D \times D$ for some quaternion algebra D over $\mathbb{Q}$, then B splits (up to isogeny) as $B_1 \times B_2$. In particular $\dim(B_i) = 2$, since D cannot be the endomorphism algebra of an elliptic curve. Hence both B_i satisfy the conditions of lemma 5.5.

  On the other hand, if $\dim(M_X) = 4$ and $C^+(M_X)$ is a quaternion algebra over a totally real quadratic extension of $\mathbb{Q}$, then there are two options.

  » If B is not absolutely simple, then all simple factors have dimension ≤ 2; since $\mathrm{End}^0(B)$ is non-commutative. Indeed, the product of an elliptic curve and a simple abelian threefold has commutative endomorphism ring (see, *e.g.,* section 2 of [21]).

  » If B is absolutely simple, then it has relative dimension 1. This abelian fourfold must be of type II(2), since type III(2) does not occur (see proposition 15 of [29], or table 1 of [20] which also proves $\mathrm{MTC}(B)$).

  In both of these cases, B satisfies the conditions of lemma 5.5.

» If $\dim(M_X) = 5$, then B is the square of an abelian fourfold C whose endomorphism algebra contains a quaternion algebra over $\mathbb{Q}$.

  » If C is not absolutely simple, then all simple factors must have dimension ≤ 2; since $\mathrm{End}^0(C)$ is non-commutative.

  » If C is simple, then we claim that C must be of type II. Indeed, the Mumford–Tate group of B surjects onto $\mathrm{G_B}(M_X)$, because $\mathrm{H_B}(M_X)$ is a sub-$\mathbb{Q}$-Hodge structure of $\mathrm{End}(\mathrm{H}^1_\mathrm{B}(B))$. Now $\dim(M_X) = 5$, hence $\mathrm{G_B}(M_X)$ is of type $\mathrm{SO}_{5,\mathbb{Q}}$, with Lie type B_2. But §6.1 of [20] shows that if C is of type III, then $\mathrm{G_B}(C)$ has Lie type $D_2 \cong A_1 \oplus A_1$. This proves our claim. Since $\mathrm{End}^0(C)$ is a quaternion algebra and C is an abelian fourfold, table 1 of [20] shows that $\mathrm{MTC}(C)$ is true and D_4 does not occur in the Lie type of $\mathrm{G_B}(C)$.

We conclude that $\mathrm{MTC}(A \times B)$ is true, and therefore $\mathrm{MTC}(M_A \oplus M_X)$ is true as well. □

The only cases left are those listed in case 7.5.3 of §7.5. Therefore, we may and do assume that F is a real quadratic field extension of $\mathbb{Q}$; and that A is an absolutely simple abelian surface with endomorphisms by F (*i.e.,* case 6.6.2). In particular $\mathscr{G}_A = \mathrm{SL}_{2,F}$.

7.8 LEMMA. — *If X falls in one of the subcases listed in case 7.5.3, then there exists a place λ of F such that $\mathscr{G}_X^{\mathrm{der}} \otimes_F F_\lambda$ does not contain a split factor.*
Proof. In case 7.5.3.1, $\mathscr{G}_X^{\mathrm{der}}$ is of Lie type A_1. In case 7.5.3.2, $\mathscr{G}_X \sim N \times N^{\mathrm{op}}$, where N is a form of $\mathrm{SL}_{2,F}$, as explained in remark 6.3. By theorem 26.9 of [15], there is an equivalence between forms of SL_2 over a field, and quaternion algebras over the same field. We find a quaternion algebra D over F corresponding to $\mathscr{G}_X^{\mathrm{der}}$, respectively N, in case 7.5.3.1, respectively case 7.5.3.2. In particular $\mathscr{G}_X^{\mathrm{der}}$ contains a split factor if and only if the quaternion algebra is split.
Let $\{\sigma, \tau\}$ be the set of embeddings $\mathrm{Hom}(F, \mathbb{R})$. Since F acts on $\mathrm{H_B}(M_X)$, we see that $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{(\sigma)} \oplus \mathbb{R}^{(\tau)}$ acts on

$$\mathrm{H_B}(M_X) \otimes_{\mathbb{Q}} \mathbb{R} \cong W^{(\sigma)} \oplus W^{(\tau)}.$$

Here $W^{(\sigma)}$ and $W^{(\tau)}$ are $\mathbb{R}$-Hodge structures of dimension $\dim_F(M_X)$. Observe that the polarisation form is definite on one of the terms, while it is non-definite on the other. Without loss of generality we may assume that the polarisation form is definite on $W^{(\sigma)}$, and non-definite on $W^{(\tau)}$.
Thus, the group $\mathrm{G_B}(M_X) \otimes_{\mathbb{Q}} \mathbb{R}$ is the product of a compact group and a non-compact group; and therefore, $\mathrm{Res}_{F/\mathbb{Q}} \mathscr{G}_X \otimes_{\mathbb{Q}} \mathbb{R}$ is the product of a compact group and a non-compact group. Indeed $\mathscr{G}_X \otimes_F \mathbb{R}^{(\sigma)}$ is compact, while $\mathscr{G}_X \otimes_F \mathbb{R}^{(\tau)}$ is non-compact. By the first paragraph of the proof, this means that $D \otimes_F \mathbb{R}^{(\sigma)}$ is non-split, while $D \otimes_F \mathbb{R}^{(\tau)}$ is split.
Since the Brauer invariants of D at the infinite places do not add up to 0, there must be a finite place λ of F such that D_λ is non-split. At this place λ, the group $\mathscr{G}_X^{\mathrm{der}} \otimes_F F_\lambda$ does not have a split factor. □

7.9 LEMMA. — *Assume that K is a number field. If X falls in one of the subcases listed in case 7.5.3, then there is a prime number ℓ for which the natural map*

$$\iota_\ell \colon \mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}} \hookrightarrow \mathrm{G}_\ell^\circ(M_A)^{\mathrm{der}} \times \mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}}$$

is an isomorphism.
Proof. The absolute rank of $\mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}}$ does not depend on ℓ, by proposition 4.4 and lemma 7.2 and remark 6.13 of [16] (or the letters of Serre to Ribet in [28], or theorem 3.19 of [14]). We now show that this absolute rank must be even, by looking at a prime ℓ that is inert in F. At such a prime ℓ all simple factors of $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A)^{\mathrm{der}} \times \mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}})$ are $\mathbb{Q}_\ell$-Lie algebras with even absolute rank (since $[F : \mathbb{Q}] = 2$). By remark 3.1, the Lie algebra of $\mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}}$ is a summand of $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A)^{\mathrm{der}} \times \mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}})$, and therefore the absolute rank of $\mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}}$ must be even.

Let λ be one of the places of F found in lemma 7.8, and let ℓ be the place of $\mathbb{Q}$ lying below λ. Since $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}})$ must surject to $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A)^{\mathrm{der}})$ (which is split, and has absolute rank 2), and $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}})$ must also surject onto $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}})$, which has a non-split factor, by lemma 7.8, we conclude that the absolute rank of $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}})$ must be at least 3. By the previous paragraph, we find that the absolute rank must be at least 4.
If $\dim_{E_X}(M_X) \neq 4$ (case 7.5.3.1) then $\mathscr{G}_X^{\mathrm{der}}$ is a group of Lie type A_1, and therefore the product $\mathrm{G}_\ell^\circ(M_A)^{\mathrm{der}} \times \mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}}$ has absolute rank 4. Hence $\mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}}$ must have absolute rank 4, which means that ι_ℓ is an isomorphism, by remark 3.1.
If $\dim_{E_X}(M_X) = 4$ (case 7.5.3.2), then $\mathscr{G}_X$ is a group of Lie type $D_2 = A_1 \oplus A_1$. (Note that in this final case $\mathrm{G_B}(M_A)$ and $\mathrm{G_B}(M_X)$ are semisimple, and therefore we may drop all the superscripts $(\_)^{\mathrm{der}}$ from the notation.) Since in this case $\mathrm{G}_\ell^\circ(M_A) \times \mathrm{G}_\ell^\circ(M_X)$ has absolute rank 6, and the absolute rank of $\mathrm{G}_\ell^\circ(M_A \oplus M_X)$ is ≥ 4, it must be 4 or 6 (since it is even).
Suppose $\mathrm{G}_\ell^\circ(M_A \oplus M_X)$ has absolute rank 4. We apply remark 3.1 to the current situation, and find Lie algebras $\mathfrak{t}$ and $\mathfrak{s}_2$ over $\mathbb{Q}_\ell$ such that $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A)) \cong \mathfrak{t}$ and $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A \oplus M_X)) \cong \mathrm{Lie}(\mathrm{G}_\ell^\circ(M_X)) \cong \mathfrak{t} \oplus \mathfrak{s}_2$. In particular, $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_X))$, which is isomorphic to $\mathrm{Lie}(\mathscr{G}_X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, has a split simple factor. By lemma 7.8 this means that ℓ splits in F as $\lambda \cdot \lambda'$. Observe that $F_\lambda \cong \mathbb{Q}_\ell \cong F_{\lambda'}$.
The fact that $E_{X,\ell} \cong F_\ell$ factors as $F_\lambda \times F_{\lambda'}$ has several implications. In the case under consideration we have $\mathrm{G_B}(M_X) \cong \mathrm{Res}_{F/\mathbb{Q}} \mathscr{G}_X$, and since $\mathrm{MTC}(M_X)$ holds, this implies $\mathrm{G}_\ell^\circ(M_X) \cong \mathscr{G}_{X,\lambda} \times \mathscr{G}_{X,\lambda'}$. Besides that, $\mathrm{H}_\ell(M_X)$ decomposes as $\mathrm{H}_\lambda(M_X) \oplus \mathrm{H}_{\lambda'}(M_X)$. The group $\mathrm{Gal}(\bar{K}/K)$ acts on $\mathrm{H}_\lambda(M_X)$ via $\mathscr{G}_{X,\lambda}$, and on $\mathrm{H}_{\lambda'}(M_X)$ via $\mathscr{G}_{X,\lambda'}$.
To summarise, our situation is now as follows. The prime number ℓ splits in F as $\lambda \cdot \lambda'$. The group $\mathscr{G}_A$ is isomorphic to $\mathrm{SL}_{2,F}$, and is split and simply connected. The group $\mathscr{G}_{X,\lambda'}$ is split, of type $\mathrm{SO}_{4,\mathbb{Q}_\ell}$, with Lie algebra $\mathfrak{t}$. The group $\mathscr{G}_{X,\lambda}$ is non-split, of type $\mathrm{SO}_{4,\mathbb{Q}_\ell}$, with Lie algebra $\mathfrak{s}_2$. Recall the natural diagram:

$$\begin{tikzcd}
 & \mathrm{G}_\ell^\circ(M_A \oplus M_X) \arrow[d, "\iota_\ell"] & \\
 & \mathrm{G}_\ell^\circ(M_A) \times \mathrm{G}_\ell^\circ(M_X) \arrow[dl] \arrow[dr] & \\
(\mathrm{SL}_{2,\mathbb{Q}_\ell} \times \mathrm{SL}_{2,\mathbb{Q}_\ell})/\langle(-1,-1)\rangle \cong \mathrm{G}_\ell^\circ(M_A) & & \mathrm{G}_\ell^\circ(M_X) \cong \mathscr{G}_{X,\lambda'} \times \mathscr{G}_{X,\lambda}
\end{tikzcd}$$

We are now set for the attack. We claim that the Galois representations $\mathrm{H}_\ell(M_A)$ and $\mathrm{H}_{\lambda'}(M_X)$ are isomorphic. Indeed, from the previous paragraph we conclude that $\mathrm{G}_\ell^\circ(M_A \oplus M_X) \cong \Gamma \times \mathscr{G}_{X,\lambda}$, where Γ is a subgroup of $\mathrm{G}_\ell^\circ(M_A) \times \mathscr{G}_{X,\lambda'}$ with surjective projections. Thus $\mathrm{H}_\ell(M_A)$ and $\mathrm{H}_{\lambda'}(M_X)$ are both orthogonal representations of $\mathrm{Gal}(\bar{K}/K)$, and the action of Galois factors via $\Gamma(\mathbb{Q}_\ell)$. We will now show that $\mathrm{H}_\ell(M_A)$ and $\mathrm{H}_\ell(M_X)$ are isomorphic as representations

of Γ, which proves the claim.
The Lie algebra of Γ is isomorphic to $\mathfrak{t}$, and $\mathrm{Lie}(\Gamma)$ is the graph of an isomorphism $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A)) \to \mathrm{Lie}(\mathscr{G}_{X,\lambda'})$. Since $\mathrm{G}_\ell^\circ(M_A)$ and $\mathscr{G}_{X,\lambda'}$ are both covered by $\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{SL}_{2,F} \cong \mathrm{Hdg}_\ell(A)$ with kernels $\{\pm 1\}$, and Γ is a subgroup of $\mathrm{G}_\ell^\circ(M_A) \times \mathscr{G}_{X,\lambda'}$, we find that Γ also has a $(2:1)$-cover by $\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{SL}_{2,F}$. Hence Γ is the graph of an isomorphism $\mathrm{G}_\ell^\circ(M_A) \to \mathscr{G}_{X,\lambda'}$. Because $\mathrm{H}_\ell(M_A)$ and $\mathrm{H}_{\lambda'}(M_X)$ are 4-dimensional faithful orthogonal representations of Γ, they must be isomorphic; for up to isomorphism, there is a unique such representation.
As a consequence, for all places v of K, the characteristic polynomial of Frob_v acting on $\mathrm{H}_\ell(M_A)$ coincides with its characteristic polynomial when acting on $\mathrm{H}_{\lambda'}(M_X)$. We conclude that $\mathrm{charpol}_{F_{\lambda'}}(\mathrm{Frob}_v|\mathrm{H}_{\lambda'}(M_X))$ has coefficients in $\mathbb{Q}$. But then the same is true for $\mathrm{charpol}_{F_\lambda}(\mathrm{Frob}_v|\mathrm{H}_\lambda(M_X))$ since their product is $\mathrm{charpol}_{\mathbb{Q}_\ell}(\mathrm{Frob}_v|\mathrm{H}_\ell(M_X))$, which has coefficients in $\mathbb{Q}$. In conclusion, $\mathrm{charpol}_{\mathbb{Q}_\ell}(\mathrm{Frob}_v|\mathrm{H}_\ell(M_X))$ factors over $\mathbb{Q}$ as

$$\mathrm{charpol}_{F_\lambda}(\mathrm{Frob}_v|\mathrm{H}_\lambda(M_X)) \cdot \mathrm{charpol}_{F_{\lambda'}}(\mathrm{Frob}_v|\mathrm{H}_{\lambda'}(M_X)).$$

This leads to a contradiction with the following facts.
Since we assumed that K is a number field, the following results hold.

» The main theorem of [4], which tells us that (up to a finite extension of K) there exists a set $\mathscr{V}$ of places of K with density 1 such that X has good and ordinary reduction at places $v \in \mathscr{V}$.
» Theorem 1 (item 1) of [6], which tells us that (up to another finite extension of K) there exists a set $\mathscr{V}$ of places of K with density 1 such that X has good reduction at places $v \in \mathscr{V}$, and the Picard number of the reduction X_v is the same as that of X (which, in our case is $22 - 8 = 14$).
» Proposition 3.2 of [34], which says that if X has good and ordinary reduction at v, then the characteristic polynomial $\mathrm{charpol}_{\mathbb{Q}_\ell}(\mathrm{Frob}_v|\mathrm{H}^2_\ell(X_v)^{\mathrm{tra}})$ is an irreducible polynomial with coefficients in $\mathbb{Q}$.

Thus $\mathrm{charpol}_{\mathbb{Q}_\ell}(\mathrm{Frob}_v|\mathrm{H}_\ell(M_X))$ is irreducible for a density 1 subset of places v; which contradicts the factorisation found above. We conclude that the absolute rank of $\mathrm{G}_\ell^\circ(M_A \oplus M_X)$ cannot be 4 and therefore it must be 6, which implies that ι_ℓ is an isomorphism. □

7.10 COROLLARY. — *If X falls in one of the subcases listed in case 7.5.3, then the Mumford–Tate conjecture is true for $M_A \oplus M_X$.*
Proof. This result follows from lemmas 4.8 and 7.9. □

7.11 PROOF OF THEOREM 1.1. — By lemma 7.2 the main theorem reduces to the Mumford–Tate conjecture for $M_A \oplus M_X$. The theorem follows from lemmas 7.4, 7.6 and 7.7 and corollary 7.10. □

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