Structure Theorem of Kummer Etale K-Group II

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Abstract. In this article, we investigate the lambda-ring structure of Kummer etale K-groups for some class of logarithmic schemes, up to torsion. In particular, we give a logarithmic analogue of Chow groups for the logarithmic schemes, and describe its structure.

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1 Introduction

In [Hag03], for a wide class of logarithmic varieties over a separably closed field, we gave an explicit description of their Kummer etale K-groups in terms of the usual K-groups of the associated stratifications. However, this description is still unsatisfactory in that it disregards λ-ring structures, with which every Kummer etale K_0-group is naturally endowed through exterior product operations. Since, already in the classical case, these structures play essential roles, for instance, for a definition of (rational) motivic cohomology, this defect should be overcome.

So in this paper, for a logarithmic variety satisfying some good conditions, we construct an isomorphism between its Kummer etale K-group and usual K-groups associated with its stratification, preserving their λ-ring structures up to torsion.

More precisely, we consider the following situation. Let k be a field of characteristic p, X a scheme smooth, separated and of finite type over k, D a strictly normal crossing divisor on X, and $\{D_i\}_{i\in I}$ its irreducible components. We also denote by X the log scheme associated with (X, D). Then the main theorem of this article is as follows:

Theorem 1.1. *(= Theorem 3.23) We have an isomorphism of rings*

$$K_0(X_{\mathrm{Ket}}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \varprojlim_{C_I} K_0(D_{J_2}) \otimes_{\mathbb{Z}} \mathbb{Q}[(\mathbb{Z}_{(p)}/\mathbb{Z})^{\oplus J_1}],$$

which is compatible with the actions of Adams operations (For their actions on the right hand side, see below).

Here, for $J = \{i_1, \cdots, i_r\} \subset I$, we put $D_J = D_{i_1} \cap \cdots \cap D_{i_r}$ (Note that $D_\emptyset = X$), and we set

$$C_I = \{(J_1, J_2) | J_1 \subset J_2 \subset I\}$$

and regard it as an ordered set by defining $(J_1, J_2) \geq (J_1', J_2')$ to be $J_1' \subset J_1 \subset J_2 \subset J_2'$. The transition morphism in the limit is induced by the natural closed inclusion $D_{J_2} \supset D_{J_2'}$ and a projection $(\mathbb{Z}_{(p)}/\mathbb{Z})^{\oplus J_1} \to (\mathbb{Z}_{(p)}/\mathbb{Z})^{\oplus J_1'}$. We consider $\mathbb{Q}[(\mathbb{Z}_{(p)}/\mathbb{Z})^{\oplus J}]$ to be a group ring endowed with ring endomorphisms $\{\Psi^m\}_{m>0}$, which we call Adams operations, defined by $\Psi^m([\alpha]) = [m\alpha]$ for $m \in \mathbb{N}, m > 0$ and $\alpha \in (\mathbb{Z}_{(p)}/\mathbb{Z})^{\oplus J}$. The actions of Adams operations on the right hand side are naturally induced from the usual ones on $K_0(D_{J_2})_{\mathbb{Q}}$ and on $\mathbb{Q}[(\mathbb{Z}_{(p)}/\mathbb{Z})^{\oplus J_1}]$ (For more details, see Subsection 3.3). By this theorem, we can determine completely the actions of Adams operations, and accordingly the λ-ring structure up to torsion, on the left hand side.

We will give some remarks on the proof. As in [Hag03], the fundamental idea of the proof is a local-global argument, that is, the reduction to the local case where we can apply various results in equivariant K-theory. However, there are some differences to be mentioned, between the strategy there and that in this article.

In [Hag03], we focused only on the (Abelian) group structure of the Kummer etale K-group, so we could use a "localisation sequence" for K'-groups and reduce the theorem to the case where the underlying scheme is of the form $\operatorname{Spec} L$ for a field L. However, this method does not work well in the analysis of the λ-ring structure. In fact, this difficulty is not so serious because we have enough tools in simplicial homotopy theory to reduce to the case of the (Henselian) local ring, where we can apply equivariant K-theory effectively.

The more serious difficulty is the high complexity of the Kummer etale K-group of each irreducible component of the divisor. This is already seen in the calculation of Kummer etale K-group of a log point. Let $(\operatorname{Spec} \mathbb{C})^{\log}$ be a log scheme associated with a monoid map from $e^{\mathbb{N}}$ (this denotes the free (multiplicative) monoid with generator e) to $\mathbb{C}$ which send every element except the unit to 0. Then its Kummer etale K-group is isomorphic to a colimit of (usual) equivariant K-groups $K_i(\operatorname{Spec} \mathbb{C}[x]/(x^n), \mu_n)$, which are very complicated because of the existence of nilpotent elements in $\mathbb{C}[x]/(x^n)$.

To avoid this difficulty, we introduce the notion of a "pellicular Kummer etale ringed topos" of an fs log scheme X, which we denote by $(X_{\mathrm{Ket}}, \overline{\mathscr{O}}_{X_{\mathrm{Ket}}})$. This has the same underlying topos as the usual Kummer etale topos, but its structure sheaf $\overline{\mathscr{O}}_X$ associates with each "open set" a ring $\Gamma(U, \mathscr{O}_U)^{\mathrm{red}}$ (not a ring

$\Gamma(U, \mathscr{O}_U)$). Upon this notion we can develop general theory of $\overline{\mathscr{O}}_X$-modules and define the corresponding K-(and K'-)groups, which we denote by $\overline{K}$ (resp. $\overline{K}'$) and call pellicular Kummer etale K-(resp. K'-)groups.

The advantage of this new K-group is that it is extremely easier to compute than the usual Kummer etale K-group. For instance, in the case of a log point above, its pellicular Kummer etale K-group is a colimit of equivariant K-groups $K_i(\operatorname{Spec}\mathbb{C}, \mu_n)$, where the action of μ_n is trivial (The name "pellicular", which means "film-like", is adopted on the ground that $\overline{\mathscr{O}}_{P_{\mathrm{Ket}}}$ is a "very thin" quotient of $\mathscr{O}_{P_{\mathrm{Ket}}}$, for instance, if P is a log point). It is easily checked that this is isomorphic to $K_i(\operatorname{Spec}\mathbb{C}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$.

Fortunately, the totality of the pellicular Kummer etale K-groups of D_J's has sufficient information for the recovery of the Kummer etale K-group of the ambient log scheme.

Another difficulty is the problem of how we carve the factor which "purely" corresponds to each D_J out of the Kummer etale K-group of the ambient log scheme. For instance, let X be a curve with a divisor D consisting of one point, and $i : D \to X$ the natural closed immersion. Then it is (or at least seems) highly difficult to separate the information corresponding only to X (i.e. the usual K-group of the underlying scheme of X) from the Kummer etale K-group of X without ignoring the λ-ring structure.

In order to get over this difficulty, we prove an "inversion formula", which enables us to interpret a Kummer etale K-group as a generalised cohomology of another Kummer etale topos with coefficients in Kummer etale K. Using it, we can construct some maps which seem to go in the "opposite" ways, but which still respect λ-ring structures, up to torsion.

For example, in the above case, we can define a map from the rational Kummer etale K-groups of X to the usual rational K-group of X compatibly with their λ-ring structures. The existence of this map already seems very non-trivial even for K_0 of curves. Note that, as careful readers can notice immediately, such maps appear implicitly in the statement of the structure theorem.

With these tools at hand, the proof (as well as the formulation) of the main theorem is only an easy exercise of simplicial homotopy theory, equivariant K-theory and logarithmic geometry.

Finally we give two remarks. First note that, as a result of the above theorem, for some logarithmic varieties we can define the notion of what we might call "Kummer etale Chow groups" as eigenspaces of Adams operations and can describe them in terms of the usual Chow groups. As an application, by using these objects we can formulate and prove a Riemann-Roch-type theorem for Kummer etale K-groups. These topics and their arithmetic applications will be given in the forthcoming paper.

Secondly, as the readers see immediately, most of our results also hold for higher Kummer etale K-theory. However, in this paper we concentrate on the K_0-case since our motivation comes from its application to number theory via the aforementioned Riemann-Roch-type theorem for the Kummer etale K_0-group and the Kummer etale Chow groups.

Now let us mention the organisation of this paper. First, in Section 2, we review logarithmic geometry and Kummer etale topos, and in Section 3, we introduce some new notions on logarithmic schemes, especially, those of pellicular Kummer etale ringed topos, and formulate the main theorem in a form suitable for some local-to-global arguments. In addition, in this section, we reduce the theorem to some key lemmata. These will be proven in Section 5, after some consideration on the general theory of pellicular Kummer etale ringed topos in Section 4.

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2 Preliminaries

For a ring A, we denote by A^{red} the quotient ring of A by its nilradical. For a scheme X, we denote by $\mathscr{O}_{X_{\mathrm{Zar}}}$ its structure sheaf, and by $\mathbf{Vect}(X_{\mathrm{Zar}})$, $\mathbf{Coh}(X_{\mathrm{Zar}})$, $\mathbf{Qcoh}(X_{\mathrm{Zar}})$ and $\mathbf{Mod}(X_{\mathrm{Zar}})$ the category of coherent locally free $\mathscr{O}_{X_{\mathrm{Zar}}}$-modules, coherent sheaves of $\mathscr{O}_{X_{\mathrm{Zar}}}$-modules, quasi-coherent sheaves of $\mathscr{O}_{X_{\mathrm{Zar}}}$-modules and $\mathscr{O}_{X_{\mathrm{Zar}}}$-modules, respectively.

In the rest of this section, we recall some definitions and propositions in logarithmic geometry. For the notions which are not given in this section and the proofs of propositions omitted here, see [Kat89], [Kat94], [Nak92], [Nak97] and [Hag03].

2.1 Logarithmic geometry

2.1.1 Monoid theory

In this paper, a monoid means a commutative one with a unit (unless otherwise mentioned), and a homomorphism of monoids is always assumed to preserve the units. The symbol $\mathbb{N}$ means the set of integers i such that $i \geq 0$ and is regarded as a monoid by addition, although we write the operations in monoids multiplicatively very often. When given a monoid P, we call a monoid Q equipped with a monoid homomorphism $P \to Q$ a P-monoid.

A subset I of a monoid P is called an ideal if $a \in P$ and $x \in I$ implies $ax \in I$. An ideal is called a prime ideal if its complement is a submonoid of P. A submonoid of P is called a face if it is the complement of a prime ideal.

We call a monoid isomorphic to $\mathbb{N}^r$ for some r a finitely generated free monoid. As is easily seen, there is a natural bijection between the set of prime ideals in a finitely generated free monoid and the power set of the basis of the monoid. We often denote the finitely generated free monoid with basis $e_1, \dots, e_r$ by $e_1^{\mathbb{N}} \cdots e_r^{\mathbb{N}}$. When we are given a monoid M and elements $m_1, \dots, m_r \in M$, the notation $m_1^{\mathbb{N}} \cdots m_r^{\mathbb{N}}$ is also used to represent the submonoid of M generated by $m_1, \dots, m_r$.

For a monoid P, we denote by $P^\times$ the submonoid of P consisting of invertible elements, that is, elements that divide the unit. P is called sharp if and only if $P^\times = \{1\}$. By P^{gp} we mean its group completion, which is naturally defined. For an Abelian group A, we often use the notation P_A instead of $P^{\mathrm{gp}} \otimes_{\mathbb{Z}} A$. In particular, $P_{\mathbb{Z}}$ means P^{gp}.

A monoid P is called integral if $ac = bc$ always implies $a = b$ for any element a, b and c in P, or equivalently, if $P \to P^{\mathrm{gp}}$ is injective. An integral monoid P is called saturated if any element $a \in P^{\mathrm{gp}}$ satisfying $a^n \in P$ for some $n \in \mathbb{N}$ belongs to P. A finitely generated saturated monoid is also called an fs monoid. For a monoid P, P^{sat} denotes its saturation, which is defined by the universality. More precisely, the functor $(-)^{\mathrm{sat}}$ is defined to be a functor left adjoint to the inclusion functor from the category of saturated monoids to that of monoids. Recall that its existence is assured and we have a natural map $P \to P^{\mathrm{sat}}$. For a saturated monoid P and saturated P-monoids Q and R, we set $Q \oplus_P^{\mathrm{sat}} R = (Q \oplus_P R)^{\mathrm{sat}}$, the cofiber product in the category of saturated monoids.

For a monoid P and a natural number n, $P^{1/n}$ is defined to be a P-monoid such that $P \to P^{1/n}$ is isomorphic to the n-th power map $(-)^n : P \to P$, and for a natural number m we set $P^{m'-\mathrm{div}} = \mathrm{colim}_{(n,m)=1} P^{1/n}$. If m is fixed (e.g. m is the characteristic exponent of a fixed scheme) and no risk of confusion is induced, we also use the notation $P^{\mathrm{div}'}$. We also set $\mathbb{Q}' = \mathbb{Z}^{\mathrm{div}'}$ and $(\mathbb{Q}/\mathbb{Z})' = \mathbb{Q}'/\mathbb{Z}$.

A monoid homomorphism $\phi : P \to Q$ from P to Q is called a projection if there exist a monoid R and an isomorphism $f : Q \times R \to P$ such that $\phi \circ f$ equals pr_1, the projection to Q.

2.1.2 Logarithmic schemes

A logarithmic scheme (or a log scheme) is a pair of a scheme X and a homomorphism of etale sheaves of monoids $\alpha_X : M_X \to \mathscr{O}_X$, with $\mathscr{O}_X$ regarded as a sheaf of monoids by multiplication, such that $\alpha_X^{-1}(\mathscr{O}_X^\times) \to \mathscr{O}_X^\times$ is an isomorphism. We always regard the sheaf of invertible functions $\mathscr{O}_X^\times$ as a subsheaf of M_X and set $\overline{M}_X = M_X/\mathscr{O}_X^\times$. We often write X to represent a log scheme (X, M_X), and then, for a log scheme X, $|X|$ denotes its underlying scheme or its underlying topological space (in the preceding paper [Hag03], the notation $\mathring{X}$ is used for underlying schemes). The morphism α_X is called the log structure of X.

When there is no risk of confusion, a log scheme X is called Noetherian, quasi-compact, regular and so on, if its underlying scheme $|X|$ is so. Similarly, we often write "$x \in X$" instead of "$x \in |X|$", and the notation X_{Zar} is used to mean $|X|_{\mathrm{Zar}}$.

A morphism from a log scheme (X, M_X) to a log scheme (Y, M_Y) is a pair of a morphism of schemes $\phi : X \to Y$ and a morphism of monoid sheaves $\phi^* M_Y \to M_X$ compatible with the maps to $\mathscr{O}_X$ (For the definition of the pullback of the log structure $\phi^* M_Y \to \mathscr{O}_X$, see [Kat89]). For a morphism f of log schemes, we denote by $|f|$ the underlying morphism between schemes. We often say, for example, "f is of finite type" instead of saying "$|f|$ is of finite type". A morphism $f : X \to Y$ between log schemes is called strict if for all $x \in X$, $\overline{M}_{Y,\overline{f(x)}} \to \overline{M}_{X,\overline{x}}$ is an isomorphism. A strictly closed (or open) subscheme of a log scheme (X, M_X) is a closed (or open) subscheme $i : Y \hookrightarrow X$ with the induced log structure $i^* M_X$.

For a monoid P we denote by $(\operatorname{Spec}\mathbb{Z}[P], \widetilde{P})$, or simply by $\operatorname{Spec}\mathbb{Z}[P]$, the log scheme whose underlying scheme is $\operatorname{Spec}\mathbb{Z}[P]$ and which is endowed with the log structure induced by the natural homomorphism of monoids $P \to \mathbb{Z}[P]$. A log scheme is called fs if etale locally it has a strict morphism to $\operatorname{Spec}\mathbb{Z}[P]$ with P an fs monoid. This strict morphism is called a chart. We denote by $\times^{\mathrm{fs}}$ the fiber product in the category **FsLogSch** of fs log schemes, to distinguish it from $\times$, the one in the category of log schemes or of schemes. Note that, in general, $|X \times^{\mathrm{fs}}_S Y| \ncong |X \times_S Y| \cong |X| \times_{|S|} |Y|$ for log schemes X and Y over S.

2.1.3 Standard coverings

Definition 1. *Let X be an fs log scheme, P a sharp fs monoid, $X \to \operatorname{Spec}\mathbb{Z}[P]$ a chart and n a natural number.*

1. *We set $X_n = X \times^{\mathrm{fs}}_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[P^{1/n}]$.*

   Note that, since the group $I_n = \operatorname{Hom}(P^{1/n}/P, \mathbb{Z}[\zeta_n]^\times)$ acts linearly on a ring $\mathbb{Z}[\zeta_n][P^{1/n}]$ by
   $$\varphi(a[p]) = (\varphi(p)a)[p],$$
   for $\varphi \in I_n$, $a \in \mathbb{Z}[\zeta_n]$ and $p \in P^{1/n}$, if $|X|$ is a scheme over $\mathbb{Z}[\zeta_n]$, we have a natural action of the group on X_n over X.

2. *We set $\widetilde{X_n} = X \times^{\mathrm{fs}}_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[\zeta_n][P^{1/n}]$.*

   Note that, letting $\Gamma_n = \operatorname{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, we have a natural action of the group $I_n \rtimes \Gamma_n$ on $\widetilde{X_n}$ over X, where the semi-direct product is constructed by the action of Γ_n on I_n such that $\gamma(\varphi) = \gamma \circ \varphi$ for $\gamma \in \Gamma_n$ and $\varphi \in I_n$.

Proposition 2.1. *In the above situation, we have*

$$\begin{aligned} \operatorname{Spec}\mathbb{Z}[P^{1/n}] \times^{\mathrm{fs}}_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[P^{1/n}] &\cong \operatorname{Spec}\mathbb{Z}[P^{1/n} \oplus^{\mathrm{sat}}_P P^{1/n}] \\ &\cong \operatorname{Spec}\mathbb{Z}[P^{1/n} \oplus (P^{1/n}/P)]. \end{aligned}$$

Here the second isomorphism is induced by a monoid isomorphism from $P^{1/n} \oplus^{\mathrm{sat}}_P P^{1/n}$ to $P^{1/n} \oplus (P^{1/n}/P)$ characterised by the property that $(a, b) \in$

$P^{1/n} \oplus P^{1/n}$ is mapped to $(ab, b \bmod P)$ via the composite with a natural map $P^{1/n} \oplus P^{1/n} \to P^{1/n} \oplus_P^{\mathrm{sat}} P^{1/n}$.
In particular,

1. *if $|X|$ is a scheme over $\operatorname{Spec} \mathbb{Z}[\zeta_n, 1/n]$, we have a canonical isomorphism*

$$X_n \times_X^{\mathrm{fs}} X_n \cong \coprod_{\alpha \in I_n} (X_n)_\alpha,$$

where $(X_n)_\alpha$ is a copy of X_n, so that the composite of the natural inclusion $i_\alpha : (X_n)_\alpha \to X_n \times_X^{\mathrm{fs}} X_n$ and the projection $\mathrm{pr}_1 : X_n \times_X^{\mathrm{fs}} X_n \to X_n$ (resp. pr_2) is the identity (resp. the action of $\alpha \in I_n$), and

2. *if $|X|$ is a scheme over $\operatorname{Spec} \mathbb{Z}[1/n]$, we have a canonical isomorphism*

$$\widetilde{X_n} \times_X^{\mathrm{fs}} \widetilde{X_n} \cong \coprod_{\alpha \in I_n \rtimes \Gamma_n} (\widetilde{X_n})_\alpha,$$

in the similar way.

Proof. Straightforward. □

2.1.4 (weak) Logarithmic regularity

Finally, we recall the notion of log regularity and generalise it slightly (cf. [Kat94] or [Hag03] Section 2.5).

Definition 2. *Let X be a locally Noetherian fs log scheme.*

1. *We say X is log regular at $x \in |X|$ if the following two conditions are satisfied:*

   (a) *$\mathscr{O}_{X,\overline{x}}/I(\overline{x}, M)$ is a regular local ring.*

   (b) *$\dim \mathscr{O}_{X,\overline{x}} = \dim \mathscr{O}_{X,\overline{x}}/I(\overline{x}, M) + \operatorname{rank}_{\mathbb{Z}} \overline{M}^{\mathrm{gp}}_{X,\overline{x}}$.*

   Here $I(\overline{x}, M)$ is an ideal of $\mathscr{O}_{X,\overline{x}}$ generated by the image of $M_{X,\overline{x}} \setminus \mathscr{O}^\times_{X,\overline{x}}$ and $\dim$ denotes the Krull dimension.

   X is said to be log regular if it is log regular at x for all $x \in X$.

2. *For a natural number r, we denote by $(\operatorname{Spec} \mathbb{Z})^{\log}$ the log scheme induced by a morphism of monoids $\mathbb{N}^r \to \mathbb{Z}$ which maps any element except the identity to zero. For a log regular log scheme T, consider the log scheme $T' = T \times_{\operatorname{Spec} \mathbb{Z}} (\operatorname{Spec} \mathbb{Z})^{\log}$, where $\operatorname{Spec} \mathbb{Z}$ is endowed with the trivial log structure.*

   X is said to be weakly log regular if it is locally isomorphic to a log scheme defined as above.

2.2 Kummer etale morphism

For the definition of logarithmic etale and smooth morphisms, see [Kat89]. Here we restrict ourselves to the review of the definition of a Kummer etale morphism. For more details see [Nak92] or [Nak97].

A morphism $\phi : P \to Q$ of integral monoids is said to be of Kummer type if it is injective and satisfies the following condition: for any element $q \in Q$ there exists a positive integer n such that $q^n \in \mathrm{Image}\,\phi$.

A morphism $f : X \to Y$ of fs log schemes is said to be of Kummer type if for all $x \in X$, $\overline{M}_{Y,\overline{f(x)}} \to \overline{M}_{X,\overline{x}}$ is of Kummer type. A morphism of fs log schemes is called Kummer etale, or shortly Ket, if it is logarithmic etale and of Kummer type.

Recall that a strict morphism is Kummer etale if and only if its underlying morphism of schemes is etale. So we often call a strict Kummer etale morphism classically etale, or more simply, etale.

The propositions below, due to Nakayama, play essential roles in the followings.

Proposition 2.2. (*[Nak92]* 6.4.2) *Let U and X fs log schemes and $f : U \longrightarrow X$ a Kummer etale morphism. Assume U is quasi-compact, X is equi-characteristic and there exists a chart $X \longrightarrow \mathrm{Spec}\,\mathbb{Z}[P]$ with P fs and sharp. Then there exists a positive integer n, invertible on U, such that the pull-back of f on X_n,$f_n : U \times^{\mathrm{fs}}_X X_n \longrightarrow X_n$ is classically etale.*

Proof. We can take a natural number n invertible in X such that $M_U^{\mathrm{gp}}/f^*M_X^{\mathrm{gp}}$ becomes zero by multiplication by n. Set $U_n = U \times^{\mathrm{fs}}_X X_n$. Then by considering stalks of $\overline{M}$-sheaves at each point and by using Proposition 2.1.1 in [Nak97], we can reduce this proposition to the next lemma. □

Lemma 2.3. *Let n be a natural number and $R \leftarrow P \to Q$ a diagram of saturated sharp monoids. Assume that*

1. *$P \to Q$ is an inclusion and $Q^n \subset P$*
2. *Every element in P becomes n-divisible when mapped to R.*

Then we have an isomorphism $R \xrightarrow{\cong} (R \oplus_P^{\mathrm{sat}} Q)/(R \oplus_P^{\mathrm{sat}} Q)^\times$.

Proof. It suffices to note that, in the category of saturated sharp monoids, $P \to Q$ is an epimorphism and that $P \to R$ factors through $P \to Q$. □

Proposition 2.4. (cf. *[Nak92]* 4.2.4.1) *Let U and X be fs log schemes and $f : U \to X$ a Kummer etale morphism. Take a point $u \in U$ and put $x = f(u)$. Assume that we are given a chart of X, $\phi : X \to \mathrm{Spec}\,\mathbb{Z}[P]$, such that the canonical map $P \to \overline{M}_{X,\overline{x}}$ is an isomorphism. Then we have an fs monoid Q and a morphism $h : P \to Q$ of Kummer type and the following diagram*

$$\begin{tikzcd}
U' \arrow[r] \arrow[d, "g"] & V \arrow[r] \arrow[d] & \mathrm{Spec}\,\mathbb{Z}[Q] \arrow[d, "\mathrm{Spec}\,\mathbb{Z}[h]"] \\
U \arrow[r, "f"] & X \arrow[r, "\phi"] & \mathrm{Spec}\,\mathbb{Z}[P],
\end{tikzcd}$$

where

1. *the right square is Cartesian,*
2. *the number of* $\operatorname{Coker} h^{\mathrm{gp}}$ *is invertible on* U',
3. u *belongs to the image of* g *and*
4. U' *is classically etale over* V *and over* U.

Proof. By the direct application of Theorem 3.5 in [Kat89] we can immediately construct the diagram satisfying all the conditions but the saturatedness of Q. In addition, investigating its proof, we also see that Q becomes saturated automatically when U is assumed to be fs. ☐

Corollary 2.5. *Let* X *be an fs log scheme and* $\{U_i \to X\}_{i\in I}$ *a Kummer etale covering of* X. *Then we have*

1. *an etale covering* $\{X_j \to X\}_{j\in J}$ *by affine log schemes* X_j *over* $\operatorname{Spec}\mathbb{Z}[1/n_j, \zeta_{n_j}]$ *with* $n_j \in \mathbb{Z}$,
2. *a map* $\phi : J \to I$,

and for each $j \in J$,

3. *a chart* $X_j \to \operatorname{Spec}\mathbb{Z}[P_j]$ *with* P_j *an fs and sharp monoid,*
4. *an etale covering* $\{V_j \to (X_j)_{n_j}\}$ *(recall that* $(X_j)_{n_j} = X_j \times^{\mathrm{fs}}_{\operatorname{Spec}\mathbb{Z}[P_j]} \operatorname{Spec}\mathbb{Z}[{P_j}^{1/n_j}]$*), and*
5. *an* X*-morphism* $V_j \to U_{\phi(j)}$

such that, for each $i \in I$, *a set of morphisms* $\{V_j \to U_i\}_{j\in\phi^{-1}(i)}$ *is a Kummer etale covering (in particular* $\{V_j \to X\}_{j\in J}$ *refines* $\{U_i \to X\}_{i\in I}$*).*

Proof. We can deduce it easily from the above proposition (cf. the proof of Corollary 2.7 in [Hag03]). ☐

2.3 Kummer etale K-theory

Let X be an fs log scheme.

Definition 3. 1. *We define a category* $\mathbf{Ket}/X$ *to be the full subcategory of* $\mathbf{FsLogSch}/X$ *consisting of fs log schemes* X' *Kummer etale over* X. *We endow it with the topology by regarding a family of Kummer etale morphisms* $\{f_i : X'_i \to X'\}$ *such that* $X' = \bigcup f_i(X'_i)$ *as a covering. Indeed, we can check that this becomes a site ([Nak92], [Nak97]), so we denote by* X_{Ket} *this site and by* $(X_{\mathrm{Ket}})^{\sim}$ *the associated topos.*

2. *The ringed topos* $((X_{\mathrm{Ket}})^{\sim}, \mathscr{O}_{X_{\mathrm{Ket}}})$ *is defined as follows:*

(a) *A topos* $(X_{\mathrm{Ket}})^{\sim}$ *is the one defined as above.*

(b) *A ring object* $\mathscr{O}_{X_{\mathrm{Ket}}}$ *in* $(X_{\mathrm{Ket}})^{\sim}$ *is the rule which associates to each* X'*, Kummer etale over* X*, the ring* $\Gamma(|X'|, \mathscr{O}_{|X'|})$ *(This is indeed a sheaf, as is shown in Proposition 3.1 in [Hag03]).*

We also denote it by $(X_{\mathrm{Ket}}, \mathscr{O}_X)$ if no confusion occurs. We have the natural notion of $\mathscr{O}_{X_{\mathrm{Ket}}}$-modules and define $\mathbf{Mod}(X_{\mathrm{Ket}})$ to be the category of $\mathscr{O}_{X_{\mathrm{Ket}}}$-modules on a ringed topos $(X_{\mathrm{Ket}}, \mathscr{O}_X)$. The natural morphism of ringed topoi from $(X_{\mathrm{Ket}}, \mathscr{O}_X)$ to $(X_{\mathrm{Zar}}, \mathscr{O}_{|X|})$ is denoted by ε_X, the subscript X being often omitted.

Now we recall the definitions of Kummer etale K-theory and its variant (cf. Subsection 3.3 of [Hag03]).

Definition 4. *Let* X *be an fs log scheme. An object* $\mathscr{F}$ *in* $\mathbf{Mod}(X_{\mathrm{Ket}})$ *is called a Kummer etale vector bundle (or a Ket vector bundle, for short) if, Kummer etale locally, it is isomorphic to a finite direct sum of* $\mathscr{O}_{X_{\mathrm{Ket}}}$*. The* $\mathscr{O}_{X_{\mathrm{Ket}}}$*-module* $\mathscr{F}$ *is said to be a Kummer etale coherent sheaf if it is so in the sense of J. -P. Serre, that is, if* $\mathscr{F}$ *is Ket locally finitely generated and any Ket locally given morphism* $\mathscr{O}^n_{U_{\mathrm{Ket}}} \to \mathscr{F}|_U$ *has a (Kummer etale locally) finitely generated kernel.*

We denote by $\mathbf{Vect}(X_{\mathrm{Ket}})$ *and* $\mathbf{Coh}(X_{\mathrm{Ket}})$ *the categories consisting of Ket vector bundles and Ket coherent sheaves, respectively.*

As is seen in Subsections 3.3 and 3.4 of [Hag03], the category $\mathbf{Vect}(X_{\mathrm{Ket}})$ becomes an exact category in a natural way, and $\mathbf{Coh}(X_{\mathrm{Ket}})$ is an Abelian category if the functor $\varepsilon_X^* : \mathbf{Mod}(X_{\mathrm{Zar}}) \to \mathbf{Mod}(X_{\mathrm{Ket}})$ is exact (e.g. if $\overline{M}_{X,\bar{x}}$ is finitely generated free for all $x \in |X|$), and $|X|$ is Noetherian.

Remark 2.6. *In fact, the definition of a Kummer etale coherent sheaf given in [Hag03] is different from the one given here. However, these two notions coincide for fs log schemes treated in this paper (cf. Corollary 3.10 and Proposition 3.12 of [Hag03]).*

Definition 5. *Let* $K(X_{\mathrm{Ket}})$ *be the simplicial set associated with the exact category* $\mathbf{Vect}(X_{\mathrm{Ket}})$ *([Qui73]) and* $K_i(X_{\mathrm{Ket}})$ *its* i*-th homotopy group. These are called a Kummer etale* K*-theory spectrum and a Kummer etale* K*-group, respectively. We also define a Kummer etale* K'*-theory spectrum* $K'(X_{\mathrm{Ket}})$ *and a Kummer etale* K'*-group* $K'_i(X_{\mathrm{Ket}})$ *by using* $\mathbf{Coh}(X_{\mathrm{Ket}})$ *in the case where* ε_X^* *is exact and* X *is Noetherian.*

Also, we denote by $\mathrm{Pic}\,(X_{\mathrm{Ket}})$ *the group of the isomorphic classes of Ket line bundles (i.e. Kummer etale vector bundles of rank one).*

The aim of this article is to determine the λ-ring structure of $K_0(X_{\mathrm{Ket}})$ for suitable fs log schemes up to torsions.

3 The statement of the main theorem

In this section we state the main theorem in Subsection 3.3 after introducing some notions in logarithmic geometry in Subsections 3.1 and 3.2.

3.1 Logarithmic schemes with standard frames

Here we recall the notion of F-framed log schemes (introduced in Section 5 of [Hag03]) and some constructions of log schemes associated with them. Throughout this subsection, fix a finitely generated free monoid F.

Definition 6. *1. An F-framed log scheme is a pair of a log scheme X and a morphism of monoids $\theta : F \longrightarrow \Gamma(X, \overline{M}_X)$ such that for all $x \in X$ the composite $F \longrightarrow \Gamma(X, \overline{M}_X) \longrightarrow \overline{M}_{X,\overline{x}}$ (which we denote by $\theta_{\bar{x}}$) is isomorphic to a projection $\mathbb{N}^m \to \mathbb{N}^n$ with $m \geq n$. Such a morphism θ is called a frame. We also call the pair (X, θ) a log scheme with a standard frame if we are not interested in F.*

2. Let (X, θ), (Y, θ') be F-framed log schemes. An F-framed morphism from (X, θ) to (Y, θ') is a morphism of log schemes $\phi : X \longrightarrow Y$ such that

$$\begin{array}{ccc} F & = & F \\ {\scriptstyle \theta'}\downarrow & & {\scriptstyle \theta}\downarrow \\ \Gamma(Y, \overline{M}_Y) & \xrightarrow{\phi^*} & \Gamma(X, \overline{M}_X) \end{array}$$

is commutative.

For example, the log scheme $\operatorname{Spec} \mathbb{Z}[F]$ can be naturally regarded as F-framed.

Remark 3.1. *1. It is easily checked that if X is F-framed, we have $\overline{M}_{X,x} \cong \overline{M}_{X,\bar{x}}$ for each $x \in X$.*

2. An F-framed morphism becomes strict automatically.

Mainly we work with the category of F-framed log schemes for a fixed monoid F. The proposition below is easily checked.

Proposition 3.2. *Let (X, θ) be an F-framed log scheme and $Y \longrightarrow X$ a strict morphism of log schemes. Then Y has a canonical F-framed log structure.*

In particular, for a subscheme Y of $|X|$ we can define an F-framed log structure canonically, and if a log scheme X has a chart $X \to \operatorname{Spec} \mathbb{Z}[F]$ with F finitely generated free, then X is naturally F-framed.
The next proposition is Proposition 3.12 of [Hag03].

Proposition 3.3. *Let (X, θ) be an fs log scheme with a standard frame. Then for all $x \in X$, $\mathscr{O}_{X,x(\log)}$ is flat over $\mathscr{O}_{X,x}$.*

Next we introduce some definitions concerning the monoid F.

Definition 7. *Let F be a finitely generated free monoid and $\mathfrak{p}$, $\mathfrak{q}$ and $\mathfrak{q}'$ its prime ideals.*

1. We define $F(\mathfrak{p})$ to be the unique face satisfying $F(\mathfrak{p}) \oplus (F \setminus \mathfrak{p}) = F$ and put $\mathfrak{p}^\vee = F \setminus F(\mathfrak{p})$. For a face N of F, we also put $N^\vee = F \setminus ((F \setminus N)^\vee)$.

2. *We denote by* $\mathfrak{q} \vee \mathfrak{q}'$ *(and* $\mathfrak{q} \wedge \mathfrak{q}'$*) the minimal prime ideal containing* $\mathfrak{q}$ *and* $\mathfrak{q}'$ *(resp. the maximal prime ideal contained in* $\mathfrak{q}$ *and* $\mathfrak{q}'$*). Note that* $\mathfrak{q} \vee \mathfrak{q}'$ *coincides with the set-theoretic join* $\mathfrak{q} \cup \mathfrak{q}'$*, while* $\mathfrak{q} \wedge \mathfrak{q}'$ *does not with* $\mathfrak{q} \cap \mathfrak{q}'$.

3. *For a natural number* m*, we set* $F^{\mathfrak{q}-m'-\mathrm{div}} = (F \setminus \mathfrak{q})^{m'-\mathrm{div}} \oplus F(\mathfrak{q})$*. We also use the notation* $F^{\mathfrak{q}-\mathrm{div}'}$*, omitting* m*, if there is no risk of confusions.*

4. *We set*
$$\Lambda[\mathfrak{q}] = \mathbb{Z}[F(\mathfrak{q})^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}].$$
As usual, for the characteristic exponent p *of a (fixed) scheme,* $\Lambda'[\mathfrak{q}]$ *is defined similarly by replacing* $\mathbb{Q}/\mathbb{Z}$ *by* $\mathbb{Q}'/\mathbb{Z} = \oplus_{l \neq p} \mathbb{Q}_l/\mathbb{Z}_l$.

The followings are immediately checked (under the assumption that the ambient monoid is finitely generated free).

Proposition 3.4. *Let* F *be as above and* $\mathfrak{q}$ *and* $\mathfrak{q}'$ *prime ideals.*

1. $\mathfrak{q}^{\vee\vee} = \mathfrak{q}$.
2. *If* $\mathfrak{q} \subset \mathfrak{q}'$*, then* $\mathfrak{q}^{\vee} \supset \mathfrak{q}'^{\vee}$ *and* $F(\mathfrak{q}) \subset F(\mathfrak{q}')$.
3. $F \setminus \mathfrak{q} = F(\mathfrak{q}^{\vee})$.
4. $F(\mathfrak{q} \vee \mathfrak{q}') = F(\mathfrak{q}) + F(\mathfrak{q}')$, $F(\mathfrak{q} \wedge \mathfrak{q}') = F(\mathfrak{q}) \cap F(\mathfrak{q}')$.
5. $(\mathfrak{q} \vee \mathfrak{q}')^{\vee} = \mathfrak{q}^{\vee} \wedge \mathfrak{q}'^{\vee}$, $(\mathfrak{q} \wedge \mathfrak{q}')^{\vee} = \mathfrak{q}^{\vee} \vee \mathfrak{q}'^{\vee}$.

Remark 3.5. *For prime ideals* $\mathfrak{q} \subset \mathfrak{q}'$*, the inclusion* $F(\mathfrak{q}) \subset F(\mathfrak{q}')$ *has a unique left inverse which is also a projection. This will also play an important role later.*

Now, for a given log scheme with a standard frame, we have two recipes for constructing other log schemes. First we "remove" the log structure of an F-framed log scheme "along" a prime ideal of F in the following way.

Definition 8. *Let* (X, θ) *be an* F*-framed fs log scheme and* $\mathfrak{q}$ *a prime ideal of* F*. Then we denote by the log scheme* $X^{\mathfrak{q}}$ *the one having the same underlying scheme as* X*, endowed with the log structure* $M_{X^{\mathfrak{q}}}$ *defined by the Cartesian diagram*
$$\begin{tikzcd} M_{X^{\mathfrak{q}}} \arrow[r, hook] \arrow[d] & M_X \arrow[d] \\ \theta'((F \setminus \mathfrak{q})_X) \arrow[r, hook] & \overline{M}_X, \end{tikzcd}$$
where $\theta'((F \setminus \mathfrak{q})_X)$ *is the image of the morphism* θ' *from the constant monoid sheaf* $(F \setminus \mathfrak{q})_X$ *to the monoid sheaf* $\overline{M}_X$ *adjoint to the morphism of monoids* $\theta : F \setminus \mathfrak{q} \to \Gamma(X, \overline{M}_X)$.

The proposition below is easily checked:

Proposition 3.6. *1. We have an isomorphism of log schemes $X^{\emptyset} \cong X$. On the other hand, if $\mathfrak{m}$ is the maximal ideal of F, then $X^{\mathfrak{m}}$ is the trivial log scheme with underlying scheme $|X|$.*

2. *For $\mathfrak{q} \subset \mathfrak{q}'$, we have a natural morphism of log schemes $X^{\mathfrak{q}} \to X^{\mathfrak{q}'}$.*

3. *The log scheme $X^{\mathfrak{q}}$ naturally has an $(F \setminus \mathfrak{q})$-framed structure. Moreover, for prime ideals $\mathfrak{q} \subset \mathfrak{q}'$ of F, we have a commutative diagram*

$$\begin{tikzcd} F \setminus \mathfrak{q}' \arrow[r, hook] \arrow[d] & F \setminus \mathfrak{q} \arrow[d] \\ \Gamma(X^{\mathfrak{q}'}, \overline{M}_{X^{\mathfrak{q}'}}) \arrow[r] & \Gamma(X^{\mathfrak{q}}, \overline{M}_{X^{\mathfrak{q}}}) \end{tikzcd}$$

of natural monoid homomorphisms.

4. *Let $\mathfrak{q}$ be a prime ideal of F, and $\mathfrak{q}'$ one of $F \setminus \mathfrak{q}$, which implies that $\mathfrak{q} \cup \mathfrak{q}'$ is a prime ideal of F and we can define $(X^{\mathfrak{q}})^{\mathfrak{q}'}$. Then we have a natural isomorphism $(X^{\mathfrak{q}})^{\mathfrak{q}'} \cong X^{\mathfrak{q} \cup \mathfrak{q}'}$.*

5. *Let X and Y be F-framed fs log schemes, $\mathfrak{q}$ a prime ideal of F and $f : X \to Y$ is a strict F-framed morphism. Then the induced morphism $X^{\mathfrak{q}} \to Y^{\mathfrak{q}}$ is also strict.*

The next proposition makes the investigations of $X^{\mathfrak{q}}$ easier.

Proposition 3.7. *Let (X, θ) be an F-framed locally Noetherian fs log scheme, $x \in |X|$ and $\mathfrak{q}$ a prime ideal of F. Then we have*

1. *a Zariski neighbourhood $U \subset |X|$ of x,*

2. *an F-framed strict morphism $U \to \operatorname{Spec} \mathbb{Z}[F]$, and*

3. *an $(F \setminus \mathfrak{q})$-framed strict morphism $U^{\mathfrak{q}} \to \operatorname{Spec} \mathbb{Z}[F \setminus \mathfrak{q}]$*

such that

$$\begin{array}{ccc} U & \longrightarrow & \operatorname{Spec} \mathbb{Z}[F] \\ \downarrow & & \downarrow \\ U^{\mathfrak{q}} & \longrightarrow & \operatorname{Spec} \mathbb{Z}[F \setminus \mathfrak{q}] \end{array}$$

is commutative, where U, $\operatorname{Spec} \mathbb{Z}[F]$, $U^{\mathfrak{q}}$ and $\operatorname{Spec} \mathbb{Z}[F \setminus \mathfrak{q}]$ are naturally endowed with an F- or $(F \setminus \mathfrak{q})$-framed structure, and the vertical maps are natural ones. In particular, $X^{\mathfrak{q}}$ is also an fs log scheme, and X has, Zariski locally, a chart.

Proof. By the definition of the frame, we have a commutative diagram

$$\begin{tikzcd} F' \arrow[r, hook, "\mathrm{incl}_1"] \arrow[drr] & F' \oplus F'' \arrow[r, "\cong"] & F \arrow[d, "\theta_{\bar{x}}"] \\ & & \overline{M}_{X,\bar{x}}, \end{tikzcd}$$

where the oblique morphism (named $\overline{\gamma}_x$) is an isomorphism, and the composite $\overline{\gamma}_x^{-1} \circ \theta_{\bar{x}}$ is a projection. Since the map $M_{X,x} \to \overline{M}_{X,x} (\cong \overline{M}_{X,\bar{x}})$ is surjective by the vanishing of $R^1 \varepsilon^{\mathrm{et}}_{X*} \mathscr{O}^{\times}_{X_{\mathrm{et}}}$, where $\varepsilon^{\mathrm{et}}_X : X_{\mathrm{et}} \to X_{\mathrm{Zar}}$ is the natural morphism of sites, the map $\overline{\gamma}_x$ can be lifted to a monoid homomorphism $\gamma_x : F' \to M_{X,x}$. Moreover, γ_x can be extended to $\gamma_U : F' \to \Gamma(U, M_X)$ for some Zariski neighbourhood $U \subset |X|$ of x such that

1. $(F'_{U_{\mathrm{et}}})^a \to M_X|_{U_{\mathrm{et}}}$, where the domain is the log structure associated with a pre-log structure $F'_{U_{\mathrm{et}}}$, and
2. $\Gamma(U, \overline{M}_X) \to \overline{M}_{X,x}$

are isomorphisms. Note here that, in particular, the composite

$$\overline{\gamma}_U : F' \to \Gamma(U, M_X) \to \Gamma(U, \overline{M}_X)$$

is also an isomorphism.

Denoting by f the chart $U \to \operatorname{Spec} \mathbb{Z}[F']$ obtained above, we define U' to be the log scheme whose underlying scheme is $|U|$ and whose log structure is induced from that on $\operatorname{Spec} \mathbb{Z}[F' \setminus (F' \cap \mathfrak{q})]$ via the morphism of schemes

$$|U| \xrightarrow{|f|} |\operatorname{Spec} \mathbb{Z}[F']| \longrightarrow |\operatorname{Spec} \mathbb{Z}[F' \setminus (F' \cap \mathfrak{q})]|.$$

In addition, we define an $(F \setminus \mathfrak{q})$-frame $F \setminus \mathfrak{q} \to \Gamma(U', \overline{M}_{U'})$ by composing the natural $F' \setminus (F' \cap \mathfrak{q})$-frame of U' (induced from that of $\operatorname{Spec} \mathbb{Z}[F' \setminus (F' \cap \mathfrak{q})]$) with the projection $F \setminus \mathfrak{q} \cong F' \setminus (F' \cap \mathfrak{q}) \oplus F'' \setminus (F'' \cap \mathfrak{q}) \to F' \setminus (F' \cap \mathfrak{q})$.

Thus we have a commutative diagram of log schemes

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{Z}[F'] \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{Z}[F] \arrow[d] \\ U' \arrow[r] & \operatorname{Spec} \mathbb{Z}[F' \setminus (F' \cap \mathfrak{q})] \arrow[r] & \operatorname{Spec} \mathbb{Z}[F \setminus \mathfrak{q}]. \end{tikzcd}$$

Here the right horizontal maps are induced from projections, so become strict, and the composite of the lower horizontal morphisms is by definition $(F \setminus \mathfrak{q})$-framed.

Moreover, the composite of the upper horizontal morphisms is F-framed, that is, the given frame $\theta : F \to \Gamma(U, \overline{M}_X)$ can be expressed as the composite of the projection $\mathrm{proj} : F \to F'$ and $\overline{\gamma}_U : F' \to \Gamma(U, \overline{M}_X)$. Indeed, the two

maps $\overline{\gamma}_U \circ \mathrm{proj}$ and θ coincide with $\theta_{\bar{x}}$ when composed with the isomorphism $\Gamma(U, \overline{M}_X) \cong \overline{M}_{X,\bar{x}}$.
Therefore it suffices to show that there exists an isomorphism $U' \cong U^{\mathfrak{q}}$ of $(F \backslash \mathfrak{q})$-framed log schemes, which is compatible with a natural morphism $U \to U^{\mathfrak{q}}$ and $U \to U'$ defined above.
From the commutative diagram

$$\begin{tikzcd}
F' \setminus (F' \cap \mathfrak{q}) \arrow[r] \arrow[d] & \Gamma(U', \overline{M}_{U'}) \arrow[d] \\
F' \arrow[r] & \Gamma(U, \overline{M}_U),
\end{tikzcd}$$

we obtain a commutative diagram of sheaves of monoids on $|U|_{\mathrm{et}}$

$$\begin{tikzcd}
(F \setminus \mathfrak{q})_{|U|} \arrow[r] \arrow[d] & F' \setminus (F' \cap \mathfrak{q})_{|U|} \arrow[r] \arrow[d] & \overline{M}_{U'} \arrow[d] \\
F_{|U|} \arrow[r] & F'_{|U|} \arrow[r] & \overline{M}_U.
\end{tikzcd}$$

The upper horizontal morphisms are surjective, so it is sufficient to prove the injectivity of $\overline{M}_{U'} \to \overline{M}_U$.
For that, it is enough to prove that, for any point $y \in |U|$, defining Q and Q' so that both squares in the diagram

$$\begin{tikzcd}
Q' \arrow[r, hook] \arrow[d, hook] & Q \arrow[r] \arrow[d, hook] & \mathscr{O}^{\times}_{U,\bar{y}} \arrow[d, hook] \\
F' \setminus (F' \cap \mathfrak{q}) \arrow[r, hook] & F' \arrow[r, "\gamma_{U,\bar{y}}"] & M_{U,\bar{y}}
\end{tikzcd}$$

are Cartesian, the natural map $(F' \setminus (F' \cap \mathfrak{q}))/Q' \to F'/Q$ is injective. It is now clear since F' is finitely generated free and both $F' \setminus (F' \cap \mathfrak{q})$ and Q are faces of F'. $\square$

Secondly, given an F-framed log scheme, we can "stratify" the log scheme "along" a prime ideal of F as follows:

Definition 9. *Let X be an F-framed log scheme and $a \in \Gamma(X, \overline{M}_X)$. We say that the element a is pseudo-zero if, for every geometric point $\bar{x}$ of X and a lift $\tilde{a} \in M_{X,\bar{x}}$ of $a_{\bar{x}} \in \overline{M}_{X,\bar{x}}$, we have $\alpha_{X,\bar{x}}(\tilde{a}) = 0$ as an element of $\mathscr{O}_{X,\bar{x}}$, where $\alpha_X : M_X \to \mathscr{O}_X$ is the logarithmic structure of X.*

Clearly, if a is pseudo-zero, for any geometric point $\bar{x}$ and any lift $\tilde{a} \in M_{X,\bar{x}}$ we have $\alpha_{X,\bar{x}}(\tilde{a}) = 0$. Note also that, for an element $b \in \Gamma(X, M_X)$, if its image $\bar{b} \in \Gamma(X, \overline{M}_X)$ is pseudo-zero, then $\alpha_X(b)$ is zero as an element of $\Gamma(X, \mathcal{O}_X)$.

Proposition-Definition 3.8. *Let (X, θ) be an F-framed fs log scheme and $\mathfrak{p}$ a prime ideal of F. Then there exists the unique strictly closed log subscheme Y such that, for every log scheme over X, $f : T \to X$, f factors (uniquely) through Y if and only if, for every element $m \in \mathfrak{p}$, $f^*(\theta(m)) \in \Gamma(T, \overline{M}_T)$ is pseudo-zero. We denote this log scheme by $X[\mathfrak{p}]$.*

Proof. Since it suffices to consider etale locally on X, we may assume that X is affine and has a chart $X \to \operatorname{Spec} \mathbb{Z}[F]$ that is F-framed (Proposition 3.7). Then the proposition follows from the following lemma, which is easily proven. □

Lemma 3.9. *Let (X, θ) be an F-framed fs log scheme with the underlying scheme $\operatorname{Spec} A$ and assume that θ is lifted to $\widetilde{\theta} : F \to \Gamma(X, M_X)$. Then $X[\mathfrak{p}]$ is $\operatorname{Spec} A/I$, where I is an ideal generated by $\alpha_X(\widetilde{\theta}(m))$ $(m \in \mathfrak{p})$, endowed with the log structure induced from X.*

Note that $X[\mathfrak{p}]$ also becomes an F-framed fs log scheme in the natural way. The followings are easily checked.

Proposition 3.10. *Let X be an F-framed fs log scheme, and $\mathfrak{q}$, $\mathfrak{p}$ and $\mathfrak{p}'$ be prime ideals of F. Then*

1. $X[\emptyset] = X$.
2. *If $\mathfrak{p} \supset \mathfrak{p}'$, then $X[\mathfrak{p}] \subset X[\mathfrak{p}']$.*
3. $(X[\mathfrak{p}])[\mathfrak{p}'] = X[\mathfrak{p} \vee \mathfrak{p}']$.
4. *Let $f : Y \to X$ be a (strict) F-framed morphism between F-framed log schemes. Then for any prime $\mathfrak{p}$ of F, we have*
$$X[\mathfrak{p}] \times^{\mathrm{fs}}_X Y = Y[\mathfrak{p}].$$
In particular, if Y be a strictly closed subscheme of X, then $Y[\mathfrak{p}] = Y \cap X[\mathfrak{p}]$.
5. *Assume that $\mathfrak{p} \wedge \mathfrak{q} = \emptyset$. Then the natural morphism $X[\mathfrak{p}]^{\mathfrak{q}} \to X^{\mathfrak{q}}$ induces an isomorphism $X[\mathfrak{p}]^{\mathfrak{q}} \cong (X^{\mathfrak{q}})[\mathfrak{p} \setminus \mathfrak{q}]$ with $\mathfrak{p} \setminus \mathfrak{q}$ regarded as a prime ideal of $F \setminus \mathfrak{q}$.*

Remark 3.11. *As is easily checked by using Lemma 3.9, the logarithmic subscheme $V_X(\mathfrak{p})$ introduced in Section 5 of [Hag03] is the maximal reduced strictly closed subscheme of $X[\mathfrak{p}]$.*

The following proposition is now easily checked.

Proposition 3.12. *Let X be a weakly log regular and regular fs log scheme, and assume that X is F-framed. Then, for prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of F, $X[\mathfrak{p}]^{\mathfrak{q}}$ is also weakly log regular and regular. If X is log regular and regular, so is $X^{\mathfrak{q}}$.*

3.2 PELLICULAR KUMMER ETALE K-THEORY

In this subsection we introduce the notions of "pellicular Kummer etale ringed topos" and its K-theory, which play essential roles throughout this paper.

DEFINITION 10. *Let X be an fs log scheme.*

1. *We define a ring object $\overline{\mathscr{O}}_{X_{\mathrm{Ket}}}$ in the topos $(X_{\mathrm{Ket}})^{\sim}$ as follows:*

$$\text{For an object } X' \text{ in } \mathbf{Ket}/X,\ \overline{\mathscr{O}}_{X_{\mathrm{Ket}}}(X') = \Gamma(|X'|, \mathscr{O}^{\mathrm{red}}_{|X'|})$$

(This is indeed a sheaf, as is proven below). We call the pair $((X_{\mathrm{Ket}})^{\sim}, \overline{\mathscr{O}}_{X_{\mathrm{Ket}}})$ a pellicular Kummer etale (or shortly, pellicular Ket) ringed topos, and often denote it simply by $(X_{\mathrm{Ket}}, \overline{\mathscr{O}}_X)$. Two natural morphisms of ringed topoi $(X_{\mathrm{Ket}}, \overline{\mathscr{O}}_X) \to (X_{\mathrm{Ket}}, \mathscr{O}_X)$ and $(X_{\mathrm{Ket}}, \overline{\mathscr{O}}_X) \to (X_{\mathrm{Zar}}, \mathscr{O}_X)$ are denoted by $\bar{\varepsilon}^{\mathrm{red}}_X$ and by $\bar{\varepsilon}_X$, respectively. We denote by $\overline{\mathbf{Mod}}(X_{\mathrm{Ket}})$ the category of $\overline{\mathscr{O}}_X$-modules on a site X_{Ket}.

2. *An object $\mathscr{F}$ in $\overline{\mathbf{Mod}}(X_{\mathrm{Ket}})$ is called a pellicular Ket vector bundle (resp. a pellicular Ket quasi-coherent sheaf) if, Kummer etale locally, it is isomorphic to a finite direct sum of $\overline{\mathscr{O}}_X$ (resp. to the module of the form $(\overline{\mathscr{O}}_X)^I \to (\overline{\mathscr{O}}_X)^J \to \mathscr{F} \to 0$). The $\overline{\mathscr{O}}_X$-module $\mathscr{F}$ is said to be a pellicular Ket coherent sheaf if it is so in the sense of J.-P. Serre, that is, if $\mathscr{F}$ is Kummer etale locally finitely generated and any Ket locally given morphism $\overline{\mathscr{O}}^n_U \to \mathscr{F}|_U$ has a (Kummer etale locally) finitely generated kernel.*

3. *We denote by $\overline{\mathbf{Vect}}(X_{\mathrm{Ket}})$, $\overline{\mathbf{Coh}}(X_{\mathrm{Ket}})$ and $\overline{\mathbf{Qcoh}}(X_{\mathrm{Ket}})$ the full subcategory of $\overline{\mathbf{Mod}}(X_{\mathrm{Ket}})$ consisting of pellicular Ket vector bundles, pellicular Ket coherent sheaves and pellicular Ket quasi-coherent sheaves, respectively.*

4. *A pellicular Ket line bundle is defined to be a pellicular Ket vector bundle of rank one and $\overline{\mathrm{Pic}}(X_{\mathrm{Ket}})$ denotes the group consisting of isomorphic classes of pellicular Ket line bundles.*

A pellicular Ket (quasi-)coherent sheaf of $\overline{\mathscr{O}}_X$-modules is simply called a (quasi-)coherent $\overline{\mathscr{O}}_X$-module, too.

DEFINITION 11. *Let X be an fs log scheme. We denote by $\overline{K}(X_{\mathrm{Ket}})$ a simplicial set made from the exact category $\overline{\mathbf{Vect}}(X_{\mathrm{Ket}})$ via Quillen's recipe. Note that it can be regarded as the 0-th space of a fibrant spectrum (for instance, use [Wal85] and a "fibrant replacement" in 5.2 in [Tho85]). We also denote this spectrum by $\overline{K}(X_{\mathrm{Ket}})$, if there is no risk of confusion, and we call it a pellicular Ket K-theory spectrum.*

As is easily seen, if X is log regular, then the category $\overline{\mathbf{Vect}}(X_{\mathrm{Ket}})$ is equivalent to $\mathbf{Vect}(X_{\mathrm{Ket}})$, so we have a natural weak equivalence $\overline{K}(X_{\mathrm{Ket}}) \cong K(X_{\mathrm{Ket}})$. Thus we will concentrate on the analysis of pellicular Ket K-theory in the following.

3.3 Structure theorem

We begin with introducing and fixing some notations. In the following, we follow Thomason's terminology in [Tho85] (See also [Jar97]).
First of all, throughout this section, we will make the following convention:

Convention 3.13. *F is a finitely generated free monoid and X is an F-framed weakly log regular fs log scheme such that $|X|$ is Noetherian, separated, regular and of finite Krull dimension, equi-characteristic of characteristic exponent p.*

Note that, under these assumptions, $X[\mathfrak{p}]$ also satisfies the same conditions for any prime ideal $\mathfrak{p}$ (Proposition 3.12).
Next, we fix a presheaf of fibrant spectra $\overline{K}_{\mathrm{Ket}}$ on $\mathbf{Ket}/X$ such that $\overline{K}_{\mathrm{Ket}}(U)$ is weakly homotopy equivalent to $\overline{K}(U_{\mathrm{Ket}})$ for each object U of $\mathbf{Ket}/X$, and for every morphism $f : U \to V$ over X, the restriction map $f^* : \overline{K}_{\mathrm{Ket}}(V) \to \overline{K}_{\mathrm{Ket}}(U)$ is homotopic to the one induced by the exact functor $f^* : \overline{\mathbf{Vect}}(V_{\mathrm{Ket}}) \to \overline{\mathbf{Vect}}(U_{\mathrm{Ket}})$. It is possible by a suitable rectification (for example, we can use the procedure in [Jar09]).
Then by the Godement-Thomason construction (i.e. Definition 1.33 in [Tho85]), we can construct a fibrant spectrum $\mathbb{H}^{\cdot}(X_{\mathrm{Ket}}, \overline{K}_{\mathrm{Ket}})$ and a natural map of fibrant spectra

$$\overline{K}(X_{\mathrm{Ket}}) \longrightarrow \mathbb{H}^{\cdot}(X_{\mathrm{Ket}}, \overline{K}_{\mathrm{Ket}}).$$

Finally, for each fibrant spectrum S, we take a fibrant spectrum $S_{\mathbb{Q}}$ and a map of spectra $S \to S_{\mathbb{Q}}$ which induce isomorphisms $\pi_i(S)_{\mathbb{Q}} \cong \pi_i(S_{\mathbb{Q}})$ on homotopy groups. Recall that this construction can be carried out functorially (for instance, it is enough to use the procedure in [BK72] and a fibrant replacement), so this notation makes sense also for a presheaf of fibrant spectra.
We first state the "inversion formula", which is one of the most important tools, not only for the proof of the structure theorem, also for its formulation.
For a prime ideal $\mathfrak{q}$ of F, we denote $(X^{\mathfrak{q}})_{\mathrm{Ket}}$ simply by $X^{\mathfrak{q}}_{\mathrm{Ket}}$, and a natural morphism of sites from X_{Ket} to $X^{\mathfrak{q}}_{\mathrm{Ket}}$ by $\varepsilon^{\mathfrak{q}}_X$. By a little abuse of notations, we also denote by $\varepsilon^{\mathfrak{q}}_X$ (resp. $\bar{\varepsilon}^{\mathfrak{q}}_X$) a morphism of ringed topoi from $(X_{\mathrm{Ket}}, \mathscr{O}_X)$ to $(X^{\mathfrak{q}}_{\mathrm{Ket}}, \mathscr{O}_{X^{\mathfrak{q}}})$ (resp. $(X_{\mathrm{Ket}}, \overline{\mathscr{O}}_X)$ to $(X^{\mathfrak{q}}_{\mathrm{Ket}}, \overline{\mathscr{O}}_{X^{\mathfrak{q}}})$). For instance, $\varepsilon^{\mathfrak{q}}_{X*}(\overline{K}_{\mathrm{Ket}})_{\mathbb{Q}}$ is a presheaf of fibrant spectra that associates to $U \in \mathbf{Ket}/X^{\mathfrak{q}}$ a spectrum weakly equivalent to $\overline{K}((U \times^{\mathrm{fs}}_{X^{\mathfrak{q}}} X)_{\mathrm{Ket}})_{\mathbb{Q}}$. We can also define $\varepsilon^{\mathfrak{q}}_{X*}(\overline{K}'_{\mathrm{Ket}})_{\mathbb{Q}}$ similarly.
Now we can construct a map of fibrant spectra

$$\overline{K}(X^{\mathfrak{q}}_{\mathrm{Ket}})_{\mathbb{Q}} \xrightarrow{\bar{\varepsilon}^{\mathfrak{q}*}_X} \overline{K}(X_{\mathrm{Ket}})_{\mathbb{Q}} \longrightarrow \mathbb{H}^{\cdot}(X^{\mathfrak{q}^{\vee}}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}^{\vee}}_{X*}(\overline{K}_{\mathrm{Ket}})_{\mathbb{Q}}).$$

Theorem 3.14. *(= Corollary 5.19) Let F and X be as in Convention 3.13. Then, for a prime ideal $\mathfrak{q}$ of F, this is a weak equivalence:*

$$\overline{K}(X^{\mathfrak{q}}_{\mathrm{Ket}})_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{H}^{\cdot}(X^{\mathfrak{q}^{\vee}}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}^{\vee}}_{X*}(\overline{K}_{\mathrm{Ket}})_{\mathbb{Q}}).$$

As a result of the above theorem, we can define an "extraordinary map" as follows:

Definition 12. *For a pair of prime ideals $\mathfrak{q}' \supset \mathfrak{q}$ and an integer $i \geq 0$, we define a map of fibrant spectra $\pi_i(\delta_X^{\mathfrak{q}',\mathfrak{q}})$ to be the composite of the following maps:*

$$\overline{K}_i(X^{\mathfrak{q}}_{\mathrm{Ket}})_{\mathbb{Q}} \overset{\overline{\varepsilon}_X^{\mathfrak{q}*}}{\to} \overline{K}_i(X_{\mathrm{Ket}})_{\mathbb{Q}} \to \pi_i \mathbb{H}^{\cdot}(X^{\mathfrak{q}'^{\vee}}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}'^{\vee}}_{X*}(\overline{K}_{\mathrm{Ket}})_{\mathbb{Q}}) \to \overline{K}_i(X^{\mathfrak{q}'}_{\mathrm{Ket}})_{\mathbb{Q}}.$$

Here the third map is the inverse of the isomorphism between the i-th homotopy groups in Theorem 3.14. The notation $\pi_i(\delta_X^{\mathfrak{q}',\mathfrak{q}})$ is a little abusive because we will not consider the map of spectra which should be denoted by "$\delta_X^{\mathfrak{q}',\mathfrak{q}}$", but it causes no confusion. We often omit the subscript X or the superscripts $\mathfrak{q}$, $\mathfrak{q}'$ if they are unnecessary.

For the next proposition, we introduce some additional notations. Observing Proposition 3.6 (2), for a pair of prime ideals $\mathfrak{q}' \supset \mathfrak{q}$, we denote by $\varepsilon_X^{\mathfrak{q},\mathfrak{q}'}$ a natural morphism of sites from $X^{\mathfrak{q}}_{\mathrm{Ket}}$ to $X^{\mathfrak{q}'}_{\mathrm{Ket}}$, and also denote by $\varepsilon_X^{\mathfrak{q},\mathfrak{q}'}$ (resp. $\overline{\varepsilon}_X^{\mathfrak{q},\mathfrak{q}'}$) a morphism of ringed topoi from $(X^{\mathfrak{q}}_{\mathrm{Ket}}, \mathscr{O}_{X^{\mathfrak{q}}})$ to $(X^{\mathfrak{q}'}_{\mathrm{Ket}}, \mathscr{O}_{X^{\mathfrak{q}'}})$ (resp. from $(X^{\mathfrak{q}}_{\mathrm{Ket}}, \overline{\mathscr{O}}_{X^{\mathfrak{q}}})$ to $(X^{\mathfrak{q}'}_{\mathrm{Ket}}, \overline{\mathscr{O}}_{X^{\mathfrak{q}'}})$).
Clearly they induce maps of fibrant spectra $\overline{\varepsilon}_X^{\mathfrak{q},\mathfrak{q}'*} : \overline{K}(X^{\mathfrak{q}'}_{\mathrm{Ket}}) \to \overline{K}(X^{\mathfrak{q}}_{\mathrm{Ket}})$, $\mathbb{H}^{\cdot}(X^{\mathfrak{q}'}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}'}_{X*}\overline{K}_{\mathrm{Ket}}) \longrightarrow \mathbb{H}^{\cdot}(X^{\mathfrak{q}}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}}_{X*}\overline{K}_{\mathrm{Ket}})$, and moreover

$$\mathbb{H}^{\cdot}(X^{\mathfrak{q}^{\vee}}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}^{\vee}}_{X*}(\overline{K}_{\mathrm{Ket}})_{\mathbb{Q}}) \longrightarrow \mathbb{H}^{\cdot}(X^{\mathfrak{q}'^{\vee}}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}'^{\vee}}_{X*}(\overline{K}_{\mathrm{Ket}})_{\mathbb{Q}})$$

(Note that we have $\mathfrak{q}'^{\vee} \subset \mathfrak{q}^{\vee}$).
The following propositions easily follow from the construction:

Proposition 3.15. *For a string of prime ideals $\mathfrak{q}'' \supset \mathfrak{q}' \supset \mathfrak{q}$ and an integer $i \geq 0$, we have*

1. *The composite $\pi_i(\delta_X^{\mathfrak{q}',\mathfrak{q}}) \circ \pi_i(\overline{\varepsilon}_X^{\mathfrak{q},\mathfrak{q}'*})$ is the identity map.*
2. *The composite $\pi_i(\delta_X^{\mathfrak{q}'',\mathfrak{q}'}) \circ \pi_i(\delta_X^{\mathfrak{q}',\mathfrak{q}})$ and $\pi_i(\delta_X^{\mathfrak{q}'',\mathfrak{q}})$ coincide.*
3. *If Y also satisfies the assumptions at the beginning of this section and $f : X \to Y$ is F-framed, then $\pi_i(\delta_X^{\mathfrak{q}',\mathfrak{q}}) \circ f^* = f^* \circ \pi_i(\delta_Y^{\mathfrak{q}',\mathfrak{q}})$ as maps from $\overline{K}_i(Y^{\mathfrak{q}}_{\mathrm{Ket}})_{\mathbb{Q}}$ to $\overline{K}_i(X^{\mathfrak{q}'}_{\mathrm{Ket}})_{\mathbb{Q}}$.*

Proposition 3.16. *The homomorphism $\pi_0(\delta_X^{\mathfrak{q}',\mathfrak{q}}) : \overline{K}_0(X^{\mathfrak{q}}_{\mathrm{Ket}})_{\mathbb{Q}} \to \overline{K}_0(X^{\mathfrak{q}'}_{\mathrm{Ket}})_{\mathbb{Q}}$ is compatible with λ-ring structures.*

Proof. First we note that the statement can be rewritten in the framework of the theory of simplicial presheaves. Then the proposition follows from the existence of the construction of product and λ-operations at the level of simplicial sets. More precisely, it is sufficient to use Gillet-Grayson's G-construction in [GG87] (plus [GG03]), and to use [Gra89] for λ-operations, and [Jar09] (for example) for the product structure. □

Next, we decompose Kummer etale K-groups "according to stratifications" by introducing an auxiliary category as below.

Definition 13. *We define a category I_F as follows:*
The set of objects consists of the pairs $(\mathfrak{p}, \mathfrak{q})$ of prime ideals of F satisfying $\mathfrak{p} \supset \mathfrak{q}$. For a clarification, we describe these objects like $(\mathfrak{p} \supset \mathfrak{q})$.
Each hom-set consists of at most one element, and we have a morphism $(\mathfrak{p} \supset \mathfrak{q}) \to (\mathfrak{p}' \supset \mathfrak{q}')$ if and only if $\mathfrak{p} \supset \mathfrak{p}' \supset \mathfrak{q}' \supset \mathfrak{q}$.

By Propositions 3.10 and 3.6 (2), for pairs of prime ideals $\mathfrak{p} \supset \mathfrak{p}'$ and $\mathfrak{q}' \supset \mathfrak{q}$, we have a morphism of sites

$$X[\mathfrak{p}]^{\mathfrak{q}}_{\mathrm{Ket}} \to X[\mathfrak{p}']^{\mathfrak{q}'}_{\mathrm{Ket}},$$

and a map of fibrant spectra

$$\mathbb{H}^{\cdot}(X[\mathfrak{p}']^{\mathfrak{q}'}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}'}_{X[\mathfrak{p}']*}\overline{K}_{\mathrm{Ket}}) \longrightarrow \mathbb{H}^{\cdot}(X[\mathfrak{p}]^{\mathfrak{q}}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}}_{X[\mathfrak{p}]*}\overline{K}_{\mathrm{Ket}}).$$

Taking their i-th homotopy groups ($i \geq 0$), we can define a contravariant functor Π_i from I_F to the category of Abelian groups, by the rule

$$\Pi_i((\mathfrak{p} \supset \mathfrak{q})) = \pi_i \mathbb{H}^{\cdot}(X[\mathfrak{p}]^{\mathfrak{q}}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}}_{X[\mathfrak{p}]*}\overline{K}_{\mathrm{Ket}}).$$

Noting that $\overline{K}_i(X_{\mathrm{Ket}})_{\mathbb{Q}} \xrightarrow{\cong} \pi_i \mathbb{H}^{\cdot}(X[\emptyset]^{\emptyset^\vee}_{\mathrm{Ket}}, \varepsilon^{\emptyset^\vee}_{X[\emptyset]*}(\overline{K}_{\mathrm{Ket}})_{\mathbb{Q}})$(Theorem 3.14), we have a morphism of Abelian groups

$$\overline{K}_i(X_{\mathrm{Ket}})_{\mathbb{Q}} \longrightarrow \lim_{\mathfrak{p} \supset \mathfrak{q}} \pi_i \mathbb{H}^{\cdot}(X[\mathfrak{p}]^{\mathfrak{q}}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}}_{X[\mathfrak{p}]*}(\overline{K}_{\mathrm{Ket}})_{\mathbb{Q}}),$$

where we use the notation "$\lim_{\mathfrak{p} \supset \mathfrak{q}}$" to mean a limit with respect to the category I_F.
The next theorem will be proven in Subsection 5.6.

Theorem 3.17. *The above map is an isomorphism for each $i \geq 0$.*

Finally, we describe each piece more explicitly.
As in [Hag03], we can define a Ket line bundle $\mathscr{O}_X(\xi)$ (more precisely, its isomorphism class) associated with an element $\xi \in \Gamma(X, \overline{M}^{\mathrm{gp}}_X) \otimes_{\mathbb{Z}} \mathbb{Q}'$ to be its image by the homomorphism

$$\Gamma(X, \overline{M}^{\mathrm{gp}}_X) \otimes_{\mathbb{Z}} \mathbb{Q}' \cong \Gamma(X_{\mathrm{Ket}}, \overline{M}^{\mathrm{gp}}_{X_{\mathrm{Ket}}}) \xrightarrow{\partial} H^1(X_{\mathrm{Ket}}, \mathscr{O}^{\times}_{X_{\mathrm{Ket}}}) \cong \mathrm{Pic}\,(X_{\mathrm{Ket}}),$$

where $\mathbb{Q}' = \mathbb{Z}_{(p)}$ with p the characteristic of $|X|$.
Here ∂ is induced by the short exact sequence of Abelian sheaves on X_{Ket}

$$0 \to \mathscr{O}^{\times}_{X_{\mathrm{Ket}}} \to M^{\mathrm{gp}}_{X_{\mathrm{Ket}}} \to \overline{M}^{\mathrm{gp}}_{X_{\mathrm{Ket}}} \to 0.$$

In addition, we can complete this diagram as follows:

$$\begin{tikzcd}
\Gamma(X,\overline{M}_X)_{\mathbb{Z}} \arrow[r, "\partial"] \arrow[d] & \mathrm{Pic}\,(X_{\mathrm{Zar}}) \arrow[d] & \\
\Gamma(X,\overline{M}_X)_{\mathbb{Q}'} \arrow[r, "\partial"] & \mathrm{Pic}\,(X_{\mathrm{Ket}}) \arrow[r, "\bar{\varepsilon}^{\mathrm{red}*}"] \arrow[d] & \overline{\mathrm{Pic}}\,(X_{\mathrm{Ket}}) \arrow[d] \\
 & K_0(X_{\mathrm{Ket}}) \arrow[r, "\bar{\varepsilon}^{\mathrm{red}*}"] & \overline{K}_0(X_{\mathrm{Ket}}).
\end{tikzcd}$$

Note that all maps are monoid homomorphisms (with K-groups regarded as monoids by the multiplication).

For $\xi \in \Gamma(X,\overline{M}_X)_{\mathbb{Q}'}$ we denote by $\overline{\mathscr{O}}_X(\xi)$ the corresponding element in $\overline{\mathrm{Pic}}\,(X_{\mathrm{Ket}})$ or $\overline{K}_0(X_{\mathrm{Ket}})$. Accordingly, using a frame $F \to \Gamma(X,\overline{M}_X)$, we can also define a pellicular Ket line bundle $\overline{\mathscr{O}}_X(\xi)$ for $\xi \in F_{\mathbb{Q}'}$.

If a prime ideal $\mathfrak{q}$ of F is given, we can make the above procedure work for $X^{\mathfrak{q}}$ to construct the morphisms

$$(F\setminus\mathfrak{q})_{\mathbb{Q}'} \longrightarrow \Gamma(X^{\mathfrak{q}}_{\mathrm{Ket}},\overline{M}^{\mathrm{gp}}_{X^{\mathfrak{q}}_{\mathrm{Ket}}}) \xrightarrow{\partial} \mathrm{Pic}\,(X^{\mathfrak{q}}_{\mathrm{Ket}}).$$

Moreover, since the commutative diagram

$$\begin{tikzcd}
(F\setminus\mathfrak{q})_{\mathbb{Q}'} \arrow[r] & \Gamma(X^{\mathfrak{q}}_{\mathrm{Ket}},\overline{M}^{\mathrm{gp}}_{X^{\mathfrak{q}}_{\mathrm{Ket}}}) \arrow[r, "\partial"] & \mathrm{Pic}\,(X^{\mathfrak{q}}_{\mathrm{Ket}}) \\
(F\setminus\mathfrak{q})_{\mathbb{Z}} \arrow[r] \arrow[u, hook] \arrow[d, hook] & \Gamma(X^{\mathfrak{q}},\overline{M}^{\mathrm{gp}}_{X^{\mathfrak{q}}}) \arrow[r, "\partial"] \arrow[u] \arrow[d] & \mathrm{Pic}\,(X^{\mathfrak{q}}_{\mathrm{Zar}}) \arrow[u] \arrow[d, "="] \\
F_{\mathbb{Z}} \arrow[r] & \Gamma(X,\overline{M}^{\mathrm{gp}}_X) \arrow[r, "\partial"] & \mathrm{Pic}\,(X_{\mathrm{Zar}})
\end{tikzcd}$$

induces a map

$$(F^{\mathfrak{q}-\mathrm{div}'})^{\mathrm{gp}} \longrightarrow \mathrm{Pic}\,(X^{\mathfrak{q}}_{\mathrm{Ket}}),$$

we can define objects $\mathscr{O}_{X^{\mathfrak{q}}_{\mathrm{Ket}}}(\xi) \in \mathrm{Pic}\,(X^{\mathfrak{q}}_{\mathrm{Ket}})$ and $\overline{\mathscr{O}}_{X^{\mathfrak{q}}_{\mathrm{Ket}}}(\xi) \in \overline{\mathrm{Pic}}\,(X^{\mathfrak{q}}_{\mathrm{Ket}})$ for $\xi \in (F^{\mathfrak{q}-\mathrm{div}'})^{\mathrm{gp}}$.

Note that this defines a group endomorphism of $\overline{K}_i(X^{\mathfrak{q}}_{\mathrm{Ket}})$ for each $i \geq 0$ via "$-\otimes\overline{\mathscr{O}}_{X^{\mathfrak{q}}_{\mathrm{Ket}}}(\xi)$".

Notice also that, for prime ideals $\mathfrak{q} \subset \mathfrak{q}'$ and an element $\xi \in (F^{\mathfrak{q}'-\mathrm{div}'})^{\mathrm{gp}}$, we see $(\varepsilon^{\mathfrak{q};\mathfrak{q}'}_X)^*\mathscr{O}_{X^{\mathfrak{q}'}}(\xi) \cong \mathscr{O}_{X^{\mathfrak{q}}}(\xi)$ and $(\bar{\varepsilon}^{\mathfrak{q};\mathfrak{q}'}_X)^*\overline{\mathscr{O}}_{X^{\mathfrak{q}'}}(\xi) \cong \overline{\mathscr{O}}_{X^{\mathfrak{q}}}(\xi)$ under the identification of $(F^{\mathfrak{q}'-\mathrm{div}'})^{\mathrm{gp}}$ with a subgroup of $(F^{\mathfrak{q}-\mathrm{div}'})^{\mathrm{gp}}$.

Definition 14. *Let $\mathfrak{q}$ be a prime ideal of F.*

1. *We denote by $J(\mathfrak{q})$ the category such that*

(a) the set of objects is $F(\mathfrak{q})_{\mathbb{Q}}$,

(b) for two objects α and β, $\mathrm{Hom}\,(\alpha,\beta)$ is the set consisting of one element $\beta-\alpha$ if $\beta-\alpha\in F(\mathfrak{q})$, and the empty set if not, and

(c) the composition is defined by using the addition in $F(\mathfrak{q})$.

2. *Let C be a category having countable coproducts and $A\in\mathrm{Ob}(C)$, and assume that we are given a monoid morphism $L:F(\mathfrak{q})\to\mathrm{Hom}_C(A,A)$, where $\mathrm{Hom}_C(A,A)$ is regarded as a (not necessarily commutative) monoid by composition.*

   Then, a covariant functor $G_{\mathfrak{q},A,L}$ from $J(\mathfrak{q})$ to C is defined by

   (a) $G_{\mathfrak{q},A,L}(\alpha)=A$, independently of the object α, and

   (b) For any pair of objects α and β such that $\beta-\alpha\in F(\mathfrak{q})$, $G_{\mathfrak{q},A,L}(\beta-\alpha)=L(\beta-\alpha)$.

   We set
$$A\rtimes^L\Lambda[\mathfrak{q}]=\operatorname{colim}G_{\mathfrak{q},A,L}.$$

3. *For a prime ideal $\mathfrak{p}$ of F, let C be the category of Abelian groups and $A=K_i(X[\mathfrak{p}]_{\mathrm{Zar}})$, and for $\gamma\in F(\mathfrak{q})$ we define $L(\gamma)$ to be the group morphism*
$$K_i(X[\mathfrak{p}]_{\mathrm{Zar}})\longrightarrow K_i(X[\mathfrak{p}]_{\mathrm{Zar}})$$
   induced by the endofunctor "$-\otimes\mathscr{O}_{X[\mathfrak{p}]}(\gamma)$" of $\mathbf{Vect}(X[\mathfrak{p}]_{\mathrm{Zar}})$. Then we set
$$K_i(X[\mathfrak{p}]_{\mathrm{Zar}})\rtimes\Lambda[\mathfrak{q}]=A\rtimes^L\Lambda[\mathfrak{q}].$$

In practice, we use its variant $K_i(X[\mathfrak{p}]_{\mathrm{Zar}})\rtimes\Lambda'[\mathfrak{q}]$ which is defined by replacing $\mathbb{Q}$ (in the definition of $J(\mathfrak{q})$) by $\mathbb{Q}'=\mathbb{Z}_{(p)}$.

Remark 3.18. *1. This notation is indeed highly abusive, in the sense that $K_i(X[\mathfrak{p}]_{\mathrm{Zar}})\rtimes\Lambda[\mathfrak{q}]$ is determined not only by $K_i(X[\mathfrak{p}]_{\mathrm{Zar}})$ and $\mathfrak{q}$, and actually the Abelian group $\Lambda[\mathfrak{q}]$ itself is not used in its definition. However, the author believes that it does not induce any risk of confusions. Anyway, the notation "$\rtimes$" symbolises the hope that we want to construct the "twisted" product of $K(X[\mathfrak{p}]_{\mathrm{Zar}})$ and $\Lambda[\mathfrak{q}]$ in a suitable sense.*

2. *As Abelian groups, we have an isomorphism*
$$K_i(X[\mathfrak{p}]_{\mathrm{Zar}})\rtimes\Lambda'[\mathfrak{q}]\cong K_i(X[\mathfrak{p}]_{\mathrm{Zar}})\otimes_{\mathbb{Z}}\Lambda'[\mathfrak{q}].$$

As is already mentioned, with every $\xi\in F(\mathfrak{q})_{\mathbb{Q}'}$, we associate a pellicular Ket line bundle $\overline{\mathscr{O}}_{X[\mathfrak{p}]^{\mathfrak{q}^\vee}_{\mathrm{Ket}}}(\xi)$ via a map
$$F(\mathfrak{q})_{\mathbb{Q}'}\hookrightarrow(F^{\mathfrak{q}^\vee-\mathrm{div}'})^{\mathrm{gp}}\to\Gamma(X[\mathfrak{p}]^{\mathfrak{q}^\vee}_{\mathrm{Ket}},\overline{M}^{\mathrm{gp}}_{X[\mathfrak{p}]^{\mathfrak{q}^\vee}_{\mathrm{Ket}}})\xrightarrow{\partial}\overline{\mathrm{Pic}}(X[\mathfrak{p}]^{\mathfrak{q}^\vee}_{\mathrm{Ket}}).$$

By using this, for $\alpha \in F(\mathfrak{q})_{\mathbb{Q}'}$, we can define a functor from $\mathbf{Vect}(X[\mathfrak{p}]_{\mathrm{Zar}})$ to $\overline{\mathbf{Vect}}(X[\mathfrak{p}]^{\mathfrak{q}^\vee}_{\mathrm{Ket}})$ which send each locally free sheaf of $\mathscr{O}_{X[\mathfrak{p}]_{\mathrm{Zar}}}$-modules $\mathscr{F}$ to a pellicular Kummer etale vector bundle $\bar{\varepsilon}^*\mathscr{F} \otimes \overline{\mathscr{O}}_{X[\mathfrak{p}]^{\mathfrak{q}^\vee}_{\mathrm{Ket}}}(-\alpha)$ on $X[\mathfrak{p}]^{\mathfrak{q}^\vee}$.

It induces a group homomorphism $K_i(X[\mathfrak{p}]_{\mathrm{Zar}})$ to $\overline{K}_i(X[\mathfrak{p}]^{\mathfrak{q}^\vee}_{\mathrm{Ket}})$, and a morphism $K_i(X[\mathfrak{p}]_{\mathrm{Zar}}) \rtimes \Lambda'[\mathfrak{q}] \longrightarrow \overline{K}_i(X[\mathfrak{p}]^{\mathfrak{q}^\vee}_{\mathrm{Ket}})$ by the universality of a colimit.

Theorem 3.19. *For prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of F satisfying $\mathfrak{p} \supset \mathfrak{q}$ and an integer $i \geq 0$, the homomorphism above is an isomorphism, that is, we have*

$$K_i(X[\mathfrak{p}]_{\mathrm{Zar}}) \rtimes \Lambda'[\mathfrak{q}] \xrightarrow{\cong} \overline{K}_i(X[\mathfrak{p}]^{\mathfrak{q}^\vee}_{\mathrm{Ket}}).$$

For a simpler description of this isomorphism, we introduce the following notation.

Definition 15. *For $\alpha \in F(\mathfrak{q})^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}'/\mathbb{Z}$, take a representative $a/n \in F(\mathfrak{q})_{\mathbb{Q}'}$ for some $a \in F(\mathfrak{q})_{\mathbb{Z}}$ and $n \in \mathbb{N}$, and denote by $x = [\mathscr{O}_X(a)]$ the image of a via a map $F^{\mathrm{gp}} \to \mathrm{Pic}\,(X_{\mathrm{Zar}}) \to K_0(X_{\mathrm{Zar}})$. Then we set*

$$\langle\langle\alpha\rangle\rangle = [\overline{\mathscr{O}}_{X^{\mathfrak{q}^\vee}_{\mathrm{Ket}}}(a/n)]\,\bar{\varepsilon}^*(\exp(-(\log x)/n)) \in \overline{K}_0(X^{\mathfrak{q}^\vee}_{\mathrm{Ket}})_{\mathbb{Q}},$$

where $\bar{\varepsilon}^*$ is a natural map from $K_0(X_{\mathrm{Zar}})_{\mathbb{Q}}$ to $K_0(X^{\mathfrak{q}^\vee}_{\mathrm{Ket}})_{\mathbb{Q}}$, and $\exp(T)$ and $\log(T)$ are formal power series in $\mathbb{Q}[[T]]$ and $\mathbb{Q}[[T-1]]$, respectively.

We can easily check its well-definedness by noticing that $x-1$ is nilpotent in $K_0(X_{\mathrm{Zar}})$, and clearly we see that $\langle\langle - \rangle\rangle$ is a monoid homomophism from $F(\mathfrak{q})^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}'/\mathbb{Z}$ to $\overline{K}_0(X^{\mathfrak{q}^\vee}_{\mathrm{Ket}})_{\mathbb{Q}}$ (The latter is regarded as a monoid by a multiplication). Note that these constructions also work for $X[\mathfrak{p}]$.
The following is only a rewrite of Theorem 3.19.

Corollary 3.20. *Let notations and assumptions be as in Theorem 3.19. Then we have a group isomorphism*

$$K_i(X[\mathfrak{p}]_{\mathrm{Zar}}) \otimes_{\mathbb{Z}} \Lambda'[\mathfrak{q}]_{\mathbb{Q}} \xrightarrow{\cong} \overline{K}_i(X[\mathfrak{p}]^{\mathfrak{q}^\vee}_{\mathrm{Ket}})_{\mathbb{Q}},$$

which maps $x \otimes [\alpha]$ to $\bar{\varepsilon}^*(x)\langle\langle\alpha\rangle\rangle$ for $x \in K_i(X[\mathfrak{p}]_{\mathrm{Zar}})$ and $\alpha \in F(\mathfrak{q})^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}'/\mathbb{Z}$.

This map enables us to make the λ-ring structure of the right hand side explicit. Here we will describe the action of Adams operations. For that we introduce that on $\Lambda'[\mathfrak{q}]$.

Definition 16.
1. *We regard $\Lambda'[\mathfrak{q}]$ as a group ring over an Abelian group $F(\mathfrak{q})^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}'/\mathbb{Z}$. In addition, for each natural number m, we define Ψ^m_Λ to be a ring endomorphism on $\Lambda'[\mathfrak{q}]$ which sends $[\alpha]$ to $[m\alpha]$ for $\alpha \in F(\mathfrak{q})^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}'/\mathbb{Z}$. We call them "Adams operations" on $\Lambda'[\mathfrak{q}]$.*
2. *On $K_0(X[\mathfrak{p}]_{\mathrm{Zar}}) \otimes_{\mathbb{Z}} \Lambda'[\mathfrak{q}]_{\mathbb{Q}}$, we introduce a ring structure as a usual tensor product and the actions of "Adams operations" as $\Psi^m_{\mathrm{Zar}} \otimes \Psi^m_\Lambda$, where Ψ^m_{Zar} on $K_0(X[\mathfrak{p}]_{\mathrm{Zar}})_{\mathbb{Q}}$ are usual Adams operations.*

3. *Let Ψ^m_{Ket} be Adams operations on $K_0(X_{\mathrm{Ket}})_{\mathbb{Q}}$ constructed by the usual procedure from its λ-ring structure.*

Remark that *a priori* we only see that the operations Ψ^m_{Ket} satisfy additivity, the usual property for line bundle elements, and the compatibility with pull-back maps.

Corollary 3.21. *In the case of $i = 0$, the isomorphism in Corollary 3.20 respects ring structures and Adams operations of both hand sides.*

Proof. First we check the compatibility with Adams operations. We have only to check $\Psi^m_{\mathrm{Ket}}(\bar{\varepsilon}^*(x)\langle\langle\alpha\rangle\rangle) = \bar{\varepsilon}^*(\Psi^m_{\mathrm{Zar}}(x))\Psi^m_{\Lambda}(\langle\langle\alpha\rangle\rangle)$ for $x \in K_0(X[\mathfrak{p}]_{\mathrm{Zar}})_{\mathbb{Q}}$ and $\alpha \in F(\mathfrak{q})^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}'/\mathbb{Z}$.
By the splitting principle (for $K_0(X[\mathfrak{p}]_{\mathrm{Zar}})$) and Corollary 3.20, we may assume that x is a line bundle element, and then the claim follows because $\bar{\varepsilon}^*(x)\langle\langle\alpha\rangle\rangle$ can be written by a linear combination of line bundle elements. The preservation of product structures can be proven similarly. $\square$

The compatibility of these maps becomes clear by the theorem below.

Theorem 3.22. *For prime ideals $\mathfrak{p} \supset \mathfrak{p}' \supset \mathfrak{q}' \supset \mathfrak{q}$, we have a commutative diagram*

$$\begin{array}{ccc} K_i(X[\mathfrak{p}']_{\mathrm{Zar}}) \otimes_{\mathbb{Z}} \Lambda'[\mathfrak{q}']_{\mathbb{Q}} & \longrightarrow & \pi_i \mathbb{H}^{\cdot}(X[\mathfrak{p}']^{\mathfrak{q}'}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}'}_{X[\mathfrak{p}']*}(\overline{K}_{\mathrm{Ket}})_{\mathbb{Q}}) \\ \downarrow & & \downarrow \\ K_i(X[\mathfrak{p}]_{\mathrm{Zar}}) \otimes_{\mathbb{Z}} \Lambda'[\mathfrak{q}]_{\mathbb{Q}} & \longrightarrow & \pi_i \mathbb{H}^{\cdot}(X[\mathfrak{p}]^{\mathfrak{q}}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}}_{X[\mathfrak{p}]*}(\overline{K}_{\mathrm{Ket}})_{\mathbb{Q}}). \end{array}$$

Here

- *the horizontal maps are the composites of isomorphisms given in Corollary 3.20 and Theorem 3.14,*
- *the right vertical one sheaf-theoretically defined, and*
- *the left vertical one induced by the inverse image functor with respect to $X[\mathfrak{p}] \to X[\mathfrak{p}']$ and the monoid homomorphism $F(\mathfrak{q}') \to F(\mathfrak{q})$ given in Remark 3.5.*

The proofs of Theorems 3.14, 3.17, 3.19 and 3.22 will be given in Section 5. Note that, when $i = 0$, the left vertical map is compatible with product and the actions of Adams operations defined above.
Recalling that $K(X_{\mathrm{Ket}}) \cong \overline{K}(X_{\mathrm{Ket}})$ in our case, we obtain the theorem and the corollary below, immediately from these theorems.

Theorem 3.23. *For each $i \geq 0$, we have an isomorphism*

$$K_i(X_{\mathrm{Ket}})_{\mathbb{Q}} \xrightarrow{\cong} \lim_{\mathfrak{p} \supset \mathfrak{q}} K_i(X[\mathfrak{p}]_{\mathrm{Zar}}) \otimes_{\mathbb{Z}} \Lambda'[\mathfrak{q}]_{\mathbb{Q}}.$$

Moreover, if $i = 0$, this isomorphism is compatible with Adams operations on both hand sides.

As a result, we can decompose $K_0(X_{\mathrm{Ket}})_{\mathbb{Q}}$ via Adams operations to define what should be called “Kummer etale Chow groups”.

Definition 17. *An element x in $K_0(X_{\mathrm{Ket}})_{\mathbb{Q}}$ is called of weight i if there exists a finite set S of prime numbers and a natural number $N \geq 1$ such that for every natural number m prime to all numbers in S, $(\Psi^m_{\mathrm{Ket}})^N(x) = m^{iN}x$. We denote by $K_0(X_{\mathrm{Ket}})^{(i)}$ the subgroup of $K_0(X_{\mathrm{Ket}})_{\mathbb{Q}}$ consisting of elements of weight i.*

Corollary 3.24. *We have an isomorphism*

$$K_0(X_{\mathrm{Ket}})_{\mathbb{Q}} \cong \bigoplus_{i=0}^{\dim |X|} K_0(X_{\mathrm{Ket}})^{(i)}.$$

Moreover, for each piece of the right hand side, we have an isomorphism of Abelian groups

$$K_0(X_{\mathrm{Ket}})^{(i)} \cong \varinjlim_{\mathfrak{p} \supset \mathfrak{q}} CH^i(X[\mathfrak{p}]_{\mathrm{Zar}}) \otimes \Lambda'[\mathfrak{q}]_{\mathbb{Q}}.$$

In the rest of this paper we will prove the theorems stated above.

4 General theory of pellicular Kummer etale sheaves

4.1 Pellicular Kummer etale ringed topos

In this subsection, we develop a general theory of pellicular Ket ringed topos introduced in Subsection 3.2. First we prove a proposition on the Kummer etale topos.

Proposition 4.1. *Let X be an fs log scheme and X_{red} its strictly closed subscheme with its underlying scheme $|X|_{\mathrm{red}}$. Then the pullback map (in the category of fs log schemes) induces an equivalence of categories*

$$(X_{\mathrm{Ket}})^{\sim} \cong ((X_{\mathrm{red}})_{\mathrm{Ket}})^{\sim}.$$

Proof. First we prove that the functor $\mathbf{Ket}/X \to \mathbf{Ket}/X_{\mathrm{red}}$ is fully faithful. This can be proven in the similar way to the classical case by using the following well-known or easily-proven facts;
Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of fs log schemes. Then

- If g and gf are Kummer etale, so is f.
- If g is Kummer etale and gf is an isomorphism, f is a strict open immersion.
- The schemes $|Y \times^{\mathrm{fs}}_X X_{\mathrm{red}}|$ and $|Y|$ have homeomorphic underlying topological spaces.

- If g is Kummer etale, f is strict and surjective, and gf is strict radicial, then g is an isomorphism.

Then the proposition follows from the next lemma and Théorème 4.1 in Exposé III of [SGA4]. □

Lemma 4.2. *Let X be an fs log scheme, $f : U \to X_{\mathrm{red}}$ a Kummer etale morphism, and $u \in U$. Then there exist an fs log scheme V Kummer etale over X and an etale morphism $g : V \times_X X_{\mathrm{red}} \to U$ such that*

$$\begin{tikzcd} V \times_X X_{\mathrm{red}} \arrow[r] \arrow[d, "g"] & V \arrow[dd] \\ U \arrow[d, "f"] & \\ X_{\mathrm{red}} \arrow[r] & X \end{tikzcd}$$

is commutative and $u \in \mathrm{Image}\,(g)$.

Proof. We may assume that X has a chart $X \to \operatorname{Spec}\mathbb{Z}[P]$ with P fs and sharp such that $P \cong \overline{M}_{X,\overline{f(u)}}$. Then, by Proposition 2.4, we can construct a diagram

$$\begin{tikzcd} & W_2 \arrow[dl, "f'"'] \arrow[d, "f''"] & \\ & W_1 \arrow[r] \arrow[d] & \operatorname{Spec}\mathbb{Z}[Q] \arrow[d, "\operatorname{Spec}\mathbb{Z}[h]"] \\ U \arrow[r, "f"] & X_{\mathrm{red}} \arrow[r] & \operatorname{Spec}\mathbb{Z}[P], \end{tikzcd}$$

where $h : P \to Q$ is of Kummer type with $\#\mathrm{Coker}\, h^{\mathrm{gp}}$ invertible on W_2, $u \in \mathrm{Image}\,(f')$, and f' and f'' are classically etale. Here we may also assume that $\#\mathrm{Coker}\, h^{\mathrm{gp}}$ is invertible on X.
If we set $V_1 = X \times^{\mathrm{fs}}_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[Q]$, then $W_1 \cong V_1 \times^{\mathrm{fs}}_X X_{\mathrm{red}}$. In addition, we can find V_2 classically etale over V_1 such that $W_2 \cong W_1 \times_{V_1} V_2$. This V_2 is a desired log scheme. □

Next we investigate some properties of pellicular Ket ringed topos.

Proposition 4.3. *Let X be an fs log scheme and $\mathscr{F}$ a quasi-coherent sheaf of $\mathscr{O}_{X_{\mathrm{Zar}}}$-modules. Then the presheaf on X_{Ket} which associates to $\pi : X' \to X$ an Abelian group $\Gamma(X', \mathscr{O}^{\mathrm{red}}_{X'} \otimes_{\mathscr{O}'_X} \pi^*\mathscr{F})$ is a sheaf.*

Proof. Note first that the sheaf condition is satisfied for strictly etale morphisms since, if $f : U \to V$ is an etale morphism of schemes, then $U_{\mathrm{red}} \to V_{\mathrm{red}}$ is also etale and $(U \times_V U)_{\mathrm{red}} \cong U_{\mathrm{red}} \times_{V_{\mathrm{red}}} U_{\mathrm{red}}$. By Corollary 2.5, it suffices to show the next lemma. □

Lemma 4.4. *Assume that we are given a reduced $\mathbb{Z}[1/n, \zeta_n]$-algebra A, an fs sharp monoid P and a monoid homomorphism $P \to A$. Then, for any A-module M,*

$$0 \longrightarrow M \longrightarrow A_n^{\mathrm{red}} \otimes_A M \longrightarrow (A_n \otimes_A^{\mathrm{fs}} A_n)^{\mathrm{red}} \otimes_A M$$

is exact. Here,

$$A_n = A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P^{1/n}] \text{ and } A_n \otimes_A^{\mathrm{fs}} A_n = A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P^{1/n} \oplus_P^{\mathrm{sat}} P^{1/n}],$$

and the middle and right arrows are induced by natural ring homomorphisms $A \to A_n$ and $A_n \rightrightarrows A_n \otimes_A^{\mathrm{fs}} A_n$, respectively.

Proof. We first recall that it is already known that

$$0 \longrightarrow M \longrightarrow A_n \otimes_A M \longrightarrow (A_n \otimes_A^{\mathrm{fs}} A_n) \otimes_A M$$

is exact (Lemma 3.3 in [Hag03]).
Set $I_n = \mathrm{Hom}\,(P^{1/n}/P, \mathbb{Z}\,[\zeta_n]^\times)$ and let N_n be the nilradical of A_n. Note that A_n is acted by I_n. By the above fact we obtain that $A \xrightarrow{\cong} (A_n)^{I_n}$ (Here, $(-)^{I_n}$ is the I_n-invariant part), which implies $(N_n)^{I_n} = 0$. By the exactness of the functor $(-)^{I_n}$ (from the category of I_n-$\mathbb{Z}[1/n]$-modules to that of Abelian groups), we also have $A \xrightarrow{\cong} (A_n^{\mathrm{red}})^{I_n}$. Since $(A_n \otimes_A^{\mathrm{fs}} A_n)^{\mathrm{red}} \cong \prod_{I_n} (A_n^{\mathrm{red}})$, the lemma follows for $M = A$.
For any A-module M, using a free resolution $A^{\oplus X} \to A^{\oplus Y} \to M \to 0$ and the exactness of $(-)^{I_n}$, we see $M \xrightarrow{\cong} (A_n^{\mathrm{red}} \otimes_A M)^{I_n}$, which proves the lemma. ☐

Corollary 4.5. *Let X be an fs log scheme.*

1. *The presheaf $\overline{\mathscr{O}}_X$ is indeed a sheaf.*
2. *For any quasi-coherent sheaf of $\mathscr{O}_{X_{\mathrm{Zar}}}$-modules $\mathscr{F}$ and a Kummer etale morphism $\pi : X' \to X$, we have*

$$\Gamma(X', \bar{\varepsilon}_X^* \mathscr{F}) \xrightarrow{\cong} \Gamma(X', \mathscr{O}_{X'}^{\mathrm{red}} \otimes_{\mathscr{O}_{X'}} \pi^* \mathscr{F}).$$

3. *For any Zariski quasi-coherent sheaf $\mathscr{F}$ on X, we have $\bar{\varepsilon}_* \bar{\varepsilon}^* \mathscr{F} \cong \mathscr{O}_{X_{\mathrm{red}}} \otimes \mathscr{F}$.*

Assume moreover that $|X|$ is reduced. Then

4. *For a quasi-coherent sheaf of $\mathscr{O}_{X_{\mathrm{Zar}}}$-modules $\mathscr{F}$, the canonical morphism $\mathscr{F} \to \bar{\varepsilon}_* \bar{\varepsilon}^* \mathscr{F}$ is an isomorphism.*
5. *The functor $\bar{\varepsilon}^* : \mathbf{Qcoh}(X_{\mathrm{Zar}}) \to \overline{\mathbf{Mod}}(X_{\mathrm{Ket}})$ is fully faithful.*

The following proposition is easily verified and will be used often.

Proposition 4.6. *Let X be an fs log scheme and $i : X_{\mathrm{red}} \to X$ the natural strict morphism. Then*

1. *The morphism i induces the two functors i_* and i^* which give equivalences of categories between $\overline{\mathbf{Mod}}(X_{\mathrm{Ket}})$ and $\overline{\mathbf{Mod}}((X_{\mathrm{red}})_{\mathrm{Ket}})$. Similar results hold for $\overline{\mathbf{Qcoh}}$, $\overline{\mathbf{Coh}}$, and $\overline{\mathbf{Vect}}$.*

2. *We have an isomorphism of functors*

$$i_*\bar{\varepsilon}^*_{X_{\mathrm{red}}} \cong \bar{\varepsilon}^*_X i^{\mathrm{Zar}}_*$$

from $\mathbf{Qcoh}((X_{\mathrm{red}})_{\mathrm{Zar}})$ to $\overline{\mathbf{Mod}}(X_{\mathrm{Ket}})$. Here i^{Zar}_* denotes the natural functor from $\mathbf{Mod}((X_{\mathrm{red}})_{\mathrm{Zar}})$ to $\mathbf{Mod}(X_{\mathrm{Zar}})$.

Proof. Use Proposition 4.1, Lemma 4.2 and Corollary 4.5. Note also that we have an isomorphism of endofunctors $i^*_{\mathrm{Zar}}i^{\mathrm{Zar}}_* \cong \mathrm{Id}$ of $\mathbf{Mod}((X_{\mathrm{red}})_{\mathrm{Zar}})$, where i^*_{Zar} denotes the natural functor from $\mathbf{Mod}(X_{\mathrm{Zar}})$ to $\mathbf{Mod}((X_{\mathrm{red}})_{\mathrm{Zar}})$. $\square$

In the following we often identify $\overline{\mathbf{Mod}}(X_{\mathrm{Ket}})$ and $\overline{\mathbf{Mod}}((X_{\mathrm{red}})_{\mathrm{Ket}})$ for simplicity.

Definition 18. *An fs log scheme X is said to have the property* (EX) *if X satisfies the following condition:*
For every Kummer etale morphism $X' \to X$, there exist a Kummer etale covering $X'' \to X'$ of X' such that the functor

$$\bar{\varepsilon}^*_{X''_{\mathrm{red}}} : \mathbf{Mod}((X''_{\mathrm{red}})_{\mathrm{Zar}}) \to \overline{\mathbf{Mod}}(X''_{\mathrm{Ket}})$$

is exact.

Clearly, for a Kummer etale morphism $f : Y \to X$, if X has the property (EX), so does Y. On the other hand, if f is Kummer etale surjective and Y has the property (EX), so does X.

Remark 4.7. *Note that the exactness of $\bar{\varepsilon}^*_X$ implies that $|X|$ is reduced.*

The importance of this condition will be made clear by the proposition below.

Proposition 4.8. *Let X be an fs log scheme.*

1. *The subcategory $\overline{\mathbf{Vect}}(X_{\mathrm{Ket}})$ has a structure of exact categories in the natural way.*

2. *We assume that $|X|$ is Noetherian and X has the property* (EX).
   (a) *For an object $\mathscr{F}$ in $\overline{\mathbf{Mod}}(X_{\mathrm{Ket}})$, the followings are equivalent;*
      i. *$\mathscr{F}$ is a pellicular Ket coherent sheaf.*
      ii. *There exists a Kummer etale covering $\{X' \to X\}$ of X such that $\mathscr{F}|_{X'}$ belongs to the essential image by $\bar{\varepsilon}^*_{X'}$ of $\mathbf{Coh}(X'_{\mathrm{Zar}})$.*
      iii. *There exists a Kummer etale covering $\{X' \to X\}$ of X such that $\mathscr{F}|_{X'}$ belongs to the essential image by $\bar{\varepsilon}^*_{X'_{\mathrm{red}}}$ of $\mathbf{Coh}((X'_{\mathrm{red}})_{\mathrm{Zar}})$, under the identification of $\overline{\mathbf{Mod}}(X'_{\mathrm{Ket}})$ with $\overline{\mathbf{Mod}}((X'_{\mathrm{red}})_{\mathrm{Ket}})$ in Proposition 4.6.*

In particular, the functor $\bar{\varepsilon}_X^* : \mathbf{Mod}(X_{\mathrm{Zar}}) \to \overline{\mathbf{Mod}}(X_{\mathrm{Ket}})$ *induces an functor* $\bar{\varepsilon}_X^* : \mathbf{Coh}(X_{\mathrm{Zar}}) \to \overline{\mathbf{Coh}}(X_{\mathrm{Ket}})$*.*

(b) The category $\overline{\mathbf{Coh}}(X_{\mathrm{Ket}})$ *is Abelian, and the inclusion functor to* $\overline{\mathbf{Mod}}(X_{\mathrm{Ket}})$ *is exact.*

3. The similar statements hold for quasi-coherent sheaves (for any fs log scheme X*).*

4. Assume moreover that the functor $\bar{\varepsilon}_X^* : \mathbf{Mod}(X_{\mathrm{Zar}}) \to \overline{\mathbf{Mod}}(X_{\mathrm{Ket}})$ *is exact. Then the functors* $\bar{\varepsilon}_X^* : \mathbf{Vect}(X_{\mathrm{Zar}}) \to \overline{\mathbf{Vect}}(X_{\mathrm{Ket}})$ *and* $\bar{\varepsilon}_X^* : \mathbf{Qcoh}(X_{\mathrm{Zar}}) \to \overline{\mathbf{Qcoh}}(X_{\mathrm{Ket}})$*, induced by the restriction, are exact. If* X *is Noetherian and has the property* (EX)*, the similar statement holds also for* $\overline{\mathbf{Coh}}(X_{\mathrm{Ket}})$*.*

Proof. We begin with the proof of (2-a). To prove (i)⇒(ii), assume $\mathscr{F} \in \overline{\mathbf{Coh}}(X_{\mathrm{Ket}})$. Then we have a Kummer etale covering U of X with U Noetherian and a morphism $f : \overline{\mathscr{O}}_U^m \to \overline{\mathscr{O}}_U^n$ such that $\mathscr{F}|_U \cong \mathrm{Coker}\, f$. Refining the covering U if necessary, we may assume that $\Gamma(U, \mathscr{O}_U) \to \Gamma(U, \mathscr{O}_U^{\mathrm{red}})$ is surjective. Then by Corollary 4.5 (5), we have a morphism $\tilde{f} : \mathscr{O}_{U_{\mathrm{Zar}}}^m \to \mathscr{O}_{U_{\mathrm{Zar}}}^n$ on U_{Zar} which induces f by $\bar{\varepsilon}_U^*$. Then we obtain $\bar{\varepsilon}_U^*(\mathrm{Coker}\, \tilde{f}) \cong \mathrm{Coker}\, f$. The implication (ii)⇒(iii) is trivial.

For the implication (iii)⇒(i), it suffices to show that, for a Noetherian fs log scheme U such that $\bar{\varepsilon}_{U_{\mathrm{red}}}^*$ is exact, a coherent sheaf of $\mathscr{O}_{U_{\mathrm{red}}}$-modules $\widetilde{\mathscr{F}}$ on $(U_{\mathrm{red}})_{\mathrm{Zar}}$, and a homomorphism $f : \overline{\mathscr{O}}_U^m \to \bar{\varepsilon}_{U_{\mathrm{red}}}^* \widetilde{\mathscr{F}}$, the kernel $\mathrm{Ker}\, f$ is Ket locally finitely generated (Here we used the property (EX)).

By Corollary 4.5 (5) and Remark 4.7 we have a homomorphism $\tilde{f} : \mathscr{O}_{(U_{\mathrm{red}})_{\mathrm{Zar}}}^m \to \widetilde{\mathscr{F}}$ such that $f = \bar{\varepsilon}_{U_{\mathrm{red}}}^*(\tilde{f})$. Since $\mathrm{Ker}\, \tilde{f}$ is Zariski locally finitely generated, we see that $\bar{\varepsilon}_{U_{\mathrm{red}}}^*(\mathrm{Ker}\, \tilde{f}) \cong \mathrm{Ker}\, f$ is also finitely generated by the exactness of $\bar{\varepsilon}_{U_{\mathrm{red}}}^*$.

Next we prove (2-b). We have only to check that if $f : \mathscr{F} \to \mathscr{F}'$ is a homomorphism of pellicular Ket coherent sheaves, then $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$ are also pellicular Ket coherent.

Take a Ket covering U and Zariski coherent sheaves $\widetilde{\mathscr{F}}$ and $\widetilde{\mathscr{F}'}$ on U_{red} such that $\mathscr{F}|_U \cong \bar{\varepsilon}_{U_{\mathrm{red}}}^* \widetilde{\mathscr{F}}$, $\mathscr{F}'|_U \cong \bar{\varepsilon}_{U_{\mathrm{red}}}^* \widetilde{\mathscr{F}'}$, and $\bar{\varepsilon}_{U_{\mathrm{red}}}^*$ is exact. By Corollary 4.5 (5), f comes from some $\tilde{f} : \widetilde{\mathscr{F}} \to \widetilde{\mathscr{F}'}$. Then $\mathrm{Ker}\, \tilde{f}$ and $\mathrm{Coker}\, \tilde{f}$ are coherent, which means that $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$ are pellicular Ket coherent.

(3) is verified similarly and more easily. (1) and (4) are obvious. □

Definition 19. *A pellicular Ket quasi-coherent sheaf (a pellicular Ket vector bundle, respectively) on* X *is said to be classical if it belongs to the essential image of* $\mathbf{Qcoh}((X_{\mathrm{red}})_{\mathrm{Zar}})$ *(*$\mathbf{Vect}((X_{\mathrm{red}})_{\mathrm{Zar}})$*, respectively) by* $\bar{\varepsilon}_{X_{\mathrm{red}}}^*$*. A pellicular Ket line bundle is defined similarly.*

If X *is Noetherian and has the property* (EX)*, the notion of classicality of pellicular Ket coherent sheaves is also defined similarly.*

Remark 4.9. *For pellicular Ket (quasi-)coherent sheaves, the notion of classicality can also be defined by using the essential image of* $\mathbf{Coh}(X_{\mathrm{Zar}})$ *(or* $\mathbf{Qcoh}(X_{\mathrm{Zar}})$*).*

Later we will consider some sufficient conditions for fs log schemes to have the property (EX). Now we introduce a pellicular K'-theory spectrum.

Definition 20. *For a Noetherian fs log scheme X satisfying* (EX)*, we define a pellicular Ket K'-theory spectrum* $\overline{K}'(X_{\mathrm{Ket}})$ *to be the spectrum constructed from the exact category* $\overline{\mathbf{Coh}}(X_{\mathrm{Ket}})$*.*

The relation to equivariant sheaf theory is given by the next proposition.

Proposition 4.10. *Let n be a positive integer, X a Noetherian fs log scheme over* $\operatorname{Spec}\mathbb{Z}[1/n]$ *and* $X \longrightarrow \operatorname{Spec}\mathbb{Z}[P]$ *a chart with P fs and sharp.*
Then we have a natural equivalence of categories between the one of pellicular Ket quasi-coherent sheaves which become classical on $\widetilde{X_n}$ *and the one of Zariski quasi-coherent sheaves of* $\mathscr{O}_{(\widetilde{X_n})_{\mathrm{red}}}$*-modules with an action of* $I_n \rtimes \Gamma_n$ *(For the notations, see Subsection 2.1.3).*
Similarly for coherent sheaves, vector bundles and line bundles.

Proof. This follows from the argument in the usual sheaf theory via Proposition 2.1 and Corollary 4.5 (5). □

Proposition 4.11. *For a pellicular Ket quasi-coherent sheaf $\mathscr{F}$ on an fs log scheme X, $\bar{\varepsilon}_*\mathscr{F}$ is a quasi-coherent sheaf on X_{Zar}. The similar statement holds for pellicular Ket coherent sheaves and (Zariski) coherent sheaves if X is Noetherian and has the property* (EX).

Proof. We can make the same argument as in Proposition 3.14 in [Hag03] work by using Corollary 4.5 (3) and Proposition 4.8. □

Proposition 4.12. *Let X be an fs log scheme and $\pi : X' \to X$ a strictly etale surjective morphism and $\mathscr{F}$ a pellicular Ket quasi-coherent sheaf. Then* $\mathscr{F} \cong \bar{\varepsilon}_X^*\widetilde{\mathscr{F}}$ *for some* $\widetilde{\mathscr{F}} \in \mathbf{Qcoh}(X_{\mathrm{Zar}})$ *if and only if* $\pi^*\mathscr{F} \cong \bar{\varepsilon}_{X'}^*\widetilde{\mathscr{F}'}$ *for some* $\widetilde{\mathscr{F}'} \in \mathbf{Qcoh}(X'_{\mathrm{Zar}})$*.*
The similar statement holds for pellicular Ket coherent sheaves if, in addition, X is Noetherian and has the property (EX).
If the natural functor $\mathbf{Vect}(X_{\mathrm{Zar}}) \to \mathbf{Vect}((X_{\mathrm{red}})_{\mathrm{Zar}})$ *is essentially surjective (e.g. $|X|$ is semi-local or reduced), the similar statement holds for pellicular Ket vector (or line) bundles.*

Proof. In the case where X (and X') is reduced, it follows from Corollary 4.5 (5) and the usual descent. In general, use Proposition 4.6 (2). □

Corollary 4.13. *Let X be a Noetherian equi-characteristic fs log scheme and* $X \to \operatorname{Spec}\mathbb{Z}[P]$ *a chart with P fs and sharp. Then, for a pellicular Ket quasi-coherent sheaf $\mathscr{F}$ on X, there exists a positive integer n, invertible on X, and*

a quasi-coherent sheaf $\widetilde{\mathscr{F}}$ *on* $(X_n)_{\mathrm{Zar}}$ *such that* $\bar{\varepsilon}^*_{X_n}\widetilde{\mathscr{F}} \cong \mathscr{F}|_{X_n}$*. Similarly for coherent sheaves if, in addition,* X *has the property* (EX).
If, moreover, the natural functor $\mathbf{Vect}(X_{\mathrm{Zar}}) \to \mathbf{Vect}((X_{\mathrm{red}})_{\mathrm{Zar}})$ *is essentially surjective, the similar statements hold for vector bundles and line bundles.*

Proof. Use Proposition 2.2 and Proposition 4.12 (cf. [Hag03] Corollary 4.10). □

4.2 Regularly stratified log schemes

In this subsection, we consider some interesting sufficient conditions for the functor $\bar{\varepsilon}^*$ to be exact.

Definition 21. *Let* X *be a locally Noetherian fs log scheme and* x *a point of* $|X|$ *such that* $\overline{M}_{X,\bar{x}}$ *is finitely generated free of rank* r*. Take a standard basis* $\{e_1, \dots, e_r\}$ *of* $\overline{M}_{X,\bar{x}} \cong \mathbb{N}^r$ *and choose their liftings* $\tilde{e}_i \in M_{X,\bar{x}} (1 \leq i \leq r)$*.*

1. *We say that* X *is weakly regularly stratified at* x *if, after a suitable permutation of* e_i*'s,* $\{\alpha(\tilde{e}_1), \dots, \alpha(\tilde{e}_s)\}$ *becomes a regular sequence of* $\mathscr{O}_{X,\bar{x}}$ *and that* $\alpha(\tilde{e}_{s+1}) = \cdots = \alpha(\tilde{e}_r) = 0$ *for some* $s \leq r$*.*

2. *We say that* X *is weakly quasi-regularly stratified at* x *if after a suitable permutation of* e_i*'s, each* $\alpha(\tilde{e}_1), \dots, \alpha(\tilde{e}_s)$ *is a non zero-divisor in* $\mathscr{O}_{X,\bar{x}}$ *and that* $\alpha(\tilde{e}_{s+1}) = \cdots = \alpha(\tilde{e}_r) = 0$ *for some* $s \leq r$*.*

If, in the condition (1)(resp. (2)), we can take $s = r$*, we say that* X *is regularly (resp. quasi-regularly) stratified at* x*. We also call* X *regularly stratified, weakly regularly stratified and so on, if* X *is so at every point.*

Of course, these conditions are independent of the choice of liftings, and the (weakly) regular stratifiedness implies the (weakly) quasi-regular stratifiedness. The following properties are easily checked:

Proposition 4.14.

1. *If* X *is (weakly) regularly stratified at* x *and* η *be a generalisation of* x *(i.e.* $x \in \overline{\{\eta\}}$*), then* X *is (weakly) regularly stratified also at* η*. Similar statements hold for (weakly) quasi-regular stratifiedness.*

2. *For a weakly quasi-regularly stratified fs log scheme* X*, the functor* $\varepsilon^*_X : \mathbf{Mod}(X_{\mathrm{Zar}}) \to \mathbf{Mod}(X_{\mathrm{Ket}})$ *is exact.*

3. *If* X *is (weakly) quasi-regularly stratified, so is* X_{red}*.*

4. *Let* $f : X \to Y$ *be a strict morphism of locally Noetherian fs log schemes such that* $|f|$ *is flat. Then, if* Y *is (weakly) quasi-regularly stratified (or (weakly) regularly stratified), so is* X*. If* $|f|$ *is faithfully flat, the converse also holds.*

5. *Assume that X is F-framed. If X is (weakly) (quasi-)regularly stratified, then so is $X^{\mathfrak{q}}$ for any prime ideal $\mathfrak{q}$ of F. If X is weakly regularly stratified, then so is $X[\mathfrak{p}]^{\mathfrak{q}}$ for any prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of F.*

Proof. The properties (1), (3) and (4) are immediately checked, and (2) is proven in Proposition 3.12 in [Hag03]. (5) is easily proven by using Proposition 3.7 and Lemma 3.9. ☐

Example 1. *1. Let A be a Noetherian ring, r a natural number, and $a_1, \dots, a_r$ elements of A. Consider a morphism of monoids $\mathbb{N}^r \to A$ which maps e_i to a_i, where $\{e_i\}$ is the standard basis of $\mathbb{N}^r$. If $\{a_1, \dots, a_r\}$ is a regular sequence, then the fs log scheme associated with this pre-log structure is regularly stratified. Similarly for the other notions.*

2. Every regular and (weakly) log regular log scheme (see Subsection 3.2) is (weakly) regularly stratified.

To check the exactness of $\bar{\varepsilon}^*_X$, the next proposition is useful.

Proposition 4.15. *Let A be a ring, n a natural number invertible in A and $a \in A$ a non zero-divisor, then we have $A_{\mathrm{red}}[T]/(T^n - a) \cong (A[T]/(T^n - a))_{\mathrm{red}}$. In particular, if A is reduced, then $A[T]/(T^n - a)$ is reduced.*

Proof. First we prove the latter part. We may assume that A is Noetherian. Then by considering localisations $A_{\mathfrak{p}_i}$ at the minimal primes $\mathfrak{p}_1, \dots, \mathfrak{p}_{\mathfrak{r}}$, we have an embedding of A into a product $\prod A_{\mathfrak{p}_i}$ of fields such that the image of a is non-zero in each component. So we can reduce to the case where A is a field, and in this case the second assertion is trivial.
Now we can easily prove the first part by using an isomorphism $(A_1 \otimes_{A_3} A_2)_{\mathrm{red}} \cong ((A_1)_{\mathrm{red}} \otimes_{(A_3)_{\mathrm{red}}} (A_2)_{\mathrm{red}})_{\mathrm{red}}$ for any ring A_3 and any rings A_1 and A_2 over A_3. ☐

Corollary 4.16. *Let X be an fs log scheme, $X \to \operatorname{Spec} \mathbb{Z}[\mathbb{N}^r]$ its chart, n a natural number invertible in X and s a natural number such that $s \leq r$. We set $N' = (\mathbb{N}^s)^{1/n} \oplus \mathbb{N}^{r-s} (\supset \mathbb{N}^r)$ and $X' = X \times^{\mathrm{fs}}_{\operatorname{Spec} \mathbb{Z}[\mathbb{N}^r]} \operatorname{Spec} \mathbb{Z}[N']$.*

1. *If X is weakly quasi-regularly stratified, then X'_{red} is also weakly quasi-regularly stratified, and $|X'|_{\mathrm{red}}$ is flat over $|X|_{\mathrm{red}}$.*

2. *If X is quasi-regularly stratified and reduced, then so is X'. In particular, $|X'|$ is flat over $|X|$.*

Proof. Note first that we may assume that X be strictly local. We then have only to use the above proposition and the induction, noting that if $a \in A$ is a non zero-divisor, then it is so also in $A[T]/(f)$ for any monic polynomial $f \in A[T]$. ☐

Proposition 4.17. *Let X be a weakly quasi-regularly stratified fs log scheme whose underlying scheme is reduced. Then,*

1. *for a log geometric point $x(\log)$ over $x \in |X|$, there exists a cofinal subset $\{U_\lambda\}_{\lambda\in\Lambda}$ in the category of all Kummer etale neighbourhoods of $x(\log)$ such that, for every $\lambda \in \Lambda$,*
   (a) *$(U_\lambda)_{\mathrm{red}}$ is weakly quasi-regularly stratified, and flat over X.*
   (b) *$\bar{\varepsilon}^*_{(U_\lambda)_{\mathrm{red}}}$ is exact.*
2. *The functor $\bar{\varepsilon}^*_X$ is exact.*

If, moreover, X is quasi-regularly stratified, then we can take a set $\{U_\lambda\}_{\lambda\in\Lambda}$ so that each U_λ is quasi-regularly stratified and reduced.

Proof. First we prove the statement (2). Take an arbitrary point $x \in |X|$. We may assume that X is local whose closed point is x, and that we have a chart $\varphi : X \to \operatorname{Spec} \mathbb{Z}[\mathbb{N}^r]$ such that $\varphi^* : \mathbb{N}^r \cong \overline{M}_{X,\bar{x}}$.
Then, for $n \geq 1$, prime to the characteristic of X, $(X_n)_{\mathrm{red}}$ is flat over X by Corollary 4.16, and from this we obtain the exactness of $\bar{\varepsilon}^*_X$ by Proposition 2.4.
Next we prove (1). We may assume that we have a chart φ as above. Then, by Corollary 4.16, for any fs log scheme V etale over U_n, where $U \subset |X|$ is a Zariski open neighbourhood of x, and n is invertible in U, we see that V_{red} is again a weakly quasi-regularly stratified and reduced, so that $\bar{\varepsilon}^*_{V_{\mathrm{red}}}$ is also exact. Now the first result also follows from Proposition 2.4.
The last statement can be proven similarly. □

COROLLARY 4.18. *Let X be a weakly quasi-regularly stratified fs log scheme. Then $\bar{\varepsilon}^*_{X_{\mathrm{red}}}$ is exact and X_{red} has the property* (EX).

COROLLARY 4.19. *For a reduced quasi-regularly stratified fs log scheme X, we have an isomorphism $\mathscr{O}_{X_{\mathrm{Ket}}} \cong \overline{\mathscr{O}}_X$.*

PROPOSITION 4.20. *Let X and Y be Noetherian fs log schemes and $f : X \to Y$ a classically etale morphism. Assume that Y is weakly quasi-regularly stratified. Then we have an isomorphism*

$$\bar{\varepsilon}^*_Y f_* \cong f_* \bar{\varepsilon}^*_X$$

of functors from $\mathbf{Qcoh}(X_{\mathrm{Zar}})$ to $\overline{\mathbf{Qcoh}}(Y_{\mathrm{Ket}})$.

Proof. Let $\mathscr{F}$ be an object in $\mathbf{Qcoh}(X_{\mathrm{Zar}})$. For any $y \in |Y|$, we have

$$\begin{aligned}(\bar{\varepsilon}^*_Y f_* \mathscr{F})_{y(\log)} &\cong \operatorname{colim}_V (\bar{\varepsilon}^*_Y f_* \mathscr{F})(V),\\ (f_* \bar{\varepsilon}^*_X \mathscr{F})_{y(\log)} &\cong \operatorname{colim}_V (f_* \bar{\varepsilon}^*_X \mathscr{F})(V).\end{aligned}$$

Here we may assume that V runs through the essentially small category of Kummer etale neighbourhoods of $y(\log)$ whose maximal reduced closed subscheme is flat over Y, which is cofinal in the category of all Ket neighbourhoods (Proposition 4.17).

For such a V we set $U = V \times^{\mathrm{fs}}_Y X$. Then, by the classical etaleness of f we obtain $|U|^{\mathrm{red}} \cong (|V| \times_{|Y|} |X|)^{\mathrm{red}} \cong |V|^{\mathrm{red}} \times_{|Y|} |X|$. So, by Corollary 4.5 (2) and the flatness of V^{red} over Y, we have isomorphisms

$$\begin{array}{rcl} \Gamma(V, \bar{\varepsilon}_Y^* f_* \mathscr{F}) & \cong & \Gamma(V^{\mathrm{red}}, \mathscr{O}_{V^{\mathrm{red}}} \otimes_{\mathscr{O}_Y} f_* \mathscr{F}) \\ & \cong & \Gamma(U^{\mathrm{red}}, \mathscr{O}_{U^{\mathrm{red}}} \otimes_{\mathscr{O}_X} \mathscr{F}) \\ & \cong & \Gamma(U, \bar{\varepsilon}_X^* \mathscr{F}) \\ & \cong & \Gamma(V, f_* \bar{\varepsilon}_X^* \mathscr{F}). \end{array}$$

Now the proposition follows. □

4.3 Some propositions

In this subsection we collect some propositions which play essential roles later. First we prove some results on pellicular Kummer etale sheaves in the case of local schemes.

Proposition 4.21. *Let X be a Noetherian local equi-characteristic fs log scheme of characteristic exponent p, equipped with a chart $X \to \operatorname{Spec} \mathbb{Z}[P]$ with P fs and sharp. Then*

1. *For each $i \geq 0$ we have an isomorphism*

$$\overline{K}_i(X_{\mathrm{Ket}}) \cong \operatorname{colim}_n K_i((\widetilde{X_n})_{\mathrm{red}}, I_n \rtimes \Gamma_n),$$

where the right hand side is a colimit of equivariant (Zariski) K-groups indexed by natural numbers prime to p. A similar statement holds for K'-groups if X is weakly quasi-regularly stratified (For the notations, see Subsection 2.1.3).

2. *For any pellicular Ket coherent module $\mathscr{F}$ on X, there exists a surjective morphism $\oplus_i \mathscr{L}_i \to \mathscr{F}$ from a finite direct sum of pellicular Ket line bundles.*

Proof. They are only direct consequences to Proposition 4.10 and Corollary 4.13. Indeed, the statement (1) follows immediately from these propositions, and (2) follows from some simple arguments in equivariant module theory for semi-local rings. □

Proposition 4.22. *Let X be a Noetherian local equi-characteristic fs log scheme of characteristic exponent p. Assume that we are given an F-frame $\theta : F \to \Gamma(X, \overline{M}_X)$, and let $F_0 \subset F$ be the (unique) direct summand of F such that $\theta : F_0 \cong \overline{M}_{X,x}$, where x is the closed point of $|X|$.*
Then the frame θ induces a canonical isomorphism

$$F_0^{\mathrm{gp}} \otimes (\mathbb{Q}/\mathbb{Z})' \cong \operatorname{Pic}(X_{\mathrm{Ket}}) \cong \overline{\operatorname{Pic}}(X_{\mathrm{Ket}})$$

(For the definition of the isomorphism, see the argument after Theorem 3.17).

Before giving the proof of this proposition, we introduce a notation.

Definition 22. *Let P be a finitely generated free monoid, n a positive integer, $T = \operatorname{Spec}\mathbb{Z}[1/n][P]$ and $\widetilde{T_n} = \operatorname{Spec}\mathbb{Z}[1/n,\zeta_n][P^{1/n}]$. Recall that the group $\widetilde{I_n} = I_n \rtimes \Gamma_n$ acts on the scheme $\widetilde{T_n}$ (Subsection 2.1.3).*
Then, for any element $\alpha \in P^{1/n}$, we have a map defined by

$$\widetilde{I_n} = I_n \rtimes \Gamma_n \to \mathbb{Z}[1/n,\zeta_n][P^{1/n}]^{\times}\ ;\ (\varphi,\gamma) \mapsto \varphi(\alpha).$$

Since this satisfies the 1-cocycle condition, it induces an $\widetilde{I_n}$-equivariant line bundle on $(\widetilde{T_n})_{\mathrm{Zar}}$, free of rank one as a line bundle. We denote by $\mathscr{O}(\alpha)$ this line bundle (or its pullback with respect to some morphism to $\widetilde{T_n}$).

The lemma below is checked by a direct calculation.

Lemma 4.23. *Let F be a finitely generated free monoid, X a Noetherian equicharacteristic fs log scheme of characteristic exponent p, and $X \to \operatorname{Spec}\mathbb{Z}[F]$ a chart. We regard X as F-framed naturally by this chart.*
Then the diagram

$$\begin{array}{ccc}
\operatorname{colim}_{(p,n)=1} F^{1/n} & \longrightarrow & \operatorname{colim}_{(p,n)=1} \operatorname{Pic}((\widetilde{X_n})_{\mathrm{Zar}}, \widetilde{I_n}) \\
\cong\downarrow & & \downarrow \\
F^{\mathrm{div}'} & \longrightarrow & \operatorname{Pic}(X_{\mathrm{Ket}})
\end{array}$$

is commutative, where

- *the upper Picard group is the equivariant one,*
- *the upper horizontal morphism is induced from $\alpha \mapsto \mathscr{O}(\alpha)$,*
- *the lower horizontal one as in the paragraphs after Theorem 3.17, and*
- *the vertical ones the natural ones (*cf. *Proposition 3.21 in [Hag03]).*

Proof. (of Proposition 4.22) We may assume that $F = F_0$, and that there exists a chart $X \to \operatorname{Spec}\mathbb{Z}[F]$ compatible with the frame θ (Proposition 3.7).
Then, since $\operatorname{Pic}((\widetilde{X_n})_{\mathrm{Zar}})$ is trivial, we have the following commutative diagram:

$$\begin{array}{ccccc}
\operatorname{colim}_n F^{1/n}/F & \longrightarrow & \operatorname{colim}_n \operatorname{Pic}(\widetilde{X_n}, \widetilde{I_n}) & \longrightarrow & \operatorname{colim}_n \operatorname{Pic}((\widetilde{X_n})_{\mathrm{red}}, \widetilde{I_n}) \\
\cong\downarrow & & \downarrow & & \downarrow \\
F^{\mathrm{div}'}/F & \longrightarrow & \operatorname{Pic}(X_{\mathrm{Ket}}) & \longrightarrow & \overline{\operatorname{Pic}}(X_{\mathrm{Ket}}),
\end{array}$$

where the upper colimit is taken with respect to positive integers prime to p, the Picard groups are equivariant (Zariski) ones, and the morphisms are

as in Lemma 4.23 or the natural ones (For the right vertical morphism, see Proposition 4.10).
Now, as is easily seen by the argument in equivariant module theory, the upper horizontal morphisms are isomorphisms, and by Proposition 4.10 and Corollary 4.13, so are the vertical ones. □

The proposition below will be used in the proof of the comparison of K- and K'-groups.

Proposition 4.24. *Let X be an F-framed Noetherian equi-characteristic divisorial fs log scheme with the property* (EX) *and $x \in |X|$. Then,*

1. *For any pellicular Ket coherent sheaves of $\overline{\mathscr{O}}_X$-modules $\mathscr{E}$, $\mathscr{F}$ and a morphism of sheaves $f : \mathscr{E}_x \to \mathscr{F}_x$ on $(\operatorname{Spec} \mathscr{O}_{X,x})_{\mathrm{Ket}}$, there exist a pellicular Ket line bundle $\mathscr{L}$ and an $s \in \Gamma(X, \mathscr{L}^{-1})$ such that $fs_x : (\mathscr{E} \otimes \mathscr{L})_x \to \mathscr{F}_x$ is extended to a global morphism $\phi : \mathscr{E} \otimes \mathscr{L} \to \mathscr{F}$.*

2. *For any pellicular Ket coherent sheaf $\mathscr{F}$, there exist a pellicular Ket vector bundle $\mathscr{E}$ and a surjective $\overline{\mathscr{O}}_X$-homomorphism $\mathscr{E} \to \mathscr{F}$.*

Proof. By Proposition 4.11, the claim (1) is proven similarly to [Hag03] Lemma 4.11, and (2) follows from (1), Proposition 4.21 (2) and Proposition 4.22 as in [Hag03] Proposition 7.2. □

Later we will also need the following proposition, which can be proven easily.

Proposition 4.25. *Let $\{X_i\}_{i \in I}$ be a system of Noetherian fs log schemes with the property* (EX)*, indexed by a filtered category I. Assume that any transition map $\pi_{ij} : X_j \to X_i$ is strict and $|\pi_{ij}|$ is affine and etale. We denote by X a projective limit $\varprojlim_{i \in I} X_i$ in the category of fs log schemes and assume that X also has the property* (EX)*.*
Then we have

$$\operatorname{colim}_{i \in I} \overline{K}'_q((X_i)_{\mathrm{Ket}}) \xrightarrow{\cong} \overline{K}'_q(X_{\mathrm{Ket}}).$$

If every X_i and every π_{ij} is F-framed, then a similar statement holds for $\overline{K}'(-^{\mathfrak{q}}_{\mathrm{Ket}})$ for any prime $\mathfrak{q}$ of F.

5 The proof of the main theorem

5.1 Localisation sequence for pellicular Ket K-theory

Definition 23. *Let X be a Noetherian fs log scheme, Y a closed subscheme of $|X|$ and U its complement. We endow them with the induced log structures. We denote by i the natural closed immersion from Y to X, and by j the natural open immersion from U to X. Then we define a ring object $\overline{\mathscr{O}}_{Y,X}$ in $(Y_{\mathrm{Ket}})\tilde{}$ to be $\varepsilon_Y^{-1}\mathscr{O}_{Y_{\mathrm{Zar}}} \otimes_{\varepsilon_Y^{-1} i^{-1} \mathscr{O}_{X_{\mathrm{Zar}}}} i^{-1}\overline{\mathscr{O}}_X$.*

The followings are easily proven.

Proposition 5.1. *For a log scheme X' Kummer etale over X, set $Y' = X' \times_X^{\mathrm{fs}} Y$. Then we have a natural isomorphism $\overline{\mathscr{O}}_{Y,X}|_{Y'} \cong \overline{\mathscr{O}}_{Y',X'}$.*

Proposition 5.2. *1. The homomorphism $i^{-1}\overline{\mathscr{O}}_X \to \overline{\mathscr{O}}_{Y,X}$ is surjective.*

2. The kernel $\overline{I}_Y$ of the above homomorphism $i^{-1}\overline{\mathscr{O}}_X \to \overline{\mathscr{O}}_{Y,X}$ is equal to $i^{-1}((\varepsilon_X^{-1} I_{Y_{\mathrm{Zar}}})\overline{\mathscr{O}}_X)$, where $I_{Y_{\mathrm{Zar}}}$ is the ideal sheaf of $\mathscr{O}_{X_{\mathrm{Zar}}}$ defining Y and $(\varepsilon_X^{-1} I_{Y_{\mathrm{Zar}}})\overline{\mathscr{O}}_X$ is the ideal sheaf of $\overline{\mathscr{O}}_X$ generated by the image of $\varepsilon_X^{-1} I_{Y_{\mathrm{Zar}}}$.

3. If $\bar{\varepsilon}_X^*$ is exact, then the homomorphism $\varepsilon_Y^{-1}\mathscr{O}_{Y_{\mathrm{Zar}}} \to \overline{\mathscr{O}}_{Y,X}$ is flat.

4. If X has the property (EX)*, the sheaf $\overline{\mathscr{O}}_{Y,X}$ is coherent as a $\overline{\mathscr{O}}_{Y,X}$-module, and so is $i_*\overline{\mathscr{O}}_{Y,X}$ as a $\overline{\mathscr{O}}_X$-module.*

Proof. (1), (2) and (3) are obvious, and (4) follows from (2), Proposition 4.8 and the above proposition. □

Remark 5.3. *In general, $\overline{\mathscr{O}}_{Y,X}$ is far from isomorphic to $\overline{\mathscr{O}}_Y$ (indeed it is so very often).*

Definition 24. *1. An $\overline{\mathscr{O}}_{Y,X}$-module $\mathscr{F}$ is said to be coherent if it is coherent with respect to $(Y_{\mathrm{Ket}}, \overline{\mathscr{O}}_{Y,X})$ in the sense of J.-P. Serre. We denote the full subcategory of $\mathbf{Mod}(Y_{\mathrm{Ket}}, \overline{\mathscr{O}}_{Y,X})$ consisting of coherent $\overline{\mathscr{O}}_{Y,X}$-modules by $\overline{\mathbf{Coh}}(Y_{\mathrm{Ket}}; X)$.*

2. For an X with the property (EX)*, we define $\overline{K}'(Y_{\mathrm{Ket}}; X)$ to be a K-theory spectrum constructed from this Abelian category.*

The next proposition is easily proven by the dévissage theorem ([Qui73]).

Proposition 5.4. *We have natural weak equivalences of spectra*

$$\overline{K}'(Y_{\mathrm{Ket}}; X) \cong \overline{K}'((Y_{\mathrm{red}})_{\mathrm{Ket}}; X) \cong \overline{K}'((Y_{\mathrm{red}})_{\mathrm{Ket}}; X_{\mathrm{red}}).$$

Notice that we have a natural exact functor $\overline{\mathbf{Coh}}(Y_{\mathrm{Ket}}; X) \to \overline{\mathbf{Coh}}(X_{\mathrm{Ket}})$. We denote by i_* the induced homomorphism $\overline{K}'_q(Y_{\mathrm{Ket}}; X) \to \overline{K}'_q(X_{\mathrm{Ket}})$.

Theorem 5.5. *Let X be a Noetherian equi-characteristic weakly quasi-regularly stratified fs log scheme, Y a strictly closed subscheme and U its open complement, which we endow with the induced log structure. Then we have a canonical long exact sequence*

$$\begin{aligned} \cdots \xrightarrow{\partial} \overline{K}'_q(Y_{\mathrm{Ket}}; X) \xrightarrow{i_*} \overline{K}'_q(X_{\mathrm{Ket}}) \xrightarrow{j^*} \overline{K}'_q(U_{\mathrm{Ket}}) \xrightarrow{\partial} \overline{K}'_{q-1}(Y_{\mathrm{Ket}}; X) \xrightarrow{i_*} \cdots \\ \cdots \xrightarrow{\partial} \overline{K}'_0(Y_{\mathrm{Ket}}; X) \xrightarrow{i_*} \overline{K}'_0(X_{\mathrm{Ket}}) \xrightarrow{j^*} \overline{K}'_0(U_{\mathrm{Ket}}) \to 0. \end{aligned}$$

Proof. First we may assume that X is reduced because taking $(-)_{\mathrm{red}}$ makes the conditions and the conclusion unchanged (Proposition 4.14 (3) and the above proposition). Then we see that $\bar{\varepsilon}_X^*$ is exact by Corollary 4.18. As usual we have only to show the following three lemmata (cf. the proof of Theorem 4.5 in [Hag03]). □

Lemma 5.6. *Let X be a Noetherian fs log scheme, Y a strictly closed subscheme and $\mathscr{F}$ a pellicular Ket coherent sheaf of $\overline{\mathscr{O}}_X$-modules. Assume that $\overline{M}_{X,\bar{x}}$ is finitely generated free for each geometric point $\bar{x} \in X$. If we have $\mathscr{F}|_U = 0$, then there exists a natural number N such that $((\varepsilon_X^{-1} I_{Y_{\mathrm{Zar}}})\overline{\mathscr{O}}_X)^N \mathscr{F} = 0$.*

Proof. First note that $\mathscr{F}_{x(\log)} = 0$ for any $x \notin |Y|$ by assumption.
Take a Kummer etale covering $f : X' \to X$ and a Zariski coherent $\mathscr{O}_{X'}$-module $\widetilde{\mathscr{F}}$ such that $\mathscr{F}|_{X'} \cong \bar{\varepsilon}^*_{X'}\widetilde{\mathscr{F}}$. Here, by assumption on $\overline{M}$, we may assume $|X'|$ is flat over $|X|$. Put $Y' = Y \times_X X'$.
Since we have $\mathscr{F}_{x'(\log)} \cong \overline{\mathscr{O}}_{X',x'(\log)} \otimes_{\mathscr{O}_{X',x'}} \widetilde{\mathscr{F}}_{x'}$ for every $x' \in |X'|$, the set $\{x' \in |X'| \mid \widetilde{\mathscr{F}}_{x'} \neq 0\}$ is contained in $|Y'|$ by the faithful flatness of $\mathscr{O}_{X',x'} \to \mathscr{O}_{X',x'(\log)}$. Thus we have a natural number N such that $I^N_{Y'_{\mathrm{Zar}}} \widetilde{\mathscr{F}} = 0$.
By the surjectivity of $\bar{\varepsilon}^*_{X'} I_{Y'_{\mathrm{Zar}}} \to ((\varepsilon_X^{-1} I_{Y_{\mathrm{Zar}}})\overline{\mathscr{O}}_X)|_{X'}$, the lemma now follows. □

Lemma 5.7. *Let X be a Noetherian equi-characteristic weakly quasi-regularly stratified fs log scheme, U a strictly open subscheme and $j : U \to X$ a natural open immersion. Then, for any pellicular Ket quasi-coherent module $\mathscr{F}$ on U, $j_*\mathscr{F}$ is pellicular Ket quasi-coherent on X.*

Proof. This is proven similarly to the proof of [Hag03] Proposition 4.6, by using Proposition 4.8 (3), Corollary 4.13 and Proposition 4.20. □

Lemma 5.8. *Let X be a Noetherian reduced weakly quasi-regularly stratified fs log scheme and $\mathscr{F}$ a pellicular Ket quasi-coherent sheaf. Then we have*

$$\mathscr{F} \cong \varinjlim \mathscr{F}_i,$$

where $\mathscr{F}_i$ run through all pellicular Ket coherent sheaves such that $\mathscr{F}_i \subset \mathscr{F}$.

Proof. (cf. [Hag03] Proposition 4.7.) Take a Kummer etale covering $\pi : X' \to X$ and a Zariski quasi-coherent $\mathscr{O}_{X'_{\mathrm{red}}}$-module $\widetilde{\mathscr{F}}$ (not a $\mathscr{O}_{X'}$-module) such that $\pi^*\mathscr{F} \cong \bar{\varepsilon}^*_{X'_{\mathrm{red}}} \widetilde{\mathscr{F}}$. By Proposition 4.17, we may assume that $\bar{\varepsilon}^*_{X'_{\mathrm{red}}}$ is also exact and that X' is Noetherian. The sheaf $\widetilde{\mathscr{F}}$ can be written as

$$\widetilde{\mathscr{F}} \cong \varinjlim \widetilde{\mathscr{F}_i},$$

using all Zariski coherent sub-$\mathscr{O}_{X'_{\mathrm{red}}}$-modules $\widetilde{\mathscr{F}_i} \subset \widetilde{\mathscr{F}}$. Then we have $\pi^*\mathscr{F} \cong \varinjlim \bar{\varepsilon}^*_{X'_{\mathrm{red}}} \widetilde{\mathscr{F}_i}$ and $\pi_*\pi^*\mathscr{F} \cong \pi_* \varinjlim \bar{\varepsilon}^*_{X'_{\mathrm{red}}} \widetilde{\mathscr{F}_i} \cong \varinjlim \pi_*\bar{\varepsilon}^*_{X'_{\mathrm{red}}} \widetilde{\mathscr{F}_i}$ (Note that as X and X' are Noetherian and $\{\bar{\varepsilon}^*_{X'_{\mathrm{red}}} \widetilde{\mathscr{F}_i}\}$ forms a direct system, inductive limit commutes with π_*). If we define $\mathscr{F}_i$ by the Cartesian diagram

$$\begin{tikzcd} \mathscr{F}_i \arrow[r] \arrow[d, hook] & \pi_*\bar{\varepsilon}^*_{X'_{\mathrm{red}}} \widetilde{\mathscr{F}_i} \arrow[d, hook] \\ \mathscr{F} \arrow[r] & \pi_*\pi^*\mathscr{F}, \end{tikzcd}$$

then we have $\mathscr{F} \cong \varinjlim \mathscr{F}_i$. By the following commutative diagram

$$\begin{tikzcd} \pi^*\mathscr{F}_i \arrow[r] \arrow[drr, hook] & \pi^*\pi_*\bar{\varepsilon}^*_{X'_{\mathrm{red}}}\widetilde{\mathscr{F}_i} \arrow[r] & \bar{\varepsilon}^*_{X'_{\mathrm{red}}}\widetilde{\mathscr{F}_i} \arrow[d] \\ & & \pi^*\mathscr{F}, \end{tikzcd}$$

we see that $\pi^*\mathscr{F}_i$ is pellicular Ket coherent, as it is a pellicular Ket quasi-coherent subsheaf of a pellicular Ket coherent sheaf $\bar{\varepsilon}^*_{X'_{\mathrm{red}}}\widetilde{\mathscr{F}_i}$, and X' is Noetherian and has the property (EX) (Use Proposition 4.8). □

As usual, we see that the presheaf of fibrant spectra $\varepsilon_{X*}\overline{K}'_{\mathrm{Ket}}$ on $\mathbf{Et}/|X|$(the category of schemes etale over $|X|$) defined by

$$\text{“}U \mapsto \overline{K}'(U_{\mathrm{Ket}})\text{”} \ \text{(where } U \text{ has the induced log structure)}$$

satisfies “a Nisnevich Brown-Gersten property” (i.e. a cd-excision property in p.286 of [Jar97]) if X is a Noetherian equi-characteristic weakly quasi-regularly stratified fs log scheme. The more precise statement is as follows.

Corollary 5.9. *Assume that we are given a Cartesian diagram of Noetherian equi-characteristic weakly quasi-regularly stratified fs log schemes*

$$\begin{tikzcd} V' \arrow[r, hook, "j'"] \arrow[d, "f'"] & U' \arrow[d, "f"] \\ V \arrow[r, hook, "j"] & U \end{tikzcd}$$

with vertical morphisms (classically) etale and horizontal ones strictly open immersions. We assume that the induced morphism $f'' : U' \setminus V' \to U \setminus V$ is isomorphism if both sides are endowed with the induced reduced log scheme structure. Then the induced square of spectra

$$\begin{tikzcd} \overline{K}'(U_{\mathrm{Ket}}) \arrow[r, "j^*"] \arrow[d, "f^*"] & \overline{K}'(V_{\mathrm{Ket}}) \arrow[d, "f'^*"] \\ \overline{K}'(U'_{\mathrm{Ket}}) \arrow[r, "j'^*"] & \overline{K}'(V'_{\mathrm{Ket}}) \end{tikzcd}$$

is homotopy Cartesian.
In particular, if X is a weakly quasi-regularly stratified fs log scheme whose underlying scheme is Noetherian, equi-characteristic, and of finite Krull dimension, we have a weak equivalence

$$\overline{K}'(X_{\mathrm{Ket}}) \xrightarrow{\cong} \mathbb{H}^{\cdot}(X_{\mathrm{Nis}}, \varepsilon_{X*}\overline{K}'_{\mathrm{Ket}}).$$

Similar statements hold for $\overline{K}'(-^{\mathfrak{q}}_{\mathrm{Ket}})$ and $\mathbb{H}^{\cdot}(X_{\mathrm{Nis}}, \varepsilon_{X*}\overline{K}'(-^{\mathfrak{q}}_{\mathrm{Ket}}))$ if U or X is endowed with an F-frame structure.

Proof. Set $W = U \setminus V$ and $W' = U' \setminus V'$. Then we have $f''^{-1}\overline{\mathscr{O}}_{W,U} \cong \overline{\mathscr{O}}_{W',U'}$. Now Corollary follows easily from the above theorem. The latter part follows from Corollary 7.68 in [Jar97]. The case for $X^{\mathfrak{q}}_{\mathrm{Ket}}$ immediately follows from that for $\mathfrak{q} = \emptyset$ by noting Proposition 3.6 (5). ◻

5.2 Comparison of pellicular Ket K-theory and pellicular Ket K'-theory

Theorem 5.10. *Let F and X be as in Convention 3.13. Then we have a natural weak equivalence*

$$\overline{K}(X_{\mathrm{Ket}}) \cong \overline{K}'(X_{\mathrm{Ket}}).$$

A similar proposition holds for $X[\mathfrak{p}]^{\mathfrak{q}}_{\mathrm{Ket}}$ for prime ideals $\mathfrak{p}$ and $\mathfrak{q}$.

Proof. The latter case is reduced to the former (by Proposition 3.12), the proof of which is completely parallel to the one of Proposition 7.1 in [Hag03]. Indeed, we have only to use Proposition 4.24 and the following lemma. ◻

Lemma 5.11. *Let X be a weakly log regular fs log scheme whose underlying scheme is Noetherian and regular.*
If we are given a chart $X \to \operatorname{Spec}\mathbb{Z}[P]$ with P fs and sharp, then, for any positive integer n, $(X_n)_{\mathrm{red}}$ is again Noetherian, regular and weakly log regular. In particular, for any log scheme X' Kummer etale over X, there exists a Kummer etale covering $X'' \to X'$ such that X''_{red} is regular and weakly log regular.

Proof. The former is straightforward, and the latter follows from the former by Proposition 2.4. ◻

Combining Theorem 5.10 and Corollary 5.9, we obtain the following corollary:

Corollary 5.12. *Let F and X be as in Convention 3.13. Then we have a weak equivalence*

$$\overline{K}(X_{\mathrm{Ket}}) \xrightarrow{\cong} \mathbb{H}^{\cdot}(X_{\mathrm{Nis}}, \varepsilon_{X*}\overline{K}_{\mathrm{Ket}}).$$

5.3 Inversion formula

We fix a finitely generated free monoid F also in this subsection. Here we prove Theorem 3.14.

Theorem 5.13. *Let X be an F-framed weakly quasi-regularly stratified fs log scheme such that $|X|$ is finite dimensional, Noetherian, separated and equi-characteristic of characteristic p. Then, for a prime ideal $\mathfrak{q}$ of F, we have a weak equivalence*

$$\overline{K}'(X^{\mathfrak{q}}_{\mathrm{Ket}})_{\mathbb{Q}} \xrightarrow{\cong} \mathbb{H}^{\cdot}(X^{\mathfrak{q}^{\vee}}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}^{\vee}}_{X*}(\overline{K}'_{\mathrm{Ket}})_{\mathbb{Q}}).$$

Proof. First we show that we may assume that X is Henselian local.
For a general X, consider the following up-to-homotopy commutative diagram

$$\begin{tikzcd}
\overline{K}'(X^{\mathfrak{q}}_{\mathrm{Ket}})_{\mathbb{Q}} \arrow[r] \arrow[d] & \mathbb{H}^{\cdot}(X_{\mathrm{Nis}}, \overline{K}'(-^{\mathfrak{q}}_{\mathrm{Ket}})_{\mathbb{Q}}) \arrow[d] \\
\mathbb{H}^{\cdot}(X^{\mathfrak{q}^\vee}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}^\vee}_{X*}(\overline{K}'_{\mathrm{Ket}})_{\mathbb{Q}}) \arrow[r] & \mathbb{H}^{\cdot}(X_{\mathrm{Nis}}, \mathbb{H}^{\cdot}(-^{\mathfrak{q}^\vee}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}^\vee}_{X*}(\overline{K}'_{\mathrm{Ket}})_{\mathbb{Q}})),
\end{tikzcd}$$

where $\mathbb{H}^{\cdot}(-^{\mathfrak{q}^\vee}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}^\vee}_{X*}(\overline{K}'_{\mathrm{Ket}})_{\mathbb{Q}})$ and $\overline{K}'(-^{\mathfrak{q}}_{\mathrm{Ket}})_{\mathbb{Q}}$ are presheaves of fibrant spectra on X_{Nis} naturally defined.
Here the upper horizontal morphism is a weak equivalence by Corollary 5.9, the lower horizontal one by the Cartan-Leray-type theorem proven formally (See Theorem 1. 56 in [Tho85]).
Therefore, the weak equivalence of the left vertical morphism follows from that of the right vertical one. Thus the theorem is reduced to the Henselian local case by Proposition 4.25.
From now on we will assume that X be Henselian local. Using the spectral sequence

$$H^i(X^{\mathfrak{q}^\vee}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}^\vee}_{X*}(\overline{K}'_{j,\mathrm{Ket}})_{\mathbb{Q}}) \Longrightarrow \pi_{j-i}\mathbb{H}^{\cdot}(X^{\mathfrak{q}^\vee}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}^\vee}_{X*}(\overline{K}'_{\mathrm{Ket}})_{\mathbb{Q}}),$$

where $\varepsilon^{\mathfrak{q}^\vee}_{X*}(\overline{K}'_{j,\mathrm{Ket}})_{\mathbb{Q}}$ is the sheaf on $X^{\mathfrak{q}^\vee}_{\mathrm{Ket}}$ associated with "$U \mapsto \overline{K}'_j((U \times^{\mathrm{fs}}_{X^{\mathfrak{q}^\vee}} X)_{\mathrm{Ket}})_{\mathbb{Q}}$", and the vanishing of the higher group cohomology of profinite groups with $\mathbb{Q}$-vector space coefficients, we have an isomorphism

$$\pi_j\mathbb{H}^{\cdot}(X^{\mathfrak{q}^\vee}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}^\vee}_{X*}(\overline{K}'_{\mathrm{Ket}})_{\mathbb{Q}}) \xrightarrow{\cong} \Gamma(X^{\mathfrak{q}^\vee}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}^\vee}_{X*}(\overline{K}'_{j,\mathrm{Ket}})_{\mathbb{Q}}).$$

Moreover, as is easily checked, the right hand side can be rewritten as follows:

$$\Gamma(X^{\mathfrak{q}^\vee}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}^\vee}_{X*}(\overline{K}'_{j,\mathrm{Ket}})_{\mathbb{Q}}) \xrightarrow{\cong} (\varinjlim_{\widetilde{X}} \overline{K}'_j((\widetilde{X} \times^{\mathrm{fs}}_{X^{\mathfrak{q}^\vee}} X)_{\mathrm{Ket}})_{\mathbb{Q}})^{\pi_1(X^{\mathfrak{q}^\vee}_{\mathrm{Ket}})},$$

where $\widetilde{X}$ runs through Kummer etale neighbourhoods of $X^{\mathfrak{q}^\vee}$ (with respect to a fixed log geometric point over the closed point), and $\pi_1(X^{\mathfrak{q}^\vee}_{\mathrm{Ket}})$ is the projective limit of their automorphism groups over $X^{\mathfrak{q}^\vee}$. The notation $(-)^{\pi_1(X^{\mathfrak{q}^\vee}_{\mathrm{Ket}})}$ denotes the invariant part for the natural action.
For a further reduction, we introduce some notations:

DEFINITION 25. *Let F be a finitely generated free monoid, and X an F-framed fs log scheme, and $\mathfrak{s}$ a prime ideal of F. Assume that we are given a chart $X \to \operatorname{Spec}\mathbb{Z}[F]$ which lifts the given frame.*
For a positive integer n invertible on X, we set

$$\widetilde{X^{\mathfrak{s}}_n} = X^{\mathfrak{s}} \times^{\mathrm{fs}}_{\operatorname{Spec}\mathbb{Z}[1/n][F\setminus\mathfrak{s}]} \operatorname{Spec}\mathbb{Z}[1/n, \mu_n][(F \setminus \mathfrak{s})^{1/n}],$$

$$I_n^{\mathfrak{s}} = \mathrm{Hom}\,((F \setminus \mathfrak{s})^{1/n}/(F \setminus \mathfrak{s}), \mathbb{Z}[\mu_n]^{\times}),$$

and $\widetilde{I_n^{\mathfrak{s}}} = I_n^{\mathfrak{s}} \rtimes \mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})$ *(cf. Subsection 2.1.3). We regard* $\widetilde{X_n^{\mathfrak{s}}}$ *as an fs log scheme by the chart* $\widetilde{X_n^{\mathfrak{s}}} \to \mathrm{Spec}\,\mathbb{Z}[1/n, \mu_n][(F \setminus \mathfrak{s})^{1/n}]$.
Note that the log scheme $\widetilde{X_n^{\mathfrak{s}}}$ *has a natural group action of* $\widetilde{I_n^{\mathfrak{s}}}$*, and that, for prime ideals* $\mathfrak{s} \subset \mathfrak{r}$*, the natural morphism* $\widetilde{X_n^{\mathfrak{s}}} \to \widetilde{X_n^{\mathfrak{r}}}$ *is compatible with the natural surjective homomorphism* $\widetilde{I_n^{\mathfrak{s}}} \to \widetilde{I_n^{\mathfrak{r}}}$.

The followings are easily checked:

Lemma 5.14. *1. The set*

$$\{\widetilde{X_m^{\mathfrak{q}^\vee}} \mid m \in \mathbb{N}, (m,p) = 1\}$$

is cofinal in the category of Kummer etale neighbourhoods of $X^{\mathfrak{q}^\vee}$ *(for a suitable choice of log geometric points over the closed points).*

2. *For a positive integer* m*, the set*

$$\{\widetilde{X_{mr}^{\mathfrak{q}^\vee}} \times^{\mathrm{fs}}_{X^{\mathfrak{m}}} \widetilde{X_n^{\mathfrak{q}}} \mid r, n \in \mathbb{N}, (r,p) = (n,p) = 1\}$$

is cofinal in the category of Kummer etale neighbourhoods of the log scheme $\widetilde{X_m^{\mathfrak{q}^\vee}} \times^{\mathrm{fs}}_{X^{\mathfrak{q}^\vee}} X = \widetilde{X_m^{\mathfrak{q}^\vee}} \times^{\mathrm{fs}}_{X^{\mathfrak{m}}} X^{\mathfrak{q}}$*, where* $\mathfrak{m}$ *is the maximal ideal of* F *(for a suitable choice of log geometric points over the closed points).*

Therefore it suffices to prove that

$$\overline{K}'_j(X^{\mathfrak{q}}_{\mathrm{Ket}})_{\mathbb{Q}} \cong (\varinjlim_m \overline{K}'_j((\widetilde{X_m^{\mathfrak{q}^\vee}} \times^{\mathrm{fs}}_{X^{\mathfrak{q}^\vee}} X)_{\mathrm{Ket}})_{\mathbb{Q}})^{\pi_1(X^{\mathfrak{q}^\vee}_{\mathrm{Ket}})},$$

or by using Proposition 4.21 and the above lemma,

$$\varinjlim_n K'_j(|\widetilde{X_n^{\mathfrak{q}}}|_{\mathrm{red}}, \widetilde{I_n^{\mathfrak{q}}})_{\mathbb{Q}} \cong \varinjlim_m \Big(\varinjlim_{n,r} K'_j(|\widetilde{X_{mr}^{\mathfrak{q}^\vee}} \times^{\mathrm{fs}}_{X^{\mathfrak{m}}} \widetilde{X_n^{\mathfrak{q}}}|_{\mathrm{red}}, \widetilde{I_{mr,m}^{\mathfrak{q}^\vee}} \times \widetilde{I_n^{\mathfrak{q}}})_{\mathbb{Q}} \Big)^{\widetilde{I_m^{\mathfrak{q}^\vee}}},$$

where $\widetilde{I_{mr,m}^{\mathfrak{q}^\vee}} = \mathrm{Kernel}\,(\widetilde{I_{mr}^{\mathfrak{q}^\vee}} \to \widetilde{I_m^{\mathfrak{q}^\vee}})$, and r, m and n runs positive integers prime to p.
Thus, Theorem 5.13 (in the Henselian local case) is reduced to the following lemma (In (1), we only need the case $\mathfrak{r} = \mathfrak{q}^\vee$):

Lemma 5.15. *Let* X *and* F *be as in Theorem 5.13, and assume that* $|X|$ *is Henselian local, and that we are given a chart* $X \to \mathrm{Spec}\,\mathbb{Z}[F]$ *lifting the frame.*

1. *For each positive integers* m *and* n *prime to* p*, and prime ideals* $\mathfrak{r}$ *and* $\mathfrak{q}$ *of* F *such that* $\mathfrak{r} \wedge \mathfrak{q} = \emptyset$*, we have a natural isomorphism*

$$K'_j(|\widetilde{X_n^{\mathfrak{q}}}|_{\mathrm{red}}, \widetilde{I_n^{\mathfrak{q}}})_{\mathbb{Q}} \cong K'_j(|\widetilde{X_m^{\mathfrak{r}}} \times_{X^{\mathfrak{m}}} \widetilde{X_n^{\mathfrak{q}}}|_{\mathrm{red}}, \widetilde{I_n^{\mathfrak{q}}})_{\mathbb{Q}}^{\widetilde{I_m^{\mathfrak{r}}}},$$

for every $j \geq 0$.

2. *For every positive integer n prime to p and a prime ideal $\mathfrak{q}$, we have a natural isomorphism*

$$\varinjlim_{m} K'_j(|\widetilde{X_m^{\mathfrak{q}^\vee}} \times^{\mathrm{fs}}_{X^{\mathfrak{m}}} \widetilde{X_n^{\mathfrak{q}}}|_{\mathrm{red}}, \widetilde{I_n^{\mathfrak{q}}}) \cong \varinjlim_{m,r} K'_j(|\widetilde{X_{mr}^{\mathfrak{q}^\vee}} \times^{\mathrm{fs}}_{X^{\mathfrak{m}}} \widetilde{X_n^{\mathfrak{q}}}|_{\mathrm{red}}, \widetilde{I_{mr,m}^{\mathfrak{q}^\vee}} \times \widetilde{I_n^{\mathfrak{q}}}),$$

for every $j \geq 0$.

To prove (2), we set $K_{m,r} = K'_j(|\widetilde{X_{mr}^{\mathfrak{q}^\vee}} \times^{\mathrm{fs}}_{X^{\mathfrak{m}}} \widetilde{X_n^{\mathfrak{q}}}|_{\mathrm{red}}, \widetilde{I_{mr,m}^{\mathfrak{q}^\vee}} \times \widetilde{I_n^{\mathfrak{q}}})$. Then we have a transition morphism $K_{m,r} \to K_{m',r'}$ if $m|m'$ and $mr|m'r'$.
In particular, for every (m,r), we have a morphism $K_{m,r} \to K_{mr,1}$, which means the set $\{K_{m,1}\}$ is cofinal. Hence (2) follows.
The claim (1) is reduced to the lemma below by the descending induction on the number of generators of $\mathfrak{r}$.

LEMMA 5.16. *Let $\mathfrak{r}' = \mathfrak{r} \vee \langle e \rangle$, where e is an element of the standard basis of F such that $e \notin \mathfrak{r}$, and suppose that $\mathfrak{q} \wedge \mathfrak{r}' = \emptyset$. Then we have*

$$K'_j(|\widetilde{X_m^{\mathfrak{r}'}} \times_{X^{\mathfrak{m}}} \widetilde{X_n^{\mathfrak{q}}}|_{\mathrm{red}}, \widetilde{I_n^{\mathfrak{q}}})_{\mathbb{Q}} \cong K'_j(|\widetilde{X_m^{\mathfrak{r}}} \times_{X^{\mathfrak{m}}} \widetilde{X_n^{\mathfrak{q}}}|_{\mathrm{red}}, \widetilde{I_n^{\mathfrak{q}}})_{\mathbb{Q}}^{\mu_n}$$

for every $j \geq 0$, and m and n prime to p.

Finally, using the following two facts proven by Proposition 4.15, we can reduce this lemma to the next proposition.

- The diagram of schemes

$$\begin{tikzcd}
|\widetilde{X_m^{\mathfrak{r}}} \times_{X^{\mathfrak{m}}} \widetilde{X_n^{\mathfrak{q}}}| \arrow[r] \arrow[d] & \operatorname{Spec} R[e^{1/m}] \arrow[d] \\
|\widetilde{X_m^{\mathfrak{r}'}} \times_{X^{\mathfrak{m}}} \widetilde{X_n^{\mathfrak{q}}}| \arrow[r, "f"] & \operatorname{Spec} R[e]
\end{tikzcd}$$

  is Cartesian.

- We have one of the followings, depending on whether $f^*(e)$ is zero or a non-zero divisor.

  1. $|\widetilde{X_m^{\mathfrak{r}}} \times_{X^{\mathfrak{m}}} \widetilde{X_n^{\mathfrak{q}}}|_{\mathrm{red}} \cong |\widetilde{X_m^{\mathfrak{r}'}} \times_{X^{\mathfrak{m}}} \widetilde{X_n^{\mathfrak{q}}}|_{\mathrm{red}}$.
  2. The diagram of schemes

$$\begin{tikzcd}
|\widetilde{X_m^{\mathfrak{r}}} \times_{X^{\mathfrak{m}}} \widetilde{X_n^{\mathfrak{q}}}|_{\mathrm{red}} \arrow[r] \arrow[d] & |\widetilde{X_m^{\mathfrak{r}}} \times_{X^{\mathfrak{m}}} \widetilde{X_n^{\mathfrak{q}}}| \arrow[d] \\
|\widetilde{X_m^{\mathfrak{r}'}} \times_{X^{\mathfrak{m}}} \widetilde{X_n^{\mathfrak{q}}}|_{\mathrm{red}} \arrow[r] & |\widetilde{X_m^{\mathfrak{r}'}} \times_{X^{\mathfrak{m}}} \widetilde{X_n^{\mathfrak{q}}}|
\end{tikzcd}$$

  is Cartesian.

Now the proof of Theorem 5.13 is completed. □

Proposition 5.17. *Let G be a finite group, Z a Noetherian and equi-characteristic scheme of characteristic exponent p with the action of G, n a positive integer prime to p, and $R = \mathbb{Z}[1/n, \mu_n]$. Suppose that we are given a G-equivariant morphism $Z \to \operatorname{Spec} R[T]$ with $\operatorname{Spec} R[T]$ trivially acted by G. Let $Y = Z \times_{\operatorname{Spec} R[T]} \operatorname{Spec} R[T^{1/n}]$, $\pi : Y \to Z$ the natural map. Note that μ_n acts on Y via the natural action "$T^{1/n} \mapsto \zeta \cdot T^{1/n}$" on $R[T^{1/n}]$. Assume, in addition, that $\pi_* \mathscr{O}_{Y_{\mathrm{red}}}$ is free as a G-equivariant $\mathscr{O}_{Z_{\mathrm{red}}}$-module (Hence, Y_{red} is flat over Z_{red}). Then we have an isomorphism of equivariant K'-groups*

$$K'_j(Z_{\mathrm{red}}, G)_{\mathbb{Q}} \longrightarrow K'_j(Y_{\mathrm{red}}, G)_{\mathbb{Q}}^{\mu_n}$$

for $j \geq 0$.

Proof. Consider the diagram

$$\{0\}_R \hookrightarrow \mathbb{A}^1_R \hookleftarrow (\mathbb{A}^1 \setminus \{0\})_R$$

and by taking the fiber product with respect to $Y \to Z \to \operatorname{Spec} R[T]$ construct the diagram

$$\begin{tikzcd} Y_0 \arrow[r, "i_Y"] \arrow[d] & Y \arrow[d] & Y \setminus Y_0 \arrow[l, "j_Y"'] \arrow[d] \\ Z_0 \arrow[r, "i_Z"] & Z & Z \setminus Z_0. \arrow[l, "j_Z"'] \end{tikzcd}$$

Then we obtain a morphism between exact sequences ([Tho87])

$$\begin{tikzcd} \cdots \arrow[r] & K'_j(Y_0, G)_{\mathbb{Q}}^{\mu_n} \arrow[r, "i_{Y*}"] & K'_j(Y, G)_{\mathbb{Q}}^{\mu_n} \arrow[r, "j_Y^*"] & K'_j(Y \setminus Y_0, G)_{\mathbb{Q}}^{\mu_n} \arrow[r] & \cdots \\ \cdots \arrow[r] & K'_j(Z_0, G)_{\mathbb{Q}} \arrow[r, "i_{Z*}"] \arrow[u] & K'_j(Z, G)_{\mathbb{Q}} \arrow[r, "j_Z^*"] \arrow[u] & K'_j(Z \setminus Z_0, G)_{\mathbb{Q}} \arrow[r] \arrow[u] & \cdots . \end{tikzcd}$$

The right vertical homomorphism is an isomorphism since $Y \setminus Y_0$ is Galois etale over $Z \setminus Z_0$ with Galois group μ_n, and so is the left one as is seen by noting that μ_n acts on $(Y_0)_{\mathrm{red}} \cong (Z_0)_{\mathrm{red}}$ trivially, and using the next lemma:

Lemma 5.18. *Let G be a finite group and W a Noetherian scheme with an action of G, and set $W[\epsilon]/(\epsilon^m) = W \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Z}[\epsilon]/(\epsilon^m)$ and define $f : W[\epsilon]/(\epsilon^m) \to W$ to be the natural map . Then the composite*

$$K'_j(W, G) \xrightarrow{f^*} K'_j(W[\epsilon]/(\epsilon^m), G) \xrightarrow{f_*} K'_j(W, G)$$

is the multiplication by m, and f_* is an isomorphism for $j \geq 0$. In particular, f^* is an isomorphism up to torsion.

Thus we deduce that the middle vertical homomorphism is also an isomorphism. Since the composites

$$K'_j(Z,G)_{\mathbb{Q}} \to K'_j(Y,G)_{\mathbb{Q}}^{\mu_n} \cong (K'_j(Y,G)_{\mathbb{Q}})_{\mu_n} \to K'_j(Z,G)_{\mathbb{Q}}$$

and

$$K'_j(Z_{\mathrm{red}},G)_{\mathbb{Q}} \to K'_j(Y_{\mathrm{red}},G)_{\mathbb{Q}}^{\mu_n} \cong (K'_j(Y_{\mathrm{red}},G)_{\mathbb{Q}})_{\mu_n} \to K'_j(Z_{\mathrm{red}},G)_{\mathbb{Q}}$$

are isomorphisms by assumptions, Proposition easily follows. □

Corollary 5.19. *Let F and X be as in Convention 3.13. Then, for a prime ideal $\mathfrak{q}$ of F, we have a weak equivalence*

$$\overline{K}(X^{\mathfrak{q}}_{\mathrm{Ket}})_{\mathbb{Q}} \xrightarrow{\cong} \mathbb{H}^{\cdot}(X^{\mathfrak{q}^\vee}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}^\vee}_{X*}(\overline{K}_{\mathrm{Ket}})_{\mathbb{Q}}).$$

Proof. It follows from Theorems 5.10 and 5.13. □

5.4 Explicit description

In this subsection we prove Theorem 3.19.

Definition 26. *Let F be a finitely generated free monoid, $\mathfrak{q}$ and $\mathfrak{r}$ its prime ideals, i a natural number, and Y an F-framed Noetherian equi-characteristic fs log scheme.*
Then we define a monoid morphism $L : F(\mathfrak{q}) \to \mathrm{End}_{\mathbf{Ab}}(\overline{K}_i(Y^{\mathfrak{r}}_{\mathrm{Ket}}))$ by associating with each $\alpha \in F(\mathfrak{q})$ the group homomorphism

$$L(\alpha) : \overline{K}_i(Y^{\mathfrak{r}}_{\mathrm{Ket}}) \longrightarrow \overline{K}_i(Y^{\mathfrak{r}}_{\mathrm{Ket}})$$

induced by the endofunctor "$- \otimes \overline{\mathscr{O}}_Y(\alpha)$" of $\overline{\mathbf{Vect}}(Y^{\mathfrak{r}}_{\mathrm{Ket}})$. Now we set

$$\overline{K}_i(Y^{\mathfrak{r}}_{\mathrm{Ket}}) \rtimes \Lambda[\mathfrak{q}] = \overline{K}_i(Y^{\mathfrak{r}}_{\mathrm{Ket}}) \rtimes^L \Lambda[\mathfrak{q}]$$

(See Definition in Subsection 3.3). Similarly, we can also define $\overline{K}_i(Y^{\mathfrak{r}}_{\mathrm{Ket}}) \rtimes \Lambda'[\mathfrak{q}]$ and $\overline{K}(Y^{\mathfrak{r}}_{\mathrm{Ket}}) \rtimes \Lambda^{(')}[\mathfrak{q}]$.

Let $\mathfrak{r}' \supset \mathfrak{r}$ be prime ideals such that $\mathfrak{q} \wedge \mathfrak{r} = \emptyset$. Then we have a natural map $\overline{\mathrm{Pic}}(X^{\mathfrak{q}^\vee}_{\mathrm{Ket}}) \to \overline{\mathrm{Pic}}(X^{\mathfrak{r}}_{\mathrm{Ket}})$. Thus we can construct a homomorphism $\overline{K}_i(Y^{\mathfrak{r}'}_{\mathrm{Ket}}) \rtimes \Lambda[\mathfrak{q}] \to \overline{K}_i(Y^{\mathfrak{r}}_{\mathrm{Ket}})$, as in Subsection 3.3 (See the argument before Theorem 3.19).

Proposition 5.20. *Let F be a finitely generated free monoid and (Y,θ) an F-framed fs log scheme such that $|Y|$ is Noetherian, local and equi-characteristic of characteristic exponent p. We denote by y its closed point. We assume that two primes $\mathfrak{r}$ and $\mathfrak{q}$ of F satisfy the following conditions:*

1. *$\mathfrak{r} \wedge \mathfrak{q} = \emptyset$.*
2. *For any element $m \in \theta_{\bar{y}}(\mathfrak{q}) \subset \overline{M}_{Y,\bar{y}}$, all (any) of its liftings $\tilde{m} \in M_{Y,\bar{y}}$ are mapped to zero by the log-structure map $\alpha_Y : M_Y \to \mathscr{O}_Y$.*

Then the natural map

$$\overline{K}_i(Y^{\mathfrak{r}\vee\mathfrak{q}}_{\mathrm{Ket}}) \rtimes \Lambda'[\mathfrak{q}] \longrightarrow \overline{K}_i(Y^{\mathfrak{r}}_{\mathrm{Ket}})$$

is an isomorphism for $i \geq 0$.

Proof. Take and fix a chart $Y \to \operatorname{Spec}\mathbb{Z}[F]$ lifting the given frame (Proposition 3.7).
By Proposition 4.21 and Lemma 4.23, it suffices to show that, for any natural number n prime to p and any integer $i \geq 0$, the morphism below is an isomorphism

$$K_i(\widetilde{|Y^{\mathfrak{r}\vee\mathfrak{q}}_n}|_{\mathrm{red}}, \widetilde{I^{\mathfrak{r}\vee\mathfrak{q}}_n}) \otimes \mathbb{Z}[F(\mathfrak{q})^{1/n}/F(\mathfrak{q})] \to K_i(\widetilde{|Y^{\mathfrak{r}}_n}|_{\mathrm{red}}, \widetilde{I^{\mathfrak{r}}_n})$$

of (Zariski) equivariant K-groups (For the notation, see Definition before Lemma 5.14).
Here this morphism maps $[\mathscr{F}] \otimes [\alpha]$ to $[\pi^*\mathscr{F} \otimes \mathscr{O}(\alpha)]$, where π is the natural morphism, and $\mathscr{O}(\alpha)$ is the pull-back of an equivariant line bundle on the scheme $\operatorname{Spec}\mathbb{Z}[\mu_n][F(\mathfrak{q})^{1/n}]$ with the action of $\operatorname{Hom}(F(\mathfrak{q})^{1/n}/F(\mathfrak{q}), \mathbb{Z}[\mu_n]^\times) \rtimes \Gamma_n$ corresponding to α (See Definition after Proposition 4.22).
Note that

$$\operatorname{Hom}(F(\mathfrak{q})^{1/n}/F(\mathfrak{q}), \mathbb{Z}[\mu_n]^\times) \times I^{\mathfrak{r}\vee\mathfrak{q}}_n \cong I^{\mathfrak{r}}_n$$

by the assumption (1).
In addition, by the assumption (2), $\widetilde{|Y^{\mathfrak{r}\vee\mathfrak{q}}_n}|_{\mathrm{red}} \cong \widetilde{|Y^{\mathfrak{r}}_n}|_{\mathrm{red}}$ and this has a trivial action of $\operatorname{Hom}(F(\mathfrak{q})^{1/n}/F(\mathfrak{q}), \mathbb{Z}[\mu_n]^\times)$, so the proposition can be deduced easily from the argument in equivariant module theory. □

Corollary 5.21. *Let F and Y be as in Proposition 5.20. Then for a prime $\mathfrak{q}$ of F satisfying the assumption (2), we have a weak equivalence*

$$\overline{K}((Y_{\mathrm{red}})_{\mathrm{Zar}}) \rtimes \Lambda'[\mathfrak{q}] \longrightarrow \overline{K}(Y^{\mathfrak{q}^\vee}_{\mathrm{Ket}}).$$

Taking Corollary 5.12 into consideration, we obtain the following corollary:

Corollary 5.22. *Let F and X be as in Convention 3.13, and $\mathfrak{q}$ a prime ideal of F. If, for any $m \in \mathfrak{q}$, $\theta(m) \in \Gamma(X, \overline{M}_X)$ is pseudo-zero, then we have a weak equivalence*

$$\overline{K}((X_{\mathrm{red}})_{\mathrm{Zar}}) \rtimes \Lambda'[\mathfrak{q}] \longrightarrow \overline{K}(X^{\mathfrak{q}^\vee}_{\mathrm{Ket}}).$$

Since $X[\mathfrak{p}]$ clearly satisfies the condition in Corollary 5.22, we obtain Theorem 3.19.

5.5 Compatibility

Here we prove Theorem 3.22. First we prove the following compatibilities.

Proposition 5.23. *Let F and X be as in Convention 3.13. Then*

1. *For prime ideals* $\mathfrak{q}$ *and* $\mathfrak{r}$ *of* F*, the diagram*

$$\begin{array}{ccc} \overline{K}_i(X^{\mathfrak{q}}_{\mathrm{Ket}}) & \xrightarrow{\pi_i(\delta)} & \overline{K}_i(X^{\mathfrak{q}\vee\mathfrak{r}}_{\mathrm{Ket}}) \\ {\scriptstyle \bar{\varepsilon}^*}\downarrow & & \downarrow{\scriptstyle \bar{\varepsilon}^*} \\ \overline{K}_i(X^{\mathfrak{q}\wedge\mathfrak{r}}_{\mathrm{Ket}}) & \xrightarrow{\pi_i(\delta)} & \overline{K}_i(X^{\mathfrak{r}}_{\mathrm{Ket}}) \end{array}$$

is commutative.

2. *The morphism* δ *is compatible with the* $\overline{K}_0$*-module structure. More precisely, for prime ideals* $\mathfrak{q} \subset \mathfrak{r}$ *of* F*, we have an equality*

$$\pi_i(\delta^{\mathfrak{r},\mathfrak{q}}_X)(x \cdot y) = \pi_i(\delta^{\mathfrak{r},\mathfrak{q}}_X)(x) \cdot \pi_0(\delta^{\mathfrak{r},\mathfrak{q}}_X)(y)$$

as elements of $\overline{K}_i(X^{\mathfrak{r}}_{\mathrm{Ket}})_{\mathbb{Q}}$*, for* $x \in \overline{K}_i(X^{\mathfrak{q}}_{\mathrm{Ket}})_{\mathbb{Q}}$ *and* $y \in \overline{K}_0(X^{\mathfrak{q}}_{\mathrm{Ket}})_{\mathbb{Q}}$*.*

Proof. (1) We set $\mathfrak{s} = \mathfrak{q} \wedge \mathfrak{r}$, $\mathfrak{t} = \mathfrak{q} \vee \mathfrak{r}$, and $\mathfrak{u} = \mathfrak{q} \vee (\mathfrak{r}^{\vee})(= \mathfrak{q} \vee (\mathfrak{t}^{\vee}))$. Then we have $(X^{\mathfrak{q}})^{(\mathfrak{t}\setminus\mathfrak{q})^{\vee}} = X^{\mathfrak{u}}$. Here recall that $X^{\mathfrak{q}}$ is $F \setminus \mathfrak{q}$-framed, and note that $\mathfrak{t} \setminus \mathfrak{q}$ is a prime ideal of $F \setminus \mathfrak{q}$ (Note also that $(\mathfrak{t} \setminus \mathfrak{q})^{\vee}$ denotes the dual in $F \setminus \mathfrak{q}$, that is, $\mathfrak{t}^{\vee} \setminus \mathfrak{q}$). Similarly we have $(X^{\mathfrak{s}})^{(\mathfrak{r}\setminus\mathfrak{s})^{\vee}} = X^{\mathfrak{u}}$.
Therefore, by using the morphisms of sites

$$\varepsilon^{(\mathfrak{t}\setminus\mathfrak{q})^{\vee}}_{X^{\mathfrak{q}}} : X^{\mathfrak{q}} \to X^{\mathfrak{u}} \text{ and } \varepsilon^{(\mathfrak{r}\setminus\mathfrak{s})^{\vee}}_{X^{\mathfrak{s}}} : X^{\mathfrak{s}} \to X^{\mathfrak{u}},$$

we can define presheaves of fibrant spectra $\varepsilon^{(\mathfrak{t}\setminus\mathfrak{q})^{\vee}}_{*}\overline{K}_{\mathrm{Ket}}$ and $\varepsilon^{(\mathfrak{r}\setminus\mathfrak{s})^{\vee}}_{*}\overline{K}_{\mathrm{Ket}}$ on $\mathbf{Ket}/X^{\mathfrak{u}}$. More concretely, we have

$$\varepsilon^{(\mathfrak{t}\setminus\mathfrak{q})^{\vee}}_{*}\overline{K}_{\mathrm{Ket}}(U) = \overline{K}((U \times^{\mathrm{fs}}_{(X^{\mathfrak{q}})^{(\mathfrak{t}\setminus\mathfrak{q})^{\vee}}} X^{\mathfrak{q}})_{\mathrm{Ket}}) = \overline{K}((U \times^{\mathrm{fs}}_{X^{\mathfrak{u}}} X^{\mathfrak{q}})_{\mathrm{Ket}})$$

for $U \in \mathbf{Ket}/(X^{\mathfrak{q}})^{(\mathfrak{t}\setminus\mathfrak{q})^{\vee}}$ and

$$\varepsilon^{(\mathfrak{r}\setminus\mathfrak{s})^{\vee}}_{*}\overline{K}_{\mathrm{Ket}}(U) = \overline{K}((U \times^{\mathrm{fs}}_{(X^{\mathfrak{s}})^{(\mathfrak{r}\setminus\mathfrak{s})^{\vee}}} X^{\mathfrak{s}})_{\mathrm{Ket}}) = \overline{K}((U \times^{\mathrm{fs}}_{X^{\mathfrak{u}}} X^{\mathfrak{s}})_{\mathrm{Ket}})$$

for $U \in \mathbf{Ket}/(X^{\mathfrak{s}})^{(\mathfrak{r}\setminus\mathfrak{s})^{\vee}}$.
Then we have a morphism of presheaves of spectra

$$\varepsilon^{(\mathfrak{t}\setminus\mathfrak{q})^{\vee}}_{*}\overline{K}_{\mathrm{Ket}} \to \varepsilon^{(\mathfrak{r}\setminus\mathfrak{s})^{\vee}}_{*}\overline{K}_{\mathrm{Ket}},$$

which induces a commutative diagram

$$\begin{array}{ccccc} \overline{K}(X^{\mathfrak{q}}_{\mathrm{Ket}}) & \longrightarrow & \mathbb{H}^{\cdot}((X^{\mathfrak{q}})^{(\mathfrak{t}\setminus\mathfrak{q})^{\vee}}_{\mathrm{Ket}}, \varepsilon^{(\mathfrak{t}\setminus\mathfrak{q})^{\vee}}_{*}(\overline{K}_{\mathrm{Ket}})_{\mathbb{Q}}) & \longleftarrow & \overline{K}((X^{\mathfrak{q}})^{(\mathfrak{t}\setminus\mathfrak{q})}_{\mathrm{Ket}})_{\mathbb{Q}} \\ {\scriptstyle \bar{\varepsilon}^*}\downarrow & & {\scriptstyle \bar{\varepsilon}^*}\downarrow & & {\scriptstyle \bar{\varepsilon}^*}\downarrow \\ \overline{K}(X^{\mathfrak{s}}_{\mathrm{Ket}}) & \longrightarrow & \mathbb{H}^{\cdot}((X^{\mathfrak{s}})^{(\mathfrak{r}\setminus\mathfrak{s})^{\vee}}_{\mathrm{Ket}}, \varepsilon^{(\mathfrak{r}\setminus\mathfrak{s})^{\vee}}_{*}(\overline{K}_{\mathrm{Ket}})_{\mathbb{Q}}) & \longleftarrow & \overline{K}((X^{\mathfrak{s}})^{(\mathfrak{r}\setminus\mathfrak{s})}_{\mathrm{Ket}})_{\mathbb{Q}}. \end{array}$$

The proposition now follows.
(2) can be proven in the similar way to Proposition 3.16. ∎

Now, according to Proposition 3.15, Theorem 3.22 is reduced to the next proposition:

Proposition 5.24. *Let F and X be as in Convention 3.13, and $\mathfrak{m}$ its maximal ideal. Take an element e of the basis of F and set $\mathfrak{n} = F \setminus e^{\mathbb{N}}$. Then, by the composite of the natural maps*

$$(e^{\mathbb{N}})_{\mathbb{Q}'} \to \mathrm{Pic}\,(X^{\mathfrak{n}}_{\mathrm{Ket}}) \to \overline{K}_0(X^{\mathfrak{n}}_{\mathrm{Ket}})_{\mathbb{Q}} \xrightarrow{\pi_0(\delta_X^{\mathfrak{m},\mathfrak{n}})} \overline{K}_0(X^{\mathfrak{m}}_{\mathrm{Ket}})_{\mathbb{Q}} \cong K_0((X_{\mathrm{red}})_{\mathrm{Zar}})_{\mathbb{Q}},$$

e^α ($\alpha \in \mathbb{Q}'$) is mapped to $\exp(\alpha \log[\mathscr{O}_X(e)])$, where $[\mathscr{O}_X(e)] \in K_0((X_{\mathrm{red}})_{\mathrm{Zar}})$ is the image of e by the composite

$$e^{\mathbb{N}} \subset F \to \Gamma(X_{\mathrm{red}}, \overline{M}_{X_{\mathrm{red}}}) \to \mathrm{Pic}\,((X_{\mathrm{red}})_{\mathrm{Zar}}) \to K_0((X_{\mathrm{red}})_{\mathrm{Zar}}).$$

Here exp *and* log *are defined by formal power series.*

Proof. We may assume that X is connected. Denote by a the image of $e^{1/n}$ in $K_0((X_{\mathrm{red}})_{\mathrm{Zar}})_{\mathbb{Q}}$. Since δ is compatible with the product structure, a^n must be equal to $[\mathscr{O}_X(e)]$. Now we can easily deduce the proposition, since the n-th power map on $K_0((X_{\mathrm{red}})_{\mathrm{Zar}})_{\mathbb{Q}}$ is an isomorphism, which follows from the fact that the map $\log : K_0((X_{\mathrm{red}})_{\mathrm{Zar}})_{\mathbb{Q}} \to K_0((X_{\mathrm{red}})_{\mathrm{Zar}})_{\mathbb{Q}}$ is an isomorphism. □

5.6 The end of the proof

The rest of this section is devoted to the proof of Theorem 3.17.

Theorem 5.25. *Let F and X be as in Convention 3.13, $\mathfrak{p}$ and $\mathfrak{q}$ prime ideals of F, e an element of the basis of F, and $\langle e\rangle$ the prime ideal generated by e. We set $\mathfrak{p}' = \mathfrak{p} \vee \langle e\rangle$ and $\mathfrak{q}' = \mathfrak{q} \vee \langle e\rangle$.*
Then the natural commutative diagram

$$\begin{tikzcd}
\mathbb{H}^{\cdot}(X[\mathfrak{p}]^{\mathfrak{q}'}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}'}_{X[\mathfrak{p}]*}(\overline{K}_{\mathrm{Ket}})_{\mathbb{Q}}) \arrow[r] \arrow[d] & \mathbb{H}^{\cdot}(X[\mathfrak{p}]^{\mathfrak{q}}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}}_{X[\mathfrak{p}]*}(\overline{K}_{\mathrm{Ket}})_{\mathbb{Q}}) \arrow[d] \\
\mathbb{H}^{\cdot}(X[\mathfrak{p}']^{\mathfrak{q}'}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}'}_{X[\mathfrak{p}']*}(\overline{K}_{\mathrm{Ket}})_{\mathbb{Q}}) \arrow[r] & \mathbb{H}^{\cdot}(X[\mathfrak{p}']^{\mathfrak{q}}_{\mathrm{Ket}}, \varepsilon^{\mathfrak{q}}_{X[\mathfrak{p}']*}(\overline{K}_{\mathrm{Ket}})_{\mathbb{Q}})
\end{tikzcd}$$

is homotopy Cartesian.

Proof. By Corollary 5.9 and Proposition 4.25, we may assume that X is local. By using Corollary 5.19 and noting that $\pi_q(\delta)$ is surjective, we have only to show that the diagram

$$\begin{tikzcd}
\overline{K}_q(X[\mathfrak{p}]^{\mathfrak{q}^\vee \wedge \langle e\rangle^\vee}_{\mathrm{Ket}})_{\mathbb{Q}} \arrow[r, "\pi_q(\delta)"] \arrow[d, "i^*"] & \overline{K}_q(X[\mathfrak{p}]^{\mathfrak{q}^\vee}_{\mathrm{Ket}})_{\mathbb{Q}} \arrow[d, "i^*"] \\
\overline{K}_q(X[\mathfrak{p} \vee \langle e\rangle]^{\mathfrak{q}^\vee \wedge \langle e\rangle^\vee}_{\mathrm{Ket}})_{\mathbb{Q}} \arrow[r, "\pi_q(\delta)"] & \overline{K}_q(X[\mathfrak{p} \vee \langle e\rangle]^{\mathfrak{q}^\vee}_{\mathrm{Ket}})_{\mathbb{Q}}
\end{tikzcd}$$

is bi-Cartesian (i.e. Cartesian and co-Cartesian), where i is the natural inclusion.

Next we consider the diagram

$$\begin{tikzcd} \overline{K}_q(X[\mathfrak{p}]^{\mathfrak{q}^\vee}_{\mathrm{Ket}})_{\mathbb{Q}} \arrow[r, "\bar{\varepsilon}^*"] \arrow[d, "i^*"] & \overline{K}_q(X[\mathfrak{p}]^{\mathfrak{q}^\vee \wedge \langle e\rangle^\vee}_{\mathrm{Ket}})_{\mathbb{Q}} \arrow[r, "\pi_q(\delta)"] \arrow[d, "i^*"] & \overline{K}_q(X[\mathfrak{p}]^{\mathfrak{q}^\vee}_{\mathrm{Ket}})_{\mathbb{Q}} \arrow[d, "i^*"] \\ \overline{K}_q(X[\mathfrak{p} \vee \langle e\rangle]^{\mathfrak{q}^\vee}_{\mathrm{Ket}})_{\mathbb{Q}} \arrow[r, "\bar{\varepsilon}^*"] & \overline{K}_q(X[\mathfrak{p} \vee \langle e\rangle]^{\mathfrak{q}^\vee \wedge \langle e\rangle^\vee}_{\mathrm{Ket}})_{\mathbb{Q}} \arrow[r, "\pi_q(\delta)"] & \overline{K}_q(X[\mathfrak{p} \vee \langle e\rangle]^{\mathfrak{q}^\vee}_{\mathrm{Ket}})_{\mathbb{Q}}. \end{tikzcd}$$

Since the outer rectangle is clearly bi-Cartesian (Proposition 3.15), it suffices to prove that so is the left square. Now we can reduce the theorem to the next lemma:

Lemma 5.26. *Suppose that F and X be as in Convention 3.13, and in addition that X be local. Let $\mathfrak{r} \subset F$ be a prime and $e \in F$ an element of the basis of F. Then the diagram*

$$\begin{tikzcd} \overline{K}_q(X^{\mathfrak{r}\vee\langle e\rangle}_{\mathrm{Ket}}) \arrow[r, "i^*"] \arrow[d, "\bar{\varepsilon}^*"'] & \overline{K}_q(X[\langle e\rangle]^{\mathfrak{r}\vee\langle e\rangle}_{\mathrm{Ket}}) \arrow[d, "\bar{\varepsilon}^*"'] \\ \overline{K}_q(X^{\mathfrak{r}}_{\mathrm{Ket}}) \arrow[r, "i^*"] & \overline{K}_q(X[\langle e\rangle]^{\mathfrak{r}}_{\mathrm{Ket}}) \end{tikzcd}$$

is bi-Cartesian for every $q \geq 0$.

Proposition 4.21 and the next proposition imply this lemma, thus the theorem is proven. $\square$

Proposition 5.27. *Let n be a positive integer, $R = \mathbb{Z}[1/n, \mu_n]$, G a finite group acted by $\Gamma_n = \mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})$, and Z a Noetherian equi-characteristic semi-local scheme of characteristic exponent p with the action of a finite group $G \rtimes \Gamma_n$.*

Suppose that p is prime to n, and that we are given a $G \rtimes \Gamma_n$-equivariant morphism $Z \to \mathbb{A}^1_R = \operatorname{Spec} R[e]$ with $G \rtimes \Gamma_n$ naturally acting on $\mathbb{A}^1_R$ through Γ_n.

We let the left square below be the fiber product of the right square with respect to $Z \to \mathbb{A}^1_R$:

$$\begin{tikzcd} W_n \arrow[r, "i'"] \arrow[d, "\pi'"'] & Z_n \arrow[d, "\pi"] \\ W \arrow[r, "i"] & Z \end{tikzcd} \qquad \begin{tikzcd} \{0\} \times_{\mathbb{A}^1_R} (\mathbb{A}^1_R)_n \arrow[r] \arrow[d] & (\mathbb{A}^1_R)_n \arrow[d] \\ \{0\} \arrow[r] & \mathbb{A}^1_R, \end{tikzcd}$$

where $(\mathbb{A}^1_R)_n = \operatorname{Spec} R[e^{1/n}]$ and $\{0\}$ is the closed subscheme consisting of the origin. Note that μ_n acts naturally on Z_n and W_n.

Suppose that

1. *Z_{red}, W_{red} and $(Z_n)_{\mathrm{red}}$ are regular (Note that $W_{\mathrm{red}} \hookrightarrow Z_{\mathrm{red}}$ and $(W_n)_{\mathrm{red}} \hookrightarrow (Z_n)_{\mathrm{red}}$ become regular immersions automatically),*

2. *$\pi_{\mathrm{red}} : (Z_n)_{\mathrm{red}} \to Z_{\mathrm{red}}$ is flat, and*

3. *the natural map $W_{\mathrm{red}} \to W \times_Z Z_{\mathrm{red}}$ is an isomorphism.*

Then the diagram consisting of (Zariski) equivariant K-groups

$$\begin{tikzcd} K_q(Z_{\mathrm{red}}, G \rtimes \Gamma_n) \arrow[r, "i^*"] \arrow[d, "\pi^*"'] & K_q(W_{\mathrm{red}}, G \rtimes \Gamma_n) \arrow[d, "\pi'^*"'] \\ K_q((Z_n)_{\mathrm{red}}, (G \times \mu_n) \rtimes \Gamma_n) \arrow[r, "i'^*"] & K_q((W_n)_{\mathrm{red}}, (G \times \mu_n) \rtimes \Gamma_n) \end{tikzcd}$$

is bi-Cartesian for every $q \geq 0$.

Proof. Since $(Z_n \setminus W_n)_{\mathrm{red}}$ is Galois etale over $(Z \setminus W)_{\mathrm{red}}$ with Galois group μ_n, $K'_r((Z \setminus W)_{\mathrm{red}}, G \rtimes \Gamma_n) \xrightarrow{\pi^*} K'_r((Z_n \setminus W_n)_{\mathrm{red}}, (G \times \mu_n) \rtimes \Gamma_n)$ is an isomorphism for $r = q, q+1$, so we see that the diagram

$$\begin{tikzcd} K'_q(W_{\mathrm{red}}, G \rtimes \Gamma_n) \arrow[r, "i_*"] \arrow[d, "\mathrm{pr}_1^*"'] & K'_q(Z_{\mathrm{red}}, G \rtimes \Gamma_n) \arrow[d, "\pi^*"'] \\ K'_q(V, (G \times \mu_n) \rtimes \Gamma_n) \arrow[r, "j_*"] & K'_q((Z_n)_{\mathrm{red}}, (G \times \mu_n) \rtimes \Gamma_n) \end{tikzcd}$$

is bi-Cartesian by Thomason's localisation sequence ([Tho87]), where $V = W_{\mathrm{red}} \times_{Z_{\mathrm{red}}} (Z_n)_{\mathrm{red}}$ and j is the natural closed immersion.
Next, by considering the composite of morphisms

$$\begin{aligned} K'_q(W_{\mathrm{red}}, G \rtimes \Gamma_n) &\xrightarrow{i_*} K'_q(Z_{\mathrm{red}}, G \rtimes \Gamma_n) \\ &\xrightarrow{(\mathrm{canonical})^{-1}} K_q(Z_{\mathrm{red}}, G \rtimes \Gamma_n) \\ &\xrightarrow{i^*} K_q(W_{\mathrm{red}}, G \rtimes \Gamma_n) \end{aligned}$$

and its counterpart for $(Z_n)_{\mathrm{red}}$, we have a commutative diagram

$$\begin{tikzcd} K'_q(W_{\mathrm{red}}, G \rtimes \Gamma_n) \arrow[r, "i^* \circ (\mathrm{can})^{-1} \circ i_*"] \arrow[d, "\mathrm{pr}_1^*"'] & K_q(W_{\mathrm{red}}, G \rtimes \Gamma_n) \arrow[d, "\pi^*"] \\ K'_q(V, (G \times \mu_n) \rtimes \Gamma_n) \arrow[r, "i^* \circ (\mathrm{can})^{-1} \circ j_*"] & K_q((W_n)_{\mathrm{red}}, (G \times \mu_n) \rtimes \Gamma_n). \end{tikzcd}$$

It suffices to show that this diagram is bi-Cartesian.

Now, since $W_{\mathrm{red}} \cong (W_n)_{\mathrm{red}}$ and the action of μ_n on $(W_n)_{\mathrm{red}}$ is trivial, we have isomorphisms

$$\begin{aligned} K'_q(W_{\mathrm{red}}, G \rtimes \Gamma_n) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] &\xrightarrow{\cong} K'_q((W_n)_{\mathrm{red}}, (G \times \mu_n) \rtimes \Gamma_n) \\ &\xrightarrow{\cong} K'_q(V, (G \times \mu_n) \rtimes \Gamma_n), \text{ and} \\ K_q(W_{\mathrm{red}}, G \rtimes \Gamma_n) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] &\xrightarrow{\cong} K_q((W_n)_{\mathrm{red}}, (G \times \mu_n) \rtimes \Gamma_n). \end{aligned}$$

Here the first isomorphism and the third are defined by the rule $x \otimes [i] \mapsto \pi'^* x \cdot [\mathscr{O}(e^{i/n})]$, where $\mathscr{O}(e^{i/n})$ is the pull-back of $\mu_n \rtimes \Gamma_n$-equivariant line bundle on $(\mathbb{A}^1_R)_n$ corresponding to $e^{i/n}$ (See Definition after Proposition 4.22), with respect to $(W_n)_{\mathrm{red}} \to \operatorname{Spec} R[e^{1/n}]/(e^{1/n}) \hookrightarrow \operatorname{Spec} R[e^{1/n}]$.
In addition, noting that V (and W_{red}) is defined by the equation "$e = 0$" in $(Z_n)_{\mathrm{red}}$ (and Z_{red}, respectively), we easily see that the above diagram can be rewritten as below, via these isomorphisms (The proofs of Proposition 1.6 and Lemma 1.7 in [Vis91] work also for our situation):

$$\begin{tikzcd} K'_q(W_{\mathrm{red}}, G \rtimes \Gamma_n) \arrow[r, "0"] \arrow[d] & K_q(W_{\mathrm{red}}, G \rtimes \Gamma_n) \arrow[d] \\ K'_q(W_{\mathrm{red}}, G \rtimes \Gamma_n) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \arrow[r, "\mathrm{can}\otimes f"] & K_q(W_{\mathrm{red}}, G \rtimes \Gamma_n) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}], \end{tikzcd}$$

where, the right vertical morphism maps x to $x \otimes [0]$, the left vertical x to $x \otimes \sum_{i=0}^{n-1} [i]$, and f is the multiplication by $[0] - [1]$. Since this diagram is clearly bi-Cartesian, Proposition follows. □

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