Cup Products, the Heisenberg Group, and Codimension Two Algebraic Cycles

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Abstract. We define higher categorical invariants (gerbes) of codimension two algebraic cycles and provide a categorical interpretation of the intersection of divisors on a smooth algebraic variety. This generalization of the classical relation between divisors and line bundles furnishes a new perspective on the Bloch-Quillen formula.

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Des buissons lumineux fusaient comme des gerbes;
Mille insectes, tels des prismes, vibraient dans l'air;
–Emile Verhaeren (1855-1916),
Le paradis (Les rythmes souverains)

1 Introduction

This aim of this paper is to define higher categorical invariants (gerbes) of codimension two algebraic cycles and provide a categorical interpretation of the intersection of divisors on a smooth proper algebraic variety. This generalization of the classical relation between divisors and line bundles sheds some light on the geometric significance of the classical Bloch-Quillen formula (5.1.3) relating Chow groups and algebraic K-theory.
Our work is motivated by the following three basic questions.

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i. Let A and B be sheaves of abelian groups on a manifold (or algebraic variety) X. Given $\alpha \in H^1(X, A)$ and $\beta \in H^1(X, B)$, one has their cup-product $\alpha \cup \beta \in H^2(X, A \otimes B)$. We recall that H^1 and H^2 classify equivalence classes of torsors and gerbes[2]

$$\begin{aligned} H^1(X, A) &\quad\longleftrightarrow\quad \text{Isomorphism classes of } A\text{-torsors} \\ H^2(X, A) &\quad\longleftrightarrow\quad \text{Isomorphism classes of } A\text{-gerbes;} \end{aligned}$$

we may pick torsors P and Q representing α and β and ask

Question 1.1. *Given P and Q, is there a natural construction of a gerbe $G_{P,Q}$ which manifests the cohomology class $\alpha \cup \beta = [P] \cup [Q]$?*

The above question admits the following algebraic-geometric analogue.

ii. Let X be a smooth proper[3] variety over a field F. Let $Z^i(X)$ be the abelian group of algebraic cycles of codimension i on X and let $CH^i(X)$ be the Chow group of algebraic cycles of codimension i modulo rational equivalence. The isomorphism

$$CH^1(X) \xrightarrow{\sim} H^1(X, \mathcal{O}^*)$$

connects (Weil) divisors and invertible sheaves (or $\mathbb{G}_m$-torsors). While divisors form a group, $\mathbb{G}_m$-torsors on X form a Picard category $\mathrm{Tors}_X(\mathbb{G}_m)$ with the monoidal structure provided by the Baer sum of torsors. Any divisor D determines a $\mathbb{G}_m$-torsor L_D -see §5.2; the torsor $L_{D+D'}$ is isomorphic to the Baer sum of L_D and $L_{D'}$. In other words, one has an additive map [23, II, Proposition 6.13]

$$Z^1(X) \to \mathrm{Tors}_X(\mathbb{G}_m) \qquad D \mapsto L_D. \tag{1.0.1}$$

Question 1.2. *What is a natural generalization of (1.0.1) to higher codimension cycles?*

Since $\mathrm{Tors}_X(\mathbb{G}_m)$ is a Picard category, one could expect the putative additive maps on $Z^i(X)$ to land in Picard categories or their generalizations.

Question 1.3. *Is there a categorification of the intersection pairing*

$$CH^1(X) \times CH^1(X) \to CH^2(X)? \tag{1.0.2}$$

More generally, one can ask for a categorical interpretation of the entire Chow ring of X.

[2] For us, the term "gerbe" signifies a stack in groupoids which is locally non-empty and locally connected (§2.1).

[3] Actually, our results which depend on the Gersten resolution (5.1.1) are valid for any separated smooth scheme of finite type over a field.

Main results

Our first result is an affirmative answer to Question 1.1; the key observation is that a certain Heisenberg group *animates* the cup-product.

Theorem 1.4. *Let A, B be sheaves of abelian groups on a topological space or scheme X.*

i. *There is a canonical functorial Heisenberg*[4] *sheaf $H_{A,B}$ on X which sits in an exact sequence*

$$0 \to A \otimes B \to H_{A,B} \to A \times B \to 0;$$

the sheaf $H_{A,B}$ (of non-abelian groups) is a central extension of $A \times B$ by $A \otimes B$.

ii. *The associated boundary map*

$$\partial : H^1(X, A) \times H^1(X, B) = H^1(X, A \times B) \to H^2(X, A \otimes B)$$

sends the class (γ, δ) to the cup-product $\gamma \cup \delta$.

iii. *Given torsors P and Q for A and B, view $P \times Q$ as a $A \times B$-torsor on X. Let $\mathcal{G}_{P,Q}$ be the gerbe of local liftings (see §2.2) of $P \times Q$ to a $H_{A,B}$-torsor; its band is $A \otimes B$ and its class in $H^2(X, A \otimes B)$ is $[P] \cup [Q]$.*

iv. *The gerbe $\mathcal{G}_{P,Q}$ is covariant functorial in A and B and contravariant functorial in X.*

v. *The gerbe $\mathcal{G}_{P,Q}$ is trivial (equivalent to the stack of $A \otimes B$-torsors) if either P or Q is trivial.*

We prove this theorem over a general site C. We also provide a natural interpretation of the (class of the) Heisenberg sheaf in terms of maps of Eilenberg-Mac Lane objects in §3.4.

Here is another rephrasing of Theorem 1.4: For abelian sheaves A and B on a site C, there is a natural bimonoidal functor

$$\mathrm{T{\small ORS}}_{\mathsf{C}}(A) \times \mathrm{T{\small ORS}}_{\mathsf{C}}(B) \longrightarrow \mathrm{G{\small ERBES}}_{\mathsf{C}}(A \otimes B) \qquad (P, Q) \mapsto \mathcal{G}_{P,Q} \qquad (1.0.3)$$

where $\mathrm{T{\small ORS}}_{\mathsf{C}}(A)$, $\mathrm{T{\small ORS}}_{\mathsf{C}}(B)$ are the Picard categories of A and B-torsors on C and $\mathrm{G{\small ERBES}}_{\mathsf{C}}(A \otimes B)$ is the Picard 2-category of $A \otimes B$-gerbes on C. Thus, Theorem 1.4 constitutes a categorification of the cup-product map

$$\cup : H^1(A) \times H^1(B) \to H^2(A \otimes B). \qquad (1.0.4)$$

Let us turn to Questions 1.2 and 1.3. Let C be the Zariski site on the smooth proper variety X.

[4]For $A = B = \mathbb{Z}$ we obtain the usual Heisenberg group over $\mathbb{Z}$.

Suppose that D and D' are divisors on X which intersect in the codimension-two cycle $D.D'$. Applying Theorem 1.4 to $\mathcal{O}_D$ and $\mathcal{O}_{D'}$ with $A = B = \mathbb{G}_m$, one has a $\mathbb{G}_m \otimes \mathbb{G}_m$-gerbe $\mathcal{G}_{D,D'}$ on X. We now invoke the isomorphisms (the second is the fundamental Bloch-Quillen isomorphism)

$$\mathbb{G}_m \xrightarrow{\sim} \mathcal{K}_1, \qquad CH^i(X) \xrightarrow[(5.1.3)]{\sim} H^i(X, \mathcal{K}_i)$$

where $\mathcal{K}_i$ is the Zariski sheaf associated with the presheaf $U \mapsto K_i(U)$. Pushforward of $\mathcal{G}_{D,D'}$ along $\mathcal{K}_1 \times \mathcal{K}_1 \to \mathcal{K}_2$ gives a $\mathcal{K}_2$-gerbe still denoted $\mathcal{G}_{D,D'}$; we call this the Heisenberg gerbe attached to the codimension-two cycle $D.D'$. This raises the possibility of relating $\mathcal{K}_2$-gerbes and codimension-two cycles on X, implicit in (5.1.3). Since a general codimension-two cycle need not be an intersection of divisors (or even rationally equivalent to an intersection of divisors), a new idea is necessary to generalize the above construction of the Heisenberg gerbe to all codimension-two cycles. Our approach proceeds via the Gersten sequence (5.1.1).

Theorem 1.5. *(i) Any codimension-two cycle $\alpha \in Z^2(X)$ determines a $\mathcal{K}_2$-gerbe $\mathcal{C}_\alpha$ on X.*
(ii) the class of $\mathcal{C}_\alpha$ in $H^2(X, \mathcal{K}_2)$ corresponds to $\alpha \in CH^2(X)$ under the Bloch-Quillen map (5.1.3).
(iii) the gerbe $\mathcal{C}_{\alpha+\alpha'}$ is equivalent to the Baer sum of $\mathcal{C}_\alpha$ and $\mathcal{C}_{\alpha'}$.
(iv) $\mathcal{C}_\alpha$ and $\mathcal{C}_{\alpha'}$ are equivalent as $\mathcal{K}_2$-gerbes if and only if $\alpha = \alpha'$ in $CH^2(X)$.

Since $\mathcal{C}_\alpha$ uses the Gersten sequence (5.1.1) in a crucial way, we call $\mathcal{C}_\alpha$ the Gersten gerbe of α; it admits a geometric description, closely analogous to that of the $\mathbb{G}_m$-torsor L_D of a divisor D of §5.2; see Remark 5.6. One has an additive map

$$Z^2(X) \to \mathrm{G\small{ERBES}}_X(\mathcal{K}_2) \qquad \alpha \mapsto \mathcal{C}_\alpha. \tag{1.0.5}$$

When $\alpha = D.D'$ is the intersection of two divisors, there are two $\mathcal{K}_2$-gerbes attached to it: the Heisenberg gerbe $\mathcal{G}_{D,D'}$ and the Gersten gerbe $\mathcal{C}_\alpha$; these are abstractly equivalent as their classes in $H^2(X, \mathcal{K}_2)$ correspond to α. More is possible.

Theorem 1.6. *If $\alpha \in Z^2(X)$ is the intersection $D.D'$ of divisors $D, D' \in Z^1(X)$, then there is a natural equivalence $\Theta : \mathcal{C}_\alpha \to \mathcal{G}_{D,D'}$ between the Gersten and Heisenberg $\mathcal{K}_2$-gerbes attached to $\alpha = D.D'$.*

Thus, Theorems 1.4, 1.5, 1.6 together provide the following commutative diagram thereby answering Question 1.3:

$$\begin{tikzcd}
Z^1(X) \times Z^1(X) \arrow[r, dotted, "\text{no map}"] \arrow[d, "(1.0.1)"] & Z^2(X) \arrow[d, "(1.0.5)"] \\
\mathrm{Tors}_X(\mathbb{G}_m) \times \mathrm{Tors}_X(\mathbb{G}_m) \arrow[r, "(1.0.3)"] \arrow[d] & \mathrm{Gerbes}_X(\mathcal{K}_2) \arrow[d] \\
CH^1(X) \times CH^1(X) \arrow[r, "(1.0.2)"] & CH^2(X).
\end{tikzcd}$$

Since the Heisenberg gerbe $\mathcal{G}_{D,D'}$ arises from a gerbe with band $\mathbb{G}_m \otimes \mathbb{G}_m$, we obtain:

PROPOSITION. *A necessary condition for a codimension-two cycle α on X to be an intersection $\alpha = D.D'$ of divisors D, D' is that the Gersten gerbe $\mathcal{C}_\alpha$ with band $\mathcal{K}_2$ lifts to a $\mathbb{G}_m \otimes \mathbb{G}_m$-gerbe.*

We begin with a review of the basic notions and tools (lifting gerbe, four-term complexes) in §2 and then present the construction and properties of the Heisenberg group in §3 before proving Theorem 1.4. After a quick discussion of various examples in §4, we turn to codimension-two algebraic cycles in §5 and construct the Gersten gerbe $\mathcal{C}_\alpha$ and prove Theorems 1.5, 1.6 using the tools in §2.

DICTIONARY FOR CODIMENSION TWO CYCLES

The above results indicate the viability of viewing $\mathcal{K}_2$-gerbes as natural invariants of codimension-two cycles on X. Additional evidence is given by the following points: [5]

- $\mathcal{K}_2$-gerbes are present (albeit implicitly) in the Bloch-Quillen formula (5.1.3) for $i = 2$.
- The Picard category $\mathfrak{P} = \mathrm{TORS}_X(\mathbb{G}_m)$ of $\mathbb{G}_m$-torsors on X satisfies

$$\pi_1(\mathfrak{P}) := H^0(X, \mathcal{O}^*) = CH^1(X, 1)\,,$$
$$\pi_0(\mathfrak{P}) := H^1(X, \mathcal{O}^*) = CH^1(X)\,.$$

  Similarly, the Picard 2-category $\mathfrak{C} = \mathrm{GERBES}_X(\mathcal{K}_2)$ of $\mathcal{K}_2$-gerbes is closely related to the higher Chow groups $CH^2(X, -)$ of Bloch [4] in codimension two, generalizing the Chow groups $CH^j(X, 0) = CH^j(X)$:

$$\pi_2(\mathfrak{C}) := H^0(X, \mathcal{K}_2) = CH^2(X, 2)\,,$$
$$\pi_1(\mathfrak{C}) := H^1(X, \mathcal{K}_2) = CH^2(X, 1)\,,$$
$$\pi_0(\mathfrak{C}) := H^2(X, \mathcal{K}_2) \overset{(5.1.3)}{=} CH^2(X)\,.$$

- The additive map arising from Theorem 1.5

$$Z^2(X) \to \mathrm{GERBES}_X(\mathcal{K}_2)\,, \quad \alpha \longmapsto \mathcal{C}_\alpha$$

  gives the Bloch-Quillen isomorphism (5.1.3) on the level of π_0. It provides an answer to Question 1.2 for codimension two cycles.
- The Gersten gerbe $\mathcal{C}_\alpha$ admits a simple algebro-geometric description (Remark 5.5): Any α determines a $K_2^\eta/\mathcal{K}_2$-torsor; then $\mathcal{C}_\alpha$ is the gerbe of liftings of this torsor to a K_2^η-torsor on X.

[5] Let $\eta\colon \mathrm{Spec}\, F_X \to X$ be the generic point of X and write K_i^η for the sheaf $\eta_* K_i(F_X)$; one has the map $\mathcal{K}_i \to K_i^\eta$.

- The gerbe $\mathcal{C}_\alpha$ is canonically trivial outside of the support of α (Remark 5.5).
- Pushing the Gersten gerbe $\mathcal{C}_\alpha$ along the map $\mathcal{K}_2 \to \Omega^2$ produces an Ω^2-gerbe which manifests the (de Rham) cycle class of α in $H^2(X, \Omega^2)$.

The map (1.0.1) is a part of the marvellous dictionary [23, II, §6] arising from the divisor sequence (5.2.1):

$$\begin{aligned}\text{Divisors} \longleftrightarrow \text{Cartier divisors} \longleftrightarrow \mathcal{K}_1\text{-torsors}&\\ \longleftrightarrow \text{Line bundles} \longleftrightarrow \text{Invertible sheaves}.&\qquad (1.0.6)\end{aligned}$$

More generally, from the Gersten sequence (5.1.1) we obtain the following:

$$Z^1(X) \xrightarrow{\cong} H^0(X, K_1^\eta/\mathcal{K}_1)(= \text{ Cartier divisors}) \twoheadrightarrow H^1(X, \mathcal{K}_1) \cong CH^1(X)$$
$$Z^2(X) \twoheadrightarrow H^1(X, K_2^\eta/\mathcal{K}_2) \xrightarrow{\cong} H^2(X, \mathcal{K}_2) \cong CH^2(X).$$

Inspired by this and [3, Definition 3.2], we call $K_2^\eta/\mathcal{K}_2$-torsors as *codimension-two Cartier cycles* on X. Thus the analog for codimension two cycles of the dictionary (1.0.6) reads

$$\text{Codimension two cycles} \quad\longleftrightarrow\quad \text{Cartier cycles} \quad\longleftrightarrow\quad \mathcal{K}_2\text{-gerbes}.$$

Since the Gersten sequence (5.1.1) exists for all $\mathcal{K}_i$, it should be possible to generalize Theorem 1.5 to higher codimensions thereby answering Question 1.2; however, this would involve *higher gerbes*. Any cycle of codimension $i > 2$ determines a higher gerbe [7] with band $\mathcal{K}_i$ (an example in the context of Parshin chains is presented below); this provides a new perspective on the Bloch-Quillen formula (5.1.3). The higher dimensional analogues of (1.0.3), (1.0.2), and Theorem 1.5 are still in progress.

Higher gerbes attached to smooth Parshin chains

By Gersten's conjecture, the localization sequence [32, §7 Proposition 3.2] breaks up into short exact sequences

$$0 \longrightarrow K_i(V) \longrightarrow K_i(V - Y) \longrightarrow K_{i-1}(Y) \longrightarrow 0, \qquad (i > 0)$$

for any smooth variety V over F and a closed smooth subvariety Y of V. Let $j : D \to X$ be a smooth closed subvariety of codimension one of X; write $\iota : X - D \to X$ for the open complement of D. Any divisor α of D is a codimension-two cycle on X; one has a map $\mathrm{Pic}(D) \to CH^2(X)$ [4, (iii), p. 269]. This gives the exact sequence (for $i > 0$)

$$0 \longrightarrow \mathcal{K}_i \longrightarrow \mathcal{F}_i \longrightarrow j_*\mathcal{K}_{i-1}^D \longrightarrow 0$$

of sheaves on X where $\mathcal{F}_i = \iota_*\mathcal{K}_i^U$ is the sheaf associated with the presheaf $U \mapsto K_i(U - D)$. We write $\mathcal{K}_i^D$ and $\mathcal{K}_i^U$ for the usual K-theory sheaves on

D and U since the notation $\mathcal{K}_i$ is already reserved for the sheaf on X. The boundary map

$$H^1(D, \mathcal{K}_1^D) = H^1(X, j_*\mathcal{K}_1^D) \longrightarrow H^2(X, \mathcal{K}_2)$$

is the map $CH^1(D) \to CH^2(X)$. For any divisor α of D, the $\mathcal{K}_1^D$-torsor $\mathcal{O}_\alpha$ determines a unique $j_*\mathcal{K}_1^D$-torsor L_α on X. The $\mathcal{K}_2$-gerbe $\mathcal{C}_\alpha$ (viewing α as a codimension two cycle on X) is the lifting gerbe of the $j_*\mathcal{K}_1^D$-torsor L_α (obstructions to lifting to a $\mathcal{F}_2$-torsor).
This generalizes to higher codimensions (and pursued in forthcoming work):

- (codimension three) If β is a codimension-two cycle of D, then the gerbe $\mathcal{C}_\beta$ on D determines a unique gerbe L_β on X (with band $j_*\mathcal{K}_2^D$). The obstructions to lifting L_β to a $\mathcal{F}_3$-gerbe is a 2-gerbe $\mathcal{G}_\beta$ [7] with band $\mathcal{K}_3$ on X. This gives an example of a higher gerbe invariant of a codimension three cycle on X. Gerbes with band $K_3^\eta/\mathcal{K}_3$ provide the *codimension-three analog* of Cartier divisors $H^0(X, K_1^\eta/\mathcal{K}_1)$.

- (Parshin chains) Recall that a chain of subvarieties

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \cdots \hookrightarrow X_n = X$$

  where each X_i is a divisor of X_{i+1} gives rise to a Parshin chain on X. We will call a Parshin chain smooth if all the subvarieties X_i are smooth. Iterating the previous construction provides a higher gerbe on $X_n = X$ with band $\mathcal{K}_j$ attached to X_{n-j} (a codimension j cycle of X_n).

Other than the classical Hartshorne-Serre correspondence between certain codimension-two cycles and certain rank two vector bundles, we are not aware of any generalizations of the dictionary (1.0.6) to higher codimension. In particular, our idea of attaching *a higher-categorical invariant to a higher codimension cycle* seems new in the literature. We expect that Picard n-categories play a role in the functorial Riemann-Roch program of Deligne [15].
Our results are related to and inspired by the beautiful work of S. Bloch [3], L. Breen [8], J.-L. Brylinski [10], P. Gajer [18, 19], A. N. Parshin [30], B. Poonen - E. Rains [31], and D. Ramakrishnan [33] (see §4). Brylinski's hope[6] [10, Introduction] for a higher-categorical geometrical interpretation of the regulator maps from algebraic K-theory to Deligne cohomology was a major catalyst. In a forthcoming paper, we will investigate the relations between the Gersten gerbe and Deligne cohomology.

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[6] "In principle such ideas will lead to a geometric description of all regulator maps, once the categorical aspects have been cleared up. Hopefully this would lead to a better understanding of algebraic K-theory itself."

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Notations and conventions

Let C be a site. We write $\mathsf{C}^\sim$ for the topos of sheaves over C, $\mathsf{C}^\sim_{\mathrm{ab}}$ the abelian group objects of $\mathsf{C}^\sim$, namely the abelian sheaves on C, and by $\mathsf{C}^\sim_{\mathrm{grp}}$ the sheaves of groups on C. Our notation for cohomology is as follows. For an abelian object A of a topos T, $H^i(A)$ denotes the cohomology of the terminal object $e \in \mathsf{T}$ with coefficients in A, namely i^{th} derived functor of $\mathrm{Hom}_{\mathsf{T}}(e, A)$. This is the same as $\mathrm{Ext}^i_{\mathsf{T}_{\mathrm{ab}}}(\mathbb{Z}, A)$. More generally, $H^i(X, A)$ denotes the cohomology of A in the topos T/X. We use $\mathbf{H}$ for hypercohomology.
A variety over a field F is an integral scheme of finite type over F.

2 Preliminaries

2.1 Abelian Gerbes [21, 16, 7]

A gerbe $\mathcal{G}$ over a site C is *a stack in groupoids which is locally non-empty and locally connected*.
$\mathcal{G}$ is locally nonempty if for every object U of C there is a cover, say a local epimorphism, $V \to U$ such that the category $\mathcal{G}(V)$ is nonempty; it is locally connected if given objects $x, y \in \mathcal{G}(U)$ as above, then, locally on U, the sheaf $\mathrm{Hom}(x, y)$ defined above has sections. For each object x over U we can introduce the automorphism sheaf $\mathrm{Aut}_{\mathcal{G}}(x)$, and by local connectedness all these automorphism sheaves are (non canonically) isomorphic.
In the sequel we will only work with *abelian* gerbes, where there is a coherent identification between the automorphism sheaves $\mathrm{Aut}_{\mathcal{G}}(x)$, for any choice of an object x of $\mathcal{G}$, and a fixed sheaf of groups G. In this case G is necessarily abelian[7], and the class of $\mathcal{G}$ determines an element in $H^2(G)$, [7, §2] (and also [26]), where $H^i(G) = \mathrm{Ext}^i_{\mathsf{C}^\sim_{\mathrm{ab}}}(\mathbb{Z}, G)$ denotes the standard cohomology with coefficients in the abelian sheaf G in the topos $\mathsf{C}^\sim$ of sheaves over C.
Let us briefly recall how the class of $\mathcal{G}$ is obtained using a Čech type argument. Assume for simplicity that the site C has pullbacks. Let $\mathcal{U} = \{U_i\}$ be a cover of an object X of C. Let x_i be a choice of an object of $\mathcal{G}(U_i)$. For simplicity, let us assume that we can find morphisms $\alpha_{ij} \colon x_j|_{U_{ij}} \to x_i|_{U_{ij}}$. The class of $\mathcal{G}$ will be represented by the 2-cocycle $\{c_{ijk}\}$ of $\mathcal{U}$ with values in G obtained in the standard way as the deviation for $\{\alpha_{ij}\}$ from satisfying the cocycle condition:

$$\alpha_{ij} \circ \alpha_{jk} = c_{ijk} \circ \alpha_{ik}.$$

In the above identity—which defines it—$c_{ijk} \in \mathrm{Aut}(x_i|_{U_{ijk}}) \cong G|_{U_{ijk}}$. It is obvious that $\{c_{ijk}\}$ is a cocycle.

[7]The automorphisms in $\mathrm{Aut}(G)$ completely decouple, hence play no role.

Returning to stacks for a moment, a stack $\mathcal{G}$ determines an object $\pi_0(\mathcal{G})$, defined as the sheaf associated to the presheaf of connected components of $\mathcal{G}$, where the latter is the presheaf that to each object U of C assigns the set of isomorphism classes of objects of $\mathcal{G}(U)$. By definition, if $\mathcal{G}$ is a gerbe, then $\pi_0(\mathcal{G}) = *$. In general, writing just π_0 in place of $\pi_0(\mathcal{G})$, by base changing to π_0, namely considering the site C/π_0, every stack $\mathcal{G}$ is (tautologically) a gerbe over π_0 [27].

Example 2.1.

i. The trivial gerbe with band G is the stack $\textsc{Tors}(G)$ of G-torsors. Moreover, for any gerbe $\mathcal{G}$, the choice of an object x in $\mathcal{G}(U)$ determines an equivalence of gerbes $\mathcal{G}|_U \cong \textsc{Tors}(G|_U)$, over C/U, where $G = \mathrm{Aut}_{\mathcal{G}}(x)$. There is an equivalence $\textsc{Tors}(G) \cong \mathrm{B}_G$, the topos of (left) G-objects of $\mathsf{C}^{\sim}$ ([21]).

ii. Any line bundle L over an algebraic variety X over $\mathbb{Q}$ determines a gerbe $\mathcal{G}_n$ with band $\boldsymbol{\mu}_n$ (the sheaf of n^{th} roots of unity) for any $n > 1$ as follows: Over any open set U, consider the category of pairs $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is a line bundle on U and $\alpha : \mathcal{L}^{\otimes n} \xrightarrow{\sim} L$ is an isomorphism of line bundles over U. These assemble to the gerbe G_n of n^{th} roots of L. This is an example of a lifting gerbe §2.2.

Remark. One also has the following interpretation, which shows that, in a fairly precise sense, a gerbe is the categorical analog of a torsor. Let $\mathcal{G}$ be a gerbe over C, let $\{U_i\}$ be a cover of $U \in \mathrm{Ob}(\mathsf{C})$, and let $\{x_i\}$ be a collection of objects $x_i \in \mathcal{G}(U_i)$. The G-torsors $E_{ij} = \mathrm{Hom}(x_j, x_i)$ are part of a "torsor cocycle" $\gamma_{ijk} \colon E_{ij} \otimes E_{jk} \to E_{ik}$, locally given by c_{ijk}, above, and subject to the obvious 2-cocycle identity [9, (5-10) on p. 201]. Let $\textsc{Tors}(G)$ be the stack of G-torsors over X. Since G is assumed abelian, $\textsc{Tors}(G)$ has a group-like composition law given by the standard Baer sum. The fact that $\mathcal{G}$ itself is locally equivalent to $\textsc{Tors}(G)$, plus the datum of the torsor cocycle $\{E_{ij}\}$, show that $\mathcal{G}$ is equivalent to a $\textsc{Tors}(G)$-torsor.

The primary examples of abelian gerbes occurring in this paper are the gerbe of local lifts associated to a central extension and four-term complexes, described in the next two sections.

2.2 The gerbe of lifts associated with a central extension

(See [21, 7, 9].) A central extension

$$0 \longrightarrow A \xrightarrow{\imath} E \xrightarrow{p} G \longrightarrow 0 \tag{2.2.1}$$

of sheaves of groups determines a homotopy-exact sequence

$$\textsc{Tors}(A) \longrightarrow \textsc{Tors}(E) \longrightarrow \textsc{Tors}(G),$$

which is an extension of topoi with characteristic class $c \in H^2(\mathrm{B}_G, A)$. (Recall that A is abelian and that $\textsc{Tors}(G)$ is equivalent to B_G.) If X is any topos over $\mathrm{Tors}(G) \cong \mathrm{B}_G$, the gerbe of lifts is the gerbe with band A

$$\mathcal{E} = \mathsf{Hom}_{\mathrm{B}_G}(\mathsf{X}, \mathrm{B}_E),$$

where Hom denotes the cartesian morphisms. The class $c(\mathcal{E}) \in H^2(\mathsf{X}, A)$ is the pullback of c along the map $\mathsf{X} \to \mathrm{B}_G$. By the universal property of B_G, the morphism $\mathsf{X} \to \mathrm{B}_G$ corresponds to a G-torsor P of X, hence the A-gerbe $\mathcal{E}$ is the gerbe whose objects are (locally) pairs of the form (Q, λ), where Q is an E-torsor and $\lambda \colon Q \to P$ an equivariant map. It is easy to see that an automorphism of an object (Q, λ) can be identified with an element of A, so that A is indeed the band of $\mathcal{E}$.
Let us take $\mathsf{X} = \mathsf{C}^\sim$, and let P be a G-torsor. With the same assumptions as the end of § 2.1, let X be an object of C with a cover $\{U_i\}$. In this case, the class of $\mathcal{E}$ is computed by choosing $E|_{U_i}$-torsors Q_i and equivariant maps $\lambda_i \colon Q_i \to P|_{U_i}$. Up to refining the cover, let $\alpha_{ij} \colon Q_j \to Q_i$ be an E-torsor isomorphism such that $\lambda_i \circ \alpha_{ij} = \lambda_j$. With these choices the class of $\mathcal{E}$ is given by the cocycle $\alpha_{ij} \circ \alpha_{jk} \circ \alpha_{ik}^{-1}$.

Remark 2.2. The above argument gives the well known boundary map [21, Proposition 4.3.4]

$$\partial^1 \colon H^1(G) \longrightarrow H^2(A)$$

(where we have omitted X from the notation). Dropping down one degree we get [ibid., Proposition 3.3.1]

$$\partial^0 \colon H^0(G) \longrightarrow H^1(A).$$

In fact these are just the boundary maps determined by the above short exact sequence when all objects are abelian. The latter can be specialized even further: if $g \colon * \to G$, then by pullback the fiber E_g is an A-torsor [22].

2.3 Four-term complexes

Let $\mathsf{C}^\sim_{\mathrm{ab}}$ be the category of abelian sheaves over the site C. Below we shall be interested in four-term exact sequences of the form:

$$0 \longrightarrow A \xrightarrow{\imath} L_1 \xrightarrow{\partial} L_0 \xrightarrow{p} B \longrightarrow 0. \tag{2.3.1}$$

Let $\mathrm{Ch}_+(\mathsf{C}^\sim_{\mathrm{ab}})$ be the category of positively graded homological complexes of abelian sheaves. The above sequence can be thought of as a (non-exact) sequence

$$0 \longrightarrow A[1] \longrightarrow [L_1 \longrightarrow L_0] \longrightarrow B \longrightarrow 0$$

of morphisms of $\mathrm{Ch}_+(\mathsf{C}^\sim_{\mathrm{ab}})$. This sequence is short-exact in the sense of Picard categories, namely as a short exact sequence of Picard stacks

$$0 \longrightarrow \mathrm{Tors}(A) \longrightarrow \mathcal{L} \xrightarrow{p} B \longrightarrow 0,$$

where $\mathcal{L}$ is the strictly commutative Picard stack associated to the complex $L_1 \to L_0$ and the abelian object B is considered as a discrete stack in the obvious way. We have isomorphisms $A \cong \pi_1(\mathcal{L})$ and $B \cong \pi_0(\mathcal{L})$, where the former is the automorphism sheaf of the object $0 \in \mathcal{L}$ and the latter the sheaf of connected components (see [7, 8, 13]). It is also well known that the projection $p\colon \mathcal{L} \to B$ makes $\mathcal{L}$ a *gerbe* over B. In this case the band of $\mathcal{L}$ over B is A_B, thereby determining a class in $H^2(B, A)$.[8]
Rather than considering $\mathcal{L}$ itself as a gerbe over B, we shall be interested in its fibers above generalized points $\beta\colon * \to B$. Let us put $\mathcal{A} = \mathrm{T{\small ORS}}(A)$. By a categorification of the arguments in [22], the fiber $\mathcal{L}_\beta$ above β is an $\mathcal{A}$-torsor, hence an abelian A-gerbe, by the observation at the end of § 2.1 (see also the equivalence described in [6]). $\mathcal{L}_\beta$ is canonically equivalent to $\mathcal{A}$ whenever $\beta = 0$. Writing

$$\mathrm{Hom}_{\mathsf{C}^\sim}(*, B) \cong \mathrm{Hom}_{\mathsf{C}^\sim_{\mathrm{ab}}}(\mathbb{Z}, B) = H^0(B),$$

we have the homomorphism

$$\partial^2 : H^0(B) \longrightarrow H^2(A), \tag{2.3.2}$$

which sends β to the class of $\mathcal{L}_\beta$ in $H^2(A)$. The sum of β and β' is sent to the Baer sum of $\mathcal{L}_\beta + \mathcal{L}_{\beta'}$, and the characteristic class is additive. In the following Lemma we show this map is the same as the one described in [21, Théorème 3.4.2].

Lemma 2.3.

i. *The map ∂^2 in (2.3.2) is the canonical cohomological map (iterated boundary map) [21, Théorème 3.4.2]*

$$d^2 : H^0(B) \longrightarrow H^1(C) \longrightarrow H^2(A)$$

(C is defined below) arising from the four-term complex (2.3.1).

ii. *The image of β under d^2 is the class of the gerbe $\mathcal{L}_\beta$.*

Proof. We keep the same notation as above. Let us split (2.3.1) as

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "\imath"] & L_1 \arrow[r, "\pi"] \arrow[dr, "\partial"'] & C \arrow[r] \arrow[d, "\jmath"] & 0 \\
 & & & L_0 \arrow[d, "p"] & \\
 & & & B \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

[8] This is *part* of the invariant classifying the four-term sequence, see the remarks in [8, §6].

with $C = \operatorname{Im}\partial$. By Grothendieck's theory of extensions [22], with $\beta\colon * \to B$, the fiber $(L_0)_\beta$ is a C-torsor (see the end of Remark 2.2). According to section 2.2, we have a morphism $\mathrm{T{\small ORS}}(L_1) \to \mathrm{T{\small ORS}}(C)$, and the object $(L_0)_\beta$ of $\mathrm{T{\small ORS}}(C)$ gives rise to the gerbe of lifts $\mathcal{E}_\beta \equiv \mathcal{E}_{L_0,\beta}$, which is an A-gerbe. Now, consider the map assigning to $\beta \in H^0(B)$ the class of $\mathcal{E}_\beta \in H^2(A)$. By construction, this map factors through $H^1(C)$ by sending β to the class of the torsor $(L_0)_\beta$. We then lift that to the class of the gerbe of lifts in $H^2(A)$. All stages are compatible with the abelian group structures. This is the homomorphism described in [21, Théorème 3.4.2].

It is straightforward that this is just the classical lift of β through the four-term sequence (2.3.1). Indeed, this is again easily seen in terms of a Čech cover $\{U_i\}$ of $*$. Lifts x_i of $\beta|_{U_i}$ are sections of the C-torsor $(L_0)_\beta$, therefore determining a standard C-valued 1-cocycle $\{c_{ij}\}$. From section 2.2 we then obtain an A-valued 2-cocycle $\{a_{ijk}\}$ arising from the choice of local L_1-torsors X_i such that $X_i \to (L_0)_\beta|_{U_i}$ is $(L_1 \to C)$-equivariant. Note that in the case at hand, $\pi\colon L_1 \to C$ being an epimorphism, the lifting of the torsor $(L_0)_\beta$ is done by choosing local trivializations, i.e. the x_i above, and then choosing $X_i = L_1|_{U_i}$.

The same argument shows that the class of $\mathcal{L}_\beta$, introduced earlier, is the same as that of $\mathcal{E}_\beta$. This is a consequence of the following well known facts: objects of $\mathcal{L}_\beta$ are locally lifts of β to L_0; morphisms between them are given by elements of L_1 acting through ∂. As a result, automorphisms are sections of A and clearly the class so obtained coincides with that of $\mathcal{E}_\beta$. Therefore $\mathcal{E}_\beta$ and $\mathcal{L}_\beta$ are equivalent and the homomorphism of [21, Théorème 3.4.2] is equal to (2.3.2), as required. $\square$

From the proof of the above lemma, we obtain the following two descriptions of the A-gerbe $\mathcal{L}_\beta$.

Corollary 2.4. *(i) For any four-term complex* (2.3.1) *and any generalized point β of B, the fiber $\mathcal{L}_\beta$ is a gerbe. Explicitly, it is the stack associated with the prestack which attaches to U the groupoid $\mathcal{L}_\beta(U)$ whose objects are elements $g \in L_0(U)$ with $p(g) = \beta$ and morphisms between g and g' given by elements h of $L_1(U)$ satisfying $\partial(h) = g - g'$.*
(ii) The A-gerbe $\mathcal{L}_\beta$ is the lifting gerbe of the C-torsor $(L_0)_\beta$ to a L_1-torsor. $\square$

We will use both descriptions in §5 especially in the comparison of the Gersten and the Heisenberg gerbe of a codimension two cycle, in the case that it is an intersection of divisors.

A slightly different point of view is the following. Recast the sequence (2.3.1) as a quasi-isomorphism

$$A[2] \xrightarrow{\cong} \bigl[L_1 \longrightarrow L_0 \longrightarrow B\bigr]$$

of three-term complexes of $\mathrm{Ch}_+(\mathsf{C}^{\sim}_{\mathrm{ab}})$, where now A has been shifted two places to the left. Also, relabel the right hand side as $L'_2 \to L'_1 \to L'_0$ (where again we employ homological degrees) for convenience. By [35], the above morphism of complexes of $\mathrm{Ch}_+(\mathsf{C}^\sim_{\mathrm{ab}})$, placed in degrees $[-2,0]$, gives an equivalence between the corresponding associated strictly commutative Picard 2-stacks

$$\mathfrak{A} \xrightarrow{\cong} \mathfrak{L}$$

over C. Here $\mathfrak{L} = [L'_2 \to L'_1 \to L'_0]^\sim$ and $\mathfrak{A} = [A \to 0 \to 0]^\sim \cong \textsc{Tors}(\mathcal{A}) \cong \textsc{Gerbes}(A)$. This time we have $\pi_0(\mathfrak{L}) = \pi_1(\mathfrak{L}) = 0$, and $\pi_2(\mathfrak{L}) \cong A$, as it follows directly from the quasi-isomorphism above. Thus $\mathfrak{L}$ is 2-connected, namely any two objects are locally (i.e. after base change) connected by an arrow; similarly, any two arrows with the same source and target are—again, locally—connected by a 2-arrow.
Locally, any object of $\mathfrak{L}$ is a section $\beta \in B = L'_0$. By the preceding argument, the Picard stack $\mathcal{L}_\beta = \mathrm{Aut}_{\mathfrak{L}}(\beta)$ is an A-gerbe, and the assignment $\beta \mapsto \mathcal{L}_\beta$ realizes (a quasi-inverse of) the equivalence between $\mathfrak{A}$ and $\mathfrak{L}$. It is easy to see that $\mathcal{L}_\beta$ is the same as the fiber over β introduced before.
In particular, for the Gersten resolution (5.1.1), (5.1.2), for $\mathcal{K}_2$, we get the equivalence of Picard 2-stacks

$$\textsc{Gerbes}(\mathcal{K}_2) \cong \left[G_2^X\right]^\sim. \tag{2.3.3}$$

3 The Heisenberg group

The purpose of this section is to describe a functor $H\colon \mathsf{Ab} \times \mathsf{Ab} \to \mathsf{Grp}$, where Ab is the category of abelian groups and Grp that of groups. If C is a site, the method immediately generalizes to the categories of abelian groups and of groups in $\mathsf{C}^\sim$, the topos of sheaves on C. For any pair A, B of abelian sheaves on C, there is a canonical Heisenberg sheaf $H_{A,B}$ (of non-commutative groups on C), a central extension of $A \times B$ by $A \otimes B$.
The definition of H is based on a generalization of the Heisenberg group construction due to Brylinski [10, §5]. A pullback along the diagonal map $A \to A \otimes A$ gives the extension constructed by Poonen and Rains [31].

3.1 The Heisenberg group

Let A and B be abelian groups. Consider the (central) extension

$$0 \to A \otimes B \to H_{A,B} \to A \times B \to 0 \tag{3.1.1}$$

where the group $H_{A,B}$ is defined by the group law:

$$(a, b, t)\,(a', b', t') = (aa', bb', t + t' + a \otimes b'). \tag{3.1.2}$$

Here a, a' are elements of A, b, b' of B, and t, t' of $A \otimes B$. The nonabelian group $H_{A,B}$ is evidently a functor of the pair (A, B), namely a pair of homomorphisms $(f\colon A \to A', g\colon B \to B')$ induces a homomorphism $H_{f,g}\colon H_{A,B} \to H_{A',B'}$. The

special case $A = B = \boldsymbol{\mu}_n$ occurs in Brylinski's treatment of the regulator map to étale cohomology [10].
The map

$$f\colon (A \times B) \times (A \times B) \longrightarrow A \otimes B, \qquad f(a, b, a', b') = a \otimes b', \tag{3.1.3}$$

is a cocycle representing the class of the extension (3.1.1) in $H^2(A \times B, A \otimes B)$ (group cohomology). Its alternation

$$\varphi_f\colon \wedge^2_{\mathbb{Z}} (A \times B) \longrightarrow A \otimes B, \qquad \varphi_f((a, b), (a', b')) = a \otimes b' - a' \otimes b,$$

coincides with the standard commutator map and represents the value of the projection of the class of f under the third map in the universal coefficient sequence

$$0 \longrightarrow \operatorname{Ext}^1(A \times B, A \otimes B) \longrightarrow H^2(A \times B, A \otimes B) \longrightarrow \operatorname{Hom}(\wedge^2_{\mathbb{Z}}(A \times B), A \otimes B).$$

As for the commutator map, it is equal to $[s, s]\colon \wedge^2_{\mathbb{Z}} (A \times B) \to A \otimes B$, where $s\colon A \times B \to H_{A,B}$ is a set-theoretic lift, but the map actually is independent of the choice of s. (For details see, e.g. the introduction to [8].)

Remark 3.1. The properties of the class of the extension $H_{A,B}$, in particular that it is a cup-product of the fundamental classes of A and B, as we can already evince from (3.1.3), are best expressed in terms of Eilenberg-Mac Lane spaces. We will do this below working in the topos of sheaves over a site.

3.2 Extension to sheaves

The construction of the Heisenberg group carries over to the sheaf context. Let C be a site, and $\mathsf{C}^\sim$ the topos of sheaves over C. Denote by $\mathsf{C}^\sim_{\mathrm{ab}}$ the abelian group objects of $\mathsf{C}^\sim$, namely the abelian sheaves on C, and by $\mathsf{C}^\sim_{\mathrm{grp}}$ the sheaves of groups on C.
For all pairs of objects A, B of $\mathsf{C}^\sim_{\mathrm{ab}}$, it is clear that the above construction of $H_{A,B}$ carries over to a functor

$$H\colon \mathsf{C}^\sim_{\mathrm{ab}} \times \mathsf{C}^\sim_{\mathrm{ab}} \longrightarrow \mathsf{C}^\sim_{\mathrm{grp}}.$$

In particular, since $H_{A,B}$ is already a sheaf of sets (isomorphic to $A \times B \times (A \otimes B)$), the only question is whether the group law varies nicely, but this is clear from its functoriality. Note further that by definition of $H_{A,B}$ the resulting epimorphism $H_{A,B} \to A \times B$ has a global section $s\colon A \times B \to H_{A,B}$ as objects of $\mathsf{C}^\sim$, namely $s = (\mathrm{id}_A, \mathrm{id}_B, 0)$, which we can use this to repeat the calculations of § 3.1.
In more detail, from § 2.2, the class of the central extenson (3.1.1) is to be found in $H^2(\mathrm{B}_{A\times B}, A \otimes B)$ ($A \otimes B$ is a trivial $A \times B$-module). This replaces the group cohomology of § 3.1 with its appropriate topos equivalent. By pulling back to the ambient topos, say $\mathsf{X} = \mathsf{C}^\sim$, this is the class of the gerbe of lifts from $B_{A\times B}$ to B_H. We are ready to give a proof of Theorem 1.4. This proof is computational.

Proof of Theorem 1.4. Let us go back to the cocycle calculations at the end of § 2.2, where X is an object of C equipped with a cover $\mathcal{U} = \{U_i\}$. An $A \times B$-torsor (P, Q) over X would be represented by a Čech cocycle (a_{ij}, b_{ij}) relative to $\mathcal{U}$. The cocycle is determined by the choice of isomorphisms $(P, Q)|_{U_i} \cong (A \times B)|_{U_i}$. Now, define $R_i = H_{A,B}|_{U_i}$ with the trivial $H_{A,B}$-torsor structure, and let $\lambda_i \colon R_i \to (P, Q)|_{U_i}$ equal the epimorphism in (3.1.1). Carrying out the calculation described at the end of 2.2 with these data gives $\alpha_{ij} \circ \alpha_{jk} \circ \alpha_{ik}^{-1} = a_{ij} \otimes b_{jk}$, which is the cup-product in Čech cohomology of the classes corresponding to the A-torsor P and the B-torsor Q. In other words, the gerbe of lifts corresponding to the central extension determined by the Heisenberg group incarnates the cup product map

$$H^1(X, A) \times H^1(X, B) \xrightarrow{\cup} H^2(X, A \otimes B).$$

For the choice $\alpha_{ij} = (a_{ij}, b_{ij}, 0)$, one has the following explicit calculation in the Heisenberg group

$$\begin{aligned}
\alpha_{ij} \circ \alpha_{jk} \circ \alpha_{ik}^{-1} &= (a_{ij}, b_{ij}, 0)(a_{jk}, b_{jk}, 0)(a_{ik}, b_{ik}, 0)^{-1} \\
&= (a_{ik}, b_{ik}, a_{ij} \otimes b_{jk})(a_{ik}^{-1}, b_{ik}^{-1}, a_{ik} \otimes b_{ik}) \\
&= (1, 1, a_{ij} \otimes b_{jk} + a_{ik} \otimes b_{ik} - a_{ik} \otimes b_{ik}) \\
&= (1, 1, a_{ij} \otimes b_{jk});
\end{aligned}$$

We used that the inverse of (a, b, t) in the Heisenberg group is $(a^{-1}, b^{-1}, -t + a \otimes b)$:

$$(a, b, t)(a^{-1}, b^{-1}, -t + a \otimes b) = (1, 1, a \otimes b^{-1} + t - t + a \otimes b) = (1, 1, 0).$$

It is well known [9, Chapter 1, §1.3, Equation (1-18), p. 29] that the Čech cup-product of $a = \{a_{ij}\}$ and $b = \{b_{ij}\}$ is given by the two-cocycle

$$\{a \cup b\}_{ijk} = \{a_{ij} \otimes b_{jk}\}.$$

This proves the first three points of the statement, whereas the fourth is built-in from the very construction. The fifth follows from the fact that the class of the gerbe of lifts is bilinear: this is evident from the expression computed above. $\square$

As hinted above, the cup product has a more intrinsic explanation in terms of maps between Eilenberg-Mac Lane objects in the topos. Passing to Eilenberg-Mac Lane objects in particular "explains" why the cup-product realizes the cup-product pairing. First, we state

Theorem 3.2. *The class of the extension (3.1.1) in $\mathsf{C}^{\sim}$ corresponds to (the homotopy class of) the cup product map*

$$K(A \times B, 1) \cong K(A, 1) \times K(B, 1) \longrightarrow K(A \otimes B, 2)$$

between the identity maps of $K(A, 1)$ and $K(B, 1)$; its expression is given by (3.1.3).

Proof. Observe the epimorphism $H_{A,B} \to A \times B$ has global set-theoretic sections. The statement follows from Propositions 3.3 and 3.4 below. □

The two main points, which we now proceed to illustrate, are that Eilenberg-Mac Lane objects represent cohomology (and hypercohomology, once we take into account simplicial objects) in a topos, and that the cohomology of a group object in a topos (such as $A \times B$ in $\mathsf{C}^\sim$) with trivial coefficients can be traded for the hypercohomology of a simplicial model of it. In this way we calculate the class of the extension as a map, and such map is identified with the cup product. We assemble the necessary results to flesh out the proof of Theorem 3.2 in the next two sections.

3.3 Simplicial computations

The class of the central extension (2.2.1) can be computed simplicially. (For the following recollections, see [25, VI.5, VI.6, VI.8] and [5, §2].)
Let T be a topos, G a group-object of T (for us it will be $\mathsf{T} = \mathsf{C}^\sim$) and $BG = K(G,1)$ the standard classifying simplicial object with $B_n G = G^n$ [14]. Let A be a trivial G-module. We will need the following well known fact.[9]

Proposition 3.3. $H^i(\mathrm{B}_G, A) \cong \mathbf{H}^i(BG, A)$.

Proof. The object on the right is the hypercohomology as a simplicial object of T. Let X be a simplicial object in a topos T. One defines

$$\mathbf{H}^i(X, A) = \mathrm{Ext}^i(\mathbb{Z}[X]^\sim, A)$$

where $M^\sim$, for any simplicial abelian object M of T, denotes the corresponding chain complex defined by $M^\sim_n = M_n$, and by taking the alternate sum of the face maps. $\mathbb{Z}X_n$ denotes the abelian object of T generated by X_n. Of interest to us is the spectral sequence [5, Example (2.10) and below]:

$$E_1^{p,q} = H^q(X_p, A) \Longrightarrow \mathbf{H}^\bullet(X, A).$$

Let X be any simplicial object of T. The levelwise topoi T/X_n, $n = 0, 1, \ldots,$ form a simplicial topos $\mathsf{X} = \mathsf{T}/X$ or equivalently a topos fibered over Δ^{op}, where Δ is the simplicial category. The topos $\mathrm{B}\mathsf{X}$ of X-objects essentially consists of descent-like data, that is, objects L of X_0 equipped with an arrow $a\colon d_1^* L \to d_0^* L$ the cocycle condition $d_0^* a \, d_2^* a = d_1^* a$ and $s_0^* a = \mathrm{id}$ (the latter is automatic if a is an isomorphism). By [25, VI.8.1.3], in the case where $X = BG$, $\mathrm{B}\mathsf{X}$ is nothing but B_G, the topos of G-objects of T. One also forms the topos $\mathrm{Tot}(\mathsf{X})$, whose objects are collections $F_n \in \mathsf{X}_n$ such that for each $\alpha\colon [m] \to [n]$ in Δ^{op} there is a morphism $F_\alpha\colon \alpha^* F_m \to F_n$, where α^* is the inverse image corresponding to the morphism $\alpha\colon \mathsf{X}_n \to \mathsf{X}_m$. There is a functor $ner\colon \mathrm{B}\mathsf{X} \to \mathrm{Tot}(\mathsf{X})$ sending

[9] Unfortunately we could not find a specific entry point in the literature to reference, therefore we assemble here the necessary prerequisites. See also [11, §§2,3] for a detailed treatment in the representable case.

(L, a) to the object of $\mathrm{Tot}(\mathsf{X})$ which at level n equals $(d_0 \cdots d_0)^* L$ (a enters through the resulting face maps), see *loc. cit.* for the actual expressions. The functor *ner* is the inverse image functor for a morphism $\mathrm{Tot}(\mathsf{X}) \to \mathrm{B}\mathsf{X}$, and, X satisfying the conditions of being a "good pseudo-category" ([25, VI 8.2]) we have an isomorphism

$$R\Gamma(\mathrm{B}\mathsf{X}, L) \xrightarrow{\cong} R\Gamma(\mathrm{Tot}(\mathsf{X}), ner(L))$$

and, in turn, a spectral sequence

$$E_1^{p,q} = H^q(X_p, ner_p(L)) \Longrightarrow H^\bullet(\mathrm{B}\mathsf{X}, L),$$

[25, VI, Corollaire 8.4.2.2]. On the left hand side we recognize the spectral sequence for the cohomology of a simplicial object in a topos [5, §2.10].
Applying the foregoing to $X = BG$, and L a left G-object of T, we obtain [25, VI.8.4.4.5]

$$E_1^{p,q} = H^q(G^p, L) \Longrightarrow H^\bullet(\mathrm{B}_G, L).$$

(We set $Y = e$, the terminal object of T, in the formulas from *loc. cit.*)
Thus if $L = A$, the trivial G-module arising from a central extension of G by A, by comparing the spectral sequences we can trade $H^2(\mathrm{B}_G, A)$ for the hypercohomology $\mathbf{H}^2(K(G,1), A)$. □

3.4 The cup product

The class of the extension extension (3.1.1) corresponds to the homotopy class of a map $K(A \times B, 1) \to K(A \otimes B, 2)$. We interpret it in terms of cup products of Eilenberg-Mac Lane objects.
Recall that for an object M of $\mathsf{C}^\sim_{\mathrm{ab}}$ we have $K(M, i) = K(M[i])$, where $M[i]$ denotes M placed in homological degree i, and $K\colon \mathrm{Ch}_+(\mathsf{C}^\sim_{\mathrm{ab}}) \to s\mathsf{C}^\sim_{\mathrm{ab}}$ is the Dold-Kan functor from nonnegative chain complexes of $\mathsf{C}^\sim_{\mathrm{ab}}$ to simplicial abelian sheaves. Explicitly:

$$K(M,i)_n = \begin{cases} 0 & 0 \leq n < i, \\ \bigoplus_{s\colon [n] \twoheadrightarrow [i]} M & n \geq i. \end{cases}$$

In particular, $K(M,i)_i = M$. K is a quasi-inverse to the normalized complex functor $N\colon s\mathsf{C}^\sim_{\mathrm{ab}} \to \mathrm{Ch}_+(\mathsf{C}^\sim_{\mathrm{ab}})$.
If X is a simplicial object X of $\mathsf{C}^\sim$, we have

$$\mathbf{H}^i(X, M) \cong [X, K(M,i)], \tag{3.4.1}$$

where the right-hand side denotes the hom-set in the homotopy category [24, 5]. In particular, there is a fundamental class $\imath^n_M \in \mathbf{H}^n(K(M,n), M)$, corresponding to the identity map.
Returning to the objects A and B of $\mathsf{C}^\sim_{\mathrm{ab}}$, also recall the morphism [5, Chapter II, Equation (2.22), p. 64]

$$\delta_{i,j}\colon K(A,i) \times K(B,j) \longrightarrow K(A \otimes B, i+j). \tag{3.4.2}$$

It is the composition of two maps. The first is:

$$K(A,i) \times K(B,j) \longrightarrow d((K(A,i) \boxtimes K(B,j)) = (K(A,i) \otimes K(B,j))),$$

where $\boxtimes$ denotes the external tensor product of simplicial objects of $\mathsf{C}^{\sim}_{\mathrm{ab}}$ and d the diagonal; the second is the map in $s\mathsf{C}^{\sim}_{\mathrm{ab}}$ corresponding to the Alexander-Whitney map under the Dold-Kan correspondence. We have:

Proposition 3.4. *The class of the extension* (3.1.1) *is equal to* $\imath^1_A \otimes \imath^1_B = \delta_{1,1}(\imath^1_A \times \imath^1_B)$.

Proof. Observe that any simplicial morphism $f \colon X \to K(M,i)$ is determined by f_i, the rest, for $n > i$, being determined by the simplicial identities. Therefore we need to compute:

$$K(A \times B, 1)_2 \cong K(A,1)_2 \times K(B,1)_2 \longrightarrow K(A \otimes B, 2)_2,$$

namely

$$(A \times B) \times (A \times B) \longrightarrow (A \times A) \times (B \times B) \longrightarrow A \otimes B.$$

From the expression of the Alexander-Whitney map, in e.g. [24], the image of the second map in $\mathrm{Ch}_+(\mathsf{C}^{\sim}_{\mathrm{ab}})$ is the sum of $d^v_0 d^v_0$, $d^h_1 d^h_1$, and $d^h_2 d^v_0$. Only the third one is nonzero, giving $((a,b),(a',b')) \to a \otimes b'$, which equals f in the construction of the extension (3.1.1). Using (3.4.1) we obtain the conclusion. $\square$

The morphism (3.4.2) represents the standard cup product in cohomology. By Proposition 3.4, for an object X of $s\mathsf{C}^{\sim}$, the cup product

$$\mathbf{H}^1(X,A) \times \mathbf{H}^1(X,B) \longrightarrow \mathbf{H}^2(X, A \otimes B)$$

factors through $X \to K(A,1) \times K(B,1)$ and the extension (3.1.1).

Remark. Proposition 3.4 and the above map provide a more conceptual proof of Theorem 1.4.

4 Examples and connections to prior results

In this section, we collect some examples and briefly indicate the connections with earlier results [3, 10, 30, 31, 33].

4.1 Self-cup products of Poonen-Rains

In [31], Poonen and Rains construct, for any abelian group A, a central extension of the form

$$0 \to A \otimes A \to UA \to A \to 0,$$

providing a functor $U \colon \mathsf{Ab} \to \mathsf{Grp}$. The group law in UA is obtained from (3.1.2) by setting $a = a'$ and $b = b'$. Hence the above extension can be obtained from (3.1.1) by pulling back along the diagonal homomorphism $\Delta_A : A \to A \times A$. Similarly, both the cocycle and its alternation for the extension constructed in *loc. cit.* are obtained from ours by pullback along Δ_A, for $A \in \mathsf{Ab}$. Similar remarks apply over an abelian sheaf A on any site C. They use UA to describe the self-cup product $\alpha \cup \alpha$ of any element $\alpha \in H^1(A)$.

4.2 Brylinski's work on regulators and étale analogues

In [10], Brylinski has proved Theorem 1.4 in the case $A = B = \boldsymbol{\mu}_n$, the étale sheaf $\boldsymbol{\mu}_n$ of n^{th} roots of unity on a scheme X over $\operatorname{Spec} \mathbb{Z}[\frac{1}{n}]$ using the Heisenberg group $H_{\boldsymbol{\mu}_n, \boldsymbol{\mu}_n}$ (in our notation). The gerbe from Theorem 1.4 in this particular case is related to the Bloch-Deligne line bundle [15, 33], in a sense made precise in [10, Proposition 5.1 and after]. Brylinski has used this special case of Theorem 1.4 to provide a geometric interpretation of certain regulator maps.

4.3 Finite flat group schemes

Let X be any variety over a perfect field F of characteristic $p > 0$. For any commutative finite flat group scheme N killed by p^n, consider the cup product pairing

$$H^1(X, N) \times H^1(X, N^D) \to H^2(C, \boldsymbol{\mu}_{p^n})$$

of flat cohomology groups where N^D is the Cartier dual of N. Theorem 1.4 provides a $\boldsymbol{\mu}_{p^n}$-gerbe on X given a N-torsor and a N^D-torsor. When N is the kernel of p^n on an abelian scheme A so that N^D is the kernel of p^n on the dual abelian scheme A^D of A, the cup-product pairing is related to the Néron-Tate pairing [28, p. 19].

4.4 The gerbe associated with a pair of divisors

Let X be a smooth variety over a field F. Let D and D' be divisors on X. Consider the non-abelian sheaf H on X obtained by pushing the Heisenberg group $H_{\mathcal{K}_1, \mathcal{K}_1}$ along the multiplication map $m : \mathcal{K}_1 \otimes \mathcal{K}_1 \to \mathcal{K}_2$. So H is a central extension of $\mathcal{K}_1 \times K_1$ by $\mathcal{K}_2$ which we write

$$0 \longrightarrow \mathcal{K}_2 \longrightarrow H \xrightarrow{\pi} \mathcal{K}_1 \times \mathcal{K}_1 \longrightarrow 0. \tag{4.4.1}$$

Let $L = L_{D,D'}$ denote the $\mathcal{K}_1 \times \mathcal{K}_1$-torsor defined by the pair D, D'. Applying Theorem 1.4 gives a $\mathcal{K}_2$-gerbe on X as follows. Since H is a central extension (so $\mathcal{K}_1 \times \mathcal{K}_1$ acts trivially on $\mathcal{K}_2$), the category of local liftings of L to a $\mathcal{K}_2$-torsor provide (§2.2, [21, IV, 4.2.2]) a canonical $\mathcal{K}_2$-gerbe $\mathcal{G}_{D,D'}$.

Definition 4.1. *The Heisenberg gerbe $\mathcal{G}_{D,D'}$ with band $\mathcal{K}_2$ is the following: For each open set U, the category $\mathcal{G}_{D,D'}(U)$ has objects pairs (P, ρ) where P is a H-torsor on U and*

$$\rho\colon P \times_\pi (\mathcal{K}_1 \times \mathcal{K}_1) \xrightarrow{\sim} L$$

is an isomorphism of $\mathcal{K}_1 \times \mathcal{K}_1$-torsors; a morphism from (P, ρ) to (P', ρ') is a map $f : P \to P'$ of H-torsors satisfying $\rho = \rho' \circ f$. It is clear that the set of morphisms from (P, ρ) to (P', ρ') is a $\mathcal{K}_2$-torsor.

Example 4.2. Assume X is a curve (smooth proper) and put $Y = X \times X$.
(i) Assume $F = \mathbb{F}_q$ is a finite field. Let D be the graph on Y of the Frobenius morphism $\pi : X \to X$ and D' be the diagonal, the image of X under the map $\Delta : X \to X \times X$. Theorem 1.4 attaches a $\mathcal{K}_2$-gerbe on Y to the zero-cycle $D.D'$, the intersection of the divisors D and D'. Since the zero cycle $D.D'$ is the pushforward $\Delta_* \beta$ of $\beta = \sum_{x \in X(\mathbb{F}_q)} x$ on X, we obtain that the set of rational points on X determines a $\mathcal{K}_2$-gerbe on $X \times X$.
(ii) Note that the diagonal Δ_Y (a codimension-two cycle on $Y \times Y$) can be written as an intersection of divisors V and V' on $Y \times Y = X \times X \times X \times X$ where V (resp. V') are the set of points of the latter of the form $\{(a, b, a, c)\}$ (resp. $\{(a, b, d, b)\}$). Theorem 1.4 says that Δ_Y determines a $\mathcal{K}_2$-gerbe on $Y \times Y$.

4.5 Adjunction formula

Let X be a smooth proper variety and D be a smooth divisor of X. The classical adjunction formula states:
The restriction of the line bundle L_D^{-1} to D is the conormal bundle N_D (a line bundle on D).
Given a pair of smooth divisors D, D' with $E = D \cap D'$ smooth of pure codimension two, write $\iota : E \hookrightarrow X$ for the inclusion. There is a map $\pi : \iota^* \mathcal{K}_2 \to \mathcal{K}_2^E$, where $\mathcal{K}_2^E$ indicates the usual K-theory sheaf $\mathcal{K}_2$ on E. An analogue of the adjunction formula for E would be a description of the $\mathcal{K}_2^E$-gerbe $\pi_* \iota^* \mathcal{G}_{D,D'}$ obtained from the $\mathcal{K}_2$-gerbe $\mathcal{G}_{D,D'}$ on X.

Proposition 4.3. *Let D and D' be smooth divisors of X with $E = D \cap D'$ smooth of pure codimension two. Consider the line bundles $V = (N_D)|_E$ and $V' = (N_{D'})|_E$ on E. Then, $\pi_* \iota^* \mathcal{G}_{D,D'}$ is equivalent to the $\mathcal{K}_2^E$-gerbe $\mathcal{G}_{V,V'}$.*

Proof. Since the restriction map $H^*(X, \mathcal{K}_i) \to H^*(E, \mathcal{K}_i^E)$ respects cup-product, this follows from the classical adjunction formula for D and D'. □

4.6 Parshin's adelic groups

Let S be a smooth proper surface over a field F. For any choice of a curve C in S and a point P on C, Parshin [30, (18)] has introduced a discrete Heisenberg group
$$0 \to \mathbb{Z} \to \tilde{\Gamma}_{P,C} \to \Gamma_{P,C} \to 0,$$
where $\Gamma_{P,C}$ is isomorphic (non-canonically) to $\mathbb{Z} \oplus \mathbb{Z}$; he has shown [30, end of §3] how a suitable product of these groups leads to an adelic description of $CH^2(S)$ and the intersection pairing (1.0.2). His constructions are closely related to an adelic resolution of the sheaf $H_{\mathcal{K}_1, \mathcal{K}_1}$ on S.

5 Algebraic cycles of codimension two

Throughout this section, X is a smooth proper variety over a field F. Let $\eta\colon \operatorname{Spec} F_X \to X$ be the generic point of X and write K_i^η for the sheaf $j_*K_i(F_X)$.
In this section, we construct the Gersten gerbe $\mathcal{C}_\alpha$ for any codimension two cycle α on X, provide various equivalent descriptions of $\mathcal{C}_\alpha$ and use them to prove Theorems 5.4, 5.10. As a consequence, we obtain Theorems 1.5 and 1.6 of the introduction.

5.1 Bloch-Quillen formula

Recall the (flasque) Gersten resolution[10] [32, §7] [17, p. 276] [20] of the Zariski sheaf $\mathcal{K}_i$ associated with the presheaf $U \mapsto K_i(U)$:

$$\begin{aligned} 0 \longrightarrow \mathcal{K}_i \longrightarrow \bigoplus_{x\in X^{(0)}} j_*K_i(x) \longrightarrow \bigoplus_{x\in X^{(1)}} j_*K_{i-1}(x) \longrightarrow \cdots \\ \cdots \longrightarrow \bigoplus_{x\in X^{(i-1)}} j_*K_1(x) \xrightarrow{\delta_{i-1}} \bigoplus_{x\in X^{(i)}} j_*K_0(x); \end{aligned} \tag{5.1.1}$$

here, any point $x \in X^{(m)}$ corresponds to a subvariety of codimension m and the map j is the canonical inclusion $x \hookrightarrow X$. So $\mathcal{K}_i$ is quasi-isomorphic to the complex

$$\begin{aligned} G_i^X = \big[K_i^\eta \longrightarrow \bigoplus_{x\in X^{(1)}} j_*K_{i-1}(x) \longrightarrow \cdots \\ \cdots \longrightarrow \bigoplus_{x\in X^{(i-1)}} j_*K_1(x) \xrightarrow{\delta_{i-1}} \bigoplus_{x\in X^{(i)}} j_*K_0(x)\big]. \end{aligned} \tag{5.1.2}$$

By (5.1.1), there is a functorial isomorphism [32, §7, Theorem 5.19] [17, Corollary 72, p. 276]

$$\bigoplus_i CH^i(X) \xrightarrow{\sim} \bigoplus_i H^i(X, \mathcal{K}_i)\,; \tag{5.1.3}$$

(Bloch-Quillen formula)

this is an isomorphism of graded rings: D. Grayson has proved that the intersection product on $CH(X) = \oplus_i CH^i(X)$ corresponds to the cup-product in cohomology [17, Theorem 77, p.278]. Thus, algebraic cycles of codimension n give n-cocycles of the sheaf $\mathcal{K}_n$ on X and that two such cocycles are cohomologous exactly when the algebraic cycles are rationally equivalent.
The final two maps in (5.1.1) arise essentially from the valuation and the tame symbol map [3, pp.351-2]. Let R be a discrete valuation ring, with fraction

[10] This resolution exists for any separated smooth scheme of finite type over F.

field L; let ord : $L^\times \to \mathbb{Z}$ be the valuation and let l be the residue field. The boundary maps from the localization sequence for Spec R are known explicitly: the map $L^\times = K_1(L) \to K_0(l) = \mathbb{Z}$ is the map ord and the map $K_2(L) \to K_1(l) = l^\times$ is the tame symbol. This applies for any normal subvariety V (corresponds to a $y \in X^{(i)}$) and a divisor x of V (corresponding to a $x \in X^{(i+1)}$).

5.2 Divisors

We recall certain well known results about divisors and line bundles for comparison with the results below for the $\mathcal{K}_2$-gerbes attached to codimension two cycles. See [23, Chapter II §6] and [17, Vol. 1, II.2] for details.
If A is a sheaf of abelian groups on X, then $\operatorname{Ext}^1_X(\mathbb{Z}, A) = H^1(X, A)$ classifies A-torsors on X. Given an extension E

$$0 \longrightarrow A \longrightarrow E \xrightarrow{\pi} \mathbb{Z} \longrightarrow 0$$

of abelian sheaves on X, the corresponding A-torsor is simply $\pi^{-1}(1)$ (a sheaf of sets). When X is a point, then $\pi^{-1}(1)$ is a coset of $\pi^{-1}(0) = A$, i.e., a A-torsor. The classical correspondence [23] between Weil divisors (codimension-one algebraic cycles) D on X, Cartier divisors, line bundles $\mathcal{L}_D$, and torsors L_D over $\mathcal{O}_X^* = \mathbb{G}_m = \mathcal{K}_1$ comes from the Gersten sequence (5.1.1) for $\mathcal{K}_1$ (see also [20, 2.2]):

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow F_X^\times \xrightarrow{d} \bigoplus_{x \in X^{(1)}} j_*\mathbb{Z} \to 0, \tag{5.2.1}$$

where F_X is the constant sheaf of rational functions on X and the sum is over all irreducible effective divisors on X, using that $K_0(L) \cong \mathbb{Z}$ and $K_1(L) = L^\times$ for any field L.
We recall that the categories of line bundles and $\mathcal{O}_X^*$-torsors are equivalent: given a line bundle $\mathcal{L}_D$, the $\mathcal{O}_X^*$-torsor L_D is the complement of the zero section of the line bundle. The functor $L_D \mapsto \mathcal{L}_D$ is induced by the natural action of $\mathbb{G}_m$ on the affine line $\mathbb{A}^1$. Pushing the principal $\mathcal{O}_X^*$-torsor along this action gives an $\mathbb{A}^1$-bundle.
As a Weil divisor $D = \Sigma_{x \in X^1} n_x x$ is a formal combination with integer coefficients of subvarieties of codimension one of X, it determines a map of sheaves

$$\psi \colon \mathbb{Z} \longrightarrow \bigoplus_{x \in X^{(1)}} j_*\mathbb{Z};$$

$\psi(1)$ is the section with components n_x. The $\mathcal{O}_X^*$-torsor L_D attached to D is given as the subset

$$d^{-1}(\psi(1)) \subset F_X^\times. \tag{5.2.2}$$

A Čech description of L_D relative to an Zariski open cover $\{U_i\}$ of X is as follows. Pick a rational function f_i on U_i with pole of order n_x along x for all $x \in U_i^{(1)}$ (so x is a irreducible subvariety of codimension one of U_i); we view

$f_i \in F_X^\times$. On $U_i \cap U_j$, one has $f_i = g_{ij} f_j$ for unique $g_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$; the collection $\{g_{ij}\}$ is a Čech one-cocycle with values in $\mathcal{O}_X^*$ representing L_D.
For any D, $\mathcal{L}_D$ (and L_D) is trivial on the complement of the support of D.

Remark 5.1. For each open U of X, one has the Picard category $\mathrm{TORS}_U(\mathcal{O}^*)$ of $\mathcal{O}^*$-torsors on U. These combine to the Picard stack $\mathrm{TORS}(\mathcal{O}^*)$ of $\mathcal{O}^*$-torsors on X. The Gersten sequence incarnates this Picard stack [12, 1.10]. □

5.3 THE GERSTEN GERBE OF A CODIMENSION TWO CYCLE

Our next task is to show that every cycle α of codimension two on X determines a gerbe $\mathcal{C}_\alpha$ with band $\mathcal{K}_2$. The Gersten complex (5.1.1) enables us to give a geometric description of $\mathcal{C}_\alpha$; see Remark 5.5 below.
First, let us make the map in (5.1.3) explicit for codimension-two cycles by showing how a codimension-two cycle determines a class in $H^2(X, \mathcal{K}_2)$. The cycle α can be viewed as an element $\mathbb{1}_\alpha$ of $H^0(X, \bigoplus_{x \in X^{(2)}} j_* K_0(x))$. Writing $\alpha = \sum_x n_x [x]$ as a sum over $x \in X^{(2)}$ (irreducible codimension two subvarieties), then the x-component of $\mathbb{1}_\alpha$ corresponds to n_x under the canonical isomorphism $K_0(x) \cong \mathbb{Z}$. This provides a natural map ϕ determined by $\phi(1) = \mathbb{1}_\alpha$:

$$\begin{tikzcd} & & & & \mathbb{Z} \arrow[d, "\phi"] & \\ 0 \arrow[r] & \mathcal{K}_2 \arrow[r, "\mu"] & K_2^\eta \arrow[r, "\nu"] & \bigoplus_{x \in X^{(1)}} j_* K_1(x) \arrow[r, "\delta"] & \bigoplus_{x \in X^{(2)}} j_* K_0(x) \arrow[r] & 0 \end{tikzcd} \tag{5.3.1}$$

and an exact sequence (by pullback)

$$0 \longrightarrow \mathcal{K}_2 \longrightarrow K_2^\eta \xrightarrow{\nu} T \xrightarrow{\delta} \mathbb{Z} \longrightarrow 0. \tag{5.3.2}$$

This two-extension of $\mathbb{Z}$ by $\mathcal{K}_2$ gives a class in $\mathrm{Ext}^2(\mathbb{Z}, \mathcal{K}_2) = H^2(X, \mathcal{K}_2)$ and so a map $Z^2(X) \to H^2(X, \mathcal{K}_2)$.
The maps δ and ν are essentially given by the valuation (or ord) and tame symbol maps; see §5.1.
Since the Gersten sequence

$$\mathcal{K}_2 \xrightarrow{\mu} K_2^\eta \xrightarrow{\nu} \bigoplus_{x \in X^{(1)}} j_* K_1(x) \xrightarrow{\delta} \bigoplus_{x \in X^{(2)}} j_* K_0(x)$$

is a four-term complex, one can apply the results of §2.3, specifically, Corollary 2.4 to it

DEFINITION 5.2. *For any codimension-two cycle α on X, the gerbe $\mathcal{C}_\alpha$ is the gerbe $\mathcal{L}_\beta$ of Corollary 2.4 from §2.3 applied to the four-term Gersten sequence*

(5.3.1) *- see above - with* $\beta = \phi(1) = \mathbb{1}_\alpha$*; and* $\mathcal{L}$ *is the Picard stack associated to the complex* $[K_2^\eta \to \bigoplus_{x \in X^{(1)}} j_* K_1(x)]$.

Remark 5.3. Corollary 2.4 provides two descriptions of $\mathcal{L}_\beta$ (and so of $\mathcal{C}_\alpha$). It should be emphasized that both descriptions are useful. One description of $\mathcal{C}_\alpha$ is the succinct description given in Remark 5.5. The other explicit description, crucial for the comparison with the Heisenberg gerbe (Theorem 5.10), consists of the following main points:

i. For any open set U of X, the category $\mathcal{C}_\alpha(U)$ has objects $u \in \bigoplus_{x \in X^{(1)}} j_* K_1(x)$ with $\delta u = \phi(1)$ and morphisms from u to u' are elements $a \in K_2^\eta$ with $\nu(a) = u' - u$.

ii. Any Hom-set $\mathrm{Hom}_{\mathcal{C}_\alpha}(u, u')$ is a $K_2(U)$-torsor.

iii. The category $\mathcal{C}_\alpha(U)$ can be described geometrically in terms of the ord and tame maps. For instance, let X be a surface. Write the zero-cycle α as a finite sum $\sum_{i \in I} n_i x_i$ of points x_i of X. We assume $n_i \neq 0$ and write V for the complement of the support of α. Any non-zero rational function f on a curve C defines an object of $\mathcal{C}_\alpha(U)$ if f is invertible on $C \cap U \cap V$ and satisfies $\mathrm{ord}_{x_i} f = n_i$ for each $x_i \in U$ (assuming, for simplicity, that x_i is a smooth point of C). A general object of $\mathcal{C}_\alpha(U)$ is a finite collection $u = \{C_j, f_j\}$ of curves C_j and non-zero rational functions f_j on C_j such that f_j is invertible on $C_j \cap U \cap V$ and $\sum \mathrm{ord}_{x_i} f_j = n_i$ (an index j occurs in the sum if $x_i \in C_j$) for each $x_i \in U$. A morphism from u to u' is an element $a \in K_2^\eta$ whose tame symbol is $u' - u$. □

Theorem 5.4. *(i)* $\mathcal{C}_\alpha$ *is a gerbe on* X *with band* $\mathcal{K}_2$.
(ii) Under (5.1.3)*, the class of* $\mathcal{C}_\alpha \in H^2(X, \mathcal{K}_2)$ *corresponds to* $\alpha \in CH^2(X)$.
(iii) $\mathcal{C}_\alpha$ *is equivalent to* $\mathcal{C}_{\alpha'}$ *(as gerbes) if and only if the cycles* α *and* α' *are rationally equivalent.*

Proof. (i) The Gersten sequence (5.3.1) is an example of a four-term complex, discussed in §2.3. As the stack $\mathcal{C}_\alpha$ is a special case of the gerbe $\mathcal{L}_\beta$ constructed in §2.3, (i) is obvious.
In more detail: We first observe that (5.3.2) provides a quasi-isomorphism between $\mathcal{K}_2$ (sheaf) and the complex (concentrated in degree zero and one)

$$\eta : \mathcal{K}_2 \to [K_2^\eta \xrightarrow{\nu} Ker(\delta)]. \tag{5.3.3}$$

Now, suppose U is disjoint from the support of α. On such an open set U, the map ϕ is zero. This means that the objects u of the category $\mathcal{C}_\alpha(U)$ are elements of $\mathrm{Ker}(\delta)$. The gerbe $\mathcal{C}_\alpha$, when restricted to U, is equivalent to the Picard stack of $\mathcal{K}_2$-torsors [1, Expose XVIII, 1.4.15]: in the complex $[K_2^\eta \xrightarrow{\nu} \mathrm{Ker}(\delta)]$, one has $\mathrm{Coker}(\nu) = 0$ and $\mathrm{Ker}(\nu) = \mathcal{K}_2|_U$. Since for any abelian sheaf G, the category

TORS(G) is the trivial G-gerbe, $\mathcal{C}_\alpha$ is the trivial gerbe with band $\mathcal{K}_2$ on the complement of the support of α.
Now, consider an arbitrary open set V of X. By the exactness of (5.3.2), there is an open covering $\{U_i\}$ of V and sections $u_i \in T(U_i)$ with $t_i = \phi(1)$. Fix i and let U be an open set contained in U_i. Then the category $\mathcal{C}_\alpha(U)$ is non-empty. The category D with objects $d \in \mathrm{Ker}(\delta) \subset T(U)$ and morphisms $\mathrm{Hom}_D(d, d') =$ elements $a \in K_2^\eta$ with $\nu(a) = d' - d$. The category D is clearly equivalent to the category of $K_2(U)$-torsors. The functor which sends d to $d + u_i$ is easily seen to be an equivalence of categories between D and $\mathcal{C}_\alpha(U)$. Thus $\mathcal{C}_\alpha$ is a gerbe with band $\mathcal{K}_2$.
(ii) The Bloch-Quillen formula (5.1.3) arises from the canonical map

$$d^2 : Z^2(X) \to H^2(X, \mathcal{K}_2)$$

of Lemma 2.3 applied to the four-term complex (5.3.1). As $\mathcal{C}_\alpha$ is a gerbe of the form $\mathcal{L}_\beta$, (ii) follows from Lemma 2.3.
(iii) This is a simple consequence of the Bloch-Quillen formula (5.1.3). □

Remark 5.5. (i) Split the sequence (5.3.1) into

$$0 \longrightarrow \mathcal{K}_2 \longrightarrow K_2^\eta \longrightarrow K_2^\eta/\mathcal{K}_2 \longrightarrow 0$$

and

$$0 \longrightarrow K_2^\eta/\mathcal{K}_2 \longrightarrow \bigoplus_{x \in X^{(1)}} j_* K_1(x) \longrightarrow \bigoplus_{x \in X^{(2)}} j_* K_0(x) \longrightarrow 0.$$

Since the Gersten resolution is by flasque sheaves, one has $H^1(X, K_2^\eta/\mathcal{K}_2) \xrightarrow{\sim} H^2(X, \mathcal{K}_2)$. Just as Cartier divisors are elements of $H^0(X, K_1^\eta/\mathcal{K}_1)$, we will now view elements of $H^1(X, K_2^\eta/\mathcal{K}_2)$ as *Cartier cycles of codimension two.* The map

$$Z^2(X) = H^0(X, \bigoplus_{x \in X^{(2)}} j_* K_0(x)\ ) \to H^1(X, K_2^\eta/\mathcal{K}_2)$$

attaches to any cycle its Cartier cycle. Lemma 2.4 provides the following succinct description of $\mathcal{C}_\alpha$:
it is the gerbe of liftings (to a K_2^η-torsor) of the $(K_2^\eta/\mathcal{K}_2)$-torsor determined by α.
(iii) The proof of Theorem 5.4 provides a canonical trivialization[11] η_α of the gerbe $\mathcal{C}_\alpha$ on the complement of the support of α.
(iv) The pushforward of $\mathcal{C}_\alpha$ along $\mathcal{K}_2 \to \Omega^2$ produces a Ω^2-gerbe which manifests the cycle class of α in de Rham cohomology $H^2(X, \Omega^2)$. If α is homologically equivalent to zero, then this latter gerbe is trivial, i.e., it is the Picard stack of Ω^2-torsors. □

Remark 5.6. It may be instructive to compare the $\mathbb{G}_m$-torsor L_D attached to a divisor D of X and the $\mathcal{K}_2$-gerbe $\mathcal{C}_\alpha$ attached to a codimension-two cycle. Let U be any open set of X. This goes, roughly speaking, as follows.

[11]This uses (5.3.3).

- L_D: The set of divisors on U rationally equivalent to zero is exactly the image of d over U in (5.2.1). So, the set of sections of L_D over U is non-empty if $D = 0$ in $CH^1(U)$. The sections of L_D over U are given by rational functions f on U whose divisor is $D|_U$. In other words, the sections are rational equivalences between the divisor D and the empty divisor. The set $L_D(U)$ is a torsor over $\mathbb{G}_m(U)$.

- $\mathcal{C}_\alpha$: We observe that the image of δ in (5.3.1) consists of codimension-two cycles rationally equivalent to zero. So $\mathcal{C}_\alpha$ is non-empty if $\alpha = 0$ in $CH^2(U)$. Each rational equivalence between α and the empty codimension-two cycle gives an object of $\mathcal{C}_\alpha(U)$. The sheaf of morphisms between two objects is a $\mathcal{K}_2$-torsor. □

The Bloch-Quillen formula (5.1.3) states that equivalence classes of $\mathcal{K}_2$-gerbes are in bijection with codimension-two cycles (modulo rational equivalence) on X. We have seen that a codimension-two cycle determines a $\mathcal{K}_2$-gerbe (an actual gerbe, not just one up to equivalence). It is natural to ask whether the converse holds: (see Proposition 5.8 in this regard)

Question 5.7. *Does a $\mathcal{K}_2$-gerbe on X determine an actual codimension-two cycle?*

Since the analogous question in codimension one has a negative answer (a $\mathbb{G}_m$-torsor does not determine a codimension-one cycle), one expects a negative answer to Question 5.7. A proof that the answer is negative can be based on the non-existence of a canonical intersection of Weil divisors: Consider the $\mathcal{K}_2$-gerbe $\mathcal{G}_{D,D'}$ attached to a pair of divisors D, D' on X. If $\mathcal{G}_{D,D'}$ determines an actual codimension-two cycle, then any pair D, D' of divisors determines a canonical codimension-two cycle on X which contradicts the non-existence of a canonical intersection of Weil divisors.

5.4 Gerbes and cohomology with support

Let F be an abelian sheaf on a site C. Recall that (see e.g. [29, §5.1]) $H^1(F)$ is the set of isomorphism classes of auto-equivalences of the trivial gerbe $\mathrm{Tors}(F)$ with band F; more generally, given gerbes $\mathcal{G}$ and $\mathcal{G}'$ with band F, then the set $\mathrm{Hom}_{\mathsf{C}}(\mathcal{G}, \mathcal{G}')$ (assumed non-empty) of maps of gerbes is a torsor for $H^1(F)$.
Recall also that, for any sheaf F on a scheme V, the cohomology $H^*_Z(V,F)$ with support in a a closed subscheme Z of V fits into an exact sequence [3, §5]

$$\cdots \longrightarrow H^i_Z(V,F) \longrightarrow H^i(V,F) \longrightarrow H^i(V-Z,F) \longrightarrow H^{i+1}_Z(V,F) \longrightarrow \cdots; \tag{5.4.1}$$

the exactness of

$$H^1(V,F) \longrightarrow H^1(V-Z,F) \longrightarrow H^2_Z(V,F) \longrightarrow H^2(V,F) \longrightarrow H^2(V-Z,F)$$

leads to an interpretation of the group $H^2_Z(V,F)$: it classifies isomorphism classes of pairs $(\mathcal{G}, \phi)$ consisting of a gerbe $\mathcal{G}$ with band F on V and a trivialization ϕ of $\mathcal{G}$ on $V-Z$, i.e., ϕ is an equivalence of $\mathcal{G}|_{V-Z}$ with $\mathrm{Tors}(F|_{V-Z})$.

5.5 Geometric interpretation of some results of Bloch

Bloch [3] has proved that:

i. [3, Proposition 5.3] Any codimension-two cycle α on X has a canonical cycle class $[\alpha] \in H^2_Z(X, \mathcal{K}_2)$; here Z is the support of α.

ii. [3, Theorem 5.11] If D is a smooth divisor of X, then $\mathrm{Pic}(D) = H^1(D, \mathcal{K}_1)$ is a direct summand of $H^2_D(X, \mathcal{K}_2)$.

For (1), we note that, by Remark 5.5, the gerbe $\mathcal{C}_\alpha$ has a trivialization η_α on $X - Z$. By the above interpretation of H^2 with support, the pair $(\mathcal{C}_\alpha, \eta_\alpha)$ defines an element of $H^2_Z(X, \mathcal{K}_2)$; this is the canonical class $[\alpha]$.
For (2), recall that Bloch constructed maps $a : \mathrm{Pic}(D) \to H^2_D(X, \mathcal{K}_2)$ and $b : H^2_D(X, \mathcal{K}_2) \to \mathrm{Pic}(D)$ with $b \circ a$ the identity on $\mathrm{Pic}(D)$. We can interpret the map a as follows. Note that any divisor E of D is a codimension-two cycle α on X. The $\mathcal{K}_2$-gerbe $\mathcal{C}_\alpha$ on X has a canonical trivialization η_α on $X - E$ (and so also on the smaller $X - D$). The association $E \mapsto (\mathcal{C}_\alpha, \eta_\alpha)$ gives the homomorphism $a : \mathrm{Pic}(D) \to H^2_D(X, \mathcal{K}_2)$.
These results of Bloch provide a partial answer to Question 5.7 summarized in the following

Proposition 5.8. *Let $\mathcal{G}$ be a $\mathcal{K}_2$-gerbe on X and let $\beta \in CH^2(X)$ correspond to $\mathcal{G}$ in the Bloch-Quillen formula (5.1.3). Let ϕ be a trivialization of $\mathcal{G}$ on the complement $X - D$ of a smooth divisor D of X. Then, β can be represented by a divisor of D (unique up to rational equivalence on D).*

Note that the data of ϕ is crucial: the map $\mathrm{Pic}(D) \to CH^2(X)$ is not injective in general [4, (iii), p. 269].

Proposition 5.9. *Let $i\colon D \to X$ and $j\colon U = X - D \to X$ be the inclusion maps. We have the following short exact sequence* [12] *of Picard 2-stacks*

$$\mathrm{Tors}(i_*\mathcal{K}_1^D) \longrightarrow \mathrm{Gerbes}(\mathcal{K}_2^X) \longrightarrow \mathrm{Gerbes}(j_*\mathcal{K}_2^U).$$

Proof. Analyzing the Gersten sequence (5.1.1), (5.1.2) for $\mathcal{K}_2$ on X and U, we get the short exact sequence:

$$0 \longrightarrow i_*G_1^D \longrightarrow G_2^X \longrightarrow j_*G_2^U \longrightarrow 0.$$

This gives a short exact sequence of Picard 2-stacks, then use (2.3.3). Note that $\mathrm{Tors}(i_*\mathcal{K}_1^D)$ is considered as a Picard 2-stack with no nontrivial 2-morphisms. $\square$

[12] For the notion of short exact sequence of Picard 2-stacks see, e.g. [2]. Briefly, we say that the diagram

$$\mathfrak{B} \xrightarrow{\imath} \mathfrak{C} \xrightarrow{p} \mathfrak{A}$$

is a short exact sequence if: (1) the composition $p \circ \imath$ is naturally isomorphic to the zero morphism; (2) p is full and essentially surjective, and (3) $\imath$ is an equivalence between $\mathfrak{B}$ and the so-called "homotopy kernel" of p, consisting of those objects whose image by p is equivalent in $\mathfrak{A}$ to the zero object. The meaning of "full" must be qualified: we want p to induce full and essentially surjective functors at the level of morphism categories (see *loc. cit.* for details).

The global long exact cohomology sequence arising from the exact sequence in the proposition gives part of the localization sequence for higher Chow groups

$$\begin{aligned} \cdots \longrightarrow CH^1(D,1) \longrightarrow CH^2(X,1) \longrightarrow CH^2(X-D,1) \longrightarrow \\ \longrightarrow \mathrm{Pic}(D) \longrightarrow CH^2(X) \longrightarrow CH^2(X-D) \longrightarrow 0. \end{aligned}$$

This uses the fact that $CH^1(D,0) = \mathrm{Pic}(D)$, that $CH^1(D,1) = H^0(D,\mathcal{O}^*)$ and $CH^1(D,j)$ is zero for $j > 1$ [4, (viii), p. 269].

5.6 The two gerbes associated with an intersection of divisors

For a codimension-two cycle α of X presented as the intersection of divisors

$$\alpha = D.D',$$

we know that the $\mathcal{K}_2$-gerbes
(i) the Gersten gerbe $\mathcal{C}_\alpha$ in Theorem 5.4 and
(ii) the Heisenberg gerbe $G_{D,D'}$ in §4.4 using Theorem 1.4
are equivalent (as their classes in $H^2(X,\mathcal{K}_2)$ corresponds to the class of the codimension-two cycle in $CH^2(X)$ via (5.1.3)). We now construct an actual equivalence between them.

Theorem 5.10. *Suppose that the codimension-two cycle α is the intersection $D.D'$ of divisors D and D' on X. There is a natural equivalence*[13]

$$\Theta : \mathcal{C}_\alpha \to \mathcal{G}_{D,D'}$$

of $\mathcal{K}_2$-gerbes on X.

Proof. By Theorem 1.4 and Theorem 5.4, the classes of the gerbes $\mathcal{G}_{D,D'}$ and $\mathcal{C}_\alpha$ in $H^2(X,\mathcal{K}_2)$ both correspond to the class of α in $CH^2(X)$. This shows that they are equivalent.

Let us exhibit an actual equivalence. We will construct a functor $\Theta_U : \mathcal{C}_\alpha(U) \to \mathcal{G}_{D,D'}(U)$, compatible with restriction maps $V \subset U \subset X$. We will need the Heisenberg group H from (4.4.1).

Consider an object $r \in \mathcal{C}_\alpha(U)$. We want to attach to r a H-torsor $\Theta_U(r)$ on U in a functorial manner. Each $\Theta_U(r)$ is a H-torsor which lifts the $\mathcal{K}_1 \times \mathcal{K}_1$-torsor $\mathcal{O}_D \times \mathcal{O}_{D'}$ on U. We will describe $\Theta_U(r)$ by means of Čech cocycles. Fix an open covering $\{U_i\}$ of U and write $\mathcal{C}^n(A)$ for Čech n-cochains with values in the sheaf A with respect to this covering.

[13] By §5.4. the set of such equivalences is a torsor under the action of the group $H^1(X,\mathcal{K}_2) = CH^2(X,1)$ [34, §2.1].

Step 1. Let $a = \{a_{i,j}\}$ and $b = \{b_{i,j}\}$ with $a, b \in \mathcal{C}^1(O^*)$ be Čech 1-cocycles for $\mathcal{O}_D$ and $\mathcal{O}_{D'}$. Pick $h = \{h_{i,j}\} \in \mathcal{C}^1(H)$ of the form

$$h_{i,j} = (a_{i,j}, b_{i,j}, c_{i,j}) \in H(U_i \cap U_j).$$

We need $c_{i,j} \in K_2(U_i \cap U_j)$ such that h is a Čech 1-cocycle (for $\Theta_U(r)$, the putative H-torsor). Since a, b are Čech cocycles, the Čech boundary ∂h is of the form

$$\partial h = \{(1, 1, y_{i,j,k})\}$$

with $y = \{y_{i,j,k}\} \in \mathcal{C}^2(\mathcal{K}_2)$ a Čech 2-cocycle. This cocycle y represents the gerbe $\mathcal{G}_{D,D'}$ on U.

Step 2. Recall that $\mathcal{C}_\alpha$ is the associated stack of the prestack $U \mapsto \mathcal{C}_\alpha(U)$ where the category $\mathcal{C}_\alpha(U)$ has objects $u \in \oplus_{x \in X^1} j_* K_1(x)$ with $\delta u = \phi(1)$ and morphisms from u to v are elements $a \in K_2^\eta$ with $\nu(a) = v - u$. Since the category $\mathcal{C}_\alpha(U)$ is non-empty, the class of the gerbe $\mathcal{C}_\alpha$ (restricted to U) in $H^2(U, \mathcal{K}_2)$ is zero. Since $\mathcal{C}_\alpha$ and $\mathcal{G}_{D,D'}$ are equivalent, so the class of $\mathcal{G}_{D,D'}$ in $H^2(U, \mathcal{K}_2)$ is also zero.

Step 3. Consider the case in which the chosen object r is given by a pair (C, g) where C is a divisor on X and g is a rational function on C. The condition $\delta(r) = \phi(1)$ says $\alpha \cap U$ is the intersection of U with the zero locus of g. Assume $g \in \mathcal{O}_C(C \cap U)$. Given any lifting $\tilde{g} \in \mathcal{O}_X(U)$ of g, let C' be its divisor on U. Notice that the codimension-two cycle $\alpha \cap U$ is the intersection of the divisors $C \cap U$ and the (principal) divisor C' in U. So we can apply the results in §4.4 to the codimension-two cycle

$$\alpha \cap U = (C \cap U).C'$$

on the smooth variety U; we obtain a $\mathcal{K}_2$-gerbe $\mathcal{G}_{C \cap U, C'}$ on U. As C' is a principal divisor, its class in $H^1(U, \mathcal{K}_1)$ is zero; also zero is the class of $\mathcal{G}_{C \cap U, C'}$ in $H^2(U, \mathcal{K}_2)$ because it is the image of the cup product of the classes of $C \cap U$ and C' in $H^1(U, \mathcal{K}_1)$.

Step 4. Let $z = \{z_{i,j,k}\} \in \mathcal{C}^2(\mathcal{K}_2)$ be a Čech 2-cocycle for $\mathcal{G}_{C \cap U, C'}$;
So $z = \partial w$ is the boundary of a Čech cochain $w = \{w_{i,j}\} \in \mathcal{C}^1(\mathcal{K}_2)$. Note that $y - z = \partial v$ for a 1-cochain v since $\mathcal{G}_{C \cap U, C'}$ and $\mathcal{G}_{D,D'}$ are equivalent as gerbes on U: both are trivial on U!
The Čech cochain $h' = \{h'_{i,j}\} \in \mathcal{C}^1(H)$ with

$$h'_{i,j} = (a_{i,j}, b_{i,j}, c_{i,j})(1, 1, -w_{i,j})(1, 1, -v_{i,j})$$

is a Čech cocycle and represents the required H-torsor $\Theta_U(r)$ on U.

Step 5. The same argument with simple modifications works for a general object of $\mathcal{C}_\alpha$. It is easy to check that Θ_U is a functor, compatible with restriction maps $V \subset U \subset X$, and that the induced morphism of gerbes is an equivalence. □

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