Bredon Homology of Partition Complexes

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Abstract. We prove that the Bredon homology or cohomology of the partition complex with fairly general coefficients is either trivial or computable in terms of constructions with the Steinberg module. The argument involves a theory of Bredon homology and cohomology approximation.

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[1]

1. Introduction

In this paper, we study the poset obtained by ordering the partitions of the set $\mathbf{n} \coloneqq \{1, \ldots, n\}$ by coarsening. The partition of $\mathbf{n}$ into one-element sets is called the *discrete*, or *trivial* partition. The partition consisting of the set $\mathbf{n}$ itself is called *indiscrete*, and partitions of $\mathbf{n}$ that are not indiscrete are called *proper*. With this terminology, let $\mathcal{P}_n$ denote the nerve of the poset of proper nontrivial partitions of $\mathbf{n}$, and let $\mathcal{P}_n^\diamond$ denote its unreduced suspension. This space, with its natural action of the symmetric group Σ_n, arises in various contexts, and in particular it plays a role in the calculus of functors. We study the Bredon homology and cohomology of $\mathcal{P}_n^\diamond$ as a Σ_n-space.

For the moment, we focus on homology. Let G be a finite group and let X be a G-space, or a simplicial G-set. We denote the Borel construction on X by $X_{\mathrm{h}G} \coloneqq (EG \times X)/G$. If X has a pointed G-action, i.e., X has a basepoint that is fixed by the G-action, we denote the reduced Borel construction by $X_{\tilde{\mathrm{h}}G} \coloneqq (EG_+ \wedge X)/G$. One type of equivariant homology for X is the ordinary twisted homology of the Borel construction $X_{\mathrm{h}G}$ with coefficients in a G-module M or, if X has a pointed action, the ordinary twisted homology of $X_{\tilde{\mathrm{h}}G}$. The

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Bredon homology of X is a finer invariant, which takes coefficients in an additive functor γ from finite G-sets to abelian groups. Our goal in this paper, in rough terms, is to sharpen the results of [1] about the mod p homology of the Borel construction on $\mathcal{P}_n^\diamond$ by proving similar results about the Bredon homology of $\mathcal{P}_n^\diamond$ with somewhat general coefficients.

To carry out this program, we need a rather detailed analysis of the fixed point spaces of various p-subgroups of Σ_n acting on $\mathcal{P}_n$. In particular, we completely classify the p-subgroups whose fixed point spaces on $\mathcal{P}_n$ are not contractible (Proposition 6.2). We also need to study equivariant approximations that induce an isomorphism on Bredon homology. Here we build on earlier work of several people, in particular Webb and the second author of the present paper [19, 20, 8, 9, 1].

We introduce some further terminology and then describe the results about $\mathcal{P}_n$ in more detail. A *Mackey functor* M for G is a pair of additive functors $(\gamma, \gamma^\natural)$ from finite G-sets to abelian groups, where γ is covariant and $\gamma^\natural$ is contravariant, and γ and $\gamma^\natural$ take common values and satisfy certain conditions (Definition 3.1). Hence γ can serve as a coefficient system for Bredon homology of G-spaces, and $\gamma^\natural$ as a coefficient system for Bredon cohomology. We denote the resulting homology and cohomology groups of a G-space X by $H_*^G(X; M)$ and $H^*_G(X; M)$. There are also reduced versions of Bredon homology and cohomology, defined for spaces with pointed G-actions. They are denoted by $\tilde{H}_*^G(X; M)$ and $\tilde{H}^*_G(X; M)$, respectively (Section 2).

The notion of a Mackey functor that is projective relative to p-subgroups is important for our main theorem, and it is reviewed in Section 3. In brief, it amounts to the following. Let M be a Mackey functor for G and let $P \subset G$ be a p-Sylow subgroup. There is a natural transformation of Mackey functors $M(G/P \times -) \longrightarrow M(-)$ induced by projection. We say M is *projective relative to p-subgroups* if this transformation is a split epimorphism.

We mentioned earlier that we use an approximation of $\mathcal{P}_n$ to compute Bredon homology and cohomology, which we refer to jointly as Bredon (co)homology, for brevity. In [1], the authors approximate the Σ_n-space $\mathcal{P}_n$ when n is a prime power by inducing up from a Tits building. We adapt that procedure for this work, as described in the next two paragraphs. Let p be a prime and suppose that $n = p^k$ for some positive integer k. Let B_k be the nerve of the poset of proper, nontrivial subgroups of the group $\Delta_k := (\mathbb{Z}/p\mathbb{Z})^k$, ordered by inclusion. The complex B_k is the Tits building for $\mathrm{GL}_k := \mathrm{GL}_k(\mathbb{F}_p)$. Since $n = p^k$, we can identify the underlying set of Δ_k, which has p^k elements, with $\mathbf{n} = \{1, \ldots, p^k\}$. We can then construct a map $B_k \longrightarrow \mathcal{P}_n$ by assigning to a subgroup $V \subseteq \Delta_k$, the partition of $\mathbf{n}$ given by the cosets of V in Δ_k. Writing $B_k^\diamond$ for the unreduced suspension of B_k, we obtain a map $B_k^\diamond \to \mathcal{P}_n^\diamond$.

The action of Δ_k on its underlying set by left translation, and the identification $\Delta_k \leftrightarrow \mathbf{n}$, allow us to identify Δ_k as a subgroup of Σ_n. The normalizer of Δ_k in Σ_n is isomorphic to the affine group $\mathrm{Aff}_k \cong \Delta_k \rtimes \mathrm{GL}_k$, which then acts on B_k (with Δ_k acting trivially). The Aff_k-equivariant map $B_k^\diamond \to \mathcal{P}_n^\diamond$ extends to a Σ_n-equivariant map $\Sigma_{n+} \wedge_{\mathrm{Aff}_k} (E\,\mathrm{GL}_{k+} \wedge B_k^\diamond) \longrightarrow \mathcal{P}_n^\diamond$, which turns out to be a good enough approximation to $\mathcal{P}_n^\diamond$ to compute Bredon homology, as stated in Theorem 1.1, below.

The following is our main theorem. If H is a subgroup of G, then $C_G(H)$ denotes the centralizer of H in G. Note that $C_G(H)$ acts on G/H by G-equivariant maps, so if M is a Mackey functor for G, there is an action of $C_G(H)$ on $M(G/H)$.

Theorem 1.1. *Fix a prime p. Let M be a Mackey functor for Σ_n that takes values in $\mathbb{Z}_{(p)}$-modules. Assume the following.*

1. *The Mackey functor M is projective relative to the collection of p-subgroups of Σ_n.*
2. *For every elementary abelian p-subgroup $D \subset \Sigma_n$ that acts freely and non-transitively on $\{1, \dots, n\}$, the kernel of the homomorphism $C_{\Sigma_n}(D) \to \pi_0 C_{\mathrm{GL}_n \mathbb{R}}(D)$ acts trivially on $M(\Sigma_n/D)$.*
3. *If p is odd and D is as above, then every odd involution in $C_{\Sigma_n}(D)$ acts on $M(\Sigma_n/D)$ by multiplication by -1.*

Then if n is not a power of p, the groups $\tilde{H}_*^{\Sigma_n}(\mathcal{P}_n^\diamond; M)$ and $\tilde{H}^*_{\Sigma_n}(\mathcal{P}_n^\diamond; M)$ vanish. If $n = p^k$, then the map

$$\Sigma_{n+} \wedge_{\mathrm{Aff}_k} (E\,\mathrm{GL}_{k\,+} \wedge B_k^\diamond) \longrightarrow \mathcal{P}_n^\diamond$$

induces an isomorphism on $\tilde{H}_*^{\Sigma_n}(-; M)$ and on $\tilde{H}^*_{\Sigma_n}(-; M)$.

The proof is found in Section 10. We will explain the assumptions and why they are needed near the end of the introduction, where we outline the proof of the main theorem.

In the case $n = p^k$, Theorem 1.1 leads to an algebraic formula for the Bredon homology and cohomology of $\mathcal{P}_n^\diamond$. Let St_k denote $\tilde{H}_{k-1}(B_k^\diamond; \mathbb{Z})$, which is the Steinberg module for $\mathrm{GL}_k(\mathbb{F}_p)$. The group $\mathrm{GL}_k(\mathbb{F}_p)$ acts on $M(\Sigma_n/\Delta_k)$. Let R denote the ring $\mathbb{Z}[\mathrm{GL}_k(\mathbb{F}_p)]$. The following is a consequence of Theorem 1.1.

Corollary 1.2. *In the setting of Theorem 1.1, suppose that $n = p^k$. Then there are isomorphisms*

$$\tilde{H}_j^{\Sigma_n}(\mathcal{P}_n^\diamond; M) \cong \begin{cases} 0 & j \neq k-1 \\ M(\Sigma_n/\Delta_k) \otimes_R \mathrm{St}_k & j = k-1 \end{cases}$$

Moreover, there are isomorphisms for all j between Bredon homology and cohomology groups: $\tilde{H}_j^{\Sigma_n}(\mathcal{P}_n^\diamond; M) \cong \tilde{H}^j_{\Sigma_n}(\mathcal{P}_n^\diamond; M)$ for all $j \geq 0$.

Example 1.3. To see an example of a Mackey functor for which Theorem 1.1 applies, recall that Σ_n acts on the one-point compactification S^n of $\mathbb{R}^n$ by permuting coordinates, and hence on the j-fold smash product $(S^n)^{\wedge j} = S^{nj}$. Fix a prime p and an integer j, where j is required to be odd if $p \neq 2$. Let E_* be a non-equivariant generalized homology theory that takes values in $\mathbb{Z}_{(p)}$-modules. There is a graded Mackey functor M_E for Σ_n that assigns to a finite Σ_n-set T the graded abelian group

$$M_E(T)_* = \tilde{E}_*\left(\Sigma^\infty T_+ \wedge S^{nj}\right)_{\tilde{\mathrm{h}}\Sigma_n}.$$

The graded constituents of M_E satisfy the hypotheses of Theorem 1.1. This is discussed in detail in Section 11, and in fact a more general statement is proved there.

Example 1.3 ties Theorem 1.1 and Corollary 1.2 to previous work. If X is a pointed Σ_n-space, then filtering X by its skeleta gives the "isotropy spectral sequences"

$$\tilde{H}_a^{\Sigma_n}\left(X;(M_E)_b\right) \Rightarrow \tilde{E}_{b+a}\left(X \wedge \Sigma^\infty S^{nj}\right)_{\tilde{\mathrm{h}}\Sigma_n}$$
$$\tilde{H}^a_{\Sigma_n}\left(X;(M_E)_b\right) \Rightarrow \tilde{E}_{b-a}\,\mathrm{map}_*\left(X, \Sigma^\infty S^{nj}\right)_{\tilde{\mathrm{h}}\Sigma_n}$$

where the second is guaranteed to converge only if X has a finite number of Σ_n-cells. These spectral sequences can be used to obtain the main theorem of [1] from Theorem 1.1. In effect, [1] calculates the abutments of these spectral sequences for $X = \mathcal{P}_n^\diamond$ and $E = H\mathbb{Z}/p$, while Theorem 1.1 and Corollary 1.2 calculate the E^2-pages in a form that implies a collapse result.
In fact, for $E_* = H_*(-;\mathbb{F}_p)$, the groups $\tilde{H}^*_{\Sigma_n}(\mathcal{P}_n^\diamond; M_E)$ were first calculated in [2] by brute force, using detailed knowledge of the homology of symmetric groups. This paper gives a new, more conceptual approach to those calculations.

Intended applications. We are particularly interested in the graded Mackey functors M_E as in Example 1.3 when E is the p-local sphere spectrum. As discussed in Section 11, this application of Theorem 1.1 leads to new proofs of some theorems of Behrens and Kuhn on the relationship between the Goodwillie tower of the identity and the symmetric power filtration of $H\mathbb{Z}$, as well as another approach to Kuhn's proof of the Whitehead Conjecture. We will pursue this in another paper.
Theorem 1.1 can also be applied to the Mackey functor

$$M(T)_* = \pi_* L_K\left(\left(E \wedge \Sigma^\infty T_+ \wedge S^{nj}\right)_{\tilde{\mathrm{h}}\Sigma_n}\right).$$

Here E is one of Morava's E-theories and L_K denotes localization with respect to the corresponding Morava K-theory. In this case Theorem 1.1 seems to offer an alternative approach to some recent calculations of Behrens and Rezk, and again we hope to discuss this elsewhere.

Connection with other work. There is a connection between this paper and the work of Grodal [11]. Grodal's paper concerns the Bredon cohomology of spaces of the form $|\mathcal{C}|$, where $\mathcal{C}$ is a poset of p-subgroups of a group G. Our space $\mathcal{P}_n$ is of a similar nature: it is Σ_n-equivariantly equivalent to the nerve of the poset of nontrivial non-transitive subgroups (not just p-subgroups) of Σ_n. The (generalized) Steinberg module also plays a central role in [11].

We devote the remainder of this introduction to outlining the proof of Theorem 1.1. Let G be a finite group and $\mathcal{C}$ a collection of subgroups of G (i.e., a set of subgroups closed under conjugation). Given a G-space X, one can associate with it a G-space $X_{\mathcal{C}}$, together with a natural map $X_{\mathcal{C}} \to X$, called the *$\mathcal{C}$-approximation to X*. The approximation is characterized up to homotopy by the fact that $X_{\mathcal{C}}$ has isotropy only in $\mathcal{C}$ and $X_{\mathcal{C}} \to X$ induces an equivalence on K-fixed points for $K \in \mathcal{C}$. The construction of $\mathcal{C}$-approximations is reviewed in Section 4. A typical example of interest is the collection of nontrivial p-subgroups of a finite group G, which we denote $\mathcal{S}_p(G)$, following Quillen [16]. The relevance of $\mathcal{C}$-approximation is that $X_{\mathcal{C}} \to X$ may induce an isomorphism on Bredon homology without being an equivalence of G-spaces. To state a first result along these lines, recall that a *family* of subgroups of G is a collection that is closed under taking subgroups as well as under conjugation. The following is a variant of the main result of Webb [19].

Proposition 4.6. *Let $\mathcal{F}$ be a family of subgroups of G, let X be a G-space, and let M be a Mackey functor for G that is projective relative to $\mathcal{F}$. Then $X_{\mathcal{F}} \to X$ induces an isomorphism on $H_*^G(-;M)$ and $H_G^*(-;M)$.*

If we add the trivial subgroup to the collection $\mathcal{S}_p(G)$, we obtain the family of all p-subgroups of G, which we denote $\overline{\mathcal{S}}_p(G)$. (We suppress the group and write $\mathcal{S}_p$ or $\overline{\mathcal{S}}_p$ if the group is clear from context.) We will apply Proposition 4.6 with $X = \mathcal{P}_n^\diamond$ and $\mathcal{F} = \overline{\mathcal{S}}_p(\Sigma_n)$, the family of all p-subgroups of Σ_n. In this way we obtain a starting approximation $(\mathcal{P}_n^\diamond)_{\overline{\mathcal{S}}_p(\Sigma_n)} \to \mathcal{P}_n^\diamond$ that induces an isomorphism on Bredon homology and cohomology.

To analyze the approximation $(\mathcal{P}_n^\diamond)_{\overline{\mathcal{S}}_p(\Sigma_n)} \to \mathcal{P}_n^\diamond$, we need to reduce the size of the approximating collection. Our key criterion for eliminating (or "discarding") elements of a collection $\mathcal{C}$ without affecting the Bredon homology of the $\mathcal{C}$-approximation is Lemma 5.1 below. (The lemma will be promoted from a statement about $*$ to a statement about a G-space X in Section 5.) We need a little more terminology to explain the criterion. If W is a finite group, then W acts on $\mathcal{S}_p(W)$ (by conjugation), and there is a natural map of spaces

$$|\mathcal{S}_p(W)|_{hW} \to *_{hW} = BW\,. \tag{1.4}$$

If M is a W-module, $\mathcal{S}_p(W)$ is called *homologically (resp. cohomologically) M-ample* if (1.4) induces an isomorphism on ordinary twisted homology (resp. cohomology) with coefficients in M.

For notation in the following lemma, suppose that D is a subgroup of G, let $N_G(D)$ denote the normalizer of D in G, and let $W_G(D) = N_G(D)/D$ be the Weyl group of D in G. Notice that $W_G(D)$ acts on G/D by G-equivariant maps.

Lemma 5.1. *Let μ be a homological (or cohomological) coefficient system for G. Let D be a subgroup of G, and let $\mathcal{D}$ denote the set of conjugates of D in G. Let $\mathcal{C}$ be a collection of p-subgroups of G for which D is a minimal element, and such that $\mathcal{C}$ contains all p-supergroups of D. Let $W = W_G(D)$. Then $(*)_{\mathcal{C}\setminus\mathcal{D}} \to (*)_{\mathcal{C}}$ gives isomorphisms on Bredon homology (resp. cohomology) with coefficients in μ if and only if $\mathcal{S}_p(W)$ is homologically (resp. cohomologically) $\mu(G/D)$-ample.*

To apply Lemma 5.1, a criterion for ampleness is needed. It is known that if M is a $\mathbb{Z}_{(p)}$-module, and W has an element of order p that acts trivially on M, then $\mathcal{S}_p(W)$ is M-ample (Proposition 5.5). Typically, such elements come from the centralizer of D, which is why condition (2) is present in Theorem 1.1.

If we start from a Bredon (co)homology isomorphism $(\mathcal{P}_n^\diamond)_{\overline{\mathcal{S}}_p(\Sigma_n)} \to \mathcal{P}_n^\diamond$, we would like to know how many subgroups must be eliminated from $\overline{\mathcal{S}}_p(\Sigma_n)$ before we obtain an identifiable calculation of the Bredon (co)homology of $\mathcal{P}_n^\diamond$. In particular, how many subgroups must be eliminated before we can conclude that $\mathcal{P}_n^\diamond$ has the Bredon (co)homology of a point? Suppose that X is a G-space and that $\mathcal{C}$ is a collection of subgroups of G such that $X_{\mathcal{C}} \to X$ is an isomorphism on Bredon (co)homology. If it happens that $X^H \simeq *$ for all $H \in \mathcal{C}$, then $X_{\mathcal{C}}$ has the same Bredon (co)homology as a point. Hence so does X, and $X^\diamond$ has trivial reduced Bredon (co)homology.

The preceding paragraph suggests that if we start from a Bredon (co)homology isomorphism $(\mathcal{P}_n^\diamond)_{\overline{\mathcal{S}}_p(\Sigma_n)} \to \mathcal{P}_n^\diamond$, it would be nice to discard from $\overline{\mathcal{S}}_p(\Sigma_n)$ the subgroups whose fixed point spaces are *not* contractible. We hope that there are not too many of them, and that they can be discarded from $\overline{\mathcal{S}}_p(\Sigma_n)$ without damaging the starting Bredon (co)homology isomorphism

We call a subgroup $H \subseteq \Sigma_n$ *problematic* if $(\mathcal{P}_n^\diamond)^H$ is not contractible.

Proposition 6.2. *Let H be a problematic p-subgroup of Σ_n. Then H is an elementary abelian p-group, and the action of H on $\mathbf{n}$ is free.*

This proposition tells us that in fact there are very few problematic subgroups. The proof of Theorem 1.1 then goes by using Lemma 5.1 to eliminate these few subgroups from the collection $\overline{\mathcal{S}}_p(\Sigma_n)$. We can usually establish the ampleness required in the hypothesis of Lemma 5.1 by using centralizing elements, as discussed just after the statement of the lemma, above. However, it turns out that in a few cases, appropriate centralizing elements do not exist. These are cases in which an isotropy group of $\mathcal{P}_n$ contains a *p-centric* problematic subgroup. Nonetheless, it turns out that the ampleness hypothesis holds in these cases, with one exception, because all the relevant homology and cohomology groups vanish. This is where we need condition (3) of the theorem.

In the end, the only problematic subgroups that cannot be eliminated using one of the methods we have described are elementary abelian p-subgroups of Σ_n that act transitively on $\mathbf{n}$. This occurs only when $n = p^k$, and in this case the Tits building B_k comes up because it is the fixed point space of the elementary abelian p-group Δ_k of Σ_{p^k} acting on $\mathcal{P}_n$.

Organization.

In Sections 2 and 3, we give background on Bredon homology and cohomology and Mackey functors, and we state the key properties that we use and prove some basic results. Section 4 reviews approximation theory from [1] and proves Proposition 4.6, the initial approximation result for Bredon homology and cohomology. Section 5 discusses how to discard subgroups from an approximating collection.

Section 6 shows that most p-subgroups of Σ_n have contractible fixed point spaces and Proposition 6.2 identifies those that may not ("problematic" subgroups). Section 7 collects some elementary results about the isotropy groups of $\mathcal{P}_n$. Then Section 8 studies centralizers of problematic subgroups inside isotropy groups of $\mathcal{P}_n$, with a view to acquiring the algebraic input for Proposition 5.5.

Section 9 establishes that the ampleness hypothesis needed to use Proposition 5.4 holds in the case of the coefficients in Theorem 1.1. This gives the data needed to prove Theorem 1.1 and Corollary 1.2 in Section 10. Finally, Section 11 looks at the coefficient functors of Example 1.3.

Notation and Terminology.

Throughout the paper, G is a finite group and p is a fixed prime. We use "space" to mean a simplicial set, and we often do not distinguish between a category and its nerve, trusting to context to indicate which is under discussion. We fix a model for a free, contractible G-space EG, and given a G-space X, we write X_{hG} for the unreduced Borel construction $(EG \times X)/G$. If X has a basepoint and the basepoint is fixed by the G-action, we write $X_{\tilde{h}G}$ for the reduced Borel construction $(EG_+ \wedge X)/G$.

If H is a subgroup of a group G, we write $C_G(H)$ for the centralizer of H in G, we write $N_G(H)$ for the normalizer of H in G, and we write $W_G(H) = N_G(H)/H$ for the Weyl group of H in G. Following Quillen [16], we denote the poset of non-identity p-subgroups of G by $\mathcal{S}_p(G)$, or just $\mathcal{S}_p$ if the group is clear from context. The poset of all p-subgroups of G, including the trivial group, is denoted by $\overline{\mathcal{S}}_p(G)$ or $\overline{\mathcal{S}}_p$. If X is a G-space, we write $\mathrm{Iso}(X)$ to denote the collection of subgroups of G that appear as isotropy groups of X.

We regard a partition λ of $\mathbf{n}$ as corresponding to equivalence classes of an equivalence relation, where $x \sim_\lambda y$ means that x and y are in the same component of λ. (We write simply $x \sim y$ if the partition is clear from context.) We write $\mathrm{cl}(\lambda)$ to denote the set of components, or equivalence classes, of a partition λ.

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2. Bredon homology and cohomology

Let G be a finite group. In this section, we give general background on G-spaces and on their Bredon homology and cohomology [6].

We work simplicially. Thus, by a G-space X we mean a simplicial set with a G-action. A G-map $f : X \to Y$ is called a *G-equivalence* if it induces an equivalence $X^K \to Y^K$ of fixed point spaces for each subgroup K of G. The geometric realization and the singular set functors preserve fixed points. It follows that if $f\colon X \to Y$ is a G-equivalence of simplicial sets, then the geometric realization of f is an equivalence in the category of G-topological spaces.

Given a G-space X, let $\mathrm{Iso}(X)$ denote the set of all subgroups of G that appear as isotropy subgroups of simplices of X. The following lemma gives an economical criterion for recognizing G-equivalences. A proof can be found in [8, 4.1], but it is older than this.

Lemma 2.1. *If $f : X \to Y$ is a map of G-spaces that induces equivalences $X^K \to Y^K$ for each $K \in \mathrm{Iso}(X) \cup \mathrm{Iso}(Y)$, then f is a G-equivalence.*

We next describe the Bredon chain and cochain complexes; to minimize redundancy, we handle both simultaneously, with the terms for cohomology in parentheses. Let μ be an additive, covariant (resp. contravariant) functor from finite G-sets to abelian groups, i.e., a functor taking coproducts of G-sets to sums (resp. products) of abelian groups. Such a functor will be called a homological (resp. cohomological) coefficient system for G. The functor μ can be extended to all G-sets by the formula

$$\mu(T) := \operatorname*{colim}_{T_\alpha} \mu\left(T_\alpha\right)$$

(resp. $\mu(T) := \lim_{T_\alpha} \mu\left(T_\alpha\right)$), where T_α ranges over the poset of finite G-subsets of T. The Bredon chains (resp. cochains) on X with coefficients in μ are obtained by applying μ degreewise to X to obtain a simplicial (resp. cosimplicial) abelian group, and then applying Dold-Kan's normalized chains functor, to obtain a chain (resp. cochain) complex. The Bredon homology (resp. cohomology) of X with coefficients in μ, denoted $H_*^G\left(X;\mu\right)$ (resp. $H_G^*\left(X;\mu\right)$), is the homology (resp. cohomology) of this chain (resp. cochain) complex.
Let $\mathcal{O}(G)$ be the orbit category of G, namely the category of transitive G-sets and G-equivariant maps. If μ is any functor from $\mathcal{O}(G)$ to abelian groups then it can be extended to an additive functor on all finite G-sets in a unique way. Thus, one may define Bredon homology (resp. cohomology) with coefficients in an arbitrary functor from $\mathcal{O}(G)$ to abelian groups.

Remark 2.2. One may view an additive covariant functor μ from G-sets to abelian groups as a functor from $\mathcal{O}(G)$ to chain complexes that takes values in complexes concentrated in degree zero. Then the Bredon chain functor is the homotopy left Kan extension of γ from $\mathcal{O}(G)$ to the category of simplicial G-sets. In the contravariant case, the Bredon cochain functor is the homotopy right Kan extension of μ from the opposite category of $\mathcal{O}(G)$ to the opposite category of simplicial G-sets.

The following example will come up later in the paper, in the proofs of Lemma 4.4 and, implicitly, Lemma 5.1.

Example 2.3. Let G be a group, and $D \subset G$ a subgroup. Note that the Weyl group $W_G(D)$ acts on G/D by G-equivariant maps. If X is a space with an action of $W_G(D)$, then $G \times_{N_G(D)} (X \times EW_G(D))$ is a space with an action of G. Let μ be a (homological or cohomological) coefficient system for G. There is an isomorphism

$$H_*^G\left(G \times_{N_G(D)} \left(X \times EW_G(D)\right); \mu\right) \cong H_*\left(X_{hW_G(D)}; \mu(G/D)\right) \tag{2.4}$$

or, in the cohomological case

$$H^*_G\left(G\times_{N_G(D)}(X\times EW_G(D));\mu\right)\cong H^*\left(X_{hW_G(D)};\mu(G/D)\right). \tag{2.5}$$

Here the groups on the left are Bredon homology or cohomology groups, while the groups on the right are ordinary homology or cohomology with twisted coefficients in the W-module $\mu(G/D)$. To see where these isomorphisms come from, let μ be a homological coefficient system for G, and let S be a set with a free W-action. There is an isomorphism $\mu\left(G\times_N S\right)\cong\mu(G/D)\otimes_{\mathbb{Z}[W]}\mathbb{Z}[S]$, which is natural in S. If μ is cohomological, then there is a natural isomorphism $\mu\left(G\times_N S\right)\cong\operatorname{Hom}_{\mathbb{Z}[W]}\left(\mathbb{Z}[S],\mu(G/D)\right)$. It follows that there is an isomorphism of simplicial abelian groups

$$\mu\left(G\times_N(X\times EW)\right)\cong\mu(G/D)\otimes_{\mathbb{Z}[W]}\mathbb{Z}[X\times EW].$$

This isomorphism implies the isomorphism in (2.4). The cohomological version of (2.5) is proved similarly.

Homotopy properties. Bredon homology has good formal properties. Proofs of the homological cases of the following two well-known lemmas are given in [9, 4.8] and [9, 4.11]. The cohomological versions can be proved similarly. The first one also follows from Remark 2.2.

Lemma 2.6. *If $f:X\to Y$ is a G-equivalence, then f induces isomorphisms on Bredon homology and cohomology (with any coefficients).*

Lemma 2.7. *A homotopy pushout square*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

of G-spaces gives long exact sequences in Bredon homology and cohomology (with any coefficients).

Remark. A homotopy pushout square of G-spaces is a square that, upon taking fixed points $(-)^K$ for any subgroup $K\subset G$, becomes a homotopy pushout square of spaces.

Restriction of coefficients. We recall that Bredon homology and cohomology have restriction of coefficients. If μ is a covariant or contravariant functor and K is a subgroup of G, we can define a coefficient functor $\mu|_K$ from K-sets to abelian groups given by

$$\mu|_K(S)=\mu(G\times_K S).$$

Then for any K-space Y, depending on the variance of μ, we have

$$\begin{aligned} H^K_*\left(Y;\mu|_K\right)&=H^G_*\left(G\times_K Y;\mu\right)\\ H^*_K\left(Y;\mu|_K\right)&=H^*_G\left(G\times_K Y;\mu\right). \end{aligned} \tag{2.8}$$

Reduced Bredon homology. For a pointed G-space $(X,*)$ and a covariant coefficient system μ, there is a split monomorphism from the Bredon chains on $*$ to the Bredon chains on X. The homology groups of the quotient complex are the reduced Bredon homology groups of X with coefficients in μ. They are denoted by $\tilde{H}^G_*(X;\mu)$. Similarly, if μ is a contravariant coefficient system, there is a split epimorphism from the cochains on X to the cochains on $*$, and the cohomology groups of the kernel are the reduced Bredon cohomology groups of X, denoted $\tilde{H}_G^*(X;\mu)$.

There is an isomorphism of Bredon homology groups

$$H^G_*(X;\mu) \cong \tilde{H}^G_*(X;\mu) \oplus H^G_*(*;\mu)$$

and a similar one for cohomology groups. There are analogues of Lemmas 2.6 and 2.7 for reduced Bredon homology and cohomology. It follows that if X is equivariantly contractible, then the reduced Bredon homology and cohomology groups of X vanish with any coefficients.

3. Mackey functors

To obtain our results, we need to work with Bredon homology theories that behave well with respect to approximation by p-subgroups. It turns out that the key property required is the presence of transfers for finite covers of G-spaces. To obtain well-behaved transfers, we use coefficient functors that extend to Mackey functors. The first part of this section collects background on Mackey functors; the second part discusses projectivity relative to a collection of subgroups, a key hypothesis for Theorem 1.1.

There are several equivalent definitions of Mackey functors in the literature. We will follow the treatments of Dress [7] and Webb [19, 20].

Definition 3.1. A *Mackey functor (for G)* is a pair of additive functors $M = (\gamma, \gamma^\natural)$ from the category of finite G-sets to abelian groups, satisfying the following conditions.

(1) The functor γ is covariant and the functor $\gamma^\natural$ is contravariant.
(2) The functors γ and $\gamma^\natural$ agree on objects. Thus for every finite G-set S, we write $M(S) := \gamma(S) = \gamma^\natural(S)$.
(3) If the diagram on the left below is a pullback diagram of G-sets, then the diagram on the right commutes.

$$\begin{array}{ccc} X & \xrightarrow{a} & U \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{b} & V \end{array} \qquad\qquad \begin{array}{ccc} M(X) & \xrightarrow{\gamma(a)} & M(U) \\ {\scriptstyle \gamma^\natural(f)}\uparrow & & \uparrow{\scriptstyle \gamma^\natural(g)} \\ M(Y) & \xrightarrow{\gamma(b)} & M(V) \end{array}$$

Because a Mackey functor has both a covariant and a contravariant part, it provides coefficient systems for both Bredon homology and Bredon cohomology. We denote the resulting homology and cohomology groups by $H^G_*(-;M)$ and $H_G^*(-;M)$.

The following remark concerns extending a Mackey functor from finite G-sets to arbitrary G-sets. Although we will only be interested in Bredon homology of simplicial sets of finite type, we include the remark for completeness.

Remark 3.2. There are two ways to extend a Mackey functor $M = (\gamma, \gamma^\natural)$ from finite set to arbitrary sets. The first is to define

$$\gamma^\natural(T) = \gamma(T) := \operatorname*{colim}_{T_\alpha} \gamma(T_\alpha)$$

where T_α ranges over finite subsets of T. With this definition, it is clear that γ is functorial with respect to all G-maps between G-sets. The contravariant functor $\gamma^\natural$ is also functorial, but with respect to *finite* covers of G-sets. Indeed, suppose $f\colon X \to Y$ is a finite cover of G-sets. Then $f^{-1}(-)$ defines a poset map from finite subsets of Y to finite subsets of X. One uses this to define a map $\gamma^\natural(Y) \to \gamma^\natural(X)$. With this definition, condition (3) of Definition 3.1 holds for an arbitrary square diagram of G-sets, provided f and g are finite covers. The other way to extend M is by the formula

$$\gamma(T) = \gamma^\natural(T) := \lim_{T_\alpha} \gamma^\natural(T_\alpha).$$

With this definition, $\gamma^\natural$ remains contravariantly functorial with respect to arbitrary G-maps, but γ is only functorial with respect to finite covers. Condition (3) of Definition 3.1 holds for an arbitrary square diagram of G-sets, provided a and b (rather than f and g) are finite covers.

For the remainder of this section we discuss projectivity of a Mackey functor with respect to a collection of subgroups. Let $M = (\gamma, \gamma^\natural)$ be a Mackey functor for G and let Z be a finite G-set. Then one may define a new Mackey functor M_Z by the formula $M_Z(T) = M(Z \times T)$. Moreover, if $f\colon Z \to Y$ is an equivariant map of finite G-sets, then γ and $\gamma^\natural$ induce natural transformations of Mackey functors $M_Z \to M_Y$ and $M_Y \to M_Z$, respectively. In particular, taking $Y = *$ we obtain natural transformations of Mackey functors $\theta_Z\colon M_Z \to M$ and $\theta^Z\colon M \to M_Z$.

Definition 3.3. A Mackey functor M is *projective relative to* Z if θ_Z is a split surjection of Mackey functors.

Recall that a *collection* is a (necessarily finite) set of subgroups of G that is closed under conjugation.

Definition 3.4. Let G be a finite group, let $\mathcal{C}$ be a collection of subgroups of G, and let M be a Mackey functor for G. We say that M is *projective relative to* $\mathcal{C}$ if M is projective relative to $Z = \coprod G/H$, where H ranges over a set of representatives of conjugacy classes of elements of $\mathcal{C}$.

The following routine lemma and corollary allow easier verification that a Mackey functor is projective relative to a collection $\mathcal{C}$, in particular relative to $\overline{\mathcal{S}}_p(G)$, the collection of all p-subgroups of G. (See Lemma 3.2 of [20].)

Lemma 3.5. *A Mackey functor is projective relative to* $\mathcal{C}$ *if and only if it is projective relative to the collection of maximal elements of* $\mathcal{C}$.

Corollary 3.6. *Let G be a finite group and let P be a p-Sylow subgroup of G. A Mackey functor is projective relative to $\overline{\mathcal{S}}_p(G)$ if and only if the natural map $\theta_{G/P}\colon M_{G/P} \to M$ is a split surjection of Mackey functors.*

The following definition and lemma allow us to recognize Mackey functors that are projective relative to $\overline{\mathcal{S}}_p(G)$. In practice, all of our examples satisfy this condition.

Definition 3.7. Suppose that M is a Mackey functor for G. We say that M *has the p-transfer property* if for every G-set Z whose cardinality is prime to p, the composition

$$\theta_Z \circ \theta^Z\colon M \to M_Z \to M$$

is an isomorphism from M to itself.

Lemma 3.8. *If a Mackey functor M has the p-transfer property, then it takes values in $\mathbb{Z}_{(p)}$-modules, and it is projective relative to $\overline{\mathcal{S}}_p$.*

Proof. For the first assertion, let Z be a set with trivial G-action. The composed map $\theta_Z \circ \theta^Z\colon M \longrightarrow M_Z \longrightarrow M$ is multiplication by the cardinality of Z. If we assume that the composition is an isomorphism for every Z of cardinality prime to p, it means exactly that M takes values in $\mathbb{Z}_{(p)}$-modules.
For the second assertion, take $Z = G/P$, where P is a p-Sylow subgroup of G. Then $\theta_{G/P} \circ \theta^{G/P}\colon M \longrightarrow M_{G/P} \longrightarrow M$ is an isomorphism of Mackey functors, which implies that $\theta_{G/P}$ is a split surjection of Mackey functors. Hence M is projective relative to p-groups. $\square$

4. Approximations

In this section we develop general tools for approximating a G-space X by other G-spaces whose Bredon homology or cohomology may be easier to calculate. First, we recall the notion of $\mathcal{C}$-approximation used in [1] and give a sufficient condition for this approximation to induce an isomorphism on Bredon homology and cohomology (Definition 4.1, Lemma 4.2). Second, we give an explicit model for an approximation that involves only one conjugacy class of subgroups (Lemma 4.4). Lastly, we observe that if M is a Mackey functor that is projective relative to a *family* $\mathcal{F}$, then $\mathcal{F}$-approximation induces an isomorphism on Bredon homology and cohomology with coefficients in M. This allows us to use the family of all p-subgroups of G as a canonical jumping-off point for approximations. In Section 5, we build on this beginning and set up an inductive process to reduce the size of the controlling collection from the family of all p-subgroups to a manageable collection, without changing the Bredon homology or cohomology.
Recall that for a G-space X, we write $\mathrm{Iso}(X)$ for the collection of subgroups of G that appear as isotropy groups of X. Let $\mathcal{C}$ be a collection of subgroups of G. A G-space X is said to have *$\mathcal{C}$-isotropy* if $\mathrm{Iso}(X) \subseteq \mathcal{C}$. A G-map $X \to Y$ is a *$\mathcal{C}$-equivalence* if $f^K : X^K \to Y^K$ is an equivalence for each $K \in \mathcal{C}$.

Definition 4.1. Given a G-space X, a *$\mathcal{C}$-approximation* to X is a $\mathcal{C}$-equivalence $X_{\mathcal{C}} \to X$ such that $X_{\mathcal{C}}$ has $\mathcal{C}$-isotropy.

We review the construction of a functorial $\mathcal{C}$-approximation, which goes back to Elmendorf [10]. Let $\mathcal{O}_{\mathcal{C}}$ be the full subcategory of transitive G-sets and G-equivariant maps whose objects have isotropy groups only in $\mathcal{C}$, and let i denote the inclusion functor from $\mathcal{O}_{\mathcal{C}}$ to G-spaces. The $\mathcal{C}$-approximation functor is the endofunctor of G-spaces obtained by taking the identity functor, restricting it to $\mathcal{O}_{\mathcal{C}}$, and then taking homotopy left Kan extension back to the category of G-spaces. In more concrete terms, $X_{\mathcal{C}}$ can be constructed as the homotopy coend $i \otimes_{\mathcal{O}_{\mathcal{C}}} \operatorname{map}_G(-, X)$. Here $\operatorname{map}_G(-, X)$ is the contravariant functor from $\mathcal{O}_{\mathcal{C}}$ to spaces given by $S \mapsto \operatorname{map}_G(S, X)$. For more detail, see [8, 4.8] and [1, Section 3].
It follows from the construction of $X_{\mathcal{C}}$ that if X has finite type, then $X_{\mathcal{C}}$ does too. The functoriality of the construction, together with Lemma 2.1, implies that $\mathcal{C}$-approximations are unique up to a canonical zigzag of G-equivalences. Further, $\mathcal{C}$-approximation commutes with homotopy pushouts of G-spaces. The Mayer-Vietoris property (Lemma 2.7) thus implies that in order to determine whether $X_{\mathcal{C}} \to X$ induces an isomorphism on Bredon homology or cohomology, it is enough to check this condition for the orbits used in building X.

Lemma 4.2. [1, 3.2 and 3.3] *Suppose that X is a G-space. If, for all $K \in \operatorname{Iso}(X)$, the map $(G/K)_{\mathcal{C}} \to G/K$ gives an isomorphism on Bredon homology with coefficients in γ (resp. cohomology with coefficients in $\gamma^\natural$), then $X_{\mathcal{C}} \to X$ gives such an isomorphism as well.*

Since $G/K \cong G \times_K *$, Lemma 4.2 involves approximating a space induced up from a subgroup. There is a general lemma available for this purpose. If K is a subgroup of G, consider the collection

$$\mathcal{C} \downarrow K := \{H \mid H \in \mathcal{C} \text{ and } H \subseteq K\}.$$

The following elementary lemma, used with $Y = *$, is helpful in applying Lemma 4.2.

Lemma 4.3. [1, 2.12] *Let K be a subgroup of G and let Y be a K-space. Then there is a canonical G-equivalence $G \times_K (Y_{\mathcal{C}\downarrow K}) \simeq (G \times_K Y)_{\mathcal{C}}$.*

Next, we give an explicit description of the approximation for a collection consisting of a single subgroup of G together with its conjugates. We also record the Bredon homology and cohomology of the approximation. Let D be a subgroup of G, and let $\mathcal{D}$ be the collection consisting of all conjugates of D in G. Recall that the normalizer $N_G(D)$ acts on the fixed point space X^D, and also on the free contractible $W_G(D)$-space $EW_G(D)$

Lemma 4.4. *With the above notation, let $F = X^D$. Then for any G-space X, the natural map*

$$Y := G \times_{N_G(D)} (EW_G(D) \times F) \to X$$

is a $\mathcal{D}$-approximation $X_{\mathcal{D}} \to X$. For any Bredon coefficient systems γ and $\gamma^{\natural}$, there are natural isomorphisms

$$\begin{aligned} H_*^G(X_{\mathcal{D}};\gamma) &\cong H_*(F_{hW_G(D)};\gamma(G/D)) \\ H^*_G\left(X_{\mathcal{D}};\gamma^{\natural}\right) &\cong H^*(F_{hW_G(D)};\gamma^{\natural}(G/D)) \end{aligned} \tag{4.5}$$

The (co)homology groups on the right side of (4.5) are the ordinary local coefficient (co)homology groups associated to the natural action of W on the coefficients through the action of W on G/D.

Proof of Lemma 4.4. The verification that $Y \simeq X_{\mathcal{D}}$ proceeds by checking the two conditions that characterize $X_{\mathcal{D}}$. First, the orbit types: all isotropy groups of Y are indeed conjugate to D. Second, the fixed point spaces: Y^D is $EW_G(D) \times F$, which is homotopy equivalent to X^D, as required of $X_{\mathcal{D}}$. The homology and cohomology calculations then follow by Example 2.3. □

To close this section, we establish a canonical starting point for approximation calculations when the Bredon coefficient system is projective relative to a family. The case $X = *$ of the following proposition can be read off as a special case of [19, Theorem A]. The case when $\mathcal{F}$ is the family of all p-subgroups of G is essentially [9, 6.4].

Proposition 4.6. *Let $\mathcal{F}$ be a family of subgroups of G, let X be a G-space, and let M be a Mackey functor for G that is projective relative to $\mathcal{F}$. Then $X_{\mathcal{F}} \to X$ induces an isomorphism on $H_*^G(-;M)$ and $H^*_G(-;M)$.*

Proof. We first assert that if $H \in \mathcal{F}$, then

$$G/H \times X_{\mathcal{F}} \to G/H \times X$$

is actually a G-equivalence by Lemma 2.1. This is because if K is an isotropy group of either the source or the target, then K is conjugate to a subgroup of H. Hence $K \in \mathcal{F}$, and $(X_{\mathcal{F}})^K \to X^K$ is a homotopy equivalence by definition of $X_{\mathcal{F}}$.

Let $Z = \coprod G/H$, where H ranges over a set of representatives of conjugacy classes in $\mathcal{F}$. Since our Mackey functor $M = (\gamma, \gamma^{\natural})$ is projective relative to Z, the homomorphisms induced on Bredon chains and cochains by the map $X_{\mathcal{F}} \to X$ are retracts of the corresponding homomorphisms induced by the map $Z \times X_{\mathcal{F}} \to Z \times X$. However, the latter map is a G-equivalence, and so induces isomorphisms on Bredon homology and cohomology. □

5. Approximations controlled by smaller collections

Recall that $\overline{\mathcal{S}}_p(G)$ is the family of all p-subgroups of a fixed group G. Let X be a G-space. It follows from Proposition 4.6 that if M is a Mackey functor projective relative to $\overline{\mathcal{S}}_p(G)$, then the approximation map $X_{\overline{\mathcal{S}}_p(G)} \to X$ induces isomorphisms on Bredon (co)homology with coefficients in M.

For a subcollection $\mathcal{C} \subseteq \overline{\mathcal{S}}_p(G)$, there is a natural factoring $X_{\mathcal{C}} \to X_{\overline{\mathcal{S}}_p(G)} \to X$. We can ask if $X_{\mathcal{C}} \to X_{\overline{\mathcal{S}}_p(G)}$ induces isomorphisms on Bredon homology and cohomology with coefficients in M, in which case the map $X_{\mathcal{C}} \to X$ induces such isomorphisms as well. Our main result along these lines is Proposition 5.4 below, which is an essential ingredient in the proof of Theorem 1.1.

Let W be a group. Recall that $\mathcal{S}_p(W)$ is the poset of non-identity p-subgroups of W. Let $|\mathcal{S}_p(W)|$ be the nerve of this poset. Let M be a W-module. Recall that $\mathcal{S}_p(W)$ is said to be homologically M-ample if the map $|\mathcal{S}_p(W)| \to *$ induces an isomorphism on homology with twisted coefficients in M:

$$H_*(|\mathcal{S}_p(W)|_{hW}; M) \xrightarrow{\cong} H_*(BW; M).$$

Similarly, we say that $\mathcal{S}_p(W)$ is cohomologically M-ample if we have an isomorphism on cohomology with twisted coefficients in M:

$$H^*(|\mathcal{S}_p(W)|_{hW}; M) \xleftarrow{\cong} H^*(BW; M).$$

The following lemma gives a useful criterion for when a group can be removed from a collection without affecting Bredon homology. It is similar to [9, Proposition 9.3]. Note that for a subgroup D of G, the Weyl group $W_G(D)$ acts by G-maps on the set G/D, so $\gamma(G/D)$ is a $W_G(D)$-module.

Lemma 5.1. *Let μ be a homological (or cohomological) coefficient system for G. Let D be a subgroup of G, and let $\mathcal{D}$ denote the set of conjugates of D in G. Let $\mathcal{C}$ be a collection of p-subgroups of G for which D is a minimal element, and such that $\mathcal{C}$ contains all p-supergroups of D. Let $W = W_G(D)$. Then $(*)_{\mathcal{C}\setminus\mathcal{D}} \to (*)_{\mathcal{C}}$ gives isomorphisms on Bredon homology (resp. cohomology) with coefficients in μ if and only if $\mathcal{S}_p(W)$ is homologically (resp. cohomologically) $\mu(G/D)$-ample.*

The proof will appear after an auxiliary lemma. If $\mathcal{C}$ is a collection of subgroups of G, a subcollection $\mathcal{D} \subset \mathcal{C}$ is *initial* if whenever $D \in \mathcal{D}$ and $C \in \mathcal{C}$ with $C \subset D$, then $C \in \mathcal{D}$. Note that the union of two collections is a collection.

Lemma 5.2. *Let $\mathcal{C}$ be a collection and let $\mathcal{D} \subset \mathcal{C}$ be an initial subcollection. Let $Y \to Z$ be a map of G-spaces that induces an equivalence $Y_{\mathcal{C}\setminus\mathcal{D}} \to Z_{\mathcal{C}\setminus\mathcal{D}}$. Then there is a homotopy pushout diagram of G-spaces*

$$\begin{array}{ccc} Y_{\mathcal{D}} & \longrightarrow & Y_{\mathcal{C}} \\ \downarrow & & \downarrow \\ Z_{\mathcal{D}} & \longrightarrow & Z_{\mathcal{C}} \end{array}.$$

Proof. It is only necessary to check that for each subgroup $H \in \mathcal{C}$, the indicated diagram becomes an ordinary homotopy pushout diagram upon taking H-fixed points, which is clear. (Note that the spaces on the left have empty H-fixed sets for $H \in \mathcal{C} \setminus \mathcal{D}$, because $\mathcal{D}$ is initial.) ☐

Proof of Lemma 5.1. The diagram of Lemma 5.2 with $Y = (*)_{\mathcal{C}\setminus\mathcal{D}}$ and $Z = *$ gives the homotopy pushout diagram

$$\begin{tikzcd} \left(*_{\mathcal{C}\setminus\mathcal{D}}\right)_{\mathcal{D}} \arrow[r] \arrow[d] & \left(*_{\mathcal{C}\setminus\mathcal{D}}\right)_{\mathcal{C}} \arrow[d] \\ (*)_{\mathcal{D}} \arrow[r] & (*)_{\mathcal{C}} \end{tikzcd} .$$

In the upper right corner, $\left(*_{\mathcal{C}\setminus\mathcal{D}}\right)_{\mathcal{C}}$ is G-equivalent to $(*)_{\mathcal{C}\setminus\mathcal{D}}$ (Lemma 2.1). Further, $\mathcal{D}$ consists of just one conjugacy class, allowing us to use Lemma 4.4 to obtain explicit formulas for $\left(*_{\mathcal{C}\setminus\mathcal{D}}\right)_{\mathcal{D}}$ and $(*)_{\mathcal{D}}$. The result is the homotopy pushout diagram of G-spaces

$$\begin{tikzcd} G \times_N \left(EW \times \left(*_{\mathcal{C}\setminus\mathcal{D}}\right)^D\right) \arrow[r] \arrow[d] & (*)_{\mathcal{C}\setminus\mathcal{D}} \arrow[d] \\ G \times_N EW \arrow[r] & (*)_{\mathcal{C}} \end{tikzcd} .$$

In view of the Mayer-Vietoris property (Lemma 2.7), the right vertical map is an isomorphism on Bredon homology or cohomology if and only if the left map is. By the second part of Lemma 4.4, the Bredon (co)homology of spaces on the left reduces to ordinary (co)homology with twisted coefficients. The only remaining point to note is that the fixed point set $((*)_{\mathcal{C}\setminus\mathcal{D}})^D$ is homotopy equivalent to $|\mathcal{S}_p(W)|$ via a W-equivariant map. This is proved in the third paragraph of [8, Pf. of 8.3]. □

Corollary 5.3. *Let μ be a homological or cohomological coefficient system for G. Let $\mathcal{C}$ be a collection of p-subgroups of G that is closed under passage to p-supergroups. Suppose that for each p-subgroup D of G with $D \notin \mathcal{C}$, the collection $\mathcal{S}_p(W_G(D))$ is $\mu(G/D)$-ample. Then the map $*_{\mathcal{C}} \to *_{\overline{\mathcal{S}}_p(G)}$ induces an isomorphism on Bredon homology or cohomology with coefficients in μ.*

Proof. Use Lemma 5.1 repeatedly to eliminate the conjugacy classes of elements of $\overline{\mathcal{S}}_p(W_G(D)) \setminus \mathcal{C}$ one by one, in such a way that smaller groups are removed before larger ones. In this way, all intermediate collections are closed under passage to p-supergroups, and at each step one removes a minimal element of the collection. Thus Lemma 5.1 applies at each step. □

The following proposition is the main result of this section. Note that if $D \subset K \subset G$ then $W_K(D)$ acts on $G/D = G \times_K K/D$ by G-equivariant maps.

Proposition 5.4. *Let μ be a homological (or cohomological) coefficient system for G. Let $\mathcal{C}$ be a collection of p-subgroups of G that is closed under passage to p-supergroups. Suppose that for each $K \in \mathrm{Iso}(X)$ and each p-subgroup $D \subset K$ with $D \notin \mathcal{C}$, the collection of non-identity p-subgroups of $W_K(D)$ is homologically (or cohomologically) $\mu(G/D)$-ample. Then the map $X_{\mathcal{C}} \to X_{\overline{\mathcal{S}}_p}$ induces an isomorphism on Bredon homology (or cohomology) with coefficients in μ.*

Proof. By Lemma 4.2, it is enough to check that for each $K \in \mathrm{Iso}(X)$ the map $(G/K)_{\mathcal{C}} \to (G/K)_{\overline{\mathcal{S}}_p}$ gives an isomorphism on Bredon homology (or cohomology). By Lemma 4.3, we need for $G \times_K (*)_{\mathcal{C}\downarrow K} \to G \times_K (*)_{\overline{\mathcal{S}}_p\downarrow K}$ to induce an isomorphism. By definition of restriction of coefficients (2.8), this amounts to showing that $(*)_{\mathcal{C}\downarrow K} \to (*)_{\overline{\mathcal{S}}_p\downarrow K}$ induces an isomorphism on Bredon homology or cohomology with coefficients in $\mu|_K$. This follows by Corollary 5.3 with K playing the role of G. □

There remains the question of how to establish ampleness. The following criterion is more or less standard and well known.

Proposition 5.5. *Assume that W is a finite group and that M is a $\mathbb{Z}_{(p)}[W]$-module, and suppose that there exists an element of order p in W that acts trivially on M. Then $\mathcal{S}_p(W)$ is homologically and cohomologically M-ample.*

Proof. The homology case when M is an $\mathbb{F}_p$-module is [8, 6.3]. The cohomology case follows from Grodal [11, Corollary 5.4 and Theorem 9.1]. It is clear that Grodal's methods can be adapted to include the homological case as well. We will give a direct proof starting from the $\mathbb{F}_p$-module case.

Let J be the kernel of the action map $W \to \mathrm{Aut}(M)$. By [8, 6.3], the natural twisted coefficient homology map

$$H_*\left(|\mathcal{S}_p(W)|_{hW}; \mathbb{F}_p[W/J]\right) \to H_*\left(BW; \mathbb{F}_p[W/J]\right)$$

is an isomorphism. By Shapiro's lemma, this is the same as the natural map

$$H_*(|\mathcal{S}_p(W)|_{hJ}; \mathbb{F}_p) \to H_*(BJ; \mathbb{F}_p)\,.$$

Since $|\mathcal{S}_p(W)|_{hJ}$ and BJ are of finite type, we conclude that the map $|\mathcal{S}_p(W)| \to *$ induces an isomorphism on homology

$$H_*(|\mathcal{S}_p(W)|_{hJ}; \mathbb{Z}_{(p)}) \xrightarrow{\cong} H_*(BJ; \mathbb{Z}_{(p)}) \tag{5.6}$$

and a similar isomorphism on cohomology. The proof is finished by comparing the Serre spectral sequences for the following diagram, whose rows are fibrations:

$$\begin{tikzcd} |\mathcal{S}_p(W)|_{hJ} \arrow[r] \arrow[d] & |\mathcal{S}_p(W)|_{hW} \arrow[r] \arrow[d] & B(W/J) \arrow[d] \\ BJ \arrow[r] & BW \arrow[r] & B(W/J) \end{tikzcd}.$$

The abutments of these two spectral sequences are the homology or cohomology of the total spaces with twisted coefficients in M, and by (5.6), the vertical maps induce isomorphisms on the E^2-pages. The proposition follows. □

6. Fixed point spaces of $\mathcal{P}_n$

In this section, we study the fixed point spaces of p-subgroups of Σ_n acting on $\mathcal{P}_n$. To motivate this problem in the current context, suppose that X is a G-space and that $\mathcal{C}$ is a collection of subgroups of G for which $X_{\mathcal{C}} \to X$ is known to be an isomorphism on Bredon (co)homology. If it happens that

$X^H \simeq *$ for all $H \in \mathcal{C}$, then $X_{\mathcal{C}}$ has the G-equivariant homotopy type of a point. A Bredon (co)homology isomorphism $X_{\mathcal{C}} \to X$ would say that X has the Bredon (co)homology of a point as well. Hence elements of $\mathcal{C}$ that have *non*-contractible fixed point spaces on X can be considered obstructions to X having the same Bredon (co)homology as a point.

To apply this idea, recall that $\overline{\mathcal{S}}_p(\Sigma_n)$ denotes the family of all p-subgroups of Σ_n, and suppose M is a Mackey functor for Σ_n that is projective relative to $\overline{\mathcal{S}}_p(\Sigma_n)$. By Proposition 4.6,

$$(\mathcal{P}_n)_{\overline{\mathcal{S}}_p(\Sigma_n)} \to \mathcal{P}_n$$

induces an isomorphism on Bredon (co)homology with coefficients in M. Our goal in this section is to show that very few p-subgroups of Σ_n have non-contractible fixed point spaces on $\mathcal{P}_n$. There are therefore very few obstructions to $\mathcal{P}_n$ having the Bredon (co)homology of a point.

Definition 6.1. If $H \subseteq \Sigma_n$, we say that H is *problematic* if $(\mathcal{P}_n)^H$ is not contractible.

Our main result in this section is the following proposition. The proof appears at the end of the section.

Proposition 6.2. *Let H be a problematic p-subgroup of Σ_n. Then H is an elementary abelian p-group, and the action of H on $\mathbf{n}$ is free.*

As a result, we conclude that very few subgroups of Σ_n are problematic. In fact, if $p^i \mid n$, then there is a unique (up to conjugacy) elementary abelian p-subgroup of rank i in Σ_n that acts freely on $\mathbf{n}$. We study the centralizers of these few problematic subgroups inside isotropy groups of $\mathcal{P}_n$ in Section 8, for the purpose of eliminating them from the approximating collection.

For the proof of the key lemma below, we need a little more notation. If $V \subseteq \Sigma_n$, let $(\mathcal{P}_n)_{V\text{−orb}}$ denote the poset of proper nontrivial partitions of $\mathbf{n}$ whose classes are unions of V-orbits. Given a partition λ that is stabilized by the action of V, let λ/V denote the coarsening of λ obtained by merging classes of λ that contain elements in the same orbit of V. Explicitly, $x \sim_{\lambda/V} y$ if there exists $v \in V$ such that $x \sim_\lambda vy$. As a result, note that the equivalence classes of λ/V are unions of orbits of V.

Lemma 6.3. *Let H be a p-subgroup of Σ_n. Suppose that there exists a nontrivial subgroup V of the center of H with the property that for all proper partitions λ that are fixed by H, the partition λ/V is proper. Then the nerve of $(\mathcal{P}_n)^H$ is contractible.*

Proof. Consider the inclusion into $(\mathcal{P}_n)^H$ of the subposet whose objects are unions of V-orbits:

$$(\mathcal{P}_n)_{V\text{−orb}} \cap (\mathcal{P}_n)^H \longrightarrow (\mathcal{P}_n)^H. \tag{6.4}$$

We assert that this inclusion has a left adjoint. Indeed, if V is central in H, then any partition λ that is stabilized by H has the property that λ/V is also stabilized by H. Provided that λ/V is always proper, a routine check shows that the functor $\lambda \mapsto \lambda/V$ is left adjoint to the inclusion (6.4). Hence (6.4) induces an equivalence on nerves. However, the left side has an initial object, namely the partition of $\mathbf{n}$ by the orbits of V, which is fixed by H because V is central in H. Therefore $(\mathcal{P}_n)_{V-\mathrm{orb}} \cap (\mathcal{P}_n)^H$ has contractible nerve, which finishes the proof. $\square$

Remark. Interpreting Lemma 6.3 in the case $n = p$ is slightly pedantic. The only p-subgroup of Σ_p is $H = \mathbb{Z}/p$, and the available candidate for V is H itself. One proper partition is fixed by V, namely the discrete partition, but V acts transitively on its classes. Hence the hypothesis of Lemma 6.3 is not satisfied, and we do not conclude that $(\mathcal{P}_p)^H$ is contractible. And, indeed, $(\mathcal{P}_p)^H$ is the empty set.

To prove that H is elementary abelian in Proposition 6.2, we need a little group theory. Given a p-group H, let $\Phi(H)$ be the Frattini subgroup of H, i.e., $\Phi(H)$ is the normal subgroup generated by commutators and p-th powers. Note that any homomorphism from H to an elementary abelian p-group factors through the quotient group $H/\Phi(H)$. The following lemma is standard.

Lemma 6.5. *If H is a p-group and is not elementary abelian, then there exists a subgroup $V \subseteq \Phi(H)$ such that V has order p and is contained in the center of H.*

Proof. If H is not elementary abelian, then $\Phi(H)$ is a nontrivial normal subgroup of H, which necessarily has nontrivial intersection with the center because H is a p-group. We can then pick out an element of order p to generate V. $\square$

The following proposition now gives the group-theoretic structure in Proposition 6.2.

Proposition 6.6. *If H is a problematic p-subgroup of Σ_n, then H is elementary abelian.*

Proof. We prove the contrapositive. Suppose that H is not elementary abelian, and let $V \subseteq \Phi(H)$ be the subgroup provided by Lemma 6.5. We want to apply Lemma 6.3 to show that H is not problematic.

Suppose $\lambda \in (\mathcal{P}_n)^H$; we need to show that λ/V is proper. If λ/V fails to be proper, then V acts transitively on the equivalence classes of λ. In this case, since λ has more than one equivalence class, it must have exactly p equivalence classes. The action of H permutes the equivalence classes of λ, giving a homomorphism $H \to \Sigma_p$. However, $H \to \Sigma_p$ necessarily factors through $H/\Phi(H)$, because H is a p-group and the only p-subgroups of Σ_p are elementary abelian. Since $V \subseteq \Phi(H)$, this says that V acts trivially on the classes of λ, a contradiction.

We have established that for any $\lambda \in (\mathcal{P}_n)^H$, the partition λ/V is proper. By Lemma 6.3, $(\mathcal{P}_n)^H$ is contractible. $\square$

Proof of Proposition 6.2. If H is problematic, then we already know from Proposition 6.6 that H is an elementary abelian p-group. We need to show that any element $h \in H$ acts freely on $\mathbf{n}$. Let $V \cong \mathbb{Z}/p$ be the subgroup generated by h. Because H is problematic, the contrapositive of Lemma 6.3 tells us that there exists a partition λ in $(\mathcal{P}_n)^H$ such that V acts transitively on the equivalence classes of λ. Since λ is not the indiscrete partition, λ must have exactly p equivalence classes, freely permuted by V, which therefore acts freely on $\mathbf{n}$. □

7. Isotropy in $\mathcal{P}_n$

In this section we collect some elementary results about the isotropy groups of the Σ_n-space $\mathcal{P}_n$. These results will be used to produce centralizing elements of problematic subgroups of isotropy groups.

We begin by setting our conventions regarding wreath products, because these groups figure prominently in the isotropy groups of $\mathcal{P}_n$. Suppose that G is a group acting (on the left) on the set $\mathbf{s}$ and that H is any group. The wreath product $H \wr G$ is the semi-direct product $H^{\mathbf{s}} \rtimes G$:

$$1 \longrightarrow H^{\mathbf{s}} \longrightarrow H \wr G \longrightarrow G \longrightarrow 1.$$

We denote a general element of this group by $(h_1, \ldots, h_s; g)$, and the group law is given by the formula

$$(h_1, \ldots, h_s; g) \cdot (h'_1, \ldots, h'_s; g') = \left(h_1 h'_{g^{-1}(1)}, \ldots, h_s h'_{g^{-1}(s)}; gg'\right).$$

Accordingly, the formula for the inverse of an element is:

$$(h_1, \ldots, h_s; g)^{-1} = \left(h^{-1}_{g(1)}, \ldots, h^{-1}_{g(s)}; g^{-1}\right).$$

There is a natural group monomorphism

$$\begin{array}{ccc} H \times G & \longrightarrow & H \wr G \\ (h, g) & \mapsto & (h, \ldots, h; g). \end{array} \tag{7.1}$$

Definition 7.2. We write $\mathrm{Diag}(H)$ for the image of $H \times \{e\}$ in $H \wr G$ under (7.1). If T is a subgroup of G, we write $\tilde{T}$ for the subgroup of $H \wr G$ given by the image of $\{e\} \times T$ under (7.1).

In particular, $\tilde{G}$ is the image of $\{e\} \times G$ under (7.1). Note that $\mathrm{Diag}(H)$ and $\tilde{G}$ centralize each other in $H \wr G$.

From Section 5, we know that eliminating a subgroup from an approximating collection requires thinking about normalizers, and we begin with an elementary calculation for a special case of a normalizer in a wreath product.

Lemma 7.3. *Suppose that a subgroup $T \subseteq G$ acts transitively on $\mathbf{s}$, and let $N = N_G(T)$. Then the normalizer of $\tilde{T}$ in $H \wr G$ is $\mathrm{Diag}(H) \times \tilde{N}$.*

Proof. An element $(h_1, \ldots, h_s; g) \in H \wr G$ normalizes $\tilde{T}$ if and only if for every $t \in T$ there exists $t' \in T$ such that

$$(h_1, \ldots, h_s; g)^{-1} \cdot (e, \ldots, e; t) \cdot (h_1, \ldots, h_s; g) = (e, \ldots, e; t').$$

Evaluating the left-hand side, we obtain

$$\begin{aligned}\Big(h_{g(1)}^{-1},\dots,h_{g(s)}^{-1}\,;\ g^{-1}\big)&\cdot(e,\dots,e\,;t)\cdot(h_1,\dots,h_s\,;g)\\ &=\Big(h_{g(1)}^{-1},\dots,h_{g(s)}^{-1}\,;g^{-1}t\Big)\cdot(h_1,\dots,h_s\,;g)\\ &=\Big(h_{g(1)}^{-1}\,h_{t^{-1}g(1)},\ \dots\ ,h_{g(s)}^{-1}\,h_{t^{-1}g(s)}\,;g^{-1}tg\Big).\end{aligned}$$

This calculation allows us to verify that elements of $\mathrm{Diag}(H)\times\tilde{N}$ normalize $\tilde{T}$ in $H\wr G$, because if $(h_1,\dots,h_s;g)=(h,\dots,h;\,g)$ where $g\in N_G(T)$, then the last line reduces to $\big(e,\dots,e\,;\,g^{-1}tg\big)$ with $g^{-1}tg\in T$.

To see that a normalizing element of $\tilde{T}$ must be in $\mathrm{Diag}(H)\times\tilde{N}$, observe that the calculation above implies that $(h_1,\dots,h_s;g)$ normalizes $\tilde{T}$ only if $g^{-1}tg\in T$ for all $t\in T$, so we must have $g\in N_G(T)$. Further, $(h_1,\dots,h_s;g)$ normalizes T only if for every $i\in\mathbf{s}$ and $t\in T$, we have $h_{g(i)}=h_{t^{-1}g(i)}$. Since the action of T on $\mathbf{s}$ is transitive, it follows that if $(h_1,\dots,h_s;g)$ normalizes T, then $h_1=h_2=\cdots=h_s$, so $(h_1,\dots,h_s)\in\mathrm{Diag}(H)$. The lemma follows. □

Next we review the standard action of the wreath product on a product set. As above, suppose that G acts on the set $\mathbf{s}$, and suppose also that H acts on the set $\mathbf{r}$. Then the wreath product $H\wr G$ acts on $\mathbf{r}\times\mathbf{s}$ as follows. If $(i,j)\in\mathbf{r}\times\mathbf{s}$, then

$$(h_1,\dots,h_s;g)(i,j)=(h_{g(j)}(i),g(j)).$$

If we visualize $\mathbf{r}\times\mathbf{s}$ as s columns, each containing the set $\mathbf{r}$, then $(h_1,\dots,h_s;g)$ acts by first letting g permute the set of columns, and then (for each $i\in\mathbf{s}$) acting by h_i on the i-th column. This action preserves the partition of $\mathbf{r}\times\mathbf{s}$ defined by the columns, that is, by pre-images of points of $\mathbf{s}$ under the projection map

$$p\colon \mathbf{r}\times\mathbf{s}\to\mathbf{s}, \tag{7.4}$$

and in fact $\Sigma_{\mathbf{r}}\wr\Sigma_{\mathbf{s}}$ is the full isotropy group of this partition.

With these preliminaries in hand, we turn to the isotropy subgroups of simplices of $\mathcal{P}_n$ under the action of Σ_n. The zero-simplices are partitions of $\mathbf{n}$. If λ is such a partition, we denote its isotropy group K_λ. By definition, elements of K_λ are bijective maps $\sigma\colon\mathbf{n}\to\mathbf{n}$ such that $x\sim_\lambda y$ if and only if $\sigma x\sim_\lambda\sigma y$ for all $x,y\in\mathbf{n}$. The action of $K_\lambda\subseteq\Sigma_n$ on $\mathbf{n}$ induces an action of K_λ on $\mathrm{cl}\,(\lambda)$, the set of equivalence classes (or "components") of λ.

There is a special type of partition that will play an important role.

Definition 7.5. We say that a partition λ of $\mathbf{n}$ is *regular* if the elements of $\mathrm{cl}\,(\lambda)$ all have the same cardinality.

Suppose λ is a regular partition of $\mathbf{n}$, with classes of cardinality r. We fix a class $\mathbf{c}\in\mathrm{cl}\,(\lambda)$, and we choose bijections between $\mathbf{c}$ and each of the other elements of $\mathrm{cl}\,(\lambda)$. These choices define a (non-canonical) bijection and corresponding isomorphism:

$$\mathbf{n} \longleftrightarrow \mathbf{c} \times \mathrm{cl}\,(\lambda) \tag{7.6}$$

$$K_\lambda \cong \Sigma_{\mathbf{c}} \wr \Sigma_{\mathrm{cl}(\lambda)}. \tag{7.7}$$

The composition of (7.6) with projection to $\mathrm{cl}\,(\lambda)$ gives a map that is K_λ-equivariant:

$$\mathbf{n} \xrightarrow{\cong} \mathbf{c} \times \mathrm{cl}\,(\lambda) \xrightarrow{p} \mathrm{cl}\,(\lambda)\,.$$

Generalizing to partitions that are not regular, suppose that λ has classes of various cardinalities $r_1 \ldots, r_l$, where $r_1 \ldots, r_l$ are pairwise distinct. If there are s_i classes of cardinality r_i, then $n = r_1 s_1 + \cdots + r_l s_l$, and K_λ is (non-canonically) isomorphic to the following product of wreath products:

$$(\Sigma_{r_1} \wr \Sigma_{s_1}) \times \cdots \times (\Sigma_{r_l} \wr \Sigma_{s_l})\,.$$

From this formula, we get the following lemma.

Lemma 7.8. *The action of K_λ on $\mathrm{cl}\,(\lambda)$ is transitive if and only if λ is regular.*

Proof. If λ has s_i classes of cardinality r_i, as above, then the action of K_λ on $\mathrm{cl}\,(\lambda)$ factors through $\Sigma_{s_1} \times \cdots \times \Sigma_{s_l}$. The action of the latter is only transitive if and only if $l = 1$, i.e., λ is regular. $\square$

Moving on to isotropy groups of higher-dimensional simplices of $\mathcal{P}_n$, a typical non-degenerate simplex is a chain

$$\Lambda = (\lambda_0 < \cdots < \lambda_j)$$

where $\lambda_0, \ldots, \lambda_j$ are partitions of $\mathbf{n}$. We write K_Λ for the subgroup of Σ_n that stabilizes all of the partitions $\lambda_0, \ldots, \lambda_j$, that is, K_Λ is the isotropy group of Λ as a simplex of the Σ_n-space $\mathcal{P}_n$. For each i, we know $K_\Lambda \subseteq K_{\lambda_i}$, so there is an action of K_Λ on $\mathrm{cl}\,(\lambda_i)$ for each i.
We need a few elementary notions for chains. First, the evident notions of restriction and isomorphism.

Definition 7.9.

(1) If λ is a partition of A and $X \subseteq A$, we define $\lambda|_X$, the *restriction of λ to X*, as the partition of X obtained by intersecting each equivalence class of λ with X.
(2) If $\Lambda = (\lambda_0 \leq \cdots \leq \lambda_j)$ is a chain of partitions of A, we write $\Lambda|_X$ for the chain $\Lambda|_X = (\lambda_0|_X \leq \cdots \leq \lambda_j|_X)\,.$
(3) Suppose given sets A and A', and suppose that we have chains of partitions $\Lambda = (\lambda_0 \leq \cdots \leq \lambda_j)$ and $\Lambda' = \left(\lambda'_0 \leq \cdots \leq \lambda'_j\right)$ of A and A', respectively. We say that Λ and Λ' are *isomorphic* if there exists a bijection $f : A \to A'$ such that for all i with $0 \leq i \leq j$, we have $x \sim_{\lambda_i} y \iff f(x) \sim_{\lambda'_i} f(y)$.

Note that the restriction of a partition λ can be discrete or indiscrete even if λ itself is neither. Likewise, the restriction of a strict inequality of partitions can

be an equality, so a nondegenerate chain (all strict inequalities) may become degenerate upon restriction.
In Section 8, subgroups of Σ_n that have transitive actions on the set of classes of a partition are of particular interest. To state the next lemma, which is the first step to understanding such situations, we need a little more notation. If $\Lambda = (\lambda_0 < \lambda_1 < ... < \lambda_j)$ and $i \leq j$, we define the shorter chain of partitions $\Lambda_{<i} = (\lambda_0 < \lambda_1 < ... < \lambda_{i-1})$. If $\mathbf{c} \in \mathrm{cl}(\lambda_i)$, then we write $(\Lambda_{<i})|_{\mathbf{c}}$ for the restriction of $\Lambda_{<i}$ to $\mathbf{c}$, and we write $K(\mathbf{c})$ for the subgroup of the symmetric group on $\mathbf{c}$ that stabilizes $(\Lambda_{<i})|_{\mathbf{c}}$.

Lemma 7.10. *Let $\Lambda = (\lambda_0 < \lambda_1 < ... < \lambda_j)$ be a chain of partitions of $\mathbf{n}$ and let $K_\Lambda \subseteq \Sigma_n$ be its isotropy group. Let i be an integer with $0 \leq i \leq j$. Then the action of K_Λ on $\mathrm{cl}(\lambda_i)$ is transitive if and only if the following two conditions hold:*

1. *The partitions $\lambda_i, \lambda_{i+1}, \ldots, \lambda_j$ are all regular.*
2. *For any $\mathbf{c}, \mathbf{c}' \in \mathrm{cl}(\lambda_i)$, the chains $(\Lambda_{<i})|_{\mathbf{c}}$ and $(\Lambda_{<i})|_{\mathbf{c}'}$ are isomorphic.*

Example 7.11. Let $n = 18$. Consider $\Lambda = (\lambda_0 < \lambda_1 < \lambda_2)$ defined as follows:

$$\begin{array}{llll}
\lambda_2: & \{1,2,3,4,5,6\} & \{7,8,9,10,11,12\} & \{13,14,15,16,17,18\} \\
\lambda_1: & \{1,2,3\}\ \{4,5,6\} & \{7,8,9\}\ \{10,11,12\} & \{13,14,15\}\ \{16,17,18\} \\
\lambda_0: & \{1,2\}\{3\}\ \{4,5\}\{6\} & \{7,8\}\{9\}\ \{10,11\}\{12\} & \{13,14\}\{15\}\ \{16,17\}\{18\}
\end{array}$$

We verify assumptions (1) and (2) of Lemma 7.10 for $i = 1$. First, the partitions λ_2 and λ_1 are regular, so (1) is satisfied for $i = 1$. Second, $\Lambda_{<1}$ is just λ_0, and the restriction of λ_0 to any class $\mathbf{c} \in \mathrm{cl}(\lambda_1)$ consists of a singleton and a two-element set. Hence restricting $\Lambda_{<1}$ to elements of $\mathrm{cl}(\lambda_1)$ gives pairwise isomorphic partitions. According to Lemma 7.10, the action of K_Λ on $\mathrm{cl}(\lambda_1)$ is transitive.
And indeed, by inspection we have $K_\Lambda \cong (\Sigma_2 \times \Sigma_1) \wr (\Sigma_2 \wr \Sigma_3)$. The action of K_Λ on $\mathrm{cl}(\lambda_1)$ is given by $(\Sigma_2 \wr \Sigma_3) \subseteq \Sigma_6 \cong \Sigma_{\mathrm{cl}(\lambda_1)}$, which is transitive. (See also Example 7.16.)

Proof of Lemma 7.10. Suppose K_Λ acts transitively on $\mathrm{cl}(\lambda_i)$. Then K_Λ also acts transitively on the set of classes of $\lambda_{i+1}, \ldots, \lambda_j$, and hence these partitions are regular by Lemma 7.8. Further, suppose that $\mathbf{c}, \mathbf{c}' \in \mathrm{cl}(\lambda_i)$. The transitive action of K_Λ on $\mathrm{cl}(\lambda_i)$ gives an element $\sigma \in K_\Lambda$ taking $\mathbf{c}$ to $\mathbf{c}'$; then σ induces the required isomorphism from $(\Lambda_{<i})|_{\mathbf{c}}$ to $(\Lambda_{<i})|_{\mathbf{c}'}$.
Now let us prove the converse. First, we assert that for any t with $i \leq t \leq j$ and for any $\mathbf{c}, \mathbf{c}' \in \mathrm{cl}(\lambda_t)$, we have isomorphic chains $(\Lambda_{<t})|_{\mathbf{c}}$ and $(\Lambda_{<t})|_{\mathbf{c}'}$. The point is that while assumption (2) only gives us such an isomorphism for $t = i$, we can reach the same conclusion inductively for $t > i$ by using regularity of λ_i, λ_{i+1},…,λ_j (assumption (1)).
To simplify notation for isotropy groups in what follows, we write K_t for the isotropy group of the chain $(\lambda_0 < \lambda_1 < ... < \lambda_t)$. We know $K_i \supset K_{i+1} \cdots \supset K_j = K_\Lambda$, and our goal is to prove that K_Λ acts transitively on $\mathrm{cl}(\lambda_i)$. Our

strategy is to first prove that K_i acts transitively on $\mathrm{cl}(\lambda_i)$, and then we prove by induction on t that K_t likewise acts transitively on $\mathrm{cl}(\lambda_i)$ for $i \le t \le j$.

For the inductive hypothesis, we need to assume more about K_{t-1} (for $t > i$) than simply a transitive action on $\mathrm{cl}(\lambda_i)$ (so we will need to verify the assumption explicitly for K_i to get the base case). To state the hypothesis for K_{t-1}, let $\mathbf{c}, \mathbf{c}'$ be arbitrary elements of $\mathrm{cl}(\lambda_i)$, and let $C, C' \in \mathrm{cl}(\lambda_{t-1})$ be the unique classes such that $\mathbf{c} \subseteq C$ and $\mathbf{c}' \subseteq C'$. We assume for the inductive hypothesis on K_{t-1} that for any choice of $\mathbf{c}, \mathbf{c}'$, there exists an element $\sigma_{t-1}(\mathbf{c}, \mathbf{c}') \in K_{t-1}$ that is a bijection from $\mathbf{c}$ to $\mathbf{c}'$, is a bijection from C to C', and is the identity on the complement of $C \sqcup C'$ in $\mathbf{n}$.

To construct $\sigma_i(\mathbf{c}, \mathbf{c}') \in K_i$, the base case, recall that by assumption (2), the chains $(\Lambda_{<i})|_{\mathbf{c}}$ and $(\Lambda_{<i})|_{\mathbf{c}'}$ are isomorphic. This means there is a bijection $f : \mathbf{c} \to \mathbf{c}'$ that induces an isomorphism of $(\Lambda_{<i})|_{\mathbf{c}}$ to $(\Lambda_{<i})|_{\mathbf{c}'}$. We define $\sigma_i(\mathbf{c}, \mathbf{c}') \in K_i$ as f on $\mathbf{c}$, as f^{-1} on $\mathbf{c}'$, and as the identity map on the complement of $\mathbf{c} \sqcup \mathbf{c}'$.

For the inductive step, we need to construct $\sigma_t(\mathbf{c}, \mathbf{c}') \in K_t$. Let $D, D' \in \mathrm{cl}(\lambda_t)$ be the classes containing $\mathbf{c}, \mathbf{c}'$, respectively. Because λ_{t-1} and λ_t are both regular, D and D' are both constructed by merging the same number of classes of λ_{t-1}, say $D = C_1 \sqcup \cdots \sqcup C_k$ and $D' = C'_1 \sqcup \cdots \sqcup C'_k$ for classes $C_1, \ldots, C_k, C'_1, \ldots, C'_k$ of λ_{t-1}. We know from what was proved above that the restrictions of $\Lambda_{<t}$ to any of the classes $C_1, \ldots, C_k, C'_1, \ldots, C'_k$ are isomorphic. Now suppose that $\mathbf{c} \subseteq C_1$ and $\mathbf{c}' \subseteq C'_1$. We construct $\sigma_t(\mathbf{c}, \mathbf{c}') \in K_t$ as follows. On the complement of $D \sqcup D'$ in $\mathbf{n}$, the element $\sigma_t(\mathbf{c}, \mathbf{c}')$ acts by the identity. On $C_1 \subseteq D$, use the bijection to $C'_1 \subseteq D'$ given by the inductive hypothesis. On the other classes, $C_2, \ldots, C_k$, use bijections to $C'_2, \ldots, C'_k$, respectively, that are isomorphisms of the restrictions of $\Lambda_{<t}$ to each class. Likewise, on $C'_1, \ldots, C'_k$ use the inverses of those isomorphisms. This completes the inductive step and finishes the proof. ☐

If, as in Lemma 7.10, the isotropy subgroup of a chain Λ acts transitively on the equivalence classes of some λ_i in the chain, then there is an explicit expression for the isotropy group K_Λ as a wreath product. To describe this, we need another definition.

Definition 7.12. Let μ and λ be partitions of a set, with $\mu \le \lambda$. We define the *partition of* $\mathrm{cl}(\mu)$ *induced by* λ, denoted $\overline{\lambda}_\mu$, or simply $\overline{\lambda}$ if μ is clear from context, by following equivalance relation: if $\mathbf{c}, \mathbf{c}'$ are equivalence classes of μ, then

$$\mathbf{c} \sim_{\overline{\lambda}_\mu} \mathbf{c}' \iff \mathbf{c} \sqcup \mathbf{c}' \text{ is contained in a single equivalence class of } \lambda.$$

For instance, in Example 7.11, we could consider $\lambda_1 < \lambda_2$. Then $\overline{\lambda}_2$, the partition of $\mathrm{cl}(\lambda_1)$ induced by λ_2, is a partition of a six-element set into three subsets, each containing two elements:

$$\overline{\lambda}_2 = \begin{cases} \{\{1,2,3\},\{4,5,6\}\} \\ \{\{7,8,9\},\{10,11,12\}\} \\ \{\{13,14,15\},\{16,17,18\}\}. \end{cases}$$

In the following lemma, and also in Section 8, we need a convention regarding the extremes in a chain of partitions $\Lambda = (\lambda_0 < \lambda_1 < ... < \lambda_j)$. We adopt the convention that λ_{-1} is the partition into singleton subsets (the discrete partition). Similarly, we interpret λ_{j+1} as the partition containing just one equivalence class (the indiscrete partition).

Lemma 7.13. *Suppose that $\Lambda = (\lambda_0 < \lambda_1 < ... < \lambda_j)$, and suppose that K_Λ acts transitively on $\mathrm{cl}(\lambda_i)$ for some $0 \leq i \leq j$. Let $\overline{\Lambda}_{>i} = \left(\overline{\lambda}_{i+1} < \cdots < \overline{\lambda}_j\right)$ be the induced chain of partitions of $\mathrm{cl}(\lambda_i)$, and let $K\left(\overline{\Lambda}_{>i}\right)$ be the isotropy subgroup of $\overline{\Lambda}_{>i}$ in $\Sigma_{\mathrm{cl}(\lambda_i)}$. Fix a class $\mathbf{c} \in \mathrm{cl}(\lambda_i)$ and let $K(\mathbf{c}) \subseteq \Sigma_{\mathbf{c}}$ denote the isotropy group of $(\Lambda_{<i})|_{\mathbf{c}}$. Then there is an isomorphism*

$$K_\Lambda \cong K(\mathbf{c}) \wr K\left(\overline{\Lambda}_{>i}\right). \tag{7.14}$$

Proof. We have a fixed class $\mathbf{c} \in \mathrm{cl}(\lambda_i)$. By Lemma 7.10, we know that for all $\mathbf{c}' \in \mathrm{cl}(\lambda_i)$, the chains $(\Lambda_{<i})|_{\mathbf{c}}$ and $(\Lambda_{<i})|_{\mathbf{c}'}$ are isomorphic. Hence for each $\mathbf{c}' \in \mathrm{cl}(\lambda_i)$, we can choose a bijection $\mathbf{c} \to \mathbf{c}'$ that respects the finer partitions $\lambda_0, \ldots, \lambda_{i-1}$. Assembling these bijections gives a bijection

$$\mathbf{n} \longrightarrow \mathbf{c} \times \mathrm{cl}(\lambda_i). \tag{7.15}$$

Under (7.15), the chain Λ of partitions of $\mathbf{n}$ corresponds to a chain of partitions of $\mathbf{c} \times \mathrm{cl}(\lambda_i)$ as described below. The lemma will follow by identifying bijections from $\mathbf{c} \times \mathrm{cl}(\lambda_i)$ to itself that preserve the chain of partitions.

We start in the middle of the chain. Subsets of $\mathbf{n}$ that are equivalence classes of λ_i correspond under (7.15) precisely to the columns of $\mathbf{c} \times \mathrm{cl}(\lambda_i)$. Explicitly, if $\mathbf{c}' \subseteq \mathbf{n}$ is an equivalence class of λ_i, then the bijection $\mathbf{c} \to \mathbf{c}'$ used to define (7.15) provides the correspondence between the column $\{(x, \mathbf{c}') : x \in \mathbf{c}\}$ and $\mathbf{c}' \subseteq \mathbf{n}$. As indicated by (7.7), the isotropy group of λ_i as a partition of $\mathbf{c} \times \mathrm{cl}(\lambda_i)$ is then identified by (7.15) with $\Sigma_{\mathbf{c}} \wr \Sigma_{\mathrm{cl}(\lambda_i)}$.

Partitions finer than λ_i take place within the columns of $\mathbf{c} \times \mathrm{cl}(\lambda_i)$. To be specific, if $t < i$, we have $(x, \mathbf{c}') \sim_{\lambda_t} (y, \mathbf{c}'')$ if and only if $\mathbf{c}' = \mathbf{c}''$ and $x \sim_{\lambda_t} y$. We know that for any $\mathbf{c}' \in \mathrm{cl}(\lambda_i)$, the bijection $\mathbf{c} \to \mathbf{c}'$ induces an isomorphism from $(\Lambda_{<i})|_{\mathbf{c}'}$ to $(\Lambda_{<i})|_{\mathbf{c}}$, and hence an isomorphism $K(\mathbf{c}) \xrightarrow{\cong} K(\mathbf{c}')$. As a consequence, the isotropy group of $\Lambda_{<i+1}$ regarded as a chain of partitions of $\mathbf{c} \times \mathrm{cl}(\lambda_i)$ is isomorphic to $K(\mathbf{c}) \wr \Sigma_{\mathrm{cl}(\lambda_i)}$.

Finally, we note that coarsenings of λ_i are in one-to-one correspondence with partitions of $\mathrm{cl}(\lambda_i)$. The coarsenings $\lambda_{i+1} < ... < \lambda_j$ of λ_i are stabilized by an element $\sigma \in \Sigma_n$ if and only if the image of σ in $\Sigma_{\mathrm{cl}(\lambda_i)}$ stabilizes $\overline{\Lambda}_{>i}$.

We conclude that under (7.15), the isotropy subgroup $K_\Lambda \subseteq \Sigma_n$ corresponds to $K_\Lambda \cong K(\mathbf{c}) \wr K\left(\overline{\Lambda}_{>i}\right)$. □

Example 7.16. We continue with the setup of Example 7.11. The action of K_Λ on $\mathrm{cl}(\lambda_1)$ is transitive, so we consider Lemma 7.13 applied with $i = 1$. The chain $\overline{\Lambda}_{>1}$ consists simply of $\overline{\lambda}_2$, which, as mentioned just before Lemma 7.13, partitions the six-element set $\mathrm{cl}(\lambda_1)$ into three two-element subsets. Therefore

$$K\left(\overline{\Lambda}_{>1}\right) \cong \Sigma_2 \wr \Sigma_3.$$

Choose $\mathbf{c} = \{1,2,3\} \in \mathrm{cl}(\lambda_1)$. Then λ_1 restricted to $\mathbf{c}$ is a singleton and a two-element set, so $K(\mathbf{c}) = \Sigma_1 \times \Sigma_2$. Thus the result of the Lemma 7.13 is what we found before by inspection:

$$K_\Lambda \cong K(\mathbf{c}) \wr K\left(\overline{\Lambda}_{>1}\right) \cong (\Sigma_2 \times \Sigma_1) \wr (\Sigma_2 \wr \Sigma_3).$$

Once again we note that the isomorphism of Lemma 7.13 is not canonical. It depends on a choice of a particular class $\mathbf{c} \in \mathrm{cl}(\lambda_i)$ and of an isomorphism

$$\mathbf{n} \cong \mathbf{c} \times \mathrm{cl}(\lambda_i).$$

A situation of particular interest comes about when a subgroup $D \subseteq K_\Lambda$ acts transitively and *freely* on $\mathrm{cl}(\lambda_i)$.

Lemma 7.17. *With the notation of Lemma 7.13, suppose that a subgroup $D \subseteq K_\Lambda$ acts freely and transitively on* $\mathrm{cl}(\lambda_i)$. *Then the isomorphism* (7.14) *identifies D with a transitive subgroup of $K\left(\overline{\Lambda}_{>i}\right)$. If $N = N_{K(\overline{\Lambda})}(D)$, then the normalizer of D in K_Λ is identified by* (7.14) *with the group*

$$\mathrm{Diag}(K(\mathbf{c})) \times \tilde{N}.$$

In particular, K_Λ contains a subgroup isomorphic to $K(\mathbf{c})$ that centralizes D.

Proof. The free and transitive action of D on $\mathrm{cl}(\lambda_i)$ allows us to choose a bijection between $\mathrm{cl}(\lambda_i)$ and the underlying set of D such that the given action of D on $\mathrm{cl}(\lambda_i)$ corresponds to the action of D on its underlying set by left translation. This choice identifies D with a transitive subgroup of $K\left(\overline{\Lambda}\right)$. The lemma's statement about the normalizer of D is now a consequence of Lemma 7.3. □

The situation described in this lemma is illustrated in Example 8.1 below.

We need another result, in a similar spirit, that will be applied in Section 8 to subgroups whose action on some $\mathrm{cl}(\lambda_i)$ in Λ is transitive but not free. Let D be an abelian group acting freely on $\mathbf{n}$, and let $\mathbf{n}/D$ be the set of orbits of that action. Then the action of D on $\mathbf{n}$ extends canonically along the diagonal inclusion $D \hookrightarrow D^{\mathbf{n}/D}$ to an action of $D^{\mathbf{n}/D}$ on $\mathbf{n}$. Further, because D acts freely on $\mathbf{n}$, the action of $D^{\mathbf{n}/D}$ on $\mathbf{n}$ is faithful, that is, $D^{\mathbf{n}/D} \to \Sigma_n$ is a monomorphism. Similarly, if $\mathbf{n}/D \twoheadrightarrow \mu$ is a surjective function of sets, then we obtain an inclusion $D^\mu \hookrightarrow D^{\mathbf{n}/D}$ that identifies D^μ as a subgroup of $D^{\mathbf{n}/D}$ via a diagonal inclusion, giving us $D^\mu \subseteq D^{\mathbf{n}/D} \subseteq \Sigma_n$. Note that it is necessary for the action of D on $\mathbf{n}$ to be free in order for $D^{\mathbf{n}/D}$ to include into Σ_n as a subgroup.

Continuing to assume that D is an abelian group acting freely on $\mathbf{n}$, suppose that $S \subseteq D$ is a subgroup. Then $S^{\mathbf{n}/D} \subseteq D^{\mathbf{n}/D}$, so in particular $S^{\mathbf{n}/D}$ commutes with $D \subseteq D^{\mathbf{n}/D}$. Hence if $\mathbf{n}/D \twoheadrightarrow \mu$ is an epimorphism, we can define $D \oplus_S S^\mu$ to be the pushout in the category of abelian groups of the diagram $D \leftarrow S \to S^\mu$. Then $D \oplus_S S^\mu$ is an abelian group with

$$D \subseteq D \oplus_S S^\mu \subseteq D^\mu \subseteq \Sigma_n.$$

Note that if S is a nontrivial subgroup and $|\mu| > 1$ then $D \oplus_S S^\mu \subseteq \Sigma_n$ strictly contains D. In particular, suppose that λ is a partition stabilized by D, and recall that λ/D denotes the minimal coarsening of λ whose classes are unions of D-orbits. (See Section 6.) There is an evident epimorphism $\mathbf{n}/D \twoheadrightarrow \mathrm{cl}\,(\lambda/D)$, and we use it to define the pushout $D \oplus_S S^{\mathrm{cl}(\lambda/D)}$.

With these preliminaries in place, we can state our next result.

Lemma 7.18. *Let $D \subseteq \Sigma_n$ be an abelian group acting freely on $\mathbf{n}$, let S be a subgroup of D, and let λ be a partition of $\mathbf{n}$ that is stabilized by D. Then:*

(1) *If μ is a refinement of λ (or λ itself) and is stabilized by D then μ is stabilized by $D \oplus_S S^{\mathrm{cl}(\lambda/D)}$.*
(2) *If μ is a coarsening of λ/S that is stabilized by D, then μ is stabilized by $D \oplus_S S^{\mathrm{cl}(\lambda/D)}$.*

Proof. In both cases μ is assumed to be stabilized by D, so it is sufficient to prove that μ is stabilized by $S^{\mathrm{cl}(\lambda/D)}$ in order to conclude that μ is stabilized by $D \oplus_S S^{\mathrm{cl}(\lambda/D)}$. In fact, because $\lambda/S \twoheadrightarrow \lambda/D$, we know that $S^{\mathrm{cl}(\lambda/D)} \subseteq S^{\mathrm{cl}(\lambda/S)}$, so it is sufficient to prove that μ is stabilized by $S^{\mathrm{cl}(\lambda/S)}$.

Suppose that μ is a refinement of λ, and that μ is stabilized by D. If $x \sim_\mu y$, then $x \sim_\lambda y$ also, and hence $x \sim_{\lambda/S} y$, say $x, y \in \mathbf{z}$ where $\mathbf{z} \in \mathrm{cl}\,(\lambda/S)$. But the action of $S^{\mathrm{cl}(\lambda/S)}$ on $\mathbf{z}$ factors through projection to the factor of S corresponding to $\mathbf{z}$. Since $S \subseteq D$ itself stabilizes μ by assumption, we know that $\sigma x \sim_\mu \sigma y$ for any $\sigma \in S$, proving that $S^{\mathrm{cl}(\lambda/S)}$ stabilizes μ.

Now suppose that μ is a coarsening of λ/S. We know that $S^{\mathrm{cl}(\lambda/S)}$ not only stabilizes λ/S, but actually acts trivially on the set of equivalence classes of λ/S. Hence $S^{\mathrm{cl}(\lambda/S)}$ stabilizes any coarsening of λ/S, and in particular, stabilizes μ. This finishes what is needed for the second statement of the lemma. □

Example 7.19. Consider the following partition

$$\lambda : \quad \{1,2\}\{3,4\}\{5\}\{6\} \quad \{7,8\}\{9,10\}\{11\}\{12\}.$$

Let $D \cong \mathbb{Z}/4$ be the subgroup of Σ_{12} generated by the following product of cycles:

$$\rho = (1,7,3,9)(2,8,4,10)(5,11,6,12).$$

Then D acts freely on the set $\{1,\ldots,12\}$, and D preserves λ. The partition λ/D is given by merging classes of λ that contain elements of the same orbit of D, so we have

$$\lambda/D : \ \{1,2,3,4,7,8,9,10\},\ \{5,6,11,12\}.$$

Let $S \subset D$ be the subgroup isomorphic to $\mathbb{Z}/2$. The group S is generated by ρ^2, which is the following product of transpositions

$$(1,3)(7,9)(2,4)(8,10)(5,6)(11,12).$$

Accordingly, $D \oplus_S S^{\mathrm{cl}(\lambda/D)}$ is the subgroup of Σ_n generated by D and the elements $(1,3)(7,9)(2,4)(8,10)$ and $(5,6)(11,12)$.

The partition λ/S is given by merging classes of λ that contain elements of the same orbit of S, so we have

$$\lambda/S: \ \ \{1,2,3,4\},\ \{5,6\},\ \{7,8,9,10\},\ \{11,12\}.$$

The reader is invited to check that $D \oplus_S S^{\lambda/D}$ is an abelian group containing D that preserves λ and λ/S, as well as any D-invariant refinement of λ or coarsening of λ/S.

8. Centralizers and involutions

Fix a prime p. In Section 4, we showed that under some hypotheses the Bredon homology of a G-space X can be calculated using the approximation $X_{\overline{\mathcal{S}}_p}$ of X. Our next goal is to further reduce the size of the approximating collection in the special case $G = \Sigma_n$ and $X = \mathcal{P}_n$. More specifically, we would like to use the methods of Section 5 to eliminate, as much as possible, subgroups in $\overline{\mathcal{S}}_p(\Sigma_n)$ whose fixed points on $\mathcal{P}_n$ are not contractible (the "problematic" subgroups). According to Proposition 6.2, these are elementary abelian p-subgroups of Σ_n that act freely on $\mathbf{n}$.

In this section, we build on the group theory developed in Section 7 to study the centralizers of problematic subgroups inside of isotropy groups of $\mathcal{P}_n$. The main result is Proposition 8.2, which gives us the algebraic data needed to eliminate problematic subgroups of Σ_n that act non-transitively on $\mathbf{n}$. Thus we will conclude in Section 9 that the only problematic subgroups that must be included in the approximating collection are the transitive ones.

Throughout this section, let D be an abelian p-group that acts freely and non-transitively on $\mathbf{n}$. (It is not necessary to assume that D is elementary.) Let $\Lambda = (\lambda_0 < \lambda_1 < ... < \lambda_j)$ be a simplex of $\mathcal{P}_n$, and let $K = K_\Lambda \subseteq \Sigma_n$ be the isotropy group of Λ. Assume that $D \subseteq K$ (so D stabilizes Λ). As usual, let $C_K(D)$, $N_K(D)$, and $W_K(D) = N_K(D)/D$ denote the centralizer, normalizer, and Weyl group of D in K, respectively. Let M be a coefficient system for Σ_n that takes values in $\mathbb{Z}_{(p)}$-modules. By Proposition 5.4, in order to eliminate D we need to show that the collection $\mathcal{S}_p(W_K(D))$ is $M(\Sigma_n/D)$-ample.

The usual strategy is to use Proposition 5.5, for which we need to show that $W_K(D)$ has an element of order p that acts trivially on $M(\Sigma_n/D)$. Typically such elements are found in $C_K(D)/D$, and therefore we would like to know that $C_K(D)/D$ has elements of order p. Subgroups that do not satisfy this condition are called *p-centric*. It turns out that in "most" cases D is not p-centric in K (so it can be eliminated using Proposition 5.5). This is the first case of Proposition 8.2. However, in some important cases D is p-centric in K, as in the following example.

Example 8.1. Let $p = 3$ and $n = 18$. Recall that a transitive elementary abelian 3-group $\Delta_2 \cong (\mathbb{Z}/3)^2$ of Σ_9 is given by the action of $(\mathbb{Z}/3)^2$ on its own elements by translation. Let Λ consist of a single partition λ_0, which partitions $\mathbf{n}$ by the nine two-element sets $\{2i-1, 2i\}$. Let $D \cong (\mathbb{Z}/3)^2$ be the diagonal embedding of Δ_2 in Σ_{18} as permutations of odd integers and even integers.

That is, D acts transtively on $\{1, 3, \ldots, 17\}$ and on $\{2, 4, \ldots, 18\}$ and satisfies $d(2i-1)+1 = d(2i)$, so D stabilizes λ_0 and acts transitively on its classes.
The isotropy group $K = K_\Lambda$ is $\Sigma_2 \wr \Sigma_9 \subseteq \Sigma_{18}$, and $D = \tilde{\Delta}_2$ (see Definition 7.2 for the notation). By Lemma 7.3, we know that the normalizer of D in K is $N_K(D) = \Sigma_2 \times \tilde{N}$, where $N = N_{\Sigma_9}(\Delta_2)$, and by inspection the centralizer is $C_K(D) = \Sigma_2 \times \tilde{C}$, where $C = C_{\Sigma_9}(\Delta_2)$. However, Δ_2 is self-centralizing in Σ_9. Hence $C_K(D)/D \cong \Sigma_2$ and has no elements of order 3, and D is 3-centric in K. Even worse, the Weyl group of D in K is $W_K(D) = \Sigma_2 \times \mathrm{GL}_2(\mathbb{F}_3)$. This means that D is also "3-radical" in K, i.e., the Weyl group has no nontrivial normal 3-subgroups. As a consequence, it is difficult to eliminate D by standard methods.

We will show in Proposition 8.2 that p-radical situations like Example 8.1 can occur only when the prime p is odd, and that in this case the Weyl group $W_K(D)$ has an odd involution that acts trivially on the poset $\mathcal{S}_p(W_K(D))$. (In Example 8.1, the involution comes from the factor Σ_2 of $\Sigma_2 \times \mathrm{GL}_2(\mathbb{F}_3)$.) By assumption (3) in Theorem 1.1, such an involution will act by -1 on $M(\Sigma_n/D)$. It will turn out that the presence of this involution implies that $\mathcal{S}_p(W)$ is ample for the trivial reason that all the relevant homology and cohomology groups vanish. (We learned from the referee that this type of argument is sometimes referred to as "center kills.")
The following proposition is the main result of this section. It says that either D is not p-centric in K, or the centralizer of D has an odd involution that acts trivially on the poset $\mathcal{S}_p(W)$. The proof of the proposition occupies the remainder of this section. By an "odd involution" we mean a permutation of order 2 that can be written as a product of an odd number of transpositions.

Proposition 8.2. *Let $K \in \operatorname{Iso}(\mathcal{P}_n) \cup \{\Sigma_n\}$, and let $D \subseteq K$ be an abelian p-subgroup of Σ_n that acts freely and non-transitively on $\mathbf{n}$. Let $C_K(D)$ be the centralizer of D in K. Then either*

1. *$p \mid [C_K(D) : D]$, or*
2. *p is odd, and there is an odd involution in $C_K(D)$ that acts trivially on the poset of p-subgroups of the normalizer of D in K.*

Proof. Recall the following construction from the end of Section 7. Suppose that S is a subgroup of $D \subseteq \Sigma_n$ and that λ is a partition of $\mathbf{n}$. We define $D \oplus_S S^{\mathrm{cl}(\lambda/D)}$ as the pushout in the category of abelian groups of the diagram $D \leftarrow S \rightarrow S^{\mathrm{cl}(\lambda/D)}$. This pushout is an abelian subgroup of Σ_n that contains D. By Lemma 7.18, $D \oplus_S S^{\mathrm{cl}(\lambda/D)}$ stabilizes any D-invariant refinement of λ and any D-invariant coarsening of λ/S.
To prove the proposition, first suppose that $K = \Sigma_n$, and let μ be the partition of $\mathbf{n}$ by the orbits of D. Because D does not act transitively on $\mathbf{n}$, the partition μ has more than one equivalence class. Then $C_K(D)$ contains the subgroup $D^{\mathrm{cl}(\mu)} \cong D \oplus_D D^{\mathrm{cl}(\mu/D)}$, which is a p-subgroup of Σ_n strictly containing D. Hence $p \mid [C_K(D) : D]$.

Now suppose $K \in \mathrm{Iso}(\mathcal{P}_n)$ is the isotropy group of a nondegenerate chain of proper, nontrivial partitions, say

$$\Lambda = (\lambda_0 < \lambda_1 < \ldots < \lambda_j) .$$

By assumption, $D \subseteq K$ so D stabilizes Λ. For each i, the group K acts on the set $\mathrm{cl}(\lambda_i)$, and therefore so does D. Let i be the smallest number for which the action of D on the set of classes of λ_i is transitive. (If D does not act transitively even on $\mathrm{cl}(\lambda_j)$, then we interpret the argument that follows with $i = j+1$, and we understand λ_{j+1} to be the indiscrete partition of $\mathbf{n}$, consisting of just one equivalence class. Likewise, if $i = 0$, we use the convention that λ_{-1} is the discrete partition of $\mathbf{n}$ into singleton sets.)
Let $S \subset D$ be the subgroup that acts trivially on the set of classes of λ_i, i.e.,

$$S := \ker\left(D \to \Sigma_{\mathrm{cl}(\lambda_i)}\right).$$

There are two cases: *(i)* S is nontrivial, in which case it turns out that $p \mid [C_K(D) : D]$; or *(ii)* $S = \{e\}$, in which case it turns out that sometimes $p \mid [C_K(D) : D]$, but if not, then p is odd and there is an odd involution in $C_K(D)$ that acts trivially on the poset of p-subgroups of the normalizer of D in K.
Suppose first that S is a nontrivial group. Let $\mu = \lambda_{i-1}/D$, that is, μ is the finest mutual coarsening of the partition λ_{i-1} and the partition of $\mathbf{n}$ by the orbits of D. Consider the group $D \oplus_S S^{\mathrm{cl}(\mu)}$. Since D acts non-transitively on $\mathrm{cl}(\lambda_{i-1})$ by assumption, we know that μ has more than one equivalence class, so $D \oplus_S S^{\mathrm{cl}(\mu)}$ is an abelian p-group that strictly contains D. Lemma 7.18 implies immediately that $D \oplus_S S^{\mathrm{cl}(\mu)}$ stabilizes $\lambda_0 < \cdots < \lambda_{i-1}$. Further, by construction S acts trivially on $\mathrm{cl}(\lambda_i)$, that is, the classes of λ_i are unions of orbits of S. Since λ_i is also a coarsening of λ_{i-1}, this means that λ_i is a D-invariant coarsening of λ_{i-1}/S. Lemma 7.18 now tells us that λ_i (as well as $\lambda_{i+1}, \ldots, \lambda_j$) is stabilized by $D \oplus_S S^{\mathrm{cl}(\mu)}$. That is, $D \oplus_S S^{\mathrm{cl}(\mu)}$ stabilizes all of Λ, so $D \oplus_S S^{\mathrm{cl}(\mu)} \subset K$. Therefore $p \mid [C_K(D) : D]$ in this case.
Now suppose that S is trivial. Then D acts freely and transitively on the classes of λ_i, and we are in the situation of Lemma 7.17. Let $\mathbf{c}$ be a fixed class of λ_i, and let $K(\mathbf{c})$ be the isotropy group of the restriction of Λ to $\mathbf{c}$. Let $G = \mathrm{im}\left(K \hookrightarrow \Sigma_{\mathrm{cl}(\lambda_i)}\right)$. Notice that since D acts freely on $\mathrm{cl}(\lambda_i)$, we can regard D as a subgroup of G. By Lemma 7.17, $K \cong K(\mathbf{c}) \wr G$. In particular, K contains a subgroup $\mathrm{Diag}(K(\mathbf{c}))$, isomorphic to $K(\mathbf{c})$, that centralizes D and has trivial intersection with D. Thus $C_K(D)/D$ contains a subgroup isomorphic to $K(\mathbf{c})$. If p divides the order of $K(\mathbf{c})$ then $p \mid [C_K(D) : D]$, and we are done.
Suppose that p does not divide the order of $K(\mathbf{c})$. Since K acts transitively on the classes of λ_i, by Lemma 7.10 the restriction of λ_0 to any class of λ_i is isomorphic to the restriction of λ_0 to $\mathbf{c}$. By assumption, the partition λ_0 is not discrete, and therefore the restriction of λ_0 to $\mathbf{c}$ is not discrete.
It follows that there exist elements $x, y \in \mathbf{c}$ that belong to the same class of λ_0. The transposition $(x\ y)$ that interchanges these two elements is an element of $K(\mathbf{c})$. In particular, the order of $K(\mathbf{c})$ is always divisible by 2, so

if $p \nmid |K(\mathbf{c})|$, then $p > 2$. Now let σ be the image of $(x\ y)$ under the diagonal embedding $K(\mathbf{c}) \hookrightarrow K \cong K(\mathbf{c}) \wr G$. The element σ is a product of disjoint transpositions, as many transpositions as there are elements of D, which is a power of p. In particular the number of transpositions is odd, so σ is an odd involution. The element σ is in $\mathrm{Diag}(K(\mathbf{c}))$ and therefore it centralizes D.
Moreover, by Lemma 7.3, we have $N_K(D) \cong K(\mathbf{c}) \times N_G(D)$. Since p does not divide the order of $K(\mathbf{c})$, every p-subgroup of $K(\mathbf{c}) \times N_G(D)$ is contained in $N_G(D)$, and is centralized by $K(\mathbf{c})$. It follows that σ centralizes every p-subgroup of $N_K(D)$, and in particular σ acts trivially on the poset of such p-subgroups. □

9. Eliminating Problematic Subgroups

We saw in Section 6 that if D is a problematic p-subgroup of Σ_n (i.e., $(\mathcal{P}_n)^D$ is not contractible), then D is an elementary abelian p-group that acts freely on $\mathbf{n}$. In this section, we introduce two conditions on a Mackey functor M for Σ_n. In the main result of this section, Proposition 9.4, we show that these conditions permit the use of Proposition 5.4 to discard the problematic subgroups when they do not act transitively on $\mathbf{n}$. The conditions hold for the Mackey functors that we have in mind for applications. (See Section 11.)
The Σ_n-spaces we need to approximate using the methods of previous sections are $\mathcal{P}_n$ and $*$. Hence throughout this section we assume an ambient isotropy subgroup $K \in \mathrm{Iso}(\mathcal{P}_n) \cup \{\Sigma_n\}$. We must show that for any K containing a problematic subgroup D, the ampleness condition of Proposition 5.4 is met. To apply the methods of Section 5, we need centralizing elements that act trivially on coefficients. We need an appropriate characterization of the coefficients that will apply in our cases of interest (Section 11) and be sufficient to guarantee the existence of the needed elements.
Recall from Example 1.3 that our prime examples of Mackey functors for applications are constructed by means of certain homotopy functors applied to S^n. Note that a permutation $\sigma \in C_{\Sigma_n}(D)$ induces a D-equivariant map $\sigma_\sharp\colon S^n \to S^n$. It turns out to be useful to look at what happens when we pass to the general linear group by embedding $\Sigma_n \hookrightarrow \mathrm{GL}_n\,\mathbb{R}$ as permutations of the standard basis. Let $C_{\mathrm{GL}_n\,\mathbb{R}}(D)$ denote the centralizer of D in $\mathrm{GL}_n\,\mathbb{R}$. Notice that if $\sigma \in \ker\left[C_{\Sigma_n}(D) \to \pi_0 C_{\mathrm{GL}_n\,\mathbb{R}}(D)\right]$, then the map $\sigma_\sharp\colon S^n \to S^n$ is actually D-equivariantly homotopic to the identity map, which is a good sign for trivial action on our coefficients.

Definition 9.1. We say that M *satisfies the centralizer condition for D* if the kernel of $C_{\Sigma_n}(D) \to \pi_0 C_{\mathrm{GL}_n\,\mathbb{R}}(D)$ acts trivially on $M\left(\Sigma_n/D\right)$.

As discussed in Section 11, our primary examples of Mackey functors satisfy the centralizer condition. The centralizer condition allows us to use Proposition 5.5 to eliminate problematic groups that are not p-centric in the ambient isotropy group K.
In the following definition, we refer to an odd permutation of order 2 as an "odd involution."

Definition 9.2. Let p be an odd prime. We say that M *satisfies the involution condition for* D if any odd involution in $C_{\Sigma_n}(D)$ acts on $M(\Sigma_n/D)$ by multiplication by -1.

The involution condition enables us to eliminate problematic subgroups in the few cases when they happen to be p-centric.

Definition 9.3. We say that the Mackey functor M *satisfies the centralizer condition* (resp. *satisfies the involution condition*) if it satisfies the corresponding condition in Definition 9.1 (resp. Definition 9.2) for all elementary abelian p-subgroups of Σ_n that act freely and non-transitively on $\mathbf{n}$.

Examples of Mackey functors that satisfy both the centralizer and the involution conditions are given in Section 11.

Our main result in this section is the following proposition. The term "ample" was defined at the beginning of Section 5.

Proposition 9.4. *Let $D \subset \Sigma_n$ be an elementary abelian p-subgroup that acts freely and non-transitively on $\mathbf{n}$, and let M be a Mackey functor for Σ_n taking values in $\mathbb{Z}_{(p)}$-modules. Assume that*

- *M satisfies the centralizer condition for D, and*
- *if p is odd, M satisfies the involution condition for D.*

Then for any $K \in \operatorname{Iso}(\mathcal{P}_n) \cup \{\Sigma_n\}$ such that $D \subseteq K$, we have that $\mathcal{S}_p(W_K(D))$ is $M(\Sigma_n/D)$-ample.

Before the proof, we need a lemma from representation theory, which follows immediately from [5, Thm 1.3.4].

Lemma 9.5. *If $D \subseteq \mathrm{GL}_n \mathbb{R}$ is finite, then $\pi_0 C_{\mathrm{GL}_n \mathbb{R}}(D)$ is an elementary abelian 2-group.*

We have everything we need for the odd-primary case, so we handle this first.

Proof of Proposition 9.4 for p odd, p dividing $|C_K(D)/D|$.
Pick $x \in C_K(D)/D$ of order p. If $\tilde{x} \in C_K(D)$ is an inverse image of x, it is clear from Lemma 9.5 that $\tilde{x}$ belongs to the kernel of $C_K(D) \to \pi_0 C_{\mathrm{GL}_n \mathbb{R}}(D)$. In view of the centralizer condition, x acts trivially on $M(\Sigma_n/D)$. The desired conclusion follows from Proposition 5.5. □

Proof of Proposition 9.4 for p odd, p not dividing $|C_K(D)/D|$.
We will show that $\mathcal{S}_p(W_K(D))$ is M-ample with $M = M(\Sigma_n/D)$ by showing that all of the relevant homology and cohomology groups vanish. To declutter the notation for quotients, let $\overline{C} = C_K(D)/D$, let $\overline{N} = N_K(D)/C_K(D)$, and let $W = W_K(D)$. The short exact sequence

$$1 \to \overline{C} \to W \to \overline{N} \to 1 \tag{9.6}$$

shows that the map $|\mathcal{S}_p(W)|_{hW} \to BW$ can be written as

$$(|\mathcal{S}_p(W)|_{h\overline{C}})_{h\overline{N}} \to \left(B\overline{C}\right)_{h\overline{N}}.$$

The Serre spectral sequence shows that for the homology case, it is enough to prove that the local coefficient groups $H_*(|\mathcal{S}_p(W)|_{h\overline{C}}; M)$ and $H_*(B\overline{C}; M)$ vanish (respectively, for cohomology, that the groups $H^*(|\mathcal{S}_p(W)|_{h\overline{C}}; M)$ and $H^*(B\overline{C}; M)$ vanish).

We will handle the case $H_*(|\mathcal{S}_p(W)|_{h\overline{C}}; M)$; the others are similar. By Proposition 8.2, there exists an odd involution $\tau \in C_K(D)$ that acts trivially on poset of p-subgroups of $N_K(D)$, and τ projects to an involution $\overline{\tau} \in \overline{C}$. The element $\overline{\tau}$ acts trivially on the space $|\mathcal{S}_p(W)|$ and, in view of the involution condition, acts by -1 on $M(\Sigma_n/D)$. Consider the Serre spectral sequence of

$$|\mathcal{S}_p(W)| \to |\mathcal{S}_p(W)|_{h\overline{C}} \to B\overline{C}\,.$$

Since M is a $\mathbb{Z}_{(p)}$-module and $\overline{C}$ has order prime to p, we know that $E^2_{i,j} = 0$ for $i > 0$, while the group $E^2_{0,j}$ is given by the coinvariants of the action of $\overline{C}$ on $H_j(|\mathcal{S}_p(W)|; M)$. However, $\overline{\tau} \in \overline{C}$ acts trivially on $|\mathcal{S}_p(W)|$ and acts on M by -1, so $\overline{\tau}$ acts on $H_j(|\mathcal{S}_p(W)|; M)$ by -1. Since the coinvariants of this $\overline{\tau}$-action vanish, the groups $H_0\left(\overline{C}; H_j(|\mathcal{S}_p(W)|; M)\right)$ vanish for all j, and the Serre spectral sequence collapses to zero at E^2. $\square$

The remainder of the proof of Proposition 9.4, for $p = 2$, requires two known lemmas. As usual, $|\mathcal{S}_p(G)|$ denotes the nerve of the poset of nontrivial p-subgroups of a finite group G, and the group G acts on $\mathcal{S}_p(G)$ by conjugation. The following lemma is due to Grodal.

Lemma 9.7 ([11], Proposition 5.7). *Let G be a finite group with a normal subgroup H of order prime to p. Then $|\mathcal{S}_p(G)|/H$ is isomorphic to $|\mathcal{S}_p(G/H)|$.*

The following well-known lemma is due to Quillen. See the proof of Proposition 2.4 in [16].

Lemma 9.8 ([16]). *If G is a finite group with a nontrivial normal p-subgroup, then $|\mathcal{S}_p(G)|$ is contractible.*

Let C_0 denote the subgroup of $C_K(D)$ generated by D and the kernel of the map $C_K(D) \to \pi_0 \operatorname{GL}_n \mathbb{R}$. It is clear from Lemma 9.5 that $C_K(D)/C_0$ is an elementary abelian 2-group.

Proof of Proposition 9.4 for $p = 2$, p dividing $|C_0/D|$. The same reasoning applies as in the case p odd, p dividing $|C_K(D)/D|$. $\square$

Proof of Proposition 9.4 for $p = 2$, p not dividing $|C_0/D|$. By Proposition 8.2, p divides $|C_K(D)/D|$, so under the assumption $p \nmid |C_0/D|$, it must be the case that $p \mid |C_K(D)/C_0|$ and $C_K(D)/C_0$ is nontrivial. Observe that C_0 is a normal subgroup of $N_K(D)$, because it is generated by the normal subgroup D and the normal subgroup obtained by intersecting $C_K(D)$ with the kernel of $N_K(D) \to \pi_0 \operatorname{GL}_n \mathbb{R}$. It follows that $C_K(D)/C_0$ is a nontrivial normal 2-subgroup of $N_K(D)/C_0$.

Let $W = W_K(D)$, and as usual, let $|\mathcal{S}_p(W)|$ denote the nerve of the poset of nontrivial p-subgroups of W. We must show that $|\mathcal{S}_p(W)|_{hW} \to BW$

induces isomorphisms on twisted M-homology and M-cohomology, where $M = M(\Sigma_n/D)$. Let $\overline{C}_0$ denote C_0/D. As in (9.6), we have a short exact sequence

$$1 \to \overline{C}_0 \to W \to N/C_0 \to 1,$$

and a Serre spectral sequence argument like that following (9.6) establishes that we need only show that the map $|\mathcal{S}_p(W)|_{h\overline{C}_0} \to B\overline{C}_0$ induces an isomorphism on M-homology and M-cohomology. Further, because of the definition of C_0 and the assumption that M satisfies the centralizer condition, the action of $\overline{C}_0$ on M is trivial, so we have untwisted coefficients. Consider the following commutative diagram comparing homotopy orbits to strict orbits:

$$\begin{array}{ccc} |\mathcal{S}_p(W)|_{h\overline{C}_0} & \longrightarrow & (*)_{h\overline{C}_0} \\ \downarrow & & \downarrow \\ |\mathcal{S}_p(W)|/\overline{C}_0 & \longrightarrow & * \end{array} .$$

Since M is a $\mathbb{Z}_{(p)}$-module, it is enough to show that all of the maps in this diagram are p-local equivalences. By Lemma 9.7, the orbit space $|\mathcal{S}_p(W)|/\overline{C}_0$ is isomorphic to the nerve of the poset of nontrivial p-subgroups of $N_K(C)/C_0$, and that poset is weakly contractible (Lemma 9.8), because $N_K(C)/C_0$ has the nontrivial normal p-subgroup $C_K(D)/C_0$. Thus the lower horizontal map is an equivalence. The right vertical map is a p-local equivalence because the order of $\overline{C}_0$ is prime to p. In the same way the isotropy groups of the action of $\overline{C}_0$ on $|\mathcal{S}_p(W)|$ are all of order prime to p, and so the isotropy spectral sequence of this action [9, 2.4] shows that the left vertical map is a p-local equivalence as well. □

Remark 9.9. Grodal has pointed out to us that [11] can be used for an alternative proof of Proposition 9.4. It is shown in [11, Ex. 8.6 and §9] that $\mathcal{S}_p(W)$ is cohomologically M-ample if and only if $\mathrm{Hom}(\mathrm{St}_*(W), M)$ is acyclic, where $\mathrm{St}_*(W)$ is the Steinberg complex of W as defined in [11]. Dually, $\mathcal{S}_p(W)$ is homologically M-ample if and only if $\mathrm{St}_*(W) \otimes M$ is acyclic. To show that the assumptions of Proposition 9.4 imply ampleness, one can use Proposition 8.2 together with the properties of the Steinberg complex from [11, §5] to establish acyclicity. This essentially representation-theoretic approach could be useful in other applications and generalizations, where the assumptions on the Mackey functor might vary.

10. Results of approximating

In this section, we assemble the results from previous sections to establish the main results announced in the introduction and restated below.

Theorem 1.1. *Fix a prime p. Let M be a Mackey functor for Σ_n that takes values in $\mathbb{Z}_{(p)}$-modules. Assume the following.*

(1) *The Mackey functor M is projective relative to the collection of p-subgroups of Σ_n.*

(2) *For every elementary abelian p-subgroup $D \subset \Sigma_n$ that acts freely and non-transitively on $\{1,\dots,n\}$, the kernel of the homomorphism $C_{\Sigma_n}(D) \to \pi_0 C_{\mathrm{GL}_n\mathbb{R}}(D)$ acts trivially on $M(\Sigma_n/D)$.*
(3) *If p is odd and D is as above, then every odd involution in $C_{\Sigma_n}(D)$ acts on $M(\Sigma_n/D)$ by multiplication by -1.*

Then if n is not a power of p, the groups $\tilde{H}_*^{\Sigma_n}(\mathcal{P}_n^\diamond; M)$ and $\tilde{H}^*_{\Sigma_n}(\mathcal{P}_n^\diamond; M)$ vanish. If $n = p^k$, then the map

$$\Sigma_{n+} \wedge_{\mathrm{Aff}_k} (E\,\mathrm{GL}_{k\,+} \wedge B_k^\diamond) \longrightarrow \mathcal{P}_n^\diamond$$

induces an isomorphism on $\tilde{H}_*^{\Sigma_n}(-; M)$ and on $\tilde{H}^*_{\Sigma_n}(-; M)$.

Recall that in the following corollary, St_k denotes $\tilde{H}_{k-1}(B_k^\diamond; \mathbb{Z})$ and R denotes the ring $\mathbb{Z}[\mathrm{GL}_k(\mathbb{F}_p)]$.

Corollary 1.2. *In the setting of Theorem 1.1, suppose that $n = p^k$. Then there are isomorphisms*

$$\tilde{H}_j^{\Sigma_n}(\mathcal{P}_n^\diamond; M) \cong \begin{cases} 0 & j \neq k-1 \\ M(\Sigma_n/\Delta_k) \otimes_R \mathrm{St}_k & j = k-1 \end{cases}$$

Moreover, there are isomorphisms for all j between Bredon homology and cohomology groups: $\tilde{H}_j^{\Sigma_n}(\mathcal{P}_n^\diamond; M) \cong \tilde{H}^j_{\Sigma_n}(\mathcal{P}_n^\diamond; M)$ for all $j \geq 0$.

First we need a small lemma. Let G be a finite group, and let $\mathbf{G}$ denote the underlying set of G. The group G acts on $\mathbf{G}$ by left translation, and hence on the poset $\mathcal{P}(\mathbf{G})$ of nontrivial, proper partitions of $\mathbf{G}$.

Lemma 10.1. *The fixed point poset $\mathcal{P}(\mathbf{G})^G$ is canonically isomorphic to the poset of proper, nontrivial subgroups of G.*

Proof. Let λ be a partition of G that is invariant under the action of G. To associate a subgroup to the partition, let $G(\lambda)$ be the equivalence class of the identity element $e \in G$. We claim that $G(\lambda)$ is a subgroup of G. Indeed, let $g_1, g_2 \in G(\lambda)$. Then $e \sim_\lambda g_1$, and since λ is G-invariant, we find that $g_2 = g_2 e \sim_\lambda g_2 g_1$. But $g_2 \sim_\lambda e$ also, and so $g_2 g_1 \sim_\lambda g_2 \sim_\lambda e$. Thus $G(\lambda)$ is a subgroup of G. If λ is neither the discrete nor the indiscrete partition of $\mathbf{n}$, then $G(\lambda)$ is a proper, nontrivial subgroup of G.
Conversely, to associate a partition of $\mathbf{G}$ to a subgroup $H \subseteq G$, we take the partition of the set $\mathbf{G}$ by the cosets of H.
It remains to check that $\lambda \mapsto G(\lambda)$ and $H \mapsto \{gH\}_{g\in G}$ are inverses. To see this, observe that

$$\begin{aligned} g_1 \sim_\lambda g_2 &\iff g_2^{-1} g_1 \sim_\lambda e \\ &\iff g_2^{-1} g_1 \in G(\lambda) \\ &\iff g_1 \text{ and } g_2 \text{ are in the same coset of } G(\lambda). \end{aligned}$$

□

Proof of Theorem 1.1. Recall that the family of all p-subgroups of Σ_n is denoted by $\overline{\mathcal{S}}_p(\Sigma_n)$. Let $\mathcal{C}$ consist of all p-subgroups of Σ_n except the elementary abelian p-subgroups that act freely on $\mathbf{n}$, and $\mathcal{D}$ the collection containing all elementary abelian p-subgroups of Σ_n that act freely and transitively on $\mathbf{n}$. The collection $\mathcal{D}$ is empty unless $n = p^k$, in which case it consists entirely of conjugates of the subgroup Δ_k. Certainly $\mathcal{C}$ is closed under passage to p-supergroups, and $\mathcal{D}$ is initial in $\mathcal{C} \cup \mathcal{D}$. Let $X = \mathcal{P}_n$, and consider the commutative diagram of Σ_n-spaces

$$\begin{array}{ccccc} X_{\mathcal{C}\cup\mathcal{D}} & \longrightarrow & X_{\overline{\mathcal{S}}_p} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ (*)_{\mathcal{C}\cup\mathcal{D}} & \longrightarrow & (*)_{\overline{\mathcal{S}}_p} & \longrightarrow & * \end{array} . \tag{10.2}$$

By Proposition 4.6, the horizontal arrows in the right-hand square induce isomorphisms on Bredon homology and cohomology with coefficients in M, because M is projective relative to $\overline{\mathcal{S}}_p(\Sigma_n)$. We are assuming that M takes values in $\mathbb{Z}_{(p)}$-modules, so Propositions 5.4 and 9.4 imply that the horizontal arrows in the left-hand square induce isomorphisms in Bredon homology and cohomology.

If $n \neq p^k$, then $\mathcal{D}$ is empty. Meanwhile, Proposition 6.2 tells us that the left vertical map is an equivalence on fixed point sets of subgroups in $\mathcal{C}$, hence a Σ_n-equivalence. It is therefore an isomorphism on Bredon homology and cohomology. Connecting the isomorphisms around the outside of the diagram gives Theorem 1.1 for $n \neq p^k$.

Suppose $n = p^k$, so that $\mathcal{D}$ consists of conjugates of Δ_k. In this case, the leftmost vertical arrow in (10.2) is not a Bredon (co)homology isomorphism, but we can still calculate it. Lemma 5.2 gives a homotopy pushout diagram

$$\begin{array}{ccc} X_{\mathcal{D}} & \longrightarrow & X_{\mathcal{C}\cup\mathcal{D}} \\ \downarrow & & \downarrow \\ (*)_{\mathcal{D}} & \longrightarrow & (*)_{\mathcal{C}\cup\mathcal{D}} \end{array} . \tag{10.3}$$

We can use the explicit formula of Lemma 4.4 to give a formula for $X_{\mathcal{D}}$ once we know X^{Δ_k}. As a Δ_k-set, $\mathbf{n}$ is isomorphic to Δ_k acting on itself by left translation, so by Lemma 10.1, we find that $X^{\Delta_k} \cong B_k$. Sticking (10.3) together with (10.2) and applying Lemma 4.4 to compute $X_{\mathcal{D}}$ and $(*)_{\mathcal{D}}$ gives us the diagram

$$\begin{array}{ccccc} \Sigma_n \times_{\mathrm{Aff}_k} (E\,\mathrm{GL}_k \times B_k) & \longrightarrow & X_{\mathcal{C}\cup\mathcal{D}} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma_n \times_{\mathrm{Aff}_k} (E\,\mathrm{GL}_k) & \longrightarrow & (*)_{\mathcal{C}\cup\mathcal{D}} & \longrightarrow & * \end{array} .$$

Here the left square is a homotopy pushout and the horizontal maps are isomorphisms in Bredon (co)homology. Taking cofibers vertically gives us the result for $n = p^k$. $\square$

Proof of Corollary 1.2. Let $G = \mathrm{GL}_k$. Since $M(\Sigma_n/\Delta_k)$ is an $\mathbb{Z}_{(p)}$-module, we may take $R = \mathbb{Z}_{(p)}[G]$ and $\mathrm{St}_k = \tilde{H}_{k-1}(B_k^{\diamond}; \mathbb{Z}_{(p)})$.
Let $Y = \Sigma_{n+} \wedge_{\mathrm{Aff}_k} (EG_+ \wedge B_k^{\diamond})$. All of the isotropy subgroups of Y are conjugate to Δ_k, so the calculation of Lemma 4.4 tells us that there are isomorphisms

$$\tilde{H}_*^G(Y; M) \cong \tilde{H}_*((B_k^{\diamond})_{\tilde{\mathrm{h}}G}; M)$$
$$\tilde{H}_G^*(Y; M) \cong \tilde{H}^*((B_k^{\diamond})_{\tilde{\mathrm{h}}G}; M)\,.$$

Consider the local coefficient homology Serre spectral sequence

$$E^2_{i,j} = \mathrm{Tor}_i^R(\tilde{H}_j(B_k^{\diamond}; \mathbb{Z}_{(p)}), M) \Rightarrow \tilde{H}_*((B_k^{\diamond})_{\tilde{\mathrm{h}}G}; M)\,.$$

The spectral sequence collapses at E^2 to give the desired homology calculation, because $\tilde{H}_j(B_k^{\diamond}; \mathbb{Z}_{(p)})$ vanishes for $j \neq k-1$ and is well known to give a projective R-module for $j = k-1$. A similar calculation with a local coefficient cohomology Serre spectral sequence, together with the fact that St_k is self-dual, completes the proof. □

11. Examples

In this section we describe some particular Mackey functors for Σ_n, and show that they satisfy the hypotheses of Theorem 1.1 and Corollary 1.2. The general construction is described in Definition 11.3 and Proposition 11.4 below.
We will consider Mackey functors with values in categories other than abelian groups. If $\mathcal{C}$ is any additive category, then one defines a Mackey functor with values in $\mathcal{C}$ to be a pair of additive functors $(\gamma, \gamma^{\natural})$ from the category of finite G-sets to $\mathcal{C}$, satisfying the same hypotheses as in Definition 3.1. In particular, we will consider Mackey functors with values in the category of graded abelian groups (graded Mackey functors).

Remark 11.1. Note that if M is a Mackey functor with values in $\mathcal{C}$, and $F\colon \mathcal{C} \to \mathcal{D}$ is an additive functor between additive categories, then $F \circ M$ is a Mackey functor with values in $\mathcal{D}$.

Our basic construction of a Mackey functor involves the homotopy category of spectra with an action of G. This is unsurprising, as the connection between Mackey functors and equivariant stable homotopy theory is well known. We will not require the full strength of this theory, and in particular we will only use a naive version of equivariant stable homotopy theory. By the *homotopy category of spectra with an action of* G, we mean the category of spectra with an action of G, localized with respect to equivariant maps that are weak equivalences of the underlying nonequivariant spectra. Equivalently, this is the stable homotopy category of G-spaces, or G-simplicial sets. This category of spectra with an action of G supports a Quillen model structure, where fibrations and weak equivalences are defined in the underlying category of spectra. (See, for example [18] or [12, Theorem 11.6.1].) Therefore, its homotopy category is well-defined. Note also that the homotopy category of spectra with an action of G is an additive category.

Define the functor γ from the category of finite G-sets to the homotopy category of spectra with an action of G by the formula $\gamma(S) = \Sigma^\infty S_+$. The following is a standard fact.

Lemma 11.2. *The functor γ extends naturally to a Mackey functor with values in the homotopy category of spectra with an action of G.*

Proof. The extended functor γ is additive by construction. The contravariant functor $\gamma^\natural$ is necessarily defined on objects to be the same as γ. To define $\gamma^\natural$ on morphisms, let the superscript $^\vee$ denote the Spanier-Whitehead dual of a spectrum. Given a finite set S, there is a weak equivalence $\Sigma^\infty S_+ \to (\Sigma^\infty S_+)^\vee$. This equivalence is natural with respect to set isomorphisms; in particular, if S is a G-set, the equivalence is G-equivariant. Let $S \to T$ be a G-map between finite G-sets. We define the map $\gamma^\natural(T) \to \gamma^\natural(S)$ as the composite

$$\Sigma^\infty T_+ \to (\Sigma^\infty T_+)^\vee \to (\Sigma^\infty S_+)^\vee \xleftarrow{\simeq} \Sigma^\infty S_+.$$

Note that the "wrong way map" is a weak equivalence, and therefore is invertible in the homotopy category of spectra with an action of G.
Consider the diagrams below. That $M = (\gamma, \gamma^\natural)$ is a Mackey functor follows from the fact that given a pullback diagram on the left of finite G-sets, the diagram on the right commutes in the homotopy category of spectra with an action of G:

$$\begin{tikzcd} S \arrow[r, "u"] \arrow[d, "\alpha"'] & T \arrow[d, "\beta"'] \\ U \arrow[r, "v"] & V \end{tikzcd} \qquad\qquad \begin{tikzcd} \Sigma^\infty S_+ \arrow[r, "\gamma(u)"] & \Sigma^\infty T_+ \\ \Sigma^\infty U_+ \arrow[r, "\gamma(v)"] \arrow[u, "\gamma^\natural(\alpha)"] & \Sigma^\infty V_+ \arrow[u, "\gamma^\natural(\beta)"] \end{tikzcd} .$$

□

We say that a functor from spectra to spectra is *additive* if it respects equivalences and preserves finite coproducts up to equivalence. Recall that Σ_n acts on the one-point compactification S^n of $\mathbb{R}^n$ by permuting coordinates, and hence on the j-fold smash product S^{nj}. The following is the general construction of Mackey functors that we wish to consider.

Definition 11.3. Suppose that j is a fixed integer, with j odd if p is odd, and that F is an additive functor from spectra to spectra. For each finite Σ_n-set T, define the graded abelian group $M_F(T)$ by

$$M_F(T) = \pi_* F\left((\Sigma^\infty T_+ \wedge S^{nj})_{\tilde{\mathrm{h}}\Sigma_n} \right) .$$

Our main result in this section is Proposition 11.4 below. Let $L_{(p)}$ denote the functor on spectra given by localization at p.

Proposition 11.4. *The assignment $T \mapsto M_F(T)$ extends naturally to a Mackey functor for Σ_n that satisfies the centralizer condition (see Definition 9.3). If $F \to F \circ L_{(p)}$ is an equivalence, then M_F takes values in $\mathbb{Z}_{(p)}$-modules, is projective relative to p-subgroups, and (if p is odd) satisfies the involution condition.*

We give the proof of Proposition 11.4 after a few examples.

Example 11.5. If $F(X) = H\mathbb{F}_p \wedge X$, then $M_F(T)$ is the $\mathbb{F}_p$-homology of the relevant reduced Borel construction. In this case Proposition 11.4 and Theorem 1.1, taken together, give a relatively conceptual approach to the homological calculations of Arone-Mahowald in [2].

Example 11.6. One advantage of our approach is that it applies in situations in which explicit calculation is impossible, e.g., when $F = L_{(p)}$ itself. Here $M_F(T)$ is the p-local *stable homotopy* of the given Borel construction. The calculation of Bredon homology in this case provides a key ingredient for a new proof of some theorems of Kuhn [13] and Kuhn-Priddy [15] on the Whitehead conjecture (see also [14] for Kuhn's latest word on the subject). The Bredon cohomology also leads to a new proof of the collapse of the homotopy spectral sequence of the Goodwillie tower of the identity functor evaluated at S^1. This was done by Behrens [3] at the prime 2 and then by Kuhn [14] at all primes. We intend to pursue this in another paper.

Example 11.7. Another interesting example to which our results apply is the functor

$$F(X) = (E \wedge X)_K.$$

Here E is the Morava E-theory and the subscript $_K$ denotes localization with respect to Morava K theory. This example, and others similar to it, were considered recently by Rezk [17] and Behrens [4]. It seems that our methods can be used to recover some of their calculations. For example, Lemma 5.6 of [4] seems to be closely related to our main theorem, applied to the functor F above.

Proof of Proposition 11.4. We saw in Lemma 11.2 that the functor $T \mapsto \Sigma^\infty T_+$ extends to a Mackey functor. The functor M_F is obtained by composing the suspension spectrum functor with the following functors: smash product with S^{nj}, taking Σ_n-homotopy orbits, F, and π_*. Each of these functors is additive (on the level of homotopy categories), and therefore M_F extends to a Mackey functor by Remark 11.1.

Next we claim that the Mackey functor $T \mapsto L_p(\Sigma^\infty T_+ \wedge S^{nj})_{\tilde{h}\Sigma_n}$ (with values in the homotopy category of spectra) has the p-transfer property. By this we mean that for every Σ_n-set Z of cardinality prime to p, the following composed map is an equivalence, i.e., an isomorphism in the homotopy category of spectra:

$$\begin{aligned} L_p\left(\Sigma^\infty T_+ \wedge S^{nj}\right)_{\tilde{h}\Sigma_n} &\longrightarrow L_p\left(\Sigma^\infty (Z\times T)_+ \wedge S^{nj}\right)_{\tilde{h}\Sigma_n} \\ &\longrightarrow L_p(\Sigma^\infty T_+ \wedge S^{nj})_{\tilde{h}\Sigma_n}. \end{aligned}$$

(See Definition 3.7.) To see this, note that our functor is equivalent to $T \mapsto \left(L_p(\Sigma^\infty T_+) \wedge S^{nj}\right)_{\tilde{h}\Sigma_n}$, so it is enough to prove that $L_p(\Sigma^\infty T_+)$ has the p-transfer property in the same sense:

$$L_p\left(\Sigma^\infty T_+\right) \longrightarrow L_p\left(\Sigma^\infty \left(Z\times T\right)_+\right) \longrightarrow L_p\left(\Sigma^\infty T_+\right) \tag{11.8}$$

should be an equivalence of Σ_n-spectra for every Σ_n-set Z of cardinality prime to p. However, the effect of the composition $\Sigma^\infty T_+ \to \Sigma^\infty (Z\times T)_+ \to \Sigma^\infty T_+$ on homology is multiplication by $|Z|$. Thus it induces an isomorphism on homology with coefficients in $\mathbb{Z}_{(p)}$, and (11.8) is an equivalence. Since $F \simeq F \circ L_p$, it follows that M_F has the p-transfer property. By Lemma 3.8, M_F takes values in $\mathbb{Z}_{(p)}$-modules, and is projective relative to p-subgroups.

Next we consider the centralizer condition. To set the stage, consider a general situation where Z is a space with a pointed action of Σ_n. Let D be a subgroup of Σ_n, with centralizer $C_{\Sigma_n}(D)$. The orbit space of the action of D on Z still has an action of $C_{\Sigma_n}(D)$, and there is a homeomorphism between two models of this $C_{\Sigma_n}(D)$-space given by:

$$\begin{aligned} \phi : Z/D &\xrightarrow{\cong} (\Sigma_n/D_+ \wedge Z)/\Sigma_n \\ Dz &\mapsto (eD, z). \end{aligned}$$

The action of $c \in C_{\Sigma_n}(D)$ on the domain is given by $c(Dz) = D(cz)$ and on the codomain is given by $c(\sigma D, z) = (\sigma c^{-1} D, z)$, and ϕ is $C_{\Sigma_n}(D)$-equivariant with respect to these actions.

We will apply the paragraph above with $Z = (E\Sigma_n)_+ \wedge S^{nj}$, where $C_{\Sigma_n}(D)$ acts on Z diagonally. Applying the discussion in the previous paragraph gives us a $C_{\Sigma_n}(D)$-equivariant homeomorphism

$$\phi : \left[(E\Sigma_n)_+ \wedge S^{nj}\right]/D \xrightarrow{\cong} \left[\Sigma_n/D_+ \wedge (E\Sigma_n)_+ \wedge S^{nj}\right]/\Sigma_n. \tag{11.9}$$

The right-hand side is $(\Sigma_n/D_+ \wedge S^{nj})_{\tilde{\mathrm{h}}\Sigma_n}$, and the centralizer condition will be satisfied if we can show that the kernel of $C_{\Sigma_n}(D) \to \pi_0 C_{\mathrm{GL}_n\mathbb{R}}(D)$ acts on this space via maps that are homotopic to the identity.

We establish what is required by using the left-hand side of (11.9) instead. Suppose $c \in \ker[C_{\Sigma_n}(D) \to \pi_0 C_{\mathrm{GL}_n\mathbb{R}}(D)]$. Then the action of c on S^{nj} is homotopic to the identity through D-equivariant maps. Likewise, translation by c on $E\Sigma_n$ is homotopic to the identity, and because c centralizes D, the homotopy is through D-equivariant maps. It follows that the action of c on $\left[(E\Sigma_n)_+ \wedge S^{nj}\right]/D$ is homotopic to the identity, and the same is true of the action on $(\Sigma_n/D_+ \wedge S^{nj})_{\tilde{\mathrm{h}}\Sigma_n}$. We conclude that the action of c on $M_F(\Sigma_n/D)$ is trivial, and hence M_F satisfies the centralizer condition.

It remains to prove that if $F \circ L_{(p)} \simeq F$, and p is odd, then M_F satisfies the involution condition. Suppose in general that τ is an involution of a spectrum X that has 2 invertible in $\pi_*(X)$. Then $\pi_*(X)$ splits as a direct sum of eigenspaces for 1 and -1. It follows that τ acts by -1 on $\pi_*(X)$ if and only if the map $\tau - 1$ is an equivalence.

For our situation, let D and $C_{\Sigma_n}(D)$ be as above, and let τ be an odd involution in $C_{\Sigma_n}(D)$. We need to show that τ acts as multiplication by -1 on $\pi_* F\left(\Sigma^\infty S^{nj}\right)_{\tilde{\mathrm{h}}D}$. Since $F \simeq F \circ L_p$, and L_p commutes with homotopy colimits, it is enough to prove that τ acts by -1 on $\pi_* F\left(L_p\left(\Sigma^\infty S^{nj}\right)_{\tilde{\mathrm{h}}D}\right)$. However, the action of τ is induced by the action of τ on S^{nj}. Since τ is an odd involution, τ acts by a map of degree -1 on S^n and hence, since j is odd, by a map of

degree -1 on S^{nj}. It follows that τ acts by -1 on $\pi_* L_p\left(\Sigma^\infty S^{nj}\right)$. Since p is odd, 2 is invertible in $\pi_* L_p\left(\Sigma^\infty S^{nj}\right)$. By the previous paragraph, the map $\tau - 1$ induces a self-equivalence on $L_p(\Sigma^\infty S^{nj})$.
Since τ is in the centralizer of D, it acts on $L_p(\Sigma^\infty S^{nj})$ by a D-equivariant map. Therefore, the map $\tau - 1$ is D-equivariant. It follows that $\tau - 1$ induces a self-equivalence of $L_p(\Sigma^\infty S^{nj})_{\tilde{\mathrm{h}}D}$. Since F is an additive functor, it follows that $\tau - 1$ induces a self-equivalence of $F(L_p(\Sigma^\infty S^{nj})_{\tilde{\mathrm{h}}D})$. Again, by the same reasoning as above, it follows that τ acts by -1 on $\pi_* F(L_p(\Sigma^\infty S^{nj})_{\tilde{\mathrm{h}}D})$, which is what we wanted to prove. $\square$

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