Hyperplane Mass Partitions via Relative Equivariant Obstruction Theory

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Abstract. The Grünbaum–Hadwiger–Ramos hyperplane mass partition problem was introduced by Grünbaum (1960) in a special case and in general form by Ramos (1996). It asks for the "admissible" triples (d, j, k) such that for any j masses in $\mathbb{R}^d$ there are k hyperplanes that cut each of the masses into 2^k equal parts. Ramos' conjecture is that the Avis–Ramos necessary lower bound condition $dk \geq j(2^k - 1)$ is also sufficient. We develop a "join scheme" for this problem, such that non-existence of an $\mathfrak{S}_k^{\pm}$-equivariant map between spheres $(S^d)^{*k} \to S(W_k \oplus U_k^{\oplus j})$ that extends a test map on the subspace of $(S^d)^{*k}$ where the hyperoctahedral group $\mathfrak{S}_k^{\pm}$ acts non-freely, implies that (d, j, k) is admissible. For the sphere $(S^d)^{*k}$ we obtain a very efficient regular cell decomposition, whose cells get a combinatorial interpretation with respect to measures on a modified moment curve. This allows us to apply relative equivariant obstruction theory successfully, even in the case when the difference of dimensions of the spheres $(S^d)^{*k}$ and $S(W_k \oplus U_k^{\oplus j})$ is greater than one. The evaluation of obstruction classes leads to counting problems for concatenated Gray codes. Thus we give a rigorous, unified treatment of the previously announced cases of the Grünbaum–Hadwiger–Ramos problem, as well as a number of new cases for Ramos' conjecture.

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1 Introduction

1.1 The Grünbaum–Hadwiger–Ramos hyperplane mass partition problem

In 1960, Grünbaum [10, Sec. 4.(v)] asked whether for any convex body in $\mathbb{R}^k$ there are k affine hyperplanes that divide it into 2^k parts of equal volume: This is now known to be true for $k \leq 3$, due to Hadwiger [11] in 1966, and remains open and challenging for $k = 4$. (A weak partition result for $k = 4$ was given in 2009 by Dimitrijević-Blagojević [8].) For $k > 4$ it is false, as shown by Avis [1] in 1984 by considering a measure on a moment curve. In 1996, Ramos [15] proposed the following generalization of Grünbaum's problem.

The Grünbaum–Hadwiger–Ramos problem. *Determine the minimal dimension $d = \Delta(j, k)$ such that for every collection of j masses $\mathcal{M}$ on $\mathbb{R}^d$ there exists an arrangement of k affine hyperplanes $\mathcal{H}$ in $\mathbb{R}^d$ that equiparts $\mathcal{M}$.*

The Ham Sandwich theorem, conjectured by Steinhaus and proved by Banach, states that $\Delta(d, 1) = d$. The Grünbaum–Hadwiger–Ramos hyperplane mass partition problem was studied by many authors. It has been an excellent testing ground for different equivariant topology methods; see to our recent survey in [3].
The first general result about the function $\Delta(j, k)$ was obtained by Ramos [15], by generalizing Avis' observation: The lower bound

$$\Delta(j, k) \geq \tfrac{2^k-1}{k} j$$

follows from considering k measures with disjoint connected supports concentrated along a moment curve in $\mathbb{R}^d$. Ramos also conjectured that this lower bound is tight.

The Ramos conjecture. $\Delta(j, k) = \lceil \frac{2^k-1}{k} j \rceil$ *for every $j \geq 1$ and $k \geq 1$.*

All available evidence up to now supports this, though it has been established rigorously only in special cases.

1.2 Product scheme and join scheme

It seems natural to use $Y_{d,k} := (S^d)^k$ as a configuration space for any k oriented affine hyperplanes/halfspaces in $\mathbb{R}^d$, which leads to the following *product scheme*: If there is no equivariant map

$$(S^d)^k \longrightarrow_{\mathfrak{S}_k^\pm} S(U_k^{\oplus j})$$

from the configuration space to the unit sphere in the space $U_k^{\oplus j}$ of values on the orthants of $\mathbb{R}^k$ that sum to 0, which is equivariant with respect to the hyperoctahedral (signed permutation) group $\mathfrak{S}_k^\pm$, then there is no counterexample for the given parameters, so $\Delta(j, k) \leq d$.

However, our critical review [3] of the main papers on the Grünbaum–Hadwiger–Ramos problem since 1998 has shown that this scheme is very hard to handle: Except for the 2006 upper bounds by Mani-Levitska, Vrećica & Živaljević [13], derived from a Fadell–Husseini index calculation, it has produced very few valid results: The group action on $(S^d)^k$ is not free, the Fadell–Husseini index is rather large and thus yields weak results, and there is no efficient cell complex model at hand.

In this paper, we provide a new approach, which proves to be remarkably clean and efficient. For this, we use a *join scheme*, as introduced by Blagojević and Ziegler [4], which takes the form

$$F: \quad (S^d)^{*k} \longrightarrow_{\mathfrak{S}_k^{\pm}} S(W_k \oplus U_k^{\oplus j}).$$

Here the domain $(S^d)^{*k} \subseteq \mathbb{R}^{(d+1)\times k}$ is a sphere of dimension $dk+k-1$, given by

$$X_{d,k} := \{(\lambda_1 x_1, \dots, \lambda_k x_k) : x_1, \dots, x_k \in S^d,\ \lambda_1, \dots, \lambda_k \geq 0,\ \lambda_1 + \dots + \lambda_k = 1\},$$

where we write $\lambda_1 x_1 + \dots + \lambda_k x_k$ as a short-hand for $(\lambda_1 x_1, \dots, \lambda_k x_k)$. The co-domain is a sphere of dimension $j(2^k-1)+k-2$. Both domain and co-domain are equipped with canonical $\mathfrak{S}_k^{\pm}$-actions. We observe that the map restricted to the points with non-trivial stabilizer (the "non-free part")

$$F': \quad X_{d,k}^{>1} \subset (S^d)^{*k} \longrightarrow_{\mathfrak{S}_k^{\pm}} S(W_k \oplus U_k^{\oplus j})$$

is the same up to homotopy for all test maps. If for any parameters (j,k,d) an equivariant extension F of F' does not exist, we get that $\Delta(j,k) \leq d$.

To decide the existence of this map, or at least obtain necessary criteria, we employ relative equivariant obstruction theory, as explained by tom Dieck [7, Sect. II.3]. This turns out to work beautifully, and have a few remarkable aspects:

- The Fox–Neuwirth [9]/Björner–Ziegler [2] combinatorial stratification method yields a simple and efficient cone stratification for the space $\mathbb{R}^{(d+1)\times k}$, which is equivariant with respect to the action of $\mathfrak{S}_k^{\pm}$ on the columns, and which respects the arrangement of k^2 subspaces of codimension d given by columns of a matrix $(x_1, \dots, x_d)$ being equal, opposite, or zero.
- This yields a small equivariant regular CW complex model for the sphere $(S^d)^{*k} \subseteq \mathbb{R}^{(d+1)\times k}$, for which the the non-free part, given by an arrangement of k^2 subspheres of codimension $d+1$, is an invariant subcomplex. The cells $D_I^S(\sigma)$ in the complex are given by combinatorial data.
- To evaluate the obstruction cocycle, we use measures on a non-standard (binomial coefficient) moment curve. For the resulting test map, the relevant cells $D_I^S(\sigma)$ can be interpreted as k-tuples of hyperplanes such that some of the hyperplanes have to pass through prescribed points of the moment curve, or equivalently, they have to bisect some extra masses.

1.3 Statement of the main results

The join scheme reduces the Grünbaum–Hadwiger–Ramos problem to a combinatorial counting problem that can be solved by hand or by means of a computer: A k-bit *Gray code* is a $k \times 2^k$ binary matrix of all column vectors of length k such that two consecutive vectors differ by only one bit. Such a k-bit code can be interpreted as a Hamiltonian path in the graph of the k-cube $[0,1]^k$. The *transition count* of a row in a binary matrix A is the number of bit-changes, *not* counting a bit change from the last to the first entry. By *transition counts* of a matrix A we refer to the vector of the transition counts of the rows of the matrix A. Two binary matrices A and A' are *equivalent*, if A can be obtained from A' by a sequence of permutations of rows and/or inversion of bits in rows.

Definition 1.1. Let $d \geq 1$, $j \geq 1$, $\ell \geq 0$ and $k \geq 1$ be integers such that $dk = (2^k - 1)j + \ell$ with $0 \leq \ell \leq d-1$. A binary matrix A of size $k \times j2^k$ is an *ℓ-equiparting matrix* if

(a) $A = (A_1, \dots, A_j)$ for Gray codes $A_1, \dots, A_j$ with the property that the last column of A_i is equal to the first column of A_{i+1} for $1 \leq i < j$; and
(b) there is one row of the matrix A with the transition count $d - \ell$, while all other rows have transition count d.

Example 1.2. If $d = 5$, $j = 2$, $\ell = 1$ and $k = 3$, then a possible 1-equiparting matrix is

$$A = (A_1, A_2) = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

In this example the first row of A has transition count 4 while the remaining two rows have transition count 5.

Theorem 1.3. *Let $d \geq 1$, $j \geq 1$, $\ell \geq 0$ and $k \geq 2$ be integers with the property that $dk = (2^k - 1)j + \ell$ and $0 \leq \ell \leq d-1$. The number of non-equivalent ℓ-equiparting matrices is the number of arrangements of k affine hyperplanes $\mathcal{H}$ that equipart a given collection of j disjoint intervals on a moment curve γ in $\mathbb{R}^d$, up to renumbering and orientation change of hyperplanes in $\mathcal{H}$, when it is forced that one of the hyperplanes passes through ℓ prescribed points on γ that lie to the left of the j disjoint intervals.*

In some situations this yields a solution for the Grünbaum–Hadwiger–Ramos problem.

Theorem 1.4. *Let $j \geq 1$ and $k \geq 3$ be integers, with $d := \lceil \frac{2^k-1}{k} j \rceil$ and $\ell := \lceil \frac{2^k-1}{k} j \rceil k - (2^k-1)j = dk - (2^k-1)j$, which implies $0 \leq \ell < k \leq d$. If the number of non-equivalent ℓ-equiparting matrices of size $k \times j2^k$ is odd, then*

$$\Delta(j,k) = \lceil \tfrac{2^k-1}{k} j \rceil.$$

Theorem 1.4 is also true for $k = 1$ (and thus $d = j$, $\ell = 0$), where it yields the Ham Sandwich theorem: In this case an equiparting matrix A is a row vector of length $2d$ and transition count d. Thus, each A_i is either $(0, 1)$ or $(1, 0)$, where A_i uniquely determines A_{i+1}. Hence, up to inversion of bits A is unique and so $\Delta(d, 1) \leq d$, and consequently $\Delta(d, 1) = d$.

While the situation for $k = 1$ hyperplane is fully understood, we seem to be far from a complete solution for the case of $k = 2$ hyperplanes. However, we do obtain the following instances.

Theorem 1.5. *Let $t \geq 1$. Then:*

(i) $\Delta(2^t - 1, 2) = 3 \cdot 2^{t-1} - 1$,

(ii) $\Delta(2^t, 2) = 3 \cdot 2^{t-1}$,

(iii) $\Delta(2^t + 1, 2) = 3 \cdot 2^{t-1} + 2$.

The statements (i) and (iii) were already known: Part (i) is the only case where the lower bound of Ramos and the upper bound of Mani-Levitska, Vrećica, and Živaljević [13, Thm. 39] coincide. Part (ii) is Hadwiger's result [11] for $t = 1$; the general case was previously claimed by Mani-Levitska et al. [13, Prop. 25]. However, the proof of the result was incorrect and not recoverable, as explained in [3, Sec. 8.1]. Here we recover this result by a different method of proof. Similarly, statement (iii) was claimed by Živaljević [17, Thm. 2.1] with a flawed proof; for an explanation of the gap see [3, Sec. 8.2], where we also gave a proof of (iii) via degrees of equivariant maps [3, Sec. 5]. Here we will prove all three cases of Theorem 1.5 in a uniform way.

In the case of $k = 3$ hyperplanes we prove using Theorem 1.4 the following instances of the Ramos conjecture.

Theorem 1.6.

(i) $\Delta(2, 3) = 5$,

(ii) $\Delta(4, 3) = 10$.

Statement (i) was previously claimed by Ramos [15, Sec. 6.1]. A gap in the method that Ramos developed and used to get this result was explained in [3, Sec. 7]. It is also claimed by Vrećica and Živaljević in the recent preprint [16] without a proof for the crucial [16, Prop. 3].

The reduction result of Hadwiger and Ramos $\Delta(j, k) \leq \Delta(2j, k - 1)$ applied to Theorem 1.6 implies the following consequences. For details on reduction results see for example [3, Sec. 3.3].

Corollary 1.7.

(i) $4 \leq \Delta(1, 4) \leq 5$,

(ii) $8 \leq \Delta(2, 4) \leq 10$.

Note that $\Delta(1, 4)$ is the open case for Grünbaum's original conjecture.

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2 The join configuration space test map scheme and equivariant obstruction theory

In this section we develop the join configuration test map scheme that was introduced in [5, Sec. 2.1]. A sufficient condition for $\Delta(j,k) \le d$ will be phrased in terms of the non-existence of a particular equivariant map between representation spheres.

2.1 Arrangements of k hyperplanes

Let $\hat{H} = \{x \in \mathbb{R}^d : \langle x, v\rangle = a\}$ be an affine hyperplane determined by a vector $v \in \mathbb{R}^d \backslash \{0\}$ and a constant $a \in \mathbb{R}$. The hyperplane $\hat{H}$ determines two (closed) halfspaces

$$\hat{H}^0 = \{x \in \mathbb{R}^d : \langle x, v\rangle \ge a\} \qquad \text{and} \qquad \hat{H}^1 = \{x \in \mathbb{R}^d : \langle x, v\rangle \le a\}.$$

Let $\mathcal{H} = (\hat{H}_1, \dots, \hat{H}_k)$ be an arrangement of k affine hyperplanes in $\mathbb{R}^d$, and let $\alpha = (\alpha_1, \dots, \alpha_k) \in (\mathbb{Z}/2)^k$. The *orthant* determined by the arrangement $\mathcal{H}$ and $\alpha \in (\mathbb{Z}/2)^k$ is the intersection

$$\mathcal{O}^{\mathcal{H}}_{\alpha} = \hat{H}_1^{\alpha_1} \cap \dots \cap \hat{H}_k^{\alpha_k}.$$

Let $\mathcal{M} = (\mu_1, \dots, \mu_j)$ be a collection of finite Borel probability measures on $\mathbb{R}^d$ such that the measure of each hyperplane is zero. Such measures will be called *masses.* The assumptions about the measures guarantee that $\mu_i(\hat{H}^0_s)$ depends continuously on $\hat{H}^0_s$.

An arrangement of affine hyperplanes $\mathcal{H} = (\hat{H}_1, \dots, \hat{H}_k)$ *equiparts* the collection of masses $\mathcal{M} = (\mu_1, \dots, \mu_j)$ if for every element $\alpha \in (\mathbb{Z}/2)^k$ and every $\ell \in \{1, \dots, j\}$

$$\mu_\ell(\mathcal{O}^{\mathcal{H}}_{\alpha}) = \tfrac{1}{2^k}.$$

2.2 The configuration spaces

The space of all oriented affine hyperplanes (or closed affine halfspaces) in $\mathbb{R}^d$ can be parametrized by the sphere S^d, where the north pole e_{d+1} and the south pole $-e_{d+1}$ represent hyperplanes at infinity. An oriented affine hyperplane in $\mathbb{R}^d$ *at infinity* is the set $\mathbb{R}^d$ or $\emptyset$, depending on the orientation. Indeed, embed $\mathbb{R}^d$ into $\mathbb{R}^{d+1}$ via the map $(\xi_1, \dots, \xi_d)^t \longmapsto (1, \xi_1, \dots, \xi_d)^t$. Then an oriented affine hyperplane $\hat{H}$ in $\mathbb{R}^d$ defines an oriented affine $(d-1)$-dimensional subspace of $\mathbb{R}^{d+1}$ that extends (uniquely) to an oriented linear hyperplane H in $\mathbb{R}^{d+1}$. The outer unit normal vector that determines the oriented linear hyperplane is a point on the sphere S^d.

We consider the following configuration spaces that parametrize arrangements of k oriented affine hyperplanes in $\mathbb{R}^d$:

(1) *The join configuration space*: $X_{d,k} := (S^d)^{*k} \cong S(\mathbb{R}^{(d+1)\times k})$,

(2) *The product configuration space*: $Y_{d,k} := (S^d)^k$.

The elements of the join $X_{d,k}$ can be presented as formal convex combinations $\lambda_1 v_1 + \dots + \lambda_k v_k$, where $\lambda_i \ge 0$, $\sum \lambda_i = 1$ and $v_i \in S^d$.

2.3 The group actions

The space of all ordered k-tuples of oriented affine hyperplanes in $\mathbb{R}^d$ has natural symmetries: Each hyperplane can change orientation and the hyperplanes can be permuted. Thus, the group $\mathfrak{S}_k^{\pm} := (\mathbb{Z}/2)^k \rtimes \mathfrak{S}_k$ encodes the symmetries of both configuration spaces.

The group $\mathfrak{S}_k^{\pm}$ acts on $X_{d,k}$ as follows. Each copy of $\mathbb{Z}/2$ acts antipodally on the appropriate sphere S^d in the join while the symmetric group $\mathfrak{S}_k$ acts by permuting factors in the join product. More precisely, for $((\beta_1,\dots,\beta_k)\rtimes\pi)\in\mathfrak{S}_k^{\pm}$ and $\lambda_1 v_1+\dots+\lambda_k v_k\in X_{d,k}$ the action is given by

$$((\beta_1,\dots,\beta_k)\rtimes\tau)\cdot(\lambda_1 v_1+\dots+\lambda_k v_k) = \lambda_{\tau^{-1}(1)}(-1)^{\beta_1}v_{\tau^{-1}(1)}+\dots+\lambda_{\tau^{-1}(k)}(-1)^{\beta_k}v_{\tau^{-1}(k)}.$$

The product space $Y_{d,k}$ is a subspace of the join $X_{d,k}$ via the diagonal embedding $Y_{d,k}\longrightarrow X_{d,k}, (v_1,\dots,v_k)\longmapsto \frac{1}{k}v_1+\dots+\frac{1}{k}v_k$. The product $Y_{d,k}$ is an invariant subspace of $X_{d,k}$ with respect to the $\mathfrak{S}_k^{\pm}$-action and consequently inherits the $\mathfrak{S}_k^{\pm}$-action from $X_{d,k}$. For $k\geq 2$, the action of $\mathfrak{S}_k^{\pm}$ is *not* free on either $X_{d,k}$ or $Y_{d,k}$.

The sets of points in the configuration spaces $X_{d,k}$ and $Y_{d,k}$ that have non-trivial stabilizer with respect to the action of $\mathfrak{S}_k^{\pm}$ can be described as follows:

$$X_{d,k}^{>1} = \{\lambda_1 v_1+\dots+\lambda_k v_k : \lambda_1\cdots\lambda_k = 0,\ \text{or } \lambda_s=\lambda_r \text{ and } v_s=\pm v_r \text{ for some } 1\leq s<r\leq k\},$$

and

$$Y_{d,k}^{>1} = \{(v_1,\dots,v_k) : v_s=\pm v_r \text{ for some } 1\leq s<r\leq k\}.$$

2.4 Test spaces

Consider the vector space $\mathbb{R}^{(\mathbb{Z}/2)^k}$, where the group element $((\beta_1,\dots,\beta_k)\rtimes\tau)\in\mathfrak{S}_k^{\pm}$ acts on a vector $(y_{(\alpha_1,\dots,\alpha_k)})_{(\alpha_1,\dots,\alpha_k)\in(\mathbb{Z}/2)^k}\in\mathbb{R}^{(\mathbb{Z}/2)^k}$ by acting on its indices as

$$((\beta_1,\dots,\beta_k)\rtimes\tau)\cdot(\alpha_1,\dots,\alpha_k) = (\beta_1+\alpha_{\tau^{-1}(1)},\dots,\beta_k+\alpha_{\tau^{-1}(k)}). \quad (1)$$

The subspace of $\mathbb{R}^{(\mathbb{Z}/2)^k}$ defined by

$$U_k = \Big\{(y_\alpha)_{\alpha\in(\mathbb{Z}/2)^k}\in\mathbb{R}^{(\mathbb{Z}/2)^k} : \sum_{\alpha\in(\mathbb{Z}/2)^k} y_\alpha = 0\Big\}$$

is $\mathfrak{S}_k^{\pm}$-invariant and therefore an $\mathfrak{S}_k^{\pm}$-subrepresentation.

Next we consider the vector space $\mathbb{R}^k$ and its subspace

$$W_k = \Big\{(z_1,\dots,z_k)\in\mathbb{R}^k : \sum_{i=1}^k z_i = 0\Big\}.$$

The group $\mathfrak{S}_k^{\pm}$ acts on $\mathbb{R}^k$ by permuting coordinates, i.e., for $((\beta_1,\dots,\beta_k)\rtimes\tau)\in\mathfrak{S}_k^{\pm}$ and $(z_1,\dots,z_k)\in\mathbb{R}^k$ we have

$$((\beta_1,\dots,\beta_k)\rtimes\tau)\cdot(z_1,\dots,z_k)=(z_{\tau^{-1}(1)},\dots,z_{\tau^{-1}(k)}). \tag{2}$$

In particular, the subgroup $(\mathbb{Z}/2)^k$ of $\mathfrak{S}_k^{\pm}$ acts trivially on $\mathbb{R}^k$. The subspace $W_k\subset\mathbb{R}^k$ is $\mathfrak{S}_k^{\pm}$-invariant and consequently a $\mathfrak{S}_k^{\pm}$-subrepresentation.

2.5 Test maps

The product test map associated to the collection of j masses $\mathcal{M}=(\mu_1,\dots,\mu_j)$ from the configuration space $Y_{d,k}$ to the *test space* $U_k^{\oplus j}$ is defined by

$$\begin{aligned}\phi_{\mathcal{M}}\colon\qquad Y_{d,k}&\longrightarrow U_k^{\oplus j},\\ (v_1,\dots,v_k)&\longmapsto\Big(\big(\mu_i(H_{v_1}^{\alpha_1}\cap\cdots\cap H_{v_k}^{\alpha_k})-\tfrac{1}{2^k}\big)_{(\alpha_1,\dots,\alpha_k)\in(\mathbb{Z}/2)^k}\Big)_{i\in\{1,\dots,j\}}.\end{aligned}$$

In this paper we mostly work with the join configuration space $X_{d,k}$. The corresponding join test map associated to a collection of j masses $\mathcal{M}=(\mu_1,\dots,\mu_j)$ maps the configuration space $X_{d,k}$ into the related *test space* $W_k\oplus U_k^{\oplus j}$. It is defined by

$$\begin{aligned}\psi_{\mathcal{M}}\colon\qquad X_{d,k}&\longrightarrow W_k\oplus U_k^{\oplus j},\\ \lambda_1v_1+\cdots+\lambda_kv_k&\longmapsto(\lambda_1-\tfrac{1}{k},\dots,\lambda_k-\tfrac{1}{k})\oplus(\lambda_1\cdots\lambda_k)\cdot\phi_{\mathcal{M}}(v_1,\dots,v_k).\end{aligned}$$

Both maps $\phi_{\mathcal{M}}$ and $\psi_{\mathcal{M}}$ are $\mathfrak{S}_k^{\pm}$-equivariant with respect to the actions defined in Sections 2.3 and 2.4. Let $S(U_k^{\oplus j})$ and $S(W_k\oplus U_k^{\oplus j})$ denote the unit spheres in the vector spaces $U_k^{\oplus j}$ and $W_k\oplus U_k^{\oplus j}$, respectively. The maps $\phi_{\mathcal{M}}$ and $\psi_{\mathcal{M}}$ are called test maps since we have the following criterion, which reduces finding an equipartition to finding zeros of the test map.

Proposition 2.1. *Let $d\ge 1$, $k\ge 1$, and $j\ge 1$ be integers.*

(i) *Let $\mathcal{M}$ be a collection of j masses on $\mathbb{R}^d$, and let*

$$\phi_{\mathcal{M}}\colon Y_{d,k}\longrightarrow U_k^{\oplus j}\quad\textit{and}\quad\psi_{\mathcal{M}}\colon X_{d,k}\longrightarrow W_k\oplus U_k^{\oplus j}$$

be the $\mathfrak{S}_k^{\pm}$-equivariant maps defined above. If $0\in\operatorname{im}\phi_{\mathcal{M}}$, or $0\in\operatorname{im}\psi_{\mathcal{M}}$, then there is an arrangement of k affine hyperplanes that equiparts $\mathcal{M}$.

(ii) *If there is no $\mathfrak{S}_k^{\pm}$-equivariant map of either type*

$$Y_{d,k}\longrightarrow S(U_k^{\oplus j})\quad\textit{or}\quad X_{d,k}\longrightarrow S(W_k\oplus U_k^{\oplus j}),$$

then $\Delta(j,k)\le d$.

It is worth pointing out that $0 \in \operatorname{im}\phi_{\mathcal{M}}$ if and only if $0 \in \operatorname{im}\psi_{\mathcal{M}}$, while the existence of an $\mathfrak{S}_k^{\pm}$-equivariant map $Y_{d,k} \longrightarrow S(U_k^{\oplus j})$ implies the existence of a $\mathfrak{S}_k^{\pm}$-equivariant map $X_{d,k} \longrightarrow S(W_k \oplus U_k^{\oplus j})$ but not vice versa.
The homotopy class of the restrictions of the test maps $\phi_{\mathcal{M}}$ and $\psi_{\mathcal{M}}$ on the set of points with non-trivial stabilizer (as maps avoiding the origin) is independent of the choice of the masses $\mathcal{M}$, by the following proposition.

Proposition 2.2. *Let $\mathcal{M}$ and $\mathcal{M}'$ be collections of j masses in $\mathbb{R}^d$. Then*

(i) $0 \notin \operatorname{im}\phi_{\mathcal{M}}|_{Y_{d,k}^{>1}}$ *and* $0 \notin \operatorname{im}\psi_{\mathcal{M}}|_{X_{d,k}^{>1}}$,

(ii) $\phi_{\mathcal{M}}|_{Y_{d,k}^{>1}}$ *and* $\phi_{\mathcal{M}'}|_{Y_{d,k}^{>1}}$ *are $\mathfrak{S}_k^{\pm}$-homotopic as maps* $Y_{d,k}^{>1} \longrightarrow U_k^{\oplus j}\backslash\{0\}$, *and*

(iii) $\psi_{\mathcal{M}}|_{X_{d,k}^{>1}}$ *and* $\psi_{\mathcal{M}'}|_{X_{d,k}^{>1}}$ *are $\mathfrak{S}_k^{\pm}$-homotopic as maps* $X_{d,k}^{>1} \longrightarrow (W_k \oplus U_k^{\oplus j})\backslash\{0\}$.

Proof. (i) If $(v_1, \dots, v_k) \in Y_{d,k}^{>1}$, then $v_s = \pm v_r$ for some $1 \le s < r \le k$. Consequently, the corresponding hyperplanes H_{v_i} and H_{v_j} coincide, possibly with opposite orientations. Thus some of the orthants associated to the collection of hyperplanes $(H_{v_1}, \dots, H_{v_k})$ are empty. Consequently, Proposition 2.1 implies that $0 \notin \operatorname{im}\phi_{\mathcal{M}}|_{Y_{d,k}^{>1}}$.
In the case where $\lambda_1 v_1 + \cdots + \lambda_k v_k \in X_{d,k}^{>1}$ the additional case $\lambda_s = 0$ for some $1 \le s \le k$ may occur. If $\lambda_s = 0$, then the s-th coordinate of $\psi(\lambda_1 v_1 + \cdots + \lambda_k v_k) \in W_k \oplus U_k^{\oplus j}$ is equal to $-\frac{1}{k}$, and hence $0 \notin \operatorname{im}\psi_{\mathcal{M}}|_{X_{d,k}^{>1}}$.
(ii) The equivariant homotopy between $\phi_{\mathcal{M}}|_{Y_{d,k}^{>1}}$ and $\phi_{\mathcal{M}'}|_{Y_{d,k}^{>1}}$ is just the linear homotopy in $U_k^{\oplus j}$. For this the linear homotopy should not have zeros, compare [3, proof of Cor. 5.4]. It suffices to prove that for each point $(v_1, \dots, v_k) \in Y_{d,k}^{>1}$, the points $\phi_{\mathcal{M}}(v_1, \dots, v_k)$ and $\phi_{\mathcal{M}'}(v_1, \dots, v_k)$ belong to some affine subspace of the test space that is not linear.
First, observe that $\mathbb{R}^{(\mathbb{Z}/2)^k}$, considered as a real $(\mathbb{Z}/2)^k$ representation, is the real regular representation of $(\mathbb{Z}/2)^k$ and therefore it decomposes into the direct sum of all real irreducible representations. For this we use the fact that all real irreducible representations of $(\mathbb{Z}/2)^k$ are 1-dimensional. The subspace U_k seen as a real $(\mathbb{Z}/2)^k$ subrepresentation of $(\mathbb{Z}/2)^k$ decomposes as follows:

$$U_k \cong \bigoplus_{\alpha \in (\mathbb{Z}/2)^k \backslash \{0\}} V_\alpha. \tag{3}$$

Here V_α is the 1-dimensional real representation of $(\mathbb{Z}/2)^k$ determined by $\beta \cdot v = -v$ for $x \in V_\alpha$ if and only if $\alpha \cdot \beta := \sum \alpha_s \beta_s = 1 \in \mathbb{Z}/2$, for $\beta \in (\mathbb{Z}/2)^k$. The isomorphism (3) is given by the direct sum of the projections $\pi_\alpha : U_k \longrightarrow V_\alpha$, $\alpha \in (\mathbb{Z}/2)^k \backslash \{0\}$,

$$(y_\beta)_{\beta \in (\mathbb{Z}/2)^k \backslash \{0\}} \longmapsto \sum_{\alpha\cdot\beta=1} y_\beta - \sum_{\alpha\cdot\beta=0} y_\beta.$$

Now let $v_s = \pm v_r$. Consider $\alpha \in (\mathbb{Z}/2)^k$ given by $\alpha_s = 1 = \alpha_r$ and $\alpha_\ell = 0$ for $\ell \notin \{s, r\}$, and the corresponding projection $\pi_\alpha^{\oplus j} \colon U_k^{\oplus j} \longrightarrow V_\alpha^{\oplus j}$. Then

$$\pi_\alpha^{\oplus j} \circ \phi_{\mathcal{M}}(v_1, \dots, v_k) = \pi_\alpha^{\oplus j} \circ \phi_{\mathcal{M}'}(v_1, \dots, v_k) = (\pm 1, \dots, \pm 1).$$

(iii) Likewise, the linear homotopy between $\psi_{\mathcal{M}}|_{X_{d,k}^{>1}}$ and $\psi_{\mathcal{M}'}|_{X_{d,k}^{>1}}$ is equivariant and avoids zero. Let $\lambda_1 v_1 + \dots + \lambda_k v_k \in X_{d,k}^{>1}$. If $\lambda := \lambda_1 \cdots \lambda_k \neq 0$, $\lambda_s = \lambda_r$ and $v_s = \pm v_r$, then

$$\begin{aligned}(\pi_\alpha^{\oplus j} \circ \eta \circ \psi_{\mathcal{M}})(\lambda_1 v_1 + \dots + \lambda_k v_k) = \\ = (\pi_\alpha^{\oplus j} \circ \eta \circ \psi_{\mathcal{M}'})(\lambda_1 v_1 + \dots + \lambda_k v_k) = (\pm\lambda, \dots, \pm\lambda),\end{aligned}$$

where $\eta \colon W_k \oplus U_k^{\oplus j} \longrightarrow U_k^{\oplus j}$ is the projection. Finally, in the case when $\lambda_s = 0$ for some $1 \le s \le k$, $\psi_{\mathcal{M}}(\lambda_1 v_1 + \dots + \lambda_k v_k)$ and $\psi_{\mathcal{M}'}(\lambda_1 v_1 + \dots + \lambda_k v_k)$ after projection to the sth coordinate of the subrepresentation W_k are equal to $-\frac{1}{k}$. □

Denote the radial projections by

$$\rho \colon U_k^{\oplus j} \backslash \{0\} \longrightarrow S(U_k^{\oplus j}) \quad \text{and} \quad \nu \colon (W_k \oplus U_k^{\oplus j}) \backslash \{0\} \longrightarrow S(W_k \oplus U_k^{\oplus j}).$$

Note that ρ and ν are $\mathfrak{S}_k^{\pm}$-equivariant maps. Now the criterion stated in Proposition 2.1 (ii) can be strengthened as follows.

Theorem 2.3. *Let $d \ge 1$, $k \ge 1$ and $j \ge 1$ be integers, and let $\mathcal{M}$ be a collection of j masses in $\mathbb{R}^d$. We have the following two criteria:*

(i) *If there is no $\mathfrak{S}_k^{\pm}$-equivariant map*

$$Y_{d,k} \longrightarrow S(U_k^{\oplus j})$$

whose restriction to $Y_{d,k}^{>1}$ is $\mathfrak{S}_k^{\pm}$-homotopic to $\rho \circ \phi_{\mathcal{M}}|_{Y_{d,k}^{>1}}$, then $\Delta(j,k) \le d$.

(ii) *If there is no $\mathfrak{S}_k^{\pm}$-equivariant map*

$$X_{d,k} \longrightarrow S(W_k \oplus U_k^{\oplus j})$$

whose restriction to $X_{d,k}^{>1}$ is $\mathfrak{S}_k^{\pm}$-homotopic to $\nu \circ \psi_{\mathcal{M}}|_{X_{d,k}^{>1}}$, then $\Delta(j,k) \le d$.

2.6 Applying relative equivariant obstruction theory

In order to prove Theorems 1.4, 1.5, and 1.6 via Theorem 2.3(ii), we study the existence of an $\mathfrak{S}_k^{\pm}$-equivariant map

$$X_{d,k} \longrightarrow S(W_k \oplus U_k^{\oplus j}), \tag{4}$$

whose restriction to $X_{d,k}^{>1}$ is $\mathfrak{S}_k^{\pm}$-homotopic to $\nu \circ \psi_{\mathcal{M}}|_{X_{d,k}^{>1}}$ for some fixed collection $\mathcal{M}$ of j masses in $\mathbb{R}^d$. If we prove that such a map cannot exist, Theorems 1.4, 1.5, and 1.6 follow.

Denote by

$$N_1 := (d+1)k - 1$$

the dimension of the sphere $X_{d,k} = (S^d)^{*k}$, and by

$$N_2 := (2^k - 1)j + k - 2$$

the dimension of the sphere $S(W_k \oplus U_k^{\oplus j})$.
If $N_1 \le N_2$, then

$$\dim X_{d,k} = N_1 \le \operatorname{conn}\big(S(W_k \oplus U_k^{\oplus j})\big) + 1 = N_2.$$

Consequently, all obstructions to the existence of an $\mathfrak{S}_k^{\pm}$-equivariant map (4) vanish and so the map exists. Here $\operatorname{conn}(\cdot)$ denotes the connectivity of a space. Therefore, we assume that $N_1 > N_2$, which is equivalent to the Ramos lower bound $d \ge \frac{2^k-1}{k} j$. Furthermore, the following prerequisites for applying equivariant obstruction theory are satisfied:

- The N_1-sphere $X_{d,k}$ can be given the structure of a relative $\mathfrak{S}_k^{\pm}$-CW complex $X := (X_{d,k}, X_{d,k}^{>1})$ with a free $\mathfrak{S}_k^{\pm}$-action on $X_{d,k} \backslash X_{d,k}^{>1}$: In Section 3 we construct an explicit relative $\mathfrak{S}_k^{\pm}$-CW complex that models $X_{d,k}$.
- The sphere $S(W_k \oplus U_k^{\oplus j})$ is path connected and N_2-simple, except in the trivial case of $k = j = 1$ when $N_2 = 0$. Indeed, the group $\pi_1(S(W_k \oplus U_k^{\oplus j}))$ is abelian for $N_2 = 1$ and trivial for $N_2 > 1$ and therefore its action on $\pi_{N_2}(S(W_k \oplus U_k^{\oplus j}))$ is trivial.
- The $\mathfrak{S}_k^{\pm}$-equivariant map $h\colon X_{d,k}^{>1} \longrightarrow S(W_k \oplus U_k^{\oplus j})$ given by the composition $h := \nu \circ \psi_{\mathcal{M}}|_{X_{d,k}^{>1}}$, for a fixed collection of j masses $\mathcal{M}$, serves as the base map for extension.

Since the sphere $S(W_k \oplus U_k^{\oplus j})$ is (N_2-1)-connected, the map h can be extended to a $\mathfrak{S}_k^{\pm}$-equivariant map from the N_2-skeleton $X^{(N_2)} \longrightarrow S(W_k \oplus U_k^{\oplus j})$. A necessary criterion for the existence of the $\mathfrak{S}_k^{\pm}$-equivariant map (4) extending h is that the $\mathfrak{S}_k^{\pm}$-equivariant map $h = \nu \circ \psi_{\mathcal{M}}|_{X_{d,k}^{>1}}$ can be extended to a map from the (N_2+1)-skeleton $X^{(N_2+1)} \longrightarrow S(W_k \oplus U_k^{\oplus j})$.
Given the above hypotheses, we can apply relative equivariant obstruction theory, as presented by tom Dieck [7, Sec. II.3], to decide the existence of such an extension.
If g is an equivariant extension of h to the N_2-skeleton $X^{(N_2)}$, then the obstruction to extending g to the (N_2+1)-skeleton is encoded by the equivariant cocycle

$$\mathfrak{o}(g) \in \mathcal{C}^{N_2+1}_{\mathfrak{S}_k^{\pm}}\big(X_{d,k}, X_{d,k}^{>1}; \pi_{N_2}(S(W_k \oplus U_k^{\oplus j}))\big).$$

The $\mathfrak{S}_k^{\pm}$-equivariant map $g\colon X^{(N_2)} \longrightarrow S(W_k \oplus U_k^{\oplus j})$ extends to $X^{(N_2+1)}$ if and only if $\mathfrak{o}(g) = 0$. Furthermore, the cohomology class

$$[\mathfrak{o}(g)] \in \mathcal{H}^{N_2+1}_{\mathfrak{S}_k^{\pm}}\big(X_{d,k}, X_{d,k}^{>1}; \pi_{N_2}(S(W_k \oplus U_k^{\oplus j}))\big),$$

vanishes if and only if the restriction $g|_{X^{(N_2-1)}}$ to the (N_2-1)-skeleton can be extended to the (N_2+1)-skeleton $X^{(N_2+1)}$. Any two extensions g and g' of h to the N_2-skeleton are equivariantly homotopic on the (N_2-1)-skeleton and therefore the cohomology classes coincide: $[\mathfrak{o}(g)] = [\mathfrak{o}(g')]$. Hence, it suffices to compute the cohomology class $[\mathfrak{o}(\nu \circ \psi_{\mathcal{M}}|_{X^{(N_2)}})]$ for a fixed collection of j masses $\mathcal{M}$ with the property that $0 \notin \operatorname{im}(\psi_{\mathcal{M}}|_{X^{(N_2)}})$.

Let f be the attaching map for an (N_2+1)-cell θ and e its corresponding basis element in the cellular chain group $C_{N_2+1}(X_{d,k}, X_{d,k}^{>1})$. Then

$$\mathfrak{o}(\nu \circ \psi_{\mathcal{M}}|_{X^{(N_2)}})(e) = [\nu \circ \psi_{\mathcal{M}} \circ f|_{\partial\theta}]$$

is the homotopy class of the map represented by the composition

$$\partial\theta_j \xrightarrow{f|_{\partial\theta}} X^{(N_2)} \xrightarrow{\nu\circ\psi_{\mathcal{M}}|_{X^{(N_2)}}} S(W_k \oplus U_k^{\oplus j}).$$

Since $\partial\theta$ and $S(W_k \oplus U_k^{\oplus j})$ are spheres of the same dimension N_2, the homotopy class $[\nu \circ \psi_{\mathcal{M}} \circ f|_{\partial\theta}]$ is determined by the degree of the map $\nu \circ \psi_{\mathcal{M}} \circ f|_{\partial\theta}$. Here we assume that the $\mathfrak{S}_k^{\pm}$-CW structure on $X_{d,k}$ is endowed with cell orientations, and in addition an orientation on the sphere $S(W_k \oplus U_k^{\oplus j})$ is fixed in advance. Therefore, the degree of the map $\nu \circ \psi_{\mathcal{M}} \circ f|_{\partial\theta}$ is well-defined.

Let $\alpha := \psi_{\mathcal{M}} \circ f|_{\partial\theta}$. In order to compute the degree of the map $\nu \circ \alpha$ and consequently the obstruction cocycle evaluated at e, fix the collection of measures as follows. Let $\mathcal{M}$ be the collection of masses $(I_1, \ldots, I_j)$ where I_r is the mass concentrated on the segment $\gamma((t_r^1, t_r^2))$ of the moment curve in $\mathbb{R}^d$

$$\gamma(t) = (t, \tbinom{t}{2}, \tbinom{t}{3}, \ldots, \tbinom{t}{d})^t,$$

such that

$$\ell < t_1^1 < t_1^2 < t_2^1 < t_2^2 < \cdots < t_j^1 < t_j^2,$$

for an integer ℓ, $0 \le \ell \le d-1$. The intervals $(I_1, \ldots, I_j)$ determined by numbers $t_r^1 < t_r^2$ can be chosen in such a way that $0 \notin \operatorname{im}(\psi_{\mathcal{M}}|_{X^{(N_2)}})$. For every concrete situation in Section 4 this is verified directly.

Now consider the following commutative diagram:

$$\begin{tikzcd}
\partial\theta \arrow[r, "f|_{\partial\theta}"] \arrow[d] & X^{(N_2)} \arrow[r, "\psi_{\mathcal{M}}|_{X^{(N_2)}}"] \arrow[d] & W_k \oplus U_k^{\oplus j}\backslash\{0\} \arrow[r, "\nu"] \arrow[d] & S(W_k \oplus U_k^{\oplus j}) \\
\theta \arrow[r, "f"] & X^{(N_2+1)} \arrow[r, "\psi_{\mathcal{M}}|_{X^{(N_2+1)}}"] & W_k \oplus U_k^{\oplus j} &
\end{tikzcd}$$

where the vertical arrows are inclusions, and the composition of the lower horizontal maps is denoted by $\beta := \psi_{\mathcal{M}}|_{X^{(N_2+1)}} \circ f$. Furthermore, let $B_\varepsilon(0)$ be a ball with center 0 in $W_k \oplus U_k^{\oplus j}$ of sufficiently small radius $\varepsilon > 0$. Set $\widetilde{\theta} := \theta\backslash\beta^{-1}(B_\varepsilon(0))$. Since $\dim\theta = \dim W_k \oplus U_k^{\oplus j}$ we can assume that the set of zeros $\beta^{-1}(0) \subset \operatorname{relint}\theta$ is finite, say of cardinality $r \ge 0$. Again, in every calculation presented in Section 4 this assumption is explicitly verified. The function β is a restriction of the test map and therefore *the points in* $\beta^{-1}(0)$ *correspond to arrangements of* k *hyperplanes* $\mathcal{H}$ *in* relint θ *that equipart* $\mathcal{M}$. Moreover, the facts that the measures are intervals on a moment curve and that each hyperplane of the arrangement from $\beta^{-1}(0)$ cuts the moment curve in d distinct points imply that each zero in $\beta^{-1}(0)$ is isolated and transversal. The boundary of $\widetilde{\theta}$ consists of the boundary $\partial\theta$ and r disjoint copies of N_2-spheres $S_1, \dots, S_r$, one for each zero of β on θ. Consequently, the fundamental class of $\partial\theta$ is equal to the sum of fundamental classes $\sum[S_i]$ in $H_{N_1}(\widetilde{\theta};\mathbb{Z})$. Here the fundamental class of $\partial\theta$ is determined by the cell orientation inherited from the $\mathfrak{S}_k^{\pm}$-CW structure on $X_{d,k}$. The fundamental classes of $[S_i]$ are determined in such a way that the equality $[\partial\theta] = \sum[S_i]$ holds. Thus

$$\sum(\nu\circ\beta|_{\widetilde{\theta}})_*([S_i]) = (\nu\circ\beta|_{\widetilde{\theta}})_*([\partial\theta]) = (\nu\circ\alpha)_*([\partial\theta]) = \deg(\nu\circ\alpha)\cdot[S(W_k\oplus U_k^{\oplus j})].$$

Recall, we have fixed the orientation on the sphere $S(W_k \oplus U_k^{\oplus j})$ and so the fundamental class $[S(W_k \oplus U_k^{\oplus j})]$ is also completely determined. On the other hand,

$$\sum(\nu \circ \beta|_{S_i})_*([S_i]) = \Big(\sum \deg(\nu \circ \beta|_{S_i})\Big) \cdot [S(W_k \oplus U_k^{\oplus j})].$$

Hence, $\deg(\nu\circ\alpha) = \sum \deg(\nu\circ\beta|_{S_i})$ where the sum ranges over all arrangements of k hyperplanes $\mathcal{H}$ in relint θ that equipart $\mathcal{M}$; consult [14, Prop. IV.4.5]. In other words,

$$\mathfrak{o}(\nu\circ\psi_{\mathcal{M}}|_{X^{(N_2)}})(e) = [\nu\circ\psi_{\mathcal{M}}\circ f|_{\partial\theta}] = \deg(\nu\circ\alpha)\cdot\zeta = \sum\deg(\nu\circ\beta|_{S_i})\cdot\zeta, \quad (5)$$

where $\zeta \in \pi_{N_2}(S(W_k \oplus U_k^{\oplus j})) \cong \mathbb{Z}$ is a generator, and the sum ranges over all arrangements of k hyperplanes $\mathcal{H}$ in relint θ that equipart $\mathcal{M}$.
If in addition we assume that all local degrees $\deg(\nu\circ\beta|_{S_i})$ are ±1 and that the number of arrangements of k hyperplanes $\mathcal{H}$ in relint θ that equipart $\mathcal{M}$ is odd, then we conclude that $\mathfrak{o}(\nu \circ \psi_{\mathcal{M}}|_{X^{(N_2)}})(e) \neq 0$. It will turn out that in many situations this information implies that the cohomology class $[\mathfrak{o}(\nu \circ \psi_{\mathcal{M}})]$ is not zero, and consequently the related $\mathfrak{S}_k^{\pm}$-equivariant map (4) does not exist, concluding the proof of corresponding Theorems 1.4, 1.5, and 1.6.

3 A regular cell complex model for the join configuration space

In this section, motivated by methods used in [2] and [6], we construct a regular $\mathfrak{S}_k^{\pm}$-CW model for the join configuration space $X_{d,k} = (S^d)^{*k} \cong S(\mathbb{R}^{(d+1)\times k})$ such that $X_{d,k}^{>1}$ is a $\mathfrak{S}_k^{\pm}$-CW subcomplex. Consequently, $(X_{d,k}, X_{d,k}^{>1})$ has the structure of a relative $\mathfrak{S}_k^{\pm}$-CW complex. For simplicity the cell complex we construct is denoted by $X := (X_{d,k}, X_{d,k}^{>1})$ as well. The cell model is obtained in two steps:

(1) the vector space $\mathbb{R}^{(d+1)\times k}$ is decomposed into a union of disjoint relatively open cones (each containing the origin in its closure) on which the $\mathfrak{S}_k^{\pm}$-action operates linearly permuting the cones, and then
(2) the open cells of a regular $\mathfrak{S}_k^{\pm}$-CW model are obtained as intersections of these relatively open cones with the unit sphere $S(\mathbb{R}^{(d+1)\times k})$.

The explicit relative $\mathfrak{S}_k^{\pm}$-CW complex we construct here is an essential object needed for the study of the existence of $\mathfrak{S}_k^{\pm}$-equivariant maps $X_{d,k} \longrightarrow S(W_k \oplus U_k^{\oplus j})$ via the relative equivariant obstruction theory of tom Dieck [7, Sec. II.3].

3.1 Stratifications by cones associated to an arrangement

The first step in the construction of the $\mathfrak{S}_k^{\pm}$-CW model is an appropriate stratification of the ambient space $\mathbb{R}^{(d+1)\times k}$. First we introduce the notion of the stratification of a Euclidean space and collect some relevant properties.

Definition 3.1. Let $\mathcal{A}$ be an arrangement of linear subspaces in a Euclidean space E. A *stratification of E (by cones) associated to $\mathcal{A}$* is a finite collection $\mathcal{C}$ of subsets of E that satisfies the following properties:

(i) $\mathcal{C}$ consists of finitely many non-empty relatively open polyhedral cones in E.
(ii) $\mathcal{C}$ is a partition of E, i.e., $E = \biguplus_{C\in\mathcal{C}} C$.
(iii) The closure $\overline{C}$ of every cone $C \in \mathcal{C}$ is a union of cones in $\mathcal{C}$.
(iv) Every subspace $A \in \mathcal{A}$ is a union of cones in $\mathcal{C}$.

An element of the family $\mathcal{C}$ is called a *stratum*.

Example 3.2. Let E be a Euclidean space of dimension d, let L be a linear subspace of codimension r, where $1 \le r \le d$, and let $\mathcal{A}$ be the arrangement $\{L\}$. Choose a flag that terminates at L, i.e., fix a sequence of linear subspaces in E

$$E = L^{(0)} \supset L^{(1)} \supset \cdots \supset L^{(r)} = L, \tag{6}$$

so that $\dim L^{(i)} = d - i$. The family $\mathcal{C}$ associated to the flag (6) consists of L and of the connected components of the successive complements

$$L^{(0)}\backslash L^{(1)},\ L^{(1)}\backslash L^{(2)},\ \dots\ , L^{(r-1)}\backslash L^{(r)}.$$

A $L^{(i)}$ is a hyperplane in $L^{(i-1)}$, each of the complements $L^{(i-1)}\backslash L^{(i)}$ has two connected components. This indeed yields a stratification by cones for the arrangement $\mathcal{A}$ in E.

Definition 3.3. Let $(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n)$ be a collection of arrangements of linear subspaces in the Euclidean space E and let $(\mathcal{C}_1, \mathcal{C}_2 \dots, \mathcal{C}_n)$ be the associated collection of stratifications of E by cones. The *common refinement* of the stratifications is the family

$$\mathcal{C} := \{C_1 \cap C_2 \cap \cdots \cap C_n \neq \emptyset : C_i \in \mathcal{C}_i \text{ for all } i\}.$$

In order to verify that the common refinement of stratifications is again a stratification, we use the following elementary lemma.

Lemma 3.4. *Let $A_1, \dots, A_n$ be relatively open convex sets in E that have non-empty intersection, $A_1 \cap \cdots \cap A_n \neq \emptyset$. Then the following relation holds for the closures:*

$$\overline{A_1 \cap \cdots \cap A_n} = \overline{A_1} \cap \cdots \cap \overline{A_n}.$$

Proof. The inclusion "$\subseteq$" follows directly. For the opposite inclusion take $x \in \overline{A_1} \cap \cdots \cap \overline{A_n}$. Choose a point $y \in A_1 \cap \cdots \cap A_n \neq \emptyset$ and consider the line segment $(x, y] := \{\lambda x + (1-\lambda)y : 0 \leq \lambda < 1\}$. As each A_i is relatively open, the segment $(x, y]$ is contained in each of the A_i and consequently it is contained in $A_1 \cap \cdots \cap A_n$. Thus we obtain a sequence in this intersection converging to x, which implies that $x \in \overline{A_1 \cap \cdots \cap A_n}$. ◻

Proposition 3.5. *Given stratifications by cones $\mathcal{C}_1, \mathcal{C}_2 \dots, \mathcal{C}_n$ associated to linear subspace arrangements $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$, their common refinement is a stratification by cones associated to the subspace arrangement $\mathcal{A} := \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n$.*

Proof. Properties (i) and (ii) of Definition 3.1 follow immediately from the definition of the common refinement. To verify property (iv), observe that a subspace $A_t \in \mathcal{A}_t$ is a union of strata from $\mathcal{C}_t$, say $A_t = \bigcup_s U_{t,s}$ where $U_{t,s} \in \mathcal{C}_t$. Hence, taking the union of intersections $C_1 \cap \cdots \cap U_{t,s} \cap \cdots \cap C_n$ for all $C_i \in \mathcal{C}_i$ where $i \neq t$, and all $U_{t,s}$ gives A_t. Property (iii) follows from Lemma 3.4. ◻

Example 3.6. Let E be a Euclidean space of dimension d and let $\mathcal{A} = \{L_1, \dots, L_s\}$ be an arrangement of linear subspaces of E. As in Example 3.2, for each of the subspaces L_i in the arrangement $\mathcal{A}$ fix a flag $L_i^{(s)}$ and form the corresponding stratifications $\mathcal{C}_1, \dots, \mathcal{C}_s$. The common refinement of stratifications $\mathcal{C}_1, \dots, \mathcal{C}_s$ is a stratification by cones associated to the subspace arrangement $\mathcal{A}$.

An arrangement of linear subspaces is *essential* if the intersection of the subspaces in the arrangement is $\{0\}$.

Proposition 3.7. *The intersection of a stratification $\mathcal{C}$ of E by cones associated to an essential linear subspace arrangement with the sphere $S(E)$ gives a regular CW-complex.*

Proof. The elements $C \in \mathcal{C}$ are relative open polyhedral cones. As $\{0\}$ is a stratum by itself, none of the strata contains a line through the origin. Thus $C \cap S(E)$ is an open cell, whose closure $\overline{C} \cap S(E)$ is a finite union of cells of the form $C' \cap S(E)$, so we get a regular CW complex. ◻

3.2 A stratification of $\mathbb{R}^{(d+1)\times k}$

Now we introduce the stratification of $\mathbb{R}^{(d+1)\times k}$ that will give us a $\mathfrak{S}_k^{\pm}$-CW model for $X_{d,k}$. One version of it, $\mathcal{C}$, arises from the construction in the previous section. However, we also give combinatorial descriptions of relatively-open convex cones in the stratification $\mathcal{C}'$ directly, for which the action of $\mathfrak{S}_k^{\pm}$ is evident. We then verify that $\mathcal{C}$ and $\mathcal{C}'$ coincide.

3.2.1 Stratification

Let elements $x \in \mathbb{R}^{(d+1)\times k}$ be written as $x = (x_1, \dots, x_k)$ where $x_i = (x_{t,i})_{t\in[d+1]}$ is the i-th column of the matrix x. Consider the arrangement $\mathcal{A}$ consisting of the following subspaces:

$$\begin{aligned} L_r &:= \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : x_r = 0\}, && 1 \le r \le k \\ L_{r,s}^{+} &:= \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : x_r - x_s = 0\}, && 1 \le r < s \le k \\ L_{r,s}^{-} &:= \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : x_r + x_s = 0\}, && 1 \le r < s \le k. \end{aligned}$$

With each subspace we associate a flag:

(i) With $L_r = \{x_r = 0\}$ we associate

$$\mathbb{R}^{(d+1)\times k} \supset \{x_{1,r} = 0\} \supset \{x_{1,r} = x_{2,r} = 0\} \supset \cdots \supset \{x_{1,r} = x_{2,r} = \cdots = x_{d+1,r} = 0\},$$

(ii) With $L_{r,s}^{+} = \{x_r - x_s = 0\}$ we associate

$$\mathbb{R}^{(d+1)\times k} \supset \{x_{1,r} - x_{1,s} = 0\} \supset \{x_{1,r} - x_{1,s} = x_{2,r} - x_{2,s} = 0\} \supset \cdots \supset \{x_{1,r} - x_{1,s} = x_{2,r} - x_{2,s} = \cdots = x_{d+1,r} - x_{d+1,s} = 0\},$$

(iii) $L_{r,s}^{-} = \{x_r + x_s = 0\}$ we associate

$$\mathbb{R}^{(d+1)\times k} \supset \{x_{1,r} + x_{1,s} = 0\} \supset \{x_{1,r} + x_{1,s} = x_{2,r} + x_{2,s} = 0\} \supset \cdots \supset \{x_{1,r} + x_{1,s} = x_{2,r} + x_{2,s} = \cdots = x_{d+1,r} + x_{d+1,s} = 0\}.$$

The construction from Example 3.2 shows how every subspace in $\mathcal{A}$ leads to a stratification by cones of $\mathbb{R}^{(d+1)\times k}$. The stratifications associated to the subspaces $L_r, L_{r,s}^{+}, L_{r,s}^{-}$ are denoted by $\mathcal{C}_r, \mathcal{C}_{r,s}^{+}, \mathcal{C}_{r,s}^{-}$, respectively. Now, if we apply Example 3.6 to this concrete situation we obtain the stratification by cones $\mathcal{C}$ of $\mathbb{R}^{(d+1)\times k}$ associated to the subspace arrangement $\mathcal{A}$. This means that each stratum of $\mathcal{C}$ is a non-empty intersection of strata from the stratifications $\mathcal{C}_r, \mathcal{C}_{r,s}^{+}, \mathcal{C}_{r,s}^{-}$ where $1 \le r < s \le k$.

3.2.2 Partition

Let us fix:

- a permutation $\sigma := (\sigma_1, \sigma_2, \dots, \sigma_k) \equiv (\sigma_1\sigma_2 \dots \sigma_k) \in \mathfrak{S}_k$, $\sigma\colon t \mapsto \sigma_t$,
- a collection of signs $S := (s_1, \dots, s_k) \in \{+1, -1\}^k$, and
- integers $I := (i_1, \dots, i_k) \in \{1, \dots, d+2\}^k$.

Furthermore, define x_0 to be the origin in $\mathbb{R}^{(d+1)\times k}$, $\sigma_0 = 0$ and $s_0 = 1$. Define

$$C_I^S(\sigma) = C_{i_1,\dots,i_k}^{s_1,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k) \subseteq \mathbb{R}^{(d+1)\times k}$$

to be the set of all points $(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k}$, $x_i = (x_{1,i},\dots,x_{d+1,i})$, such that for each $1 \le t \le k$,

- if $1 \le i_t \le d+1$, then $s_{t-1}x_{i_t,\sigma_{t-1}} < s_t x_{i_t,\sigma_t}$ with $s_{t-1}x_{i',\sigma_{t-1}} = s_t x_{i',\sigma_t}$ for every $i' < i_t$,
- if $i_t = d+2$, then $s_{i_{t-1}}x_{\sigma_{t-1}} = s_{i_t}x_{\sigma_t}$.

Any triple $(\sigma|I|S) \in \mathfrak{S}_k \times \{1,\dots,d+2\}^k \times \{+1,-1\}^k$ is called a *symbol.* In the notation of symbols we abbreviate signs $\{+1,-1\}$ by $\{+,-\}$. The defining set of "inequalities" for the stratum $C_I^S(\sigma)$ is briefly denoted by:

$$\begin{aligned} C_I^S(\sigma) &= C_{i_1,\dots,i_k}^{s_1,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k) \\ &= \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : 0 <_{i_1} s_1x_{\sigma_1} <_{i_2} s_2x_{\sigma_2} <_{i_3} \cdots <_{i_k} s_kx_{\sigma_k}\}, \end{aligned}$$

where $y <_i y'$, for $1 \le i \le d+1$, means that y and y' agree in the first $i-1$ coordinates and at the i-th coordinate $y_i < y'_i$. The inequality $y <_{d+2} y'$ denotes that $y = y'$. Each set $C_I^S(\sigma)$ is the relative interior of a polyhedral cone in $(\mathbb{R}^{d+1})^k$ of codimension $(i_1-1)+\cdots+(i_k-1)$, i.e.,

$$\dim C_{i_1,\dots,i_k}^{s_1,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k) = (d+2)k - (i_1+\cdots+i_k).$$

Let $\mathcal{C}'$ denote the family of strata $C_I^S(\sigma)$ defined by all symbols, i.e.,

$$\mathcal{C}' = \{C_I^S(\sigma) : (\sigma|I|S) \in \mathfrak{S}_k \times \{1,\dots,d+2\}^k \times \{+1,-1\}^k\}.$$

Different symbols can define the same set, and

$$C_I^S(\sigma) \cap C_{I'}^{S'}(\sigma) \neq \emptyset \iff C_I^S(\sigma) = C_{I'}^{S'}(\sigma).$$

In order to verify that the family $\mathcal{C}'$ is a partition of $\mathbb{R}^{(d+1)\times k}$ it remains to prove that it is a covering.

Lemma 3.8. $\bigcup \mathcal{C}' = \mathbb{R}^{(d+1)\times k}$.

Proof. Let $(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k}$. First, choose signs $r_1,\dots,r_k \in \{+1,-1\}$ so that the vectors $r_1x_1,\dots,r_kx_k$ are greater or equal to $0 \in \mathbb{R}^{(d+1)\times k}$ with respect to the lexicographic order, i.e., the first non-zero coordinate of each of the vectors r_ix_i is greater than zero. The choice of signs is not unique if one of the vectors x_i is zero. Next, record a permutation $\sigma \in \mathfrak{S}_k$ such that

$$0 <_{\text{lex}} r_{\sigma_1}x_{\sigma_1} <_{\text{lex}} r_{\sigma_2}x_{\sigma_2} <_{\text{lex}} \cdots <_{\text{lex}} r_{\sigma_k}x_{\sigma_k},$$

where $<_{\text{lex}}$ denotes the lexicographic order. The permutation σ is not unique if $r_ix_i = r_tx_t$ for some $i \neq t$. Define $s_i := r_{\sigma_i}$. Finally, collect coordinates i_t where vectors $s_{t-1}x_{\sigma_{t-1}}$ and $s_tx_{\sigma_t}$ first differ, or put $i_t = d+2$ if they coincide. Thus, $(x_1,\dots,x_k) \in C_{i_1,\dots,i_k}^{s_1,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k)$. $\square$

Example 3.9. Let $d = 0$ and $k = 2$. Then the plane $\mathbb{R}^2$ is decomposed into the following cones. There are 8 open cones of dimension 2:

$$\begin{array}{rcl}
C^{+;+}_{1,1}(12) &=& \{(x_1,x_2) \in \mathbb{R}^2 : 0 < x_1 < x_2\},\\
C^{-;+}_{1,1}(12) &=& \{(x_1,x_2) \in \mathbb{R}^2 : 0 < -x_1 < x_2\},\\
C^{+;-}_{1,1}(12) &=& \{(x_1,x_2) \in \mathbb{R}^2 : 0 < x_1 < -x_2\},\\
C^{-;-}_{1,1}(12) &=& \{(x_1,x_2) \in \mathbb{R}^2 : 0 < -x_1 < -x_2\},\\
C^{+;+}_{1,1}(21) &=& \{(x_1,x_2) \in \mathbb{R}^2 : 0 < x_2 < x_1\},\\
C^{-;+}_{1,1}(21) &=& \{(x_1,x_2) \in \mathbb{R}^2 : 0 < -x_2 < x_1\},\\
C^{+;-}_{1,1}(21) &=& \{(x_1,x_2) \in \mathbb{R}^2 : 0 < x_2 < -x_1\},\\
C^{-;-}_{1,1}(21) &=& \{(x_1,x_2) \in \mathbb{R}^2 : 0 < -x_2 < -x_1\}.
\end{array}$$

Furthermore, there are 8 cones of dimension 1:

$$\begin{array}{rcccl}
C^{+;+}_{1,2}(12) &=& C^{+;+}_{1,2}(21) &=& \{(x_1,x_2) \in \mathbb{R}^2 : 0 < x_1 = x_2\},\\
C^{-;+}_{1,2}(12) &=& C^{+;-}_{1,2}(21) &=& \{(x_1,x_2) \in \mathbb{R}^2 : 0 < -x_1 = x_2\},\\
C^{+;-}_{1,2}(12) &=& C^{-;+}_{1,2}(21) &=& \{(x_1,x_2) \in \mathbb{R}^2 : 0 < x_1 = -x_2\},\\
C^{-;-}_{1,2}(12) &=& C^{-;-}_{1,2}(21) &=& \{(x_1,x_2) \in \mathbb{R}^2 : 0 < -x_1 = -x_2\},\\
C^{+;+}_{2,1}(12) &=& C^{-;+}_{2,1}(12) &=& \{(x_1,x_2) \in \mathbb{R}^2 : 0 = x_1 < x_2\},\\
C^{+;-}_{2,1}(12) &=& C^{-;-}_{2,1}(12) &=& \{(x_1,x_2) \in \mathbb{R}^2 : 0 = x_1 < -x_2\},\\
C^{+;+}_{2,1}(21) &=& C^{-;+}_{2,1}(21) &=& \{(x_1,x_2) \in \mathbb{R}^2 : 0 = x_2 < x_1\},\\
C^{+;-}_{2,1}(21) &=& C^{-;-}_{2,1}(21) &=& \{(x_1,x_2) \in \mathbb{R}^2 : 0 = x_2 < -x_1\}.
\end{array}$$

The origin in $\mathbb{R}^2$ is given by $C^{\pm;\pm}_{2,2}(12) = C^{\pm;\pm}_{2,2}(21)$. The example illustrates a property of our decomposition of $\mathbb{R}^{(d+1)\times k}$: There is a surjection from symbals to cones that is not a bijection, i.e., different symbols can define the same cones.

Example 3.10. Let $d = 2$ and $k = 4$. The stratum associated to the symbol $(2143\,|\,2,3,1,4\,|+1,-1,+1,-1)$ can be described explicitly as follows.

$$\left\{ \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} \end{pmatrix} \in (\mathbb{R}^3)^4 : \right.$$
$$\left. \begin{array}{ccccccccc} 0 & = & x_{1,2} & = & -x_{1,1} & < & x_{1,4} & = & -x_{1,3} \\ 0 & < & x_{2,2} & = & -x_{2,1} & & x_{2,4} & = & -x_{2,3} \\ & & x_{3,2} & < & -x_{3,1} & & x_{3,4} & = & -x_{3,3} \end{array} \right\}.$$

In particular,

$$C^{+,-,+,-}_{2,3,1,4}(2143) = C^{+,-,-,+}_{2,3,1,4}(2134).$$

3.2.3 $\mathcal{C}$ and $\mathcal{C}'$ coincide

We proved that $\mathcal{C}$ is a stratification by cones of $\mathbb{R}^{(d+1)\times k}$, and that $\mathcal{C}'$ is a partition of $\mathbb{R}^{(d+1)\times k}$. Since both $\mathcal{C}$ and $\mathcal{C}'$ are partitions it suffices to prove

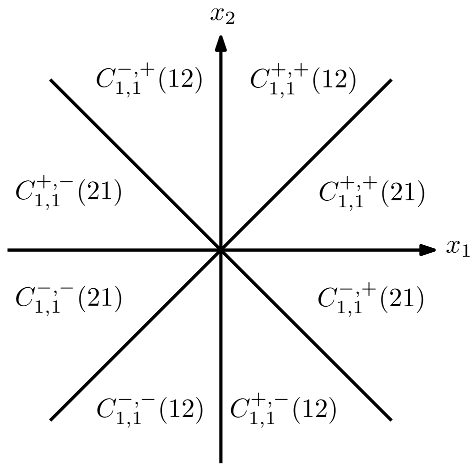

Figure 1: Illustration of the stratification in Example 3.9

that for every symbol $(\sigma|I|S) \in \mathfrak{S}_k \times \{1,\dots,d+2\}^k \times \{+1,-1\}^k$ the cone $C_I^S(\sigma) \in \mathcal{C}'$ also belongs to $\mathcal{C}$.
Consider the cone $C_I^S(\sigma)$ in $\mathcal{C}'$. It is determined by

$$\begin{aligned} C_I^S(\sigma) &= C_{i_1,\dots,i_k}^{s_1,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k) \\ &= \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : 0 <_{i_1} s_1 x_{\sigma_1} <_{i_2} s_2 x_{\sigma_2} <_{i_3} \cdots <_{i_k} s_k x_{\sigma_k}\}. \end{aligned}$$

The defining inequalities for $C_I^S(\sigma)$ imply that $(x_1,\dots,x_k) \in C_I^S(\sigma)$ if and only if

- $0 <_{\min\{i_1,\dots,i_a\}} s_a x_a$ for $1 \le a \le k$, and
- $s_a x_a <_{\min\{i_{a+1},\dots,i_b\}} s_b x_b$ for $1 \le a < b \le k$,

if and only if

- $(x_1,\dots,x_k)$ belongs to the appropriate one of two strata in the complement

$$L_a{}^{(\min\{i_1,\dots,i_a\}-1)} \setminus L_a{}^{(\min\{i_1,\dots,i_a\}-2)}$$

  of the stratification $\mathcal{C}_a$ depending on the sign s_a where $1 \le a \le k$, and
- $(x_1,\dots,x_k)$ belongs to the appropriate one of two strata in the complement

$$L_{a,b}^{s_a s_b\,(\min\{i_{a+1},\dots,i_b\}-1)} \setminus L_{a,b}^{s_a s_b\,(\min\{i_{a+1},\dots,i_b\}-2)}$$

  of the stratification $\mathcal{C}_{a,b}^{s_a s_b}$ depending on the sign of the product $s_a s_b$ where $1 \le a < b \le k$. The product $s_a s_b$, appearing in the "exponent notation" of $L_{a,b}^{s_a s_b}$, is either "+" when the product $s_a s_b = 1$, or "−" when $s_a s_b = -1$.

Here we use the notation of Examples 3.2 and 3.6.
Thus we have proved that $C_I^S(\sigma) \in \mathcal{C}$ and consequently $\mathcal{C} = \mathcal{C}'$.

3.3 The $\mathfrak{S}_k^\pm$-CW model for $X_{d,k}$

The action of the group $\mathfrak{S}_k^\pm$ on the space $\mathbb{R}^{(d+1)\times k}$ induces an action on the family of strata $\mathcal{C}$ by as follows:

$$\begin{aligned} \pi \cdot C_I^S(\sigma) &= C_I^S(\pi\sigma), & (7)\\ \varepsilon_t \cdot C_I^S(\sigma) &= \varepsilon_t \cdot C_{i_1,\dots,i_k}^{s_1,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k) \\ &= C_{i_1,\dots,i_k}^{s_1,\dots,-s_t,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k), & (8)\end{aligned}$$

where $\pi \in \mathfrak{S}_k$ and $\varepsilon_1,\dots,\varepsilon_k$ are the canonical generators of the subgroup $(\mathbb{Z}/2)^k$ of $\mathfrak{S}_k^\pm$.

The $\mathfrak{S}_k^\pm$-CW complex that models $X_{d,k} = S(\mathbb{R}^{(d+1)\times k})$ is obtained by intersecting each stratum $C_I^S(\sigma)$ with the unit sphere $S(\mathbb{R}^{(d+1)\times k})$. Each stratum is a relatively open cone that does not contain a line. Therefore the intersection

$$D_I^S(\sigma) = D_{i_1,\dots,i_k}^{s_1,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k) := C_{i_1,\dots,i_k}^{s_1,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k) \cap S(\mathbb{R}^{(d+1)\times k})$$

is an open cell of dimension $(d+2)k - (i_1+\dots+i_k) - 1$. The action of $\mathfrak{S}_k^\pm$ is induced by (7) and (8):

$$\begin{aligned} \pi \cdot D_I^S(\sigma) &= D_I^S(\pi\sigma), & (9)\\ \varepsilon_t \cdot D_I^S(\sigma) &= \varepsilon_t \cdot D_{i_1,\dots,i_k}^{s_1,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k) \\ &= D_{i_1,\dots,i_k}^{s_1,\dots,-s_t,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k). & (10)\end{aligned}$$

Thus we have obtained a regular $\mathfrak{S}_k^\pm$-CW model for $X_{d,k}$. In particular, the action of the group $\mathfrak{S}_k^\pm$ on the space $\mathbb{R}^{(d+1)\times k}$ induces a cellular action on the model.

Theorem 3.11. *Let $d \geq 1$ and $k \geq 1$ be integers, and $N_1 = (d+1)k - 1$. The family of cells*

$$\{D_I^S(\sigma) : (\sigma|I|S) \neq (\sigma|d+2,\dots,d+2|S)\}$$

forms a finite regular N_1-dimensional $\mathfrak{S}_k^\pm$-CW complex $X := (X_{d,k}, X_{d,k}^{>1})$ that models the join configuration space $X_{d,k} = S(\mathbb{R}^{(d+1)\times k})$. It has
- *one full $\mathfrak{S}_k^\pm$-orbit in maximal dimension N_1, and*
- *k full $\mathfrak{S}_k^\pm$-orbits in dimension $N_1 - 1$.*

The (cellular) $\mathfrak{S}_k^\pm$-action on $X_{d,k}$ is given by (9) and (10). Furthermore the collection of cells

$$\{D_I^S(\sigma) : i_s = d+2 \text{ for some } 1 \leq s \leq k\}$$

is a $\mathfrak{S}_k^\pm$-CW subcomplex and models $X_{d,k}^{>1}$.

Example 3.12. Let $d \geq 1$ and $k \geq 2$ be integers with $dk = (2^k-1)j + \ell$, where $0 \leq \ell \leq d-1$. Consider the cell $\theta := D_{\ell+1,1,1,\dots,1}^{+,+,+,\dots,+}(1,2,3,\dots,k)$ of dimension $N_1 - \ell = N_2 + 1$ in $X_{d,k}$. It is determined by the following inequalities:

$$0 <_{\ell+1} x_1 <_1 x_2 <_1 \cdots <_1 x_k.$$

For the process of determining the boundary of θ, depending on value of ℓ, we distinguish the following cases.

(1) Let $\ell = 0$. Then $\theta := D^{+,+,+,\dots,+}_{1,1,1,\dots,1}(1,2,3,\dots,k)$. The cells of codimension 1 in the boundary of θ are obtained by introducing one of the following extra equalities:

$$x_{1,1} = 0\,, \quad x_{1,1} = x_{1,2}\,, \quad \dots \quad x_{1,k-1} = x_{1,k}.$$

Each of these equalities will give two cells of dimension N_2, hence in total $2k$ cells of codimension 1, in the boundary of θ.

(A) The equality $x_{1,1} = 0$ induces cells:

$$\gamma_1 := D^{+,+,+,\dots,+}_{2,1,1,\dots,1}(1,2,3,\dots,k), \qquad \gamma_2 := D^{-,+,+,\dots,+}_{2,1,1,\dots,1}(1,2,3,\dots,k)$$

that are related, as sets, via $\gamma_2 = \varepsilon_1 \cdot \gamma_1$. Both cells γ_1 and γ_2 belong to the linear subspace

$$V_1 = \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : x_{1,1} = 0\}.$$

(B) The equality $x_{1,r-1} = x_{1,r}$ for $2 \le r \le k$ gives cells:

$$\gamma_{2r-1} := D^{+,+,+,\dots,+}_{1,\dots,1,2,1,\dots,1}(1,\dots,r-1,r,r+1,\dots,k),$$

$$\gamma_{2r} := D^{+,+,+,\dots,+}_{1,\dots,1,2,1,\dots,1}(1,\dots,r,r-1,r+1,\dots,k)$$

satisfying $\gamma_{2r} = \tau_{r-1,r}\cdot\gamma_{2r-1}$. In these cells the index 2 in the subscript $1,\dots,1,2,1,\dots,1$ appears at the position r. These cells belong to the linear subspace

$$V_r = \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : x_{1,r-1} = x_{1,r}\}.$$

Let e_θ denote a generator in $C_{N_2+1}(X_{d,k}, X^{>1}_{d,k})$ that corresponds to the cell θ. Furthermore let $e_{\gamma_1},\dots,e_{\gamma_{2k}}$ denote generators in $C_{N_2}(X_{d,k}, X^{>1}_{d,k})$ related to the cells $\gamma_1,\dots,\gamma_{2k}$.

The boundary of the cell θ is contained in the union of the linear subspaces $V_1,\dots,V_k$. Therefore we can orient the cells $\gamma_{2i-1},\gamma_{2i}$ consistently with the orientation of V_i, $1 \le i \le k$, that is given in such a way that

$$\partial e_\theta = (e_{\gamma_1} + e_{\gamma_2}) + (e_{\gamma_3} + e_{\gamma_4}) + \dots + (e_{\gamma_{2k-1}} + e_{\gamma_{2k}}).$$

Consequently,

$$\partial e_\theta = (1 + (-1)^d \varepsilon_1)\cdot e_{\gamma_1} + \sum_{i=2}^{k} (1 + (-1)^d \tau_{i-1,i})\cdot e_{\gamma_{2i-1}}. \tag{11}$$

(2) Let $\ell = 1$. Then $\theta := D^{+,+,+,\dots,+}_{2,1,1,\dots,1}(1,2,3,\dots,k)$. Now the cells in the boundary of θ are obtained by introducing extra equalities:

$$x_{2,1} = 0\,, \quad (0 =)\, x_{1,1} = x_{1,2}\,, \quad \dots \quad x_{1,k-1} = x_{1,k}.$$

Each of these equalities, except for the second one, will give two cells of dimension N_2, which yields $2(k-1)$ cells in total, in the boundary of θ. The equality $x_{1,1} = x_{1,2}$ will give additional four cells in the boundary of θ.

(A) The equality $x_{2,1} = 0$ induces cells:

$$\gamma_1 := D^{+,+,+,\dots,+}_{3,1,1,\dots,1}(1,2,3,\dots,k), \qquad \gamma_2 := D^{-,+,+,\dots,+}_{3,1,1,\dots,1}(1,2,3,\dots,k)$$

that are related, as sets, via $\gamma_2 = \varepsilon_1 \cdot \gamma_1$. Notice that both cells γ_1 and γ_2 belong to the linear subspace

$$V_1 = \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : x_{1,1} = x_{2,1} = 0\}.$$

(B) The equality $x_{1,1} = x_{1,2}$ yields the cells

$$\gamma_3 := D^{+,+,+,\dots,+}_{2,2,1,\dots,1}(1,2,3,\dots,k), \quad \gamma_{31} := D^{+,-,+,\dots,+}_{2,2,1,\dots,1}(1,2,3,\dots,k),$$

$$\gamma_{32} := D^{+,+,+,\dots,+}_{2,2,1,\dots,1}(2,1,3,\dots,k), \quad \gamma_{33} := D^{-,+,+,\dots,+}_{2,2,1,\dots,1}(2,1,3,\dots,k).$$

They satisfy set identities $\gamma_{31} = \varepsilon_2 \cdot \gamma_3$, $\gamma_{32} = \tau_{1,2} \cdot \gamma_3$, and $\gamma_{33} = \varepsilon_1 \tau_{1,2} \cdot \gamma_3$. All four cells belong to the linear subspace

$$V_2 = \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : 0 = x_{1,1} = x_{1,2}\}.$$

(C) The equality $x_{1,r-1} = x_{1,r}$ for $3 \le r \le k$ gives cells:

$$\gamma_{2r-1} := D^{+,+,+,\dots,+}_{2,\dots,1,2,1,\dots,1}(1,\dots,r-1,r,r+1,\dots,k),$$

$$\gamma_{2r} := D^{+,+,+,\dots,+}_{2,\dots,1,2,1,\dots,1}(1,\dots,r,r-1,r+1,\dots,k)$$

satisfying $\gamma_{2r} = \tau_{r-1,r} \cdot \gamma_{2r-1}$. In these cells the second index 2 in the subscript $2,\dots,1,2,1,\dots,1$ appears at the position r. These cells belong to the linear subspace

$$V_r = \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : x_{1,1} = 0,\ x_{1,r-1} = x_{1,r}\}.$$

Again e_θ denotes a generator in $C_{N_2+1}(X_{d,k}, X^{>1}_{d,k})$ corresponding to θ. Let $e_{\gamma_1}, e_{\gamma_2}, e_{\gamma_3}, e_{\gamma_{31}}, e_{\gamma_{32}}, e_{\gamma_{33}}, e_{\gamma_4} \dots, e_{\gamma_{2k}}$ denote generators in $C_{N_2}(X_{d,k}, X^{>1}_{d,k})$ related to the cells $\gamma_1, \gamma_2, \gamma_3, \gamma_{31}, \gamma_{32}, \gamma_{33}, \dots, \gamma_{2k}$.
The boundary of the cell θ, as before, is contained in the union of the linear subspaces $V_1, \dots, V_k$. Therefore we can orient cells consistently with the orientation of V_i, $1 \le i \le k$, that is given in such a way that

$$\partial e_\theta = (e_{\gamma_1} + e_{\gamma_2}) + (e_{\gamma_3} + e_{\gamma_{31}} + e_{\gamma_{32}} + e_{\gamma_{33}}) + \dots + (e_{\gamma_{2k-1}} + e_{\gamma_{2k}}).$$

Consequently,

$$\begin{aligned} \partial e_\theta \;=\; & (1 + (-1)^{d-1}\varepsilon_1) \cdot e_{\gamma_1} + \\ & (1 + (-1)^d \varepsilon_2 + (-1)^d \tau_{1,2} + (-1)^{d+d} \varepsilon_1 \tau_{1,2}) \cdot e_{\gamma_3} + \\ & \sum_{i=3}^{k} (1 + (-1)^d \tau_{i-1,i}) \cdot e_{\gamma_{2i-1}}. \end{aligned} \tag{12}$$

(3) Let $2 \le \ell \le d-1$. Then $\theta := D^{+,+,+,\dots,+}_{\ell+1,1,1,\dots,1}(1,2,3,\dots,k)$. The cells in the boundary of θ are now obtained by introducing following equalities:

$$x_{\ell+1,1} = 0\,,\quad (0=)x_{1,1} = x_{1,2}\,,\quad \dots \quad x_{1,k-1} = x_{1,k}.$$

Each of them will give two cells of dimension N_2 in the boundary of θ, all together $2k$.

(A) The equality $x_{\ell+1,1} = 0$ induces cells:

$$\gamma_1 := D^{+,+,+,\dots,+}_{\ell+2,1,1,\dots,1}(1,2,3,\dots,k), \qquad \gamma_2 := D^{-,+,+,\dots,+}_{\ell+2,1,1,\dots,1}(1,2,3,\dots,k)$$

that are related, as sets, via $\gamma_2 = \varepsilon_1 \cdot \gamma_1$. Both cells γ_1 and γ_2 belong to the linear subspace

$$V_1 = \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : x_{1,1} = \dots = x_{\ell+1,1} = 0\}.$$

(B) The equality $(0=)x_{1,1} = x_{1,2}$ gives the cells

$$\gamma_3 := D^{+,+,+,\dots,+}_{\ell+1,2,1,\dots,1}(1,2,3,\dots,k), \qquad \gamma_4 := D^{+,-,+,\dots,+}_{\ell+1,2,1,\dots,1}(1,2,3,\dots,k)$$

that satisfy $\gamma_4 = \varepsilon_2 \cdot \gamma_3$. Both cells belong to the linear subspace

$$V_2 = \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : x_{1,1} = \dots = x_{\ell,1} = 0,\, x_{1,1} = x_{1,2}\}.$$

(C) The equality $x_{1,r-1} = x_{1,r}$ for $3 \le r \le k$ gives cells:

$$\gamma_{2r-1} := D^{+,+,+,\dots,+}_{\ell+1,\dots,1,2,1,\dots,1}(1,\dots,r-1,r,r+1,\dots,k),$$
$$\gamma_{2r} := D^{+,+,+,\dots,+}_{\ell+1,\dots,1,2,1,\dots,1}(1,\dots,r,r-1,r+1,\dots,k)$$

satisfying $\gamma_{2r} = \tau_{r-1,r} \cdot \gamma_{2r-1}$. In these cells the index 2 in the subscript $\ell+1,\dots,1,2,1,\dots,1$ appears at the position r. These cells belong to the linear subspace

$$V_r = \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : x_{1,1} = \dots = x_{\ell,1} = 0,\, x_{1,r-1} = x_{1,r}\}.$$

Again e_θ denotes a generator in $C_{N_2+1}(X_{d,k}, X^{>1}_{d,k})$ that corresponds to the cell θ. Furthermore $e_{\gamma_1},\dots,e_{\gamma_{2k}}$ denote generators in $C_{N_2}(X_{d,k}, X^{>1}_{d,k})$ related to the cells $\gamma_1,\dots,\gamma_{2k}$.
As before, the boundary of the cell θ is contained in the union of the linear subspaces $V_1,\dots,V_k$. Thus we can orient cells $\gamma_{2i-1}, \gamma_{2i}$ consistently with the orientation of V_i, $1 \le i \le k$, that is given in such a way that

$$\partial e_\theta = (e_{\gamma_1} + e_{\gamma_2}) + (e_{\gamma_3} + e_{\gamma_4}) + \dots + (e_{\gamma_{2k-1}} + e_{\gamma_{2k}}).$$

Hence,

$$\partial e_\theta = (1+(-1)^{d-\ell}\varepsilon_1)\cdot e_{\gamma_1} + (1+(-1)^d\varepsilon_2)\cdot e_{\gamma_3} + \sum_{i=3}^{k}(1+(-1)^d\tau_{i-1,i})\cdot e_{\gamma_{2i-1}}. \tag{13}$$

The relations (11), (12) and (13) that we have now derived will be essential in the proofs of Theorems 1.4 and 1.5.

3.4 The arrangements parametrized by a cell

In this section we describe all arrangements of k hyperplanes parametrized by the cell

$$\theta := D^{+,+,+,\dots,+}_{\ell+1,1,1,\dots,1}(1,2,3,\dots,k),$$

where $1 \le \ell \le d-1$. This description will be one of the key ingredients in Section 4 when the obstruction cocycle is evaluated on the cell θ.
Recall that the cell θ is defined as the intersection of the sphere $S(\mathbb{R}^{(d+1)\times k})$ and the cone given by the inequalities:

$$0 <_{\ell+1} x_1 <_1 x_2 <_1 \cdots <_1 x_k.$$

Consider the binomial coefficient moment curve $\hat{\gamma}\colon \mathbb{R} \longrightarrow \mathbb{R}^d$ defined by

$$\hat{\gamma}(t) = (t, \tbinom{t}{2}, \tbinom{t}{3}, \dots, \tbinom{t}{d})^t. \tag{14}$$

After embedding $\mathbb{R}^d \longrightarrow \mathbb{R}^{d+1}, (\xi_1,\dots,\xi_d)^t \longmapsto (1,\xi_1,\dots,\xi_d)^t$ it corresponds to the curve $\gamma\colon \mathbb{R} \longrightarrow \mathbb{R}^{d+1}$ given

$$\gamma(t) = (1, t, \tbinom{t}{2}, \tbinom{t}{3}, \dots, \tbinom{t}{d})^t.$$

Consider the following points on the moment curve γ:

$$q_1 := \gamma(0), \dots, q_{\ell+1} := \gamma(\ell). \tag{15}$$

Next, recall that each oriented affine hyperplane $\hat{H}$ in $\mathbb{R}^d$ (embedded in $\mathbb{R}^{d+1}$) determines the unique linear hyperplane H such that $\hat{H} = H \cap \mathbb{R}^d$, and almost vice versa. Now, the family of arrangements parametrized by the (open) cell θ is described as follows:

Lemma 3.13. *The cell $\theta = D^{+,+,+,\dots,+}_{\ell+1,1,1,\dots,1}(1,2,3,\dots,k)$ parametrizes all arrangements $\mathcal{H} = (H_1,\dots,H_k)$ of k linear hyperplanes in $\mathbb{R}^{d+1}$, where the order and orientation are fixed appropriately such that*

- $Q := \{q_1,\dots,q_\ell\} \subset H_1$,
- $q_{\ell+1} \notin H_1$,
- $q_1 \notin H_2, \dots, q_1 \notin H_k$, *and*
- $H_2,\dots,H_k$ *have unit normal vectors with different (positive) first coordinates, that is,* $|\{\langle x_2,q_1\rangle, \langle x_3,q_1\rangle,\dots,\langle x_k,q_1\rangle\}| = k-1$.

Here $x_i \in S(\mathbb{R}^{(d+1)\times k})$ is a unit normal vector of the hyperplane H_i, for $1 \le i \le k$.

Proof. Observe that $\{q_1,\dots,q_\ell\} \subset H_1$ holds if and only if $\langle x_1,q_1\rangle = \langle x_1,q_2\rangle = \cdots = \langle x_1,q_\ell\rangle = 0$ if and only if $x_{1,1} = x_{2,1} = \cdots = x_{\ell,1} = 0$: This is true since we have the binomial moment curve, so $q_i = \gamma(i-1)$ has only the first i coordinates non-zero.
Furthermore, $q_{\ell+1} \notin H_1$ holds if and only if $x_{\ell+1,1} \neq 0$; choosing an appropriate orientation for H_1 we can assume that $x_{\ell+1,1} > 0$.

The third condition is equivalent to $0 \notin \{\langle x_2, q_1\rangle, \langle x_3, q_1\rangle, \dots, \langle x_k, q_1\rangle\}$, that is, $x_{1,2}, x_{1,3}, \dots, x_{1,k} \neq 0$. Choosing orientations of $H_2, \dots, H_k$ suitably this yields $x_{1,2}, x_{1,3}, \dots, x_{1,k} > 0$.
Since the values $x_{1,2} = \langle x_2, q_1\rangle$, $x_{1,3} = \langle x_3, q_1\rangle$, $\dots$, $x_{1,k} = \langle x_k, q_1\rangle$ are positive and distinct, we get $0 < x_{1,2} < x_{1,3} < \dots < x_{1,k}$ by choosing the right order on $H_2, \dots, H_k$. □

4 Proofs

4.1 Proof of Theorem 1.3

Let $d \geq 1$, $j \geq 1$, $\ell \geq 0$ and $k \geq 2$ be integers with the property that $dk = j(2^k - 1) + \ell$ for $0 \leq \ell \leq d-1$.
Consider a collection of j ordered disjoint intervals $\mathcal{M} = (I_1, \dots, I_j)$ along the moment curve γ. Let $Q = \{q_1, \dots, q_\ell\} \subset \gamma$ be a set of ℓ predetermined points that lie to the left of the interval I_1. We prove Theorem 1.3 in two steps.

Lemma 4.1. *Let A be an ℓ-equiparting matrix, that is, a binary matrix of size $k \times j2^k$ with one row of transition count $d - \ell$ and all other rows of transition count d such that $A = (A_1, \dots, A_j)$ for Gray codes $A_1, \dots, A_j$ with the property that the last column of A_i is equal to the first column of A_{i+1} for $1 \leq i < j$. Then A determines an arrangement $\mathcal{H}$ of k affine hyperplanes that equipart $\mathcal{M} = (I_1, \dots, I_j)$ and one of the hyperplanes passes through each point in Q.*

Proof. Without loss of generality we assume that the first row of the matrix A has transition count $d - \ell$ while rows 2 through k have transition count d. For a row a_s of the matrix A, denote by t_s its transition count, $1 \leq s \leq k$.
Place $j(2^k + 1)$ ordered points $q_{\ell+1}, \dots, q_{\ell+j(2^k+1)}$ on γ, such that

$$I_i = [q_{\ell+(i-1)2^k+i},\ q_{\ell+i2^k+i}]$$

and each sequence of 2^k+1 points divides I_i into 2^k subintervals of equal length. *Ordered* refers to the property that $q_r = \gamma(t_r)$ if $t_1 < t_2 < \dots < t_{j(2^k+1)}$.
We now define the hyperplanes in $\mathcal{H}$ by specifying which of the points they pass through and then choosing their orientations. Force the affine hyperplane H_1 to pass through all of the points in Q. For $s = 1, \dots, i$, the affine hyperplane H_s passes through $x_{\ell+r+i}$ if there is a bit change in row a_s from entry r to entry $r+1$ for $(i-1)2^k < r \leq i2^k$. Orient H_s such that the subinterval $[q_r, q_{r+1}]$ is on the positive side of H_s if it corresponds to a 0-entry in a_s. Since each $A_1, \dots, A_j$ is a Gray code, the arrangement $\mathcal{H}$ is indeed an equipartition. □

Lemma 4.2. *Every arrangement of k affine hyperplanes $\mathcal{H}$ that equiparts $\mathcal{M} = (I_1, \dots, I_j)$ and where one of the hyperplanes passes through each point of Q induces a unique binary matrix A as in Lemma 4.1.*

Proof. Since $dk = j(2^k - 1) + \ell$ and $0 \leq \ell \leq d-1$, the hyperplanes in $\mathcal{H}$ must pass through the points $q_{\ell+(i-1)2^k+i+1}, \dots, q_{\ell+i2^k+i-1}$ of the intervals I_i

for $i \in \{1, \dots, j\}$. Recording the position of the subintervals $[q_{\ell+r}, q_{\ell+r+1}]$, for $r \neq i2^k + i$, with respect to each hyperplane leads to a matrix as in described in Lemma 4.1. $\square$

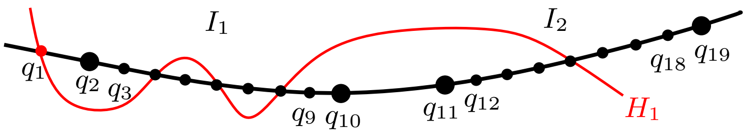

Figure 2: Illustration of one step in the proof of Lemma 4.1. Here H_1 is an affine hyperplane bisecting two intervals I_1 and I_2 on the 5-dimensional moment curve.

Thus the number of non-equivalent ℓ-equiparting matrices is the same as the number of arrangements of k affine hyperplanes $\mathcal{H}$ that equipart the collection of j disjoint intervals on the moment curve in $\mathbb{R}^d$, up to renumbering and orientation change of hyperplanes in $\mathcal{H}$, when one of the hyperplanes is forced to pass through ℓ prescribed points on the moment curve lying to the left of the intervals. This concludes the proof of Theorem 1.3. $\square$

4.2 Proof of Theorem 1.4

Let $j \geq 1$ and $k \geq 3$ be integers with $d = \lceil \frac{2^k-1}{k} j \rceil$ and $\ell = dk - (2^k - 1)j$. In addition, assume that the number of non-equivalent ℓ-equiparting matrices of size $k \times j2^k$ is odd.

In order to prove that $\Delta(j,k) \leq d$ it suffices by Theorem 2.3 to prove that there is no $\mathfrak{S}_k^{\pm}$-equivariant map

$$X_{d,k} \longrightarrow S(W_k \oplus U_k^{\oplus j}),$$

whose restriction to $X_{d,k}^{>1}$ is $\mathfrak{S}_k^{\pm}$-homotopic to $\nu \circ \psi_{\mathcal{M}}|_{X_{d,k}^{>1}}$ for $\mathcal{M} = (I_1, \dots, I_j)$. Following Section 2.6 we verify that the cohomology class

$$[\mathfrak{o}(g)] \in \mathcal{H}_{\mathfrak{S}_k^{\pm}}^{N_2+1}\big(X_{d,k}, X_{d,k}^{>1}; \pi_{N_2}(S(W_k \oplus U_k^{\oplus j}))\big),$$

does not vanish, where $g = \nu \circ \psi_{\mathcal{M}}|_{X^{(N_2)}}$.

Consider the cell $\theta := D_{\ell+1,1,1,\dots,1}^{+,+,+,\dots,+}(1,2,3,\dots,k)$ of dimension $(d+1)k - 1 - \ell = N_2 + 1$ in $X_{d,k}$, as in Example 3.12. Let e_θ denote the corresponding basis element of the cell θ in the cellular chain group $C_{N_2+1}(X_{d,k}, X_{d,k}^{>1})$, and let h_θ be the attaching map of θ. This cell is cut out from the unit sphere $S(\mathbb{R}^{(d+1)\times k})$ by the following inequalities:

$$0 <_{\ell+1} x_1 <_1 x_2 <_1 \cdots <_1 x_k.$$

In particular, this means that the first ℓ coordinates of x_1 are zero, i.e., $x_{1,1} = x_{2,1} = x_{3,1} = \cdots = x_{\ell,1} = 0$, and $x_{\ell+1,1} > 0$.

Let us fix ℓ points $Q = \{q_1, \dots, q_\ell\}$ on the moment curve (14) in $\mathbb{R}^{d+1}$ as it was done in (15): $q_1 := \gamma(0), \dots, q_\ell := \gamma(\ell-1)$. Then, by Lemma 3.13, the relative interior of $D^{+,+,+,\dots,+}_{\ell+1,1,1,\dots,1}(1,2,3,\dots,k)$ parametrizes the arrangements $\mathcal{H} = (H_1, \dots, H_k)$ for which orientations and order of the hyperplanes are fixed with H_1 containing all the points from Q. According to the formula (5) we have that

$$\mathfrak{o}(g)(e_\theta) = [\nu \circ \psi_{\mathcal{M}} \circ h_\theta|_{\partial\theta}] = \sum \deg(\nu \circ \psi_{\mathcal{M}}|_{X^{(N_2+1)}} \circ h_\theta|_{S_i}) \cdot \zeta,$$

where as before $\zeta \in \pi_{N_2}(S(W_k \oplus U_k^{\oplus j})) \cong \mathbb{Z}$ is a generator, and the sum ranges over all arrangements of k hyperplanes in relint θ that equipart $\mathcal{M}$. Here, as before, S_i denotes a small N_2-sphere around a root of the function $\psi_{\mathcal{M}}|_{X^{(N_2+1)}} \circ h_\theta$, i.e., the point that parametrizes an arrangements of k hyperplanes in relint θ that equipart $\mathcal{M}$.
Now, the local degrees of the function $\nu \circ \psi_{\mathcal{M}}|_{X^{(N_2+1)}} \circ h_\theta$ are ± 1. Indeed, in a small neighborhood $U \subseteq$ relint θ around any root the test map $\psi_{\mathcal{M}}$ is a continuous bijection. Thus $\psi_{\mathcal{M}}|_{\partial U}$ is a continuous bijection into some N_2-sphere around the origin in $W_k \oplus U_k^{\oplus j}$ and by compactness of ∂U is a homeomorphism. Consequently,

$$\mathfrak{o}(g)(e_\theta) = \sum \deg(\nu \circ \psi_{\mathcal{M}}|_{X^{(N_2+1)}} \circ h_\theta|_{S_i}) \cdot \zeta = \big(\sum \pm 1\big) \cdot \zeta = a \cdot \zeta, \qquad (16)$$

where the sum ranges over all arrangements of k hyperplanes in relint θ that equipart $\mathcal{M}$. According to Theorem 1.3 the number of (± 1)'s in the sum (16) is equal to the number of non-equivalent ℓ-equiparting matrices of size $k \times j2^k$. By our assumption this number is odd and consequently $a \in \mathbb{Z}$ is an odd integer. We obtained that

$$\mathfrak{o}(g)(e_\theta) = a \cdot \zeta, \qquad (17)$$

where $a \in \mathbb{Z}$ is an odd integer.

Remark 4.3. It is important to point out that the calculations and formulas up to this point also hold for $k = 2$. The assumption $k \geq 3$ affects the $\mathfrak{S}_k^{\pm} = (\mathbb{Z}/2)^k \rtimes \mathfrak{S}_k$ module structure on $\pi_{N_2}(S(W_k \oplus U_k^{\oplus j})) \cong \mathbb{Z}$. For $k \geq 2$ every generator ε_i of the subgroup $(\mathbb{Z}/2)^k$ acts trivially, while each transposition $\tau_{i,t}$, a generator of the subgroup $\mathfrak{S}_k$, acts as multiplication by -1 in the case $k \geq 3$, and as multiplication by $(-1)^{j+1}$ in the case $k = 2$.

Finally, we prove that $[\mathfrak{o}(g)]$ does not vanish and conclude the proof. This will be achieved by proving that the cocycle $\mathfrak{o}(g)$ is not a coboundary.
Let us assume to the contrary that $\mathfrak{o}(g)$ is a coboundary. Thus there exists a cochain

$$\mathfrak{h} \in \mathcal{C}^{N_2}_{\mathfrak{S}_k^{\pm}}\big(X_{d,k}, X_{d,k}^{>1}\,;\, \pi_{N_2}(S(W_k \oplus U_k^{\oplus j}))\big)$$

such that $\mathfrak{o}(g) = \delta\mathfrak{h}$, where δ denotes the coboundary operator. In the case when

(1) $\ell=0$ the relation (11) implies that

$$\begin{array}{rcl}a\cdot\zeta &=& \mathfrak{o}(g)(e_\theta)=\delta\mathfrak{h}(e_\theta)=\mathfrak{h}(\partial e_\theta)\\ &=& (1+(-1)^d\varepsilon_1)\cdot\mathfrak{h}(e_{\gamma_1})+\sum_{i=2}^{k}(1+(-1)^d\tau_{i-1,i})\cdot\mathfrak{h}(e_{\gamma_{2i-1}})\\ &=& (1+(-1)^d)\cdot\mathfrak{h}(e_{\gamma_1})+\sum_{i=2}^{k}(1+(-1)^{d+1})\cdot\mathfrak{h}(e_{\gamma_{2i-1}})\\ &=& 2b\cdot\zeta,\end{array}$$

for some integer b. Since a is an odd integer this is not possible, and therefore $\mathfrak{o}(g)$ is not a coboundary.

(2) $\ell=1$ the relation (12) implies that

$$\begin{array}{rcl}a\cdot\zeta &=& \mathfrak{o}(g)(e_\theta)=\delta\mathfrak{h}(e_\theta)=\mathfrak{h}(\partial e_\theta)\\ &=& (1+(-1)^{d-1}\varepsilon_1)\cdot\mathfrak{h}(e_{\gamma_1})+\\ && (1+(-1)^d\varepsilon_2+(-1)^d\tau_{1,2}+(-1)^{d+d}\varepsilon_1\tau_{1,2})\cdot\mathfrak{h}(e_{\gamma_3})+\\ && \sum_{i=3}^{k}(1+(-1)^d\tau_{i-1,i})\cdot\mathfrak{h}(e_{\gamma_{2i-1}})\\ &=& (1+(-1)^{d-1})\cdot\mathfrak{h}(e_{\gamma_1})+(1+(-1)^d+(-1)^{d+1}-1)\cdot\mathfrak{h}(e_{\gamma_3})+\\ && \sum_{i=3}^{k}(1+(-1)^{d+1})\cdot\mathfrak{h}(e_{\gamma_{2i-1}})\\ &=& (1+(-1)^{d-1})\cdot\mathfrak{h}(e_{\gamma_1})+\sum_{i=3}^{k}(1+(-1)^{d+1})\cdot\mathfrak{h}(e_{\gamma_{2i-1}})\\ &=& 2b\cdot\zeta,\end{array}$$

for $b\in\mathbb{Z}$. Again we reached a contradiction, so $\mathfrak{o}(g)$ is not a coboundary.

(3) $2\le\ell\le d-1$ the relation (13) implies that

$$\begin{array}{rcl}a\cdot\zeta &=& \mathfrak{o}(g)(e_\theta)=\delta\mathfrak{h}(e_\theta)=\mathfrak{h}(\partial e_\theta)\\ &=& (1+(-1)^{d-\ell}\varepsilon_1)\cdot\mathfrak{h}(e_{\gamma_1})+(1+(-1)^d\varepsilon_2)\cdot\mathfrak{h}(e_{\gamma_3})+\\ && \sum_{i=3}^{k}(1+(-1)^d\tau_{i-1,i})\cdot\mathfrak{h}(e_{\gamma_{2i-1}})\\ &=& (1+(-1)^{d-\ell})\cdot\mathfrak{h}(e_{\gamma_1})+(1+(-1)^d)\cdot\mathfrak{h}(e_{\gamma_3})+\\ && \sum_{i=3}^{k}(1+(-1)^{d+1})\cdot\mathfrak{h}(e_{\gamma_{2i-1}})\\ &=& 2b\cdot\zeta,\end{array}$$

for an integer b. Since a is an odd integer this is not possible. Again, $\mathfrak{o}(g)$ is not a coboundary. □

4.3 Proof of Theorem 1.5

Let $j \geq 1$ be an integer with $d = \lceil \frac{3}{2}j \rceil$ and $\ell = 2d - 3j \leq 1$.
The proof of this theorem is done in the footsteps of the proof of Theorem 1.4. In all three cases we rely on Theorem 2.3 and prove

- the non-existence of $\mathfrak{S}_2^{\pm}$-equivariant map $X_{d,2} \longrightarrow S(W_2 \oplus U_2^{\oplus j})$ whose restriction to $X_{d,2}^{>1}$ is $\mathfrak{S}_2^{\pm}$-homotopic to $\nu \circ \psi_{\mathcal{M}}|_{X_{d,2}^{>1}}$ for $\mathcal{M} = (I_1, \dots, I_j)$
- by evaluating the obstruction cocycle $\mathfrak{o}(g)$ for $g = \nu \circ \psi_{\mathcal{M}}|_{X^{(N_2)}}$ on cells $D_{1,1}^{+;+}(1,2)$ or $D_{2,1}^{+;+}(1,2)$, depending on ℓ being 0 or 1, using Theorem 1.3; and then
- prove that the cocycle $\mathfrak{o}(g)$ cannot be a coboundary, utilizing boundary formulas from Example 3.12.

4.3.1 2-bit Gray codes

In order to evaluate the obstruction cocycle $\mathfrak{o}(g)$ on the relevant cells in the case $k = 2$ we need to understand (2×4)-Gray codes. These correspond to equipartitions of an interval I on the moment curve into four equal orthants by intersecting with two hyperplanes H_1 and H_2 in altogether three points of the interval. There are two such configurations: either H_1 cuts through the midpoint of I and H_2 separates both halves of I into equal pieces by two additional intersections, or the roles of H_1 and H_2 are reversed. In terms of Gray codes we can express this as follows.

Lemma 4.4. *There are two different* 2*-bit Gray codes that start with the zero column (or any other fixed binary vector of length* 2*):*

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Proof. The second column of the Gray code determines the rest of the code, and there are only two choices for a bit flip. $\square$

This means that in the case $k = 2$ an ℓ-equiparting matrix A has a more compact representation: it is determined by the first column – a binary vector of length 2 – and j additional bits, one for each A_i, encoding whether the first bit flip in A_i is in the first or second row. These j bits *cannot be chosen independently* since there are restrictions imposed by the transition count.

Lemma 4.5. *Let $j \geq 1$ be an integer with $d = \lceil \frac{3}{2}j \rceil$ and $\ell = 2d - 3j \leq 1$.*

(1) *If $\ell = 0$, then the number of non-equivalent* 0*-equiparting matrices is equal to*
$$\tfrac{1}{2}\binom{j}{\frac{j}{2}}.$$

(2) *If $\ell = 1$, then the number of non-equivalent* 1*-equiparting matrices is equal to*
$$\binom{j}{\frac{j+1}{2}}.$$

Proof. We count the number of non-equivalent ℓ-equiparting matrices of the form $A = (A_1, \dots, A_j)$ where A_i is a 2-bit Gray code. A (2×4)-Gray code with the first bit flip in the first row has in total two bit flips in the first row and one bit flip in the second row.
(1) Let $\ell = 0$. Then $2d = 3j$ and consequently j has to be even. The matrix A must have transition count d in each row. Thus, half of the A_i's have the first bit flip in the first row. Consequently, 0-equiparting matrices A with a fixed first column are in bijection with $\frac{j}{2}$-element subsets of a set with j elements. By inverting the bits in each row we can fix the first column of A to be the zero vector. Additionally, we are allowed to interchange the rows. Up to this equivalence there are $\frac{1}{2}\binom{j}{j/2}$ such matrices.
(2) Let $\ell = 1$. Then $2d = 3j + 1$ and so j is odd. The matrix A must have transition count d in one row while transition count $d - 1$ in the remaining row. Without loss of generality we can assume that A have transition count d in the first row. Assume that r of the A_i's have the first bit flip in the first row. Consequently, $j - r$ of the A_i's have the first bit flip in the second row. Now the transition count of the first row is $2r + j - r$ while the transition count of the second row is $r + 2(j - r)$. The system of equations $2r + j - r = d$, $r + 2(j - r) = d - 1$ yields that $r = \frac{j+1}{2}$. Therefore, up to equivalence, there are $\binom{j}{r}$ such matrices. $\square$

4.3.2 The case $\ell = 0 \Leftrightarrow 2d = 3j$

Let $\theta := D_{1,1}^{+,+}(1,2)$, and let e_θ denote the related basis element of the cell θ in the top cellular chain group $C_{2d+1}(X_{d,2}, X_{d,2}^{>1})$ which, in this case, is equivariantly generated by θ. According to equation (16), which also holds for $k = 2$ as explained in Remark 4.3,

$$\mathfrak{o}(g)(e_\theta) = \big(\sum \pm 1\big) \cdot \zeta = a \cdot \zeta, \tag{18}$$

where $\zeta \in \pi_{2d+1}(S(W_2 \oplus U_2^{\oplus j})) \cong \mathbb{Z}$ is a generator, and the sum ranges over all arrangements of two hyperplanes in $\operatorname{relint}\theta$ that equipart $\mathcal{M}$. Since θ parametrizes all arrangements $\mathcal{H} = (H_1, H_2)$ where orientations and order of hyperplanes are fixed, the sum in (18) ranges over all arrangements of two hyperplanes that equipart $\mathcal{M}$ where orientation and order of hyperplanes are fixed. Therefore, by Theorem 1.3, the number of (± 1)'s in the sum of (18) is equal to the number of non-equivalent 0-equiparting matrices of size $2 \times 4j$. Now, Lemma 4.5 implies that the number of (± 1)'s in the sum of (18) is $\frac{1}{2}\binom{j}{j/2}$. Consequently, integer a is odd if and only if $\frac{1}{2}\binom{j}{j/2}$ is odd.
Assume that the cocycle $\mathfrak{o}(g)$ is a coboundary. Hence, there exists a cochain

$$\mathfrak{h} \in \mathcal{C}^{2d}_{\mathfrak{S}_2^{\pm}}\big(X_{d,2}, X_{d,2}^{>1}\,;\, \pi_{2d}(S(W_2 \oplus U_2^{\oplus j}))\big)$$

with the property that $\mathfrak{o}(g) = \delta\mathfrak{h}$. The relation (11) for $k = 2$ transforms into

$$\partial e_\theta = (1 + (-1)^d \varepsilon_1) \cdot e_{\gamma_1} + (1 + (-1)^d \tau_{1,2}) \cdot e_{\gamma_3}.$$

Thus we have that

$$\begin{aligned}
a\cdot\zeta &= \mathfrak{o}(g)(e_\theta)=\delta\mathfrak{h}(e_\theta)=\mathfrak{h}(\partial e_\theta)\\
&= (1+(-1)^d\varepsilon_1)\cdot\mathfrak{h}(e_{\gamma_1})+(1+(-1)^d\tau_{1,2})\cdot\mathfrak{h}(e_{\gamma_3})\\
&= (1+(-1)^d)\cdot\mathfrak{h}(e_{\gamma_1})+(1+(-1)^{d+j+1})\cdot\mathfrak{h}(e_{\gamma_3})\\
&= 2b\cdot\zeta.
\end{aligned}$$

Consequently, $\mathfrak{o}(g)$ is not a coboundary if and only if a is odd if and only if $\frac{1}{2}\binom{j}{j/2}$ is odd. Having in mind the Kummer criterion stated below we conclude that: A $\mathfrak{S}_2^{\pm}$-equivariant map $X_{d,2}\longrightarrow S(W_2\oplus U_2^{\oplus j})$ whose restriction to $X_{d,2}^{>1}$ is $\mathfrak{S}_2^{\pm}$-homotopic to $\nu\circ\psi_{\mathcal{M}}|_{X_{d,2}^{>1}}$ does not exists *if and only is* $\mathfrak{o}(g)$ is not a coboundary *if and only if* a is an odd integer *if and only if* $\frac{1}{2}\binom{j}{j/2}$ is odd *if and only if* $j=2^t$ for $t\geq 1$.

Lemma 4.6 (Kummer [12]). *Let $n\geq m\geq 0$ be integers and let p be a prime. The maximal integer k such that p^k divides $\binom{n}{m}$ is the number of carries when m and $n-m$ are added in base p.*

Thus we have proved the case (ii) of Theorem 1.5. Moreover, since the primary obstruction $\mathfrak{o}(g)$ is the only obstruction, we have proved that a $\mathfrak{S}_2^{\pm}$-equivariant map $X_{d,2}\longrightarrow S(W_2\oplus U_2^{\oplus j})$ whose restriction to $X_{d,2}^{>1}$ is $\mathfrak{S}_2^{\pm}$-homotopic to $\nu\circ\psi_{\mathcal{M}}|_{X_{d,2}^{>1}}$ exists if and only if j, an even integer, is not a power of 2.

4.3.3 The case $\ell=1\Leftrightarrow 2d=3j+1$

Let $\theta:=D_{2,1}^{+;+}(1,2)$, and again let e_θ denote the related basis element of the cell θ in the cellular chain group $C_{2d}(X_{d,2},X_{d,2}^{>1})$ which, in this case, is equivariantly generated by two cells $D_{2,1}^{+;+}(1,2)$ and $D_{1,2}^{+;+}(1,2)$. Again, the equation (16) implies that

$$\mathfrak{o}(g)(e_\theta)=\big(\sum\pm 1\big)\cdot\zeta=a\cdot\zeta, \tag{19}$$

where $\zeta\in\pi_{2d+1}(S(W_2\oplus U_2^{\oplus j}))\cong\mathbb{Z}$ is a generator, and the sum ranges over all arrangements of k hyperplanes in $\operatorname{relint}\theta$ that equipart $\mathcal{M}$. The cell θ parametrizes all arrangements $\mathcal{H}=(H_1,H_2)$ where H_1 passes through the given point on the moment curve and orientations and order of hyperplanes are fixed. Thus, the sum in (19) ranges over all arrangements of two hyperplanes that equipart $\mathcal{M}$ where H_1 passes through the given point on the moment curve with order and orientation of hyperplanes being fixed. Therefore, by Theorem 1.3, the number of (±1)'s in the sum of (19) is the same as the number of non-equivalent 1-equiparting matrices of size $2\times 4j$. Again, Lemma 4.5 implies that the number of (±1)'s in the sum of (19) is $\binom{j}{(j+1)/2}$. The integer a is odd if and only if $\binom{j}{(j+1)/2}$ is odd if and only if $j=2^t-1$ for $t\geq 1$.
Assume that the cocycle $\mathfrak{o}(g)$ is a coboundary. Then there exists a cochain

$$\mathfrak{h}\in\mathcal{C}^{2d-1}_{\mathfrak{S}_2^{\pm}}\big(X_{d,2},X_{d,2}^{>1}\,;\,\pi_{2d-1}(S(W_2\oplus U_2^{\oplus j}))\big)$$

with the property that $\mathfrak{o}(g) = \delta\mathfrak{h}$. Now, the relation (12) for $k = 2$ transforms into

$$\partial e_\theta = (1 + (-1)^{d-1}\varepsilon_1) \cdot e_{\gamma_1} + (1 + (-1)^d\varepsilon_2 + (-1)^d\tau_{1,2} + (-1)^{d+d}\varepsilon_1\tau_{1,2}) \cdot e_{\gamma_3}.$$

Thus, having in mind that j has to be odd, we have

$$\begin{aligned} a \cdot \zeta &= \mathfrak{o}(g)(e_\theta) = \delta\mathfrak{h}(e_\theta) = \mathfrak{h}(\partial e_\theta) \\ &= (1 + (-1)^{d-1}\varepsilon_1) \cdot \mathfrak{h}(e_{\gamma_1}) + \\ &\quad\ (1 + (-1)^d\varepsilon_2 + (-1)^d\tau_{1,2} + (-1)^{d+d}\varepsilon_1\tau_{1,2}) \cdot \mathfrak{h}(e_{\gamma_3}) \\ &= (1 + (-1)^{d-1}) \cdot \mathfrak{h}(e_{\gamma_1}) + (1 + (-1)^d + (-1)^{d+j+1} + (-1)^{j+1}) \cdot \mathfrak{h}(e_{\gamma_3}) \\ &= (1 + (-1)^{d-1}) \cdot \mathfrak{h}(e_{\gamma_1}) + (1 + (-1)^d + (-1)^d + 1) \cdot \mathfrak{h}(e_{\gamma_3}) \\ &= \begin{cases} 2\mathfrak{h}(e_{\gamma_1}), & d \text{ odd} \\ 4\mathfrak{h}(e_{\gamma_3}), & d \text{ even.} \end{cases} \end{aligned} \tag{20}$$

Now, we separately consider cases depending on parity of d and value of j.

(1) Let d be odd. Recall that a is odd if and only if $j = 2^t - 1$ for $t \geq 1$. Since $d = \frac{1}{2}(3j + 1) = 3 \cdot 2^{t-1} - 1$ and d is odd we have that for $j = 2^t - 1$, with $t \geq 2$, the integer a is odd and consequently $\mathfrak{o}(g)$ is not a coboundary. Thus a $\mathfrak{S}_2^{\pm}$-equivariant map $X_{d,2} \longrightarrow S(W_2 \oplus U_2^{\oplus j})$ whose restriction to $X_{d,2}^{>1}$ is $\mathfrak{S}_2^{\pm}$-homotopic to $\nu \circ \psi_{\mathcal{M}}|_{X_{d,2}^{>1}}$ does not exists. We have proved the case (ii) of Theorem 1.5 for $t \geq 2$.

(2) Let $d = 2$ and $j = 1 = 2^1 - 1$. Then the integer a is again odd and consequently cannot be divisible by 4 implying again that $\mathfrak{o}(g)$ is not a coboundary. Therefore a $\mathfrak{S}_2^{\pm}$-equivariant map $X_{2,2} \longrightarrow S(W_2 \oplus U_2)$ whose restriction to $X_{2,2}^{>1}$ is $\mathfrak{S}_2^{\pm}$-homotopic to $\nu \circ \psi_{\mathcal{M}}|_{X_{2,2}^{>1}}$ does not exists. This concludes the proof of the case (ii) of Theorem 1.5.

(3) Let $d \geq 4$ be even. Now we determine the integer a by computing local degrees $\deg(\nu \circ \psi_{\mathcal{M}}|_{X^{(N_2+1)}} \circ h_\theta|_{S_i})$; see (16) and (19). We prove, almost identically as in [3, Proof of Lem. 5.6], that all local degrees equal, either 1 or -1.

That local degrees of $\nu \circ \psi_{\mathcal{M}}|_\theta$ are ± 1 is simple to see since in a small neighborhood U in relint θ around any root $\lambda u + (1 - \lambda)v$ the test map $\psi_{\mathcal{M}}|_\theta$ is a continuous bijection. Indeed, for any vector $w \in W_2 \oplus U_2^{\oplus j}$, with sufficiently small norm, there is exactly one $\lambda u' + (1 - \lambda)v' \in U$ with $\psi_{\mathcal{M}}(\lambda u' + (1 - \lambda)v') = w$. Thus $\psi_{\mathcal{M}}|_{\partial U}$ is a continuous bijection into some $3j$-sphere around the origin of $W_2 \oplus U_2^{\oplus j}$ and by compactness of ∂U is a homeomorphism.

Next we compute the signs of the local degrees. First we describe a neighborhood of every root of the test map $\psi_{\mathcal{M}}$ in relint θ. Let $\lambda u + (1 - \lambda)v \in \operatorname{relint}\theta$ with $\psi_{\mathcal{M}}(\lambda u + (1 - \lambda)v) = 0$. Consequently $\lambda = \frac{1}{2}$. Denote the intersections of the hyperplane H_u with the moment curve by $x_1, \ldots, x_d$ in the correct order along the moment curve. Similarly, let $y_1, \ldots, y_d$ be the intersections of H_v with the moment curve. In particular, x_1 is the point q_1 that determines the cell θ, see Lemma 3.13. Choose an $\epsilon > 0$ such that ϵ-balls around $x_2, \dots, x_d$ and around $y_1, \dots, y_d$ are pairwise disjoint with the property that these balls intersect the moment curve only in precisely one of the intervals $I_1, \dots, I_j$. Pairs of hyperplanes $(H_{u'}, H_{v'})$ with $\lambda u' + (1-\lambda)v' \in \operatorname{relint}\theta$ that still intersect the moment curve in the corresponding ϵ-balls parametrize a neighborhood of $\frac{1}{2}u + \frac{1}{2}v$. The local neighborhood consisting of pairs of hyperplanes with the same orientation still intersecting the moment curve in the corresponding ϵ-balls where the parameter λ is in some neighborhood of $\frac{1}{2}$. For sufficiently small $\epsilon > 0$ the neighborhood can be naturally parametrized by the product

$$(\tfrac{1}{2}-\epsilon, \tfrac{1}{2}+\epsilon) \times \prod_{i=2}^{2d} (-\epsilon, \epsilon),$$

where the first factor relates to λ, the next $d-1$ factors correspond to neighborhoods of the $x_2, \dots, x_d$ and the last d factors to ϵ-balls around $y_1, \dots, y_d$. A natural basis of the tangent space at $\frac{1}{2}u + \frac{1}{2}v$ is obtained via the push-forward of the canonical basis of $\mathbb{R}^{2d}$ as tangent space at $(\frac{1}{2}, 0, \dots, 0)^t$.
Consider the subspace $Z \subseteq \operatorname{relint}\theta$ that consists all points $\lambda u + (1-\lambda)v$ associated to the pairs of hyperplanes (H_u, H_v) such that both hyperplanes intersect the moment curve in d points. In the space Z the local degrees only depend on the orientations of the hyperplanes H_u and H_v, but these are fixed since $Z \subseteq \operatorname{relint}\theta$. Indeed, any two neighborhoods of distinct roots of the test map $\psi_{\mathcal{M}}$ can be mapped onto each other by a composition of coordinate charts since their domains coincide. This is a smooth map of degree 1: the Jacobian at the root is the identity map. Let $\frac{1}{2}u + \frac{1}{2}v$ and $\frac{1}{2}u' + \frac{1}{2}v'$ be roots in Z of the test map $\psi_{\mathcal{M}}$ and let Ψ be the change of coordinate chart described above. Then $\psi_{\mathcal{M}}$ and $\psi_{\mathcal{M}} \circ \Psi$ differ in a neighborhood of $\frac{1}{2}u + \frac{1}{2}v$ just by a permutation of coordinates. This permutation is always even by the following:

Claim. *Let A and B be finite sets of the same cardinality. Then the cardinality of the symmetric sum $A \vartriangle B$ is even.*

The orientations of the hyperplanes H_u and H_v are fixed by the condition that $\frac{1}{2}u + \frac{1}{2}v \in \operatorname{relint}\theta$. Thus, H_u and H_v are completely determined by the set of intervals that H_u cuts once. Let $A \subseteq \{1, \dots, j\}$ be the set of indices of intervals $I_1, \dots, I_h$ that H_u intersects once, and let $B \subseteq \{1, \dots, j\}$ be the same set for H_v. Then Ψ is a composition of a multiple of $A \vartriangle B$ transpositions and, hence, an even permutation. This means that all the local degrees (± 1's) in the sum (19) are of the same sign, and consequently $a = \pm\binom{j}{(j+1)/2}$.
Now, since d is even the equality (20) implies that

$$a \cdot \zeta = 4b \cdot \zeta.$$

Thus, if $\mathfrak{o}(g)$ is a coboundary a is divisible by 4. In the case $j = 2^t + 1$ where $t \geq 2$, and $d = 3 \cdot 2^{t-1} + 2$ the Kummer criterion implies that the binomial coefficient $\binom{j}{(j+1)/2}$ is divisible by 2 but *not* by 4. Hence, $\mathfrak{o}(g)$ is

not a coboundary and a $\mathfrak{S}_2^{\pm}$-equivariant map $X_{d,2} \longrightarrow S(W_2 \oplus U_2^{\oplus j})$ whose restriction to $X_{d,2}^{>1}$ is $\mathfrak{S}_2^{\pm}$-homotopic to $\nu \circ \psi_{\mathcal{M}}|_{X_{d,2}^{>1}}$ does not exist.
This concludes the final instance (iii) of Theorem 1.5. □

4.4 Proof of Theorem 1.6

We prove both instances of the Ramos conjecture $\Delta(2,3) = 5$ and $\Delta(4,3) = 10$ using Theorem 1.4. Thus in order to prove that

- $\Delta(2,3) = 5$ it suffices to show that the number of non-equivalent 1-equiparting matrices of size $3 \times 2 \cdot 2^3$ is odd, Proposition 4.8;
- $\Delta(4,3) = 10$ it suffices to show that the number of non-equivalent 2-equiparting matrices of size $3 \times 4 \cdot 2^3$ is also odd, Enumeration 4.9.

Consequently we turn our attention to 3-bit Gray codes. It is not hard to see that the following lemma holds.

Lemma 4.7. *Let $c_1 \in \{0,1\}^3$ be a choice of first column.*

(i) *There are 18 different 3-bit Gray codes $A = (c_1, c_2, \ldots, c_8) \in \{0,1\}^{3\times 8}$ that start with c_1. They have transition counts $(3,2,2)$, $(3,3,1)$, or $(4,2,1)$.*

(ii) *There are 3 equivalence classes of Gray codes that start with with c_1. The three classes can be distinguished by their transition counts.*

Proof. (i): Starting at a given vertex of the 3-cube, there are precisely 18 Hamiltonian paths. This can be seen directly or by computer enumeration.
(ii): Follows directly from (i), as all equivalence classes have size 6: If $c_1 = (0,0,0)^t$ then all elements in a class are obtained by permutation of rows. For other choices of c_1, they are obtained by arbitrary permutations of rows followed by the "correct" row bit-inversions to obtain c_1 in the first column. □

Proposition 4.8. *There are 13 non-equivalent 1-equiparting matrices that are of size $3 \times (2 \cdot 2^3)$.*

Proof. Let $A = (A_1, A_2)$ be a 1-equiparting matrix. This means that both A_1 and A_2 are 3-bit Gray codes and the last column of A_1 is equal to the first column of A_2. In addition, the transition counts cannot exceed 5 and must sum up to 14. Having in mind that A is a 1-equiparting matrix it follows that A must have transition counts $\{5,5,4\}$. Hence two of its rows must have transition count 5 and one row must have transition count 4. In the following a *realization* of transition counts is a Gray code with the prescribed transition counts.
Since we are counting 1-equiparting matrices up to equivalence we may fix the first column of A, and hence first column of A_1, to be $(0,0,0)^t$ and choose for A_1 one of the matrices from each of the 3 classes of 3-bit Gray codes described in Lemma 4.7(ii).
If A_1 has transition counts $(3,2,2)$, i.e., the first row has transition count 3 while remaining rows have transition count 2, then its last column is $(1,0,0)^t$. The

next Gray code A_2 in the matrix a can have transition counts $(2,3,2)$, $(2,2,3)$, or $(1,3,3)$, each having 2 realizations A_2, each with first column $(1,0,0)^t$.
If A_1 has transition $(3,3,1)$, then its last column is $(1,1,0)^t$. The Gray code A_2 can have transition counts $(2,2,3)$, having 2 realizations, or $(1,2,4)$, having 1 realization, or $(2,1,4)$, having 1 realization, each with first column $(1,1,0)^t$.
If A_1 has transition counts $(4,2,1)$, then its last column is $(0,0,1)^t$. The Gray code A_2 can have transition counts $(1,2,4)$, having 1 realization, or $(1,3,3)$, having 2 realizations, each with first column $(0,0,1)^t$.
In total we have $6+4+3=13$ non-equivalent 1-equiparting matrices $A=(A_1,A_2)$. □

Enumeration 4.9. *There are* 2015 *non-equivalent* 2*-equiparting matrices that are of size* $3\times 4\cdot 2^3$.

Proof. Using Lemma 4.7 we enumerate non-equivalent 2-equiparting matrices by computer. Let $A=(A_1,A_2,A_3,A_4)$ be a 2-equiparting matrix. It must have transition counts $\{10,10,8\}$. Similarly as above, A is constructed by fixing the first column to be $(0,0,0)^t$ and A_1 to be one representative from each of the 3 classes of Gray codes. Then all possible Gray codes for A_2,A_3,A_4 are checked, making sure that the last column of A_i is equal to the first column of A_{i+1} and that the transition counts of $A_1,\dots,A_4$ sum up to $\{10,10,8\}$. This leads to 2015 possibilities. □

This concludes the proof of Theorem 1.6.

Remark 4.10. By means of a computer we were able to calculate the number $N(j,k,d)$ of non-equivalent ℓ-equiparting matrices for several values of $j\geq 1$ and $k\geq 3$, where $d=\lceil\frac{2^k-1}{k}j\rceil$ and $\ell=dk-(2^k-1)j$. See Table 1.

Number $N(j,k,d)$ of non-equiv ℓ-equiparting matrices given $j\geq 2$, and $k\geq 3$.				
j	k	ℓ	d	$N(j,k,d)$
2	3	1	5	13
3	3	0	7	60
4	3	2	10	2015
5	3	1	12	35040
6	3	0	14	185130
7	3	2	17	7572908
8	3	1	19	132909840
9	3	0	21	732952248
1	4	1	4	16
2	4	2	8	37964

Table 1: Number $N(j,k,d)$ of non-equivalent ℓ-equiparting matrices given $j\geq 2$ and $k\geq 3$, where $d=\lceil\frac{2^k-1}{k}j\rceil$ and $\ell=dk-(2^k-1)j$.

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