On Zeta Elements for $\mathbb{G}_m$

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Abstract. In this paper, we present a unifying approach to the general theory of abelian Stark conjectures. To do so we define natural notions of 'zeta element', of 'Weil-étale cohomology complexes' and of 'integral Selmer groups' for the multiplicative group $\mathbb{G}_m$ over finite abelian extensions of number fields. We then conjecture a precise connection between zeta elements and Weil-étale cohomology complexes, we show this conjecture is equivalent to a special case of the equivariant Tamagawa number conjecture and we give an unconditional proof of the analogous statement for global function fields. We also show that the conjecture entails much detailed information about the arithmetic properties of generalized Stark elements including a new family of integral congruence relations between Rubin-Stark elements (that refines recent conjectures of Mazur and Rubin and of the third author) and explicit formulas in terms of these elements for the higher Fitting ideals of the integral Selmer groups of $\mathbb{G}_m$, thereby obtaining a clear and very general approach to the theory of abelian Stark conjectures. As first applications of this approach, we derive, amongst other things, a proof of (a refinement of) a conjecture of Darmon concerning cyclotomic units, a proof of (a refinement of) Gross's 'Conjecture for Tori' in the case that the base field is $\mathbb{Q}$, explicit conjectural formulas for both annihilating elements and, in certain cases, the higher Fitting ideals (and hence explicit structures) of ideal class groups and a strong refinement of many previous results concerning abelian Stark conjectures.

Contents

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1. Introduction

The study of the special values of zeta functions and, more generally, of L-functions is a central theme in number theory that has a long tradition stretching back to Dirichlet and Kummer in the nineteenth century. In particular, much work has been done concerning the arithmetic properties of the special values of L-functions and their incarnations in appropriate arithmetic cohomology groups, or 'zeta elements' as they are commonly known.

The aim of our project is to systematically study the fine arithmetic properties of such zeta elements and thereby to obtain both generalizations and refinements of a wide range of well-known results and conjectures in the area. In this first article we shall concentrate, for primarily pedagogical reasons, on the classical and very concrete case of the L-functions that are attached to the multiplicative group $\mathbb{G}_m$ over a finite abelian extension K/k of global fields. In subsequent articles we will then investigate the key Iwasawa-theoretic aspects of our approach (see [9]) and also explain how the conjectures and results presented here naturally extend both to the case of Galois extensions that are not abelian and to the case of the zeta elements that are associated (in general conjecturally) to a wide class of motives over number fields.

The main results of the present article are given below as Theorems 1.1, 1.5 and 1.10. In the rest of this introduction we state these results and also discuss a selection of interesting consequences.

To do this we fix a finite abelian extension of global fields K/k with Galois group $G = \mathrm{Gal}(K/k)$.

We then fix a finite non-empty set of places S of k containing both the set $S_{\mathrm{ram}}(K/k)$ of places which ramify in K/k and the set $S_\infty(k)$ of archimedean places (if any).

Lastly we fix an auxiliary finite non-empty set of places T of k which is disjoint from S and such that the group $\mathcal{O}_{K,S,T}^\times$ of S-units of K that are endowed with a trivialization at each place of K above a place in T is $\mathbb{Z}$-torsion-free (for the precise definition of $\mathcal{O}_{K,S,T}^\times$, see §1.7).

1.1. The leading term conjecture and Rubin-Stark elements. As a first step we shall define a canonical 'T-modified Weil-étale cohomology' complex for $\mathbb{G}_m$ and then formulate (as Conjecture 3.6) a precise 'leading term conjecture' $\mathrm{LTC}(K/k)$ for the extension K/k. This conjecture predicts that the canonical *zeta element* $z_{K/k,S,T}$ interpolating the leading terms at $s=0$ of the (S-truncated T-modified) L-functions $L_{k,S,T}(\chi,s)$ generates the determinant module over G of the T-modified Weil-étale cohomology complex for $\mathbb{G}_m$ over K.

The main result of the first author in [5] implies that $\mathrm{LTC}(K/k)$ is valid if k is a global function field.

In the number field case our formulation of $\mathrm{LTC}(K/k)$ is motivated by the 'Tamagawa Number Conjecture' formulated by Bloch and Kato in [1] and by the 'generalized Iwasawa main conjecture' studied by Kato in [24] and [25]. In particular, we shall show that for extensions K/k of number fields $\mathrm{LTC}(K/k)$

is equivalent to the relevant special case of the 'equivariant Tamagawa number conjecture' formulated in the article [7] of Flach and the first author. Taken in conjunction with previous work of several authors, this fact implies that $\mathrm{LTC}(K/k)$ is also unconditionally valid for several important families of number fields.

We assume now that S contains a subset $V = \{v_1, \ldots, v_r\}$ of places which split completely in K. In this context, one can use the values at $s = 0$ of the r-th derivatives of S-truncated T-modified L-functions to define a canonical element

$$\epsilon^V_{K/k,S,T}$$

in the exterior power module $\bigwedge^r_{\mathbb{Z}[G]} \mathcal{O}^\times_{K,S,T} \otimes \mathbb{R}$ (for the precise definition see §5.1).

As a natural generalization of a classical conjecture of Stark (dealing with the case $r = 1$) Rubin conjectured in [45] that the elements $\epsilon^V_{K/k,S,T}$ should always satisfy certain precise integrality conditions (for more details see Remark 1.6). As is now common in the literature, in the sequel we shall refer to $\epsilon^V_{K/k,S,T}$ as the 'Rubin-Stark element' (relative to the given data) and to the central conjecture of Rubin in [45] as the 'Rubin-Stark Conjecture'.

In some very special cases $\epsilon^V_{K/k,S,T}$ can be explicitly computed and the Rubin-Stark Conjecture verified. For example, this is the case if $r = 0$ (so $V = \emptyset$) when $\epsilon^V_{K/k,S,T}$ can be described in terms of Stickelberger elements and if $k = \mathbb{Q}$ and $V = \{\infty\}$ when $\epsilon^V_{K/k,S,T}$ can be described in terms of cyclotomic units.

As a key step in our approach we show that in all cases the validity of $\mathrm{LTC}(K/k)$ implies that $\epsilon^V_{K/k,S,T}$ can be computed as 'the canonical projection' of the zeta element $z_{K/k,S,T}$.

This precise result is stated as Theorem 5.14 and its proof will also incidentally show that $\mathrm{LTC}(K/k)$ implies the validity of the Rubin-Stark conjecture for K/k. The latter implication was in fact already observed by the first author in [3] (and the techniques developed in loc. cit. have since been used by several other authors) but we would like to point out that the proof presented here is very much simpler than that given in [3] and is therefore much more amenable to subsequent generalization.

1.2. Refined class number formulas for $\mathbb{G}_m$. The first consequence of Theorem 5.14 that we record here concerns a refined version of a conjecture that was recently formulated independently by Mazur and Rubin in [37] (where it is referred to as a 'refined class number formula for $\mathbb{G}_m$') and by the third author in [46].

To discuss this we fix an intermediate field L of K/k and a subset $V' = \{v_1, \ldots, v_{r'}\}$ of S which contains V and is such that every place in V' splits completely in L.

In this context it is known that the elements $\epsilon^V_{K/k,S,T}$ naturally constitute an Euler system of rank r and the elements $\epsilon^{V'}_{L/k,S,T}$ an Euler system of rank r'. If $r < r'$, then the image of $\epsilon^V_{K/k,S,T}$ under the map induced by the field theoretic

norm $K^\times \to L^\times$ vanishes. However, in this case Mazur and Rubin (see [37, Conj. 5.2]) and the third author (see [46, Conj. 3]) independently observed that the reciprocity maps of local class field theory lead to an important conjectural relationship between the elements $\epsilon^V_{K/k,S,T}$ and $\epsilon^{V'}_{L/k,S,T}$.
We shall here formulate an interesting refinement $\mathrm{MRS}(K/L/k,S,T)$ of the central conjectures of [37] and [46] (see Conjecture 5.4 and the discussion of Remark 5.7) and we shall then prove the following result.

Theorem 1.1. $\mathrm{LTC}(K/k)$ *implies the validity of* $\mathrm{MRS}(K/L/k,S,T)$.

This result is both a generalization and strengthening of the main result of the third author in [46, Th. 3.22] and provides strong evidence for $\mathrm{MRS}(K/L/k,S,T)$.
As already remarked earlier, if k is a global function field, then the validity of $\mathrm{LTC}(K/k)$ is a consequence of the main result of [5]. In addition, if $k=\mathbb{Q}$, then the validity of $\mathrm{LTC}(K/k)$ follows from the work of Greither and the first author in [8] and of Flach in [14].
Theorem 1.1 therefore has the following consequence.

Corollary 1.2. $\mathrm{MRS}(K/L/k,S,T)$ *is valid if* $k=\mathbb{Q}$ *or if* k *is a global function field.*

This result is of particular interest since it verifies the conjectures of Mazur and Rubin [37] and of the third author [46] even in cases for which one has $r>1$.
In a sequel [9] to this article we will also prove a partial converse to Theorem 1.1 and show that this converse can be used to derive significant new evidence in support of the conjecture $\mathrm{LTC}(K/k)$ (for more details see §1.6 below).
Next we recall that in [12] Darmon used the theory of cyclotomic units to formulate a refined version of the class number formula for the class groups of real quadratic fields. We further recall that Mazur and Rubin in [36], and later the third author in [46], have proved the validity of the central conjecture of [12] but only after inverting the prime 2.
We shall formulate in §6 a natural refinement of Darmon's conjecture. By using Corollary 1.2 we shall then give a full proof of our refined version of Darmon's conjecture, thereby obtaining the following result (for a precise version of which see Theorem 6.1).

Corollary 1.3. *A natural refinement of Darmon's conjecture in* [12] *is valid.*

Let now K/k be an abelian extension as above and choose intermediate fields L and $\widetilde{L}$ with $[L:k]=2$, $L\cap\widetilde{L}=k$ and $K=L\widetilde{L}$. In this context Gross has formulated in [21] a 'conjecture for tori' regarding the value of the canonical Stickelberger element associated to K/k modulo a certain ideal constructed from class numbers and a canonical integral regulator map. This conjecture has been widely studied in the literature, perhaps most notably by Hayward in [22] and by Greither and Kučera in [16, 17].

We shall formulate (as Conjecture 6.3) a natural refinement of Gross's conjecture for tori and we shall then prove (in Theorem 6.5) that the validity of this refinement is a consequence of $\mathrm{MRS}(K/L/k, S, T)$.

As a consequence of Corollary 1.2 we shall therefore obtain the following result.

Corollary 1.4. *A natural refinement of Gross's conjecture for tori is valid if $k = \mathbb{Q}$ or if k is a global function field.*

This result is a significant improvement of the main results of Greither and Kučera in [16, 17]. In particular, whilst the latter articles only study the case that $k = \mathbb{Q}$, L is an imaginary quadratic field, and $\widetilde{L}/\mathbb{Q}$ is an abelian extension satisfying several technical conditions (see Remark 6.6), Corollary 1.4 now proves Gross's conjecture completely in the case $k = \mathbb{Q}$ and with no assumption on either L or $\widetilde{L}$.

1.3. Selmer groups and their higher Fitting ideals. In order to state our second main result, we introduce two new Galois modules which are each finitely generated abelian groups and will play a key role in the arithmetic theory of zeta elements.

The first of these is a canonical '(Σ-truncated T-modified) integral dual Selmer group' $\mathcal{S}_{\Sigma,T}(\mathbb{G}_{m/K})$ for the multiplicative group over K for each finite non-empty set of places Σ of K that contains $S_\infty(K)$ and each finite set of places T of K that is disjoint from Σ.

If $\Sigma = S_\infty(K)$ and T is empty, then $\mathcal{S}_{\Sigma,T}(\mathbb{G}_{m/K})$ is simply defined to be the cokernel of the map

$$\prod_w \mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K^\times, \mathbb{Z}), \quad (x_w)_w \mapsto (a \mapsto \sum_w \mathrm{ord}_w(a) x_w),$$

where in the product and sum w runs over all finite places of K, and in this case constitutes a canonical integral structure on the Pontryagin dual of the Bloch-Kato Selmer group $H^1_f(K, \mathbb{Q}/\mathbb{Z}(1))$ (see Remark 2.3(i)).

In general, the group $\mathcal{S}_{\Sigma,T}(\mathbb{G}_{m/K})$ is defined to be a natural analogue for $\mathbb{G}_m$ of the 'integral Selmer group' that was introduced for abelian varieties by Mazur and Tate in [38] and, in particular, lies in a canonical exact sequence of G-modules of the form

(1)
$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}^T_\Sigma(K), \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathcal{S}_{\Sigma,T}(\mathbb{G}_{m/K}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}^\times_{K,\Sigma,T}, \mathbb{Z}) \longrightarrow 0$$

where $\mathrm{Cl}^T_\Sigma(K)$ is the ray class group of $\mathcal{O}_{K,\Sigma}$ modulo the product of all places of K above T (see §1.7).

This Selmer group is also philosophically related to the theory of Weil-étale cohomology that is conjectured to exist by Lichtenbaum in [34], and in this direction we show that in all cases there is a natural identification

$$\mathcal{S}_{\Sigma,T}(\mathbb{G}_{m/K}) = H^2_{c,T}((\mathcal{O}_{K,\Sigma})_{\mathcal{W}}, \mathbb{Z})$$

where the right hand group denotes the cohomology in degree two of a canonical 'T-modified compactly supported Weil-étale cohomology complex' that we introduce in §2.2.

The second module $\mathcal{S}^{\mathrm{tr}}_{\Sigma,T}(\mathbb{G}_{m/K})$ that we introduce is a canonical 'transpose' (in the sense of Jannsen's homotopy theory of modules [23]) for $\mathcal{S}_{\Sigma,T}(\mathbb{G}_{m/K})$. In terms of the complexes introduced in §2.2 this module can be described as a certain 'T-modified Weil-étale cohomology group' of $\mathbb{G}_m$

$$\mathcal{S}^{\mathrm{tr}}_{\Sigma,T}(\mathbb{G}_{m/K}) = H^1_T((\mathcal{O}_{K,\Sigma})_{\mathcal{W}}, \mathbb{G}_m)$$

and can also be shown to lie in a canonical exact sequence of G-modules of the form

$$0 \longrightarrow \mathrm{Cl}^T_\Sigma(K) \longrightarrow \mathcal{S}^{\mathrm{tr}}_{\Sigma,T}(\mathbb{G}_{m/K}) \longrightarrow X_{K,\Sigma} \longrightarrow 0. \tag{2}$$

Here $X_{K,\Sigma}$ denotes the subgroup of the free abelian group on the set Σ_K of places of K above Σ comprising elements whose coefficients sum to zero.
We can now state our second main result.
In this result we write $\mathrm{Fitt}^r_G(M)$ for the r-th Fitting ideal of a finitely generated G-module M, though the usual notation is $\mathrm{Fitt}_{r,\mathbb{Z}[G]}(M)$, in order to make the notation consistent with the exterior power $\bigwedge^r_{\mathbb{Z}[G]} M$. Note that we will review the definition of higher Fitting ideals in §7.1 and also introduce there for each finitely generated G-module M and each pair of non-negative integers r and i a natural notion of 'higher relative Fitting ideal'

$$\mathrm{Fitt}^{(r,i)}_G(M) = \mathrm{Fitt}^{(r,i)}_G(M, M_{\mathrm{tors}}).$$

We write $x \mapsto x^\#$ for the $\mathbb{C}$-linear involution of $\mathbb{C}[G]$ which inverts elements of G.

Theorem 1.5. *Let $K/k, S, T, V$ and r be as above, and assume that* $\mathrm{LTC}(K/k)$ *is valid. Then all of the following claims are also valid.*

(i) *One has*

$$\mathrm{Fitt}^r_G(\mathcal{S}_{S,T}(\mathbb{G}_{m/K})) = \{\Phi(\epsilon^V_{K/k,S,T})^\# : \Phi \in \bigwedge^r_{\mathbb{Z}[G]} \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G])\}.$$

(ii) *Let $\mathcal{P}_k(K)$ be the set of all places which split completely in K. Fix a non-negative integer i and set*

$$\mathcal{V}_i = \{V' \subset \mathcal{P}_k(K) : |V'| = i \text{ and } V' \cap (S \cup T) = \emptyset\}.$$

Then one has

$$\begin{aligned} &\mathrm{Fitt}^{(r,i)}_G(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K})) \\ = \;& \{\Phi(\epsilon^{V\cup V'}_{K/k,S\cup V',T}) : V' \in \mathcal{V}_i \text{ and } \Phi \in \bigwedge^{r+i}_{\mathbb{Z}[G]} \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^\times_{K,S\cup V',T}, \mathbb{Z}[G])\}. \end{aligned}$$

In particular, if $i = 0$, then one has

$$\mathrm{Fitt}^r_G(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K})) = \{\Phi(\epsilon^V_{K/k,S,T}) : \Phi \in \bigwedge^r_{\mathbb{Z}[G]} \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G])\}.$$

Remark 1.6. In terms of the notation of Theorem 1.5, the Rubin-Stark Conjecture asserts that $\Phi(\epsilon^V_{K/k,S,T})$ belongs to $\mathbb{Z}[G]$ for every Φ in $\bigwedge^r_{\mathbb{Z}[G]}\mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^\times_{K,S,T},\mathbb{Z}[G])$. The property described in Theorem 1.5 is deeper in that it shows the ideal generated by $\Phi(\epsilon^V_{K/k,S,T})$ as Φ runs over $\bigwedge^r_{\mathbb{Z}[G]}\mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^\times_{K,S,T},\mathbb{Z}[G])$ should encode significant arithmetic information relating to integral Selmer groups. (See also Remark 5.13 in this regard.)

1.4. Galois structures of ideal class groups. In this subsection, in order to better understand the content of Theorem 1.5, we discuss several interesting consequences concerning the explicit Galois structure of ideal class groups.

To do this we fix an odd prime p and suppose that K/k is any finite abelian extension of global fields. We write L for the (unique) intermediate field of K/k such that K/L is a p-extension and $[L:k]$ is prime to p. Then the group $\mathrm{Gal}(K/k)$ decomposes as a direct product $\mathrm{Gal}(L/k)\times\mathrm{Gal}(K/L)$ and we fix a non-trivial faithful character χ of $\mathrm{Gal}(L/k)$. We set $\mathrm{Cl}^T(K):=\mathrm{Cl}^T_\emptyset(K)$ and define its '(p,χ)-component' by setting

$$A^T(K)^\chi:=(\mathrm{Cl}^T(K)\otimes\mathbb{Z}_p)\otimes_{\mathbb{Z}_p[\mathrm{Gal}(L/k)]}\mathcal{O}_\chi.$$

Here we write $\mathcal{O}_\chi$ for the module $\mathbb{Z}_p[\mathrm{im}(\chi)]$ upon which $\mathrm{Gal}(L/k)$ acts via χ so that $A^T(K)^\chi$ has an induced action of the group ring $R^\chi_K:=\mathcal{O}_\chi[\mathrm{Gal}(K/L)]$.

Then in Theorem 8.1 we shall derive the following results about the structure of $A^T(K)^\chi$ from the final assertion of Theorem 1.5(ii).

In this result we write '$\chi(v)\neq 1$' to mean that the decomposition group of v in $\mathrm{Gal}(L/k)$ is non-trivial.

Corollary 1.7. *Let V be the set of archimedean places of k that split completely in K and set $r:=|V|$. Assume that any ramifying place v of k in K satisfies $\chi(v)\neq 1$. Assume also that the equality of $\mathrm{LTC}(K/k)$ is valid after applying the functor $-\otimes_{\mathbb{Z}_p[\mathrm{Gal}(L/k)]}\mathcal{O}_\chi$.*

Then for any non-negative integer i one has an equality

$$\mathrm{Fitt}^i_{R^\chi_K}(A^T(K)^\chi)=\{\Phi(\epsilon^{V\cup V',\chi}_{K/k,S\cup V',T}):V'\in\mathcal{V}_i \text{ and } \Phi\in\bigwedge^{r+i}_{R^\chi_K}\mathcal{H}_\chi\}$$

where we set $S:=S_\infty(k)\cup S_{\mathrm{ram}}(K/k)$ and $\mathcal{H}_\chi:=\mathrm{Hom}_{R^\chi_K}((\mathcal{O}^\times_{K,S\cup V',T}\otimes\mathbb{Z}_p)^\chi,R^\chi_K)$.

We remark that Corollary 1.7 specializes to give refinements of several results in the literature.

For example, if $k=\mathbb{Q}$ and K is equal to the maximal totally real subfield $\mathbb{Q}(\zeta_m)^+$ of $\mathbb{Q}(\zeta_m)$ where ζ_m is a fixed choice of primitive m-th root of unity for some natural number m, then $\mathrm{LTC}(K/k)$ is known to be valid and so Corollary 1.7 gives an explicit description of the higher Fitting ideals of ideal class groups in terms of cyclotomic units (which are the relevant Rubin-Stark elements in this case). In particular, if $m=p^n$ for any non-negative integer n, then the

necessary condition on χ is satisfied for all non-trivial χ and Corollary 1.7 gives a strong refinement of Ohshita's theorem in [41] for the field $K = \mathbb{Q}(\zeta_{p^n})^+$.
The result is also stronger than that of Mazur and Rubin in [35, Th. 4.5.9] since the latter describes structures over a discrete valuation ring whilst Corollary 1.7 describes structures over the group ring R_K^χ.
In addition, if K is a CM extension of a totally real field k, then Corollary 1.7 constitutes a generalization of the main results of the second author in both [28] and [30]. To explain this we suppose that K/k is a CM-extension and that χ is an odd character. Then classical Stickelberger elements can be used to define for each non-negative integer i a 'higher Stickelberger ideal'

$$\Theta^i(K/k) \subseteq \mathbb{Z}_p[\mathrm{Gal}(K/k)]$$

(for details see §8.3). By taking T to be empty we can consider the (p,χ)-component of the usual ideal class group

$$A(K)^\chi := (\mathrm{Cl}^T(K) \otimes \mathbb{Z}_p) \otimes_{\mathbb{Z}_p[\mathrm{Gal}(L/k)]} \mathcal{O}_\chi.$$

Then, by using both Theorem 1.1 and Corollary 1.7 we shall derive the following result as a consequence of the more general Theorem 8.6.
In this result we write ω for the Teichmüller character giving the Galois action on the group of p-th roots of unity.

Corollary 1.8. *Let K be a CM-field, k totally real, and χ an odd character with $\chi \neq \omega$. We assume that any ramifying place v of k in K satisfies $\chi(v) \neq 1$ and that $\mathrm{LTC}(F/k)$ is valid for certain extensions F of K (see Theorem 8.6 for the precise conditions on F).*
Then for any non-negative integer i one has an equality

$$\mathrm{Fitt}^i_{R_K^\chi}(A(K)^\chi) = \Theta^i(K/k)^\chi.$$

In the notation of Corollary 1.8 suppose that K is the n-th layer of the cyclotomic $\mathbb{Z}_p$-extension of L for some non-negative integer n and that every place $\mathfrak{p}$ above p satisfies $\chi(\mathfrak{p}) \neq 1$. Then the conditions on $\chi(v)$ and $\mathrm{LTC}(F/k)$ that are stated in Corollary 1.8 are automatically satisfied and so Corollary 1.8 generalizes the main results of the second author in [30].
To get a better feeling for Corollary 1.8, consider the simple case that $[K:k]$ is prime to p. In this case $K = L$, the ring $\mathbb{Z}_p[\mathrm{Gal}(K/k)]$ is semi-local and $A(K)^\chi$ is a module over the discrete valuation ring $\mathcal{O}_\chi = R_K^\chi$. Then the conclusion in Corollary 1.8 in the case $i = 0$ implies that

$$|A(K)^\chi| = |\mathcal{O}_\chi / L_k(\chi^{-1}, 0)| \tag{3}$$

where $L_k(\chi^{-1}, s)$ is the usual Artin L-function. If every place $\mathfrak{p}$ above p satisfies $\chi(\mathfrak{p}) \neq 1$, then this equality is known to be a consequence of the main conjecture for totally real fields proved by Wiles [54]. However, without any such restriction on the values $\chi(\mathfrak{p})$, the equality (3) is as yet unproved.
In addition, in this case the result of Corollary 1.8 is much stronger than (3) in that it shows the explicit structure of $A(K)^\chi$ as a Galois module to be

completely determined (conjecturally at least) by Stickelberger elements by using the obvious (non-canonical) isomorphism of $\mathcal{O}_\chi$-modules

$$A(K)^\chi \simeq \bigoplus_{i\geq 1} \mathrm{Fitt}^i_{\mathcal{O}_\chi}(A(K)^\chi)/\,\mathrm{Fitt}^{i-1}_{\mathcal{O}_\chi}(A(K)^\chi) = \bigoplus_{i\geq 1} \Theta^i(K/k)^\chi/\Theta^{i-1}(K/k)^\chi.$$

Next we note that the proof of Corollary 1.8 also combines with the result of Theorem 1.16 below to give the following result (which does not itself assume the validity of $\mathrm{LTC}(K/k)$).

This result will be proved in Corollaries 8.4 and 8.8. In it we write $\mu_{p^\infty}(k(\zeta_p))$ for the p-torsion subgroup of $k(\zeta_p)^\times$.

Corollary 1.9. *Assume that K/k is a CM-extension, that the degree $[K:k]$ is prime to p, and that χ is an odd character of G such that there is at most one p-adic place $\mathfrak{p}$ of k with $\chi(\mathfrak{p}) = 1$. Assume also that the p-adic μ-invariant of K_∞/K vanishes.*

Then one has both an equality

$$|A(K)^\chi| = \begin{cases} |\mathcal{O}_\chi/L_k(\chi^{-1},0)| & \text{if } \chi \neq \omega, \\ |\mathcal{O}_\chi/(|\mu_{p^\infty}(k(\zeta_p))|\cdot L_k(\chi^{-1},0))| & \text{if } \chi = \omega \end{cases}$$

and a (non-canonical) isomorphism of $\mathcal{O}_\chi$-modules

$$A(K)^\chi \simeq \bigoplus_{i\geq 1} \Theta^i(K/k)^\chi/\Theta^{i-1}(K/k)^\chi.$$

This result is a generalization of the main theorem of the second author in [28] where it is assumed that $\chi(\mathfrak{p}) \neq 1$ for all p-adic places $\mathfrak{p}$. It also generalizes the main result of Rubin in [44] which deals only with the special case $K = \mathbb{Q}(\mu_p)$ and $k = \mathbb{Q}$.

To end this subsection we note Remark 1.13 below explains why Theorem 1.5(ii) also generalizes and refines the main result of Cornacchia and Greither in [10].

1.5. Annihilators and Fitting ideals of class groups for small Σ.

In this subsection we discuss further connections between Rubin-Stark elements and the structure of class groups of the form $\mathrm{Cl}^T_\Sigma(K)$ for 'small' sets Σ which do not follow from Theorem 1.5. In particular, we do not assume here that Σ contains $S_{\mathrm{ram}}(K/k)$.

To do this we denote the annihilator ideal of a G-module M by $\mathrm{Ann}_G(M)$.

Theorem 1.10. *Assume $\mathrm{LTC}(K/k)$ is valid.*

Fix Φ in $\bigwedge^r_{\mathbb{Z}[G]}\mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^\times_{K,S,T},\mathbb{Z}[G])$ and any place v in $S\setminus V$.

Then one has

$$\Phi(\epsilon^V_{K/k,S,T}) \in \mathrm{Ann}_G(\mathrm{Cl}^T_{V\cup\{v\}}(K))$$

and, if G is cyclic, also

$$\Phi(\epsilon^V_{K/k,S,T}) \in \mathrm{Fitt}^0_G(\mathrm{Cl}^T_{V\cup\{v\}}(K)).$$

Remark 1.11. The first assertion of Theorem 1.10 provides a common refinement and wide-ranging generalization (to L-series of arbitrary order of vanishing) of several well-known conjectures and results in the literature. To discuss this we write $\mathrm{Cl}^T(K)$ for the full ray class group modulo T (namely, $\mathrm{Cl}^T(K) = \mathrm{Cl}^T_\emptyset(K)$, see §1.7).

(i) We first assume that $r = 0$ (so V is empty) and that k is a number field. Then, without loss of generality (for our purposes), we can assume that k is totally real and K is a CM field. In this case $\epsilon^{\emptyset}_{K/k,S,T}$ is the Stickelberger element $\theta_{K/k,S,T}(0)$ of the extension K/k (see §3.1). We take v to be an archimedean place in S. Then $\mathrm{Cl}^T_{\{v\}}(K) = \mathrm{Cl}^T(K)$ and so the first assertion of Theorem 1.10 shows that $\mathrm{LTC}(K/k)$ implies the classical Brumer-Stark Conjecture,

$$\theta_{K/k,S,T}(0) \cdot \mathrm{Cl}^T(K) = 0.$$

(ii) We next consider the case that K is totally real and take V to be $S_\infty(k)$ so that $r = |V| = [k : \mathbb{Q}]$. In this case Corollary 1.10 implies that for any non-archimedean place v in S, any element σ_v of the decomposition subgroup G_v of v in G and any element Φ of $\bigwedge^{[k:\mathbb{Q}]}_{\mathbb{Z}[G]} \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G])$, one has

$$(1 - \sigma_v) \cdot \Phi(\epsilon^{S_\infty}_{K/k,S,T}) \in \mathrm{Ann}_G(\mathrm{Cl}^T(K)). \tag{4}$$

To make this containment even more explicit we further specialize to the case that $k = \mathbb{Q}$ and that K is equal to $\mathbb{Q}(\zeta_m)^+$ for some natural number m. We recall that $\mathrm{LTC}(K/k)$ has been verified in this case. We take S to be the set comprising the (unique) archimedean place ∞ and all prime divisors of m, and V to be $S_\infty = \{\infty\}$ (so $r = 1$). For a set T which contains an odd prime, we set $\delta_T := \prod_{v\in T}(1 - \mathrm{N}v\mathrm{Fr}_v^{-1})$, where $\mathrm{Fr}_v \in G$ denotes the Frobenius automorphism at a place of K above v. In this case, we have

$$\epsilon^{\{\infty\}}_{K/k,S,T} = \epsilon_{m,T} := (1 - \zeta_m)^{\delta_T} \in \mathcal{O}^\times_{K,S,T}$$

(see, for example, [50, p.79] or [42, §4.2]) and so (4) implies that for any $\sigma_v \in G_v$ and any $\Phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G])$ one has

$$(1 - \sigma_v) \cdot \Phi(\epsilon_{m,T}) \in \mathrm{Ann}_G(\mathrm{Cl}^T(K)).$$

Now the group G is generated by the decomposition subgroups G_v of each prime divisor v of m, and so for any $\sigma \in G$ one has an equality $\sigma - 1 = \Sigma_{v|m} x_v$ for suitable elements x_v of the ideal $I(G_v)$ of $\mathbb{Z}[G]$ that is generated by $\{\sigma_v - 1 : \sigma_v \in G_v\}$. Hence, since $\epsilon^{\sigma-1}_{m,T}$ belongs to $\mathcal{O}^\times_K$ one finds that for any $\varphi \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^\times_K, \mathbb{Z}[G])$ one has $\varphi(\epsilon^{\sigma-1}_{m,T}) = \Sigma_{v|m} x_v \widetilde{\varphi}(\epsilon_{m,T})$ where $\widetilde{\varphi}$ is any lift of φ to $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G])$. Therefore, for any φ in $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^\times_K, \mathbb{Z}[G])$ and any σ in G, one has

$$\varphi(\epsilon^{\sigma-1}_{m,T}) \in \mathrm{Ann}_G(\mathrm{Cl}^T(K)). \tag{5}$$

This containment is actually finer than the annihilation result proved by Rubin in [43, Th. 2.2 and the following Remark] since it deals with the group $\mathrm{Cl}^T(K)$ rather than $\mathrm{Cl}(K)$.

Remark 1.12. We next consider the case that K/k is a cyclic CM-extension and V is empty. As remarked above, in this case the Rubin-Stark element $\epsilon^{\emptyset}_{K/k,S,T}$ coincides with the Stickelberger element $\theta_{K/k,S,T}(0)$.
The second assertion of Theorem 1.10 therefore combines with the argument in Remark 1.11(i) to show that $\mathrm{LTC}(K/k)$ implies a containment

$$\theta_{K/k,S,T}(0) \in \mathrm{Fitt}^0_G(\mathrm{Cl}^T(K)).$$

This is a strong refinement of the Brumer-Stark conjecture. To see this note that $\mathrm{Cl}^T(K)$ is equal to the ideal class group $\mathrm{Cl}(K)$ of K when T is empty. The above containment thus combines with [50, Chap. IV, Lem. 1.1] to imply that if G is cyclic, then one has

$$\theta_{K/k,S}(0) \cdot \mathrm{Ann}_G(\mu(K)) \subset \mathrm{Fitt}^0_G(\mathrm{Cl}(K))$$

where $\mu(K)$ denotes the group of roots of unity in K. It is known that this inclusion is not in general valid without the hypothesis that G is cyclic (see [18]). The possibility of such an explicit refinement of Brumer's Conjecture was discussed by the second author in [29] and [31]. In fact, in the terminology of [29], the above argument shows that both properties (SB) and (DSB) follow from $\mathrm{LTC}(K/k)$ whenever G is cyclic. For further results in the case that G is cyclic see Corollary 7.10.

Remark 1.13. Following the discussion of Remark 1.11(ii) we can also now consider Theorem 1.5 further in the case that $k = \mathbb{Q}$, $K = \mathbb{Q}(\zeta_{p^n})^+$ for an odd prime p and natural number n and $S = \{\infty, p\}$.
In this case the G-module $X_{K,S}$ is free of rank one and so the exact sequence (2) combines with the final assertion of Theorem 1.5(ii) (with $r = 1$) to give equalities

$$\begin{aligned}\mathrm{Fitt}^0_G(\mathrm{Cl}^T_S(K)) &= \mathrm{Fitt}^1_G(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K}))\\ &= \{\Phi(\epsilon_{p^n,T}) \mid \Phi \in \mathrm{Hom}_G(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G])\}\\ &= \mathrm{Fitt}^0_G(\mathcal{O}^\times_{K,S,T}/(\mathbb{Z}[G]\cdot \epsilon_{p^n,T}))\end{aligned}$$

where the last equality follows from the fact that G is cyclic.
Since (in this case) $\mathrm{Cl}_S(K) = \mathrm{Cl}(K)$ a standard argument shows that the above displayed equality implies $\mathrm{Fitt}^0_G(\mathrm{Cl}(K)) = \mathrm{Fitt}^0_G(\mathcal{O}^\times_K/\mathcal{C}_K)$ with $\mathcal{C}_K$ denoting the group $\mathbb{Z}[G]\cdot\{1-\zeta_{p^n}, \zeta_{p^n}\} \cap \mathcal{O}^\times_K$ of cyclotomic units of K, and this is the main result of Cornacchia and Greither in [10]. Our results therefore constitute an extension of the main result in [10] for $K = \mathbb{Q}(\zeta_{p^n})^+$.

For any finite group Γ and any Γ-module M we write $M^\vee$ for its Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, endowed with the natural contragredient action of Γ.
In §7.5 we show that the proof of Theorem 1.10 also implies the following result. In this result we fix an odd prime p and set $\mathrm{Cl}^T(K)^\vee_p := \mathrm{Cl}^T(K)^\vee \otimes \mathbb{Z}_p$.

Corollary 1.14. *Let K/k be any finite abelian CM-extension and p any odd prime. If $\mathrm{LTC}(K/k)$ is valid, then one has a containment*

$$\theta_{K/k,S,T}(0)^{\#} \in \mathrm{Fitt}^0_{\mathbb{Z}_p[G]}(\mathrm{Cl}^T(K)^\vee_p).$$

Remark 1.15.
(i) In [19] Greither and Popescu prove the validity of the displayed containment in Corollary 1.14 under the hypotheses that S contains all p-adic places of k (so that the Stickelberger element $\theta_{K/k,S,T}(0)$ is in general 'imprimitive') and that the p-adic μ-invariant of K vanishes. In [9, §3.5] we give a new proof of their result by using the natural Selmer modules for $\mathbb{G}_m$ defined in §2 below in place of the Galois modules 'related to 1-motives' that are explicitly constructed for this purpose in [19]. In addition, by combining Corollary 1.14 with the result of Theorem 1.16 below we can also now prove the containment in Corollary 1.14, both unconditionally and without the assumption that S contains all p-adic places, for important families of examples. For more details see [9, §3.5 and §5].
(ii) For any odd prime p the group $\mathrm{Cl}(K)^\vee_p := \mathrm{Cl}(K)^\vee \otimes \mathbb{Z}_p$ is not a quotient of $\mathrm{Cl}^T(K)^\vee_p$ and so Corollary 1.14 does not imply that $\theta_{K/k,S,T}(0)$ belongs to $\mathrm{Fitt}^0_{\mathbb{Z}_p[G]}(\mathrm{Cl}(K)^\vee_p)$.
(iii) For any odd prime p write $\mathrm{Cl}(K)^{\vee,-}_p$ for the submodule of $\mathrm{Cl}(K) \otimes \mathbb{Z}_p$ upon which complex conjugation acts as multiplication by -1. Then, under a certain technical hypothesis on $\mu(K)$, the main result of Greither in [15] shows that $\mathrm{LTC}(K/k)$ also implies an explicit description of the Fitting ideal $\mathrm{Fitt}^0_{\mathbb{Z}_p[G]}(\mathrm{Cl}(K)^{\vee,-}_p)$ in terms of suitably normalized Stickelberger elements. By replacing the role of 'Tate sequences for small S' in the argument of Greither by the 'T-modified Weil-étale cohomology' complexes that we introduce in §2.2 one can in fact prove the same sort of result without any hypothesis on $\mu(K)$.

1.6. New verifications of the leading term conjecture. In a sequel [9] to this article we investigate the natural Iwasawa-theoretic aspects of our general approach.

In particular, we show in [9, Th. 5.2] that, without any restriction to CM extensions (or to the 'minus parts' of conjectures), under the assumed validity of a natural main conjecture of higher rank Iwasawa theory, the validity of the p-part of $\mathrm{MRS}(L/K/k,S,T)$ for all finite abelian extensions L/k implies the validity of the p-part of $\mathrm{LTC}(K/k)$. Such a result provides an important partial converse to Theorem 1.1 and can also be used to derive new evidence in support of $\mathrm{LTC}(K/k)$.

For example, in [9, Th. 4.9] we show that, in all relevant cases, the validity of $\mathrm{MRS}(K/L/k,S,T)$ is implied by a well-known leading term formula for p-adic L-series that has been conjectured by Gross (the 'Gross-Stark conjecture' [20]). By combining this observation with significant recent work of Darmon, Dasgupta and Pollack and of Ventullo concerning the Gross-Stark conjecture we are then able to give (in [9, Cor. 5.8]) the following new evidence in support of the conjectures $\mathrm{LTC}(K/k)$ and $\mathrm{MRS}(K/L/k,S,T)$.

Theorem 1.16. *Assume that k is a totally real field, that K is an abelian CM extension of k (with maximal totally real subfield K^+) and that p is an odd prime. If the p-adic Iwasawa μ-invariant of K vanishes and at most one p-adic place of k splits in K/K^+, then for any finite subextension K'/K of the* *cyclotomic $\mathbb{Z}_p$-extension of K the minus parts of the p-parts of both* $\mathrm{LTC}(K'/k)$ *and* $\mathrm{MRS}(K'/K/k,S,T)$ *are valid.*

For examples of explicit families of extensions K/k that satisfy all of the hypotheses of Theorem 1.16 with respect to any given odd prime p see [9, Examples 5.9].

1.7. Notation. In this final subsection of the Introduction we collect together some important notation which will be used in the article.

For an abelian group G, a $\mathbb{Z}[G]$-module is simply called a G-module. Tensor products, Hom, exterior powers, and determinant modules over $\mathbb{Z}[G]$ are denoted by $\otimes_G$, Hom_G, $\bigwedge_G$, and $\det_G$, respectively. We use similar notation for Ext-groups, Fitting ideals, etc. The augmentation ideal of $\mathbb{Z}[G]$ is denoted by $I(G)$. For any G-module M and any subgroup $H \subset G$, we denote M^H for the submodule of M comprising elements fixed by H. The norm element of H is denoted by N_H, namely,

$$\mathrm{N}_H = \sum_{\sigma \in H} \sigma \in \mathbb{Z}[G].$$

Let E denote either $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. For an abelian group A, we denote $E \otimes_{\mathbb{Z}} A$ by EA. The maximal $\mathbb{Z}$-torsion subgroup of A is denoted by A_{tors}. We write A/A_{tors} by A_{tf}. The Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z})$ of A is denoted by $A^\vee$ for discrete A.

Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. For a positive integer n, we denote by μ_n the group of n-th roots of unity in $\overline{\mathbb{Q}}^\times$.

Let k be a global field. The set of all infinite places of k is denoted by $S_\infty(k)$ or simply by S_∞ when k is clear from the context. (If k is a function field, then $S_\infty(k)$ is empty.) Consider a finite Galois extension K/k, and denote its Galois group by G. The set of all places of k which ramify in K is denoted by $S_{\mathrm{ram}}(K/k)$ or simply by S_{ram} when K/k is clear from the context. For any non-empty finite set S of places of k, we denote by S_K the set of places of K lying above places in S. The ring of S-integers of K is defined by

$$\mathcal{O}_{K,S} := \{a \in K : \mathrm{ord}_w(a) \geq 0 \text{ for all finite places } w \text{ of } K \text{ not contained in } S_K\},$$

where ord_w denotes the normalized additive valuation at w. The unit group of $\mathcal{O}_{K,S}$ is called the S-unit group of K. Let T be a finite set of finite places of k, which is disjoint from S. The (S,T)-unit group of K is defined by

$$\mathcal{O}_{K,S,T}^\times := \{a \in \mathcal{O}_{K,S}^\times : a \equiv 1 \pmod{w} \text{ for all } w \in T_K\}.$$

The ideal class group of $\mathcal{O}_{K,S}$ is denoted by $\mathrm{Cl}_S(K)$. This is called the S-class group of K. The (S,T)-class group of K, which we denote by $\mathrm{Cl}_S^T(K)$, is defined to be the ray class group of $\mathcal{O}_{K,S}$ modulo $\prod_{w \in T_K} w$ (namely, the quotient of the group of fractional ideals whose supports are coprime to all places above $S \cup T$ by the subgroup of principal ideals with a generator congruent to 1 modulo all places in T_K). When $S \subset S_\infty$, we omit S and write $\mathrm{Cl}^T(K)$ for $\mathrm{Cl}_S^T(K)$. When

$S \subset S_\infty$ and $T = \emptyset$, we write $\mathrm{Cl}(K)$ which is the class group of the integer ring $\mathcal{O}_K$.

We denote by $X_{K,S}$ the augmentation kernel of the divisor group $Y_{K,S} := \bigoplus_{w \in S_K} \mathbb{Z}w$. If S contains $S_\infty(k)$, then the Dirichlet regulator map

$$\lambda_{K,S} : \mathbb{R}\mathcal{O}_{K,S}^\times \longrightarrow \mathbb{R}X_{K,S},$$

defined by $\lambda_{K,S}(a) := -\sum_{w \in S_K} \log |a|_w w$, is an isomorphism.

For a place w of K, the decomposition subgroup of w in G is denoted by G_w. If w is finite, the residue field of w is denoted by $\kappa(w)$. The cardinality of $\kappa(w)$ is denoted by $\mathrm{N}w$. If the place v of k lying under w is unramified in K, then the Frobenius automorphism at w is denoted by $\mathrm{Fr}_w \in G_w$. When G is abelian, then G_w and Fr_w depend only on v, so in this case we often denote them by G_v and Fr_v respectively. The $\mathbb{C}$-linear involution $\mathbb{C}[G] \to \mathbb{C}[G]$ induced by $\sigma \mapsto \sigma^{-1}$ with $\sigma \in G$ is denoted by $x \mapsto x^\#$.

A complex of G-modules is said to be 'perfect' if it is quasi-isomorphic to a bounded complex of finitely generated projective G-modules.

We denote by $D(\mathbb{Z}[G])$ the derived category of G-modules, and by $D^{\mathrm{p}}(\mathbb{Z}[G])$ the full subcategory of $D(\mathbb{Z}[G])$ consisting of perfect complexes.

2. Canonical Selmer groups and complexes for $\mathbb{G}_m$

In this section, we give a definition of 'integral dual Selmer groups for $\mathbb{G}_m$', as analogues of Mazur-Tate's 'integral Selmer groups' defined for abelian varieties in [38]. We shall also review the construction of certain natural arithmetic complexes, which are used for the formulation of the leading term conjecture.

2.1. Integral dual Selmer groups. Let K/k be a finite Galois extension of global fields with Galois group G. Let S be a non-empty finite set of places which contains $S_\infty(k)$. Let T be a finite set of places of k which is disjoint from S.

Definition 2.1. We define the '(S-relative T-trivialized) integral dual Selmer group for $\mathbb{G}_m$' by setting

$$\mathcal{S}_{S,T}(\mathbb{G}_{m/K}) := \mathrm{coker}\Big(\prod_{w \notin S_K \cup T_K} \mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K_T^\times, \mathbb{Z})\Big),$$

where $K_T^\times$ is the subgroup of $K^\times$ defined by

$$K_T^\times := \{a \in K^\times : \mathrm{ord}_w(a-1) > 0 \text{ for all } w \in T_K\},$$

and the homomorphism on the right hand side is defined by

$$(x_w)_w \mapsto (a \mapsto \sum_{w \notin S_K \cup T_K} \mathrm{ord}_w(a) x_w).$$

When T is empty, we omit the subscript T from this notation.

By the following proposition we see that our integral dual Selmer groups are actually like usual dual Selmer groups (see also Remark 2.3 below).

Proposition 2.2. *There is a canonical exact sequence*

$$0 \longrightarrow \mathrm{Cl}_S^T(K)^\vee \longrightarrow \mathcal{S}_{S,T}(\mathbb{G}_{m/K}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}) \longrightarrow 0$$

of the form (1) in §1.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod_{w\notin S_K\cup T_K}\mathbb{Z} & \longrightarrow & \prod_{w\notin S_K\cup T_K}\mathbb{Q} & \longrightarrow & \prod_{w\notin S_K\cup T_K}\mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(K_T^\times, \mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(K_T^\times, \mathbb{Q}) & \longrightarrow & (K_T^\times)^\vee, & &
\end{array}$$

where each row is the natural exact sequence, and each vertical arrow is given by $(x_w)_w \mapsto (a \mapsto \sum_{w\notin S_K\cup T_K} \mathrm{ord}_w(a)x_w)$. Using the exact sequence

$$0 \longrightarrow \mathcal{O}_{K,S,T}^\times \longrightarrow K_T^\times \stackrel{\bigoplus \mathrm{ord}_w}{\longrightarrow} \bigoplus_{w\notin S_K\cup T_K} \mathbb{Z} \longrightarrow \mathrm{Cl}_S^T(K) \longrightarrow 0$$

and applying the snake lemma to the above commutative diagram, we obtain the exact sequence

$$0 \longrightarrow \mathrm{Cl}_S^T(K)^\vee \longrightarrow \mathcal{S}_{S,T}(\mathbb{G}_{m/K}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S,T}^\times, \mathbb{Q}) \longrightarrow (\mathcal{O}_{K,S,T}^\times)^\vee.$$

Since the kernel of the last map is $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S,T}^\times, \mathbb{Z})$, we obtain the desired conclusion. □

Remark 2.3.
(i) The Bloch-Kato Selmer group $H_f^1(K, \mathbb{Q}/\mathbb{Z}(1))$ is defined to be the kernel of the diagonal map

$$K^\times \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow \bigoplus_w K_w^\times/\mathcal{O}_{K_w}^\times \otimes \mathbb{Q}/\mathbb{Z}$$

where w runs over all finite places, and so lies in a canonical exact sequence

$$0 \longrightarrow \mathcal{O}_K^\times \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow H_f^1(K, \mathbb{Q}/\mathbb{Z}(1)) \longrightarrow \mathrm{Cl}(K) \longrightarrow 0.$$

Its Pontryagin dual $H_f^1(K, \mathbb{Q}/\mathbb{Z}(1))^\vee$ is a finitely generated $\hat{\mathbb{Z}}$-module and our integral dual Selmer group $\mathcal{S}_{S_\infty}(\mathbb{G}_{m/K})$ provides a canonical finitely generated $\mathbb{Z}$-structure on $H_f^1(K, \mathbb{Q}/\mathbb{Z}(1))^\vee$.
(ii) In general, the exact sequence (1) also means that $\mathcal{S}_{S,T}(\mathbb{G}_{m/K})$ is a natural analogue (relative to S and T) for $\mathbb{G}_m$ over K of the 'integral Selmer group' that is defined for abelian varieties by Mazur and Tate in [38, p.720].

In the next subsection we shall give a natural cohomological interpretation of the group $\mathcal{S}_{S,T}(\mathbb{G}_{m/K})$ (see Proposition 2.4(iii)) and also show that it has a canonical 'transpose' (see Definition 2.6).

2.2. 'Weil-étale cohomology' complexes. In the following, we construct two canonical complexes of G-modules, and use them to show that there exists a canonical transpose of the module $\mathcal{S}_{S,T}(\mathbb{G}_{m/K})$. The motivation for our choice of notation (and terminology) for these complexes is explained in Remark 2.5 below.

We fix data $K/k, G, S, T$ as in the previous subsection. We also write $\mathbb{F}_{T_K}^\times$ for the direct sum $\bigoplus_{w\in T_K}\kappa(w)^\times$ of the multiplicative groups of the residue fields of all places in T_K.

Proposition 2.4. *There exist canonical complexes of G-modules $R\Gamma_c((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})$ and $R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})$ which satisfy all of the following conditions.*

(i) *There exists a canonical commutative diagram of exact triangles in $D(\mathbb{Z}[G])$*

(6)
$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S}^\times,\mathbb{Q})[-3] & \xrightarrow{\theta} & R\Gamma_c(\mathcal{O}_{K,S},\mathbb{Z}) & \to & R\Gamma_c((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z}) & \to \\
\downarrow & & \| & & \downarrow & \\
(\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S}^\times,\mathbb{Q})\oplus(\mathbb{F}_{T_K}^\times)^\vee)[-3] & \xrightarrow{\theta'} & R\Gamma_c(\mathcal{O}_{K,S},\mathbb{Z}) & \to & R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z}) & \to \\
\downarrow & & & & \downarrow & \\
(\mathbb{F}_{T_K}^\times)^\vee[-3] & & & & (\mathbb{F}_{T_K}^\times)^\vee[-2] & \\
\downarrow & & & & \downarrow{\scriptstyle\theta''} &
\end{array}$$

in which the first column is induced by the obvious exact sequence

$$0\longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S}^\times,\mathbb{Q})\longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S}^\times,\mathbb{Q})\oplus(\mathbb{F}_{T_K}^\times)^\vee\longrightarrow(\mathbb{F}_{T_K}^\times)^\vee\longrightarrow 0$$

and $H^2(\theta'')$ is the Pontryagin dual of the natural injective homomorphism

$$H^3(R\Gamma_c((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z}))^\vee=\mathcal{O}_{K,\mathrm{tors}}^\times\longrightarrow\mathbb{F}_{T_K}^\times.$$

(ii) *If S' is a set of places of k which contains S and is disjoint from T, then there is a canonical exact triangle in $D(\mathbb{Z}[G])$*

$$R\Gamma_{c,T}((\mathcal{O}_{K,S'})_{\mathcal{W}},\mathbb{Z})\longrightarrow R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})\longrightarrow\bigoplus_{w\in S'_K\setminus S_K}R\Gamma((\kappa(w))_{\mathcal{W}},\mathbb{Z}),$$

where $R\Gamma((\kappa(w))_{\mathcal{W}},\mathbb{Z})$ is the complex of G_w-modules which lies in the exact triangle

$$\mathbb{Q}[-2]\longrightarrow R\Gamma(\kappa(w),\mathbb{Z})\longrightarrow R\Gamma((\kappa(w))_{\mathcal{W}},\mathbb{Z})\longrightarrow,$$

where the H^2 of the first arrow is the natural map

$$\mathbb{Q}\longrightarrow\mathbb{Q}/\mathbb{Z}=H^2(\kappa(w),\mathbb{Z}).$$

(iii) *The complex $R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})$ is acyclic outside degrees one, two and three, and there are canonical isomorphisms*

$$H^i(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})) \simeq \begin{cases} Y_{K,S}/\Delta_S(\mathbb{Z}) & \text{if } i=1,\\ \mathcal{S}_{S,T}(\mathbb{G}_{m/K}) & \text{if } i=2,\\ (K^\times_{T,\mathrm{tors}})^\vee & \text{if } i=3,\end{cases}$$

where Δ_S is the natural diagonal map.

(iv) *If S contains $S_{\mathrm{ram}}(K/k)$, then $R\Gamma_c((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})$ and $R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})$ are both perfect complexes of G-modules.*

Proof. In this argument we use the fact that morphisms in $D(\mathbb{Z}[G])$ between bounded above complexes $K_1^\bullet$ and $K_2^\bullet$ can be computed by means of the spectral sequence

$$E_2^{p,q} = \prod_{a\in\mathbb{Z}} \mathrm{Ext}^p_G(H^a(K_1^\bullet), H^{q+a}(K_2^\bullet)) \Rightarrow H^{p+q}(R\,\mathrm{Hom}_G(K_1^\bullet, K_2^\bullet)) \tag{7}$$

constructed by Verdier in [53, III, 4.6.10].

Set $C^\bullet = C_S^\bullet := R\Gamma_c(\mathcal{O}_{K,S},\mathbb{Z})$ and $W := \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S}^\times,\mathbb{Q})$ for simplicity. Then we recall first that $C^\bullet$ is acyclic outside degrees one, two and three, that there are canonical isomorphisms

$$H^i(C^\bullet) \cong \begin{cases} Y_{K,S}/\Delta_S(\mathbb{Z}) & \text{if } i=1,\\ \mathrm{Cl}_S(K)^\vee & \text{if } i=2,\\ (\mathcal{O}_{K,S}^\times)^\vee & \text{if } i=3,\end{cases} \tag{8}$$

where Δ_S is the map that occurs in the statement of claim (iii) and that, when S contains $S_{\mathrm{ram}}(K/k)$, $C^\bullet$ is isomorphic to a bounded complex of cohomologically-trivial G-modules.

It is not difficult to see that the groups $\mathrm{Ext}^i_G(W, H^{3-i}(C^\bullet))$ vanish for all $i>0$, and so the spectral sequence (7) implies that the 'passage to cohomology' homomorphism

$$H^0(R\,\mathrm{Hom}_G(W[-3],C^\bullet)) = \mathrm{Hom}_{D(\mathbb{Z}[G])}(W[-3],C^\bullet) \longrightarrow \mathrm{Hom}_G(W,(\mathcal{O}_{K,S}^\times)^\vee)$$

is bijective. We may therefore define θ to be the unique morphism in $D(\mathbb{Z}[G])$ for which $H^3(\theta)$ is equal to the natural map

$$W = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S}^\times,\mathbb{Q}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S}^\times,\mathbb{Q}/\mathbb{Z}) = (\mathcal{O}_{K,S}^\times)^\vee$$

and then take $C_{\mathcal{W}}^\bullet := R\Gamma_c((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})$ to be any complex which lies in an exact triangle of the form that occurs in the upper row of (6). An analysis of the long exact cohomology sequence of this triangle then shows that $C_{\mathcal{W}}^\bullet$ is acyclic outside degrees one, two and three, that $H^1(C_{\mathcal{W}}^\bullet) = H^1(C^\bullet)$, that $H^2(C_{\mathcal{W}}^\bullet)_{\mathrm{tors}} = H^2(C^\bullet)$, that $H^2(C_{\mathcal{W}}^\bullet)_{\mathrm{tf}} = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S}^\times,\mathbb{Z})$ and that $H^3(C_{\mathcal{W}}^\bullet) = (\mathcal{O}_{K,\mathrm{tors}}^\times)^\vee$. In particular, when S contains $S_{\mathrm{ram}}(K/k)$, since each of these groups is finitely generated and both of the complexes $W[-3]$ and $C^\bullet$ are represented by bounded complexes of cohomologically-trivial G-modules, this implies that $C_{\mathcal{W}}^\bullet$ is perfect.

To define the morphism θ' we first choose a finite set S'' of places of k which is disjoint from $S \cup T$ and such that $\mathrm{Cl}_{S'}(K)$ vanishes for $S' := S \cup S''$. Note that (8) with S replaced by S' implies $C^\bullet_{S'}$ is acyclic outside degrees one and three. We also note that, since each place in T is unramified in K/k, there is also an exact sequence of G-modules

$$0 \longrightarrow \bigoplus_{v\in T}\mathbb{Z}[G] \xrightarrow{(1-\mathrm{N}v\mathrm{Fr}_w)_v} \bigoplus_{v\in T}\mathbb{Z}[G] \longrightarrow (\mathbb{F}_{T_K}^\times)^\vee \longrightarrow 0 \tag{9}$$

where w is any choice of place of K above v. This sequence shows both that $(\mathbb{F}_{T_K}^\times)^\vee[-3]$ is a perfect complex of G-modules and also that the functor $\mathrm{Ext}^i_G((\mathbb{F}_{T_K}^\times)^\vee, -)$ vanishes for all $i > 1$. In particular, the spectral sequence (7) implies that in this case the passage to cohomology homomorphism

$$\mathrm{Hom}_{D(\mathbb{Z}[G])}((\mathbb{F}_{T_K}^\times)^\vee[-3], C^\bullet_{S'}) \longrightarrow \mathrm{Hom}_G((\mathbb{F}_{T_K}^\times)^\vee, (\mathcal{O}_{K,S'}^\times)^\vee)$$

is bijective. We may therefore define θ' to be the morphism which restricts on $W[-3]$ to give θ and on $(\mathbb{F}_{T_K}^\times)^\vee[-3]$ to give the composite morphism

$$(\mathbb{F}_{T_K}^\times)^\vee[-3] \xrightarrow{\theta'_1} R\Gamma_c(\mathcal{O}_{K,S'},\mathbb{Z}) \xrightarrow{\theta'_2} R\Gamma_c(\mathcal{O}_{K,S},\mathbb{Z})$$

where θ'_1 is the unique morphism for which $H^3(\theta'_1)$ is the Pontryagin dual of the natural map $\mathcal{O}_{K,S'}^\times \to \mathbb{F}_{T_K}^\times$ and θ'_2 occurs in the canonical exact triangle

$$R\Gamma_c(\mathcal{O}_{K,S'},\mathbb{Z}) \xrightarrow{\theta'_2} R\Gamma_c(\mathcal{O}_{K,S},\mathbb{Z}) \longrightarrow \bigoplus_{w\in S''_K} R\Gamma(\kappa(w),\mathbb{Z}) \longrightarrow \tag{10}$$

constructed by Milne in [39, Chap. II, Prop. 2.3 (d)].

We now take $C^\bullet_{\mathcal{W},T} := R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})$ to be any complex which lies in an exact triangle of the form that occurs in the second row of (6) and then, just as above, an analysis of this triangle shows that $C^\bullet_{\mathcal{W},T}$ is a perfect complex of G-modules when S contains $S_{\mathrm{ram}}(K/k)$. Note also that since for this choice of θ' the upper left hand square of (6) commutes the diagram can then be completed to give the right hand vertical exact triangle. The claim (ii) follows easily from the above constructions.

It only remains to prove claim (iii). It is easy to see that the groups $H^i(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z}))$ for $i = 1$ and 3 are as described in claim (iii), so we need only prove that there is a natural isomorphism

$$H^2(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})) \simeq \mathcal{S}_{S,T}(\mathbb{G}_{m/K}).$$

To do this we first apply claim (ii) for a set S' that is large enough to ensure that $\mathrm{Cl}^T_{S'}(K)$ vanishes. Since in this case

$$H^2(R\Gamma_{c,T}((\mathcal{O}_{K,S'})_{\mathcal{W}},\mathbb{Z})) = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S',T}^\times,\mathbb{Z}),$$

we obtain in this way a canonical isomorphism

$$H^2(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})) \simeq \mathrm{coker}(\bigoplus_{w\in S'_K\setminus S_K}\mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S',T}^\times,\mathbb{Z})). \tag{11}$$

Consider next the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \prod_{w\notin S'_K\cup T_K}\mathbb{Z} \arrow[r] \arrow[d, equal] & \prod_{w\notin S_K\cup T_K}\mathbb{Z} \arrow[r] \arrow[d] & \bigoplus_{w\in S'_K\setminus S_K}\mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \prod_{w\notin S'_K\cup T_K}\mathbb{Z} \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(K_T^\times,\mathbb{Z}) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S',T}^\times,\mathbb{Z}) \arrow[r] & 0 \end{tikzcd}$$

with exact rows, where the first exact row is the obvious one, the second is the dual of the exact sequence

$$0 \longrightarrow \mathcal{O}_{K,S',T}^\times \longrightarrow K_T^\times \stackrel{\bigoplus \mathrm{ord}_w}{\longrightarrow} \bigoplus_{w\notin S'_K\cup T_K}\mathbb{Z} \longrightarrow 0,$$

and the vertical arrows are given by $(x_w)_w \mapsto (a \mapsto \sum_w \mathrm{ord}_w(a)x_w)$. From this we have the canonical isomorphism

$$\mathcal{S}_{S,T}(\mathbb{G}_{m/K}) \simeq \mathrm{coker}(\bigoplus_{w\in S'_K\setminus S_K}\mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S',T}^\times,\mathbb{Z})). \tag{12}$$

From (11) and (12) our claim follows. □

Given the constructions in Proposition 2.4, in each degree i we set

$$H^i_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z}) := H^i(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})).$$

We also define a complex

$$R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{G}_m) := R\,\mathrm{Hom}_{\mathbb{Z}}(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z}),\mathbb{Z})[-2].$$

We endow this complex with the natural contragredient action of G and then in each degree i set

$$H^i_T((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{G}_m) := H^i(R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{G}_m)).$$

Remark 2.5. Our notation for the above cohomology groups and complexes is motivated by the following facts.
(i) Assume that k is a function field. Write C_k for the corresponding smooth projective curve, $C_{k,W\text{ét}}$ for the Weil-étale site on C_k that is defined by Lichtenbaum in [33, §2] and j for the open immersion $\mathrm{Spec}(\mathcal{O}_{k,S}) \longrightarrow C_k$. Then the group $H^i_c((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})$ defined above is canonically isomorphic to the Weil-étale cohomology group $H^i(C_{k,W\text{ét}},j_!\mathbb{Z})$.
(ii) Assume that k is a number field. In this case there has as yet been no construction of a 'Weil-étale topology' for $Y_S := \mathrm{Spec}(\mathcal{O}_{K,S})$ with all of the properties that are conjectured by Lichtenbaum in [34]. However, if $\overline{Y_S}$ is a compactification of Y_S and ϕ is the natural inclusion $Y_S \subset \overline{Y_S}$, then the approach of [4] can be used to show that, should such a topology exist with all of the expected properties, then the groups $H^i_c((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})$ defined above would be canonically isomorphic to the group $H^i_c(Y_S,\mathbb{Z}) := H^i(\overline{Y_S},\phi_!\mathbb{Z})$ that is discussed in [34].
(iii) The definition of $R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{G}_m)$ as the (shifted) linear dual of the complex $R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})$ is motivated by [4, Rem. 3.8] and hence by the

duality theorem in Weil-étale cohomology for curves over finite fields that is proved by Lichtenbaum in [33].

An analysis of the complex $R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})$ as in the proof of Lemma 2.8 below then leads us to give the following definition. In this definition we use the notion of 'transpose' in the sense of Jannsen's homotopy theory of modules [23].

Definition 2.6. The 'transpose' of $\mathcal{S}_{S,T}(\mathbb{G}_{m/K})$ is the group

$$\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K}) := H^1_T((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{G}_m) = H^{-1}(R\operatorname{Hom}_{\mathbb{Z}}(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z}),\mathbb{Z})).$$

When T is empty, we omit the subscript T from this notation.

Remark 2.7. By using the spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_{\mathbb{Z}}(H^{-q}_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z}),\mathbb{Z}) \Rightarrow H^{p+q+2}_T((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{G}_m),$$

which is obtained from (7), one can check that $R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{G}_m)$ is acyclic outside degrees zero and one, that there is a canonical isomorphism

$$H^0_T((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{G}_m) \simeq \mathcal{O}^\times_{K,S,T},$$

and that there is a canonical exact sequence

$$0 \longrightarrow \mathrm{Cl}^T_S(K) \longrightarrow \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K}) \longrightarrow X_{K,S} \longrightarrow 0$$

of the form (2) in §1.

In the sequel we shall say that a G-module M has a 'locally-quadratic presentation' if it lies in an exact sequence of finitely generated G-modules of the form

$$P \to P' \to M \to 0$$

in which P and P' are projective and the $\mathbb{Q}[G]$-modules $\mathbb{Q}P$ and $\mathbb{Q}P'$ are isomorphic.

Lemma 2.8. *Assume that G is abelian, that S contains $S_\infty(k) \cup S_{\mathrm{ram}}(K/k)$, and that $\mathcal{O}^\times_{K,S,T}$ is $\mathbb{Z}$-torsion-free. Then each of the groups $\mathcal{S}_{S,T}(\mathbb{G}_{m/K})$ and $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K})$ have locally-quadratic presentations, and for each non-negative integer i one has an equality*

$$\operatorname{Fitt}^i_G(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K})) = \operatorname{Fitt}^i_G(\mathcal{S}_{S,T}(\mathbb{G}_{m/K}))^{\#}.$$

Proof. Set

$$C^\bullet := R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})$$

and

$$C^{\bullet,*} := R\operatorname{Hom}_{\mathbb{Z}}(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z}),\mathbb{Z}).$$

From Proposition 2.4 we also know that $C^\bullet$ is a perfect complex of G-modules that is acyclic outside degree one and two and $\mathbb{Z}$-torsion-free in degree one. This implies, by a standard argument, that $C^\bullet$ can be represented by a complex $P \xrightarrow{\delta} P'$ of G-modules, where P and P' are finitely generated and projective and the first term is placed in degree one, and hence that there is a tautological exact sequence of G-modules

$$0 \longrightarrow H^1(C^\bullet) \longrightarrow P \xrightarrow{\delta} P' \longrightarrow H^2(C^\bullet) \longrightarrow 0. \tag{13}$$

The descriptions in Proposition 2.4(iii) imply that the linear dual of the Dirichlet regulator map $\lambda_{K,S}$ induces an isomorphism of $\mathbb{R}[G]$-modules

$$\lambda_{K,S}^* : \mathbb{R}H^1(C^\bullet) \cong \mathbb{R}H^2(C^\bullet). \tag{14}$$

Taken in conjunction with the sequence (13) this isomorphism implies that the $\mathbb{Q}[G]$-modules $\mathbb{Q}P$ and $\mathbb{Q}P'$ are isomorphic and hence that $\mathcal{S}_{S,T}(\mathbb{G}_{m/K}) = H^2(C^\bullet)$ has a locally-quadratic presentation, as claimed.

The complex $C^{\bullet,*}[-2]$ is represented by $\mathrm{Hom}_\mathbb{Z}(P',\mathbb{Z}) \xrightarrow{\delta^*} \mathrm{Hom}_\mathbb{Z}(P,\mathbb{Z})$ where the linear duals are endowed with contragredient action of G, the first term is placed in degree zero and δ^* is the map induced by δ. There is therefore a tautological exact sequence

$$0 \longrightarrow H^0(C^{\bullet,*}[-2]) \longrightarrow \mathrm{Hom}_\mathbb{Z}(P',\mathbb{Z}) \xrightarrow{\delta^*} \mathrm{Hom}_\mathbb{Z}(P,\mathbb{Z}) \longrightarrow H^1(C^{\bullet,*}[-2]) \longrightarrow 0, \tag{15}$$

and, since the above observations imply that $\mathrm{Hom}_\mathbb{Z}(P',\mathbb{Z})$ and $\mathrm{Hom}_\mathbb{Z}(P,\mathbb{Z})$ are projective G-modules that span isomorphic $\mathbb{Q}[G]$-spaces, this sequence implies that the module $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K}) = H^1(C^{\bullet,*}[-2])$ has a locally-quadratic presentation.

It now only remains to prove the final claim and it is enough to prove this after completion at each prime p. We shall denote for any abelian group A the p-completion $A\otimes\mathbb{Z}_p$ of A by A_p. By Swan's Theorem (cf. [11, (32.1)]) one knows that for each prime p the $\mathbb{Z}_p[G]$-modules P_p and P'_p are both free of rank, d say, that is independent of p. In particular, after fixing bases of P_p and P'_p the homomorphism $P_p \xrightarrow{\delta} P'_p$ corresponds to a matrix $A_{\delta,p}$ in $\mathrm{M}_d(\mathbb{Z}_p[G])$ and the sequence (13) implies that the ideal $\mathrm{Fitt}^i_G(H^2(C^\bullet))_p$ is generated over $\mathbb{Z}_p[G]$ by the determinants of all $(d-i)\times(d-i)$ minors of $A_{\delta,p}$. The corresponding dual bases induce identifications of both $\mathrm{Hom}_\mathbb{Z}(P',\mathbb{Z})_p$ and $\mathrm{Hom}_\mathbb{Z}(P,\mathbb{Z})_p$ with $\mathbb{Z}_p[G]^{\oplus d}$, with respect to which the homomorphism $\mathrm{Hom}_\mathbb{Z}(P',\mathbb{Z})_p \xrightarrow{\delta^*} \mathrm{Hom}_\mathbb{Z}(P,\mathbb{Z})_p$ is represented by the matrix $A^{\mathrm{tr},\#}_{\delta,p}$ that is obtained by applying the involution # to each entry of the transpose of $A_{\delta,p}$. The exact sequence (15) therefore implies that $\mathrm{Fitt}^i_G(H^1(C^{\bullet,*}[-2]))_p$ is generated over $\mathbb{Z}_p[G]$ by the determinants of all $(d-i)\times(d-i)$ minors of $A^{\mathrm{tr},\#}_{\delta,p}$. Hence one has an equality

$$\mathrm{Fitt}^i_G(H^2(C^\bullet))_p = \mathrm{Fitt}^i_G(H^1(C^{\bullet,*}[-2]))^\#_p,$$

as required. □

2.3. Tate sequences. In this subsection we review the construction of Tate's exact sequence, which is used in the formulation of the leading term conjecture in the next section. Let $K/k, G, S$ be as in the previous subsection. We assume that $S_{\mathrm{ram}}(K/k) \subset S$. We assume only in this subsection that S is large enough so that $\mathrm{Cl}_S(K)$ vanishes.

In this setting, Tate constructed a 'fundamental class' $\tau_{K/k,S} \in \mathrm{Ext}^2_G(X_{K,S}, \mathcal{O}^\times_{K,S})$ using the class field theory [49]. This class $\tau_{K/k,S}$ has the following property: if we regard $\tau_{K/k,S}$ as an element of $H^2(G, \mathrm{Hom}_{\mathbb{Z}}(X_{K,S}, \mathcal{O}^\times_{K,S}))$ via the canonical isomorphism

$$\begin{aligned}\mathrm{Ext}^2_G(X_{K,S}, \mathcal{O}^\times_{K,S}) \simeq \mathrm{Ext}^2_G(\mathbb{Z}, \mathrm{Hom}_{\mathbb{Z}}(X_{K,S}, \mathcal{O}^\times_{K,S}))&\\ = H^2(G, \mathrm{Hom}_{\mathbb{Z}}(X_{K,S}, \mathcal{O}^\times_{K,S})),&\end{aligned}$$

then, for every integer i, the map between Tate cohomology groups

$$\widehat{H}^i(G, X_{K,S}) \xrightarrow{\sim} \widehat{H}^{i+2}(G, \mathcal{O}^\times_{K,S})$$

that is defined by taking cup product with $\tau_{K/k,S}$ is bijective.
The Yoneda extension class of $\tau_{K/k,S}$ is therefore represented by an exact sequence of the following sort:

$$0 \longrightarrow \mathcal{O}^\times_{K,S} \longrightarrow A \longrightarrow B \longrightarrow X_{K,S} \longrightarrow 0, \tag{16}$$

where A and B are finitely generated cohomologically-trivial G-modules (see [50, Chap. II, Th. 5.1]). We call this sequence a 'Tate sequence' for K/k.

Proposition 2.9. *The complex $R\Gamma((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m)$ defines an element of*

$$\mathrm{Ext}^2_G(\mathcal{S}^{\mathrm{tr}}_S(\mathbb{G}_{m/K}), \mathcal{O}^\times_{K,S}).$$

This element is equal to Tate's fundamental class $\tau_{K/k,S}$.

Proof. The first assertion follows directly from the discussion of Remark 2.7. The assumed vanishing of $\mathrm{Cl}_S(K)$ combines with the exact sequence (2) to imply that $\mathcal{S}^{\mathrm{tr}}_S(\mathbb{G}_{m/K}) = X_{K,S}$. Given this, the second claim is proved by the first author in [4, Prop. 3.5(f)] □

3. Zeta elements and the leading term conjecture

In this section, we suppose that K/k is a finite abelian extension of global fields with Galois group G.
We fix a finite non-empty set of places S of k which contains both $S_\infty(k)$ and $S_{\mathrm{ram}}(K/k)$ and an auxiliary finite set of places T of k that is disjoint from S.

3.1. L-functions. We recall the definition of (abelian) L-functions of global fields. For any linear character $\chi \in \widehat{G} := \mathrm{Hom}(G, \mathbb{C}^\times)$, we define the S-truncated T-modified L-function for K/k and χ by setting

$$L_{k,S,T}(\chi, s) := \prod_{v \in T}(1 - \chi(\mathrm{Fr}_v)\mathrm{N}v^{1-s}) \prod_{v \notin S}(1 - \chi(\mathrm{Fr}_v)\mathrm{N}v^{-s})^{-1}.$$

This is a complex function defined on $\mathrm{Re}(s) > 1$ and is well-known to have a meromorphic continuation on $\mathbb{C}$ and to be holomorphic at $s = 0$. We denote by $r_{\chi,S}$ the order of vanishing of $L_{k,S,T}(\chi, s)$ at $s = 0$ (this clearly does not depend on T). We denote the leading coefficient of the Taylor expansion of $L_{k,S,T}(\chi, s)$ at $s = 0$ by

$$L^*_{k,S,T}(\chi, 0) := \lim_{s \to 0} s^{-r_{\chi,S}} L_{k,S,T}(\chi, s).$$

We then define the S-truncated T-modified equivariant L-function for K/k by setting

$$\theta_{K/k,S,T}(s) := \sum_{\chi\in\widehat{G}} L_{k,S,T}(\chi^{-1},s)e_\chi,$$

where $e_\chi := \frac{1}{|G|}\sum_{\sigma\in G}\chi(\sigma)\sigma^{-1}$, and we define its leading term to be

$$\theta^*_{K/k,S,T}(0) := \sum_{\chi\in\widehat{G}} L^*_{k,S,T}(\chi^{-1},0)e_\chi.$$

It is then easy to see that $\theta^*_{K/k,S,T}(0)$ belongs to $\mathbb{R}[G]^\times$.

When $T=\emptyset$, we simply denote $L_{k,S,\emptyset}(\chi,s)$, $\theta_{K/k,S,\emptyset}(s)$, etc., by $L_{k,S}(\chi,s)$, $\theta_{K/k,S}(s)$, etc., respectively, and refer to them as the S-truncated L-function for K/k, S-truncated equivariant L-function for K/k, etc., respectively.

3.2. The leading term lattice. In this section we recall the explicit formulation of a conjectural description of the lattice $\theta^*_{K/k,S,T}(0)\cdot\mathbb{Z}[G]$ which involves Tate sequences. In particular, up until Remark 3.3, we always assume (without further explicit comment) that S is large enough to ensure the group $\mathrm{Cl}_S(K)$ vanishes.

At the outset we also note that, as observed by Knudsen and Mumford in [27], to avoid certain technical difficulties regarding signs, determinant modules must be regarded as graded invertible modules. Nevertheless, for simplicity of notation, in the following we have preferred to omit explicit reference to the grading of any graded invertible modules. Thus, for a finitely generated projective G-module P, we have abbreviated the graded invertible G-module $(\det_G(P),\mathrm{rk}_G(P))$ to $\det_G(P)$, where $\mathrm{rk}_G(P)$ is the rank of P. Since the notation $\det_G(P)$ explicitly indicates P, which in turn determines $\mathrm{rk}_G(P)$, we feel that this abbreviation should not cause difficulties.

We shall also use the following general notation. Suppose that we have a perfect complex $C^\bullet$ of G-modules, which is concentrated in degree i and $i+1$ with some integer i, and an isomorphism $\lambda:\mathbb{R}H^i(C^\bullet)\xrightarrow{\sim}\mathbb{R}H^{i+1}(C^\bullet)$. Then we define an isomorphism

$$\vartheta_\lambda : \mathbb{R}\mathrm{det}_G(C^\bullet)\xrightarrow{\sim}\mathbb{R}[G]$$

as follows:

$$\begin{aligned}
\mathbb{R}\mathrm{det}_G(C^\bullet) &\xrightarrow{\sim} \bigotimes_{j\in\mathbb{Z}} \mathrm{det}^{(-1)^j}_{\mathbb{R}[G]}(\mathbb{R}C^j)\\
&\xrightarrow{\sim} \bigotimes_{j\in\mathbb{Z}} \mathrm{det}^{(-1)^j}_{\mathbb{R}[G]}(\mathbb{R}H^j(C^\bullet))\\
&= \mathrm{det}^{(-1)^i}_{\mathbb{R}[G]}(\mathbb{R}H^i(C^\bullet))\otimes_{\mathbb{R}[G]}\mathrm{det}^{(-1)^{i+1}}_{\mathbb{R}[G]}(\mathbb{R}H^{i+1}(C^\bullet))\\
&\xrightarrow{\sim} \mathrm{det}^{(-1)^i}_{\mathbb{R}[G]}(\mathbb{R}H^{i+1}(C^\bullet))\otimes_{\mathbb{R}[G]}\mathrm{det}^{(-1)^{i+1}}_{\mathbb{R}[G]}(\mathbb{R}H^{i+1}(C^\bullet))\\
&\xrightarrow{\sim} \mathbb{R}[G],
\end{aligned}$$

where the fourth isomorphism is induced by $\lambda^{(-1)^i}$.
Let A and B be the G-modules which appear in the Tate sequence (16). Since we have the regulator isomorphism

$$\lambda_{K,S} : \mathbb{R}\mathcal{O}_{K,S}^{\times} \xrightarrow{\sim} \mathbb{R}X_{K,S},$$

the above construction for $C^{\bullet} = (A \to B)$, where A is placed in degree 0, gives the isomorphism

$$\vartheta_{\lambda_{K,S}} : \mathbb{R}\mathrm{det}_G(A) \otimes_{\mathbb{R}[G]} \mathbb{R}\mathrm{det}_G^{-1}(B) \xrightarrow{\sim} \mathbb{R}[G].$$

We study the following conjecture.

Conjecture 3.1. *In $\mathbb{R}[G]$ one has*

$$\vartheta_{\lambda_{K,S}}(\mathrm{det}_G(A) \otimes_G \mathrm{det}_G^{-1}(B)) = \theta_{K/k,S}^{*}(0) \cdot \mathbb{Z}[G].$$

Remark 3.2. This conjecture coincides with the conjecture C(K/k) stated in [3, §6.3]. The observations made in [3, Rem. 6.2] therefore imply that Conjecture 3.1 is equivalent in the number field case to the 'equivariant Tamagawa number conjecture' [7, Conj. 4 (iv)] for the pair $(h^0(\mathrm{Spec}\, K), \mathbb{Z}[G])$, that the validity of Conjecture 3.1 is independent of S and of the choice of Tate sequence and that its validity for the extension K/k implies its validity for all extensions F/E with $k \subseteq E \subseteq F \subseteq K$.

Remark 3.3. Conjecture 3.1 is known to be valid in each of the following cases:

(i) K is an abelian extension of $\mathbb{Q}$ (by Greither and the first author [8] and Flach [14]),
(ii) k is a global function field (by the first author [5]),
(iii) $[K : k] \leq 2$ (by Kim [26, §2.4, Rem. i)]).

In the following result we do not assume that the group $\mathrm{Cl}_S(K)$ vanishes and we interpret the validity of Conjecture 3.1 in terms of the 'Weil-étale cohomology' complexes $R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{Z})$ and $R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m)$ defined in §2.2.
We note at the outset that $R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{Z})$ (resp. $R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m)$) is represented by a complex which is concentrated in degrees one and two (resp. zero and one), and so we can define the isomorphism

$$\vartheta_{\lambda_{K,S}^{*}} : \mathbb{R}\mathrm{det}_G(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{Z})) \xrightarrow{\sim} \mathbb{R}[G]$$

$$(\text{resp. } \vartheta_{\lambda_{K,S}} : \mathbb{R}\mathrm{det}_G(R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m)) \xrightarrow{\sim} \mathbb{R}[G]).$$

Proposition 3.4. *Let S be any finite non-empty set of places of k containing both $S_\infty(k)$ and $S_{\mathrm{ram}}(K/k)$ and let T be any finite set of places of k that is disjoint from S. Then the following conditions on K/k are equivalent.*

(i) *Conjecture 3.1 is valid.*
(ii) *In $\mathbb{R}[G]$ one has an equality*

$$\vartheta_{\lambda_{K,S}^{*}}(\mathrm{det}_G(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{Z}))) = \theta_{K/k,S,T}^{*}(0)^{-1\#} \cdot \mathbb{Z}[G].$$

(iii) *In $\mathbb{R}[G]$ one has an equality*

$$\vartheta_{\lambda_{K,S}}(\mathrm{det}_G(R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m))) = \theta_{K/k,S,T}^{*}(0) \cdot \mathbb{Z}[G].$$

Proof. For any finitely generated projective G-module P of (constant) rank d there is a natural identification

$$\bigwedge_G^d \mathrm{Hom}_{\mathbb{Z}}(P,\mathbb{Z}) \cong \bigwedge_G^d \mathrm{Hom}_G(P,\mathbb{Z}[G])^{\#} \cong \mathrm{Hom}_G(\bigwedge_G^d P,\mathbb{Z}[G])^{\#},$$

where G acts on $\mathrm{Hom}_{\mathbb{Z}}(P,\mathbb{Z})$ contragrediently and on $\mathrm{Hom}_G(P,\mathbb{Z}[G])$ via right multiplication. The equivalence of the equalities in claims (ii) and (iii) is therefore a consequence of the fact that for any element Δ of the mutliplicative group of invertible $\mathbb{Z}[G]$-lattices in $\mathbb{R}[G]$ the evaluation pairing identifies $\mathrm{Hom}_G(\Delta,\mathbb{Z}[G])^{\#}$ with the image under the involution $\#$ of the inverse lattice Δ^{-1}.

To relate the equalities in claims (ii) and (iii) to Conjecture 3.1 we note first that the third column of (6) implies that

$$\begin{aligned}&\vartheta_{\lambda^*_{K,S}}(\mathrm{det}_G(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})))\\ &= \mathrm{det}_G((\mathbb{F}_{T_K}^{\times})^{\vee}[-2])\cdot \vartheta_{\lambda^*_{K,S}}(\mathrm{det}_G(R\Gamma_c((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z}))),\end{aligned}$$

whilst the resolution (9) implies that

$$\begin{aligned}\mathrm{det}_G((\mathbb{F}_{T_K}^{\times})^{\vee}[-2]) &= (\prod_{v\in T}(1-\mathrm{N}v\mathrm{Fr}_w))^{-1}\cdot\mathbb{Z}[G]\\ &= (\theta^*_{K/k,S,T}(0)/\theta^*_{K/k,S}(0))^{-1\#}\cdot\mathbb{Z}[G].\end{aligned}$$

The equality in claim (ii) is therefore equivalent to an equality

$$\vartheta_{\lambda^*_{K,S}}(\mathrm{det}_G(R\Gamma_c((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z}))) = \theta^*_{K/k,S}(0)^{-1\#}\cdot\mathbb{Z}[G]. \tag{17}$$

We now choose an auxiliary set of places S'' as in the proof of Proposition 2.4 and set $S' := S\cup S''$. By Chebotarev density theorem we can even assume that all places in S'' split completely in K/k and, for simplicity, this is what we shall do. Then, in this case, the exact triangle (10) combines with the upper triangle in (6) to give an exact triangle in $D(\mathbb{Z}[G])$ of the form

(18)

$$Y_{K,S''}[-1]\oplus Y_{K,S''}[-2] \xrightarrow{\alpha} R\Gamma_c((\mathcal{O}_{K,S'})_{\mathcal{W}},\mathbb{Z}) \xrightarrow{\beta} R\Gamma_c((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z}) \longrightarrow .$$

After identifying the cohomology groups of the second and the third occurring complexes by using Proposition 2.4(iii) the long exact cohomology sequence of this triangle induces (after scaler extension) the sequence

$$\begin{aligned}0\longrightarrow \mathbb{Q}Y_{K,S''}\longrightarrow \mathbb{Q}Y_{K,S'}/\Delta_{S'}(\mathbb{Q})\longrightarrow \mathbb{Q}Y_{K,S}/\Delta_S(\mathbb{Q})&\\ \xrightarrow{0}\mathbb{Q}Y_{K,S''}\xrightarrow{\mathrm{ord}^*_{S''}}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S'}^{\times},\mathbb{Q})\xrightarrow{\pi_{S''}}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S}^{\times},\mathbb{Q})\longrightarrow 0.&\end{aligned}$$

Here $\mathrm{ord}^*_{S''}$ is induced by the linear dual of the map $\mathcal{O}_{K,S'}^{\times}\to Y_{K,S''}$ induced by taking valuations at each place in S''_K and $\pi_{S''}$ by the linear dual of the inclusion $\mathcal{O}_{K,S}^\times \subseteq \mathcal{O}_{K,S'}^\times$ and all other maps are obvious. This sequence implies that there is an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R}Y_{K,S''} \arrow[r, "H^1(\alpha)"] \arrow[d, "\eta_{S''}"'] & \mathbb{R}H_c^1((\mathcal{O}_{K,S'})_{\mathcal{W}},\mathbb{Z}) \arrow[r, "H^1(\beta)"] \arrow[d, "\lambda^*_{K,S'}"'] & \mathbb{R}H_c^1((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z}) \arrow[r] \arrow[d, "\lambda^*_{K,S}"'] & 0 \\
0 \arrow[r] & \mathbb{R}Y_{K,S''} \arrow[r, "H^2(\alpha)"] & \mathbb{R}H_c^2((\mathcal{O}_{K,S'})_{\mathcal{W}},\mathbb{Z}) \arrow[r, "H^2(\beta)"] & \mathbb{R}H_c^2((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

where $\eta_{S''}$ sends each sum $\sum_{v\in S''}\sum_{w|v} x_w w$ to $\sum_{v\in S''}\sum_{w|v}\log(\mathrm{N}v)x_w w$. This diagram combines with the triangle (18) to imply that

$$\begin{aligned}
&\vartheta_{\lambda^*_{K,S'}}(\det_G(R\Gamma_c((\mathcal{O}_{K,S'})_{\mathcal{W}},\mathbb{Z})))\\
&=\det_{\mathbb{R}[G]}(\eta_{S''})^{-1}\vartheta_{\lambda^*_{K,S}}(\det_G(R\Gamma_c((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})))\\
&=\Big(\prod_{v\in S''}\log(\mathrm{N}v)\Big)^{-1}\vartheta_{\lambda^*_{K,S}}(\det_G(R\Gamma_c((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z}))).
\end{aligned}$$

Since $\theta^*_{K/k,S'}(0) = \big(\prod_{v\in S''}\log(\mathrm{N}v)\big)\theta^*_{K/k,S}(0)$ this equality shows that (after changing S if necessary) we may assume that $\mathrm{Cl}_S(K)$ vanishes when verifying (17). Given this, the proposition follows from Proposition 2.9. □

3.3. Zeta elements. We now use the above results to reinterpret Conjecture 3.1 in terms of the existence of a canonical 'zeta element'. This interpretation will then play a key role in the proofs of Theorem 5.12, 5.16 and 7.5 given below.

The following definition of zeta element is in the same spirit as that used by Kato in [24] and [25].

Definition 3.5. The 'zeta element' $z_{K/k,S,T}$ of $\mathbb{G}_m$ relative to the data K/k, S and T is the unique element of

$$\mathbb{R}\mathrm{det}_G(R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{G}_m)) \cong \det_{\mathbb{R}[G]}(\mathbb{R}\mathcal{O}_{K,S}^\times)\otimes_{\mathbb{R}[G]}\det^{-1}_{\mathbb{R}[G]}(\mathbb{R}X_{K,S})$$

which satisfies $\vartheta_{\lambda_{K,S}}(z_{K/k,S,T}) = \theta^*_{K/k,S,T}(0)$.

The following 'leading term conjecture' is then our main object of study.

Conjecture 3.6 (LTC(K/k)). *In* $\mathbb{R}\mathrm{det}_G(R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{G}_m))$ *one has an equality*

$$\mathbb{Z}[G]\cdot z_{K/k,S,T} = \det_G(R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{G}_m)).$$

Given the definition of $z_{K/k,S,T}$, Proposition 3.4 implies immediately that this conjecture is equivalent to Conjecture 3.1 and hence is independent of the choices of S and T.

4. Preliminaries concerning exterior powers

In this section, we recall certain useful constructions concerning exterior powers and also prove algebraic results that are to be used in later sections.

4.1. Exterior powers. Let G be a finite abelian group. For a G-module M and $f \in \mathrm{Hom}_G(M, \mathbb{Z}[G])$, there is a G-homomorphism

$$\bigwedge_G^r M \longrightarrow \bigwedge_G^{r-1} M$$

for all $r \in \mathbb{Z}_{\geq 1}$, defined by

$$m_1 \wedge \cdots \wedge m_r \mapsto \sum_{i=1}^{r} (-1)^{i-1} f(m_i) m_1 \wedge \cdots \wedge m_{i-1} \wedge m_{i+1} \wedge \cdots \wedge m_r.$$

This morphism is also denoted by f.
This construction gives a homomorphism

$$\bigwedge_G^s \mathrm{Hom}_G(M, \mathbb{Z}[G]) \longrightarrow \mathrm{Hom}_G(\bigwedge_G^r M, \bigwedge_G^{r-s} M) \tag{19}$$

for all $r, s \in \mathbb{Z}_{\geq 0}$ such that $r \geq s$, defined by

$$f_1 \wedge \cdots \wedge f_s \mapsto (m \mapsto f_s \circ \cdots \circ f_1(m)).$$

By using this homomorphism we often regard an element of $\bigwedge_G^s \mathrm{Hom}_G(M, \mathbb{Z}[G])$ as an element of $\mathrm{Hom}_G(\bigwedge_G^r M, \bigwedge_G^{r-s} M)$.
For a G-algebra Q and a homomorphism f in $\mathrm{Hom}_G(M, Q)$, there is a G-homomorphism

$$\bigwedge_G^r M \longrightarrow (\bigwedge_G^{r-1} M) \otimes_G Q$$

defined by

$$m_1 \wedge \cdots \wedge m_r \mapsto \sum_{i=1}^{r} (-1)^{i-1} m_1 \wedge \cdots \wedge m_{i-1} \wedge m_{i+1} \wedge \cdots \wedge m_r \otimes f(m_i).$$

By the same method as the construction of (19), we have a homomorphism

$$\bigwedge_G^s \mathrm{Hom}_G(M, Q) \longrightarrow \mathrm{Hom}_G(\bigwedge_G^r M, (\bigwedge_G^{r-s} M) \otimes_G Q). \tag{20}$$

In the sequel we will find an explicit description of this homomorphism to be useful. This description is well-known and given by the following proposition, the proof of which we omit.

Proposition 4.1. *Let $m_1, \ldots, m_r \in M$ and $f_1, \ldots, f_s \in \mathrm{Hom}_G(M, Q)$. Then we have*

$$\begin{aligned} &(f_1 \wedge \cdots \wedge f_s)(m_1 \wedge \cdots \wedge m_r) \\ &= \sum_{\sigma \in \mathfrak{S}_{r,s}} \mathrm{sgn}(\sigma) m_{\sigma(s+1)} \wedge \cdots \wedge m_{\sigma(r)} \otimes \det(f_i(m_{\sigma(j)}))_{1 \leq i,j \leq s}, \end{aligned}$$

where

$$\mathfrak{S}_{r,s} := \{\sigma \in \mathfrak{S}_r : \sigma(1) < \cdots < \sigma(s) \text{ and } \sigma(s+1) < \cdots < \sigma(r)\}.$$

In particular, if $r = s$, then we have

$$(f_1 \wedge \cdots \wedge f_r)(m_1 \wedge \cdots \wedge m_r) = \det(f_i(m_j))_{1\le i,j\le r}.$$

We will also find the technical observations that are contained in the next two results to be very useful.

Lemma 4.2. *Let E be a field and A an n-dimensional E-vector space. If we have an E-linear map*

$$\Psi : A \longrightarrow E^{\oplus m},$$

where $\Psi = \bigoplus_{i=1}^m \psi_i$ with $\psi_1, \ldots, \psi_m \in \mathrm{Hom}_E(A, E)$ ($m \le n$), then we have

$$\mathrm{im}(\bigwedge_{1\le i\le m} \psi_i : \bigwedge_E^n A \longrightarrow \bigwedge_E^{n-m} A) = \begin{cases} \bigwedge_E^{n-m} \ker(\Psi), & \text{if } \Psi \text{ is surjective,} \\ 0, & \text{if } \Psi \text{ is not surjective.} \end{cases}$$

Proof. Suppose first that Ψ is surjective. Then there exists a subspace $B \subset A$ such that $A = \ker(\Psi) \oplus B$ and Ψ maps B isomorphically onto $E^{\oplus m}$. We see that $\bigwedge_{1\le i\le m} \psi_i$ induces an isomorphism

$$\bigwedge_E^m B \xrightarrow{\sim} E.$$

Hence we have an isomorphism

$$\bigwedge_{1\le i\le m} \psi_i : \bigwedge_E^n A = \bigwedge_E^{n-m} \ker(\Psi) \otimes_E \bigwedge_E^m B \xrightarrow{\sim} \bigwedge_E^{n-m} \ker(\Psi).$$

In particular, we have

$$\mathrm{im}(\bigwedge_{1\le i\le m} \psi_i : \bigwedge_E^n A \longrightarrow \bigwedge_E^{n-m} A) = \bigwedge_E^{n-m} \ker(\Psi).$$

Next, suppose that Ψ is not surjective. Then $\psi_1, \ldots, \psi_m \in \mathrm{Hom}_E(A, E)$ are linearly dependent. In fact, since each ψ_i is contained in $\mathrm{Hom}_E(A/\ker(\Psi), E)$, we have

$$\dim_E(\langle \psi_1, \ldots, \psi_m \rangle) \le \dim_E(A/\ker(\Psi)) = \dim_E(\mathrm{im}(\Psi)),$$

so $\dim_E(\langle \psi_1, \ldots, \psi_m \rangle) < m$ if $\dim_E(\mathrm{im}(\Psi)) < m$. This shows that the element $\bigwedge_{1\le i\le m} \psi_i$ vanishes, as required. □

Using the same notation as in Lemma 4.2, we now consider an endomorphism $\psi \in \mathrm{End}_E(A)$. We write r_ψ for the dimension over E of $\ker(\psi)$ and consider the composite isomorphism

$$\begin{aligned} F_\psi : \bigwedge_E^n A \otimes_E \bigwedge_E^n \mathrm{Hom}_E(A, E) &\simeq \mathrm{det}_E(A) \otimes_E \mathrm{det}_E^{-1}(A) \\ &\xrightarrow{\sim} \mathrm{det}_E(\ker(\psi)) \otimes_E \mathrm{det}_E^{-1}(\mathrm{coker}(\psi)) \\ &\simeq \bigwedge_E^{r_\psi} \ker(\psi) \otimes_E \bigwedge_E^{r_\psi} \mathrm{Hom}_E(\mathrm{coker}(\psi), E), \end{aligned}$$

where the second isomorphism is induced by the tautological exact sequence

$$0 \longrightarrow \ker(\psi) \longrightarrow A \xrightarrow{\psi} A \longrightarrow \operatorname{coker}(\psi) \longrightarrow 0.$$

Then the proof of Lemma 4.2 leads directly to the following useful description of this isomorphism F_ψ.

Lemma 4.3. *With E, A and ψ as above, we fix an E-basis $\{b_1, \ldots, b_n\}$ of A so that $\operatorname{im}(\psi) = \langle b_{r_\psi+1}, \ldots, b_n\rangle$ and write $\{b_1^*, \ldots, b_n^*\}$ for the corresponding dual basis of $\operatorname{Hom}_E(A, E)$. For each index i we also set $\psi_i := b_i^* \circ \psi$.*
Then for every a in $\bigwedge_E^n A$ the element $(\bigwedge_{r_\psi < i \le n} \psi_i)(a)$ belongs to $\bigwedge_E^{r_\psi} \ker(\psi)$ and one has

$$F_\psi(a \otimes (b_1^* \wedge \cdots \wedge b_n^*)) = (-1)^{r_\psi(n-r_\psi)}(\bigwedge_{r_\psi < i \le n} \psi_i)(a) \otimes (b_1^* \wedge \cdots \wedge b_{r_\psi}^*).$$

Here, on the right hand side of the equation, we use the equality $\operatorname{im}(\psi) = \langle b_{r_\psi+1}, \ldots, b_n\rangle$ to regard b_i^* for each i with $1 \le i \le r_\psi$ as an element of $\operatorname{Hom}_E(\operatorname{coker}(\psi), E)$.

4.2. Rubin lattices. The following definition is due to Rubin [45, §1.2]. We adopt the notation in [46] for the lattice. Note in particular that the notation '$\bigcap$' does not refer to an intersection.

Definition 4.4. For a finitely generated G-module M and a non-negative integer r we define the 'r-th Rubin lattice' by setting

$$\bigcap_G^r M = \{m \in \mathbb{Q}\bigwedge_G^r M : \Phi(m) \in \mathbb{Z}[G] \text{ for all } \Phi \in \bigwedge_G^r \operatorname{Hom}_G(M, \mathbb{Z}[G])\}.$$

In particular, one has $\bigcap_G^0 M = \mathbb{Z}[G]$.

Remark 4.5. We define the homomorphism $\iota : \bigwedge_G^r \operatorname{Hom}_G(M, \mathbb{Z}[G]) \to \operatorname{Hom}_G(\bigwedge_G^r M, \mathbb{Z}[G])$ by sending each element $\varphi_1 \wedge \cdots \wedge \varphi_r$ to $\varphi_r \circ \cdots \circ \varphi_1$ (see (19)). Then it is not difficult to see that the map

$$\bigcap_G^r M \xrightarrow{\sim} \operatorname{Hom}_G(\operatorname{im}(\iota), \mathbb{Z}[G]);\ m \mapsto (\Phi \mapsto \Phi(m))$$

is an isomorphism (see [45, §1.2]).

By this remark, one obtains the following result.

Proposition 4.6. *Let P be a finitely generated projective G-module. Then we have*

$$\bigcap_G^r P = \bigwedge_G^r P$$

for all non-negative integers r.

Lemma 4.7. *Let M be a G-module. Suppose that there is a finitely generated projective G-module P and an injection $j : M \hookrightarrow P$ whose cokernel is $\mathbb{Z}$-torsion-free.*

(i) *The map*

$$\mathrm{Hom}_G(P, \mathbb{Z}[G]) \longrightarrow \mathrm{Hom}_G(M, \mathbb{Z}[G])$$

induced by j is surjective.

(ii) *If we regard M as a submodule of P via j, then we have*

$$\bigcap_G^r M = (\mathbb{Q} \bigwedge_G^r M) \cap \bigwedge_G^r P.$$

Proof. The assertion (i) follows from [45, Prop. 1.1 (ii)]. Note that

$$\bigwedge_G^r \mathrm{Hom}_G(P, \mathbb{Z}[G]) \longrightarrow \bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G])$$

is also surjective. This induces a surjection

$$\mathrm{im}(\iota_P) \longrightarrow \mathrm{im}(\iota_M),$$

where ι_P and ι_M denote the maps defined in Remark 4.5 for P and M, respectively. Hence, taking the dual, we have an injection

$$\bigcap_G^r M \simeq \mathrm{Hom}_G(\mathrm{im}(\iota_M), \mathbb{Z}[G]) \longrightarrow \mathrm{Hom}_G(\mathrm{im}(\iota_P), \mathbb{Z}[G]) \simeq \bigcap_G^r P.$$

Since P is projective, we have $\bigcap_G^r P = \bigwedge_G^r P$ by Proposition 4.6. Hence we have

$$\bigcap_G^r M \subset \bigwedge_G^r P.$$

Next, we show the reverse inclusion '$\supset$'. To do this we fix a in $(\mathbb{Q} \bigwedge_G^r M) \cap \bigwedge_G^r P$ and Φ in $\bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G])$. By (i), we can take a lift $\widetilde{\Phi} \in \bigwedge_G^r \mathrm{Hom}_G(P, \mathbb{Z}[G])$ of Φ. Since $a \in \bigwedge_G^r P$, we have

$$\Phi(a) = \widetilde{\Phi}(a) \in \mathbb{Z}[G].$$

This shows that a belongs to $\bigcap_G^r M$, as required. □

Remark 4.8. The proof of Lemma 4.7 shows that the cokernel of the injection

$$\bigcap_G^r M \longrightarrow \bigwedge_G^r P$$

is $\mathbb{Z}$-torsion-free. This implies that for any abelian group A, the map

$$(\bigcap_G^r M) \otimes_{\mathbb{Z}} A \longrightarrow (\bigwedge_G^r P) \otimes_{\mathbb{Z}} A$$

is injective.

4.3. Homomorphisms between Rubin lattices. In the sequel we fix a subgroup H of G and an ideal J of $\mathbb{Z}[H]$. Recall that we denote the augmentation ideal of $\mathbb{Z}[H]$ by $I(H)$. Put $J_H := J/I(H)J$. We also put $\mathcal{J} := J\mathbb{Z}[G]$, and $\mathcal{J}_H := \mathcal{J}/I(H)\mathcal{J}$.

Proposition 4.9. *We have a natural isomorphism of G/H-modules*

$$\mathcal{J}_H \simeq \mathbb{Z}[G/H] \otimes_{\mathbb{Z}} J_H.$$

Proof. Define a homomorphism

$$\mathbb{Z}[G/H] \otimes_{\mathbb{Z}} J_H \longrightarrow \mathcal{J}_H$$

by $\tau \otimes \overline{a} \mapsto \overline{\widetilde{\tau} a}$, where $\tau \in G/H$, $a \in J$, and $\widetilde{\tau} \in G$ is a lift of τ. One can easily check that this homomorphism is well-defined, and bijective. □

Definition 4.10. Let M be a G-module. For $\varphi \in \mathrm{Hom}_G(M, \mathbb{Z}[G])$, we define $\varphi^H \in \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H])$ by

$$M^H \xrightarrow{\varphi} \mathbb{Z}[G]^H \simeq \mathbb{Z}[G/H],$$

where the last isomorphism is given by $\mathrm{N}_H = \sum_{\sigma \in H} \sigma \mapsto 1$. Let r be a non-negative integer. For $\Phi \in \bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G])$, we define $\Phi^H \in \bigwedge_{G/H}^r \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H])$ to be the image of Φ under the map

$$\varphi_1 \wedge \cdots \wedge \varphi_r \mapsto \varphi_1^H \wedge \cdots \wedge \varphi_r^H.$$

For convention, if $r = 0$, then we define $\Phi^H \in \mathbb{Z}[G/H]$ to be the image of $\Phi \in \mathbb{Z}[G]$ under the natural map : $\mathbb{Z}[G] \longrightarrow \mathbb{Z}[G/H]$.

Proposition 4.11. *Let M be a G-module and $r \in \mathbb{Z}_{\geq 0}$. For any $m \in \mathbb{Q}\bigwedge_G^r M$ and $\Phi \in \bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G])$, we have*

$$\Phi(m) = \Phi^H(\mathrm{N}_H^r m) \text{ in } \mathbb{Q}[G/H],$$

where N_H^r denote the map $\mathbb{Q}\bigwedge_G^r M \to \mathbb{Q}\bigwedge_{G/H}^r M^H$ induced by $\mathrm{N}_H : M \to M^H$.

Proof. This follows directly from the definition of Φ^H. □

We consider the canonical map

$$\nu : \bigcap_{G/H}^{r} M^H \longrightarrow \bigcap_{G}^{r} M$$

which is defined as follows. Let

$$\iota : \bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G]) \longrightarrow \mathrm{Hom}_G(\bigwedge_G^r M, \mathbb{Z}[G])$$

and

$$\iota_H : \bigwedge_{G/H}^r \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]) \longrightarrow \mathrm{Hom}_{G/H}(\bigwedge_{G/H}^r M^H, \mathbb{Z}[G/H])$$

be the homomorphisms defined in Remark 4.5. The map

$$\mathrm{im}(\iota) \longrightarrow \mathrm{im}(\iota_H);\ \iota(\Phi) \mapsto \iota_H(\Phi^H)$$

induces a map

$$\alpha : \mathrm{Hom}_G(\mathrm{im}(\iota_H), \mathbb{Z}[G]) \longrightarrow \mathrm{Hom}_G(\mathrm{im}(\iota), \mathbb{Z}[G]) \simeq \bigcap_G^r M.$$

Note that we have a canonical isomorphism

$$\beta : \mathrm{Hom}_G(\mathrm{im}(\iota_H), \mathbb{Z}[G]) \xrightarrow{\sim} \mathrm{Hom}_{G/H}(\mathrm{im}(\iota_H), \mathbb{Z}[G/H]) \simeq \bigcap_{G/H}^r M^H;\ \varphi \mapsto \varphi^H.$$

We define a map ν by

$$\nu := \alpha \circ \beta^{-1} : \bigcap_{G/H}^r M^H \longrightarrow \bigcap_G^r M.$$

Proposition 4.12. *Let M be a finitely generated G-module which is $\mathbb{Z}$-torsion-free. For any $r \in \mathbb{Z}_{\geq 0}$, the map $\nu : \bigcap_{G/H}^r M^H \to \bigcap_G^r M$ is injective. Furthermore, the maps*

$$(\bigcap_{G/H}^r M^H) \otimes_{\mathbb{Z}} J_H \longrightarrow (\bigcap_G^r M) \otimes_{\mathbb{Z}} J_H \longrightarrow (\bigcap_G^r M) \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)J$$

are both injective, where the first map is induced by ν, and the second by inclusion $J_H \hookrightarrow \mathbb{Z}[H]/I(H)J$.

Proof. The proof is the same as [46, Lem. 2.11], so we omit it. □

Remark 4.13. The inclusion $M^H \subset M$ induces a map

$$\xi : \bigcap_{G/H}^r M^H \longrightarrow \bigcap_G^r M.$$

We note that this map *does not coincide* with the above map ν if $r > 1$. Indeed, one can check that $\mathrm{im}(\xi) \subset |H|^{\max\{0,r-1\}} \bigcap_G^r M$ (see [37, Lem. 4.8]), and

$$\nu = |H|^{-\max\{0,r-1\}}\xi.$$

Remark 4.14. Let P be a finitely generated projective G-module. Then, any element of P^H is written as $\mathrm{N}_H\, a$ with some $a \in P$, since P is cohomologically trivial. One can check that, if $r > 0$ (resp. $r = 0$), then the map $\nu : \bigwedge_{G/H}^r P^H \to \bigwedge_G^r P$ constructed above coincides with the map

$$\mathrm{N}_H\, a_1 \wedge \cdots \wedge \mathrm{N}_H\, a_r \mapsto \mathrm{N}_H\, a_1 \wedge \cdots \wedge a_r$$

$$(\text{resp. } \mathbb{Z}[G/H] \simeq \mathbb{Z}[G]^H \hookrightarrow \mathbb{Z}[G]).$$

In particular, we know that $\mathrm{im}(\nu) = \mathrm{N}_H \bigwedge_G^r P$.

Proposition 4.15. *Let M be a finitely generated G-module which is $\mathbb{Z}$-torsion-free, and $r \in \mathbb{Z}_{\geq 0}$. Then the map*

$$(\bigcap_{G/H}^{r} M^H) \otimes_{\mathbb{Z}} J_H \longrightarrow \mathrm{Hom}_G(\bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G]), \mathcal{J}_H);\ \alpha \mapsto (\Phi \mapsto \Phi^H(\alpha))$$

is injective. (We regard $\Phi^H(\alpha) \in \mathbb{Z}[G/H] \otimes_{\mathbb{Z}} J_H$ as an element of $\mathcal{J}_H$ via the isomorphism $\mathcal{J}_H \simeq \mathbb{Z}[G/H] \otimes_{\mathbb{Z}} J_H$ in Proposition 4.9.)

Proof. The proof is the same as [46, Th. 2.12]. □

4.4. Congruences between exterior powers. The following definition is originally due to Darmon [12], and used in [46, Def. 2.13] and [37, Def. 5.1].

Definition 4.16. Let M be a G-module. For $m \in M$, define

$$\mathcal{N}_H(m) = \sum_{\sigma \in H} \sigma m \otimes \sigma^{-1} \in M \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)J.$$

The following proposition is an improvement of the result of the third author in [46, Prop. 2.15].

Proposition 4.17. *Let P be a finitely generated projective G-module, $r \in \mathbb{Z}_{\geq 0}$, and*

$$\nu : (\bigwedge_{G/H}^{r} P^H) \otimes_{\mathbb{Z}} J_H \longrightarrow (\bigwedge_G^r P) \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)J$$

be the injection in Proposition 4.12. For an element $a \in \bigwedge_G^r P$, the following are equivalent.

(i) $a \in \mathcal{J} \cdot \bigwedge_G^r P$,
(ii) $\mathcal{N}_H(a) \in \mathrm{im}(\nu)$,
(iii) $\Phi(a) \in \mathcal{J}$ *for every* $\Phi \in \bigwedge_G^r \mathrm{Hom}_G(P, \mathbb{Z}[G])$.

Furthermore, if the above equivalent conditions are satisfied, then for every $\Phi \in \bigwedge_G^r \mathrm{Hom}_G(P, \mathbb{Z}[G])$ we have

$$\Phi(a) = \Phi^H(\nu^{-1}(\mathcal{N}_H(a))) \text{ in } \mathcal{J}_H,$$

where we regard $\Phi^H(\nu^{-1}(\mathcal{N}_H(a))) \in \mathbb{Z}[G/H] \otimes_{\mathbb{Z}} J_H$ as an element of $\mathcal{J}_H$ via the isomorphism $\mathcal{J}_H \simeq \mathbb{Z}[G/H] \otimes_{\mathbb{Z}} J_H$ in Proposition 4.9.

Proof. By Swan's Theorem (see [11, (32.1)]), for every prime p, P_p is a free $\mathbb{Z}_p[G]$-module of rank, d say, independent of p. Considering locally, we may assume that P is a free G-module of rank d. We may assume $r \leq d$. Clearly, (i) implies (iii). We shall show that (iii) implies (ii). Suppose $\Phi(a) \in \mathcal{J}$ for all $\Phi \in \bigwedge_G^r \mathrm{Hom}_G(P, \mathbb{Z}[G])$. Fix a basis $\{b_1, \ldots, b_d\}$ of P. Write

$$a = \sum_{\mu \in \mathfrak{S}_{d,r}} x_\mu b_{\mu(1)} \wedge \cdots \wedge b_{\mu(r)},$$

with some $x_\mu \in \mathbb{Z}[G]$. For each μ, by Proposition 4.1, we have

$$x_\mu = (b_{\mu(1)}^* \wedge \cdots \wedge b_{\mu(r)}^*)(a) \in \mathcal{J},$$

where $b_i^* \in \mathrm{Hom}_G(P, \mathbb{Z}[G])$ is the dual basis of b_i. For each $\tau \in G/H$, fix a lift $\widetilde{\tau} \in G$. Note that we have a direct sum decomposition

$$\mathcal{J} = \bigoplus_{\tau \in G/H} J\widetilde{\tau}.$$

Therefore, we can write each x_μ as follows:

$$x_\mu = \sum_{\tau \in G/H} y_{\tau\mu}\widetilde{\tau},$$

where $y_{\tau\mu} \in J$. Hence we have

$$\begin{aligned}
\mathcal{N}_H(a) &= \sum_{\sigma \in H}\sum_{\mu \in \mathfrak{S}_{d,r}}\sum_{\tau \in G/H} \sigma y_{\tau\mu}\widetilde{\tau} b_{\mu(1)} \wedge \cdots \wedge b_{\mu(r)} \otimes \sigma^{-1} \\
&= \sum_{\sigma \in H}\sum_{\mu \in \mathfrak{S}_{d,r}}\sum_{\tau \in G/H} \sigma\widetilde{\tau} b_{\mu(1)} \wedge \cdots \wedge b_{\mu(r)} \otimes \sigma^{-1} y_{\tau\mu} \\
&= \sum_{\mu \in \mathfrak{S}_{d,r}}\sum_{\tau \in G/H} \mathrm{N}_H\, \widetilde{\tau} b_{\mu(1)} \wedge \cdots \wedge b_{\mu(r)} \otimes y_{\tau\mu} \\
&\in \mathrm{N}_H \bigwedge_G^r P \otimes_{\mathbb{Z}} J_H = \mathrm{im}(\nu)
\end{aligned}$$

(see Remark 4.14). This shows (ii). We also see by Remark 4.14 that

$$\nu^{-1}(\mathcal{N}_H(a)) = \sum_{\mu \in \mathfrak{S}_{d,r}}\sum_{\tau \in G/H} \tau\, \mathrm{N}_H\, b_{\mu(1)} \wedge \cdots \wedge \mathrm{N}_H\, b_{\mu(r)} \otimes y_{\tau\mu} \in (\bigwedge_{G/H}^r P^H) \otimes_{\mathbb{Z}} J_H.$$

Hence, by Proposition 4.11, we have

$$\Phi(a) = \Phi^H(\nu^{-1}(\mathcal{N}_H(a))) \quad in \quad \mathcal{J}_H$$

for all $\Phi \in \bigwedge_G^r \mathrm{Hom}_G(P, \mathbb{Z}[G])$.

Finally, we show that (ii) implies (i). Suppose $\mathcal{N}_H(a) \in \mathrm{im}(\nu) = (\mathrm{N}_H \bigwedge_G^r P) \otimes_{\mathbb{Z}} J_H$. As before, we write

$$a = \sum_{\mu \in \mathfrak{S}_{d,r}}\sum_{\tau \in G/H} y_{\tau\mu}\widetilde{\tau} b_{\mu(1)} \wedge \cdots \wedge b_{\mu(r)}$$

with $y_{\tau\mu} \in \mathbb{Z}[H]$. We have

$$\mathcal{N}_H(a) = \sum_{\sigma \in H}\sum_{\mu \in \mathfrak{S}_{d,r}}\sum_{\tau \in G/H} \sigma\widetilde{\tau} b_{\mu(1)} \wedge \cdots \wedge b_{\mu(r)} \otimes \sigma^{-1} y_{\tau\mu} \in (\mathrm{N}_H \bigwedge_G^r P) \otimes_{\mathbb{Z}} J_H.$$

Since $(\bigwedge_G^r P) \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)J \simeq \bigoplus_{\sigma,\mu,\tau} \mathbb{Z}[H]/I(H)J$ as abelian groups, we must have $y_{\tau\mu} \in J$. This shows that $a \in \mathcal{J} \cdot \bigwedge_G^r P$. □

5. Congruences for Rubin-Stark elements

For a finite abelian extension K/k, and an intermediate field L, a conjecture which describes a congruence relation between two Rubin-Stark elements for K/k and L/k was formulated by the third author in [46, Conj. 3]. Mazur and Rubin also formulated in [37, Conj. 5.2] essentially the same conjecture. In this section, we formulate a refined version (see Conjecture 5.4) of these conjectures. We also recall a conjecture formulated by the first author, which was studied in [22], [3], [16], [17], [48], and [46] (see Conjecture 5.9). In [46, Th. 3.15], the third author proved a link between Conjecture 5.4 and Conjecture 5.9. We now improve the argument given there to show that Conjecture 5.4 and Conjecture 5.9 are in fact equivalent (see Theorem 5.10). Finally we prove that the natural equivariant leading term conjecture (Conjecture 3.6) implies both Conjecture 5.4 and Conjecture 5.9 (see Theorem 5.16).

5.1. The Rubin-Stark conjecture. In this subsection, we recall the formulation of the Rubin-Stark conjecture [45, Conj. B′].

Let $K/k, G, S, T$ be as in §3, namely, K/k is a finite abelian extension of global fields, G is its Galois group, S is a non-empty finite set of places of k such that $S_\infty(k) \cup S_{\rm ram}(K/k) \subset S$, and T is a finite set of places of k which is disjoint from S. In this section, we assume that $\mathcal{O}_{K,S,T}^\times$ is $\mathbb{Z}$-torsion-free.

Following Rubin [45, Hyp. 2.1] we assume that S satisfies the following hypothesis with respect to some chosen integer r with $0 \le r < |S|$: there exists a subset $V \subset S$ of order r such that each place in V splits completely in K/k.

Recall that for any $\chi \in \widehat{G}$ we denote by $r_{\chi,S}$ the order of vanishing of $L_{k,S,T}(\chi,s)$ at $s=0$. We know by [50, Chap. I, Prop. 3.4] that

$$(21)\quad r_{\chi,S} = \dim_{\mathbb{C}}(e_\chi \mathbb{C}X_{K,S}) = \begin{cases} |\{v \in S : \chi(G_v)=1\}| & \text{if } \chi \neq 1, \\ |S|-1 & \text{if } \chi = 1. \end{cases}$$

Therefore, the existence of V ensures that $r \le r_{\chi,S}$ for every χ and hence the function $s^{-r}L_{k,S,T}(\chi,s)$ is holomorphic at $s=0$. We define the 'r-th order Stickelberger element' by

$$\theta_{K/k,S,T}^{(r)} := \lim_{s\to 0} \sum_{\chi \in \widehat{G}} s^{-r} L_{k,S,T}(\chi^{-1},s) e_\chi \in \mathbb{R}[G].$$

Note that the 0-th order Stickelberger element $\theta_{K/k,S,T}^{(0)} (= \theta_{K/k,S,T}(0))$ is the usual Stickelberger element.

Recall that we have the regulator isomorphism

$$\lambda_{K,S} : \mathbb{R}\mathcal{O}_{K,S,T}^\times \xrightarrow{\sim} \mathbb{R}X_{K,S}$$

defined by

$$\lambda_{K,S}(a) = -\sum_{w \in S_K} \log|a|_w w.$$

This map $\lambda_{K,S}$ induces the isomorphism

$$\bigwedge_{\mathbb{R}[G]}^{r} \mathbb{R}\mathcal{O}_{K,S,T}^{\times} \xrightarrow{\sim} \bigwedge_{\mathbb{R}[G]}^{r} \mathbb{R}X_{K,S},$$

which we also denote by $\lambda_{K,S}$. For each place $v \in S$, fix a place w of K lying above v. Take any $v_0 \in S \setminus V$, and define the '(r-th order) Rubin-Stark element'

$$\epsilon_{K/k,S,T}^{V} \in \bigwedge_{\mathbb{R}[G]}^{r} \mathbb{R}\mathcal{O}_{K,S,T}^{\times} = \mathbb{R} \bigwedge_{G}^{r} \mathcal{O}_{K,S,T}^{\times}.$$

by

$$\lambda_{K,S}(\epsilon_{K/k,S,T}^{V}) = \theta_{K/k,S,T}^{(r)} \bigwedge_{v \in V} (w - w_0), \tag{22}$$

where $\bigwedge_{v \in V}(w - w_0)$ is arranged by some chosen order of the elements in V. One can show that the Rubin-Stark element $\epsilon_{K/k,S,T}^{V}$ does not depend on the choice of $v_0 \in S \setminus V$.

We consider the Rubin lattice

$$\bigcap_{G}^{r} \mathcal{O}_{K,S,T}^{\times} \subset \mathbb{Q} \bigwedge_{G}^{r} \mathcal{O}_{K,S,T}^{\times}$$

(see Definition 4.4). The Rubin-Stark conjecture claims

Conjecture 5.1 (The Rubin-Stark conjecture for $(K/k, S, T, V)$). *One has*

$$\epsilon_{K/k,S,T}^{V} \in \bigcap_{G}^{r} \mathcal{O}_{K,S,T}^{\times}.$$

Remark 5.2. One can check that the above Rubin-Stark conjecture is equivalent to [45, Conj. B′] for the data $(K/k, S, T, V)$, and that our Rubin-Stark element $\epsilon_{K/k,S,T}^{V}$ coincides with the unique element predicted by [45, Conj. B′]. This shows, in particular, that the validity of the conjecture does not depend on the choice of the places lying above $v \in S$ or on the ordering of the elements in V.

Remark 5.3. The Rubin-Stark conjecture for $(K/k, S, T, V)$ is known to be true in the following cases:

(i) $r = 0$. In this case $\epsilon_{K/k,S,T}^{\emptyset} = \theta_{K/k,S,T}^{(0)} = \theta_{K/k,S,T}(0) \in \mathbb{R}[G]$ so the Rubin-Stark conjecture claims only that $\theta_{K/k,S,T}(0) \in \mathbb{Z}[G]$ which is a celebrated result of Deligne-Ribet, Cassou-Noguès, and Barsky.
(ii) $[K : k] \leq 2$. This is due to Rubin [45, Cor. 3.2 and Th. 3.5].
(iii) K is an abelian extension over $\mathbb{Q}$. This is due to the first author [3, Th. A].
(iv) k is a global function field. This is due to the first author [3, Th. A].

5.2. Conventions for Rubin-Stark elements. The notation $\epsilon^V_{K/k,S,T}$ has some ambiguities, since $\epsilon^V_{K/k,S,T}$ depends on the choice of the places lying above $v \in S$, and on the choice of the order of the elements in V. To avoid this ambiguity, we use the following convention: when we consider the Rubin-Stark element $\epsilon^V_{K/k,S,T}$, we always fix a place w of K lying above each $v \in S$, and label the elements of S as

$$S = \{v_0, v_1, \ldots, v_n\}$$

so that $V = \{v_1, \ldots, v_r\}$, and thus we fix the order of the elements in V. So, under this convention, the Rubin-Stark element $\epsilon^V_{K/k,S,T}$ is the element characterized by

$$\lambda_{K,S}(\epsilon^V_{K/k,S,T}) = \theta^{(r)}_{K/k,S,T} \bigwedge_{1\le i\le r}(w_i - w_0).$$

5.3. Conjectures on Rubin-Stark elements. In this subsection, we give a refinement of the conjecture formulated by the third author [46, Conj. 3], and Mazur and Rubin [37, Conj. 5.2]. Let $K/k, G, S, T$ be as before, and we assume that, for a non-negative integer r, there exists a subset $V \subset S$ of order r such that each place in V splits completely in K. We fix a subgroup H of G for which, for some integer r' with $r' \ge r$, there exists a subset $V' \subset S$ of order r', which contains V, and satisfies that each place in V' splits completely in the field $L := K^H$.

Following the convention in §5.2, we fix, for each place $v \in S$, a place w of K lying above v, and label the elements of S as $S = \{v_0, \ldots, v_n\}$ so that $V = \{v_1, \ldots, v_r\}$ and $V' = \{v_1, \ldots, v_{r'}\}$. We consider the Rubin-Stark elements $\epsilon^V_{K/k,S,T}$ and $\epsilon^{V'}_{L/k,S,T}$ characterized by

$$\lambda_{K,S}(\epsilon^V_{K/k,S,T}) = \theta^{(r)}_{K/k,S,T} \bigwedge_{1\le i\le r}(w_i - w_0)$$

and

$$\lambda_{L,S}(\epsilon^{V'}_{L/k,S,T}) = \theta^{(r')}_{L/k,S,T} \bigwedge_{1\le i\le r'}(w_i - w_0)$$

respectively, where we denote the place of L lying under w also by w.

For each integer i with $1 \le i \le n$ we write G_i for the decomposition group of v_i in G. For any subgroup $U \subset G$, recall that the augmentation ideal of $\mathbb{Z}[U]$ is denoted by $I(U)$. Put $\mathcal{I}_i := I(G_i)\mathbb{Z}[G]$ and $I_H := I(H)\mathbb{Z}[G]$. We define

$$\mathrm{Rec}_i : \mathcal{O}^\times_{L,S,T} \longrightarrow (\mathcal{I}_i)_H = \mathcal{I}_i / I_H \mathcal{I}_i$$

by

$$\mathrm{Rec}_i(a) = \sum_{\tau \in G/H} \tau^{-1}(\mathrm{rec}_{w_i}(\tau a) - 1).$$

Here, rec_{w_i} is the reciprocity map $L^\times_{w_i} \to G_i$ at w_i. Note that $\tau^{-1}(\mathrm{rec}_{w_i}(\tau a) - 1)$ is well-defined for $\tau \in G/H$ in $(\mathcal{I}_i)_H$.

We put $W := V' \setminus V = \{v_{r+1}, \ldots, v_{r'}\}$. We define an ideal J_W of $\mathbb{Z}[H]$ by

$$J_W := \begin{cases} (\prod_{r<i\leq r'} I(G_i))\mathbb{Z}[H], & \text{if } W \neq \emptyset, \\ \mathbb{Z}[H], & \text{if } W = \emptyset, \end{cases}$$

and put $(J_W)_H := J_W/I(H)J_W$. We also define an ideal $\mathcal{J}_W$ of $\mathbb{Z}[G]$ by

$$\mathcal{J}_W := \begin{cases} \prod_{r<i\leq r'} \mathcal{I}_i, & \text{if } W \neq \emptyset, \\ \mathbb{Z}[G], & \text{if } W = \emptyset, \end{cases}$$

and put $(\mathcal{J}_W)_H := \mathcal{J}_W/I_H\mathcal{J}_W$. Note that $\mathcal{J}_W = J_W\mathbb{Z}[G]$. By Proposition 4.9, we have a natural isomorphism of G/H-modules $\mathbb{Z}[G/H] \otimes_{\mathbb{Z}} (J_W)_H \simeq (\mathcal{J}_W)_H$. We consider the graded G/H-algebra

$$\mathcal{Q}_W := \bigoplus_{a_1,\ldots,a_{r'-r}\in\mathbb{Z}_{\geq 0}} (\mathcal{I}_{r+1}^{a_1} \cdots \mathcal{I}_{r'}^{a_{r'-r}})_H,$$

where

$$(\mathcal{I}_{r+1}^{a_1} \cdots \mathcal{I}_{r'}^{a_{r'-r}})_H := \mathcal{I}_{r+1}^{a_1} \cdots \mathcal{I}_{r'}^{a_{r'-r}}/I_H\mathcal{I}_{r+1}^{a_1} \cdots \mathcal{I}_{r'}^{a_{r'-r}},$$

and we define the 0-th power of any ideal of $\mathbb{Z}[G]$ to be $\mathbb{Z}[G]$.

For any integer i with $r < i \leq r'$ we regard Rec_i as an element of $\mathrm{Hom}_{G/H}(\mathcal{O}_{L,S,T}^\times, \mathcal{Q}_W)$ via the natural embedding $(\mathcal{I}_i)_H \hookrightarrow \mathcal{Q}_W$. Then by the same method as in [46, Prop. 2.7] (or [37, Cor. 2.1]), one shows that $\bigwedge_{r<i\leq r'} \mathrm{Rec}_i \in \bigwedge_{G/H}^{r'-r} \mathrm{Hom}_{G/H}(\mathcal{O}_{L,S,T}^\times, \mathcal{Q}_W)$ induces the map

$$\bigcap_{G/H}^{r'} \mathcal{O}_{L,S,T}^\times \longrightarrow (\bigcap_{G/H}^{r} \mathcal{O}_{L,S,T}^\times) \otimes_{G/H} (\mathcal{J}_W)_H \simeq (\bigcap_{G/H}^{r} \mathcal{O}_{L,S,T}^\times) \otimes_{\mathbb{Z}} (J_W)_H, \tag{23}$$

which we denote by Rec_W.

Following Definition 4.16, we define

$$\mathcal{N}_H : \bigcap_G^r \mathcal{O}_{K,S,T}^\times \longrightarrow (\bigcap_G^r \mathcal{O}_{K,S,T}^\times) \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)J_W$$

by $\mathcal{N}_H(a) = \sum_{\sigma\in H} \sigma a \otimes \sigma^{-1}$.

Note that since $(\mathcal{O}_{K,S,T}^\times)^H = \mathcal{O}_{L,S,T}^\times$, there is a natural injective homomorphism

$$\nu : (\bigcap_{G/H}^r \mathcal{O}_{L,S,T}^\times) \otimes_{\mathbb{Z}} (J_W)_H \longrightarrow (\bigcap_G^r \mathcal{O}_{K,S,T}^\times) \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)J_W$$

by Proposition 4.12.

To state the following conjecture we assume the validity of the Rubin-Stark conjecture (Conjecture 5.1) for both $(K/k, S, T, V)$ and $(L/k, S, T, V')$.

Conjecture 5.4 ($\mathrm{MRS}(K/L/k, S, T, V, V')$). *The element $\mathcal{N}_H(\epsilon_{K/k,S,T}^V)$ belongs to* $\mathrm{im}(\nu)$, *and satisfies*

$$\mathcal{N}_H(\epsilon_{K/k,S,T}^V) = (-1)^{r(r'-r)} \cdot \nu(\mathrm{Rec}_W(\epsilon_{L/k,S,T}^{V'})).$$

Remark 5.5. In this article we write 'MRS($K/L/k, S, T$) is valid' to mean that the statement of Conjecture 5.4 is valid for all possible choices of V and V'.

Remark 5.6. In §6 we show that Conjecture 5.4 constitutes a natural refinement and generalization of both a conjecture of Darmon from [12] and of several conjectures of Gross from [21]. In addition, in a subsequent article [9] we will show that the validity of Conjecture 5.4 also implies the 'Gross-Stark conjecture' formulated by Gross in [20] and a refinement of the main result of Solomon in [47] concerning the 'wild Euler system' that he constructs in loc. cit.

Remark 5.7. One has $I(G_i)\mathbb{Z}[H] \subset I(H)$, so $J_W \subseteq I(H)^e$ where $e := r' - r \geq 0$. Thus there is a natural homomorphism

$$(\bigcap_G^r \mathcal{O}_{K,S,T}^\times) \otimes_{\mathbb{Z}} (J_W)_H \longrightarrow (\bigcap_G^r \mathcal{O}_{K,S,T}^\times) \otimes_{\mathbb{Z}} I(H)^e/I(H)^{e+1}.$$

Conjecture 5.4 is therefore a strengthening of the central conjecture of the third author in [46, Conj. 3] and of the conjecture formulated by Mazur and Rubin in [37, Conj. 5.2], both of which claim only that the given equality is valid after projection to the group $(\bigcap_G^r \mathcal{O}_{K,S,T}^\times) \otimes_{\mathbb{Z}} I(H)^e/I(H)^{e+1}$. This refinement is in the same spirit as Tate's strengthening in [51] of the 'refined class number formula' formulated by Gross in [21].

Remark 5.8. Note that, when $r = 0$, following [46, Def. 2.13] $\mathcal{N}_H$ would be defined to be the natural map $\mathbb{Z}[G] \to \mathbb{Z}[G]/I_H\mathcal{J}_W$, but this does not make any change because of the observation of Mazur and Rubin in [37, Lem. 5.6 (iv)]. Note also that, by Remark 4.13, the map $\mathbf{j}_{L/K}$ in [37, Lem. 4.9] (where our K/L is denoted by L/K) is essentially the same as our homomorphism ν. Finally we note that Mazur and Rubin do not use the fact that $\mathbf{j}_{L/K}$ is injective, so the formulation of [46, Conj. 3] is slightly stronger than the conjecture [37, Conj. 5.2].

We next state a refinement of a conjecture that was formulated by the first author in [3] (the original version of which has been studied in many subsequent articles of different authors including [22], [16], [17], [48], and [46]).

Conjecture 5.9. (B($K/L/k, S, T, V, V'$)). *For every*

$$\Phi \in \bigwedge_G^r \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G]),$$

we have

$$\Phi(\epsilon_{K/k,S,T}^V) \in \mathcal{J}_W$$

and an equality

$$\Phi(\epsilon_{K/k,S,T}^V) = (-1)^{r(r'-r)}\Phi^H(\mathrm{Rec}_W(\epsilon_{L/k,S,T}^{V'})) \text{ in } (\mathcal{J}_W)_H.$$

In this article we improve an argument of the third author in [46] to prove the following result.

Theorem 5.10. *The conjectures*

$$\mathrm{MRS}(K/L/k,S,T,V,V') \text{ and } \mathrm{B}(K/L/k,S,T,V,V')$$

are equivalent.

The proof of this result will be given in §5.5.

5.4. An explicit resolution. As a preliminary step, we choose a useful representative of the complex

$$D^\bullet_{K,S,T} := R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{G}_m) \in D^{\mathrm{p}}(\mathbb{Z}[G]).$$

To do this we follow the method used in [3, §7].

Let d be a sufficiently large integer, and F be a free G-module of rank d with basis $b = \{b_i\}_{1\le i\le d}$. We define a surjection

$$\pi : F \longrightarrow \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K})(= H^1(D^\bullet_{K,S,T}))$$

as follows. Recall that $S = \{v_0,\dots,v_n\}$. Let $F_{\le n}$ be a free $\mathbb{Z}[G]$-module generated by $\{b_i\}_{1\le i\le n}$. First, choose a homomorphism

$$\pi_1 : F_{\le n} \longrightarrow \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K})$$

such that the composition map

$$F_{\le n} \xrightarrow{\pi_1} \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K}) \longrightarrow X_{K,S}$$

sends b_i to $w_i - w_0$. (Such a homomorphism exists since $F_{\le n}$ is free.) Next, let A denote the kernel of the composition map

$$\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K}) \longrightarrow X_{K,S} \longrightarrow Y_{K,S\setminus\{v_0\}},$$

where the last map sends the places above v_0 to 0. Since d is sufficiently large, we can choose a surjection

$$\pi_2 : F_{>n} \longrightarrow A,$$

where $F_{>n}$ is the free $\mathbb{Z}[G]$-module generated by $\{b_i\}_{n<i\le d}$. Define

$$\pi := \pi_1 \oplus \pi_2 : F = F_{\le n} \oplus F_{>n} \longrightarrow \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K}).$$

One can easily show that π is surjective.

$D^\bullet_{K,S,T}$ defines a Yoneda extension class in $\mathrm{Ext}^2_G(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K}),\mathcal{O}^\times_{K,S,T})$. Since $D^\bullet_{K,S,T}$ is perfect, this class is represented by an exact sequence of the following form:

$$0 \longrightarrow \mathcal{O}^\times_{K,S,T} \longrightarrow P \xrightarrow{\psi} F \xrightarrow{\pi} \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K}) \longrightarrow 0, \tag{24}$$

where π is the above map and P is a cohomologically-trivial G-module. Since $\mathcal{O}^\times_{K,S,T}$ is $\mathbb{Z}$-torsion-free, it follows that P is also $\mathbb{Z}$-torsion-free. Hence, P is projective. Note that the complex

$$P \xrightarrow{\psi} F,$$

where P is placed in degree 0, is quasi-isomorphic to $D^\bullet_{K,S,T}$. Hence we have an isomorphism

$$\det_G(D^\bullet_{K,S,T}) \simeq \det_G(P) \otimes_G \det_G^{-1}(F). \tag{25}$$

For each $1 \le i \le d$, we define

$$\psi_i := b_i^* \circ \psi \in \mathrm{Hom}_G(P, \mathbb{Z}[G]),$$

where $b_i^* \in \mathrm{Hom}_G(F, \mathbb{Z}[G])$ is the dual basis of $b_i \in F$.

5.5. THE EQUIVALENCE OF CONJECTURES 5.4 AND 5.9. In this subsection, we prove Theorem 5.10.

Proof of Theorem 5.10. We regard $\mathcal{O}^\times_{K,S,T} \subset P$ by the exact sequence (24). Note that, since $P/\mathcal{O}^\times_{K,S,T} \simeq \mathrm{im}(\psi) \subset F$ is $\mathbb{Z}$-torsion-free, we can apply Lemma 4.7 and Remark 4.8 for $M = \mathcal{O}^\times_{K,S,T}$. If $\mathcal{N}_H(\epsilon^V_{K/k,S,T}) \in \mathrm{im}(\nu)$, then we have

$$\Phi(\epsilon^V_{K/k,S,T}) = \Phi^H(\nu^{-1}(\mathcal{N}_H(\epsilon^V_{K/k,S,T}))) \text{ in } (\mathcal{J}_W)_H \tag{26}$$

for every $\Phi \in \bigwedge^r_G \mathrm{Hom}_G(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G])$, by Proposition 4.17. Hence Conjecture 5.4 implies Conjecture 5.9.

Conversely, suppose that Conjecture 5.9 is valid. Then we have $\Phi(\epsilon^V_{K/k,S,T}) \in \mathcal{J}_W$ for every $\Phi \in \bigwedge^r_G \mathrm{Hom}_G(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G])$, so again we use Proposition 4.17 to deduce that

$$\mathcal{N}_H(\epsilon^V_{K/k,S,T}) \in \mathrm{im}\left(\nu : (\bigwedge_{G/H}^r P^H) \otimes_{\mathbb{Z}} (J_W)_H \to (\bigwedge_G^r P) \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)J_W\right),$$

and that the equality (26) holds for every $\Phi \in \bigwedge^r_G \mathrm{Hom}_G(P, \mathbb{Z}[G])$. By Proposition 4.15, we see that the equality

$$\nu^{-1}(\mathcal{N}_H(\epsilon^V_{K/k,S,T})) = (-1)^{r(r'-r)}\mathrm{Rec}_W(\epsilon^{V'}_{L/k,S,T})$$

holds in $(\bigwedge^r_{G/H} P^H) \otimes_{\mathbb{Z}} (J_W)_H$. Since the natural map

$$(\bigcap_{G/H}^r \mathcal{O}^\times_{L,S,T}) \otimes_{\mathbb{Z}} (J_W)_H \longrightarrow (\bigwedge_{G/H}^r P^H) \otimes_{\mathbb{Z}} (J_W)_H$$

is injective by Remark 4.8, we see that the above equality holds in $(\bigcap^r_{G/H} \mathcal{O}^\times_{L,S,T}) \otimes_{\mathbb{Z}} (J_W)_H$. Thus Conjecture 5.9 implies Conjecture 5.4. $\square$

REMARK 5.11. Although in the proof of Theorem 5.10 we used the exact sequence (24) to verify the existence of a finitely generated projective G-module P and an injection $\mathcal{O}^\times_{K,S,T} \hookrightarrow P$ whose cokernel is $\mathbb{Z}$-torsion-free, the referee pointed out that it is unnecessary to use (24) at this point. Indeed, choosing a projective module P' and a surjection $f : P' \to \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}^\times_{K,S,T}, \mathbb{Z})$, we have an embedding $\mathcal{O}^\times_{K,S,T} \hookrightarrow P := \mathrm{Hom}_{\mathbb{Z}}(P', \mathbb{Z})$ by dualizing f, whose cokernel is $\mathbb{Z}$-torsion-free.

5.6. The leading term conjecture implies the Rubin-Stark conjecture. The following result was first proved by the first author in [3, Cor. 4.1] but the proof given here is very much simpler than that given in loc. cit.

Theorem 5.12. LTC(K/k) *implies the Rubin-Stark conjecture for both* $(K/k, S, T, V)$ *and* $(L/k, S, T, V')$.

Proof. Assume that LTC(K/k) is valid so the zeta element $z_{K/k,S,T}$ is a $\mathbb{Z}[G]$-basis of $\det_G(D^\bullet_{K,S,T})$. In this case one also knows that P must be free of rank d and we define $z_b \in \bigwedge_G^d P$ to be the element corresponding to the zeta element $z_{K/k,S,T} \in \det_G(D^\bullet_{K,S,T})$ via the isomorphism

$$\bigwedge_G^d P \xrightarrow{\sim} \bigwedge_G^d P \otimes \bigwedge_G^d \mathrm{Hom}_G(F, \mathbb{Z}[G]) \simeq \det_G(D^\bullet_{K,S,T}),$$

where the first isomorphism is defined by

$$a \mapsto a \otimes \bigwedge_{1 \le i \le d} b_i^*,$$

and the second isomorphism is given by (25).

Then Theorem 5.12 follows immediately from the next theorem (see also Corollary 5.15 below for $(L/k, S, T, V')$). □

Remark 5.13. In [52] Vallières closely follows the proof of [3, Cor. 4.1] to show that Conjecture 3.1 (and hence also LTC(K/k) by virtue of Proposition 3.4) implies the extension of the Rubin-Stark Conjecture formulated by Emmons and Popescu in [13]. The arguments used here can be used to show that LTC(K/k) implies a refinement of the main result of Vallières, and hence also of the conjecture of Emmons and Popescu, that is in the spirit of Theorem 1.5. This result is to be explained in forthcoming work of Livingstone-Boomla.

The following theorem was essentially obtained in [3] by the first author. This theorem describes the Rubin-Stark element in terms of the zeta elements. It is a key to prove Theorem 5.12, and also plays important roles in the proofs of Theorem 5.16 and Theorem 7.5 given below.

Theorem 5.14. *Assume that* LTC(K/k) *holds. Then, regarding* $\mathcal{O}^\times_{K,S,T}$ *as a submodule of* P, *one has*

$$(\bigwedge_{r<i\le d} \psi_i)(z_b) \in \bigcap_G^r \mathcal{O}^\times_{K,S,T} (\subset \bigwedge_G^r P)$$

(see Lemma 4.7 (ii)) and also

$$(-1)^{r(d-r)} (\bigwedge_{r<i\le d} \psi_i)(z_b) = \epsilon^V_{K/k,S,T}.$$

Proof. Take any $\chi \in \widehat{G}$. Recall from (21) that

$$r_{\chi,S} = \dim_{\mathbb{C}}(e_\chi \mathbb{C} X_{K,S}) = \dim_{\mathbb{C}}(e_\chi \mathbb{C}\mathcal{O}^\times_{K,S,T})$$

(the last equality follows from $\mathbb{C}\mathcal{O}^\times_{K,S,T} \simeq \mathbb{C}X_{K,S}$). Consider the map

$$\Psi := \bigoplus_{r<i\le d} \psi_i : e_\chi \mathbb{C}P \longrightarrow e_\chi \mathbb{C}[G]^{\oplus(d-r)}.$$

This map is surjective if and only if $r_{\chi,S} = r$. Indeed, if $r_{\chi,S} = r$, then $\{e_\chi(w_i - w_0)\}_{1\le i\le r}$ is a $\mathbb{C}$-basis of $e_\chi\mathbb{C}X_{K,S}$, so $e_\chi\mathbb{C}\,\mathrm{im}(\psi) = e_\chi\mathbb{C}\ker(\pi) = \bigoplus_{r<i\le d} e_\chi\mathbb{C}[G]b_i$. In this case, Ψ is surjective. If $r_{\chi,S} > r$, then $\dim_{\mathbb{C}}(e_\chi\mathbb{C}\,\mathrm{im}(\psi)) = d - r_{\chi,S} < d-r$, so Ψ is not surjective. Applying Lemma 4.2, we have

$$e_\chi(\bigwedge_{r<i\le d}\psi_i)(z_b)\begin{cases}\in e_\chi\mathbb{C}\bigwedge_G^r \mathcal{O}^\times_{K,S,T}, & \text{if } r_{\chi,S} = r,\\ = 0, & \text{if } r_{\chi,S} > r.\end{cases}$$

From this and Lemma 4.7 (ii), we have

$$(\bigwedge_{r<i\le d}\psi_i)(z_b) \in (\mathbb{Q}\bigwedge_G^r \mathcal{O}^\times_{K,S,T}) \cap \bigwedge_G^r P = \bigcap_G^r \mathcal{O}^\times_{K,S,T}.$$

By Lemma 4.3 and the definition of z_b, we have

$$\lambda_{K,S}((-1)^{r(d-r)}(\bigwedge_{r<i\le d}\psi_i)(z_b)) = \theta^{(r)}_{K/k,S,T}\bigwedge_{1\le i\le r}(w_i - w_0).$$

By the characterization of the Rubin-Stark element, we have

$$(-1)^{r(d-r)}(\bigwedge_{r<i\le d}\psi_i)(z_b) = \epsilon^V_{K/k,S,T}.$$

This completes the proof. □

By the same argument as above, one obtains the following result.

Corollary 5.15. *Assume that* LTC(K/k) *holds. Then we have an equality*

$$(-1)^{r'(d-r')}(\bigwedge_{r'<i\le d}\psi_i^H)(\mathrm{N}_H^d\, z_b) = \epsilon^{V'}_{L/k,S,T}$$

in $\bigcap^{r'}_{G/H}\mathcal{O}^\times_{L,S,T}$.

5.7. The leading term conjecture implies Conjecture 5.4. In this subsection we prove the following result.

Theorem 5.16. LTC(K/k) *implies* MRS$(K/L/k,S,T,V,V')$.

By Remark 3.3, this directly implies the following result.

Corollary 5.17. MRS$(K/L/k,S,T,V,V')$ *is valid if K is an abelian extension over $\mathbb{Q}$ or if k is a function field.*

Remark 5.18. Theorem 5.16 is an improvement of the main result in [46, Th. 3.22] by the third author, which asserts that under some hypotheses LTC(K/k) implies most of Conjecture 5.4. In [3, Th. 3.1], the first author proved that LTC(K/k) implies most of Conjecture 5.9. Since we know by Theorem 5.10

that Conjecture 5.4 and Conjecture 5.9 are equivalent, Theorem 5.16 is also an improvement of [3, Th. 3.1].

Remark 5.19. In [46, §4], by using a weak version of Corollary 5.17, the third author gave another proof of the 'except 2-part' of Darmon's conjecture on cyclotomic units [12], which was first proved by Mazur and Rubin in [36] via Kolyvagin systems. In §6, we shall use Corollary 5.17 to give a full proof of a refined version of Darmon's conjecture, and also give a new evidence for Gross's conjecture on tori [21], which was studied by Hayward [22], Greither and Kučera [16], [17].

We prove Theorem 5.16 after proving some lemmas. The following lemma is a restatement of [3, Lem. 7.4].

Lemma 5.20. *If $1 \leq i \leq n$, then we have an inclusion*

$$\mathrm{im}(\psi_i) \subset \mathcal{I}_i.$$

In particular, $\psi_i = 0$ for $1 \leq i \leq r$.

Proof. Take any $a \in P$. Write

$$\psi(a) = \sum_{j=1}^{d} x_j b_j$$

with some $x_j \in \mathbb{Z}[G]$. For each i with $1 \leq i \leq n$, we show that $x_i \in \mathcal{I}_i$, or equivalently, $\mathrm{N}_{G_i} x_i = 0$. Noting that F^{G_i} is a free G/G_i-module with basis $\{\mathrm{N}_{G_i} b_j\}_{1 \leq j \leq d}$, it is sufficient to show that

$$\sum_{j=1}^{d} \mathrm{N}_{G_i} x_j b_j \in \langle \mathrm{N}_{G_i} b_j : 1 \leq j \leq d, j \neq i \rangle_{G/G_i}.$$

The left hand side is equal to $\psi(\mathrm{N}_{G_i} a)$. By the exact sequence (24), this is contained in $\ker(\pi|_{F^{G_i}})$. Note that we have a natural isomorphism

$$\mathrm{N}_{G_i} X_{K,S} \simeq X_{K^{G_i},S}.$$

Since v_i splits completely in K^{G_i}, the G/G_i-submodule of $\mathrm{N}_{G_i} X_{K,S}$ generated by $\mathrm{N}_{G_i}(w_i - w_0)$ is isomorphic to $\mathbb{Z}[G/G_i]$. This shows that

$$\ker(\pi|_{F^{G_i}}) \subset \langle \mathrm{N}_{G_i} b_j : 1 \leq j \leq d, j \neq i \rangle_{G/G_i}.$$

□

For each integer i with $r < i \leq r'$, we define a map

$$\widetilde{\mathrm{Rec}}_i : P^H \longrightarrow (\mathcal{I}_i)_H$$

as follows. For $a \in P^H$, take $\widetilde{a} \in P$ such that $\mathrm{N}_H \widetilde{a} = a$ (this is possible since P is cohomologically-trivial). Define

$$\widetilde{\mathrm{Rec}}_i(a) := \psi_i(\widetilde{a}) \bmod I_H \mathcal{I}_i \ \in (\mathcal{I}_i)_H.$$

(Note that $\mathrm{im}(\psi_i) \subset \mathcal{I}_i$ by Lemma 5.20.) One can easily check that this is well-defined.

Lemma 5.21. *On $\mathcal{O}_{L,S,T}^{\times}$, which we regard as a submodule of P^H, $\widetilde{\mathrm{Rec}}_i$ coincides with the map Rec_i. In particular, by the construction of (20), we can extend the map*

$$\mathrm{Rec}_W : \bigcap_{G/H}^{r'} \mathcal{O}_{L,S,T}^{\times} \longrightarrow (\bigcap_{G/H}^{r} \mathcal{O}_{L,S,T}^{\times}) \otimes_{\mathbb{Z}} (J_W)_H$$

to

$$\widetilde{\mathrm{Rec}}_W := \bigwedge_{r<i\leq r'} \widetilde{\mathrm{Rec}}_i : \bigwedge_{G/H}^{r'} P^H \to (\bigwedge_{G/H}^{r} P^H) \otimes_{G/H} (\mathcal{J}_W)_H \simeq (\bigwedge_{G/H}^{r} P^H) \otimes_{\mathbb{Z}} (J_W)_H \ .$$

Proof. The proof is essentially the same as [3, Prop. 10.1] and [2, Lem. 8]. For $a \in \mathcal{O}_{L,S,T}^{\times}$, take $\widetilde{a} \in P$ such that $\mathrm{N}_H\, \widetilde{a} = a$ in P^H. For each $\tau \in G/H$, fix a lift $\widetilde{\tau} \in G$. Regard F as the free H-module with basis $\{\widetilde{\tau} b_i\}_{i,\tau}$. It is sufficient to show that

$$(\widetilde{\tau} b_i)^* \circ \psi(\widetilde{a}) = \mathrm{rec}_{\widetilde{\tau} w_i}(a) - 1 = \mathrm{rec}_{w_i}(\tau^{-1} a) - 1, \tag{27}$$

for every $r < i \leq r'$, where $(\widetilde{\tau} b_i)^* \in \mathrm{Hom}_H(F, \mathbb{Z}[H])$ is the dual basis of F as a free H-module. Indeed, using

$$\widetilde{\mathrm{Rec}}_i(a) = \psi_i(\widetilde{a}) = \sum_{\tau \in G/H} \widetilde{\tau}((\widetilde{\tau} b_i)^* \circ \psi(\widetilde{a})),$$

we know from (27) that

$$\widetilde{\mathrm{Rec}}_i(a) = \sum_{\tau \in G/H} \widetilde{\tau}(\mathrm{rec}_{w_i}(\tau^{-1} a) - 1) = \mathrm{Rec}_i(a).$$

We shall show (27). For simplicity, set $w := \widetilde{\tau} w_i$ and $b := \widetilde{\tau} b_i$. We denote the decomposition group of w by $G_w (= G_i)$. As in the proof of Proposition 2.4, one can show that there is a unique morphism

$$\theta_w : \mathbb{Q}[-2] \longrightarrow R\Gamma(K_w, \mathbb{G}_m)$$

in $D(\mathbb{Z}[G_w])$ such that $H^2(\theta_w)$ is equal to the natural map

$$\mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \simeq H^2(K_w, \mathbb{G}_m),$$

where the last isomorphism is the invariant map in the local class field theory. We define the complex $R\Gamma((K_w)_{\mathcal{W}}, \mathbb{G}_m)$ by

$$R\Gamma((K_w)_{\mathcal{W}}, \mathbb{G}_m) := \mathrm{Cone}(\theta_w)$$

for the local field K_w. We have natural identifications $H^0((K_w)_{\mathcal{W}}, \mathbb{G}_m) = K_w^{\times}$ and $H^1((K_w)_{\mathcal{W}}, \mathbb{G}_m) = \mathbb{Z}$. The complex $R\Gamma((K_w)_{\mathcal{W}}, \mathbb{G}_m)$ defines a Yoneda extension class τ_w in $\mathrm{Ext}^2_{G_w}(\mathbb{Z}, K_w^{\times})$, and [6, Prop. 3.5(a)] shows that τ_w coincides with the local fundamental class in $H^2(G_w, K_w^{\times})$. The class τ_w is represented by a 2-extension of the form

$$0 \longrightarrow K_w^{\times} \longrightarrow P_w \xrightarrow{\psi_w} \mathbb{Z}[G_w] \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where P_w is a cohomologically trivial G_w-module. Define

$$\rho_w : L_w^\times \longrightarrow I(G_w)/I(G_w)^2$$

by $\rho(x) := \psi_w(\widetilde{x})$, where $\widetilde{x} \in P_w$ is taken so that $\mathrm{N}_{G_w}\, \widetilde{x} = x$ (note that $\psi_w(\widetilde{x})$ is well-defined in $I(G_w)/I(G_w)^2$). Then, the existence of a natural localization morphism

$$D^\bullet_{K,S,T} = R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m) \longrightarrow \mathbb{Z}[H] \otimes^{\mathbb{L}}_{\mathbb{Z}[G_w]} R\Gamma((K_w)_{\mathcal{W}}, \mathbb{G}_m)$$

and our choice of a representative of $D^\bullet_{K,S,T}$ implies

$$b^* \circ \psi(\widetilde{a}) = \rho_w(a) \text{ in } I(G_w)/I(H)I(G_w).$$

Hence, (27) is reduced to the equality

$$\rho_w = \mathrm{rec}_w - 1.$$

Consider the map

$$I(G_w)/I(G_w)^2 = \widehat{H}^{-1}(G_w, I(G_w)) \simeq \widehat{H}^{-2}(G_w, \mathbb{Z}) \xrightarrow{\sim} \widehat{H}^0(G_w, K_w^\times), \tag{28}$$

where the first isomorphism is the connecting homomorphism with respect to the short exact sequence

$$0 \longrightarrow I(G_w) \longrightarrow \mathbb{Z}[G_w] \longrightarrow \mathbb{Z} \longrightarrow 0,$$

and the last is given by the cup product with τ_w. The map (28) is the inverse of $\mathrm{rec}_w - 1$ by definition. One can also check that (28) coincides with the δ-map of the snake lemma applied to the diagram

$$\begin{tikzcd}
 & (K_w^\times)_{G_w} \arrow[r] \arrow[d, "\mathrm{N}_{G_w}"] & (P_w)_{G_w} \arrow[r, "\psi_w"] \arrow[d, "\mathrm{N}_{G_w}"] & I(G_w)/I(G_w)^2 \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & L_w^\times \arrow[r] & P_w^{G_w} \arrow[r, "\psi_w"] & I(G_w)^{G_w}, &
\end{tikzcd}$$

i.e. the inverse of ρ_w. Thus we have $\rho_w = \mathrm{rec}_w - 1$, which completes the proof. □

Note that, by Lemma 5.20, $\bigwedge_{r<i\le d} \psi_i$ defines a map

$$\bigwedge_G^d P \longrightarrow \mathcal{J}_W \bigwedge_G^r P.$$

Let ν be the injection

$$\nu : (\bigwedge_{G/H}^r P^H) \otimes_{\mathbb{Z}} (J_W)_H \longrightarrow (\bigwedge_G^r P) \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)J_W$$

in Proposition 4.12. By Proposition 4.17, we have

$$\mathcal{N}_H(\mathcal{J}_W \bigwedge_G^r P) \subset \mathrm{im}(\nu),$$

so we can define a map

$$\nu^{-1} \circ \mathcal{N}_H : \mathcal{J}_W \bigwedge_G^r P \longrightarrow (\bigwedge_{G/H}^r P^H) \otimes_{\mathbb{Z}} (J_W)_H.$$

Lemma 5.22. *We have the following commutative diagram:*

$$\begin{tikzcd}
\bigwedge_G^d P \arrow[r] \arrow[d, "\mathrm{N}_H^d"'] & \mathcal{J}_W \bigwedge_G^r P \arrow[d, "\nu^{-1}\circ\mathcal{N}_H"] \\
\bigwedge_{G/H}^d P^H \arrow[r] & (\bigwedge_{G/H}^r P^H) \otimes_{\mathbb{Z}} (J_W)_H,
\end{tikzcd}$$

where the top arrow is $(-1)^{r(d-r)} \bigwedge_{r<i\le d} \psi_i$*, and the bottom arrow is the composition of* $(-1)^{r(r'-r)}\widetilde{\mathrm{Rec}}_W$ *and* $(-1)^{r'(d-r')} \bigwedge_{r'<i\le d} \psi_i^H$.

Proof. We can prove this lemma by explicit computations, using Proposition 4.1, Proposition 4.11, and Remark 4.14. □

Proof of Theorem 5.16. By Remark 4.8 we may compute in $(\bigwedge_{G/H}^r P^H) \otimes_{\mathbb{Z}} (J_W)_H$. Using Corollary 5.15, Lemma 5.21, Lemma 5.22, and Theorem 5.14 in this order, we compute

$$\begin{aligned}
& (-1)^{r(r'-r)}\mathrm{Rec}_W(\epsilon_{L/k,S,T}^{V'}) \\
= \;& (-1)^{r(r'-r)}\widetilde{\mathrm{Rec}}_W((-1)^{r'(d-r')}(\bigwedge_{r'<i\le d} \psi_i^H)(\mathrm{N}_H^d z_b)) \\
= \;& (-1)^{r(d-r)}\nu^{-1}(\mathcal{N}_H((\bigwedge_{r<i\le d} \psi_i)(z_b))) \\
= \;& \nu^{-1}(\mathcal{N}_H(\epsilon_{K/k,S,T}^{V})).
\end{aligned}$$

This completes the proof of Theorem 5.16. □

6. Conjectures of Darmon and of Gross

In this section we use Corollary 5.17 to prove a refined version of the conjecture formulated by Darmon in [12] and to obtain important new evidence for a refined version of the 'conjecture for tori' formulated by Gross in [21].

6.1. Darmon's Conjecture. We formulate a slightly modified and refined version of Damon's conjecture ([12],[36]).

Let L be a real quadratic field. Let f be the conductor of L. Let χ be the Dirichlet character defined by

$$\chi : (\mathbb{Z}/f\mathbb{Z})^\times = \mathrm{Gal}(\mathbb{Q}(\mu_f)/\mathbb{Q}) \longrightarrow \mathrm{Gal}(L/\mathbb{Q}) \simeq \{\pm 1\},$$

where the first map is the restriction map. Fix a square-free positive integer n which is coprime to f, and let K be the maximal real subfield of $L(\mu_n)$. Set $G := \mathrm{Gal}(K/\mathbb{Q})$ and $H := \mathrm{Gal}(K/L)$. Put $n_\pm := \prod_{\ell \mid n, \chi(\ell)=\pm 1} \ell$, and

$\nu_\pm := |\{\ell | n_\pm\}|$ (in this section, ℓ always denotes a prime number). We fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Define a cyclotomic unit by

$$\beta_n := \mathrm{N}_{L(\mu_n)/K}\Big(\prod_{\sigma\in \mathrm{Gal}(\mathbb{Q}(\mu_{nf})/\mathbb{Q}(\mu_n))} \sigma(1-\zeta_{nf})^{\chi(\sigma)}\Big) \in K^\times,$$

where $\zeta_{nf} = e^{\frac{2\pi i}{nf}}$. Let τ be the generator of $G/H = \mathrm{Gal}(L/\mathbb{Q})$. Write $n_+ = \ell_1 \cdots \ell_{\nu_+}$. Note that $(1-\tau)\mathcal{O}_L[1/n]^\times$ is a free abelian group of rank $\nu_+ + 1$ (see [36, Lem. 3.2 (ii)]). Take $u_0, \ldots, u_{\nu_+} \in \mathcal{O}_L[1/n]^\times$ so that $\{u_0^{1-\tau}, \ldots, u_{\nu_+}^{1-\tau}\}$ is a basis of $(1-\tau)\mathcal{O}_L[1/n]^\times$ and that $\det(\log|u_i^{1-\tau}|_{\lambda_j})_{0\le i,j\le \nu_+} > 0$, where each λ_j $(1 \le j \le \nu_+)$ is a (fixed) place of L lying above ℓ_j, and λ_0 is the infinite place of L determined by the embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ fixed above. Define

$$R_n := \Big(\bigwedge_{1\le i\le \nu_+} (\mathrm{rec}_{\lambda_i} - 1)\Big)(u_0^{1-\tau} \wedge \cdots \wedge u_{\nu_+}^{1-\tau}) \in L^\times \otimes_{\mathbb{Z}} (J_{n_+})_H,$$

where

$$J_{n_+} := \begin{cases} (\prod_{i=1}^{i=\nu_+} I(G_{\ell_i}), & \text{if } \nu_+ \neq 0, \\ \mathbb{Z}[H], & \text{if } \nu_+ = 0, \end{cases}$$

where G_{ℓ_i} is the decomposition group of ℓ_i in G (note that since ℓ_i splits in L, we have $G_{\ell_i} \subset H$), and $(J_{n_+})_H := J_{n_+}/I(H)J_{n_+}$. We set $h_n := |\mathrm{Pic}(\mathcal{O}_L[\frac{1}{n}])|$. For any element $a \in K^\times$, following Definition 4.16 we define

$$\mathcal{N}_H(a) := \sum_{\sigma\in H} \sigma a \otimes \sigma^{-1} \in K^\times \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)J_{n_+}.$$

Note that, since $K^\times/L^\times$ is $\mathbb{Z}$-torsion-free, the natural map

$$(L^\times/\{\pm 1\}) \otimes_{\mathbb{Z}} (J_{n_+})_H \longrightarrow (K^\times/\{\pm 1\}) \otimes_{\mathbb{Z}} \mathbb{Z}[H]I(H)J_{n_+}$$

is injective.

Our refined Darmon's conjecture is formulated as follows.

Theorem 6.1. *One has*

$$\mathcal{N}_H(\beta_n) = -2^{\nu_-} h_n R_n \quad \textit{in} \quad (L^\times/\{\pm 1\}) \otimes_{\mathbb{Z}} (J_{n_+})_H.$$

Remark 6.2. Let I_n be the augmentation ideal of $\mathbb{Z}[\mathrm{Gal}(L(\mu_n)/L)]$. Note that there is a natural isomorphism

$$I_n^{\nu_+}/I_n^{\nu_+ + 1} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] \xrightarrow{\sim} I(H)^{\nu_+}/I(H)^{\nu_+ + 1} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}].$$

It is not difficult to see that the following statement is equivalent to [36, Th. 3.9]:

$$\mathcal{N}_H(\beta_n) = -2^{\nu_-} h_n R_n \quad \textit{in} \quad (L^\times/\{\pm 1\}) \otimes_{\mathbb{Z}} I(H)^{\nu_+}/I(H)^{\nu_+ + 1} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$$

(see [46, Lem. 4.7]). Since there is a natural map $(J_{n_+})_H \longrightarrow I(H)^{\nu_+}/I(H)^{\nu_+ + 1}$, Theorem 6.1 refines [36, Th. 3.9]. Note also that, in the original statement of Darmon's conjecture, the cyclotomic unit is defined by

$$\alpha_n := \prod_{\sigma\in \mathrm{Gal}(\mathbb{Q}(\mu_{nf})/\mathbb{Q}(\mu_n))} \sigma(1-\zeta_{nf})^{\chi(\sigma)},$$

whereas our cyclotomic unit is $\beta_n = \mathrm{N}_{L(\mu_n)/K}(\alpha_n)$. Since cyclotomic units, as Stark elements, lie in real fields, it is natural to consider β_n. Thus, modifying the original statement of Darmon's conjecture in the '2-part', we obtained Theorem 6.1, which does not exclude the '2-part'.

Proof of Theorem 6.1. We show that Darmon's conjecture is a consequence of Conjecture 5.4, and use Corollary 5.17 to prove it. We fit notation in this section into that in §5. Set $S := \{\infty\} \cup \{\ell | nf\}$. Take a prime v_0 of $\mathbb{Q}$, which divides f. We denote by w_1 the infinite place of K (and also L) which corresponds to the fixed embedding $\mathbb{Q} \hookrightarrow \mathbb{C}$. For $2 \le i \le \nu_+ + 1$, set $w_i := \lambda_{i-1}$. Let T be a finite set of primes that is disjoint from S and satisfying that $\mathcal{O}^\times_{K,S,T}$ is $\mathbb{Z}$-torsion-free. (In the sequel, we refer such a set of primes as 'T'.) Since K and L are abelian over $\mathbb{Q}$, the Rubin-Stark conjecture for $K/\mathbb{Q}$ and $L/\mathbb{Q}$ holds (see Remark 5.3 (iii)). Set $V := \{\infty\}$ and $V' := \{\infty, \ell_1, \ldots, \ell_{\nu_+}\}$. We denote $\epsilon_{K,T} = \epsilon^V_{K/\mathbb{Q},S,T} \in \mathcal{O}^\times_{K,S,T}$ and $\epsilon_{L,T} = \epsilon^{V'}_{L/\mathbb{Q},S,T} \in \bigcap^{\nu_+ + 1}_{G/H} \mathcal{O}^\times_{L,S,T}$ for the Rubin-Stark elements, characterized by

$$\lambda_{K,S}(\epsilon_{K,T}) = \theta^{(1)}_{K/\mathbb{Q},S,T}(w_1 - w_0),$$

$$\lambda_{L,S}(\epsilon_{L,T}) = \theta^{(\nu_+ + 1)}_{L/\mathbb{Q},S,T} \bigwedge_{1 \le i \le \nu_+ + 1} (w_i - w_0).$$

We take $\mathscr{T}$, a finite family of 'T', such that

$$\sum_{T \in \mathscr{T}} a_T \delta_T = 2$$

for some $a_T \in \mathbb{Z}[G]$, where $\delta_T := \prod_{\ell \in T}(1 - \ell \mathrm{Fr}_\ell^{-1})$ (see [50, Chap. IV, Lem. 1.1]). By [46, Lem. 4.6], we have

$$(1-\tau) \sum_{T \in \mathscr{T}} a_T \epsilon_{K,T} = \beta_n \quad in \quad K^\times/\{\pm 1\},$$

(where $\tau \in \mathrm{Gal}(L/\mathbb{Q})$ is regarded as an element of $\mathrm{Gal}(K/\mathbb{Q}(\mu_n)^+)$) and

$$(1-\tau) \sum_{T \in \mathscr{T}} a_T \epsilon_{L,T} = (-1)^{\nu_+ + 1} 2^{\nu_-} h_n (1-\tau) u_0 \wedge \cdots \wedge u_{\nu_+} \quad in \quad \mathbb{Q} \bigwedge^{\nu_+ + 1}_{G/H} \mathcal{O}^\times_{L,S}.$$

As in §5.3, for $1 < i \le \nu_+ + 1$ we denote by Rec_i the homomorphism

$$\mathrm{Rec}_i : \mathcal{O}^\times_{L,S,T} \longrightarrow (\mathcal{J}_{n_+})_H = J_{n_+}\mathbb{Z}[G]/I_H J_{n_+}\mathbb{Z}[G]$$

defined by

$$\mathrm{Rec}_i(a) = \mathrm{rec}_{\lambda_{i-1}}(a) - 1 + \tau(\mathrm{rec}_{\lambda_{i-1}}(\tau a) - 1).$$

$\bigwedge_{1 < i \le \nu_+ + 1} \mathrm{Rec}_i$ induces a homomorphism

$$\bigcap^{\nu_+ + 1}_{G/H} \mathcal{O}^\times_{L,S,T} \longrightarrow (\bigcap^{1}_{G/H} \mathcal{O}^\times_{L,S,T}) \otimes_{\mathbb{Z}} (J_{n_+})_H = \mathcal{O}^\times_{L,S,T} \otimes_{\mathbb{Z}} (J_{n_+})_H,$$

which we denote by Rec_{n_+}. We compute

$$\begin{aligned} & (1-\tau)\sum_{T\in\mathscr{T}} a_T \mathrm{Rec}_{n_+}(\epsilon_{L,T}) \\ = & \sum_{T\in\mathscr{T}} \mathrm{Rec}_{n_+}(a_T(1-\tau)\epsilon_{L,T}) \\ = & \sum_{T\in\mathscr{T}} (\bigwedge_{1\le i\le \nu_+} (\mathrm{rec}_{\lambda_i}-1))((1-\tau)^{\nu_++1} a_T \epsilon_{L,T}) \\ = & (\bigwedge_{1\le i\le \nu_+} (\mathrm{rec}_{\lambda_i}-1))((-1)^{\nu_++1} 2^{\nu_-} h_n u_0^{1-\tau}\wedge\cdots\wedge u_{\nu_+}^{1-\tau}) \\ = & (-1)^{\nu_++1} 2^{\nu_-} h_n R_n. \end{aligned}$$

By Corollary 5.17, we have

$$\mathcal{N}_H(\epsilon_{K,T}) = (-1)^{\nu_+}\mathrm{Rec}_{n_+}(\epsilon_{L,T})$$

(note that the map ν in Conjecture 5.4 is the natural inclusion map in this case.) Hence, we have

$$\begin{aligned} \mathcal{N}_H(\beta_n) &= (1-\tau)\sum_{T\in\mathscr{T}} a_T \mathcal{N}_H(\epsilon_{K,T}) \\ &= (-1)^{\nu_+}(1-\tau)\sum_{T\in\mathscr{T}} a_T \mathrm{Rec}_{n_+}(\epsilon_{L,T}) \\ &= -2^{\nu_-} h_n R_n, \end{aligned}$$

as required. □

6.2. Gross's conjecture for tori. In this section we use Corollary 5.17 to obtain some new evidence in support of the 'conjecture for tori' formulated by Gross in [21].

We review the formulation of Gross's conjecture for tori. We follow [22, Conj. 7.4]. Let k be a global field, and L/k be a quadratic extension. Let $\widetilde{L}/k$ be a finite abelian extension, which is disjoint to L, and set $K := L\widetilde{L}$. Set $G := \mathrm{Gal}(K/k)$, and $H := \mathrm{Gal}(K/L) = \mathrm{Gal}(\widetilde{L}/k)$. Let τ be the generator of $G/H = \mathrm{Gal}(L/k)$. Let S be a non-empty finite set of places of k such that $S_\infty(k)\cup S_{\mathrm{ram}}(K/k)\subset S$. Let T be a finite set of places of k that is disjoint from S and satisfies that $\mathcal{O}_{K,S,T}^\times$ is $\mathbb{Z}$-torsion-free. Let $v_1,\ldots,v_{r'}$ be all places in S which split in L. We assume $r' < |S|$. Then, by [45, Lem. 3.4 (i)], we see that $h_{k,S,T} := |\mathrm{Cl}_S^T(k)|$ divides $h_{L,S,T} := |\mathrm{Cl}_S^T(L)|$. Take $u_1,\ldots,u_{r'}\in\mathcal{O}_{L,S,T}^\times$ such that $\{u_1^{1-\tau},\ldots,u_{r'}^{1-\tau}\}$ is a basis of $(1-\tau)\mathcal{O}_{L,S,T}^\times$, which is isomorphic to $\mathbb{Z}^{\oplus r'}$, and $\det(-\log|u_i^{1-\tau}|_{w_j})_{1\le i,j\le r'} > 0$, where w_j is a (fixed) place of L lying above v_j. Put $W := \{v_1,\ldots,v_{r'}\}$. As in §5.3, we define

$$J_W := \begin{cases} (\prod_{0<i\le r'} I(G_i))\mathbb{Z}[H], & \text{if } W\neq\emptyset, \\ \mathbb{Z}[H], & \text{if } W=\emptyset, \end{cases}$$

where $G_i \subset H$ denotes the decomposition group of v_i, and $I(G_i)$ is the augmentation ideal of $\mathbb{Z}[G_i]$. Set

$$R_{S,T} := \det(\mathrm{rec}_{w_j}(u_i^{1-\tau}) - 1)_{1\le i,j\le r'} \in (J_W)_H.$$

Let χ be the non-trivial character of G/H. The map

$$\mathbb{Z}[G] = \mathbb{Z}[H \times G/H] \longrightarrow \mathbb{Z}[H]$$

induced by χ is also denoted by χ.
Gross's tori conjecture is formulated as follows.

Conjecture 6.3.

$$\chi(\theta_{K/k,S,T}(0)) = 2^{|S|-1-r'}\frac{h_{L,S,T}}{h_{k,S,T}}R_{S,T} \quad in \quad (J_W)_H.$$

Remark 6.4. The statement that the equality of Conjecture 6.3 holds in $\mathbb{Z}[H]/I(H)^{r'+1}$ is equivalent to [22, Conj. 7.4] (if we neglect the sign). Indeed, we see that

$$R_{S,T} = ((\mathcal{O}_{L,S,T}^\times)^- : (1-\tau)\mathcal{O}_{L,S,T}^\times)R_H^-,$$

where R_H^- is the 'minus-unit regulator' defined in [22, §7.2] (where our H is denoted by G). Since there is a natural map $(J_W)_H \to \mathbb{Z}[H]/I(H)^{r'+1}$, Conjecture 6.3 refines [22, Conj. 7.4].

Theorem 6.5. *Conjecture 5.4 implies Conjecture 6.3. In particular, Conjecture 6.3 is valid if K is an abelian extension over $\mathbb{Q}$ or k is a global function field.*

Proof. First, note that the Rubin-Stark conjecture for $(K/k,S,T,\emptyset)$ and $(L/k,S,T,W)$ is true by Remark 5.3 (i) and (ii), respectively. By Conjecture 5.4, we have

$$\theta_{K/k,S,T}(0) = \mathrm{Rec}_W(\epsilon_{L/k,S,T}^W) \quad in \quad (\mathcal{J}_W)_H (= \mathbb{Z}[G/H]\otimes_{\mathbb{Z}} (J_W)_H).$$

(Note that $\nu^{-1}(\mathcal{N}_H(\theta_{K/k,S,T}(0))) = \theta_{K/k,S,T}(0)$ in $(\mathcal{J}_W)_H$ by [37, Lem. 5.6 (iv)].) Note that $\chi \circ \mathrm{Rec}_i = \mathrm{rec}_{w_i}((1-\tau)(\cdot)) - 1$. So we have

$$\chi(\mathrm{Rec}_W(\epsilon_{L/k,S,T}^W)) = (\bigwedge_{1\le i\le r'}(\mathrm{rec}_{w_i} - 1))((1-\tau)^{r'}\epsilon_{L/k,S,T}^W).$$

We know by the proof of [45, Th. 3.5] that

$$(1-\tau)^{r'}\epsilon_{L/k,S,T}^W = 2^{|S|-1-r'}\frac{h_{L,S,T}}{h_{k,S,T}}u_1^{1-\tau}\wedge\cdots\wedge u_{r'}^{1-\tau}.$$

Hence we have

$$\begin{aligned}\chi(\theta_{K/k,S,T}(0)) &= \chi(\mathrm{Rec}_W(\epsilon^W_{L/k,S,T}))\\ &= (\bigwedge_{1\le i\le r'}(\mathrm{rec}_{w_i}-1))((1-\tau)^{r'}\epsilon^W_{L/k,S,T})\\ &= 2^{|S|-1-r'}\frac{h_{L,S,T}}{h_{k,S,T}}(\bigwedge_{1\le i\le r'}(\mathrm{rec}_{w_i}-1))(u_1^{1-\tau}\wedge\cdots\wedge u_{r'}^{1-\tau})\\ &= 2^{|S|-1-r'}\frac{h_{L,S,T}}{h_{k,S,T}}R_{S,T},\end{aligned}$$

as required.

Having now proved the first claim, the second claim follows directly from Corollary 5.17. □

Remark 6.6. The strongest previous evidence in favour of the conjecture for tori is that obtained by Greither and Kučera in [16, 17], in which it is referred to as the 'Minus Conjecture' and studied in a slightly weaker form in order to remove any occurence of the auxiliary set T. More precisely, by using rather different methods they were able to prove that this conjecture was valid in the case that $k=\mathbb{Q}$, $K=FK^+$ where F is imaginary quadratic of conductor f and class number h_F and $K^+/\mathbb{Q}$ is tamely ramified, abelian of exponent equal to an odd prime ℓ and ramified at precisely s primes $\{p_i\}_{1\le i\le s}$ each of which splits in $F/\mathbb{Q}$; further, any of the following conditions are satisfied

- $s=1$ and $\ell\nmid f$ [16, Th. 8.8], or
- $s=2, \ell\nmid fh_F$ and either $K^+/\mathbb{Q}$ is cyclic or p_1 is congruent to an ℓ-th power modulo p_2 [16, Th. 8.9], or
- $\ell\ge 3(s+1)$ and $\ell\nmid h_F$ [17, Th. 3.7].

It is straightforward to show that the conjecture for tori implies their 'Minus conjecture', using [50, Chap. IV, Lem. 1.1] to eliminate the dependence on 'T' (just as in the proof of Theorem 6.1). The validity of the 'Minus conjecture' in the case $k=\mathbb{Q}$ is thus also a consequence of Theorem 6.5.

7. Higher Fitting ideals of Selmer groups

In this section, we introduce a natural notion of 'higher *relative* Fitting ideals' in §7.1, and then study the higher Fitting ideals of the transposed Selmer group $\mathcal{S}^{\rm tr}_{S,T}(\mathbb{G}_{m/K})$. In this way we prove Theorems 1.5 and 1.10 and Corollary 1.14.

7.1. Relative Fitting ideals. In this subsection, we recall the definition of Fitting ideals and also introduce a natural notion of 'higher *relative* Fitting ideals'.

Suppose that R is a noetherian ring and M is a finitely generated R-module. Take an exact sequence

$$R^{\oplus m}\xrightarrow{f} R^{\oplus n}\to M\to 0,$$

and denote by A_f the matrix with n rows and m columns corresponding to f. Then for $i \in \mathbb{Z}_{\geq 0}$ the i-th Fitting ideal of M, denoted by $\mathrm{Fitt}_R^i(M)$, is defined to be the ideal generated by all $(n-i) \times (n-i)$ minors of A_f if $0 \leq i < n$ and R if $i \geq n$. In this situation we call A_f a *relation matrix* of M. These ideals do not depend on the choice of the above exact sequence (see [40, Chap. 3]). The usual notation is $\mathrm{Fitt}_{i,R}(M)$, but we use the above notation which is consistent with the exterior power $\bigwedge_R^i M$. If we can take a presentation

$$R^{\oplus m} \xrightarrow{f} R^{\oplus n} \to M \to 0$$

of M with $m = n$, then we say M has a *quadratic presentation*.
We now fix a submodule N of M, and non-negative integers a and b. We write ν for the minimal number of generators of N.
If $b > \nu$, then we simply set

$$\mathrm{Fitt}_R^{(a,b)}(M, N) := \mathrm{Fitt}_R^a(M/N).$$

However, if $b \leq \nu$ then we consider a relation matrix for M of the form

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

where A_1 is a relation matrix of N. We suppose that A_1 is a matrix with n_1 rows and m_1 columns and A_3 is a matrix with n_2 rows and m_2 columns. We remove b rows from among the first row to the n_1-th row of A to get a matrix A', and remove a rows from A' to get A''. We denote by $F(A'')$ the ideal generated by all $c \times c$ minors of A'' where $c = n_1 + n_2 - a - b$ if $c > 0$ and $F(A'') = R$ otherwise. We consider all such A'' obtained from A and then define the relative Fitting ideal by setting

$$\mathrm{Fitt}_R^{(a,b)}(M, N) := \sum_{A''} F(A'').$$

By the standard method using the elementary operations of matrices (see the proof of [40, p.86, Th. 1]), one can show that this sum does not depend on the choice of relation matrix A.
The following result gives an alternative characterization of this ideal.

Lemma 7.1. *Let X be an R-submodule of M that is generated by $(a+b)$ elements $x_1, \ldots, x_{a+b}$ such that the elements $x_1, \ldots, x_b$ belong to N. Let $\mathcal{X}$ be the set of such R-submodules of M. Then we have*

$$\mathrm{Fitt}_R^{(a,b)}(M, N) = \sum_{X \in \mathcal{X}} \mathrm{Fitt}_R^0(M/X).$$

Proof. If $b > \nu$, both sides equal $\mathrm{Fitt}_R^a(M/N)$, so we may assume $b \leq \nu$. Let $e_1, \ldots, e_n$ be the generators of M corresponding to the above matrix A where $n = n_1 + n_2$. Suppose that A'' is obtained from A by removing $(a+b)$ rows, from the i_1-th row to the i_{a+b}-th row with $1 \leq i_1, \ldots, i_b \leq n_1$. Let X be a submodule of M generated by $e_{i_1}, \ldots, e_{i_{a+b}}$. Then by definitions $X \in \mathcal{X}$ and

$F(A'') = \mathrm{Fitt}_R^0(M/X)$. This shows that the left hand side of the equation in Lemma 7.1 is in the right hand side.
On the other hand, suppose that X is in $\mathcal{X}$ and $x_1, \ldots, x_{a+b}$ are generators of X. Regarding $e_1, \ldots, e_{n_1}, x_1, \ldots, x_b, e_{n_1+1}, \ldots, e_n, x_{b+1}, \ldots, x_{a+b}$ as generators of M, we have a relation matrix of M of the form

$$B = \begin{pmatrix} A_1 & B_1 & A_2 & B_2 \\ 0 & I_b & 0 & 0 \\ 0 & 0 & A_3 & B_3 \\ 0 & 0 & 0 & I_a \end{pmatrix}$$

where I_a, I_b are the identity matrices of degree a, b, respectively. Then

$$C = \begin{pmatrix} A_1 & B_1 & A_2 & B_2 \\ 0 & 0 & A_3 & B_3 \end{pmatrix}$$

is a relation matrix of M/X. Since C is obtained from B by removing $(a+b)$ rows in the way of obtaining A'' from A, it follows from the definition of the relative Fitting ideal that $\mathrm{Fitt}_R^0(M/X) \subset \mathrm{Fitt}_R^{(a,b)}(M,N)$. This shows the other inclusion. □

In the next result we record some useful properties of higher relative Fitting ideals.

Lemma 7.2.
(i) $\mathrm{Fitt}_R^{(a,b)}(M,N) \subset \mathrm{Fitt}_R^{a+b}(M)$.
(ii) $\mathrm{Fitt}_R^{(a,0)}(M,N) = \mathrm{Fitt}_R^{a}(M)$.
(iii) *Suppose that there exists an exact sequence* $0 \to M' \to M \to R^{\oplus r} \to 0$ *of* R*-modules and that* $N \subset M'$. *Then one has*

$$\mathrm{Fitt}_R^{(a,b)}(M,N) = \begin{cases} \mathrm{Fitt}_R^{(a-r,b)}(M',N), & \textit{if } a \geq r, \\ 0, & \textit{otherwise.} \end{cases}$$

(iv) *Assume that* M/N *has a quadratic presentation. Then one has*

$$\mathrm{Fitt}_R^{(0,b)}(M,N) = \mathrm{Fitt}_R^{b}(N)\,\mathrm{Fitt}_R^0(M/N).$$

Proof. Claims (i), (ii) and (iii) follow directly from the definition of the higher relative Fitting ideal. To prove claim (iv), we consider a relation matrix

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

as above, where A_1 is a matrix with n_1 rows and A_3 is a square matrix of n_2 rows. Put $n = n_1 + n_2$. Then a matrix A'' obtained from A as above is of the form

$$A'' = \begin{pmatrix} A_1'' & A_2'' \\ 0 & A_3 \end{pmatrix}.$$

This is a matrix with $(n-b)$ rows and so a nonzero $(n-b) \times (n-b)$ minor of A'' must be $\det(A_3)$ times a $(n_1 - b) \times (n_1 - b)$ minor of A_1''. This implies the required conclusion. □

7.2. Statement of the conjecture. Let $K/k, G, S, T, V$ be as in §5.1. For the element $\epsilon^V_{K/k,S,T}$, the Rubin-Stark conjecture asserts that $\Phi(\epsilon^V_{K/k,S,T})$ belongs to $\mathbb{Z}[G]$ for every Φ in $\bigwedge_G^r \mathrm{Hom}_G(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G])$.
We next formulate a much stronger conjecture which describes the arithmetic significance of the ideal generated by the elements $\Phi(\epsilon^V_{K/k,S,T})$ when Φ runs over $\bigwedge_G^r \mathrm{Hom}_G(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G])$.

Conjecture 7.3. *One has an equality*

$$\mathrm{Fitt}^r_G(\mathcal{S}_{S,T}(\mathbb{G}_{m/K})) = \{\Phi(\epsilon^V_{K/k,S,T})^\# : \Phi \in \bigwedge_G^r \mathrm{Hom}_G(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G])\},$$

or equivalently (by Lemma 2.8),

$$\mathrm{Fitt}^r_G(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K})) = \{\Phi(\epsilon^V_{K/k,S,T}) : \Phi \in \bigwedge_G^r \mathrm{Hom}_G(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G])\}.$$

The following result shows that this conjecture refines the first half of the statement of Conjecture 5.9.

Proposition 7.4. *For a finite set Σ of places, we put $\mathcal{J}_\Sigma = \prod_{v\in\Sigma} I(G_v)\mathbb{Z}[G]$. Assume Conjecture 7.3 is valid. Then, for every $\Phi \in \bigwedge_G^r \mathrm{Hom}_G(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G])$ and $v \in S \setminus V$, one has*

$$\Phi(\epsilon^V_{K/k,S,T}) \in \mathcal{J}_{S\setminus(V\cup\{v\})}.$$

Proof. It is sufficient to show that $\mathrm{Fitt}^r_G(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K})) \subset \mathcal{J}_{S\setminus(V\cup\{v\})}$. Since there is a canonical surjective homomorphism

$$\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K}) \longrightarrow X_{K,S} \simeq \mathbb{Z}[G]^{\oplus r} \oplus X_{K,S\setminus V},$$

we have

$$\mathrm{Fitt}^r_G(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K})) \subset \mathrm{Fitt}^r_G(X_{K,S}) = \mathrm{Fitt}^0_G(X_{K,S\setminus V}).$$

The existence of the surjective homomorphism $X_{K,S\setminus V} \to Y_{K,S\setminus(V\cup\{v\})}$ implies that $\mathrm{Fitt}^0_G(X_{K,S\setminus V}) \subset \mathrm{Fitt}^0_G(Y_{K,S\setminus(V\cup\{v\})}) = \mathcal{J}_{S\setminus(V\cup\{v\})}$. This completes the proof. □

7.3. The leading term conjecture implies Conjecture 7.3. The following result combines with Lemma 2.8 to imply the statement of Theorem 1.5(i).

Theorem 7.5. LTC(K/k) *implies Conjecture 7.3. In particular, Conjecture 7.3 is valid if either K is an abelian extension over $\mathbb{Q}$ or k is a function field or $[K : k] \leq 2$.*

Proof. The second claim is a consequence of Remark 3.3.
To prove the first claim we assume the validity of LTC(K/k). Then the module P that occurs in the exact sequence (24) is free of rank d, as we noted before.

Hence we may assume $P = F$. Let $z_b \in \bigwedge_G^d F$ be as in §5.6. By LTC(K/k), z_b is a G-basis of $\bigwedge_G^d F$. Write z_b as

$$z_b = x \bigwedge_{1 \le i \le d} b_i$$

with some $x \in \mathbb{Z}[G]^\times$. By Theorem 5.14 and Proposition 4.1, we have

$$\epsilon^V_{K/k,S,T} = \pm x \sum_{\sigma \in \mathfrak{S}_{d,r}} \mathrm{sgn}(\sigma) \det(\psi_i(b_{\sigma(j)}))_{r<i,j\le d} b_{\sigma(1)} \wedge \cdots \wedge b_{\sigma(r)}.$$

Take $\Phi \in \bigwedge_G^r \mathrm{Hom}_G(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G])$. Since $F/\mathcal{O}^\times_{K,S,T} \simeq \mathrm{im}(\psi) \subset F$ is $\mathbb{Z}$-torsion-free, we know by Lemma 4.7(ii) that the map

$$\mathrm{Hom}_G(F, \mathbb{Z}[G]) \longrightarrow \mathrm{Hom}_G(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G])$$

induced by the inclusion $\mathcal{O}^\times_{K,S,T} \subset F$ is surjective. Hence, we can take a lift $\widetilde{\Phi}$ of Φ to $\bigwedge_G^r \mathrm{Hom}_G(F, \mathbb{Z}[G])$. We have

$$\begin{aligned} \Phi(\epsilon^V_{K/k,S,T}) &= \pm x \sum_{\sigma \in \mathfrak{S}_{d,r}} \mathrm{sgn}(\sigma) \det(\psi_i(b_{\sigma(j)}))_{r<i,j\le d} \widetilde{\Phi}(b_{\sigma(1)} \wedge \cdots \wedge b_{\sigma(r)}) \\ &\in \langle \det(\psi_i(b_{\sigma(j)}))_{r<i,j\le d} : \sigma \in \mathfrak{S}_{d,r} \rangle_G. \end{aligned}$$

We consider the matrix A corresponding to the presentation

$$F \to F \to \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K}) \to 0$$

which comes from the exact sequence (24). By Lemma 5.20, $\psi_i = 0$ for $1 \le i \le r$. If we write elements in F as column vectors, this implies that the i-th row of A is zero for all i such that $1 \le i \le r$. Hence we have

$$\mathrm{Fitt}^r_G(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K})) = \langle \det(\psi_i(b_{\sigma(j)}))_{r<i,j\le d} : \sigma \in \mathfrak{S}_{d,r} \rangle_G.$$

Therefore, we get an inclusion

$$\{\Phi(\epsilon^V_{K/k,S,T}) : \Phi \in \bigwedge_G^r \mathrm{Hom}_G(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G])\} \subset \mathrm{Fitt}^r_G(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K})).$$

We obtain the reverse inclusion from

$$(b^*_{\sigma(1)} \wedge \cdots \wedge b^*_{\sigma(r)})(\epsilon^V_{K/k,S,T}) = \pm x \det(\psi_i(b_{\sigma(j)}))_{r<i,j\le d}$$

and the fact that x is a unit in $\mathbb{Z}[G]$. □

7.4. The proof of Theorem 1.10. For any G-module M we write M^* for the linear dual $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ endowed with the natural contragredient action of G. We also set $V' := V \cup \{v\}$.

We start with a useful technical observation.

Lemma 7.6. *For each integer i with $1 \le i \le r$ let φ_i be an element of $(\mathcal{O}^\times_{K,S,T})^*$. Then for any given integer N there is a subset $\{\varphi'_i : 1 \le i \le r\}$ of $(\mathcal{O}^\times_{K,S,T})^*$ which satisfies the following properties.*

(i) *For each i one has $\varphi'_i \equiv \varphi_i$ modulo $N \cdot (\mathcal{O}^\times_{K,S,T})^*$.*

(ii) *The image in* $(\mathcal{O}_{K,V',T}^{\times})^*$ *of the submodule of* $(\mathcal{O}_{K,S,T}^{\times})^*$ *that is generated by the set* $\{\varphi_i' : 1 \le i \le r\}$ *is free of rank* r.

Proof. Our choice of V implies that we may choose a free G-submodule $\mathcal{F}$ of $(\mathcal{O}_{K,V',T}^{\times})^*$ of rank r. We then choose a subset $\{f_i : 1 \le i \le r\}$ of $(\mathcal{O}_{K,S,T}^{\times})^*$ which the natural surjection $\rho : (\mathcal{O}_{K,S,T}^{\times})^* \to (\mathcal{O}_{K,V',T}^{\times})^*$ sends to a basis of $\mathcal{F}$. For any integer m we set $\varphi_{i,m} := \varphi_i + mNf_i$ and note it suffices to show that for any sufficiently large m the elements $\{\rho(\varphi_{i,m}) : 1 \le i \le r\}$ are linearly independent over $\mathbb{Q}[G]$.

Consider the composite homomorphism of G-modules $\mathcal{F} \to \mathbb{Q}(\mathcal{O}_{K,V',T}^{\times})^* \to \mathbb{Q}\mathcal{F}$ where the first arrow sends each $\rho(f_i)$ to $\rho(\varphi_{i,m})$ and the second is induced by a choice of $\mathbb{Q}[G]$-equivariant section to the projection $\mathbb{Q}(\mathcal{O}_{K,V',T}^{\times})^* \to \mathbb{Q}((\mathcal{O}_{K,V',T}^{\times})^*/\mathcal{F})$. Then, with respect to the basis $\{\rho(f_i) : 1 \le i \le r\}$, this linear map is represented by a matrix of the form $A + mNI_r$ for a matrix A in $\mathrm{M}_r(\mathbb{Q}[G])$ that is independent of m. In particular, if m is large enough to ensure that $-mN$ is not an eigenvalue of $e_\chi A$ for any χ, then the composite homomorphism is injective and so the elements $\{\rho(\varphi_{i,m}) : 1 \le i \le r\}$ are linearly independent over $\mathbb{Q}[G]$, as required. □

For each integer i with $1 \le i \le r$ let φ_i be an element of $(\mathcal{O}_{K,S,T}^{\times})^*$. Then, for any non-zero integer N which belongs to $\mathrm{Fitt}_G^0(\mathrm{Cl}(K))$ we choose homomorphisms φ_i' as in Lemma 7.6. Then the congruences in Lemma 7.6(i) imply that

$$(\bigwedge_{1\le i\le r} \varphi_i)(\epsilon_{K/k,S,T}^V) \equiv (\bigwedge_{1\le i\le r} \varphi_i')(\epsilon_{K/k,S,T}^V) \text{ modulo } N\cdot\mathbb{Z}[G].$$

Given this, Lemma 7.6(ii) implies that Theorem 1.10 is true provided that it is true for all Φ of the form $\bigwedge_{1\le i\le r}\varphi_i$ where the images in $(\mathcal{O}_{K,V',T}^{\times})^*$ of the homomorphisms φ_i span a free module of rank r.

We shall therefore assume in the sequel that Φ is of this form.

For each index i we now choose a lift $\widetilde{\varphi}_i$ of φ_i to $\mathcal{S}_{S,T}(\mathbb{G}_{m/K})$ and then write $\mathcal{E}_\Phi$ for the G-module that is generated by the set $\{\widetilde{\varphi}_i : 1 \le i \le r\}$.

Proposition 7.7. *If* $\mathrm{LTC}(K/k)$ *is valid, then for every* Φ *as above one has*

$$\Phi(\epsilon_{K/k,S,T}^V)^\# \in \mathrm{Fitt}_G^0(\mathcal{S}_{S,T}(\mathbb{G}_{m/K})/\mathcal{E}_\Phi).$$

Proof. We use the existence of an exact triangle in $D^{\mathrm{p}}(\mathbb{Z}[G])$ of the form

$$\mathbb{Z}[G]^{\oplus r,\bullet} \xrightarrow{\theta} R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z}) \xrightarrow{\theta'} C^\bullet \to \mathbb{Z}[G]^{\oplus r,\bullet}[1]. \tag{29}$$

Here $\mathbb{Z}[G]^{\oplus r,\bullet}$ denotes the complex $\mathbb{Z}[G]^{\oplus r}[-1] \oplus \mathbb{Z}[G]^{\oplus r}[-2]$ and, after choosing an ordering $\{v_i : 1 \le i \le r\}$ of the places in V, the morphism θ is uniquely specified by the condition that $H^1(\theta)$ sends each element b_i of the canonical basis $\{b_i : 1 \le i \le r\}$ of $\mathbb{Z}[G]^{\oplus r}$ to w_i^* in $(Y_{K,V})^* \subset (X_{K,S})^* = H_{c,T}^1((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})$ and $H^2(\theta)$ sends each b_i to $\widetilde{\varphi}_i$ in $\mathcal{S}_{S,T}(\mathbb{G}_{m/K})$.

Note that the long exact cohomology sequence of this triangle implies $C^\bullet$ is acyclic outside degrees one and two and identifies $H^1(C^\bullet)$ and $H^2(C^\bullet)$ with $(X_{K,S\setminus V})^*$ and $\mathcal{S}_{S,T}(\mathbb{G}_{m/K})/\mathcal{E}_\Phi$, respectively.

In particular, if we now write e_r for the idempotent of $\mathbb{Q}[G]$ obtained as $\sum_{r_{\chi,S}=r} e_\chi$, then the space $e_r\mathbb{Q}H^i(C^\bullet)$ vanishes for both $i=1$ and $i=2$. We may therefore choose a commutative diagram of $\mathbb{R}[G]$-modules

(30)

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R}[G]^{\oplus r} \arrow[r, "H^1(\theta)"] \arrow[d, "\lambda_1"'] & \mathbb{R}H^1_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z}) \arrow[r, "H^1(\theta')"] \arrow[d, "\lambda_2"'] & \mathbb{R}H^1(C^\bullet) \arrow[r] \arrow[d, "\lambda_3"'] & 0 \\ 0 \arrow[r] & \mathbb{R}[G]^{\oplus r} \arrow[r, "H^2(\theta)"] & \mathbb{R}H^2_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z}) \arrow[r, "H^2(\theta')"] & \mathbb{R}H^2(C^\bullet) \arrow[r] & 0 \end{tikzcd}$$

such that $e_r\lambda_2 = e_r\lambda^*_{K,S}$.

This diagram combines with the triangle (29) to imply that there is an equality of lattices

$$\vartheta_{\lambda_2}(\mathrm{det}_G(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})))^{-1} = \det(\lambda_1)\cdot\vartheta_{\lambda_3}(\mathrm{det}_G(C^\bullet))^{-1}. \tag{31}$$

We now assume that the conjecture LTC(K/k) is valid. Then Proposition 3.4 implies that $\mathrm{det}_G(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z}))^{-1}$ is a free rank one $\mathbb{Z}[G]$-module and further that if we choose any basis ξ for this module, then both $e_r\xi$ and $e_r\theta^*_{K/k,S,T}(0)^{\#} = \theta^{(r),\#}_{K/k,S,T}$ are bases of the $e_r\mathbb{Z}[G]$-module

$$e_r\vartheta_{\lambda_2}(\mathrm{det}_G(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})))^{-1} = e_r\vartheta_{\lambda^*_{K,S}}(\mathrm{det}_G(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})))^{-1}.$$

Bass's Theorem (cf. [32, Chap. 7, (20.9)]) implies that for each prime p the projection map $\mathbb{Z}_{(p)}[G]^\times \to e_r\mathbb{Z}_{(p)}[G]^\times$ is surjective. The above equality thus implies that the $\mathbb{Z}_{(p)}[G]$-module $\vartheta_{\lambda_2}(\mathrm{det}_G(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})))^{-1}\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$ has a basis ξ_p for which one has $e_r\xi_p = e_r\theta^*_{K/k,S,T}(0)^{\#} = \theta^{(r),\#}_{K/k,S,T}$. For each prime p the equality (31) therefore implies that

$$\begin{aligned} &e_r\vartheta_{\lambda_3}(\mathrm{det}_G(C^\bullet))^{-1}\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)} \\ =&\det(\lambda_1)^{-1}e_r\vartheta_{\lambda_2}(\mathrm{det}_G(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})))^{-1}\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)} \\ =&\mathbb{Z}_{(p)}[G]\cdot e_r\det(\lambda_1)^{-1}\theta^{(r),\#}_{K/k,S,T}. \end{aligned} \tag{32}$$

Now the commutativity of (30) implies that $e_r\det(\lambda_1)$ is equal to the determinant of the matrix which represents $e_r\lambda^*_{K,S}$ with respect to the bases $\{e_rw^*_i : 1\le i\le r\}$ and $\{e_r\varphi_i : 1\le i\le r\}$ and hence that

$$e_r\bigwedge_{1\le i\le r}\lambda^*_{K,S}(w^*_i) = e_r\det(\lambda_1)\Phi.$$

Since the element $\epsilon^V_{K/k,S,T}$ is defined via the equality

$$\theta^{(r)}_{K/k,S,T}\bigwedge_{1\le i\le r}\lambda^{-1}_{K,S}(w_i-w) = \epsilon^V_{K/k,S,T},$$

one therefore has

$$
\begin{aligned}
&\Phi(\epsilon^V_{K/k,S,T})^{\#}\\
(33)\qquad &= (e_r\mathrm{det}(\lambda_1))^{-1}\bigwedge_{1\le i\le r}\lambda^*_{K,S}(w_i^*)(\theta^{(r),\#}_{K/k,S,T}(\bigwedge_{1\le i\le r}\lambda^{-1}_{K,S}(w_i-w)))\\
&= (e_r\mathrm{det}(\lambda_1))^{-1}\theta^{(r),\#}_{K/k,S,T}\\
&\in e_r\vartheta_{\lambda_3}(\mathrm{det}_G(C^\bullet))^{-1}\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}
\end{aligned}
$$

where the last containment follows from (32).
Now by the same reasoning as used in the proof of Lemma 2.8, we know that the p-localized complex $\mathbb{Z}_{(p)}\otimes C^\bullet$ is represented by a complex $P\xrightarrow{\delta}P$, where P is a finitely generated free $\mathbb{Z}_{(p)}[G]$-module and the first term is placed in degree one. In particular, since for any character χ of G the space $e_\chi\mathbb{C}H^1(C^\bullet)=e_\chi\mathbb{C}\ker(\delta)$ does not vanish if $e_\chi e_r=0$, one has

$$
\begin{aligned}
(34)\qquad e_r\vartheta_{\lambda_3}(\mathrm{det}_G(C^\bullet))^{-1}_{(p)} &= \mathbb{Z}_{(p)}[G]e_r\mathrm{det}(\delta)\\
&= \mathbb{Z}_{(p)}[G]\mathrm{det}(\delta)\\
&\subset \mathrm{Fitt}^0_{\mathbb{Z}_{(p)}[G]}((H^2_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})/\mathcal{E}_\Phi)\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)})\\
&= \mathrm{Fitt}^0_G(H^2_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})/\mathcal{E}_\Phi)\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}.
\end{aligned}
$$

The inclusion here follows from the tautological exact sequence

$$P\xrightarrow{\delta}P\longrightarrow H^2(\mathbb{Z}_{(p)}\otimes C^\bullet)\longrightarrow 0$$

and the identification $H^2(\mathbb{Z}_{(p)}\otimes C^\bullet)=(H^2_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}},\mathbb{Z})/\mathcal{E}_\Phi)\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$.
The claimed result now follows from (33) and (34). □

Now we proceed to the proof of Theorem 1.10. The existence of a surjective homomorphism of G-modules $f:\mathcal{S}_{S,T}(\mathbb{G}_{m/K})\to\mathcal{S}_{V'\cup S_\infty,T}(\mathbb{G}_{m/K})$ (see Proposition 2.4(ii)) combines with Proposition 7.7 to imply that

$$(35)\qquad \Phi(\epsilon^V_{K/k,S,T})^{\#}\in\mathrm{Fitt}^0_G(\mathcal{S}_{V'\cup S_\infty,T}(\mathbb{G}_{m/K})/f(\mathcal{E}_\Phi)).$$

This implies the first assertion of Theorem 1.10 since the natural map $\mathrm{Cl}^T_{V'}(K)^\vee\to\mathcal{S}_{V'\cup S_\infty,T}(\mathbb{G}_{m/K})$ induces an injection

$$\mathrm{Cl}^T_{V'}(K)^\vee\to\mathcal{S}_{V'\cup S_\infty,T}(\mathbb{G}_{m/K})/f(\mathcal{E}_\Phi).$$

In addition, if G is cyclic, then the latter injection combines with (35) to imply that

$$\Phi(\epsilon^V_{K/k,S,T})\in\mathrm{Fitt}^0_G(\mathrm{Cl}^T_{V'}(K)^\vee)^{\#}=\mathrm{Fitt}^0_G(\mathrm{Cl}^T_{V'}(K)),$$

as claimed by the second assertion of Theorem 1.10.
This completes the proof of Theorem 1.10.

7.5. The proof of Corollary 1.14. Let K/k be a CM-extension, $S = S_\infty(k)$, and p an odd prime. For a $\mathbb{Z}_p[G]$-module M, we denote by M^- the submodule on which the complex conjugation acts as -1.
Then, since complex conjugation acts trivially on $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}) \otimes \mathbb{Z}_p$, the exact sequence

$$0 \longrightarrow \mathrm{Cl}^T(K)^\vee \longrightarrow \mathcal{S}_{S,T}(\mathbb{G}_{m/K}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}) \longrightarrow 0,$$

implies that in this case there is an equality

$$((\mathrm{Cl}^T(K) \otimes \mathbb{Z}_p)^\vee)^- = (\mathcal{S}_{S,T}(\mathbb{G}_{m/K}) \otimes \mathbb{Z}_p)^-.$$

In addition, in this case the containment of Proposition 7.7 applies with V empty (so $r = 0$ and $\mathcal{E}_\Phi$ vanishes) to imply that

$$\theta_{K/k,S,T}(0)^\# \in \mathrm{Fitt}_G^0(\mathcal{S}_{S,T}(\mathbb{G}_{m/K})),$$

and hence one has

$$\theta_{K/k,S,T}(0)^\# \in \mathrm{Fitt}^0_{\mathbb{Z}_p[G]}(((\mathrm{Cl}^T(K) \otimes \mathbb{Z}_p)^\vee)^-).$$

Since $\theta_{K/k,S,T}(0)$ lies in the minus component of $\mathbb{Z}_p[G]$, this is in turn equivalent to the required containment

$$\theta_{K/k,S,T}(0)^\# \in \mathrm{Fitt}^0_{\mathbb{Z}_p[G]}((\mathrm{Cl}^T(K) \otimes \mathbb{Z}_p)^\vee).$$

This completes the proof of Corollary 1.14.

7.6. The higher relative Fitting ideals of the dual Selmer group. We write M_{tors} for the $\mathbb{Z}$-torsion submodule of a G-module M and abbreviate the higher relative Fitting ideal $\mathrm{Fitt}^{(a,b)}_{\mathbb{Z}[G]}(M, M_{\mathrm{tors}})$ to $\mathrm{Fitt}^{(a,b)}_G(M)$.
In this subsection, we study the ideals $\mathrm{Fitt}^{(r,i)}_G(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K}))$ and, in particular, prove Theorem 1.5(ii). We note that the exact sequence (2) identifies $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K})_{\mathrm{tors}}$ with the group $\mathrm{Cl}^T_S(K)$.
For each non-negative integer i we define the set $\mathcal{V}_i$ as in Theorem 1.5(ii).

Conjecture 7.8. *For each non-negative integer i one has an equality*

$$\begin{aligned}&\mathrm{Fitt}^{(r,i)}_G(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K}))\\ &= \{\Phi(\epsilon^{V\cup V'}_{K/k,S\cup V',T}) : V' \in \mathcal{V}_i \text{ and } \Phi \in {\bigwedge}_G^{r+i} \mathrm{Hom}_G(\mathcal{O}^\times_{K,S\cup V',T}, \mathbb{Z}[G])\}.\end{aligned}$$

The following result is a generalization of Theorem 7.5 in §7.3.

Theorem 7.9. *If* $\mathrm{LTC}(K/k)$ *is valid, then so is Conjecture 7.8.*

Proof. We consider the composition of the two canonical homomorphisms

$$\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K}) \to X_{K,S} \to Y_{K,V},$$

and denote its kernel by $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K})'$. By Lemma 7.2 (iii), we have

$$\mathrm{Fitt}^{(r,i)}_G(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K})) = \mathrm{Fitt}^{(0,i)}_G(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K})'). \tag{36}$$

We also note that the sequence (2) gives rise to an exact sequence of G-modules

$$0 \longrightarrow \mathrm{Cl}_S^T(K) \longrightarrow \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K})' \longrightarrow X_{K,S\setminus V} \longrightarrow 0. \tag{37}$$

For $V' \in \mathcal{V}_i$, we denote by $\mathcal{S}^{\mathrm{tr}}_{S\cup V',T}(\mathbb{G}_{m/K})'$ the kernel of the natural composition

$$\mathcal{S}^{\mathrm{tr}}_{S\cup V',T}(\mathbb{G}_{m/K}) \to X_{K,S\cup V'} \to Y_{K,V\cup V'}$$

so that the following sequence is exact

$$0 \longrightarrow \mathrm{Cl}_{S\cup V'}^T(K) \longrightarrow \mathcal{S}^{\mathrm{tr}}_{S\cup V',T}(\mathbb{G}_{m/K})' \longrightarrow X_{K,S\setminus V} \longrightarrow 0.$$

Let $X_{V'}$ be the subgroup of $\mathrm{Cl}_S^T(K)$ generated by the classes of places of K above V' in $\mathrm{Cl}_S^T(K)$. Since $\mathrm{Cl}_S^T(K)/X_{V'} = \mathrm{Cl}_{S\cup V'}^T(K)$, there is an isomorphism $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K})'/X_{V'} \simeq \mathcal{S}^{\mathrm{tr}}_{S\cup V',T}(\mathbb{G}_{m/K})'$. By Chebotarev density theorem and Lemma 7.1, we obtain

$$\begin{aligned} \mathrm{Fitt}_G^{(0,i)}(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K})') &= \sum_{V'\in\mathcal{V}'_i} \mathrm{Fitt}_G^0(\mathcal{S}^{\mathrm{tr}}_{S\cup V',T}(\mathbb{G}_{m/K})') \\ &= \sum_{V'\in\mathcal{V}'_i} \mathrm{Fitt}_G^{r+i}(\mathcal{S}^{\mathrm{tr}}_{S\cup V',T}(\mathbb{G}_{m/K})) \end{aligned} \tag{38}$$

where we used Lemma 7.2 (iii) again to get the last equality.
Now Theorem 7.9 follows from (36), (38) and Theorem 7.5. □

Corollary 7.10. *We assume that* LTC(K/k) *is valid and that the group* $G = \mathrm{Gal}(K/k)$ *is cyclic. Then for each non-negative integer i one has an equality*

$$\begin{aligned} &\mathrm{Fitt}_G^i(\mathrm{Cl}_S^T(K))\,\mathrm{Fitt}_G^0(X_{K,S\setminus V}) \\ &\quad = \{\Phi(\epsilon^{V\cup V'}_{K/k,S\cup V',T}) : V' \in \mathcal{V}_i \text{ and } \Phi \in {\textstyle\bigwedge_G^{r+i}} \mathrm{Hom}_G(\mathcal{O}^\times_{K,S\cup V',T}, \mathbb{Z}[G])\}. \end{aligned}$$

Proof. Since G is cyclic, the G-module $X_{K,S\setminus V}$ has a quadratic presentation. We may therefore apply Lemma 7.2(iv) to the exact sequence (37) to obtain an equality

$$\mathrm{Fitt}_G^i(\mathrm{Cl}_S^T(K))\,\mathrm{Fitt}_G^0(X_{K,S\setminus V}) = \mathrm{Fitt}_G^{(0,i)}(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K})').$$

Given this equality, the claimed result follows from Theorem 7.9 and the equality (36). □

An application of Theorem 7.9 to character components of ideal class groups will be given in §8.

8. Higher Fitting ideals of character components of class groups

In this section, as an application of Theorem 7.9, we study the higher Fitting ideals of character components of class groups.

8.1. General abelian extensions. We suppose that K/k is a finite abelian extension as in §7. We take and fix an odd prime p in this section. We put $A_S^T(K) = \mathrm{Cl}_S^T(K) \otimes \mathbb{Z}_p$, $A^T(K) = \mathrm{Cl}^T(K) \otimes \mathbb{Z}_p$, and $A(K) = \mathrm{Cl}(K) \otimes \mathbb{Z}_p$.
We take a character χ of $G = \mathrm{Gal}(K/k)$. Throughout this section, we assume that *the order of* χ *is prime to* p.
We decompose $G = \Delta_K \times \Gamma_K$ where $|\Delta_K|$ is prime to p and Γ_K is a p-group. By our assumption χ is regarded as a character of Δ_K. For any $\mathbb{Z}_p[\Delta_K]$-module M, we define the χ-component M^χ by setting

$$M^\chi := M \otimes_{\mathbb{Z}_p[\Delta_K]} \mathcal{O}_\chi$$

where $\mathcal{O}_\chi = \mathbb{Z}_p[\mathrm{im}(\chi)]$ on which Δ_K acts via χ. This is an exact functor from the category of $\mathbb{Z}_p[G]$-modules to that of $\mathcal{O}_\chi[\Gamma_K]$-modules.
Let k_χ be the subfield of K corresponding to the kernel of χ, namely, χ induces a faithful character of $\mathrm{Gal}(k_\chi/k)$. Put $K_{(\Delta)} := K^{\Gamma_K}$, then $k_\chi \subset K_{(\Delta)}$. We also put $\Delta_{K,\chi} := \mathrm{Gal}(K_{(\Delta)}/k_\chi)$ which is a subgroup of Δ_K. We consider $K(\chi) := K^{\Delta_{K,\chi}}$, then $\mathrm{Gal}(K(\chi)/k_\chi) = \Gamma_K$. We consider $A_S^T(K)^\chi$ which we regard as an $\mathcal{O}_\chi[\Gamma_K]$-module. By the standard norm argument, we know the natural map $A_S^T(K(\chi))^\chi \to A_S^T(K)^\chi$ is bijective, so when we consider the χ-component $A_S^T(K)^\chi$, we may assume that χ is a faithful character of Δ_K by replacing K with $K(\chi)$. In the following, we assume this. We write $\chi(v) \neq 1$ if the decomposition group of Δ_K at v is non-trivial.
We denote the χ-component of $\epsilon^V_{K/k,S,T}$ by $\epsilon^{V,\chi}_{K/k,S,T} \in ((\bigcap_G^r \mathcal{O}^\times_{K,S,T}) \otimes \mathbb{Z}_p)^\chi$.
Let $\mathcal{V}_i$ be the set as in Theorem 1.5(ii) for $i \geq 0$.
Finally we assume that the following condition is satisfied

(∗) any ramifying place v of k in K does not split completely in $K_{(\Delta)}$.

Theorem 8.1. *Let V be the set of the archimedean places of k that split completely in K and set $r := |V|$. We assume that $\chi \neq 1$ is a faithful character of Δ_K, and consider the χ-component of the class group $A^T(K)^\chi$ which is an $\mathcal{O}_\chi[\Gamma_K]$-module. We assume that the χ-component of* LTC(K/k) *is valid and that the condition (∗) is satisfied.*
Then for any non-negative integer i one has an equality

$$\mathrm{Fitt}^i_{\mathcal{O}_\chi[\Gamma_K]}(A^T(K)^\chi) = \{\Phi(\epsilon^{V\cup V',\chi}_{K/k,S\cup V',T}) : V' \in \mathcal{V}_i \text{ and } \Phi \in \bigwedge^{r+i}_{\mathcal{O}_\chi[\Gamma_K]} \mathcal{H}_\chi\}$$

where $S = S_\infty(k) \cup S_{\mathrm{ram}}(K/k)$ *and* $\mathcal{H}_\chi = \mathrm{Hom}_{\mathcal{O}_\chi[\Gamma_K]}((\mathcal{O}^\times_{K,S\cup V',T} \otimes \mathbb{Z}_p)^\chi, \mathcal{O}_\chi[\Gamma_K])$.

Proof. Since $v \in S_{\mathrm{ram}}(K/k)$ does not split completely in $K_{(\Delta)}$, one has $\chi(v) \neq 1$ and hence $(Y_{K,S_{\mathrm{ram}}} \otimes \mathbb{Z}_p)^\chi = 0$.
As $\chi \neq 1$, we therefore also have $(X_{K,S_{\mathrm{ram}}} \otimes \mathbb{Z}_p)^\chi = (Y_{K,S_{\mathrm{ram}}} \otimes \mathbb{Z}_p)^\chi = 0$. Hence $(X_{K,S} \otimes \mathbb{Z}_p)^\chi = (Y_{K,S_\infty} \otimes \mathbb{Z}_p)^\chi$ is isomorphic to $\mathcal{O}_\chi[\Gamma_K]^{\oplus r}$. This implies that

$$\mathrm{Fitt}^{(r,i)}_{\mathcal{O}_\chi[\Gamma_K]}((\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/K}) \otimes \mathbb{Z}_p)^\chi), A_S^T(K)^\chi) = \mathrm{Fitt}^i_{\mathcal{O}_\chi[\Gamma_K]}(A_S^T(K)^\chi)$$

and so the claim follows from Theorem 7.9. □

In the case $K = k_\chi$, the condition $(*)$ is automatically satisfied. We denote the group $A^T(k_\chi)^\chi$ by $(A^T)^\chi$, which is determined only by χ.

Corollary 8.2. *Let χ be a non-trivial linear character of k of order prime to p, and V the set of the archimedean places of k that split completely in k_χ and set $r := |V|$. We assume the χ-component of $\mathrm{LTC}(k_\chi/k)$ to be valid. Then for any non-negative integer i one has an equality*

$$\mathrm{Fitt}^i_{\mathcal{O}_\chi}((A^T)^\chi) = \{\Phi(\epsilon^{V\cup V',\chi}_{k_\chi/k,S\cup V',T}) : V' \in \mathcal{V}_i \text{ and } \Phi \in \bigwedge_{\mathcal{O}_\chi}^{r+i} \mathcal{H}_\chi\}$$

where $S = S_\infty(k) \cup S_{\mathrm{ram}}(k_\chi/k)$ and $\mathcal{H}_\chi = \mathrm{Hom}_{\mathcal{O}_\chi}((\mathcal{O}^\times_{k_\chi,S\cup V',T} \otimes \mathbb{Z}_p)^\chi, \mathcal{O}_\chi)$.

8.2. The order of character components in CM abelian extensions. In this subsection, we assume that k is totally real, K is a CM-field, and χ is an odd character. In this case, we can compute the right hand side of Theorem 8.1 more explicitly. First of all, note that $r = 0$ in this case.

We first consider the case $K = k_\chi$ and $i = 0$. When $S = S_\infty(k) \cup S_{\mathrm{ram}}(k_\chi/k)$, we denote the L-function $L_{k,S,T}(\chi^{-1}, s)$ by $L^T_k(\chi^{-1}, s)$. When T is empty, we denote $L^T_k(\chi^{-1}, s)$ by $L_k(\chi^{-1}, s)$. In this case, we know

$$\epsilon^{\emptyset,\chi}_{k_\chi/k,S,T} = \theta_{k_\chi/k,S,T}(0)^\chi = L^T_k(\chi^{-1}, 0)$$

(see §5.3). Therefore, Corollary 8.2 with $i = 0$ implies

Corollary 8.3. *Let k be totally real, and χ a one dimensional odd character of k of order prime to p. We assume the χ-component of $\mathrm{LTC}(k_\chi/k)$ to be valid.*

(i) *One has $|(A^T)^\chi| = |\mathcal{O}_\chi/L^T_k(\chi^{-1}, 0)|$.*

(ii) *Let $\mathrm{Cl}(k_\chi)$ be the ideal class group of k_χ, $A(k_\chi) = \mathrm{Cl}(k_\chi) \otimes \mathbb{Z}_p$, and $A^\chi = A(k_\chi)^\chi$. We denote by ω the Teichmüller character giving the Galois action on μ_p, the group of p-th roots of unity, and by $\mu_{p^\infty}(k(\mu_p))$ the group of roots of unity of p-power order in $k(\mu_p)$. Then one has*

$$|A^\chi| = \begin{cases} |\mathcal{O}_\chi/L_k(\chi^{-1}, 0)| & \text{if } \chi \neq \omega, \\ |\mathcal{O}_\chi/(|\mu_{p^\infty}(k(\mu_p))|L_k(\chi^{-1}, 0))| & \text{if } \chi = \omega. \end{cases}$$

Proof. Claim (i) is an immediate consequence of Corollary 8.2 and a remark before this corollary. We shall now prove claim (ii).

When $\chi \neq \omega$, we take a finite place v such that v is prime to p and $\mathrm{N}v \not\equiv \chi(\mathrm{Fr}_v) \pmod{p}$. We put $T = \{v\}$. Then $(A^T)^\chi = A^\chi$ and $\mathrm{ord}_p L^T_k(\chi^{-1}, 0) = \mathrm{ord}_p L_k(\chi^{-1}, 0)$. Therefore, claim (i) implies the equality in claim (ii).

When $\chi = \omega$, using Chebotarev density theorem we take a finite place v such that v splits completely in $k_\chi = k_\omega = k(\mu_p)$ and $\mathrm{ord}_p|\mu_{p^\infty}(k_\chi)| = \mathrm{ord}_p(\mathrm{N}v - 1)$. We take $T = \{v\}$, then we also have $(A^T)^\chi = A^\chi$ from the exact sequence

$$\mu_{p^\infty}(k_\chi) \longrightarrow (\bigoplus_{w|v} \kappa(w)^\times \otimes \mathbb{Z}_p)^\chi \longrightarrow (A^T)^\chi \longrightarrow A^\chi \longrightarrow 0$$

where w runs over all places of k_χ above v. Therefore, claim (ii) follows from claim (i) in this case, too. □

By combining the argument of Corollary 8.3 with the result of Theorem 1.16 one also directly obtains the following result.

Corollary 8.4. *Assume that at most one p-adic place $\mathfrak{p}$ of k satisfies $\chi(\mathfrak{p}) = 1$. Then the same conclusion as Corollary 8.3 holds.*

Remark 8.5. We note that the formula on A^χ in Corollary 8.3 has not yet been proved in general even in such a semi-simple case (namely the case that the order of χ is prime to p). If no p-adic place $\mathfrak{p}$ satisfies $\chi(\mathfrak{p}) = 1$, this is an immediate consequence of the main conjecture proved by Wiles [54]. Corollary 8.4 shows that this holds even if the set $\{\mathfrak{p} : p\text{-adic place of } k \text{ with } \chi(\mathfrak{p}) = 1\}$ has cardinality one.

8.3. The structure of the class group of a CM field. Now we consider a general CM-field K over a totally real number field k (in particular, we do not assume that $K = k_\chi$).

We assume the condition $(*)$ stated just prior to Theorem 8.1.

We fix a strictly positive integer N. Suppose that v is a place of k such that v is prime to p, v splits completely in K and there is a cyclic extension $F(v)/k$ of degree p^N, which is unramified outside v and in which v is totally ramified. (Note that $F(v)$ is not unique.) We denote by $\mathcal{S}(K)$ the set of such places v and recall that $\mathcal{S}(K)$ is infinite (see [30, Lem. 3.1]).

Suppose now that $V = \{v_1, \ldots, v_t\}$ is a subset of $\mathcal{S}(K)$ consisting of t distinct places. We take a cyclic extension $F(v_j)/k$ as above, and put $F = F(v_1)\cdots F(v_t)$ the compositum of fields $F(v_j)$. In particular, F is totally real. We denote by $\mathcal{F}_{t,N}$ the set of all fields F constructed in this way. When $t = 0$, we define $\mathcal{F}_{0,N} = \{k\}$.

We set

$$H := \mathrm{Gal}(KF/K) \cong \mathrm{Gal}(F/k) \cong \prod_{j=1}^{t} \mathrm{Gal}(F(v_j)/k),$$

where the first (restriction) isomorphism is due to the fact that $K \cap F = k$ and the second to the fact that each extension $F(v_j)/k$ is totally ramified at v_j and unramified at all other places.

We fix a generator σ_j of $\mathrm{Gal}(F(v_j)/k)$ and set $S_j := \sigma_j - 1 \in \mathbb{Z}[\mathrm{Gal}(KF/k)]$. Noting that $\mathrm{Gal}(KF/k) = G \times H$ where $G = \mathrm{Gal}(K/k)$, for each element x of $\mathbb{Z}[\mathrm{Gal}(KF/k)] = \mathbb{Z}[G][H]$ we write $x = \sum x_{n_1,\ldots,n_t} S_1^{n_1} \cdots S_t^{n_t}$ where each $x_{n_1,\ldots,n_t}$ belongs to $\mathbb{Z}[G]$. We then define a map

$$\varphi_V : \mathbb{Z}[\mathrm{Gal}(KF/k)] \to \mathbb{Z}/p^N[G]$$

by sending x to $x_{1,\ldots,1}$ modulo p^N and we note that this map is a well-defined homomorphism of G-modules.

We consider $\theta_{KF/k,S\cup V,T}(0) \in \mathbb{Z}[\mathrm{Gal}(KF/k)]$. We define $\Theta^i_{N,S,T}(K/k)$ to be the ideal of $\mathbb{Z}/p^N[G]$ generated by all $\varphi_V(\theta_{KF/k,S\cup V,T}(0)) \in \mathbb{Z}/p^N[G]$ where F

runs over $\mathcal{F}_{t,N}$ such that $t \le i$. We note that we can compute $\theta_{KF/k,S\cup V,T}(0)$, and hence also $\varphi_V(\theta_{KF/k,S\cup V,T}(0))$, numerically. Taking $F = k$, we know that $\theta_{K/k,S,T}(0) \bmod p^N$ is in $\Theta^i_{N,S,T}(K/k)$ for any $i \ge 0$.
We set $\mathcal{F}_N := \bigcup_{t\ge 0} \mathcal{F}_{t,N}$.
For any abelian extension M/k, if $S = S_\infty(k) \cup S_{\mathrm{ram}}(M/k)$ and T is the empty set, we write $\theta_{M/k}(0)$ for $\theta_{M/k,S,T}(0)$.
We take a character χ of Δ_K such that $\chi \neq \omega$, at first. We take $S = S_\infty(k) \cup S_{\mathrm{ram}}(K/k)$ and $T = \emptyset$. In this case, we know that the χ-component $\theta_{KF/k}(0)^\chi$ is integral, namely is in $O_\chi[\Gamma_K \times H]$. We simply denote the χ-component $\Theta^i_{N,S,\emptyset}(K/k)^\chi$ by $\Theta^i_N(K/k)^\chi$ ($\subset O_\chi[\Gamma_K]$). This ideal $\Theta^i_N(K/k)^\chi$ coincides with the higher Stickelberger ideal $\Theta^{(\delta,N),\chi}_{i,K}$ defined in [30, §8.1].
When $\chi = \omega$, we assume that $K = k(\mu_{p^m})$ for some $m \ge 1$. By using the Chebotarev density theorem we can choose a place v which satisfies all of the following conditions

(i) v splits completely in $k(\mu_p)/k$,
(ii) each place above v of $k(\mu_p)$ is inert in $K/k(\mu_p)$, and
(iii) each place w of K above v satisfies $\mathrm{ord}_p|\mu_{p^\infty}(K)| = \mathrm{ord}_p(\mathrm{N}w - 1)$.

We set $T := \{v\}$. We consider the ω-component $\Theta^i_{N,S,\{v\}}(K/k)^\omega$, which we denoted by $\Theta^i_{N,\{v\}}(K/k)^\omega$

Theorem 8.6. *Let K/k be a finite abelian extension, K a CM-field, and k totally real. Suppose that χ is an odd faithful character of Δ_K, and consider the χ-component of the class group $A(K)^\chi$ which is an $\mathcal{O}_\chi[\Gamma_K]$-module. We assume the condition $(*)$ stated just prior to Theorem 8.1 and the validity of the χ-component of* LTC(FK/k) *for every field F in $\mathcal{F}_N$.*

(i) *Suppose that $\chi \neq \omega$. For any integer $i \ge 0$, we have*
$$\mathrm{Fitt}^i_{\mathcal{O}_\chi[\Gamma_K]/p^N}(A(K)^\chi \otimes \mathbb{Z}/p^N) = \Theta^i_N(K/k)^\chi.$$
(ii) *We assume that $K = k(\mu_{p^m})$ for some $m \ge 1$. For $\chi = \omega$, using a place v as above, we have*
$$\mathrm{Fitt}^i_{\mathcal{O}_\omega[\Gamma_K]/p^N}(A(K)^\omega \otimes \mathbb{Z}/p^N) = \Theta^i_{N,\{v\}}(K/k)^\omega$$
for any $i \ge 0$.

Proof. We first prove claim (i). Since the image of $\theta_{KF/k}(0)$ in $\mathbb{Z}[G]$ is a multiple of $\theta_{K/k}(0)$, $\Theta^0_N(K/k)^\chi$ is a principal ideal generated by $\theta_{K/k}(0)^\chi$. Therefore, this theorem for $i = 0$ follows from Theorem 8.1.
Now suppose that $i > 0$. For a place $v \in \mathcal{S}(K)$, we take a place w of K above v. Put $H(v) = \mathrm{Gal}(F(v)K/K) = \mathrm{Gal}(F(v)/k) \simeq \mathbb{Z}/p^N$. We take a generator σ_v of $H(v)$ and fix it. We define ϕ_v by
$$\begin{array}{rcl}\phi_v : K^\times & \xrightarrow{\mathrm{Rec}_v} & (I(H(v))\mathbb{Z}[\mathrm{Gal}(F(v)K/k)]/I(H(v))^2\mathbb{Z}[\mathrm{Gal}(F(v)K/k)]) \\ & = & \mathbb{Z}[G] \otimes_{\mathbb{Z}} I(H(v))/I(H(v))^2 \simeq \mathbb{Z}/p^N[G]\end{array}$$

Here, the last isomorphism is defined by $\sigma_v - 1 \mapsto 1$, and Rec_v is defined by

$$\mathrm{Rec}_v(a) = \sum_{\tau \in G} \tau^{-1}(\mathrm{rec}_w(\tau a) - 1)$$

as in §5.3 by using the reciprocity map $\mathrm{rec}_w : K_w^\times \to H(v)$ at w. Taking the χ-component of ϕ_v, we obtain

$$\phi_v : (K^\times \otimes \mathbb{Z}/p^N)^\chi \longrightarrow \mathcal{O}_\chi[\Gamma_K]/p^N,$$

which we also denote by ϕ_v.

We take $S = S_\infty(k) \cup S_{\mathrm{ram}}(K/k)$, $T = \emptyset$, and $V = \{v_1, \ldots, v_i\} \in \mathcal{V}_i$. Suppose that $\Phi = \varphi_1 \wedge \cdots \wedge \varphi_i$ where

$$\varphi_j \in \mathcal{H}_\chi = \mathrm{Hom}_{\mathcal{O}_\chi[\Gamma_K]}((\mathcal{O}_{K,S\cup V}^\times \otimes \mathbb{Z}_p)^\chi, \mathcal{O}_\chi[\Gamma_K]/p^N)$$

for $j = 1, \ldots, i$. We take a place w_j of K above v_j for j such that $1 \leq j \leq i$. We denote by $[w_j]$ the class of w_j in $A(K)^\chi$.

By [30, Lem. 10.1], for each integer $j = 1,...,i$ we can choose a place $v'_j \in \mathcal{S}(K)$ that satisfies all of the following conditions;

(a) $[w'_j] = [w_j]$ in $A(K)^\chi$ where w'_j is a place of K above v'_j,

(b) $\varphi_j(x) = \phi_{v'_j}(x)$ for any $x \in (\mathcal{O}_{K,S\cup V}^\times \otimes \mathbb{Z}/p^N)^\chi$,

Here, we used the fact that the natural map $(\mathcal{O}_{K,S\cup V}^\times \otimes \mathbb{Z}/p^N)^\chi \to (K^\times \otimes \mathbb{Z}/p^N)^\chi$ is injective.

Set $V' = \{v'_1, \ldots, v'_i\}$. By property (b), we have

$$\Phi(\epsilon_{K/k,S\cup V,\emptyset}^{V,\chi}) = (\phi_{v'_1} \wedge \cdots \wedge \phi_{v'_i})(\epsilon_{K/k,S\cup V,\emptyset}^{V,\chi}).$$

By property (a), there exists an x_j in $\mathcal{O}_{K,S\cup V\cup V'}^\times$ whose prime decomposition is $(x_j) = w_j(w'_j)^{-1}$ for any j such that $1 \leq j \leq i$. Put $V_{i-1} = \{v_1, ..., v_{i-1}\}$ and $V'_i = \{v_1, ..., v_{i-1}, v'_i\}$. Then

$$\epsilon_{K/k,S\cup V,\emptyset}^{V,\chi} = \epsilon_{K/k,S\cup V'_i,\emptyset}^{V'_i,\chi} + \epsilon_{K/k,S\cup V_{i-1},\emptyset}^{V_{i-1},\chi} \wedge x_i$$

and by using this kind of equation recursively, one deduces that $\epsilon_{K/k,S\cup V,\emptyset}^{V,\chi} - \epsilon_{K/k,S\cup V',\emptyset}^{V',\chi}$ is a sum of elements of the form $\epsilon_{K/k,S\cup W,\emptyset}^{W,\chi} \wedge b_j$ with $|W| = i-1$. Now, by induction on i, we know $\Psi(\epsilon_{K/k,S\cup W,\emptyset}^{W,\chi})$ is in $\Theta_N^{i-1}(K/k)^\chi$ for any Ψ in $\mathrm{Hom}_{\mathcal{O}_\chi[\Gamma_K]}((\mathcal{O}_{K,S\cup W}^\times \otimes \mathbb{Z}_p)^\chi, \mathcal{O}_\chi[\Gamma_K]/p^N)$. Therefore we have

$$(\phi_{v'_1} \wedge \cdots \wedge \phi_{v'_i})(\epsilon_{K/k,S\cup V,\emptyset}^{V,\chi}) \equiv (\phi_{v'_1} \wedge \cdots \wedge \phi_{v'_i})(\epsilon_{K/k,S\cup V',\emptyset}^{V',\chi}) \pmod{\Theta_N^{i-1}(K/k)^\chi}.$$

Set $F = F(v'_1) \cdots F(v'_i)$ and $H = \mathrm{Gal}(FK/K) = \mathrm{Gal}(F/k)$. Then as in §5.3 we can define $\mathrm{Rec}_{V'}(\epsilon_{K/k,S\cup V',\emptyset}^{V',\chi}) \in \mathbb{Z}[G] \otimes (J_{V'})_H$. Let $\varphi_{V'} : \mathbb{Z}[G \times H] \to \mathbb{Z}/p^N[G]$ be the homomorphism defined before Theorem 8.6 by using the generators $\sigma_{v'_i}$ we fixed. This $\varphi_{V'}$ induces a homomorphism

$$\mathbb{Z}[G \times H]/I(H)^{i+1}\mathbb{Z}[G \times H] = \mathbb{Z}[G] \otimes \mathbb{Z}[H]/I(H)^{i+1} \to \mathbb{Z}/p^N[G]$$

and we also denote the composite homomorphism

$$\mathbb{Z}[G] \otimes (J_{V'})_H \to \mathbb{Z}[G] \otimes \mathbb{Z}[H]/I(H)^{i+1} \xrightarrow{\varphi_{V'}} \mathbb{Z}/p^N[G]$$

by $\varphi_{V'}$.
Then by the definitions of these homomorphisms, we have

$$(\phi_{v'_1} \wedge \cdots \wedge \phi_{v'_i})(\epsilon^{V',\chi}_{K/k,S\cup V',\emptyset}) = \varphi_{V'}(\mathrm{Rec}_{V'}(\epsilon^{V',\chi}_{K/k,S\cup V',\emptyset})).$$

By Conjecture 5.4 which is a theorem under our assumptions (Theorem 5.16), we get

$$\varphi_{V'}(\mathrm{Rec}_{V'}(\epsilon^{V',\chi}_{K/k,S\cup V',\emptyset})) = \varphi_{V'}(\theta_{KF/k}(0)^\chi).$$

Combining the above equations, we get

$$\Phi(\epsilon^{V,\chi}_{K/k,S\cup V,\emptyset}) \equiv \varphi_{V'}(\theta_{KF/k}(0)^\chi) \pmod{\Theta^{i-1}_N(K/k)^\chi}.$$

Since $\varphi_{V'}(\theta_{KF/k}(0)^\chi)$, $\Theta^{i-1}_N(K/k)^\chi$ are in $\Theta^i_N(K/k)^\chi$, we get $\Phi(\epsilon^{V,\chi}_{K/k,S\cup V,\emptyset}) \in \Theta^i_N(K/k)^\chi$. It follows from Theorem 8.1 that the left hand side of the equation in Theorem 8.6 (i) is in the right hand side.
On the other hand, suppose that F is in $\mathcal{F}_{t,N}$ with $t \le i$, and that $V = \{v_1, \ldots, v_t\}$ is the set of ramifying place in F/k. As above, by Theorem 5.16 we have

$$\varphi_V(\theta_{KF/k}(0)^\chi) = \varphi_V(\mathrm{Rec}_V(\epsilon^{V,\chi}_{K/k,S\cup V,\emptyset})) = (\phi_{v_1} \wedge \cdots \wedge \phi_{v_t})(\epsilon^{V,\chi}_{K/k,S\cup V,\emptyset}).$$

Therefore, by Theorem 8.1 we have

$$\varphi_V(\theta_{KF/k}(0)^\chi) \in \mathrm{Fitt}^t_{\mathcal{O}_\chi[\Gamma_K]/p^N}(A(K)^\chi \otimes \mathbb{Z}/p^N).$$

Since $\mathrm{Fitt}^t_{\mathcal{O}_\chi[\Gamma_K]/p^N}(A(K)^\chi \otimes \mathbb{Z}/p^N) \subset \mathrm{Fitt}^i_{\mathcal{O}_\chi[\Gamma_K]/p^N}(A(K)^\chi \otimes \mathbb{Z}/p^N)$, we get

$$\varphi_V(\theta_{KF/k}(0)^\chi) \in \mathrm{Fitt}^i_{\mathcal{O}_\chi[\Gamma_K]/p^N}(A(K)^\chi \otimes \mathbb{Z}/p^N).$$

Thus, the right hand side of the equation in Theorem 8.6 (i) is in the left hand side.
We can prove claim (ii) by the same method. The condition on v is used to show the injectivity of the natural homomorphism $(\mathcal{O}^\times_{K,S\cup V,T} \otimes \mathbb{Z}/p^N)^\omega \to (K^\times \otimes \mathbb{Z}/p^N)^\omega$ with $T = \{v\}$. □

Corollary 8.7. *Let K/k and χ be as in Theorem 8.6. We assume the condition $(*)$ stated just prior to Theorem 8.1 and that there is at most one place $\mathfrak{p}$ of k above p such that $\chi(\mathfrak{p}) = 1$. Then the same conclusion as in Theorem 8.6 holds.*

Proof. It suffices to note that, under the stated conditions, Theorem 1.16 implies that the χ-component of $\mathrm{LTC}(FK/k)$ is valid. □

To give an example of Corollary 8.7 we suppose that K is the m-th layer of the cyclotomic $\mathbb{Z}_p$-extension of $K_{(\Delta)}$ for some strictly positive integer m, and assume that $\chi(\mathfrak{p}) \neq 1$ for any $\mathfrak{p} \mid p$.
Then this assumption implies that the condition $(*)$ is satisfied and so all of the assumptions in Corollary 8.7 are satisfied. Therefore, by taking the projective limit of the conclusion, Corollary 8.7 implies the result of the second author in [30, Th. 2.1].
In this sense, Corollary 8.7 is a natural generalization of the main result in [30].

To state our final result we now set

$$\Theta^i(K/k)^\chi = \varprojlim_N \Theta^i_N(K/k)^\chi \subseteq \mathcal{O}_\chi[\Gamma_K].$$

Then Theorem 8.6 implies that $\mathrm{Fitt}^i_{\mathcal{O}_\chi[\Gamma]}(A(K)^\chi) = \Theta^i(K/k)^\chi$.

Let k_χ be the field corresponding to the kernel of χ as in Corollary 8.3. We denote $\Theta^i(k_\chi/k)^\chi$ by $\Theta^{i,\chi}$, which is an ideal of $\mathcal{O}_\chi$. For $\chi = \omega$, we denote $\varprojlim_N \Theta^i_{N,\{v\}}(k_\chi/k)^\omega$ by $\Theta^{i,\omega}$.

Then Corollary 8.7 implies the following result, which is a generalization of the main result of the second author in [28].

Corollary 8.8. *Set $A^\chi := (\mathrm{Cl}(k_\chi) \otimes \mathbb{Z}_p)^\chi$ as in Corollary 8.3. Assume that there is at most one p-adic place $\mathfrak{p}$ of k such that $\chi(\mathfrak{p}) = 1$ and that the p-adic Iwasawa μ-invariant of K vanishes.*
Then there is an isomorphism of $\mathcal{O}_\chi$-modules of the form $A^\chi \simeq \bigoplus_{i\geq 1} \Theta^{i,\chi}/\Theta^{i-1,\chi}$.

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