On Asymptotic Bounds for the Number of Irreducible Components of the Moduli Space of Surfaces of General Type II

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Abstract. In this paper we investigate the asymptotic growth of the number of irreducible and connected components of the moduli space of surfaces of general type corresponding to certain families of surfaces isogenous to a higher product with group $(\mathbb{Z}/2\mathbb{Z})^k$. We obtain a significantly higher growth than the one in our previous paper [LP14].

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1 Introduction

It is well known (see [Gie77]) that once two positive integers x, y are fixed there exists a quasi-projective coarse moduli space $\mathcal{M}_{y,x}$ of canonical models of surfaces of general type with $x = \chi(S) = \chi(\mathcal{O}_S)$ and $y = K_S^2$. The number $\iota(x, y)$, resp. $\gamma(x, y)$, of irreducible, resp. connected, components of $\mathcal{M}_{y,x}$ is bounded from above by a function of y. In fact, Catanese proved that the number $\iota^0(x, y)$ of components containing regular surfaces, i.e., $q(S) = 0$, has an exponential upper bound in K^2. More precisely [Cat92, p.592] gives the following inequality

$$\iota^0(x, y) \leq y^{77y^2}.$$

This result is not known to be sharp and in recent papers [M97, Ch96, GP14, LP14] inequalities are proved which tell how close one can get to this bound from below. In particular, in the last two papers the authors considered families of surfaces isogenous to a product in order to construct many irreducible components of the moduli space of surfaces of general type. The reason why one works with these surfaces, is the fact that the number of families of these surfaces can be easily computed using group theoretical and combinatorial methods.

In our previous work [LP14] we constructed many such families with many different 2−groups. There, we exploited the fact that the number of 2−groups with given order grows very fast in function of the order. In this paper we obtain a significantly better lower bound for $\iota^0(x,y)$ using only the groups $(\mathbb{Z}/2\mathbb{Z})^k$ and again some properties of the moduli space of surfaces isogenous to a product. Our main result is the following theorem.

Theorem 1.1. *Let $h = h_{k,l}$ be number of connected components of the moduli space of surfaces of general type which contain regular surfaces isogenous to a product of holomorphic Euler number $x_{k,l} = 2^{l-3}(k^2+k-4)$ given by a $(\mathbb{Z}/2\mathbb{Z})^k$ action with ramification structure of type $(2^{k(k+1)}, 2^{4+2^{l-k+1}})$.*
If k, l are positive integers with $l > 2k$, then

$$h \quad \geq \quad 2^{\sqrt[2+\nu]{x_k}} \qquad \textit{for} \quad k \to \infty,$$

where ν is the positive real number such that $l = (2+\nu)k$. In particular, given any sequence α_i which is positive, increasing and bounded by $\frac{1}{2}$ from above, we obtain increasing sequences x_i and $y_i = 8x_i$ with

$$\iota^0(x_i, y_i) \quad \geq \quad y_i^{(y_i^{\alpha_i})}.$$

Let us explain now the way in which this paper is organized.

In the next section *Preliminaries* we recall the definition and the properties of surfaces isogenous to a higher product and the its associated group theoretical data. Moreover, we recall a result of Bauer–Catanese [BC] which allows us to count the number of connected components of the moduli space of surfaces isogenous to a product with given group and type of ramification structure.

In the last section we give the proof of the Theorem 1.1.

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Notation and conventions. We work over the field $\mathbb{C}$ of complex numbers. By *surface* we mean a projective, non-singular surface S. For such a surface $\omega_S = \mathcal{O}_S(K_S)$ denotes the canonical bundle, $p_g(S) = h^0(S, \omega_S)$ is the *geometric genus*, $q(S) = h^1(S, \omega_S)$ is the *irregularity*, $\chi(\mathcal{O}_S) = \chi(S) = 1 - q(S) + p_g(S)$ is the *Euler-Poincaré characteristic* and $e(S)$ is the *topological Euler number* of S.

2 Preliminaries

Definition 2.1. *A surface S is said to be* isogenous to a higher product of curves *if and only if, S is a quotient $(C_1 \times C_2)/G$, where C_1 and C_2 are curves of genus at least two, and G is a finite group acting freely on $C_1 \times C_2$.*

Using the same notation as in Definition 2.1, let S be a surface isogenous to a product, and $G^\circ := G \cap (Aut(C_1) \times Aut(C_2))$. Then G° acts on the two factors C_1, C_2 and diagonally on the product $C_1 \times C_2$. If G° acts faithfully on both curves, we say that $S = (C_1 \times C_2)/G$ is a *minimal realization*. In [Cat00] it is also proven that any surface isogenous to a product admits a unique minimal realization.

Assumptions. In the following we always assume:

1. Any surface S isogenous to a product is given by its unique minimal realization;
2. $G^\circ = G$, this case is also known as *unmixed type*, see [Cat00].

Under these assumption we have.

Proposition 2.2. *[Cat00] Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a higher product of curves, then S is a minimal surface of general type with the following invariants:*

$$\chi(S) = \frac{(g(C_1) - 1)(g(C_2) - 1)}{|G|}, \quad e(S) = 4\chi(S), \quad K_S^2 = 8\chi(S). \tag{1}$$

The irregularity of these surfaces is computed by

$$q(S) = g(C_1/G) + g(C_2/G). \tag{2}$$

Among the nice features of surfaces isogenous to a product, one is that their deformation class can be obtained in a purely algebraic way. Let us briefly recall this in the particular case when S is regular, i.e., $q(S) = 0$, $C_i/G \cong \mathbb{P}^1$.

Definition 2.3. *Let G be a finite group and $r \in \mathbb{N}$ with $r \geq 2$.*

- *An $r-$tuple $T = (v_1, \dots, v_r)$ of elements of G is called a* spherical system of generators *of G if $\langle v_1, \dots, v_r \rangle = G$ and $v_1 \cdot \ldots \cdot v_r = 1$.*
- *We say that T has an* unordered type *$\tau := (m_1, \dots, m_r)$ if the orders of $(v_1, \dots, v_r)$ are $(m_1, \dots, m_r)$ up to a permutation, namely, if there is a permutation $\pi \in \mathfrak{S}_r$ such that*

$$\mathrm{ord}(v_1) = m_{\pi(1)}, \dots, \mathrm{ord}(v_r) = m_{\pi(r)}.$$

- *Moreover, two spherical systems* $T_1 = (v_{1,1}, \dots, v_{1,r_1})$ *and* $T_2 = (v_{2,1}, \dots, v_{2,r_2})$ *are said to be* disjoint*, if:*

$$\Sigma(T_1) \bigcap \Sigma(T_2) = \{1\}, \tag{3}$$

where

$$\Sigma(T_i) := \bigcup_{g \in G} \bigcup_{j=0}^{\infty} \bigcup_{k=1}^{r_i} g \cdot v_{i,k}^j \cdot g^{-1}.$$

We shall also use the shorthand, for example $(2^4, 3^2)$, to indicate the tuple $(2, 2, 2, 2, 3, 3)$.

DEFINITION 2.4. *Let* $2 < r_i \in \mathbb{N}$ *for* $i = 1, 2$ *and* $\tau_i = (m_{i,1}, \dots, m_{i,r_i})$ *be two sequences of natural numbers such that* $m_{k,i} \geq 2$. *A* (spherical-) ramification structure *of type* (τ_1, τ_2) *and size* (r_1, r_2) *for a finite group* G, *is a pair* (T_1, T_2) *of disjoint spherical systems of generators of* G, *whose types are* τ_i, *such that:*

$$\mathbb{Z} \ni \frac{|G|(-2 + \sum_{l=1}^{r_i}(1 - \frac{1}{m_{i,l}}))}{2} + 1 \geq 2, \qquad \text{for } i = 1, 2. \tag{4}$$

REMARK 2.5. Following e.g., the discussion in [LP14, Section 2] we obtain that the datum of the deformation class of a regular surface S isogenous to a higher product of curves of unmixed type together with its minimal realization $S = (C_1 \times C_2)/G$ is determined by the datum of a finite group G together with two disjoint spherical systems of generators T_1 and T_2 (for more details see also [BCG06]).

REMARK 2.6. Recall that from Riemann Existence Theorem a finite group G acts as a group of automorphisms of some curve C of genus g such that $C/G \cong \mathbb{P}^1$ if and only if there exist integers $m_r \geq m_{r-1} \geq \dots \geq m_1 \geq 2$ such that G has a spherical system of generators of type $(m_1, \dots, m_r)$ and the following Riemann-Hurwitz relation holds:

$$2g - 2 = |G|(-2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i})). \tag{5}$$

REMARK 2.7. Note that a group G and a ramification structure determine the main numerical invariants of the surface S. Indeed, by (1) and (5) we obtain:

$$\begin{aligned} 4\chi(S) &= |G| \cdot \left(-2 + \sum_{k=1}^{r_1}(1 - \frac{1}{m_{1,k}})\right) \cdot \left(-2 + \sum_{k=1}^{r_2}(1 - \frac{1}{m_{2,k}})\right) \\ &=: 4\chi(|G|, (\tau_1, \tau_2)). \end{aligned} \tag{6}$$

Let S be a surface isogenous to a product of unmixed type with group G and a pair of two disjoint spherical systems of generators of types (τ_1, τ_2). By (6) we have $\chi(S) = \chi(G, (\tau_1, \tau_2))$, and hence, by (1), $K_S^2 = K^2(G, (\tau_1, \tau_2)) = 8\chi(S)$.

Let us fix a group G and a pair of unmixed ramification types (τ_1, τ_2), and denote by $\mathcal{M}_{(G,(\tau_1,\tau_2))}$ the moduli space of isomorphism classes of surfaces isogenous to a product admitting these data, by [Cat00, Cat03] the space $\mathcal{M}_{(G,(\tau_1,\tau_2))}$ consists of a finite number of connected components. Indeed, there is a group theoretical procedure to count these components. In case G is abelian it is described in [BC].

Theorem 2.8. *[BC, Theorem 1.3] . Let S be a surface isogenous to a higher product of unmixed type and with $q = 0$. Then to S we attach its finite group G (up to isomorphism) and the equivalence classes of an unordered pair of disjoint spherical systems of generators (T_1, T_2) of G, under the equivalence relation generated by:*

i. Hurwitz equivalence for T_1;

ii. Hurwitz equivalence for T_2;

iii. Simultaneous conjugation for T_1 and T_2, i.e., for $\phi \in \mathrm{Aut}(G)$ we let $\big(T_1 := (v_{1,1}, \dots, v_{1,r_1}), \quad T_2 := (v_{2,1}, \dots, v_{2,r_2})\big)$ be equivalent to

$$\big(\phi(T_1) := (\phi(v_{1,1}), \dots, \phi(v_{1,r_1})), \quad \phi(T_2) := (\phi(v_{2,1}), \dots, \phi(v_{2,r_2}))\big).$$

Then two surfaces S, S' are deformation equivalent if and only if the corresponding equivalence classes of pairs of spherical generating systems of G are the same.

The Hurwitz equivalence is defined precisely in e.g., [P13]. In the cases that we will treat the Hurwitz equivalence is given only by the braid group action on T_i defined as follows. Recall the Artin presentation of the Braid group of r_1 strands

$$\mathbf{B}_{r_1} := \langle \gamma_1, \dots, \gamma_{r_1-1} |\, \gamma_i\gamma_j = \gamma_j\gamma_i \text{ for } |i-j| \geq 2,\, \gamma_{i+1}\gamma_i\gamma_{i+1} = \gamma_i\gamma_{i+1}\gamma_i\rangle.$$

For $\gamma_i \in \mathbf{B}_{r_1}$ then:

$$\gamma_i(T_1) = \gamma_i(v_1, \dots, v_{r_1}) = (v_1, \dots, v_{i+1}, v_{i+1}^{-1}v_iv_{i+1}, \dots, v_{r_1}).$$

Moreover, notice that, since we deal here with abelian groups only, the braid group action is indeed only by permutation of the elements on the spherical system of generators.

Once we fix a finite abelian group G and a pair of types (τ_1, τ_2) (of size (r_1,r_2)) of an unmixed ramification structure for G, counting the number of connected components of $\mathcal{M}_{(G,(\tau_1,\tau_2))}$ is then equivalent to the group theoretical problem of counting the number of classes of pairs of spherical systems of generators of G of type (τ_1, τ_2) under the equivalence relation given by the action of $\mathbf{B}_{r_1} \times \mathbf{B}_{r_2} \times \mathrm{Aut}(G)$, given by:

$$(\gamma_1, \gamma_2, \phi) \cdot (T_1, T_2) := \big(\phi(\gamma_1(T_1)), \phi(\gamma_2(T_2))\big), \tag{7}$$

where $\gamma_1 \in \mathbf{B}_{r_1}$, $\gamma_2 \in \mathbf{B}_{r_1}$ and $\phi \in \mathrm{Aut}(G)$, see for more details e.g., [P13].

3 Proof of Theorem 1.1

Let us consider the group $G := (\mathbb{Z}/2\mathbb{Z})^k$, with $k >> 0$ and an integer l. We want to give to G many classes of ramification structures of size $(r_1, r_2) = (k(k+1), 2^{l-k+1} + 4)$. Since the elements of G have only order two we will produce in the end ramification structure of type $((2^{r_1}), (2^{r_2}))$.
Let $T_1 := (v_1, v_2, \dots, v_{k(k+1)})$ with the following elements v_i of G

$$\begin{aligned} v_1 &= (1, 0, 0, \dots, 0) \\ v_2 &= (1, 0, 0, \dots, 0) \\ v_3 &= (0, 1, 0, \dots, 0) \\ v_4 &= (0, 1, 0, \dots, 0) \\ v_5 &= (0, 1, 0, \dots, 0) \\ v_6 &= (0, 1, 0, \dots, 0) \\ &\vdots \\ v_{k(k+1)} &= (0, 0, \dots, 0, 1) \end{aligned}$$

By construction the product of the elements in T_1 is 1_G and $< T_1 >\cong G$. Define the set $M := G \setminus \{0, v_1, \dots, v_{k(k+1)}\}$. Since M has $\#M = 2^k - k - 1$ elements we can choose a bijection

$$\varphi : \quad \{n \in \mathbb{N} | n \leq 2^k - k - 1\} \longrightarrow M.$$

Let B the set of $(2^k - k - 1)$-tuples $(n_1, \dots, n_{\#M})$ of positive integers of sum $n_1 + n_2 + \cdots + n_{\#M}$ equal to $2^{l-k} + 2$. For every element in B we get a $2^{l-k+1} + 4$-tuple T_2 of elements of G by the map

$$(n_1, \dots, n_{2^k-k-1}) \quad \mapsto \quad T_2$$

where

$$T_2 \quad = \quad \big(\underbrace{\varphi(1), \dots, \varphi(1)}_{2n_1}, \dots, \underbrace{\varphi(2^k - k - 1), \dots, \varphi(2^k - k - 1)}_{2n_{2^k-k-1}}\big).$$

By construction again $< T_2 >\cong G$ (for $k > 2$) and the product of the elements in T_2 is 1_G. Hence, T_2 is a spherical system of generators for G of size $2^{l-k+1}+4$. Since $G = \mathbb{Z}/2^k$ is abelian with all non-trivial elements of order two, the pair (T_1, T_2) is a ramification structure for G of the desired type, for any element in B.

Now we count how many inequivalent ramification structures of this kind we have under the action of the group defined in Theorem 2.8 and Equation (7). First, by construction, the only element in $\mathrm{Aut}(G)$, which stabilises T_1 is the identity. Next, accordingly, (T_1, T_2') and (T_1, T_2'') are equivalent if and only

if T_2' and T_2'' are in the same braid orbit. So finally, by construction, we get inequivalent pairs associated to different elements of B.
Hence the number of inequivalent such ramification structures is equal to the number of (2^k-k-1)-tuples in B of positive integers whose sum is $2^{l-k}+2$ $= r_2/2$.
Any element in B corresponds uniquely to the sequence of 2^k-k-1 integers $n_1, n_1+n_2, \dots, n_1+\cdots+n_{\#M} = r_2/2$. The sequence is strongly increasing by $n_i > 0$, so the elements in B correspond bijectively to the number of choices of 2^k-k-2 integers (all but the last integer in the sequence) in the range from 1 to $r_2/2-1$, cf [F68, Section II.5, p.38]. Hence,

$$\#B \quad = \quad \binom{\frac{r_2}{2}-1}{2^k-k-2} \quad = \quad \binom{2^{l-k}+1}{2^k-k-2},$$

Let $\nu > 0$ be a rational number and let us suppose that $l = (\nu+2)\cdot k$, then we exploit Stirling's approximation of the binomial coefficient - more exactly a corresponding lower bound. To bound the binomial coefficient $\binom{m}{n}$ we use the lower bound $(m-n)^n < m!/(m-n)!$ and an upper bound on $n!$ related to Stirling's formula. In fact, cf. [F68, eqn. (9.6), p.53],

$$n! \quad = \quad e^{d_n} n^{n+\frac{1}{2}} e^{-n} \qquad \text{for} \quad d_n \quad = \quad \ln n! - (n+\frac{1}{2})\ln n + n$$

and we get an upper bound on replacing d_n by $d_1 = 1$ due to the observation [F68, eqn. (9.9), p.53] that (d_n) is decreasing. Thus we obtain

$$\begin{aligned}\binom{2^{l-k}+1}{2^k-k-2} \quad &> \quad \frac{(\frac{2^{l-k}+1}{2^k-k-2}-1)^{2^k-k-2} e^{2^k-k-2}}{e\sqrt{(2^k-k-2)}} \\ &> \quad (2^{\nu k})^{(2^k-k-2)} \cdot \frac{e^{2^k-k-2}}{e\sqrt{(2^k-k-2)}} \quad > \quad 2^{\nu k (2^k)}. \qquad (8)\end{aligned}$$

Now $y_k = 8\chi(S) = 2|G|(-2+\frac{1}{2}r_1)(-2+\frac{1}{2}r_2)$ by (6) implies $y_k = 2^k \cdot 2^{l-k} \cdot (k^2+k-4) = 2^{(\nu+2)k}(k^2+k-4)$ according to $l = (2+\nu)k$. Hence we have

$$(y_k)^{\frac{1}{\nu+2}} \cdot \frac{k}{(k^2+k-4)^{\frac{1}{\nu+2}}} \quad = \quad k2^k.$$

Using this, we obtain for k large enough in the second inequality

$$h > 2^{\nu (y_k)^{\frac{1}{\nu+2}} \cdot \frac{k}{(k^2+k-4)^{\frac{1}{\nu+2}}}} \quad > \quad 2^{(y_k^{\frac{1}{\nu+2}})}. \qquad (9)$$

We can bound further for k large enough

$$2^{(y_k^{\frac{1}{\nu+2}})} \quad > \quad 2^{(y_k^{\frac{1}{2\nu+2}})\frac{\ln y_k}{\ln 2}}. \qquad (10)$$

We use the identity $x^{f(x)} = e^{f(x)\ln x} = 2^{f(x)\frac{1}{\ln 2}\ln x}$ to get for all $\alpha < \frac{1}{2}$

$$h > y_k^{(y_k)^\alpha}$$

if k is large enough, depending on α. □

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