Milne's Correcting Factor and Derived De Rham Cohomology

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Received: June 15, 2015
Revised: November 13, 2015

Communicated by Stephen Lichtenbaum

Abstract. Milne's correcting factor is a numerical invariant playing an important role in formulas for special values of zeta functions of varieties over finite fields. We show that Milne's factor is simply the Euler characteristic of the derived de Rham complex (relative to $\mathbb{Z}$) modulo the Hodge filtration.

2010 Mathematics Subject Classification: 14G10, 14F40, 11S40, 11G25
Keywords and Phrases: Zeta functions, Special values, Derived de Rham cohomology

A result of Milne ([9] Theorem 0.1) describes the special values of the zeta function of a smooth projective variety X over a finite field satisfying the Tate conjecture. A very natural reformulation of this result was given by Lichtenbaum and Geisser (see [2], [7], [8] and [10]) using Weil-étale cohomology of motivic complexes. They conjecture that

$$\lim_{t\to q^{-n}} Z(X,t)\cdot(1-q^n t)^{\rho_n} = \pm\chi(H^*_W(X,\mathbb{Z}(n)),\cup e)\cdot q^{\chi(X/\mathbb{F}_q,\mathcal{O}_X,n)} \tag{1}$$

and show that (1) holds whenever the groups $H^i_W(X,\mathbb{Z}(n))$ are finitely generated. Here $H^*_W(X,\mathbb{Z}(n))$ denotes Weil-étale motivic cohomology, $e \in H^1(W_{\mathbb{F}_q},\mathbb{Z})$ is a fundamental class and $\chi(H^*_W(X,\mathbb{Z}(n)),e)$ is the Euler characteristic of the complex

$$\cdots \xrightarrow{\cup e} H^i_W(X,\mathbb{Z}(n)) \xrightarrow{\cup e} H^{i+1}_W(X,\mathbb{Z}(n)) \xrightarrow{\cup e} \cdots \tag{2}$$

More precisely, the cohomology groups of the complex (2) are finite and $\chi(H^*_W(X,\mathbb{Z}(n)),\cup e)$ is the alternating product of their orders. Finally, Milne's correcting factor $q^{\chi(X/\mathbb{F}_q,\mathcal{O},n)}$ was defined in [9] by the formula

$$\chi(X/\mathbb{F}_q,\mathcal{O}_X,n)=\sum_{i\le n,j}(-1)^{i+j}\cdot(n-i)\cdot\dim_{\mathbb{F}_q}H^j(X,\Omega^i_{X/\mathbb{F}_q}).$$

The author was supported by ANR-12-BS01-0002 and ANR-12-JS01-0007.

It is possible to generalize (1) in order to give a conjectural description of special values of zeta functions of all separated schemes of finite type over $\mathbb{F}_q$ (see [3] Conjecture 1.4), and even of all motivic complexes over $\mathbb{F}_q$ (see [11] Conjecture 1.2). The statement of those more general conjectures is in any case very similar to formula (1). The present note is motivated by the hope for a further generalization, which would apply to zeta functions of all algebraic schemes over Spec($\mathbb{Z}$). As briefly explained below, the special-value conjecture for (flat) schemes over Spec($\mathbb{Z}$) must take a rather different form than formula (1). Going back to the special case of smooth projective varieties over finite fields, this leads to a slightly different restatement of formula (1).
Let $\mathcal{X}$ be a regular scheme proper over Spec($\mathbb{Z}$). The "fundamental line"

$$\Delta(\mathcal{X}/\mathbb{Z},n) := \det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n$$

should be a well defined invertible $\mathbb{Z}$-module endowed with a canonical trivialization

$$\mathbb{R} \xrightarrow{\sim} \Delta(\mathcal{X}/\mathbb{Z},n) \otimes_{\mathbb{Z}} \mathbb{R}.$$

involving a fundamental class $\theta \in H^1(\mathbb{R},\mathbb{R}) =$ "$H^1(W_{\mathbb{F}_1},\mathbb{R})$" analogous to $e \in H^1(W_{\mathbb{F}_q},\mathbb{Z})$. Here $R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n))$ denotes Weil-étale cohomology with compact support. However, there is no natural trivialization $\mathbb{R} \xrightarrow{\sim} \det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{R}$. Consequently, it is not possible to define an Euler characteristic generalizing $\chi(H^*_W(X,\mathbb{Z}(n)),\cup e)$, neither to define a correcting factor generalizing Milne's correcting factor: one is forced to consider the fundamental line as a whole. Let us go back to the case of smooth projective varieties $X/\mathbb{F}_q$, which we now see as schemes over $\mathbb{Z}$. Accordingly, we replace $Z(X,t)$ with $\zeta(X,s) = Z(X,q^{-s})$, the fundamental class e with θ and the cotangent sheaf $\Omega^1_{X/\mathbb{F}_q} \simeq L_{X/\mathbb{F}_q}$ with the cotangent complex $L_{X/\mathbb{Z}}$. Assuming that $H^i_W(X,\mathbb{Z}(n))$ is finitely generated for all i, the fundamental line

$$\Delta(X/\mathbb{Z},n) := \det_{\mathbb{Z}} R\Gamma_W(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} R\Gamma(X, L\Omega^*_{X/\mathbb{Z}}/F^n) \tag{3}$$

is well defined and cup-product with θ gives a trivialization

$$\lambda : \mathbb{R} \xrightarrow{\sim} \Delta(X/\mathbb{Z},n) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Here $L\Omega^*_{X/\mathbb{Z}}/F^n$ is Illusie's derived de Rham complex modulo the Hodge filtration (see [6] VIII.2.1). The aim of this note is to show that the Euler characteristic of $R\Gamma(X, L\Omega^*_{X/\mathbb{Z}}/F^n)$ equals $q^{\chi(X/\mathbb{F}_q,\mathcal{O}_X,n)}$, hence that Milne's correcting factor is naturally part of the fundamental line. We denote by $\zeta^*(X,n)$ the leading coefficient in the Taylor development of $\zeta(X,s)$ near $s=n$.

Theorem. *Let X be a smooth proper scheme over $\mathbb{F}_q$ and let $n \in \mathbb{Z}$ be an integer. Then we have*

$$\prod_{i\in\mathbb{Z}} |\, H^i(X, L\Omega^*_{X/\mathbb{Z}}/F^n)\,|^{(-1)^i} \;=\; q^{\chi(X/\mathbb{F}_q,\mathcal{O}_X,n)}.$$

Assume moreover that X is projective and that the groups $H^i_W(X,\mathbb{Z}(n))$ are finitely generated for all i. Then one has

$$\begin{aligned}\Delta(X/\mathbb{Z},n) &= \mathbb{Z}\cdot\lambda\left(\log(q)^{\rho_n}\cdot\chi(H^*_W(X,\mathbb{Z}(n)),\cup e)^{-1}\cdot q^{-\chi(X/\mathbb{F}_q,\mathcal{O}_X,n)}\right)\\ &= \mathbb{Z}\cdot\lambda\left(\zeta^*(X,n)^{-1}\right)\end{aligned}$$

where $\rho_n := -\mathrm{ord}_{s=n}\zeta(X,s)$ is the order of the pole of $\zeta(X,s)$ at $s=n$.

Before giving the proof, we need to fix some notations. For an object C in the derived category of abelian groups such that $H^i(C)$ is finitely generated for all i and $H^i(C)=0$ for almost all i, we set

$$\det\nolimits_{\mathbb{Z}}(C) := \bigotimes_{i\in\mathbb{Z}}\det\nolimits_{\mathbb{Z}}^{(-1)^i}H^i(C).$$

If $H^i(C)$ is moreover finite for all i, then we call the following isomorphism

$$\det\nolimits_{\mathbb{Z}}(C)\otimes_{\mathbb{Z}}\mathbb{Q}\xrightarrow{\sim}\bigotimes_{i\in\mathbb{Z}}\det\nolimits_{\mathbb{Q}}^{(-1)^i}\left(H^i(C)\otimes_{\mathbb{Z}}\mathbb{Q}\right)\xrightarrow{\sim}\bigotimes_{i\in\mathbb{Z}}\det\nolimits_{\mathbb{Q}}^{(-1)^i}(0)\xrightarrow{\sim}\mathbb{Q}$$

the *canonical* $\mathbb{Q}$*-trivialization* of $\det_{\mathbb{Z}}(C)$. Let A be a finite abelian group, which we see as a complex concentrated in degree 0. Then the canonical $\mathbb{Q}$-trivialization $\det_{\mathbb{Z}}(A)\otimes_{\mathbb{Z}}\mathbb{Q}\simeq\mathbb{Q}$ identifies $\det_{\mathbb{Z}}(A)$ with $|A|^{-1}\cdot\mathbb{Z}\subset\mathbb{Q}$, where $|A|$ denotes the order of A.
Given a ring R and an R-module M, we denote by $\Gamma_R(M)$ the universal divided power R-algebra of M, and by $\Gamma^i_R(M)$ its submodule of homogeneous elements of degree i. We refer to ([1] Appendix A) for the definition of $\Gamma_R(M)$ and its main properties. There is a canonical map $\gamma^i : M\to\Gamma^i_R(M)$, such that composition with γ^i induces a bijection $\mathrm{Hom}_R(\Gamma^i_R(M),N)\xrightarrow{\sim}P^i(M,N)$, where $P^i(M,N)$ is the set of "homogeneous polynomial functions of degree i". The functor Γ^i_R sends free modules of finite type to free modules of finite type. Moreover Γ^i_R commutes with filtered colimits, hence sends flat modules to flat modules. If M is free of rank one, then so is $\Gamma^i_R(M)$. If (T,R) is a ringed topos and M an R-module, then $\Gamma_R(M)$ is the sheafification of $U\mapsto\Gamma_{R(U)}(M(U))$. We also denote by Λ^i_R the (non-additive) exterior power functor and by $L\Lambda^i_R$ its left derived functor (see [5] I.4.2). We often omit the subscript R and simply write Γ^iM, Λ^iM and $L\Lambda^iM$.
Let X be a scheme. The notation $R\Gamma(X,-)$ refers to hypercohomology with respect to the Zariski topology.

Proof. Since Milne's correcting factor is insensitive to restriction of scalars (i.e. $q^{\chi(X/\mathbb{F}_q,\mathcal{O}_X,n)}=p^{\chi(X/\mathbb{F}_p,\mathcal{O}_X,n)}$), we may consider X over $\mathbb{F}_p$. We need the following

Lemma 1. *Let $E^{*,*}_* = (E^{p,q}_r, d^{p,q}_r)^{p,q}_r$ be a cohomological spectral sequence of abelian groups with abutment H^*. Assume that there exists an index r_0 such that $E^{p,q}_{r_0}$ is finite for all $(p,q)\in\mathbb{Z}^2$ and $E^{p,q}_{r_0}=0$ for all but finitely many*

(p,q). Then we have a canonical isomorphism

$$\iota : \bigotimes_{p,q} \det_{\mathbb{Z}}^{(-1)^{p+q}} E_{r_0}^{p,q} \xrightarrow{\sim} \bigotimes_{n} \det_{\mathbb{Z}}^{(-1)^{n}} H^n$$

such that the square of isomorphisms

$$\begin{array}{ccc} \left(\bigotimes_{p,q} \det_{\mathbb{Z}}^{(-1)^{p+q}} E_{r_0}^{p,q}\right) \otimes \mathbb{Q} & \xrightarrow{\iota\otimes\mathbb{Q}} & \left(\bigotimes_{n} \det_{\mathbb{Z}}^{(-1)^{n}} H^n\right) \otimes \mathbb{Q} \\ \downarrow & & \downarrow \\ \mathbb{Q} & \xrightarrow{\mathrm{Id}} & \mathbb{Q} \end{array}$$

commutes, where the vertical maps are the canonical $\mathbb{Q}$-trivializations.

Proof. For any $t \geq r_0$, consider the bounded cochain complex C_t^* of finite abelian groups:

$$\cdots \longrightarrow \bigoplus_{p+q=n-1} E_t^{p,q} \longrightarrow \bigoplus_{p+q=n} E_t^{p,q} \xrightarrow{\oplus d_t^{p,q}} \bigoplus_{p+q=n+1} E_t^{p,q} \longrightarrow \cdots$$

The fact that the cohomology of C_t^* is given by $H^n(C_t^*) = \bigoplus_{p+q=n} E_{t+1}^{p,q}$ gives an isomorphism

$$\bigotimes_{p,q} \det_{\mathbb{Z}}^{(-1)^{p+q}} E_t^{p,q} \xrightarrow{\sim} \bigotimes_{p,q} \det_{\mathbb{Z}}^{(-1)^{p+q}} E_{t+1}^{p,q}$$

compatible with the canonical $\mathbb{Q}$-trivializations. By assumption, there exists an index $r_1 \geq r_0$ such that the spectral sequence degenerates at the r_1-page, i.e. $E_{r_1}^{*,*} = E_{\infty}^{*,*}$. The induced filtration on each H^n is such that $\mathrm{gr}^p H^n = E_{\infty}^{p,n-p}$. We obtain isomorphisms

$$\bigotimes_{p,q} \det_{\mathbb{Z}}^{(-1)^{p+q}} E_{r_0}^{p,q} \xrightarrow{\sim} \bigotimes_{p,q} \det_{\mathbb{Z}}^{(-1)^{p+q}} E_{\infty}^{p,q} \xrightarrow{\sim}$$
$$\xrightarrow{\sim} \bigotimes_{n} \bigotimes_{p} \det_{\mathbb{Z}}^{(-1)^{n}} E_{\infty}^{p,n-p} \xrightarrow{\sim} \bigotimes_{n} \det_{\mathbb{Z}}^{(-1)^{n}} H^n$$

compatible with the canonical $\mathbb{Q}$-trivializations. $\square$

Consider the Hodge filtration F^* on the derived de Rham complex $L\Omega^*_{X/\mathbb{Z}}$. By ([6] VIII.2.1.1.5) we have

$$\mathrm{gr}(L\Omega^*_{X/\mathbb{Z}}) \simeq \bigoplus_{p\geq 0} L\Lambda^p L_{X/\mathbb{Z}}[-p].$$

This gives a (convergent) spectral sequence

$$E_1^{p,q} = H^q(X, L\Lambda^{p<n} L_{X/\mathbb{Z}}) \Longrightarrow H^{p+q}(X, L\Omega^*_{X/\mathbb{Z}}/F^n)$$

where $L\Lambda^{p<n} L_{X/\mathbb{Z}} := L\Lambda^{p} L_{X/\mathbb{Z}}$ for $p<n$ and $L\Lambda^{p<n} L_{X/\mathbb{Z}} := 0$ otherwise. The scheme X is proper and $L\Lambda^{p} L_{X/\mathbb{Z}}$ is isomorphic, in the derived category $\mathcal{D}(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules, to a bounded complex of coherent sheaves (see (6) below). It

follows that $E_1^{p,q}$ is a finite dimensional $\mathbb{F}_p$-vector space for all (p,q) vanishing for almost all (p,q). By Lemma 1, this yields isomorphisms

$$\begin{aligned}\det_{\mathbb{Z}} R\Gamma(X, L\Omega^*_{X/\mathbb{Z}}/F^n) &\xrightarrow{\sim} \bigotimes_i \det_{\mathbb{Z}}^{(-1)^i} H^i(X, L\Omega^*_{X/\mathbb{Z}}/F^n)\\ &\xrightarrow{\sim} \bigotimes_{p<n,q} \det_{\mathbb{Z}}^{(-1)^{p+q}} H^q(X, L\Lambda^p L_{X/\mathbb{Z}})\\ &\xrightarrow{\sim} \bigotimes_{p<n} \det_{\mathbb{Z}}^{(-1)^p} R\Gamma(X, L\Lambda^p L_{X/\mathbb{Z}})\end{aligned}$$

which are compatible with the canonical $\mathbb{Q}$-trivializations. The transitivity triangle (see [5] II.2.1) for the composite map $X \xrightarrow{f} \mathrm{Spec}(\mathbb{F}_p) \to \mathrm{Spec}(\mathbb{Z})$ reads as follows (using [5] III.3.1.2 and [5] III.3.2.4(iii)):

$$Lf^*(p\mathbb{Z}/p^2\mathbb{Z})[1] \to L_{X/\mathbb{Z}} \to \Omega^1_{X/\mathbb{F}_p}[0] \xrightarrow{\omega} Lf^*(p\mathbb{Z}/p^2\mathbb{Z})[2]. \tag{4}$$

We set $\mathcal{L} := Lf^*(p\mathbb{Z}/p^2\mathbb{Z})$, a trivial invertible $\mathcal{O}_X$-module. By ([5] Théorème III.2.1.7), the class

$$\omega \in \mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_{X/\mathbb{F}_p}, \mathcal{L}) \simeq H^2(X, T_{X/\mathbb{F}_p})$$

is the obstruction to the existence of a lifting of X over $\mathbb{Z}/p^2\mathbb{Z}$. If such a lifting does exist then we have $\omega = 0$, in which case the following lemma is superfluous. For an object C of $\mathcal{D}(\mathcal{O}_X)$ with bounded cohomology, we set

$$\mathrm{gr}_\tau C := \bigoplus_{i\in\mathbb{Z}} H^i(C)[-i].$$

Lemma 2. *We have an isomorphism*

$$\det_{\mathbb{Z}} R\Gamma(X, L\Lambda^p L_{X/\mathbb{Z}}) \simeq \det_{\mathbb{Z}} R\Gamma(X, L\Lambda^p(\mathrm{gr}_\tau L_{X/\mathbb{Z}}))$$

compatible with the canonical $\mathbb{Q}$-trivializations.

Proof. The map $X \to \mathrm{Spec}(\mathbb{Z})$ is a local complete intersection, hence the complex $L_{X/\mathbb{Z}}$ has perfect amplitude $\subset [-1,0]$ (see [5] III.3.2.6). In other words, $L_{X/\mathbb{Z}}$ is locally isomorphic in $\mathcal{D}(\mathcal{O}_X)$ to a complex of free modules of finite type concentrated in degrees -1 and 0. By ([4] 2.2.7.1) and ([4] 2.2.8), $L_{X/\mathbb{Z}}$ is globally isomorphic to such a complex, i.e. there exists an isomorphism $L_{X/\mathbb{Z}} \simeq [M \to N]$ in $\mathcal{D}(\mathcal{O}_X)$, where M and N are finitely generated locally free $\mathcal{O}_X$-modules put in degrees -1 and 0 respectively. Consider the exact sequences

$$0 \to \mathcal{L} \to M \to F \to 0 \text{ and } 0 \to F \to N \to \Omega \to 0 \tag{5}$$

where $\mathcal{L} := Lf^*(p\mathbb{Z}/p^2\mathbb{Z})$ and $\Omega := \Omega^1_{X/\mathbb{F}_p}$ are finitely generated and locally free. It follows that F is also finitely generated and locally free. One has an isomorphism in $\mathcal{D}(\mathcal{O}_X)$

$$L\Lambda^p L_{X/\mathbb{Z}} \simeq [\Gamma^p M \to \Gamma^{p-1} M \otimes N \to \cdots \to M \otimes \Lambda^{p-1} N \to \Lambda^p N] \tag{6}$$

where the right hand side sits in degrees $[-p, 0]$ (see [6] VIII.2.1.2 and [5] I.4.3.2.1). Moreover, in view of (4) we may choose an isomorphism

$$\mathrm{gr}_\tau L_{X/\mathbb{Z}} \simeq [\mathcal{L} \xrightarrow{0} \Omega]$$

in $\mathcal{D}(\mathcal{O}_X)$, the right hand side being concentrated in degrees $[-1, 0]$. Hence the complex $L\Lambda^p(\mathrm{gr}_\tau L_{X/\mathbb{Z}}) \in \mathcal{D}(\mathcal{O}_X)$ is represented by a complex of the form

$$(7) \quad \begin{aligned} L\Lambda^p(\mathrm{gr}_\tau L_{X/\mathbb{Z}}) &\simeq L\Lambda^p([\mathcal{L} \to \Omega]) \simeq \\ &\simeq [\Gamma^p\mathcal{L} \to \Gamma^{p-1}\mathcal{L} \otimes \Omega \to \cdots \to \mathcal{L} \otimes \Lambda^{p-1}\Omega \to \Lambda^p\Omega] \end{aligned}$$

where the right hand side sits in degrees $[-p, 0]$. Lemma 1 and (6) give an isomorphism

$$(8) \qquad \det_{\mathbb{Z}} R\Gamma(X, L\Lambda^p L_{X/\mathbb{Z}}) \simeq \bigotimes_{0 \le q \le p} \det_{\mathbb{Z}}^{(-1)^{p-q}} R\Gamma(X, \Gamma^{p-q}M \otimes \Lambda^q N)$$

compatible with the $\mathbb{Q}$-trivializations. The second exact sequence in (5) endows $\Lambda^q N$ with a finite decreasing filtration Fil^* such that $\mathrm{gr}^i_{\mathrm{Fil}}(\Lambda^q N) = \Lambda^i F \otimes \Lambda^{q-i}\Omega$. Since $\Gamma^{p-q}M$ is flat, Fil^* induces a similar filtration on $\Gamma^{p-q}M \otimes \Lambda^q N$ such that

$$\mathrm{gr}^i_{\mathrm{Fil}}(\Gamma^{p-q}M \otimes \Lambda^q N) = \Gamma^{p-q}M \otimes \Lambda^i F \otimes \Lambda^{q-i}\Omega.$$

This filtration induces an isomorphism

$$(9) \quad \det_{\mathbb{Z}} R\Gamma(X, \Gamma^{p-q}M \otimes \Lambda^q N) \simeq \bigotimes_{0 \le i \le q} \det_{\mathbb{Z}} R\Gamma(X, \Gamma^{p-q}M \otimes \Lambda^i F \otimes \Lambda^{q-i}\Omega)$$

compatible with the $\mathbb{Q}$-trivializations. Lemma 1 and (7) give an isomorphism

$$(10) \qquad \det_{\mathbb{Z}} R\Gamma(X, L\Lambda^p(\mathrm{gr}_\tau L_{X/\mathbb{Z}})) \simeq \bigotimes_{0 \le i \le p} \det_{\mathbb{Z}}^{(-1)^{p-i}} R\Gamma(X, \Gamma^{p-i}\mathcal{L} \otimes \Lambda^i\Omega)$$

compatible with the $\mathbb{Q}$-trivializations. Moreover, we have an isomorphism (see [5] I.4.3.1.7)

$$\Gamma^{p-i}\mathcal{L} \simeq [\Gamma^{p-i}M \to \Gamma^{p-i-1}M \otimes F \to \cdots \to M \otimes \Lambda^{p-i-1}F \to \Lambda^{p-i}F]$$

where the right hand side sits in degrees $[0, p-i]$. Since $\Lambda^i\Omega$ is flat, we have an isomorphism between $\Gamma^{p-i}\mathcal{L} \otimes \Lambda^i\Omega$ and

$$\begin{aligned} &[\Gamma^{p-i}M \otimes \Lambda^i\Omega \to \Gamma^{p-i-1}M \otimes F \otimes \Lambda^i\Omega \to \cdots \\ &\qquad \cdots \to M \otimes \Lambda^{p-i-1}F \otimes \Lambda^i\Omega \to \Lambda^{p-i}F \otimes \Lambda^i\Omega]. \end{aligned}$$

By Lemma 1, we have

(11)
$$\det_{\mathbb{Z}} R\Gamma(X, \Gamma^{p-i}\mathcal{L} \otimes \Lambda^i\Omega) \simeq \bigotimes_{0 \le j \le p-i} \det_{\mathbb{Z}}^{(-1)^j} R\Gamma(X, \Gamma^{p-i-j}M \otimes \Lambda^j F \otimes \Lambda^i\Omega).$$

Putting (10), (11), (9) and (8) together, we obtain isomorphisms

$$\begin{aligned}
&\det_{\mathbb{Z}} R\Gamma(X, L\Lambda^p(\mathrm{gr}_\tau L_{X/\mathbb{Z}})) \simeq \\
&\simeq \bigotimes_{0\leq i\leq p} \det_{\mathbb{Z}}^{(-1)^{p-i}} R\Gamma(X, \Gamma^{p-i}\mathcal{L}\otimes\Lambda^i\Omega) \\
&\simeq \bigotimes_{0\leq i\leq p}\left(\bigotimes_{0\leq j\leq p-i} \det_{\mathbb{Z}}^{(-1)^{p-i-j}} R\Gamma(X, \Gamma^{p-i-j}M\otimes\Lambda^j F\otimes\Lambda^i\Omega)\right) \\
&= \bigotimes_{0\leq q\leq p}\left(\bigotimes_{0\leq i,j\ ;\, i+j=q} \det_{\mathbb{Z}}^{(-1)^{p-q}} R\Gamma(X, \Gamma^{p-q}M\otimes\Lambda^j F\otimes\Lambda^i\Omega)\right) \\
&\simeq \bigotimes_{0\leq q\leq p} \det_{\mathbb{Z}}^{(-1)^{p-q}} R\Gamma(X, \Gamma^{p-q}M\otimes\Lambda^q N) \\
&\simeq \det_{\mathbb{Z}} R\Gamma(X, L\Lambda^p L_{X/\mathbb{Z}})
\end{aligned}$$

compatible with the canonical $\mathbb{Q}$-trivializations. □

Recall from (7) that the complex $L\Lambda^p(\mathrm{gr}_\tau L_{X/\mathbb{Z}})$ is isomorphic in $\mathcal{D}(\mathcal{O}_X)$ to a complex of the form

$$0\to\Gamma^p\mathcal{L}\to\Gamma^{p-1}\mathcal{L}\otimes\Omega^1_{X/\mathbb{F}_p}\to\cdots\to\Gamma^1\mathcal{L}\otimes\Omega^{p-1}_{X/\mathbb{F}_p}\to\Omega^p_{X/\mathbb{F}_p}\to 0$$

put in degrees $[-p,0]$. An isomorphism of $\mathbb{F}_p$-vector spaces $\mathbb{F}_p\simeq p\mathbb{Z}/p^2\mathbb{Z}$ induces an identification $\mathcal{O}_X\simeq\mathcal{L}$, and more generally $\mathcal{O}_X\simeq\Gamma^i\mathcal{L}$ for any $i\geq 0$. Hence $(L\Lambda^p(\mathrm{gr}_\tau L_{X/\mathbb{Z}}))[-p]\in\mathcal{D}(\mathcal{O}_X)$ is represented by a complex of the form

$$0\to\mathcal{O}_X\to\Omega^1_{X/\mathbb{F}_p}\to\cdots\to\Omega^p_{X/\mathbb{F}_p}\to 0 \tag{12}$$

sitting in degrees $[0,p]$. We obtain a spectral sequence

$$E_1^{i,j}=H^j(X,\Omega^{i\leq p}_{X/\mathbb{F}_p})\Longrightarrow H^{i+j}(X,(L\Lambda^p(\mathrm{gr}_\tau L_{X/\mathbb{Z}}))[-p])$$

where $\Omega^{i\leq p}:=\Omega^i$ for $i\leq p$ and $\Omega^{i\leq p}:=0$ for $i>p$. By Lemma 1 again, we get an identification

$$\begin{aligned}
\bigotimes_{i\leq p,j}\det_{\mathbb{Z}}^{(-1)^{i+j}} H^j(X,\Omega^i_{X/\mathbb{F}_p}) &\xrightarrow{\sim} \det_{\mathbb{Z}} R\Gamma(X,(L\Lambda^p(\mathrm{gr}_\tau L_{X/\mathbb{Z}}))[-p]) \\
&\xrightarrow{\sim} \det_{\mathbb{Z}}^{(-1)^p} R\Gamma(X, L\Lambda^p(\mathrm{gr}_\tau L_{X/\mathbb{Z}})).
\end{aligned}$$

In summary, we have the following isomorphisms

$$\det_{\mathbb{Z}} R\Gamma(X, L\Omega^*_{X/\mathbb{Z}}/F^n) \xrightarrow{\sim} \bigotimes_{p<n}\det_{\mathbb{Z}}^{(-1)^p} R\Gamma(X, L\Lambda^p L_{X/\mathbb{Z}}) \tag{13}$$

$$\xrightarrow{\sim} \bigotimes_{p<n}\det_{\mathbb{Z}}^{(-1)^p} R\Gamma(X, L\Lambda^p(\mathrm{gr}_\tau L_{X/\mathbb{Z}})) \tag{14}$$

$$\xrightarrow{\sim} \bigotimes_{p<n}\left(\bigotimes_{i\leq p,j}\det_{\mathbb{Z}}^{(-1)^{i+j}} H^j(X,\Omega^i_{X/\mathbb{F}_p})\right) \tag{15}$$

such that the square

$$\begin{tikzcd}
\left(\det_{\mathbb{Z}} R\Gamma(X, L\Omega^*_{X/\mathbb{Z}}/F^n)\right)\otimes\mathbb{Q} \arrow[r] \arrow[d, "\gamma"] & \left(\bigotimes_{p<n}\bigotimes_{i\leq p,j}\det_{\mathbb{Z}}^{(-1)^{i+j}} H^j(X, \Omega^i_{X/\mathbb{F}_p})\right)\otimes\mathbb{Q} \arrow[d, "\gamma'"] \\
\mathbb{Q} \arrow[r, "\mathrm{Id}"] & \mathbb{Q}
\end{tikzcd}$$

commutes, where the top horizontal map is induced by (15), and the vertical isomorphisms are the canonical trivializations. The first assertion of the theorem follows:

$$\begin{aligned}
\mathbb{Z}\cdot\left(\prod_{i\in\mathbb{Z}} \mid H^i(X, L\Omega^*_{X/\mathbb{Z}}/F^n)\mid^{(-1)^i}\right)^{-1} &= \\
&= \gamma\left(\det_{\mathbb{Z}} R\Gamma(X, L\Omega^*_{X/\mathbb{Z}}/F^n)\right) \\
&= \gamma'\left(\bigotimes_{p<n}\bigotimes_{i\leq p,j}\det_{\mathbb{Z}}^{(-1)^{i+j}} H^j(X, \Omega^i_{X/\mathbb{F}_p})\right) \\
&= \mathbb{Z}\cdot p^{-\chi(X/\mathbb{F}_p,\mathcal{O}_X,n)}.
\end{aligned}$$

We now explain why the second assertion of the theorem is a restatement of ([2] Theorem 1.3). We assume that $H^i_W(X,\mathbb{Z}(n))$ is finitely generated for all $i\in\mathbb{Z}$ (X and n being fixed). Recall from [2] that this assumption implies the following: $H^i_W(X,\mathbb{Z}(n))$ is in fact finite for $i\neq 2n, 2n+1$, the complex (2) has finite cohomology groups and one has

$$\rho_n := -\mathrm{ord}_{s=n}\zeta(X,s) = \mathrm{rank}_{\mathbb{Z}} H^{2n}_W(X,\mathbb{Z}(n)).$$

In particular the complex

$$\cdots \xrightarrow{\cup e} H^i_W(X,\mathbb{Z}(n))\otimes\mathbb{Q} \xrightarrow{\cup e} H^{i+1}_W(X,\mathbb{Z}(n))\otimes\mathbb{Q} \xrightarrow{\cup e} \cdots \tag{16}$$

is acyclic. This gives a trivialization

$$\begin{aligned}
\beta : \mathbb{Q} \xrightarrow{\sim} \bigotimes_i \det_{\mathbb{Q}}^{(-1)^i}\left(H^i_W(X,\mathbb{Z}(n))\otimes\mathbb{Q}\right) &\xrightarrow{\sim} \\
&\xrightarrow{\sim} \left(\bigotimes_i \det_{\mathbb{Z}}^{(-1)^i} H^i_W(X,\mathbb{Z}(n))\right)\otimes\mathbb{Q}
\end{aligned}$$

such that

$$\mathbb{Z}\cdot\beta\left(\chi(H^*_W(X,\mathbb{Z}(n)),\cup e)^{-1}\right) = \bigotimes_i \det_{\mathbb{Z}}^{(-1)^i} H^i_W(X,\mathbb{Z}(n)).$$

The class $e \in H^1(W_{\mathbb{F}_q}, \mathbb{Z}) = \mathrm{Hom}(W_{\mathbb{F}_q}, \mathbb{Z})$ maps the Frobenius Frob $\in W_{\mathbb{F}_q}$ to $1 \in \mathbb{Z}$. We define the map

$$W_{\mathbb{F}_q} = \mathbb{Z} \cdot \mathrm{Frob} \longrightarrow \mathbb{R} =: W_{\mathbb{F}_1}$$

as the map sending Frob to $\log(q)$, while $\theta \in H^1(W_{\mathbb{F}_1}, \mathbb{R}) = \mathrm{Hom}(\mathbb{R}, \mathbb{R})$ is the identity map. It follows that the acyclic complex

$$\cdots \xrightarrow{\cup\theta} H^i_W(X, \mathbb{Z}(n)) \otimes \mathbb{R} \xrightarrow{\cup\theta} H^{i+1}_W(X, \mathbb{Z}(n)) \otimes \mathbb{R} \xrightarrow{\cup\theta} \cdots$$

induces a trivialization

$$\alpha : \mathbb{R} \xrightarrow{\sim} \bigotimes_i \det_{\mathbb{R}}^{(-1)^i} \left(H^i_W(X, \mathbb{Z}(n)) \otimes \mathbb{R}\right) \xrightarrow{\sim} \left(\bigotimes_i \det_{\mathbb{Z}}^{(-1)^i} H^i_W(X, \mathbb{Z}(n))\right) \otimes \mathbb{R}$$

such that

$$\mathbb{Z} \cdot \alpha \left(\chi(H^*_W(X, \mathbb{Z}(n)), \cup e)^{-1} \cdot \log(q)^{\rho_n}\right) = \bigotimes_i \det_{\mathbb{Z}}^{(-1)^i} H^i_W(X, \mathbb{Z}(n)).$$

The trivialization λ is the product of α with the canonical trivialization

$$\mathbb{R} \xrightarrow{\sim} \det_{\mathbb{Z}} R\Gamma(X, L\Omega^*_{X/\mathbb{Z}}/F^n) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Hence we have

$$\mathbb{Z} \cdot \lambda \left(\log(q)^{\rho_n} \cdot \chi(H^*_W(X, \mathbb{Z}(n)), \cup e)^{-1} \cdot q^{-\chi(X, \mathcal{O}_X, n)}\right) = \Delta(X/\mathbb{Z}, n).$$

Moreover, formula (1) gives

$$\zeta^*(X, s) = \pm \log(q)^{-\rho_n} \cdot \chi(H^*_W(X, \mathbb{Z}(n)), \cup e) \cdot q^{\chi(X, \mathcal{O}_X, n)}$$

hence the result follows from ([2] Theorem 1.3). □

Acknowledgments. I would like to thank Matthias Flach, Stephen Lichtenbaum and Niranjan Ramachandran for their interest and comments.

References

[1] Berthelot, P.; Ogus, A.: *Notes on crystalline cohomology.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978.

[2] Geisser, T.: *Weil-étale cohomology over finite fields.* Math. Ann. 330 (4) (2004), 665–692.

[3] Geisser, T.: *Arithmetic cohomology over finite fields and special valyues of ζ-functions.* Duke Math. J. 133 (1) (2006), 27–57.

[4] Illusie, L.: *Existence de Résolutions Globales.* Théorie des Intersections et Théorème de Riemann-Roch (SGA6), Lecture Notes in Mathematics 225, (1971), 160–221.

[5] Illusie, L.: *Complexe cotangent et déformations. I.* Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, Berlin-New York, 1971.

[6] Illusie, L.: *Complexe cotangent et déformations. II.* Lecture Notes in Mathematics, Vol. 283. Springer-Verlag, Berlin-New York, 1972.

[7] Lichtenbaum, S.: *Values of zeta-functions at nonnegative integers.* In: Number theory, Noordwijkerhout 1983, Lecture Notes in Mathematics, Vol. 1068, 127–138, Springer, Berlin, 1984.
[8] Lichtenbaum, S.: *The Weil-étale topology on schemes over finite fields.* Compositio Math. 141 (3) (2005), 689–702.
[9] Milne, J.S.: *Values of zeta functions of varieties over finite fields.* Amer. J. Math. 108 (1986), no. 2, 297–360.
[10] Milne, J.S.: *Addendum to: Values of zeta functions of varieties over finite fields, Amer. J. Math. 108, (1986), 297-360.* (2013), available at http://www.jmilne.org/.
[11] Milne, J.S., Ramachandran, N.: *Motivic complexes and special values of zeta functions.* Preprint 2013, arXiv:1311.3166.

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