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Eisenstein Series and the Trace Formula for GL(2) over a Function Field

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Abstract. We write out and prove the trace formula for a convolution operator on the space of cusp forms on GL(2) over the function field F of a smooth projective absolutely irreducible curve over a finite field. The proof – which follows Drinfeld – is complete and all terms in the formula are explicitly computed. The structure of the homogeneous space $\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A})$ is studied in section 2 by means of locally free sheaves of $\mathcal{O}_X$-modules. Section 3 deals with the regularization and computation of the geometric terms, over conjugacy classes. Section 4 develops the theory of intertwining operators and Eisenstein Series, and the trace formula is proven in section 5.

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1 Introduction and statement of the Trace Formula

1.1 Introduction

The (non-invariant) trace formula for GL(2) over a number field was stated and its proof sketched in chapter 15 of the influential book of Jacquet and Langlands [JL70] of 1970. It was used there for comparison of automorphic representations of the multiplicative group of a quaternion algebra, with automorphic representations of GL(2).
Drinfeld used the trace formula for GL(2) over a function field F to prove Langlands' conjecture for $\mathrm{GL}(2,F)$, and to count in [D81] the number of two

dimensional irreducible representations of the fundamental group of a smooth projective geometrically irreducible curve X over a finite field. To check the statement of the trace formula of [JL70] in the function field case, Drinfeld gave a detailed (but unpublished) proof. It differs from the one sketched in [JL70]. It is this proof of Drinfeld which is given in this paper.
The main reason why this proof is still interesting is the elementary and unconventional treatment of Eisenstein series (see subsections 4.7-4.8 below), and the computation of traces in the spirit of Tate [T68], see subsection 5.2. In both cases it is based on a "baby model" (see Proposition 4.31, Corollary 4.32, Lemma 5.11), which cries out for generalization.

Let us describe the contents of this article.
The trace formula itself is stated in subsection 1.2 with a few comments. More comments, including informal ones, are given in section 3.
Section 2 contains a dictionary between the language of adèles and the language of vector bundles on the smooth projective curve X corresponding to F. In particular, the set of rank n vector bundles on X is identified with $\mathrm{GL}(n,F)\backslash \mathrm{GL}(n,\mathbb{A})/\mathrm{GL}(n,O_{\mathbb{A}})$, where $O_{\mathbb{A}} \subset \mathbb{A}$ is the ring of integral adèles. This dictionary goes back to A. Weil [W38], although in an older language. It underlies Drinfeld's Geometric Langlands program [BD].
The terms which appear in the geometric part of the trace formula – orbital integrals and weighted orbital integrals – are estimated and regularized in section 3.
In section 4 intertwining operators, Eisenstein series, and L-functions are introduced. The rationality of the intertwining operator $M(\mu_1,\mu_2,t)$ and the functional equation $M^2 = 1$ are first proven using local computations: normalization of the intertwining operators by L-functions and ε-factors, and the functional equation of the L-functions.
In subsections 4.7-4.8 these facts are proven using an alternative, global approach. The ideas might go back to Selberg. But technically the exposition is quite different and more elementary: in the case of function fields the analytic problems disappear.
The trace formula is proven in section 5. The logarithmic derivative of the intertwining operator appears as a result of a computation of the trace of some operator in a power series space, see Lemma 5.11. This computation is probably related to Tate's article [T68].

Here are some questions.
1. Could the methods of subsections 4.7-4.8 and section 5 be extended to prove the functional equation for Eisenstein series, and the trace formula, for an arbitrary reductive group over a function field?
2. Is there a modification of the technique from subsections 4.7-4.8 that would work in the case of number fields, e.g., for $\mathrm{GL}(2,\mathbb{Q})$? One could try to replace the space of formal power series used in subsections 4.7-4.8 by some space of holomorphic functions.
3. What is the precise relationship between Lemma 5.11 and Tate's [T68]?

4. What is the relationship between the approach to Eisenstein series of subsections 4.7-4.8, and the classical approaches: that of Selberg-Langlands-Arthur, and that of scattering theory (see [FP72] or [LP76])?

This author's initial motivation to write out Drinfeld's expression and proof of the trace formula for GL(2) over a function field stems from his search for higher rank analogues of Drinfeld's formula [D81]. This led us to count with Deligne [DF13] the number of rank n (≥ 2) local systems with principal unipotent local monodromy at least at two places. There we use the trace formula in the compact quotient case, and the transfer of automorphic representations from a compact form to GL(n). This explains the condition: "at least at two places". The case of [D81] is rank $n = 2$, no monodromy. To complete the study of [D81] and of [DF13] in rank two one has to consider the case of principal unipotent local monodromy at a single place. This is done in [F], using the explicit computations of the trace formula for GL(2) over a function field of the present work. This was our initial motivation to write out this formula. Drinfeld's proof in the case of rank two, no ramification, is also given in [F].

Of course there are numerous expositions of the trace formula of [JL70], e.g. [GJ79], geared to explain the lifting application of [JL70], mainly in the number field case. But none computes explicitly (and accurately, cf. [D81]) all the terms which appear in the trace formula. The latter is precisely what is needed for the counting applications of [D81] and [F]. An attempt at a complete exposition of the computations for GL(2) in the number field case is at [AFOO].

Of course the trace formula of [JL70] was generalized to the higher rank case by Arthur, see e.g. [A05], in the number field case, and by Lafforgue, see e.g. [Lf97], in the function field case. But the important applications of these works did not require explicit evaluation of all the terms which appear in the trace formula, so our results are not included in those of [Lf97], even in the case of GL(2) considered here.

In the number field case, the Remark on p. 112 of [A05] states: "As a matter of fact, it is only in the case of GL(2) that the general coefficients have been evaluated. It would be very interesting to understand them better in other examples, although this does not seem to be necessary for presently conceived applications of the trace formula". Indeed the applications of [D81], [DF13], [F] – counting rather than comparing – are of different nature than those of [JL70], [A05], [Lf97], where most terms can be erased a-priori in the comparison so they need not be computed.

To repeat what is explained above, we also think the approach of subsections 4.7-4.8 and section 5 is original, substantially different from the currently known methods (which are developed in [A05], [Lf97]), interesting and warrants further development.

I am deeply grateful to V. Drinfeld for making available to me his unpublished notes, for teaching me lots of mathematics in the process, and for his permission to publish this paper; to A. Beilinson for telling me at IHES about Drinfeld's notes; to the referee for the very careful reading. The author was a Schonbrunn visiting Professor at the Hebrew University, Jerusalem. Work partially supported by a grant (#267097) from the Simons Foundation.

1.2 STATEMENT OF THE TRACE FORMULA

Let us write the trace formula for GL(2) over a function field F of a smooth projective geometrically connected curve X over a finite field $\mathbb{F}_q$, and a test function f in $C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$ (subscript c for "compactly supported", superscript ∞ for "locally constant", $\mathbb{A}$ denotes the ring of adèles of F).

Let r_0 be the representation of $\mathrm{GL}(2,\mathbb{A})$ by right translation on the space $A_{0,\alpha}$ of cusp forms on $\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$, and $r_0(f) = \int f(g)r_0(g)dg$ ($g \in \mathrm{GL}(2,\mathbb{A})$) the convolution operator; $dg = \otimes_v dg_v$ is a Haar measure. Here α is a fixed idèle of degree 1, whose components are almost all equal to 1.

A cusp form is a function $\phi : \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}) \to E$ (E is a fixed algebraically closed subfield of $\mathbb{C}$) which is invariant on the right by some open compact subgroup of $\mathrm{GL}(2,\mathbb{A})$, and $\int_{N(F)\backslash N(\mathbb{A})} \phi(nx)dn = 0$ for all x in $\mathrm{GL}(2,\mathbb{A})$. Here N denotes the unipotent upper triangular subgroup of GL(2). We also write A for the diagonal subgroup, and $A' = A - Z$ where Z is the center of GL(2). By a well known result of G. Harder, when F is a function field (but not a number field) a cusp form is compactly supported modulo $Z(\mathbb{A})$.

THEOREM 1.1. *For any $f \in C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$ we have* $\mathrm{tr}\, r_0(f) = \sum_{1\le i\le 8} S_i(f)$. *The terms are:*

$$S_1(f) = \left|\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})\right| \sum_{\gamma\in\alpha^{\mathbb{Z}}\cdot F^\times} f(\gamma).$$

$$S_2(f) = \sum_{F_2} S_{2,F_2}(f),$$

$$S_{2,F_2}(f) = |\operatorname{Aut}_F F_2|^{-1} \sum_{\gamma\in\alpha^{\mathbb{Z}}(F_2-F)} \int_{\mathrm{GL}(2,\mathbb{A})/\alpha^{\mathbb{Z}}\cdot \mathbb{A}_{F_2}^\times} f(x\gamma x^{-1})dx.$$

Here F_2 ranges over the set of isomorphism classes of quadratic extensions of the field F. For each F_2 we fix an embedding $F_2 \hookrightarrow M(2,F)$ into the ring of 2×2 matrices over F.

$$S_3(f) = \sum_{\gamma\in\alpha^{\mathbb{Z}}\cdot A'(F)} \int_{A(\mathbb{A})\backslash \mathrm{GL}(2,\mathbb{A})} f(x^{-1}\gamma x)v(x)dx.$$

Any $x \in \mathrm{GL}(2,\mathbb{A})$ can be written in the form ank, $a \in A(\mathbb{A})$, $k \in \mathrm{GL}(2,O_{\mathbb{A}})$, $n = \left(\begin{smallmatrix}1&b\\0&1\end{smallmatrix}\right)$, b is determined uniquely by x up to $b \mapsto ub+w$, $u \in O_{\mathbb{A}}^\times$, $w \in O_{\mathbb{A}}$. Put $v(x) = \sum_v \log_q(\max(1,|b_v|_v))$.

$$S_4(f) = \sum_{a\in F^\times\alpha^{\mathbb{Z}}} \tilde\theta_{a,f}(1), \qquad \tilde\theta_{a,f}(t) = \frac{1}{2}(\theta_{a,f}(t) + \theta_{a,f}(t^{-1})),$$

$$\theta_{a,f}(t) = \int_{F^\times\alpha^{\mathbb{Z}}N(F)\backslash \mathrm{GL}(2,\mathbb{A})} f\left(x^{-1}\left(\begin{smallmatrix}a&a\\0&a\end{smallmatrix}\right)x\right) t^{\mathrm{ht}^+(x)}dx,$$

$\mathrm{ht}^+ : \mathrm{GL}(2,\mathbb{A}) \to \mathbb{Z}$ is defined by $\mathrm{ht}^+((\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix})k) = \deg a - \deg b$ ($k \in \mathrm{GL}(2, O_{\mathbb{A}})$; $a, b \in \mathbb{A}^\times$; $c \in \mathbb{A}$).

$$S_5(f) = \frac{-1}{4\pi i} \sum_{\mu_1,\mu_2} \oint_{|z|=1} \mathrm{tr}\, I(\mu_1\nu_z, \mu_2\nu_{z^{-1}}, f) \frac{m'(\mu_1/\mu_2, z)}{m(\mu_1/\mu_2, z)} 2z dz.$$

Here $m(\mu, z) = L(\mu, z)/L(\mu, z/q)$. The μ_1, μ_2 range over the set of characters of $\mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}}$, $\nu_z(x) = z^{\deg(x)}$. Also $I(\mu_1, \mu_2)$ is the space of right locally constant functions ϕ on $\mathrm{GL}(2,\mathbb{A})$ with

$$\phi\left((\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix})x\right) = |a/b|^{1/2}\mu_1(a)\mu_2(b)\phi(x) \qquad (x \in \mathrm{GL}(2,\mathbb{A});\ \ a, b \in \mathbb{A}^\times;\ \ c \in \mathbb{A}).$$

It is a $\mathrm{GL}(2,\mathbb{A})$-module by right translation, and $\mathrm{tr}\, I(\mu_1\nu_z, \mu_2\nu_{z^{-1}}, f)$ is the trace of the indicated convolution operator.

$$S_6(f) = \frac{-1}{4\pi i} \sum_{\mu_1,\mu_2} \oint_{|z|=1} \mathrm{tr}[I(\mu_1\nu_z, \mu_2\nu_{z^{-1}}, f) \cdot R(\mu_1, \mu_2, z)^{-1} \frac{d}{dz} R(\mu_1, \mu_2, z)] dz.$$

Notations are as in $S_5(f)$, and $R(\mu_1, \mu_2, z) : I(\mu_1\nu_z, \mu_2\nu_{z^{-1}}) \to I(\mu_2\nu_{z^{-1}}, \mu_1\nu_z)$ is an operator, rational in z, defined as a product $\otimes_v R(\mu_{1v}, \mu_{2v}, z_v)$, $z_v = z^{\deg(v)}$. The product is well defined as the local operator maps the function in the source whose restriction to $\mathrm{GL}(2, O_v)$ is 1 to such function in the target. Further, $R(\mu_{1v}, \mu_{2v}, z)$ is defined to be

$$[L(\mu_{1v}/\mu_{2v}, z^2/q_v)/L(\mu_{1v}/\mu_{2v}, z^2)] M(\mu_{1v}, \mu_{2v}, z).$$

The operator $M(\mu_{1v}, \mu_{2v}, z) = M(\mu_{1v}\nu_z, \mu_{2v}\nu_{z^{-1}})$ is defined first by an integral

$$\phi \mapsto \int \phi\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right)x\right) dy \quad \text{if} \quad |(\mu_{1v}/\mu_{2v})(\boldsymbol{\pi}_v)z^2| < 1,$$

then by analytic continuation, as it is a rational function in z. The operators $I(\mu_1\nu_z, \mu_2\nu_{z^{-1}}, f)$ and $R(\mu_1, \mu_2, z)$ are considered as operators on

$$I_0(\mu_1, \mu_2) = \{\phi \in C^\infty(\mathrm{GL}(2, O_{\mathbb{A}}));\ \phi\left((\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix})x\right) = \mu_1(a)\mu_2(b)\phi(x);$$
$$x \in \mathrm{GL}(2, O_{\mathbb{A}}),\ \ a, b \in O_{\mathbb{A}}^\times;\ \ c \in O_{\mathbb{A}}\}.$$

$$S_7(f) = \frac{1}{4} \sum_{\mu} \mathrm{tr}\, I(\mu, \mu, f), \qquad S_8(f) = -\sum_{\mu} \int_{\mathrm{GL}(2,\mathbb{A})} f(x)\mu(\det x) dx.$$

Both sums range over all characters μ of $\mathbb{A}^\times/F^\times \cdot \alpha^{2\mathbb{Z}}$. The sum of S_8 is over all automorphic one dimensional representations ($\mu \circ \det$) of $\alpha^{\mathbb{Z}} \backslash \mathrm{GL}(2,\mathbb{A})$. The integral there represents the trace of the convolution operator associated with f.

The terms $S_1(f)$ and $S_2(f)$ are finite by Proposition 3.5, 3.6, 3.9. The argument used in the proof of Proposition 3.9 shows that for any $\gamma \in \alpha^{\mathbb{Z}}(A(F) - Z(F))$ the function $x \mapsto f(x^{-1}\gamma x)$ on $A(\mathbb{A}) \backslash \mathrm{GL}(2,\mathbb{A})$ has compact support, hence the integral in $S_3(f)$ converges.

By Proposition 3.11 the function $\theta_{a,f}(t)$ is rational and may have at $t = 1$ a pole of order at most 1, for each $a \in \mathbb{A}^\times$. Hence $\tilde{\theta}_{a,f}(t)$ is regular at $t = 1$. From Proposition 3.5 it follows that the sums in $S_3(f)$ and $S_4(f)$ are finite, so these terms are well defined.

For any $f = \otimes f_v$ in $C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$, the operator $I(\mu_1, \mu_2, f)$ is zero unless μ_i are unramified at each v where f_v is $\mathrm{GL}(2, O_v)$ biinvariant. This implies that the sums in $S_i(f)$ $(5 \le i \le 8)$ are finite, for a given f. To see that $S_5(f)$ and $S_6(f)$ are well defined, note that the rational functions $m(\mu, t)$, $R(\mu_1, \mu_2, t)$, $R(\mu_1, \mu_2, t)^{-1}$ are regular on $|t| = 1$ for all characters μ, μ_1, μ_2 of $\mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}}$. For $m(\mu, t)$ this follows from Proposition 4.11, for R and R^{-1} from Corollary 4.28.

The distributions [linear forms on $C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$] $f \mapsto \operatorname{tr} r_0(f)$, $S_i(f)$ $(i = 1, 2, 5, 7, 8)$ are invariant, namely take the same value at f and $f^h(x) = f(h^{-1}xh)$, $h \in \mathrm{GL}(2,\mathbb{A})$. For $i = 3, 4, 6$ we have $S_i(f^h) = S_i(f)$ if $h \in \mathrm{GL}(2, O_{\mathbb{A}})$, but S_i is not invariant.

If $f \in C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$ takes values in $\mathbb{Q}$ then $\operatorname{tr} r_0(f) \in \mathbb{Q}$, since the representation r_0 is defined over $\mathbb{Q}$. For $i = 1, 2, 3, 4, 8$ it is clear that $S_i(f) \in \mathbb{Q}$. For $i = 7$ the integrand contains the factor $\mu(ab)|a/b|^{1/2}$ which involves $\sqrt{q}$. However the sum includes with μ also $\mu\varepsilon$, $\varepsilon(\alpha) = -1$, and so the sum of the terms indexed by μ and $\mu\varepsilon$ can be written as an integral over the domain where $|a/b|$ is in $q^{2\mathbb{Z}}$.

To see that $S_5(f)$ is rational, we put $a(\mu_1, \mu_2) = \frac{1}{2\pi i} \oint_{|t|=1} f(\mu_1, \mu_2, t) dt$ where

$$f(\mu_1, \mu_2, t) = \operatorname{tr} I(\mu_1\nu_t, \mu_2\nu_{t^{-1}}, f) \cdot \frac{d}{dt} \ln m(\mu_1/\mu_2, t^2),$$

and claim that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ one has $\sigma(a(\mu_1, \mu_2)) = a({}^\sigma\mu_1, {}^\sigma\mu_2)$. Note that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the group of characters on $\mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}}$ as they are all $\mathbb{Q}$-valued. Now $a(\mu_1, \mu_2)$ is the sum of the residues of $f(\mu_1, \mu_2, t)$ at the points of the unit disc. We have that $\sigma(f(\mu_1, \mu_2, t)) = f({}^\sigma\mu_1, {}^\sigma\mu_2, \varepsilon(\sigma) \cdot {}^\sigma t)$ with $\varepsilon(\sigma) = \sigma(\sqrt{q})/\sqrt{q}$. However, if $f(\mu_1, \mu_2, t)$ has a pole at $t = t_0$ and $|t_0| < 1$, then by Proposition 4.11, $|\sigma(t_0)| < 1$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Hence $S_5(f) \in \mathbb{Q}$. To see that $S_6(f) \in \mathbb{Q}$ one proceeds similarly, using the results of Corollary 4.28 on the poles of $R(\mu_1, \mu_2, t)$ and $R(\mu_1, \mu_2, t)^{-1}$.

2 Locally free sheaves of $\mathcal{O}_X$-modules

2.1 Stable bundles

Let X be a smooth geometrically connected projective curve over $\mathbb{F}_q$ (we take minimal q). Denote by $\mathcal{O}_X$ the structure sheaf of X. Denote by Bun_n the set of isomorphism classes of rank n locally free sheaves of $\mathcal{O}_X$-modules. By a (vector) bundle we mean here simply a locally free sheaf. In particular, $\mathrm{Bun}_1 = \operatorname{Pic} X$. The Picard group $\operatorname{Pic} X$ of invertible, or rank 1, locally free sheaves $\mathcal{L}$ of $\mathcal{O}_X$-modules, is naturally isomorphic to the group of classes $\overline{D}$ of (Weil) divisors $D = \sum_v n_v v$ $(n_v \in \mathbb{Z},\ v \in |X|)$. Here $|X|$ is the set of closed points of X, and the divisors D, D' lie in the same class (are linearly equivalent) if their difference is the (principal) divisor $(f) = \sum_v \mathrm{ord}_v(f)v$ where f is a nonzero rational function on X and $\mathrm{ord}_v(f)$ is the order of f at $v \in |X|$ ($\mathrm{ord}_v(f) > 0$ if v is a zero, $\mathrm{ord}_v(f) < 0$ if v is a pole, $\mathrm{ord}_v(f) = 0$ otherwise). If $\mathcal{L}$, $\mathcal{M} \in \operatorname{Pic} X$ correspond to the divisors D, D' then $\mathcal{L} \otimes \mathcal{M}$ corresponds to $D + D'$.

There is a degree map deg on $\operatorname{Pic} X$: $\deg(\sum_v n_v v) = \sum_v n_v \deg(v)$ defines $\deg(\mathcal{L}) = \deg(D)$, where $\deg(v) = [k_v : \mathbb{F}_q]$. Here k_v is the residue field of the function field $F = \mathbb{F}_q(X)$ of X over $\mathbb{F}_q$ at v; assume $\mathbb{F}_q$ is algebraically closed in F. We write F_v for the completion of F at v, O_v for its ring of integers. The cardinality of the residue field $k_v = \mathbb{F}_{q_v}$ at v is denoted by q_v, thus $q_v = q^{\deg(v)}$. We also write $\deg(\overline{D})$ for $\deg(D)$, as the degree of a principal divisor is 0; recall that $\overline{D}$ denotes the class of D.

Denote by $\chi(\mathcal{L}) = \dim_{\mathbb{F}_q} H^0(X, \mathcal{L}) - \dim_{\mathbb{F}_q} H^1(X, \mathcal{L})$ the Euler-Poincaré characteristic of $\mathcal{L} \in \operatorname{Pic} X$. Here $H^i(X, \mathcal{L})$ are finite dimensional vector spaces over $\mathbb{F}_q$. Then $\chi(\mathcal{O}_X) = 1 - g$ where $g = \dim_{\mathbb{F}_q} H^1(X, \mathcal{O}_X)$ is named the genus of X. The Riemann-Roch theorem asserts that $\chi(\mathcal{L}) - \deg(\mathcal{L}) = \chi(\mathcal{O}_X)$ is independent of $\mathcal{L} \in \operatorname{Pic} X$.

Define the *degree* of a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules of rank n to be $\deg \mathcal{E} = \chi(\mathcal{E}) - n\chi(\mathcal{O}_X)$. The *determinant* of $\mathcal{E}$ is $\det \mathcal{E} = \bigwedge^n \mathcal{E} \in \operatorname{Pic} X$. We have $\deg \mathcal{E} = \deg \det \mathcal{E}$. This gives an alternative definition of the degree. A proof of this equality is as follows. If $\mathcal{E}$ is a line bundle, then there is nothing to prove. In the general case, use the fact that both $\deg \mathcal{E}$ and $\deg \det \mathcal{E}$ are additive (if $\mathcal{E}' \subset \mathcal{E}$ is a subbundle, then $\deg \mathcal{E} = \deg \mathcal{E}' + \deg(\mathcal{E}/\mathcal{E}')$ and similarly for $\deg \det \mathcal{E}$), and that each vector bundle has a flag, $\mathcal{E}_i$, such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ are line bundles.

The *height* of a rank two locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules is the integer $\mathrm{ht}(\mathcal{E}) = \max_{\mathcal{L}}(2 \deg \mathcal{L} - \deg \mathcal{E})$, $\mathcal{L}$ ranges over all invertible subsheaves of $\mathcal{E}$.

Proposition 2.1. *We have* $-2g \le \mathrm{ht}(\mathcal{E}) < \infty$.

Proof. Let $\mathcal{L}$ be an invertible subsheaf of $\mathcal{E}$. From the Riemann-Roch theorem $\chi(\mathcal{L}) = \deg \mathcal{L} + 1 - g$ we obtain $\dim_{\mathbb{F}_q} H^0(X, \mathcal{L}) \ge \deg \mathcal{L} + 1 - g$, whence $\deg \mathcal{L} \le \dim_{\mathbb{F}_q} H^0(X, \mathcal{L}) + g - 1 \le \dim_{\mathbb{F}_q} H^0(X, \mathcal{E}) + g - 1$, so $\mathrm{ht}(\mathcal{E})$ is finite. Let $\mathcal{L}$ be an invertible subsheaf of $\mathcal{E}$ of maximal degree. Let $\mathcal{M}$ be an invertible sheaf with $\deg \mathcal{M} = \deg \mathcal{L} + 1$. Then $\mathrm{Hom}(\mathcal{M}, \mathcal{E}) = 0$. Also, by Riemann-Roch for the rank 2 sheaf $\mathcal{E}$, $\dim_{\mathbb{F}_q} \mathrm{Hom}(\mathcal{M}, \mathcal{E}) = \dim_{\mathbb{F}_q} H^0(X, \mathcal{M}^{-1}\mathcal{E}) \ge \deg(\mathcal{M}^{-1}\mathcal{E}) + 2 - 2g = \deg \mathcal{E} - 2 \deg \mathcal{M} + 2 - 2g = \deg \mathcal{E} - 2 \deg \mathcal{L} - 2g$, so $2 \deg \mathcal{L} - \deg \mathcal{E} \ge -2g$. □

A rank two locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules is called *stable* if $\mathrm{ht}(\mathcal{E}) < 0$ and *semistable* if $\mathrm{ht}(\mathcal{E}) \le 0$. In general, the *slope* $\mu(\mathcal{E})$ of a locally free sheaf $\mathcal{E}$ over an algebraic curve is defined to be $\deg \mathcal{E} / \operatorname{rk} \mathcal{E}$, and $\mathcal{E}$ is called stable if $\mu(\mathcal{F}) < \mu(\mathcal{E})$ for all proper nonzero subbundles $\mathcal{F}$ of $\mathcal{E}$ (semistable if $\le$). A locally free sheaf $\mathcal{E}$ of rank two is called *almost stable* if $\mathrm{ht}(\mathcal{E}) < 2g - 1$, and *very unstable* if $\mathrm{ht}(\mathcal{E}) \ge 2g - 1$. If $g = 0$, every $\mathcal{E}$ is very unstable.

Remark 1. A very unstable vector bundle $\mathcal{E}$ of rank 2 splits into the direct sum of two line bundles. We give here a relatively elementary treatment. An extension can be found in the work of Harder and Narasimhan. If $\mathcal{E}$ is very unstable, $\mathcal{L}$ is an invertible subsheaf of $\mathcal{E}$ of maximal degree, and $\mathcal{M} = \mathcal{E}/\mathcal{L}$, then $\mathcal{M}$ is invertible and $\mathrm{Ext}(\mathcal{M}, \mathcal{L}) = H^1(X, \mathcal{M}^{-1}\mathcal{L})$ is 0 since

$$\deg \mathcal{M}^{-1}\mathcal{L} = \deg \mathcal{L} - \deg \mathcal{M} = 2 \deg \mathcal{L} - \deg \mathcal{E} = \mathrm{ht}\, \mathcal{E} \geq 2g - 1.$$

Indeed, by Serre duality $H^1(X, \mathcal{M}^{-1}\mathcal{L}) = H^0(X, \mathcal{L}^{-1}\mathcal{M}\omega)$ where ω denotes the canonical bundle. But $\deg \mathcal{L}^{-1}\mathcal{M}\omega \leq 2g - 2 - (2g - 1) < 0$, and $H^0(X, \mathcal{F}) = 0$ for an invertible sheaf $\mathcal{F}$ with negative degree.

Proposition 2.2. *The number of isomorphism classes of almost stable rank two locally free sheaves $\mathcal{E}$ of $\mathcal{O}_X$-modules with a fixed degree is finite.*

Proof. The height of an almost stable sheaf lies in $[-2g, 2g - 2]$. Hence it suffices to show the finiteness for $\mathcal{E}$ with a fixed degree n and height h. Every such sheaf lies in an exact sequence $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \to 0$, where $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves and $2 \deg \mathcal{L} - \deg \mathcal{E} = h$. Then $\deg \mathcal{L} = (n + h)/2$, $\deg \mathcal{M} = (n - h)/2$. Since the degrees of $\mathcal{L}$ and $\mathcal{M}$ are fixed, there are only finitely many possibilities for $\mathcal{L}$ and $\mathcal{M}$ (set of cardinality of the $\mathbb{F}_q$-points on the abelian variety $\mathrm{Pic}^0(X)$). With $\mathcal{L}$ and $\mathcal{M}$ fixed there are only finitely many choices for $\mathcal{E}$ as $\mathrm{Ext}(\mathcal{L}, \mathcal{M})$ is finite. □

The group $\mathrm{Pic}\, X$ acts on $\mathrm{Bun}_2 : (\mathcal{L} \in \mathrm{Pic}\, X, \mathcal{E} \in \mathrm{Bun}_2) \mapsto \mathcal{L} \otimes \mathcal{E}$. As

$$\deg(\mathcal{L} \otimes \mathcal{E}) = 2 \deg(\mathcal{L}) + \deg(\mathcal{E}),$$

the set of almost stable sheaves is invariant under this action. In a $\mathrm{Pic}\, X$-orbit we may choose $\mathcal{E}$ to have $\deg(\mathcal{E})$ in $\{0, 1\}$. Hence we deduce

Corollary 2.3. *The number of* $\mathrm{Pic}\, X$*-orbits on the set of isomorphism classes of almost stable rank two locally free sheaves of $\mathcal{O}_X$-modules is finite.*

2.2 Bundles and lattices

Let $\mathcal{E}$ be a rank n locally free sheaf of $\mathcal{O}_X$-modules. Denote by $\mathcal{E}_\eta$ the fiber (= stalk) of $\mathcal{E}$ over the generic point η of X. Let $\mathcal{E}_{(v)}$ be the stalk of $\mathcal{E}$ at the closed point $v \in |X|$. Let $O_{(v)}$ be the local ring of X at v. Then $\mathcal{E}_\eta$ is an n-dimensional vector space over F, and $\mathcal{E}_{(v)}$ is an $O_{(v)}$-lattice in $\mathcal{E}_\eta$, namely a rank n free $O_{(v)}$-submodule of $\mathcal{E}_\eta$.

A set M of $O_{(v)}$-lattices $M_{(v)}$ in a finite dimensional vector space V over F, v ranges over the set $|X|$ of closed points in X, is called *adelic* if there exists a basis $\{e_1, \ldots, e_n\}$ in V such that $M_{(v)} = O_{(v)}e_1 + \cdots + O_{(v)}e_n$ for almost all v in $|X|$. "Almost all" means "with at most finitely many exceptions". If M is adelic then it is adelic with respect to any basis $\{e_1, \ldots, e_n\}$ of V.

The set of stalks $\{\mathcal{E}_{(v)}; v \in |X|\}$ of a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules is adelic. Conversely, an adelic set of lattices $M = \{M_{(v)}; v \in |X|\}$ in a finite dimensional vector space V over F is the set of stalks of the locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules defined by

$$H^0(\mathrm{U}, \mathcal{E}) = \{s \in V; \forall v \in \mathrm{U}, s \in M_{(v)}\}$$

for any open subset U of X. Obtained is an equivalence of the category of finite rank locally free sheaves of $\mathcal{O}_X$-modules, with the category of finite dimensional vector spaces over F with adelic sets of $O_{(v)}$-lattices.

Let O_v be the completion of $O_{(v)}$. The completion of F at v is denoted F_v. Let V be a finite dimensional vector space over F. Put $V_v = V \otimes_F F_v$. There is a natural bijection between the set of $\mathcal{O}_{(v)}$-lattices in V, and O_v-lattices in V_v: an $O_{(v)}$-lattice $M \subset V$ corresponds to the lattice $M \otimes_{\mathcal{O}_{(v)}} \mathcal{O}_v$ in V_v; an O_v-lattice $N \subset V_v$ corresponds to the $O_{(v)}$-lattice $N \cap V$.

The category $\mathcal{C}$ whose objects are finite dimensional F-vector spaces V with adelic sets $\{M_v; v \in |X|\}$ of O_v-lattices M_v in V_v is equivalent to the category of finite rank locally free sheaves of $\mathcal{O}_X$-modules $\mathcal{E}$, by $\mathcal{E} \mapsto (\mathcal{E}_\eta, \{\mathcal{E}_v\})$, where $\mathcal{E}_\eta$ is the generic fiber of $\mathcal{E}$ and $\mathcal{E}_v$ is the completion of the stalk of $\mathcal{E}$ at the closed point $v \in |X|$.

Let R_n be the set of isomorphism classes of pairs $(\mathcal{E}, i)$ where $\mathcal{E}$ is a rank n locally free sheaf of $\mathcal{O}_X$-modules, and i is an isomorphism from the generic fiber of $\mathcal{E}$ to F^n. The pairs $(\mathcal{E}, i)$ and $(\mathcal{E}', i')$ are isomorphic if there is an isomorphism $\mathcal{E} \tilde{\to} \mathcal{E}'$ which induces a commutative diagram when restricted to the generic fiber with sides i and i' and the identity $F^n \to F^n$. The group $\mathrm{GL}(n, F)$ acts on R_n by $g : (\mathcal{E}, i) \mapsto (\mathcal{E}, g \circ i)$. Then $\mathrm{GL}(n, F) \backslash R_n = \mathrm{Bun}_n$ is the set of isomorphism classes of rank n locally free sheaves of $\mathcal{O}_X$-modules.

The set R_n is the set of adelic collections of O_v-lattices $M_v \subset F_v^n$, $v \in |X|$. The group $\mathrm{GL}(n, F_v)$ acts transitively on the set of O_v-lattices in F_v^n. The stabilizer of the standard lattice O_v^n in F_v^n is $\mathrm{GL}(n, O_v)$. Thus the set of O_v-lattices in F_v^n is $\mathrm{GL}(n, F_v)/\mathrm{GL}(n, O_v)$, and R_n is $\mathrm{GL}(n, \mathbb{A})/\mathrm{GL}(n, O_\mathbb{A})$, where $\mathbb{A}$ is the ring of adèles in F and $O_\mathbb{A} = \prod_{v \in |X|} O_v$. Thus

$$\mathrm{Bun}_n = \mathrm{GL}(n, F) \backslash \mathrm{GL}(n, \mathbb{A}) / \mathrm{GL}(n, O_\mathbb{A}).$$

The elements of $\mathrm{GL}(n, \mathbb{A})/\mathrm{GL}(n, O_\mathbb{A})$ are called matrix divisors, and the elements of $\mathrm{GL}(n, F) \backslash \mathrm{GL}(n, \mathbb{A})/\mathrm{GL}(n, O_\mathbb{A})$ classes of matrix divisors. For $n = 1$, the identification of $\mathrm{GL}(n, F) \backslash \mathrm{GL}(n, \mathbb{A})/\mathrm{GL}(n, O_\mathbb{A})$ with Bun_n is the identification of classes of divisors with invertible sheaves.

The group $\mathrm{GL}(n, \mathbb{A})$ can be identified with the set of triples

$$(\mathcal{E}, i_\eta : \mathcal{E}_\eta \tilde{\to} F^n, (i_v : \mathcal{E}_v \tilde{\to} O_v^n)).$$

Given a rank n locally free sheaf $\mathcal{E}$, an isomorphism $i_\eta : \mathcal{E}_\eta \tilde{\to} F^n$, and for each closed point v in $|X|$ an isomorphism $i_v : \mathcal{E}_v \tilde{\to} O_v^n$ of the completion $\mathcal{E}_v$ of the stalk $\mathcal{E}_{(v)}$ at v with O_v^n, let us define the corresponding $g = (g_v)$ in $\mathrm{GL}(n, \mathbb{A})$. Each g_v has to be an automorphism $F_v^n \to F_v^n$, with $g_v(O_v^n) = O_v^n$ for almost all v. Construct g_v as the composition $i_v \circ i_\eta^{-1}$:

$$F_v^n = F^n \otimes_F F_v \overset{i_\eta}{\overset{\sim}{\leftarrow}} \mathcal{E}_\eta \otimes_F F_v = \mathcal{E}_{F_v} = \mathcal{E}_v \otimes_{O_v} F_v \overset{i_v}{\overset{\sim}{\rightarrow}} O_v^n \otimes_{O_v} F_v = F_v^n.$$

Note that since $\mathcal{E}$ is locally free, for almost all v the map $g_v = i_v \circ i_\eta^{-1}$ takes $O_v^n \subset F_v^n$ to $\mathcal{E}_v \subset \mathcal{E}_\eta \otimes_F F_v$ via i_η^{-1}, and then to O_v^n via i_v. To show that the map $\{(\mathcal{E}, i_\eta, (i_v))\} \to \mathrm{GL}(n, \mathbb{A})$ is bijective one shows that $\mathrm{GL}(n, \mathbb{A})$ acts on the set of triples, simply transitively. Viewing the trivial locally free sheaf as $O_\mathbb{A}^n$ (space of columns), $g(\mathcal{E}, i_\eta, (i_v))$ is defined to be $(g\mathcal{E}, i_\eta, (i_v \circ g_v^{-1}))$, where $i_v \circ g_v^{-1}$ maps the stalk $g_v\mathcal{E}_v$ of $g\mathcal{E}$ at v to O_v^n. The set of pairs $\{(\mathcal{E}, i_\eta)\}$ then corresponds to $\mathrm{GL}(n, \mathbb{A})/\mathrm{GL}(n, O_\mathbb{A})$, the set of pairs $\{(\mathcal{E}, (i_v))\}$ to $\mathrm{GL}(n, F)\backslash \mathrm{GL}(n, \mathbb{A})$, and the set $\{\mathcal{E}\}$ to $\mathrm{GL}(n, F)\backslash \mathrm{GL}(n, \mathbb{A})/\mathrm{GL}(n, O_\mathbb{A})$.
To an idèle $a = (\boldsymbol{\pi}_v^{-n_v} u_v; v \in |X|)$, where $\boldsymbol{\pi}_v$ denotes a generator of the maximal ideal in the ring O_v of integers in F_v, $u_v \in O_v^\times$ and $n_v \in \mathbb{Z}$, we associate the divisor $D = \sum_v n_v v$, and the degree

$$\deg(a) = \deg(D) = \sum_v n_v \deg(v), \qquad \deg(v) = [\mathbb{F}_v : \mathbb{F}_q],$$

where $\mathbb{F}_v$ is the residue field of F at v, a finite field of $q_v = q^{\deg(v)}$ elements. For $g \in \mathrm{GL}(2, \mathbb{A})$ write $\deg g$ for $\deg \det g$. Recall that $O_\mathbb{A} = \prod_v O_v$ $(v \in |X|)$. For $t \in \mathbb{C}^\times$ we write

$$\nu_t(a) = t^{-\deg(a)} = \prod_v t_v^{-n_v}$$

where $t_v = t^{\deg(v)}$. Then $\nu_{q^{-1}}(a) = \prod_v q_v^{n_v} = |a|$ is equal to $\nu(a) = q^{\deg(a)}$. Also $\nu_t(\boldsymbol{\pi}_v) = t_v$, $\nu_{q^{-1}}(\boldsymbol{\pi}_v) = |\boldsymbol{\pi}_v|$.
Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves. Fix isomorphisms $i_\mathcal{L}$, $i_\mathcal{M}$ of their generic fibers with F. Each of $(\mathcal{L}, i_\mathcal{L})$ and $(\mathcal{M}, i_\mathcal{M})$ defines an element of $\mathbb{A}^\times/O_\mathbb{A}^\times$, namely a divisor on X. Choose representatives a, b in $\mathbb{A}^\times$, for example $\sum_v n_v v$ is represented by $(\boldsymbol{\pi}_v^{-n_v})$. Given an exact sequence $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \to 0$ of locally free sheaves, choose an isomorphism φ between the generic fiber of $\mathcal{E}$ and F^2 so that the induced exact sequence of generic fibers $0 \to F \to F^2 \to F \to 0$ is standard $\left(x \mapsto \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto y\right)$. The isomorphism φ is defined uniquely up to left multiplication by an automorphism of F^2 of the form $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$, $t \in F$. The pair $(\mathcal{E}, \varphi)$ determines an element of $\mathrm{GL}(2, \mathbb{A})/\mathrm{GL}(2, O_\mathbb{A})$, of the form $u = \left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, with z in $\mathbb{A}$. Since u is defined up to right multiplication by an element of $\mathrm{GL}(2, O)$, z is uniquely defined up to addition of an element of $\frac{a}{b}O_\mathbb{A}$. Replacing φ by $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)\varphi$ with $t \in F$ replaces z by $z + t$. Thus we get a bijection

$$\mathrm{Ext}(\mathcal{M}, \mathcal{L}) \to \mathbb{A}/(F + \frac{a}{b}O_\mathbb{A}).$$

This is an isomorphism of $\mathbb{F}_q$-vector spaces.
In summary, if the invertible sheaves $\mathcal{L}$ and $\mathcal{M}$ correspond to idèles a and b, then $\mathrm{Ext}(\mathcal{M}, \mathcal{L}) \simeq \mathbb{A}/(F + \frac{a}{b}O_\mathbb{A})$, and the map $\mathrm{Ext}(\mathcal{M}, \mathcal{L}) \to \mathrm{Bun}_2$ which associates to the exact sequence $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \to 0$ its middle term,

coincides with the map $\mathbb{A}/(F+\frac{a}{b}O_{\mathbb{A}}) \simeq H^1(X,\mathcal{M}^{-1}\mathcal{L})$, see [S97], II. 5. The isomorphism $\mathbb{A}/(F+\frac{a}{b}O_{\mathbb{A}}) \xrightarrow{\sim} \mathrm{Ext}(\mathcal{M},\mathcal{L})$ is $H^1(X,\mathcal{M}^{-1}\mathcal{L})\tilde{\to}\,\mathrm{Ext}(\mathcal{M},\mathcal{L})$.

2.3 The space $\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A})$

Proposition 2.4. *Given $a \in \mathbb{A}^\times$, $\deg a \geq 2g-1$, then $aO_{\mathbb{A}}+F=\mathbb{A}$.*

Proof. If $\mathcal{L}$ is an invertible sheaf on X associated with a, then $\mathbb{A}/(F+aO_{\mathbb{A}}) = H^1(X,\mathcal{L})$. By Serre duality $H^1(X,\mathcal{L}) \simeq H^0(X,\mathcal{L}^{-1}\omega)$, where ω is the canonical bundle of degree $2g-2$. Then $\deg(\mathcal{L}^{-1}\omega) \leq (2g-2)-(2g-1) = -1 < 0$, hence $H^0(X,\mathcal{L}^{-1}\omega)=\{0\}$. $\square$

Define a function

$$\mathrm{ht}^+ : \mathrm{GL}(2,\mathbb{A}) \to \mathbb{Z} \qquad \text{by} \qquad \mathrm{ht}^+\left(\left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right)k\right) = \deg a - \deg b$$

for all $a,b \in \mathbb{A}^\times$, $c \in \mathbb{A}$, $k \in \mathrm{GL}(2,O_{\mathbb{A}})$. It is clearly a well defined function on $B(F)\backslash\mathrm{GL}(2,\mathbb{A})$. For $x \in \mathrm{GL}(2,\mathbb{A})$, put

$$\mathrm{ht}(x) = \max_{\gamma\in\mathrm{GL}(2,F)} \mathrm{ht}^+(\gamma x).$$

On $\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A})$ it is well defined.

Proposition 2.5. *For any $x \in \mathrm{GL}(2,\mathbb{A})$ we have $-2g \leq \mathrm{ht}(x) < \infty$.*

Proof. This follows from Proposition 2.1 as if $\mathcal{E}$ is a rank two locally free sheaf of $\mathcal{O}_X$-modules associated to the image of x in $\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A})/\mathrm{GL}(2,O_{\mathbb{A}})$, then $\mathrm{ht}(x)=\mathrm{ht}(\mathcal{E})$. $\square$

Put $H_B = \{x \in B(F)\backslash\mathrm{GL}(2,\mathbb{A}); \mathrm{ht}^+(x) > 0\}$ and

$$H = \{x \in \mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A}); \mathrm{ht}(x) > 0\}.$$

Proposition 2.6. (1) *The restriction p to H_B of the natural projection p' : $B(F)\backslash\mathrm{GL}(2,\mathbb{A}) \to \mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A})$ is a homeomorphism $H_B \to H$.*
(2) *The set $\{x \in \mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A}); \mathrm{ht}(x) \leq n\}$ is compact modulo the center $Z(\mathbb{A})$ of $\mathrm{GL}(2,\mathbb{A})$ for every integer n.*

Proof. (1) The map p is clearly onto. To show that p is injective it suffices to show for any x in $\mathrm{GL}(2,\mathbb{A}), \gamma \in \mathrm{GL}(2,F)$, that $\mathrm{ht}^+(x) > 0$ and $\mathrm{ht}^+(\gamma x) > 0$ implies $\gamma \in B(F)$. This is a typical application of the Harder-Narasimhan filtration. In simple, explicit terms, this follows from

Lemma 2.7. *If $g \in \mathrm{GL}(2,F) - B(F)$ then $\mathrm{ht}^+(x) + \mathrm{ht}^+(gx) \leq 0$.*

Proof. Write g as g_1wg_2 with g_1,g_2 in $B(F)$, $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Put $x' = g_2x$. Then $\mathrm{ht}^+(x) = \mathrm{ht}^+(x')$, $\mathrm{ht}^+(gx) = \mathrm{ht}^+(wx')$. Thus we need to show that $\mathrm{ht}^+(x') + \mathrm{ht}^+(wx') \leq 0$. Suppose $x' = \left(\begin{smallmatrix} a_1 & c_1 \\ 0 & b_1 \end{smallmatrix}\right)k_1$, $wx' = \left(\begin{smallmatrix} a_2 & c_2 \\ 0 & b_2 \end{smallmatrix}\right)k_2$ with

$k_1, k_2 \in \mathrm{GL}(2, O_{\mathbb{A}})$. Put $k_2 k_1^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then $\begin{pmatrix} a_2 & c_2 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = w \begin{pmatrix} a_1 & c_1 \\ 0 & b_1 \end{pmatrix} = \begin{pmatrix} 0 & b_1 \\ a_1 & c_1 \end{pmatrix}$. Hence $b_2\gamma = a_1$. Thus $\deg a_1 \le \deg b_2$ (as $\deg \gamma \le 0$, since $\gamma \in O_{\mathbb{A}}$). But $\deg a_2 b_2 = \deg a_1 b_1$. Hence $\deg a_2 \le \deg b_1$. Then $\mathrm{ht}^+(x') + \mathrm{ht}^+(wx') = \deg a_1 - \deg b_1 + \deg a_2 - \deg b_2 \le 0$. □

Now the natural projection $p' : B(F)\backslash \mathrm{GL}(2, \mathbb{A}) \to \mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A})$ is open and H_B is an open subset of $B(F)\backslash \mathrm{GL}(2, \mathbb{A})$. Hence the bijection $p = p'|H_B : H_B \to H$ is open. Since it is also continuous, p is a homeomorphism.

(2) The image of the set $S = \{x \in B(F)\backslash \mathrm{GL}(2, \mathbb{A}); -2g \le \mathrm{ht}^+(x) \le n\}$ in $\mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A})$ under p' contains the set $\{x \in \mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A}); \mathrm{ht}(x) \le n\}$. So it suffices to show that S is compact mod $Z(\mathbb{A})$. Choose a compact C in $\mathbb{A}^\times$ with

$$CF^\times = \{t \in \mathbb{A}^\times; -2g \le \deg t \le n\}.$$

Choose an idèle d with $\deg d \ge 2g - 1$. Put

$$Y = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} k; \quad k \in \mathrm{GL}(2, O_{\mathbb{A}}),\ \ a, b \in \mathbb{A}^\times,\ \ \frac{a}{b} \in C,\ \ c \in dO_{\mathbb{A}} \right\}.$$

Lemma 2.8. *The map $Y \to S$ is surjective.*

Proof. Let $x \in \mathrm{GL}(2, \mathbb{A}), -2g \le \mathrm{ht}^+(x) \le n$. We need to show that x can be written as hy with $y \in Y$ and $h \in B(F)$. Write x as $\begin{pmatrix} r & s \\ 0 & t \end{pmatrix} K$ with $k \in \mathrm{GL}(2, O_{\mathbb{A}}), r, t \in \mathbb{A}^\times, s \in \mathbb{A}$. It remains to show that $\begin{pmatrix} r & s \\ 0 & t \end{pmatrix}$ can be expressed as $\begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with $a, b \in \mathbb{A}^\times$, $\frac{a}{b} \in C$, $c \in dO_{\mathbb{A}}$, $\alpha, \beta \in F^\times$, $\gamma \in F$. Thus we need to show the existence of $a, b, c, \alpha, \beta, \gamma$ such that

(*) $\quad a\alpha = r, \ \ \beta b = t, \ \ a, b \in \mathbb{A}^\times, \ \ \alpha, \beta \in F^\times, \ \ \frac{a}{b} \in C,$

(**) $\quad b(\alpha c + \gamma) = s, \ \ c \in dO_{\mathbb{A}}, \ \ \gamma \in F.$

By definition of x, $\deg r - \deg t$ lies in $[-2g, n]$, so the existence of a, b, α, β satisfying (*) follows from the definition of C. The existence of $c \in dO_{\mathbb{A}}$ and $\gamma \in F$ satisfying $\alpha c + \gamma = s/b$ follows from: $cO_{\mathbb{A}} + F = \mathbb{A}$ if $\deg c \ge 2g - 1$. □

Since Y is compact mod $Z(\mathbb{A})$, so is S, and (2) follows. □

In summary, the homogeneous space $\mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A})$ is the union of the compact mod $Z(\mathbb{A})$ set $\{x \in \mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A});\ \mathrm{ht}(x) \le 0\}$, and the set $H = \{x \in \mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A}); \mathrm{ht}(x) > 0\}$, whose structure is simpler. The set H_B, hence also the sets H and $\mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A})$, are noncompact modulo $Z(\mathbb{A})$. Indeed the function ht^+ takes arbitrary large values.

The image of H in $\mathrm{Bun}_2 = \mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A})/\mathrm{GL}(2, O_{\mathbb{A}})$ is the set of non-semistable locally free sheaves.

The set $\mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A})/\mathrm{GL}(2, O_{\mathbb{A}})$ is analogous to the set

$$\mathrm{SL}(2, \mathbb{Z})\backslash \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2) = \mathrm{SL}(2, \mathbb{Z})\backslash \mathfrak{h},$$

where $\mathfrak{h} = \{z \in \mathbb{C}; \mathrm{Im}\, z > 0\}$, the upper half plane, is isomorphic to $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$, by

$$g \mapsto g(i) = (ai + b)/(ci + d).$$

The set $B(F)\backslash \mathrm{GL}(2,\mathbb{A})/\mathrm{GL}(2,O_{\mathbb{A}})$ is analogous to $N\backslash\mathfrak{h}$ where N is the group of transformations $z \mapsto z+n$ $(n \in \mathbb{Z})$ on $\mathfrak{h}$. The function ht^+ is analogous to the function $z \mapsto \ln \mathrm{Im}\, z$ on $N\backslash\mathfrak{h}$. The statement $-2g \leq \mathrm{ht}(x) < \infty$ corresponds to the statement that the natural map from the half plane $\{z \in \mathbb{C}; \mathrm{Im}\, z \geq \sqrt{3}/2\}$ to $\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}$ is onto. The statement that $p : H_B \to H$ is homeomorphism corresponds to the statement that the map $N\backslash\{z \in \mathbb{C}; \mathrm{Im}\, z > 1\} \to \mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}$ is injective, and the compactness of $\{x \in \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A});\ \mathrm{ht}(x) \leq n\}$ corresponds to the statement that the complement in $\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}$ of the image of the half plane $\{z \in \mathbb{C}; \mathrm{Im}\, z > h\}$ is compact.

2.4 ℓ-groups

An *ℓ-space* is a Hausdorff topological space such that each of its points has a fundamental system of open compact neighborhoods.

We shall consider on ℓ-spaces only measures for which every open compact subset is measurable, and its volume is a rational number. If dx is such a measure on an ℓ-space Y, and f is a locally constant compactly supported function on Y with values in a field E of characteristic zero, then $\int_Y f(x)dx$ reduces to a finite sum, and it is well defined.

On topological groups we consider only left- or right-invariant measures.

An ℓ-group is a topological group with an ℓ-space structure.

Proposition 2.9. *Let G be an ℓ-group. Then* (1) *there exists a fundamental system of neighborhoods of the identity in G consisting of open compact subgroups*;
(2) *there exists a left Haar measure on G such that the volume of each open compact set is a rational number.*

Proof. (1) Let U be a neighborhood of the identity in G. We shall show that U contains an open compact subgroup. Since G is ℓ-space, we may assume that U is open and compact. Put $V = \{x \in G; xU \subset U\}$. Then $V = \cap_{u\in U} Uu^{-1}$, hence it is compact. Now for each v in V and u in U, by continuity of multiplication m there exists an open subset W_u containing v, and U_u in U containing u, such that $m(W_u, U_u) \subset U$. As U is compact and $U = \cup_{u\in U} U_u$, there are finitely many $u_1, \dots, u_n$ in U with $U = \cup_{1\leq i\leq n} U_{u_i}$. Then $W = \cap_{1\leq i\leq n} W_{u_i}$ is open in V and it contains v. Thus V is an open neighborhood of the identity, and $V \cdot V = V$. Then $V \cap V^{-1}$ is an open compact subgroup in U.
(2) Fix some left Haar measure on G. Denote the volume of an open compact subgroup U by $|U|$. For two such groups, U_1 and U_2 we have

$$\frac{|U_1|}{|U_2|} = \frac{|U_1|}{|U_1 \cap U_2|} \Big/ \frac{|U_2|}{|U_1 \cap U_2|} = \frac{[U_1 : U_1 \cap U_2]}{[U_2 : U_1 \cap U_2]} \in \mathbb{Q}.$$

Consequently the Haar measure on G can be chosen to assign rational volume to every open compact subgroup of G. But then the volume of every open compact subset K in G is rational, since as in (1) for such K there is a compact open subgroup U of G with $KU \subset K$, and then $|K| = [K : U]|U|$ is rational, where K is a disjoint union of $[K : U]$ translates of U. $\square$

Fix an ℓ-group G and a left Haar measure on G such that the volume of any open compact set is a rational number. Fix a field E of characteristic zero. The E-vector space H_G of compactly supported locally constant functions $f : G \to E$ is an algebra under the convolultion $(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1}g)dh$. For an open compact subgroup U in G the set of U-biinvariant functions in H_G is a subalgebra H_G^U, called the *Hecke algebra* of (G, U). Although H_G has no unit (unless G is discrete, when the δ-function is in H_G), H_G^U does: it is $\delta_U : G \to \mathbb{Q}$, the characteristic function of U divided by $|U|$.
A representation π of the group G on a vector space V is called *smooth* if the stabilizer of any vector of V is open, and *admissible* if it is smooth and for any open subgroup U of G the space V^U of U-fixed vectors in V is finite dimensional.
If π is a smooth representation of an ℓ-group G on a vector space V over E, for each $f \in H_G$ define the operator $\pi(f) : V \to V$ by $\pi(f)v = \int_G f(g)\pi(g)vdg$. This integral reduces to a finite sum since π is smooth, and $\pi(f_1 * f_2) = \pi(f_1) \circ \pi(f_2)$. Then V is naturally an H_G-module, and for any open compact subgroup U of G, the space V^U is a unital module over H_G^U.

PROPOSITION 2.10. (1) *A smooth G-module $V \neq \{0\}$ is irreducible iff for every open compact subgroup U of G either $V^U = 0$ or V^U is an irreducible H_G^U-module.*
(2) *Given an open compact subgroup U of G and an irreducible unital H_G^U-module M, there exists a smooth irreducible G-module V such that V^U is isomorphic to M as an H_G^U-module, and V is determined by this property up to isomorphism.*

For a proof see [BZ76], 2.10. See [BZ76], 2.11 for
Schur's Lemma. *Let π be an irreducible admissible representation of G in a vector space V over an algebraically closed field E. Then any nonzero G-module morphism* (*intertwining operator*) *$V \to V$ is a scalar.*

PROPOSITION 2.11. *Let π be an irreducible admissible representation of G in a vector space V over an algebraically closed field E. For any field extension E' of E, the representation of G in $V \otimes_E E'$ is also irreducible.*

Proof. By Proposition 2.10, the statement reduces to a similar statement for finite dimensional algebras, since π is assumed to be admissible. $\square$

Let E be a subfield of $\mathbb{C}$ invariant with respect to complex conjugation. A representation of G on a vector space V over E is *unitary* if there is a G-invariant scalar product on V (thus a bilinear function $(\cdot, \cdot) : V \times V \to E$ with $\overline{(v, w)} = (w, v)$ and $(v, v) = 0$ iff $v = 0$, and $(gv, gw) = (v, w)$ for all v, w in V and g in G).

Note that we do not require V to be complete with respect to the scalar product, even in the case $E = \mathbb{C}$. If E is algebraically closed and the representation of G in E is irreducible and admissible, then the G-invariant inner product on V is unique up to a scalar multiple, if it exists.

Proposition 2.12. *Let π be an admissible unitary representation of G in the E-space V. Fix a G-invariant scalar product on V. Let L be an invariant subspace of V, and $L^{\perp}$ its orthogonal complement. Then $V = L \oplus L^{\perp}$.*

Proof. Given $x \in V$, we need to express it as $x_1 + x_2$ with $x_1 \in L$ and $x_2 \in L^{\perp}$. Since π is smooth there exists a compact open subgroup U of G with $x \in V^U$. Since π is admissible, $\dim_E V^U$ is finite. Thus $x = x_1 + x_2$ for some $x_1 \in L^U$, $x_2 \in V^U$, x_2 orthogonal to L^U. It remains to show that x_2 is orthogonal to the entire space L. Let δ_U be the unit in H_G^U. Then $\pi(\delta_U)$ is the orthogonal projector $V \mapsto V^U$. Hence for every y in L, $(x_2, y) = (\pi(\delta_U)x_2, y) = (x_2, \pi(\delta_U)y) = 0$ since $\pi(\delta_U)y \in L^U$. ∎

It follows that every admissible unitary representation of G is a direct sum of irreducible representations. This sum is not necessarily finite. However, given an open compact subgroup U of G, only finitely many summands contain nonzero U-invariant vectors.

2.5 Automorphic forms

Let E be an algebraically closed field of characteristic zero. An *automorphic form* is a smooth function $\phi : \mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A}) \to E$, where by *smooth* we mean that there is an open subgroup U_ϕ of $\mathrm{GL}(2, \mathbb{A})$ such that $\phi(xu) = \phi(x)$ for all $u \in \mathrm{U}_\phi$ and $x \in \mathrm{GL}(2, \mathbb{A})$. A *cusp form* is an automorphic form ϕ with $\int_{\mathbb{A}/F} \phi\left(\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) x\right) dz = 0$ for all $x \in \mathrm{GL}(2, \mathbb{A})$.
Since ϕ is right locally constant (= smooth) and $\mathbb{A}/F$ is compact, the integral here is well defined and reduces to a finite sum.
Let A_0^E be the space of cusp forms $\phi : \mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A}) \to E$. The group $\mathrm{GL}(2, \mathbb{A})$ acts on A_0^E by right translation: $(r(h)\phi)(g) = \phi(gh)$. By a *character* of an ℓ-group G with values in E we mean a locally constant homomorphism $\chi : G \to E^\times$. If $E \subset \mathbb{C}$ such χ is called a *unitary* character if $|\chi(g)| = 1$ for all g in G.
Denote by $A_0^E(\chi)$ the space of $\phi \in A_0^E$ with $\phi(ax) = \chi(a)\phi(x), a \in \mathbb{A}^\times$ (identified with the center of $\mathrm{GL}(2, \mathbb{A})$), $x \in \mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A})$. The space $A_0^E(\chi)$ is invariant under the $\mathrm{GL}(2, \mathbb{A})$-action.
Let π be an irreducible representation of $\mathrm{GL}(2, \mathbb{A})$ over E. By Schur's lemma, there is a character $\chi : \mathbb{A}^\times \to E^\times$ such that for every a in $\mathbb{A}^\times, \pi(a)$ is multiplication by $\chi(a)$. This χ is called the *central character* of π.
If $V \subset A_0^E$ is an irreducible admissible representation π of $\mathrm{GL}(2, \mathbb{A})$ and χ is the central character of V, then $V \subset A_0^E(\chi)$. Since the center of $\mathrm{GL}(2, F)$ acts trivially on A_0^E, χ is trivial on $F^\times$. Thus every irreducible admissible $\pi \subset A_0^E$ lies in $A_0^E(\chi)$, where χ is the central character of π, which is a character of $\mathbb{A}^\times/F^\times$. The following is known also e.g. for $\mathrm{GL}(n)$.

Proposition 2.13. *Fix an open subgroup U of* $\mathrm{GL}(2,\mathbb{A})$*. There exists a compact mod $Z(\mathbb{A})$ subset K of* $\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A})$ *such that the support of any U-invariant cusp form is contained in K.*

Proof. We first show that there is an integer n such that given $z\in\mathbb{A}$ and $x\in\mathrm{GL}(2,\mathbb{A})$ with $\mathrm{ht}^+(x)\geq n$, there exist $u\in U$ and $\beta\in F$ with $\left(\begin{smallmatrix}1&z\\0&1\end{smallmatrix}\right)x=\left(\begin{smallmatrix}1&\beta\\0&1\end{smallmatrix}\right)xu$.
To see this, fix an effective divisor $-D=\sum_{v\in|X|}n_vv$ on X, put $d=(\boldsymbol{\pi}_v^{n_v})$ and let $J_D=dO_{\mathbb{A}}$ be the corresponding ideal in $O_{\mathbb{A}}$. The groups $\Gamma(D)=\{\gamma\in\mathrm{GL}(2,O_{\mathbb{A}});\gamma\equiv I\mod J_D\}$ make a basis of neighborhoods of the identity in $\mathrm{GL}(2,\mathbb{A})$. Thus we may assume in this proof that $U=\Gamma(D)$. In this case we shall show that $n=2g-1-\deg(d)$. Indeed, fix $z\in\mathbb{A}$ and $x=\left(\begin{smallmatrix}a&c\\0&b\end{smallmatrix}\right)k$ with $k\in\mathrm{GL}(2,O_{\mathbb{A}})$ and $\mathrm{ht}^+(x)=\deg a-\deg b\geq 2g-1-\deg(d)$ (note: $\deg(d)=-\deg D=\sum_v n_v\deg v$). Then $\frac{ad}{b}O_{\mathbb{A}}+F=\mathbb{A}$ and $z=\frac{ad}{b}t+\beta$ for some $\beta\in F$ and $t\in O_{\mathbb{A}}$. Put $u=k^{-1}\left(\begin{smallmatrix}1&td\\0&1\end{smallmatrix}\right)k$. Then $u\in\Gamma(D)$ and $\left(\begin{smallmatrix}1&z\\0&1\end{smallmatrix}\right)x=\left(\begin{smallmatrix}1&\beta\\0&1\end{smallmatrix}\right)xu$.
We claim the proposition holds with $K=\{x\in\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A});\mathrm{ht}(x)<n\}$. This K is compact modulo $Z(\mathbb{A})$. Let ϕ be a U-invariant cusp form, $x\in\mathrm{GL}(2,\mathbb{A})$, $\mathrm{ht}(x)\geq n$. We shall show that $\phi(x)=0$. Replacing x by γx for suitable $\gamma\in\mathrm{GL}(2,F)$, we assume that $\mathrm{ht}^+(x)\geq n$. By our choice of n, $\phi(\left(\begin{smallmatrix}1&z\\0&1\end{smallmatrix}\right)x)=\phi(x)$ for all z in $\mathbb{A}$. Since ϕ is a cusp form, $\phi(x)=0$. □

Corollary 2.14. *The representation of* $\mathrm{GL}(2,\mathbb{A})$ *in $A_0^E(\chi)$ is admissible.*

Proposition 2.15. *Let E' be an extension of E, and $\chi:\mathbb{A}^\times/F^\times\to E^\times$ a character. Then $A_0^{E'}(\chi)=A_0^E(\chi)\otimes_E E'$.*

Proof. The space $A_0^E(\chi)\otimes_E E'$ consists of the functions ϕ in $A_0^{E'}(\chi)$ whose values span a finite dimensional space over E, since $\phi\in A_0^E(\chi)$ takes finite number of values times the set Γ of values of χ. But every ϕ in $A_0^{E'}(\chi)$ has this property, since the set of its values lies in finitely many cosets of Γ. □

Given a representation π of $\mathrm{GL}(2,\mathbb{A})$ over E and a character $\omega:\mathbb{A}^\times\to E^\times$, write $\omega\pi$ or $\pi\omega$ or $\omega\otimes\pi$ or $\pi\otimes\omega$ for the representation $(\pi\omega)(x)=\omega(\det x)\pi(x)$ in the space of π.

Proposition 2.16. *For any characters χ, $\omega:\mathbb{A}^\times/F^\times\to E^\times$, we have* $A_0^E(\chi)\otimes\omega=A_0^E(\chi\omega^2)$.

Proof. We need to construct an invertible linear map $L:A_0^E(\chi)\to A_0^E(\chi\omega^2)$ such that for every $\phi\in A_0^E(\chi)$ and $h\in\mathrm{GL}(2,\mathbb{A})$ we have $r(h)L(\phi)=\omega(\det h)L(r(h)\phi)$, where $(r(h)\phi)(x)=\phi(xh)$.
Such L is $(L\phi)(x)=\phi(x)\omega(\det x)$. □

Proposition 2.17. *Given a character $\chi:\mathbb{A}^\times/F^\times\to E^\times$ there exists a character $\omega:\mathbb{A}^\times/F^\times\to E^\times$ such that $\chi(x)\omega(x)^2$ is a root of unity for every x in $\mathbb{A}^\times/F^\times$.*

Proof. Fix $\alpha \in \mathbb{A}^\times/F^\times$ with $\deg \alpha = 1$. Such α exists since in the finite field extension $F/\mathbb{F}_q(t)$, where $t \in F$ is transcendental over $\mathbb{F}_q$, there are always primes which split completely. Fix c in the algebraically closed field E with $c^2 = \chi(\alpha)$. Define $\omega : \mathbb{A}^\times/F^\times \to E^\times$ by $\omega(x) = c^{-\deg(x)}$, put $\chi_1(x) = \chi(x)\omega^2(x)$, put $\alpha^{\mathbb{Z}} = \{\alpha^n; n \in \mathbb{Z}\}$. Then χ_1 is a character of the profinite group $\mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}}$, hence the values of χ_1 are roots of 1. □

Proposition 2.18. *Let E be a subfield of $\mathbb{C}$ invariant under complex conjugation, χ an $E^\times$-valued unitary character of $\mathbb{A}^\times/F^\times$. Then the representation of* $\mathrm{GL}(2,\mathbb{A})$ *in $A_0^E(\chi)$ is unitary.*

Proof. The function $x \mapsto \phi_1(x)\overline{\phi}_2(x)$ on $\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A})$, where ϕ_1, $\phi_2 \in A_0^E(\chi)$, is invariant under $Z(\mathbb{A})$ and is compactly supported as a function on $\mathrm{PGL}(2,F)\backslash\mathrm{PGL}(2,\mathbb{A})$. Let dx be an invariant measure on $\mathrm{PGL}(2,F)\backslash\mathrm{PGL}(2,\mathbb{A})$. It exists since $\mathrm{PGL}(2,F)$ is a discrete subgroup of $\mathrm{PGL}(2,\mathbb{A})$, a group with a two-sided invariant measure. Then

$$(\phi_1,\phi_2) = \int \phi_1(x)\overline{\phi}_2(x)dx \qquad (x \in \mathrm{PGL}(2,F)\backslash\mathrm{PGL}(2,\mathbb{A}))$$

is an invariant scalar product on $A_0^E(\chi)$. □

Corollary 2.19. *The representation of* $\mathrm{GL}(2,\mathbb{A})$ *in $A_0^E(\chi)$ is a direct sum of irreducible subrepresentations.*

Note that we may assume that all values of χ are roots of unity, and that $E = \overline{\mathbb{Q}}$.
The *multiplicity one theorem* asserts that in $A_0^E(\chi)$ any irreducible representation of $\mathrm{GL}(2,\mathbb{A})$ occurs with multiplicity one.
An irreducible representation of $\mathrm{GL}(2,\mathbb{A})$ over an algebraically closed field E is called *cuspidal* if it is isomorphic to a subrepresentation of A_0^E.

2.6 Factorizability

Irreducible admissible representations of $\mathrm{GL}(2,\mathbb{A})$ are factorizable, as we proceed to show. Let E denote an algebraically closed subfield of $\mathbb{C}$. An irreducible representation of $\mathrm{GL}(2,F_v)$ in an E-space V is *unramified* if V contains a nonzero $\mathrm{GL}(2,O_v)$-invariant vector.

Proposition 2.20. *The space of* $\mathrm{GL}(2,O_v)$*-invariant vectors $V^{\mathrm{GL}(2,O_v)}$ in an unramified representation (π,V) of* $\mathrm{GL}(2,F_v)$ *is one dimensional.*

Proof. Denote by $H_v = C_c(\mathrm{GL}(2,O_v)\backslash\mathrm{GL}(2,F_v)/\mathrm{GL}(2,O_v))$ the Hecke convolution algebra of compactly supported $\mathrm{GL}(2,O_v)$-biinvariant E-valued functions on $\mathrm{GL}(2,F_v)$. We claim it is a commutative algebra. Indeed, for any $f \in H_v$, the function ${}^tf(x) = f({}^tx)$, where ${}^tx$ is the transpose of x, is also in H_v. Since ${}^t(xy) = {}^ty{}^tx$, we have ${}^t(f_1 * f_2) = {}^tf_2 * {}^tf_1$ for all $f_1, f_2 \in H_v$. By Cartan decomposition every $\mathrm{GL}(2,O_v)$-double coset in $\mathrm{GL}(2,F_v)$ contains a diagonal matrix. Hence ${}^t f = f$ for all $f \in H_v$, and $f_1 * f_2 = {}^t(f_1 * f_2) = {}^t f_2 * {}^t f_1 = f_2 * f_1$ for all $f_1,\ f_2 \in H_v$. If V is unramified, $V^{\mathrm{GL}(2,O_v)}$ is a nonzero irreducible H_v-module. But H_v is commutative, so $\dim_E V^{\mathrm{GL}(2,O_v)}$ is 1. □

Given an irreducible admissible representation π_v of $\mathrm{GL}(2,F_v)$ in a space V_v for every closed point $v \in |X|$ such that π_v is unramified for all $v \in S$, $S \subset |X|$ finite, construct a representation $\pi = \otimes \pi_v$ of $\mathrm{GL}(2,\mathbb{A})$ as follows. For each $v \in |X| - S$ choose a nonzero vector $\xi_v^0 \in V_v^{\mathrm{GL}(2,O_v)}$. For any finite set $S' \supset S$ of closed points of X put $V_{S'} = \otimes_{v \in S'} V_v$. If $S'' \supset S' \supset S$, define an inclusion $V_{S'} \hookrightarrow V_{S''}$ by $x \mapsto (\otimes_{v \in S''-S'} \xi_v^0) \otimes x$. Put $V = \lim\limits_{\overrightarrow{S' \supset S}} V_{S'}$. It is the span of the vectors $\otimes_{v \in |X|} \xi_v$, $\xi_v = \xi_v^0$ for almost all v, and $\xi_v \in V_v$ for all $v \in |X|$. Then V is a $\mathrm{GL}(2,\mathbb{A})$-module in a natural way; denote by π the corresponding representation of $\mathrm{GL}(2,\mathbb{A})$. The vectors ξ_v^0 are determined uniquely up to a scalar multiple, hence π is uniquely determined by the π_v for all $v \in |X|$.
Reducing to irreducible finite dimensional representations of tensor products of algebras, we have

Proposition 2.21. *Given an irreducible admissible representation π_v of $\mathrm{GL}(2,F_v)$ for every v in $|X|$ which is unramified for almost all v, $\pi = \otimes_v \pi_v$ is an irreducible admissible representation of $\mathrm{GL}(2,\mathbb{A})$. Every irreducible admissible representation π of $\mathrm{GL}(2,\mathbb{A})$ equals $\otimes_v \pi_v$ for some irreducible admissible representations π_v of $\mathrm{GL}(2,F_v)$ which are almost all unramified. The representations π_v are determined by π uniquely up to isomorphism.*

3 Looking for a trace formula

3.1 Trace formula in the compact case

Let X be an ℓ-space. Denote by $C^\infty(X)$ the space of E-valued locally constant (= smooth) functions on X. Here E is a fixed algebraically closed subfield of $\mathbb{C}$. Let $C_c^\infty(X)$ be the space of smooth compactly supported E-valued functions on X. Let r be an admissible representation of an ℓ-group G in an E-space V. Fix a Haar measure dx on G. Given $f \in C_c^\infty(G)$, define $r(f) = \int_G f(x) r(x) dx$, an endomorphism of V. Since f is C^∞, that is smooth, it is right invariant under an open subgroup U of G. Then $\operatorname{Im} r(f) \subset V^U$, so $\operatorname{Im} r(f)$ is finite dimensional, and the trace $\operatorname{tr} r(f)$ is well defined. Let r be now the representation of G on $C^\infty(\Gamma\backslash G)$ by right translation, where Γ is a discrete cocompact subgroup of G. Since r is admissible, $\operatorname{tr} r(f)$ is defined.

Proposition 3.1. *Let G be an ℓ-group. Let Γ be a discrete cocompact sugroup of G. Then G has a two sided invariant measure and $\Gamma\backslash G$ has a G-invariant measure.*

Proof. Since (see [BZ76]) $\Gamma\backslash G$ admits a measure which when translated by x in G is multiplied by $\Delta(x)$, where Δ is the modulus of G, we have $|\Gamma\backslash G| = \Delta(x)|\Gamma\backslash G|$, thus $\Delta = 1$. □

Proposition 3.2. *Let X be an ℓ-space, dx a measure on X, $K \in C_c^\infty(X \times X)$. Define a linear endomorphism A of $C^\infty(X)$ by $(A\phi)(y) = \int_X K(x,y)\phi(x)dx$. Then the image of A is finite dimensional and* $\operatorname{tr} A = \int_X K(x,x)dx$.

Proof. We may assume that $K(x,y)$ is of the form $\varphi(x)\psi(y)$, as such functions span $C_c^\infty(X \times X)$. In this case the claim is clear. □

Proposition 3.3. *Let G be an ℓ-group, Γ a discrete cocompact subgroup, r the representation of G in $C^\infty(\Gamma\backslash G)$ by right translation, dx a Haar measure on $G, f \in C_c^\infty(G), S$ a set of representatives of the conjugacy classes in $\Gamma, Z_\Gamma(\gamma)$ the centralizer of γ in Γ. Then* $\operatorname{tr} r(f) = \sum_{\gamma \in S} \int_{G/Z_\Gamma(\gamma)} f(x\gamma x^{-1})dx$.

Proof. We first show that for each $\gamma \in \Gamma$ the function $x \mapsto f(x\gamma x^{-1})$ on $G/Z_\Gamma(\gamma)$ is compactly supported, and that there are at most finitely many $\gamma \in S$ for which $x \mapsto f(x\gamma x^{-1})$ is not identically zero. For this, fix a compact subset K in G with $K\Gamma = G$. Given $x \in G$ there are $k \in K, \delta \in \Gamma$, with $x = k\delta$. Fix $\gamma \in \Gamma$. If $f(x\gamma x^{-1}) \neq 0$ then $k\delta\gamma\delta^{-1}k^{-1}$ lies in $\operatorname{supp} f$, thus $\delta\gamma\delta^{-1} \in K_f = K{\cdot}\operatorname{supp} f \cdot K$. Since K_f is compact $K_f \cap \Gamma$ is finite, and there are only finite number of possibilities for $\delta\gamma\delta^{-1}$. Hence there are only a finite number of possibilities $\delta_1, \dots, \delta_n$ for δ modulo $Z_\Gamma(\gamma)$. Then $f(x\gamma x^{-1}) \neq 0$ implies that $x \in K'Z_\Gamma(\gamma)$, where $K' = \cup_{1 \le i \le n} K\delta_i$ is compact. If $f(x\gamma x^{-1}) \neq 0$, the conjugacy class of γ in Γ intersects the finite set $K_f \cap \Gamma$. The number of such classes is finite. Thus the sum is finite and the integrals converge.
Now given ϕ in $C^\infty(\Gamma\backslash G)$, for any y in G we have

$$(r(f)\phi)(y) = \int_G f(x)\phi(yx)dx = \int_G f(y^{-1}x)\phi(x)dx = \int_{\Gamma\backslash G} K_f(x,y)\phi(x)dx$$

where $K_f(x,y) = \sum_{\gamma \in \Gamma} f(y^{-1}\gamma x)$. Then

$$\operatorname{tr} r(f) = \int_{\Gamma\backslash G} K_f(x,x)dx = \int_{\Gamma\backslash G} \sum_{\gamma \in \Gamma} f(x^{-1}\gamma x)dx$$

$$= \int_{\Gamma\backslash G} \sum_{\gamma \in S} \sum_{\delta \in Z_\Gamma(\gamma)\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)dx = \sum_{\gamma \in S} \int_{\Gamma\backslash G} \sum_{\delta \in Z_\Gamma(\gamma)\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)dx$$

$$= \sum_{\gamma \in S} \int_{Z_\Gamma(\gamma)\backslash G} f(x^{-1}\gamma x)dx.$$

□

3.2 Case of GL(2), oversimplified

Let now A_0^E denote the space of E-valued cusp forms on $\mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$. The right-shifts representation of $\mathrm{GL}(2,\mathbb{A})$ on A_0^E is not admissible since the center $Z(\mathbb{A})$ of $\mathrm{GL}(2,\mathbb{A})$ is not compact. Fix a degree-one idèle α and put $\alpha^{\mathbb{Z}} = \{\alpha^n; n \in \mathbb{Z}\}$. It is a cyclic subgroup of $\mathbb{A}^\times$, and we view $\mathbb{A}^\times$ as the center of $\mathrm{GL}(2,\mathbb{A})$. Denote by $A^E_{0,\alpha}$ the space of cusp forms in A^E_0 invariant under α, and by r_0 the representation of $\mathrm{GL}(2,\mathbb{A})$ on $A^E_{0,\alpha}$ by right translation. Since $\mathbb{A}^\times/F^\times\alpha^{\mathbb{Z}}$ is compact and every U-invariant cusp form – where U is an open subgroup of $\mathrm{GL}(2,\mathbb{A})$ – is supported on some compact modulo $Z(\mathbb{A})$ set $K \subset \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$, the representation r_0 is admissible. Hence $\operatorname{tr} r_0(f)$ is defined for every $f \in C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$.

Put $A_{c,\alpha} = C_c^\infty(\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}))$. Fix $f \in C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$. Let r be the right representation of $\mathrm{GL}(2,\mathbb{A})$ on $A_{c,\alpha}$. We proceed to compute $\operatorname{tr} r(f)$ as if the space $\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$ were compact, to see what needs to be corrected. This space is not compact and r is not admissible, so that in fact $\operatorname{tr} r(f)$ makes no sense.

For any ring R define $A(R) = \{\mathrm{diag}(a,b); a,b \in R^\times\}$, $A'(R) = \{\mathrm{diag}(a,b); a,b \in R^\times,\ a \neq b\}$, $N(R) = \{\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right); a \in R\}$. Let Q be the set of quadratic extensions of the field F. For each $L \in Q$ choose an embedding $L \hookrightarrow M(2,F)$; it exists and is unique up to an automorphism of $M(2,F)$; all automorphisms of $M(2,F)$ are inner. Given $\gamma \in \alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)$, denote by $Z(\gamma)$ the centralizer of γ in $\alpha^{\mathbb{Z}}\,\mathrm{GL}(2,F)$.

PROPOSITION 3.4. *Every conjugacy class of* $\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)$ *intersects precisely one of* : $F^\times \cdot \alpha^{\mathbb{Z}}$; $a\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $a \in F^\times \cdot \alpha^{\mathbb{Z}}$; $\alpha^{\mathbb{Z}} \cdot A'(F)$; $\alpha^{\mathbb{Z}} \cdot (L^\times - F^\times)$ *for some* $L \in Q$. *In the first two cases the number of intersection points is* 1, *in the* 3*rd case* 2, *in the* 4*th case*: *the number of automorphisms of* L *over* F. *The centralizers* $Z(\gamma)$ *are* $\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)$, $\alpha^{\mathbb{Z}} F^\times N(F)$, $\alpha^{\mathbb{Z}} \cdot A(F)$, $\alpha^{\mathbb{Z}} L^\times$, *respectively.*

Immitating the trace formula in the compact case, one may expect

$$\operatorname{tr} r(f) = S_1(f) + \sum_{L \in Q} S_{2,L}(f) + S_3(f) + S_4(f)$$

with

$$S_1(f) = |\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})|,$$

$$S_{2,L}(f) = |\operatorname{Aut}_F(L)|^{-1} \sum_{\gamma \in \alpha^{\mathbb{Z}} \cdot (L^\times - F^\times)} \int_{\alpha^{\mathbb{Z}} \cdot L^\times \backslash \mathrm{GL}(2,\mathbb{A})} f(x^{-1}\gamma x)dx,$$

$$S_3(f) = \frac{1}{2} \sum_{\gamma \in \alpha^{\mathbb{Z}} A'(F)} \int_{\alpha^{\mathbb{Z}} A(F)\backslash \mathrm{GL}(2,\mathbb{A})} f(x^{-1}\gamma x)dx,$$

$$S_4(f) = \sum_{a \in \alpha^{\mathbb{Z}} \cdot F^\times} \int_{\alpha^{\mathbb{Z}} F^\times N(F)\backslash \mathrm{GL}(2,\mathbb{A})} f(x^{-1} a \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) x)dx.$$

The left side of this wrong trace formula is divergent. So is $S_3(f)$, since the homogeneous space $A(\mathbb{A})/\alpha^{\mathbb{Z}} \cdot A(F)$ is not compact. We shall show that $S_1(f)$ and $\sum_{L \in Q} S_{2,L}(f)$ converge, and although $S_4(f)$ diverges, we shall show in which way it does.

PROPOSITION 3.5. *Given* $f \in C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$, *the number of conjugacy classes of* $\gamma \in \alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)$ *with* $x \in \mathrm{GL}(2,\mathbb{A})$ *and* $f(x\gamma x^{-1}) \neq 0$ *is finite.*

Proof. The sets $K_1 = \{\operatorname{tr} h; h \in \operatorname{supp} f\} \subset \mathbb{A}$, $K_2 = \{\det h; h \in \operatorname{supp} f\} \subset \mathbb{A}^\times$ are compact. It suffices to show that the set $\{\gamma \in \alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F); \operatorname{tr}\gamma \in K_1, \det\gamma \in K_2\}$ is a union of finitely many conjugacy classes. Put $\gamma = \alpha^n x$ for some $x \in \mathrm{GL}(2,F)$. Then $2n = \deg\gamma$, so n lies in a finite set. Fix n. Then $\operatorname{tr} x \in \alpha^{-n}K_1, \det x \in \alpha^{-2n}K_2$. But the sets $F \cap \alpha^{-n}K_1$ and $F^\times \cap \alpha^{-2n}K_2$ are finite. Hence the trace and determinant of x can take only finitely many values. As the number of conjugacy classes of elements in $\mathrm{GL}(2,F)$ with fixed trace and determinant is at most two, we are done. □

3.3 Central elements

Proposition 3.6. *The volume* $|\,\mathrm{GL}(2,F)\cdot\alpha^{\mathbb{Z}}\backslash\,\mathrm{GL}(2,\mathbb{A})|$ *is finite.*

Proof. This volume is equal to (below $x \in \alpha^{\mathbb{Z}}\,\mathrm{GL}(2,F)\backslash\,\mathrm{GL}(2,\mathbb{A})/\,\mathrm{GL}(2,O_{\mathbb{A}})$)

$$\begin{aligned}&\textstyle\sum_x |\alpha^{\mathbb{Z}}\,\mathrm{GL}(2,F) \cap x\,\mathrm{GL}(2,O_{\mathbb{A}})x^{-1}\backslash x\,\mathrm{GL}(2,O_{\mathbb{A}})|\\ &= |\,\mathrm{GL}(2,O_{\mathbb{A}})| \textstyle\sum_x |\alpha^{\mathbb{Z}}\,\mathrm{GL}(2,F) \cap x\,\mathrm{GL}(2,O_{\mathbb{A}})x^{-1}|^{-1}.\end{aligned}$$

For x in $\mathrm{GL}(2,\mathbb{A})/\,\mathrm{GL}(2,O_{\mathbb{A}})$, let $\mathcal{E} = xO_{\mathbb{A}}^2$ be the associated rank 2 locally free sheaf on X. Then $\mathrm{Aut}(\mathcal{E})$ consists of the $g \in \mathrm{GL}(2,\mathbb{A})$ which map $(\mathcal{E} =)xO_{\mathbb{A}}^2$ to $xO_{\mathbb{A}}^2$ and the generic fiber F^2 to itself, thus $\operatorname{Aut}\mathcal{E}$ is

$$\mathrm{GL}(2,F) \cap x\,\mathrm{GL}(2,O_{\mathbb{A}})x^{-1} = \alpha^{\mathbb{Z}}\,\mathrm{GL}(2,F) \cap x\,\mathrm{GL}(2,O_{\mathbb{A}})x^{-1}.$$

We then need to show the convergence of

$$\sum_{\mathcal{E}\in\mathrm{Bun}_2/J} |\operatorname{Aut}\mathcal{E}|^{-1},$$

J being the image of $\alpha^{\mathbb{Z}}$ under the natural homomorphism $\mathbb{A}^\times \to \operatorname{Pic} X$. The number of J-orbits on the set of stable rank two locally free sheaves on X is finite, so it remains to show that the sum of $|\operatorname{Aut}\mathcal{E}|^{-1}$ over the set $\mathrm{Bun}_2^{\mathrm{un}}$ of J-orbits of unstable rank two locally free sheaves on X is convergent.

Lemma 3.7. (1) *A rank two locally free sheaf* $\mathcal{E}$ *on* X *is very unstable* ($\mathrm{ht}(\mathcal{E}) \geq 2g-1$) *iff* $\mathcal{E} \simeq \mathcal{L}\oplus\mathcal{M}$ *where* $\mathcal{L}$, $\mathcal{M}$ *are invertible sheaves with* $\deg\mathcal{L} - \deg\mathcal{M} \geq 2g-1$.
(2) *If* $\mathcal{L},\mathcal{M} \in \operatorname{Pic} X$ *and* $\deg\mathcal{L} - \deg\mathcal{M} \geq \max(2g-1,1)$ *then*

$$|\operatorname{Aut}(\mathcal{L}\oplus\mathcal{M})| = (q-1)^2 q^{\deg\mathcal{L}-\deg\mathcal{M}+1-g}.$$

(3) *If* $\mathcal{L}\oplus\mathcal{M} \simeq \mathcal{L}'\oplus\mathcal{M}'$ *with* $\deg\mathcal{L} > \deg\mathcal{M}$, $\deg\mathcal{L}' > \deg\mathcal{M}'$ *then* $\mathcal{L} \simeq \mathcal{L}', \mathcal{M} \simeq \mathcal{M}'$.

Proof. (1) If $\mathcal{L}$ is an invertible sheaf of $\mathcal{E}$ of maximal degree and $\mathcal{M} = \mathcal{E}/\mathcal{L}$, then $\mathcal{M}$ is invertible, and $\mathrm{Ext}(\mathcal{M},\mathcal{L}) = H^1(X,\mathcal{M}^{-1}\mathcal{L})$ is 0 (by Serre duality) as

$$\deg\mathcal{M}^{-1}\mathcal{L} = \deg\mathcal{L} - \deg\mathcal{M} = 2\deg\mathcal{L} - \deg\mathcal{E} = \mathrm{ht}(\mathcal{E}) \geq 2g-1.$$

The exact sequence

$$0 \to \mathrm{Hom}(\mathcal{M}, \mathcal{L}) \to \mathrm{Aut}(\mathcal{L} \oplus \mathcal{M}) \to \mathrm{Aut}\,\mathcal{L} \times \mathrm{Aut}\,\mathcal{M} \to 0$$

implies (2) since $\mathrm{Hom}(\mathcal{M}, \mathcal{L}) = H^0(X, \mathcal{M}^{-1}\mathcal{L})$ and $H^1(X, \mathcal{M}^{-1}\mathcal{L}) = \{0\}$, so the Riemann-Roch theorem implies that $\dim H^0(X, \mathcal{M}^{-1}\mathcal{L}) = \deg(\mathcal{M}^{-1}\mathcal{L}) + 1-g$. If the invertible sheaf $\mathcal{L}$ corrsponds to $aO_{\mathbb{A}}$, then $\mathrm{Aut}\,\mathcal{L}$ consists of $g \in \mathbb{A}^\times$ which map the generic fiber F onto itself (thus $g \in F^\times$) and map $aO_{\mathbb{A}}$ onto itself (thus $g \in O_{\mathbb{A}}^\times$). Then $\mathrm{Aut}\,\mathcal{L} = F^\times \cap O_{\mathbb{A}}^\times = \mathbb{F}_q^\times$ has cardinality $q-1$.
For (3), put $\mathcal{E} = \mathcal{L} \oplus \mathcal{M} \xrightarrow{\sim} \mathcal{L}' \oplus \mathcal{M}'$. Since $\deg \mathcal{L} > (\deg \mathcal{E})/2 > \deg \mathcal{M}'$, we have $\mathrm{Hom}(\mathcal{L}, \mathcal{M}') = \{0\}$. Hence the image of $\mathcal{L}$ under the isomorphism $\mathcal{L} \oplus \mathcal{M} \xrightarrow{\sim} \mathcal{L}' \oplus \mathcal{M}'$ lies in $\mathcal{L}'$. Hence $\mathcal{L} \simeq \mathcal{L}'$ and $\mathcal{M} \simeq \mathcal{E}/\mathcal{L} \simeq \mathcal{E}/\mathcal{L}' \simeq \mathcal{M}'$. □

Assume $g \geq 1$, so that $2g-1 \geq 1$ (the case $g=0$ is similar). The lemma implies

$$\sum_{\mathcal{E} \in \mathrm{Bun}_2^{\mathrm{un}}/J} |\,\mathrm{Aut}\,\mathcal{E}|^{-1} = (q-1)^{-2} |\,\mathrm{Pic}^0(X)| \sum_{n \geq 2g-1} q^{g-1-n} < \infty.$$

□

Corollary 3.8. *If the Haar measure on* $\mathrm{GL}(2,\mathbb{A})$ *is normalized so that* $|\,\mathrm{GL}(2, O_{\mathbb{A}})|$ *is a rational number, then* $|\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F) \backslash \mathrm{GL}(2,\mathbb{A})| \in \mathbb{Q}$.

This follows from the proof of the last proposition.

3.4 Elliptic elements

Proposition 3.9. *Let* L *be a quadratic extension of* F, $\gamma \in \alpha^{\mathbb{Z}} \cdot (L^\times - F^\times) \subset \mathrm{GL}(2,\mathbb{A})$, *and* $f \in C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$. *Then the function* $x \mapsto f(x\gamma x^{-1})$ *on* $\mathrm{GL}(2,\mathbb{A})/\alpha^{\mathbb{Z}} \cdot L^\times$ *has compact support.*

Proof. We need to show that the map $x \mapsto x\gamma x^{-1}$ on $\mathrm{GL}(2,\mathbb{A})/\alpha^{\mathbb{Z}} \cdot L^\times$ is proper (the preimage of a compact is compact). Since $(L \otimes_F \mathbb{A})^\times/\alpha^{\mathbb{Z}} \cdot L^\times$ is compact, it suffices to show that the map $\psi(x) = x\gamma x^{-1}$, $\psi : \mathrm{GL}(2,\mathbb{A})/\mathbb{A}_L^\times \to \mathrm{GL}(2,\mathbb{A})$, is proper ($\mathbb{A}_L = L \otimes_F \mathbb{A}$ is the ring of adèles of L).

Lemma 3.10. *Let* F *be a local field in this lemma. Suppose* $\gamma \in M(2,F)$ *is regular, i.e. the subalgebra* $E = F[\gamma]$ *generated by* γ *is a field or is* $F \times F$. *Then the map* $\psi : \mathrm{GL}(2,F)/E^\times \to \mathrm{GL}(2,F)$, $x \mapsto x\gamma x^{-1}$, *is proper. Moreover, if* $\gamma \in \mathrm{GL}(2,O)$ *and the ring* $O[\gamma]$ *is integrally closed, then* $\psi^{-1}(\mathrm{GL}(2,O)) = \mathrm{GL}(2,O)/E^\times \cap \mathrm{GL}(2,O)$.

Proof. The conjugacy class C of γ is a closed subset of $\mathrm{GL}(2,F)$, since γ is regular. So it suffices to show that ψ maps $\mathrm{GL}(2,F)/E^\times$ homeomorphically onto C. It is clear that ψ is continuous, injective and $\mathrm{Im}\,\psi = C$. It remains to show that the map $\psi' : \mathrm{GL}(2,F) \to C$, $x \mapsto x\gamma x^{-1}$, is open. For this, it suffices to show that C is the set of F-points of a smooth variety $\mathbf{C}$ over F, and that ψ'

is smooth, that is its differential is everywhere onto. Since $\mathbf{C}$ is a homogeneous space under a connected group $\mathbf{G}$ is suffices to show that the tangent map $d\psi'$ of ψ' at the identity is onto. When verifying these properties of $\mathbf{C}$ and ψ', we may replace F with an extension, thus we may assume that γ is of the form $\mathrm{diag}(a,b)$ with $a \neq b$, or $\left(\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right)$ (if E is nonseparable over F). To compute the tangent map $d\psi' : \mathrm{Lie}\ G \to T_\gamma(\mathbf{C})$ of $\psi'(x) = x\gamma x^{-1}$ near the identity $x = 1$, let Y be in Lie G, and put $x = 1 + \epsilon Y$, where $\epsilon^2 = 0$. Then $x^{-1} = 1 - \epsilon Y$ and $\psi'(x) = (1+\epsilon Y)\gamma(1-\epsilon Y) = 1 + \epsilon(Y\gamma - \gamma Y)$, so $d\psi'(Y) = Y\gamma - \gamma Y$ is onto the tangent space $T_\gamma(\mathbf{C})$ of $\mathbf{C}$ at γ, and ψ is proper.
If $x \in \mathrm{GL}(2,F)$ and $x\gamma x^{-1} \in \mathrm{GL}(2,O)$, put $M = x^{-1}O^2$. Then $\gamma M \subset M$. In addition, $\gamma \in \mathrm{GL}(2,O)$, so $\gamma O^2 \subset O^2$. Thus M and O^2 are $O[\gamma]$-submodules in F^2. Both modules are of finite type. As F^2 is a rank one free $E = F[\gamma]$-module, and we assume that $O[\gamma]$ is integrally closed, namely it is the ring of integers in $E = F[\gamma]$, both M and O^2 are rank one torsion free over the discrete valuation ring $O[\gamma]$ (being rank two over O). Hence there exists $a \in E^\times$ with $M = aO^2$. Thus $xaO^2 = O^2$, that is $xa \in \mathrm{GL}(2,O)$. □

Now for γ as in the proposition, for almost all closed points in X the component of α at v is 1, $\gamma \in \mathrm{GL}(2,O_v)$, and the ring $O_v[\gamma]$ is integrally closed. This and the lemma imply the proposition. □

3.5 Regularization of the unipotent terms

To study the integral which occurs in $S_4(f)$, we regularize it as

$$\theta_{a,f}(t) = \int_{\alpha^{\mathbb{Z}}\cdot F^\times N(F)\backslash \mathrm{GL}(2,F)} f(ax^{-1}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)x)t^{\mathrm{ht}^+(x)}dx.$$

Proposition 3.11. (1) *For every $f \in C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$ and $a \in \mathbb{A}^\times$, the integral $\theta_{a,f}(t)$ converges as an element of $\mathbb{C}((t))$, and $\zeta_F(q^{-1}t)^{-1}\theta_{a,f}(t) \in \mathbb{C}[t,t^{-1}]$, where $\zeta_F(t) = \prod_{v\in|X|}(1-t_v)^{-1}$, $t_v = t^{\deg v}$.*
(2) *If f is the characteristic function of $\mathrm{GL}(2,O_\mathbb{A})$ in $\mathrm{GL}(2,\mathbb{A})$, then*

$$\theta_{1,f}(t) = |\,\mathrm{GL}(2,O_\mathbb{A})| \cdot (q-1)^{-1}q^{g-1} \cdot |\,\mathrm{Pic}^0(X)|\zeta_F(q^{-1}t).$$

Proof. (1) It suffices to consider $f(x) = \prod_v f_v(x_v)$, $x = (x_v) \in \mathrm{GL}(2,\mathbb{A})$, where $f_v \in C_c^\infty(\mathrm{GL}(2,F_v))$ for all $v \in |X|$ and f_v is the characteristic function f_v^0 of $\mathrm{GL}(2,O_v)$ at almost all v, since such functions span $C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$. Normalize the measures on $F_v^\times$ and F_v so that $|O_v^\times| = 1 = |O_v|$. Denote by $\mathrm{val}_v(x_v)$ the valuation of $x_v \in F_v^\times$, normalized by $\mathrm{val}_v(\boldsymbol{\pi}_v) = 1$. Define a function

$$h_v^+ : \mathrm{GL}(2,F_v) \to \mathbb{Z} \text{ by } h_v^+(\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)k) = \mathrm{val}_v(a) - \mathrm{val}_v(c), \quad k \in \mathrm{GL}(2,O_v).$$

Then h_v^+ is well-defined and $\mathrm{ht}^+(x) = \sum_{v\in|X|} h_v^+(x_v)\deg(v)$. We have

$$\theta_{a,f}(t) = |\mathbb{A}^\times/\alpha^{\mathbb{Z}} \cdot F^\times| \cdot |\mathbb{A}/F| \prod_v \theta_{a_v,f_v}(t_v)$$

where

$$\theta_{a_v,f_v}(t_v)=\int_{F_v^\times N(F_v)\backslash \mathrm{GL}(2,F_v)} f_v(a_v x^{-1}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)x)t^{h_v^+(x)\deg v}dx$$

and $t_v=t^{\deg(v)}$. To compute it, note that $p_{n,v}=\operatorname{diag}(\boldsymbol{\pi}_v^n,1)$ $(n\in\mathbb{Z})$ make a set of representatives of the two sided coset space

$$F_v^\times N(F_v)\backslash \mathrm{GL}(2,F_v)/\mathrm{GL}(2,O_v).$$

Then

$$\theta_{a_v,f_v}(t_v)=\sum_{n\in\mathbb{Z}}t_v^n\int_{F_v^\times N(F_v)\cap p_{n,v}^{-1}\mathrm{GL}(2,O_v)p_{n,v}\backslash p_{n,v}^{-1}\mathrm{GL}(2,O_v)} f_v(a_v x^{-1}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)x)dx$$

$$=\sum_{n\in\mathbb{Z}}t_v^n|F_v^\times N(F_v)\cap p_{n,v}^{-1}\mathrm{GL}(2,O_v)p_{n,v}|^{-1}\int_{p_{n,v}^{-1}\mathrm{GL}(2,O_v)} f_v(a_v x^{-1}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)x)dx$$

$$=\sum_{n\in\mathbb{Z}}q_v^{-n}t_v^n\int_{\mathrm{GL}(2,O_v)} f_v(a_v y p_{n,v}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)p_{n,v}^{-1}y^{-1})dy=\sum_{n\in\mathbb{Z}}\tau_n(f_v)q_v^{-n}t_v^n,$$

where $\tau_n(f_v)=\int_{\mathrm{GL}(2,O_v)}f_v(a_v y\left(\begin{smallmatrix}1&\boldsymbol{\pi}_v^n\\0&1\end{smallmatrix}\right)y^{-1})dy$ is 0 if $n<<0$ and $\tau_n(f_v)=f_v(a_v)$ for $n>>0$.
If $a_v\in O_v^\times$ and f_v is the characteristic function of $\mathrm{GL}(2,O_v)$, then $\tau_n(f_v)=|\mathrm{GL}(2,O_v)|$ for $n\geq 0$ and $u_{n,v}=0$ for $n<0$, so

$$\theta_{a_v,f_v}(t_v)=|\mathrm{GL}(2,O_v)|(1-t_v/q_v)^{-1}.$$

(2) It remains to compute (note that $|O_{\mathbb{A}}^\times|=1$ and $|O_{\mathbb{A}}|=1$) :

$$|\mathbb{A}^\times N(\mathbb{A})/\alpha^{\mathbb{Z}}F^\times N(F)|=(|\mathbb{A}^\times/\alpha^{\mathbb{Z}}F^\times|/|O_{\mathbb{A}}^\times|)(|\mathbb{A}/F|/|O_{\mathbb{A}}|).$$

The exact sequence $1\to\mathbb{F}_q^\times\to O_{\mathbb{A}}^\times\to\mathbb{A}^\times/\alpha^{\mathbb{Z}}F^\times\to\operatorname{Pic}X/\alpha^{\mathbb{Z}}(=\operatorname{Pic}^0(X))\to 1$ implies that the first factor on the right is $|\operatorname{Pic}^0(X)|/(q-1)$. The exact sequence

$$0\to\mathbb{F}_q\to O_{\mathbb{A}}\to\mathbb{A}/F\to H^1(X,O_X)\to 0$$

implies that the second factor on the right is q^{g-1}. □

4 Intertwining operators and Eisenstein series

4.1 Intertwining operators

Let E be an algebraically closed field of characteristic zero, and $v\in|X|$ a closed point of X. Denote by $|a|_v$ the absolute value of $a\in F_v^\times$ normalized by $|\boldsymbol{\pi}_v|=q_v^{-1}$. It is an $E^\times$-valued character of $F_v^\times$. Fix a square root $\sqrt{q}=q^{1/2}$ of q in E. If $E\subset\mathbb{C}$ we choose $q^{1/2}>0$. For E-valued characters

μ_1, μ_2 of $F_v^\times$ *denote by* $I(\mu_1,\mu_2)$ *both the space of right locally constant functions* $\phi : \mathrm{GL}(2,F_v) \to E$ with $\phi(\left(\begin{smallmatrix} a_1 & b \\ 0 & a_2 \end{smallmatrix}\right) x) = |a_1/a_2|_v^{1/2}\mu_1(a_1)\mu_2(a_2)\phi(x)$ $(x \in \mathrm{GL}(2,F_v); a_1,a_2 \in F_v^\times; b \in F_v)$, *and the action of the group* $\mathrm{GL}(2,F_v)$ by right translation on $I(\mu_1,\mu_2)$. The induced representation $I(\mu_1,\mu_2)$ is admissible by the Iwasawa decomposition $G = BK$. It is unitarizable when μ_1, μ_2 are unitary. It is possible to work with $I(|\cdot|_v^{1/2}\mu_1, |\cdot|_v^{1/2}\mu_2)$, in whose definition the factor $|a_1/a_2|_v^{1/2}\mu_1(a_1)\mu_2(a_2)$ becomes $|a_1|_v\mu_1(a_1)\mu_2(a_2)$, but later we shall need to multiply back by $|\cdot|_v^{-1/2}$. The following is a standard basic result.

PROPOSITION 4.1. *If* $\mu_1/\mu_2 \neq |\cdot|_v, |\cdot|_v^{-1}$, *then the representations of* $\mathrm{GL}(2,F_v)$ *in* $I(\mu_1,\mu_2)$ *and* $I(\mu_2,\mu_1)$ *are irreducible and isomorphic. If* $\mu_1/\mu_2 = |\cdot|_v$ *or* $|\cdot|_v^{-1}$ *then* $I(\mu_1,\mu_2)$ *contains a unique proper invariant subspace* $I'(\mu_1,\mu_2)$ *and there is a* $\mathrm{GL}(2,F_v)$*-isomorphism* $I'(\mu_1,\mu_2) \simeq I(\mu_2,\mu_1)/I'(\mu_2,\mu_1)$. *If* $\mu_2/\mu_1 = |\cdot|_v$, *the subspace* $I'(\mu_1|\cdot|_v^{-1/2}, \mu_1|\cdot|_v^{1/2})$ *is one dimensional*; $x \in \mathrm{GL}(2,F_v)$ *acts on* $I'(\mu_1|\cdot|_v^{-1/2}, \mu_1|\cdot|_v^{1/2})$ *via multiplication by* $\mu_1(x)$. *The subspace*

$$I'(\mu_2|\cdot|_v^{1/2}, \mu_2|\cdot|_v^{-1/2}) \quad \textit{is denoted by} \quad \mathrm{St}(\mu_2) = \mathrm{St}(\mu_2|\cdot|_v^{1/2}, \mu_2|\cdot|_v^{-1/2}).$$

It is isomorphic to $I(\mu_2|\cdot|_v^{-1/2}, \mu_2|\cdot|_v^{1/2})/I'(\mu_2|\cdot|_v^{-1/2}, \mu_2|\cdot|_v^{1/2})$. *It consists of*

$$\phi \in I(\mu_2|\cdot|_v^{1/2}, \mu_2|\cdot|_v^{-1/2}) \quad \textit{with} \quad \int_{\mathrm{GL}(2,O_v)} \mu_2(\det x)^{-1}\phi(x)dx = 0.$$

If $I(\mu_1,\mu_2) \simeq I(\mu_1',\mu_2')$ *then* $\{\mu_1,\mu_2\} = \{\mu_1',\mu_2'\}$, *the representations* $I(\mu_1,\mu_2)$ $(\mu_1/\mu_2 \neq |\cdot|_v$ *or* $|\cdot|_v^{-1})$ *and* $\mathrm{St}(\mu_2')$ *are infinite dimensional and inequivalent, and* $\mathrm{St}(\mu_1) \simeq \mathrm{St}(\mu_2)$ *implies* $\mu_1 = \mu_2$.

We proceed to describe the operator intertwining $I(\mu_1,\mu_2)$ and $I(\mu_2,\mu_1)$.

PROPOSITION 4.2. *If* $|\mu_1(\boldsymbol{\pi}_v)/\mu_2(\boldsymbol{\pi}_v)| < 1$ *the integral*

$$(M\phi)(x) = \int_{F_v} \phi(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) x)dy$$

converges for each $\phi \in I(\mu_1,\mu_2)$ *and* $x \in \mathrm{GL}(2,F_v)$, *and* $M\phi \in I(\mu_2,\mu_1)$.

Proof. As $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) = \begin{pmatrix} y^{-1} & -1 \\ 0 & y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y^{-1} & 1 \end{pmatrix}$, the integrand is

$$\mu_2(y)\mu_1(y)^{-1}|y|_v^{-1}\phi\left(\begin{pmatrix} 1 & 0 \\ y^{-1} & 1 \end{pmatrix} x\right),$$

which is 0 if $|y|_v$ is small, and $\mu_2(y)\mu_1(y)^{-1}|y|_v^{-1}\phi(x)$ if $|y|_v$ is big enough. For sufficiently large n then the part of the integral over $|y|_v \geq q_v^n$ is bounded by $\phi(x)$ times

$$\int_{|y|_v \geq q_v^n} |\mu_2(y)/\mu_1(y)| \cdot |y|_v^{-1}dy = |O_v^\times| \sum_{k \geq n} |\mu_1(\boldsymbol{\pi}_v)/\mu_2(\boldsymbol{\pi}_v)|^k < \infty.$$

It is clear that $(M\phi)((\begin{smallmatrix}1&c\\0&1\end{smallmatrix})x) = (M\phi)(x)$ $(c \in F_v)$ and $(M\phi)((\begin{smallmatrix}a&0\\0&b\end{smallmatrix})x)$ equals

$$\int_{F_v} \phi((\begin{smallmatrix}b&0\\0&a\end{smallmatrix})(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix})(\begin{smallmatrix}1&yb/a\\0&1\end{smallmatrix})x)dy = \mu_1(b)\mu_2(a)\left|\frac{b}{a}\right|_v^{1/2}\left|\frac{a}{b}\right|_v (M\phi)(x).$$

□

We obtained, if $|\mu_1(\boldsymbol{\pi}_v)/\mu_2(\boldsymbol{\pi}_v)| < 1$, a $\mathrm{GL}(2, F_v)$-equivariant map

$$M = M(\mu_1, \mu_2) : I(\mu_1, \mu_2) \to I(\mu_2, \mu_1).$$

Let ν_t *be the unramified character of* $F_v^\times$ with $\nu_t(\boldsymbol{\pi}_v) = t$. Put $M(\mu_1, \mu_2, t) = M(\mu_1\nu_t, \mu_2\nu_{t^{-1}})$. It converges for any μ_1, μ_2, provided $t \in \mathbb{C}$ is small enough in absolute value. To define $M(\mu_1, \mu_2)$ as the value at $t = 1$ of the analytic continuation of $M(\mu_1, \mu_2, t)$, we need these operators to be defined on the same space, which we will take to be

$$\begin{aligned} I_0(\mu_1, \mu_2) = \quad & \{\phi \in C^\infty(\mathrm{GL}(2, O_v)); \phi((\begin{smallmatrix}a_1&b\\0&a_2\end{smallmatrix})x) = \mu_1(a_1)\mu_2(a_2)\phi(x), \\ & \qquad a_1, a_2 \in O_v^\times,\ \ b \in O_v,\ \ x \in \mathrm{GL}(2, O_v)\}. \end{aligned}$$

By the Iwasawa decomposition $G = BK$, the restriction map $I(\mu_1\nu_t, \mu_2\nu_{t^{-1}}) \to I_0(\mu_1, \mu_2)$ is bijective for any t. Identifying these spaces, the operator $M(\mu_1, \mu_2, t)$ becomes a map $I_0(\mu_1, \mu_2) \to I_0(\mu_2, \mu_1)$.
Write $L(\mu, t)$ for $(1 - \mu(\boldsymbol{\pi}_v)t)^{-1}$ if μ *is unramified*, and $L(\mu, t) = 1$ if μ *is a ramified character of* $F_v^\times$.

PROPOSITION 4.3. *The operator valued function* $M(\mu_1, \mu_2, t)$ *is rational in* $t \in \mathbb{C}^\times$. *In fact the function* $t \mapsto L(\mu_1/\mu_2, t^2)^{-1}(M(\mu_1, \mu_2, t)\phi)(x)$ *is a polynomial in* t *for all* $\phi \in I_0(\mu_1, \mu_2)$, $x \in \mathrm{GL}(2, O_v)$. *If* μ_1, μ_2 *are unramified and the restrictions of* $\phi \in I(\mu_1\nu_t, \mu_2\nu_{t^{-1}})$ *and* $\psi \in I(\mu_2\nu_{t^{-1}}, \mu_1\nu_t)$ *to* $\mathrm{GL}(2, O_v)$ *are* 1, *then* $M(\mu_1, \mu_2, t)\phi = \frac{L(\mu_1/\mu_2, t^2)}{L(\mu_1/\mu_2, q_v^{-1}t^2)}\psi$.

Proof. Put $\phi_t = M(\mu_1, \mu_2, t)\phi$ and $a_1 = \int_{|y|_v \le 1} \phi((\begin{smallmatrix}0&-1\\1&y\end{smallmatrix})x)dy$ where $x \in \mathrm{GL}(2, O_v)$. Then

$$\phi_t(x) = a_1 + \int_{|y|_v > 1} \mu_2(y)\mu_1(y)^{-1}|y|_v^{-1}\nu_t(y)^{-2}\phi\left(\begin{pmatrix}1&0\\y^{-1}&1\end{pmatrix}x\right)dy.$$

We shall show that this is the Taylor series of a rational function.
If n is large enough, $\phi\left(\begin{pmatrix}1&0\\y^{-1}&1\end{pmatrix}x\right) = \phi(x)$ for $|y|_v \ge q_v^n$. Then $\phi_t(x) = a_1 + a_2(t) + a_3(t)$ with

$$\begin{aligned} a_2(t) &= \int_{1<|y|_v<q_v^n} \mu_2(y)\mu_1(y)^{-1}|y|_v^{-1}\nu_t(y)^{-2}\phi\left(\begin{pmatrix}1&0\\y^{-1}&1\end{pmatrix}x\right)dy, \\ a_3(t) &= \phi(x)\int_{|y|_v \ge q_v^n} \mu_2(y)\mu_1(y)^{-1}|y|_v\nu_t(y)^{-2}dy. \end{aligned}$$

Clearly $a_2(t)$ is a polynomial in t (since $\nu_t(\boldsymbol{\pi}_v^{-1})^{-1} = t$) and $a_3(t) = ct^{2n}L(\mu_1/\mu_2, t^2)$.
If μ_1, μ_2 are unramified and $x \in \mathrm{GL}(2, O_v)$, $a_1 = 1$ and the expression for $\phi_t(x)$ is

$$\begin{aligned}\phi_t(x) = 1 \quad &+ \quad \int_{|y|_v>1} \mu_2(y)\mu_1(y)^{-1}|y|_v^{-1}\nu_t(y)^{-2}dy \\ &= \quad 1 - (1-q_v^{-1})\sum_{k\geq 1}(\mu_1(\boldsymbol{\pi}_v)/\mu_2(\boldsymbol{\pi}_v))^k t^{2k} \\ &= \quad 1 + \frac{(1-q_v^{-1})(\mu_1(\boldsymbol{\pi}_v)/\mu_2(\boldsymbol{\pi}_v))t^2}{1-(\mu_1(\boldsymbol{\pi}_v)/\mu_2(\boldsymbol{\pi}_v))t^2} = \frac{L(\mu_1/\mu_2, t^2)}{L(\mu_1/\mu_2, q_v^{-1}t^2)}.\end{aligned}$$

□

The operator $M(\mu_1, \mu_2, t) : I(\mu_1\nu_t, \mu_2\nu_{t^{-1}}) \to I(\mu_2\nu_{t^{-1}}, \mu_1\nu_t)$ intertwines the $\mathrm{GL}(2, F_v)$-modules for every t where it is defined. It can be regarded as a rational function of t (in fact, of t^2) with values in the set of operators $I_0(\mu_1, \mu_2) \to I_0(\mu_2, \mu_1)$. Indeed,

$$M(\mu_1, \mu_2, t) = M(\mu_1\nu_t, \mu_2\nu_{t^{-1}}) = M(\mu_1\nu_{t^2}, \mu_2).$$

Define

$$R(\mu_1, \mu_2, t) = \frac{L(\mu_1/\mu_2, q_v^{-1}t^2)}{L(\mu_1/\mu_2, t^2)} M(\mu_1, \mu_2, t).$$

Corollary 4.4. *Suppose μ_1 and μ_2 are unramified and $\varphi \in I(\mu_1\nu_t, \mu_2\nu_{t^{-1}})$, $\psi \in I(\mu_2\nu_{t^{-1}}, \mu_1\nu_t)$ are the functions whose restrictions to $\mathrm{GL}(2, O_v)$ are one, then $R(\mu_1, \mu_2, t)\varphi = \psi$.* □

Given characters μ_1, μ_2 of $\mathbb{A}^\times$, write $I(\mu_1, \mu_2)$ *for the space of right locally constant functions* ϕ on $\mathrm{GL}(2, \mathbb{A})$ which satisfy

$$\phi\left(\left(\begin{smallmatrix} a_1 & b \\ 0 & a_2 \end{smallmatrix}\right) x\right) = \mu_1(a_1)\mu_2(a_2)|a_1/a_2|^{1/2}\phi(x). \quad \text{Put} \quad \nu(a) = q^{\deg(a)}.$$

Then $I(\mu_1, \mu_2)$ is the restricted tensor product of the spaces $I(\mu_{1v}, \mu_{2v})$ where μ_{iv} is the component of μ_i at v (the restriction of μ_i to $F_v^\times \hookrightarrow \mathbb{A}^\times$); it is spanned by $\otimes_v\phi_v$ with $\phi_v \in I(\mu_{1v}, \mu_{2v})$ for all v and $\phi_v|\,\mathrm{GL}(2, O_v) = 1$ for almost all v, where $\mu_{iv}|O_v^\times = 1$, i.e. μ_{iv} are unramified. *Define the character* ν_t of $\mathbb{A}^\times$ by $\nu_t(a) = t^{\deg(a)}$. Then the restriction of ν_t to $F_v^\times$ is ν_{t_v}, the unramified character of $F_v^\times$ with $\nu_{t_v}(\boldsymbol{\pi}_v) = t_v(= t^{\deg(v)})$. As in the local case, we identify the spaces $I(\mu_1\nu_t, \mu_2\nu_{t^{-1}})$ with $I_0(\mu_1, \mu_2)$ for all t. The operator $R(\mu_1, \mu_2, t)$ from $I(\mu_1\nu_t, \mu_2\nu_{t^{-1}})$ to $I(\mu_2\nu_{t^{-1}}, \mu_1\nu_t)$ *defined by* $R(\mu_1, \mu_2, t) = \otimes_v R(\mu_{1v}, \mu_{2v}, t_v)$ is rational in t. On any element in $I(\mu_1\nu_t, \mu_2\nu_{t^{-1}})$ at most finitely many components $R(\mu_{1v}, \mu_{2v}, t_v)$ do not act as the identity. Also *write* $m(\mu, t)$ for $L(\mu, t)/L(\mu, t/q)$.

4.2 Eisenstein series

Write $A_\alpha = C^\infty(\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}))$,

$$A_{c,\alpha} = C_c^\infty(\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})), \quad Y = A(F)N(\mathbb{A})\backslash \mathrm{GL}(2,\mathbb{A})$$

and $Y_\alpha = Y/\alpha^{\mathbb{Z}}$. Normalize the Haar measure on $N(\mathbb{A}) \simeq \mathbb{A}$ by $|N(\mathbb{A})/N(F)| = |\mathbb{A}/F| = 1$. The Haar measure on $N(\mathbb{A})$ is invariant with respect to conjugation by the elements of $A(F)$ by the product formula. So it extends to a two-sided invariant measure on the space $\alpha^{\mathbb{Z}} \cdot A(F)N(\mathbb{A})$. This, and the two-sided Haar measure on $\mathrm{GL}(2,\mathbb{A})$ induce an invariant measure on Y_α.

Let φ and ψ be locally constant functions on Y_α, at least one of which is compactly supported. Put $(\varphi,\psi) = \int_{Y_\alpha} \varphi(x)\overline{\psi}(x)dx$. On $\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$ a scalar product is similarly defined. *Define the map* $E^* : A_\alpha \to C^\infty(Y_\alpha)$ by

$$\phi \mapsto \phi_N, \quad \phi_N(x) = \int_{N(F)\backslash N(\mathbb{A})} \phi(nx)dn, \quad x \in \mathrm{GL}(2,\mathbb{A}).$$

Note that $N(F)\backslash N(\mathbb{A})$ is compact, so the integral converges. Note that $\ker E^*$ is the space $A_{0,\alpha}$ of cusp forms invariant under α. For any $f \in C_c^\infty(Y_\alpha)$ *define a function* Ef on $\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$ by

$$(Ef)(x) = \sum_{\gamma \in A(F)N(F)\backslash \mathrm{GL}(2,F)} f(\gamma x), \quad x \in \mathrm{GL}(2,\mathbb{A}).$$

Proposition 4.5. *The sum defining* $(Ef)(x)$ *converges. For* $f \in C_c^\infty(Y_\alpha)$ *and* $\phi \in A_\alpha$ *we have* $(Ef,\phi) = (f, E^*\phi)$.

Proof. Consider the diagram

$$Y_\alpha \xleftarrow{r} \alpha^{\mathbb{Z}} \cdot A(F)N(F)\backslash \mathrm{GL}(2,\mathbb{A}) \xrightarrow{s} \alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}).$$

Since $N(F)\backslash N(\mathbb{A})$ is compact, the map r is proper. Hence the natural embedding r^* maps $C_c^\infty(Y_\alpha)$ to $C_c^\infty(\alpha^{\mathbb{Z}} \cdot A(F)N(F)\backslash \mathrm{GL}(2,\mathbb{A}))$. Given

$$\psi \in C_c^\infty(\alpha^{\mathbb{Z}} A(F)N(F)\backslash \mathrm{GL}(2,\mathbb{A})),$$

define a function $s_*\psi$ on $\alpha^{\mathbb{Z}}\,\mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$ by

$$(s_*\psi)(x) = \sum_{\gamma \in A(F)N(F)\backslash \mathrm{GL}(2,F)} \psi(\gamma x), \quad x \in \mathrm{GL}(2,\mathbb{A}).$$

The sum is finite since ψ is compactly supported, and

$$s_*\psi \in C_c^\infty(\alpha^{\mathbb{Z}}\,\mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})).$$

The sum which defines $(Ef)(x)$ converges since $E = s_* r^*$.

Now define $E^* = r_* s^*$, where s^* is the natural embedding, and

$$r_* : C^\infty(\alpha^{\mathbb{Z}} A(F)N(F)\backslash \mathrm{GL}(2,\mathbb{A})) \to C^\infty(Y_\alpha)$$

is defined by $(r_*h)(x) = \int_{N(F)\backslash N(\mathbb{A})} h(nx)dn$, $x \in \mathrm{GL}(2,\mathbb{A})$. Since (r^*, r_*) and (s_*, s^*) are adjoint pairs, so is $(E = s_*r^*, E^* = r_*s^*)$. ☐

The image $A_{E,\alpha}$ of the *Eisenstein map* $E = s_* r^* : C_c^\infty(Y_\alpha) \to A_{c,\alpha}$ is called the *Eisenstein part* of $A_{c,\alpha}$. The maps E and E^* intertwine the $\mathrm{GL}(2,\mathbb{A})$-action; $A_{E,\alpha}$ is an invariant subspace of $A_{c,\alpha}$.

PROPOSITION 4.6. *The space $A_{c,\alpha}$ is an orthogonal direct sum of the space $A_{0,\alpha}$ of cusp forms and of $A_{E,\alpha}$.*

Proof. Cusp forms are compactly supported. Since $A_{0,\alpha} = \ker E^*$ and $A_{E,\alpha} = \operatorname{im} E$, we have $A_{0,\alpha} \perp A_{E,\alpha}$. Given a compact open subgroup U in $\mathrm{GL}(2,\mathbb{A})$, put A_α^U for the space of U-invariant functions in A_α, and

$$A_{c,\alpha}^U = A_{c,\alpha} \cap A_\alpha^U, \quad A_{0,\alpha}^U = A_{0,\alpha} \cap A_\alpha^U, \quad A_{E,\alpha}^U = A_{E,\alpha} \cap A_\alpha^U.$$

It remains to show that $A_{0,\alpha}^U + A_{E,\alpha}^U = A_{c,\alpha}^U$. If not there exists a nonzero linear form $\ell : A_{c,\alpha}^U \to \mathbb{C}$ which is zero on $A_{0,\alpha}^U + A_{E,\alpha}^U$. There exists $f \in A_\alpha^U$ such that $\ell(\phi) = (\phi, f)$ for every $\phi \in A_{c,\alpha}^U$. For any U-invariant function $\psi \in C_c^\infty(Y_\alpha)$ we have $(\psi, E^* f) = (E\psi, f) = \ell(E\psi) = 0$. Hence $E^* f = 0$, thus $f \in A_{0,\alpha}^U$. This however is impossible since f is orthogonal to the space $A_{0,\alpha}^U$ of U-invariant cusp forms. $\square$

Given $\phi \in C_c^\infty(Y_\alpha)$ and $x \in \mathrm{GL}(2,\mathbb{A})$, put $(M\phi)(x) = \int_{N(\mathbb{A})} \phi(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx) dn$. The integral converges, by

PROPOSITION 4.7. *The map $N(\mathbb{A}) \to Y_\alpha$, $n \mapsto \alpha^{\mathbb{Z}} A(F) N(\mathbb{A}) \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx$, is proper.*

Proof. It suffices to consider the case of $x = 1$. The function

$$\mathrm{ht}^+ : Y_\alpha \to \mathbb{Z}, \qquad \left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right) k \mapsto \deg a - \deg b,$$

is continuous. Thus it suffices to show that the map $\varphi(a) = \mathrm{ht}^+(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right))$, $\varphi : \mathbb{A} \to \mathbb{Z}$, is proper. But $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & a_v \\ 0 & 1 \end{smallmatrix}\right)$ is in $\mathrm{GL}(2, O_v)$ if $|a_v|_v \le 1$; otherwise it is $= \begin{pmatrix} a_v^{-1} & -1 \\ 0 & a_v \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a_v^{-1} & 1 \end{pmatrix}$. If $a = (a_v)$, then $\varphi(a) = -2\sum_v \max(0, \log_q |a_v|_v)$, as $\log_q |a_v|_v = -\mathrm{val}_v(a_v)\deg(v)$. Hence φ is proper. $\square$

By definition, $x \mapsto (M\phi)(x)$ is invariant under left translation by $N(\mathbb{A})$, and also by $\alpha^{\mathbb{Z}} \cdot A(F)$. Indeed,

$$(M\phi)(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) x) = \int_{\mathbb{A}} \phi(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) n \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) x) dy = \left|\frac{a}{b}\right| \int_{N(\mathbb{Z})} \phi(\left(\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx) dn$$

and $|a/b| = q^{\deg(a/b)}$. Thus M maps $C_c^\infty(Y_\alpha)$ to $C^\infty(Y_\alpha)$.

PROPOSITION 4.8. *Denote by I the natural embedding of $C_c^\infty(Y_\alpha)$ in $C^\infty(Y_\alpha)$. Then*

$$E^* E = I + M.$$

Proof. By the Bruhat decomposition, an element of $\mathrm{GL}(2,F)$ which is not in $A(F)N(F)$ has a unique decomposition $n_1 a \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) n_2$ with $n_i \in N(F)$, $a \in A(F)$. Thus, for any $\phi \in C_c^\infty(Y_\alpha)$, $x \in \mathrm{GL}(2,\mathbb{A})$, we have

$$(E\phi)(x) = \sum_{\gamma \in A(F)N(F)\backslash \mathrm{GL}(2,F)} \phi(\gamma x) = \phi(x) + \sum_{\nu \in N(F)} \phi(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \nu x).$$

Hence

$$\begin{aligned}(E^* E\phi)(x) &= |N(\mathbb{A})/N(F)|\phi(x) + \int_{N(F)\backslash N(\mathbb{A})} \sum_{\nu \in N(F)} \phi(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \nu n x) dn \\ &= \phi(x) + \int_{N(\mathbb{A})} \phi(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) n x) dn \quad = \quad \phi(x) + (M\phi)(x).\end{aligned}$$

□

PROPOSITION 4.9. *Let μ_1, μ_2 be characters of $\mathbb{A}^\times/F^\times$. If t is sufficiently small, for all $\phi \in I(\mu_1\nu_t, \mu_2\nu_{t^{-1}})$ and $x \in \mathrm{GL}(2,\mathbb{A})$, the integral $(M(\mu_1,\mu_2,t)\phi)(x) = \int_{N(\mathbb{A})} \phi(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) n x) dn$ converges and defines a function in $I(\mu_2\nu_{t^{-1}}, \mu_1\nu_t)$. Moreover, $M(\mu_1,\mu_2,t) = q^{1-g} m(\mu_1/\mu_2, t^2) R(\mu_1,\mu_2,t)$.*

Proof. Recall that $|a| = q^{\deg(a)}$ and that $I(\mu_1,\mu_2)$ consists of the ϕ in $C^\infty(\mathrm{GL}(2,\mathbb{A}))$ with

$$\phi(\left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right) x) = |a_1/a_2|^{1/2} \mu_1(a_1)\mu_2(a_2)\phi(x),$$

while $\nu_t(a) = t^{\deg a}$. We put $t_v = t^{\deg(v)}$. We may assume that $\phi(x) = \prod_v \phi_v(x_v)$ with $\phi_v \in I(\mu_{1v}\nu_{t_v}, \mu_{2v}\nu_{t_v^{-1}})$. For almost all v, the restriction of ϕ_v to $\mathrm{GL}(2,O_v)$ is 1. We may replace ϕ_v, μ_i, t by their complex absolute values to assume $t > 0$ and ϕ_v, μ_i take real nonnegative values. Then $(M(\mu_1,\mu_2,t)\phi)(x) = c\prod_v \tau_v$, with $\tau_v = \int_{N(F_v)} \phi_v(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) n x_v) dn = \int_{F_v} \phi_v(\left(\begin{smallmatrix} 0 & -1 \\ 1 & z \end{smallmatrix}\right) x_v) dz$. The measure dn_v on $N(F_v)$ is normalized by $|N(O_v)| = 1$, and $c = |N(\mathbb{A})/N(F)|$ in the measure $\otimes_v dn_v$ on $N(\mathbb{A})$.
We saw that for small enough t the integral which defines τ_v converges for all v. For almost all v we have $\tau_v = L(\mu_{1v}/\mu_{2v}, t_v^2)/L(\mu_{1v}/\mu_{2v}, q_v^{-1}t_v^2)$, so the product $\prod_v \tau_v$ converges for small t. Now $M(\mu_1,\mu_2,t) = c\prod_v M(\mu_{1v},\mu_{2v},t_v)$. Each factor here is $\frac{L(\mu_{1v}/\mu_{2v}, t_v^2)}{L(\mu_{1v}/\mu_{2v}, q_v^{-1}t_v^2)} R(\mu_{1v},\mu_{2v},t_v)$. Put $R(\mu_1,\mu_2,t) = \otimes_v R(\mu_{1v},\mu_{2v},t_v)$, and $m(\mu,t) = \frac{L(\mu,t)}{L(q^{-1}t,\mu)}$, where $L(\mu,t) = \prod_v L(\mu_v,t_v)$. Note that c is $|O| = q^{1-g}$, using $0 \to \mathbb{F}_q \to O \to \mathbb{A}/F \to H^1(X,O_X) \to 0$. □

It follows (since $L(\mu,t)$ is a rational function of t) that after identifying the spaces $I(\mu_1\nu_t, \mu_2\nu_{t^{-1}})$ for all t, the operator

$$M(\mu_1,\mu_2,t) : I(\mu_1\nu_t, \mu_2\nu_{t^{-1}}) \to I(\mu_2\nu_{t^{-1}}, \mu_1\nu_t)$$

(defined for small t) depends on t rationally. Hence $M(\mu_1,\mu_2,t)$ is defined for almost all t, and it commutes with the action of $\mathrm{GL}(2,\mathbb{A})$.

4.3 L-functions

Let us review the theory of L-functions for GL(2). Let E be an algebraically closed field of characteristic zero. The valuation $\mathrm{val}_v(a)$ of $a \in F_v^\times$ is the largest integer n with $a \in \boldsymbol{\pi}_v^n O_v$. For any character $\psi : F_v \to E^\times$, $\psi \neq 1$, let $r(\psi)$ be the largest n such that $\psi(\boldsymbol{\pi}_v^{-n} O_v) = 1$. Normalize the Haar measure on F_v by $|O_v| = 1$. The *conductor* of a character $\chi : F_v^\times \to E^\times$ is $n = 0$ if $\chi(O_v^\times) = 1$, i.e., χ is unramified; otherwise it is the smallest $n \geq 1$ such that $\chi(1 + \boldsymbol{\pi}_v^n O_v) = 1$. Given χ, put $L(t, \chi) = (1 - \chi(\boldsymbol{\pi}_v)t)^{-1}$ if χ is unramified, $L(t, \chi) = 1$ is χ is ramified. Given $\psi \neq 1$, put

$$\Gamma(\chi, \psi, t) = \int_{F_v^\times} \chi(x)^{-1} \psi(x) t^{-\mathrm{val}_v(x)} dx, \quad \psi : F_v \to E^\times.$$

This $\Gamma(\chi, \psi, t)$ is a formal power series in t which contains positive and negative powers of t. Tate's thesis (see [Lg94], VII, section 3-4) establishes

Proposition 4.10. *The formal series $\Gamma(\chi, \psi, t)$ has finitely many positive powers of t. It is a rational function of t, namely a Laurent series of a rational function of t at $t = \infty$. Put $\varepsilon(\chi, \psi, t) = \frac{L(\chi,t)\Gamma(\chi,\psi,t)}{L(\chi^{-1}, q_v^{-1}t^{-1})}$. It has the form $c(\chi, \psi)t^{n(\chi,\psi)}$. If $r(\psi) = 0$ then $n(\chi, \psi)$ is the conductor of χ. If in addition χ is unramified then $\varepsilon(\chi, \psi, t)$ is 1. If $a \in F_v^\times, \psi_a(x) = \psi(ax)$, then $\varepsilon(\chi, \psi_a, t) = \chi(a)(q_v t)^{\mathrm{val}_v(a)} \varepsilon(\chi, \psi, t)$.*

Note that L and ε are usually considered, in the case where $E = \mathbb{C}$, as functions of s, where $t = q_v^{-s}$, rather than of t. The Haar measure on F_v is usually normalized by $|O_v| = q_v^{-r(\psi)/2}$, as this measure is self-dual with respect to the pairing $F_v \times F_v \to E^\times, (x, y) \mapsto \psi(xy)$. This choice of measure is not convenient if $E \neq \mathbb{C}$ since E has no distinguished square root of q.
Given a character χ of $\mathbb{A}^\times$, denote its restriction to $F_v^\times$ by χ_v. The restriction to F_v of a character ψ of $\mathbb{A}$ is denoted ψ_v. For a closed point v of X, we write $\deg(v)$ for the dimension of the residue field at v over $\mathbb{F}_q$, and $q_v = q^{\deg(v)}$. Given a character $\chi : \mathbb{A}^\times / F^\times \to E^\times$, put $L(\chi, t) = \prod_v L(\chi_v, t_v)$, where $t_v = t^{\deg(v)}$; the product converges in $E[[t]]$. Let $\psi : \mathbb{A}/F \to E^\times$ be a character $\neq 1$. Then $\varepsilon(\chi, t) = q^{1-g} \prod_v \varepsilon(\chi_v, \psi_v, t_v)$ converges as almost all factors are 1, and $\varepsilon(\chi, t)$ is independent of ψ by Proposition 4.10.

Proposition 4.11. *For any character $\chi : \mathbb{A}^\times / F^\times \to E^\times$ the formal series $L(\chi, t)$ is rational in t, and $L(\chi, t) = \varepsilon(\chi, t) L(\chi^{-1}, q^{-1}t^{-1})$. If the restriction of χ to the group of $x \in \mathbb{A}^\times / F^\times$ with $\deg(x) = 0$ is nontrivial, then $L(\chi, t)$ is a polynomial. If the restriction is trivial, χ is given by $\chi(x) = u^{\deg(x)}$, and then $L(\chi, t)$ has precisely two poles: $t = u^{-1}$ and $t = q^{-1}u^{-1}$, both poles are simple. If $\chi : \mathbb{A}^\times / F^\times \to \mathbb{C}^\times$ is a unitary character ($|\chi(x)| = 1$ for all x) then the zeroes of $L(\chi, t)$ lie in the doughnut $\{t \in \mathbb{C}; q^{-1} < |t| < 1\}$.*

The proof of this is also in [Lg94], Chapter VII, sections 7-8. The following is due to [W45].

THEOREM 4.12. (*A. Weil*). *For any unitary character $\chi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$, all zeroes of $L(\chi,t)$ lie on the circle $|t| = q^{-1/2}$.*

Given a character $\psi : \mathbb{A}/F \to E^\times, \psi \neq 1$, let $W(\psi)$ be the space of locally constant functions $\phi : \mathrm{GL}(2,F_v) \to E$ with $\phi((\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix})x) = \psi(z)\phi(x)$ for all $z \in F_v$, $x \in \mathrm{GL}(2,F_v)$. The group $\mathrm{GL}(2,F_v)$ acts on $W(\psi)$ by right translation. Fix a Haar measure $d^\times x$ on $F_v^\times$. For any $\phi \in W(\psi)$ put

$$\Lambda_\phi(t) = \int \phi((\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}))(q_v t)^{\mathrm{val}_v(a)} d^\times a, \quad \tilde{\Lambda}_\phi(t) = \int \phi((\begin{smallmatrix} 0 & 1 \\ a & 0 \end{smallmatrix}))(q_v t)^{\mathrm{val}_v(a)} d^\times a.$$

Both $\Lambda_\phi(t)$ and $\tilde{\Lambda}_\pi(t)$ are formal power series in t, containing positive and negative powers of t.
Let π be an irreducible admissible representation of $\mathrm{GL}(2,F_v)$ over E. Then $\pi((\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}))$ is the operator of multiplication by a scalar $\eta(a) \in E^\times$. The character $\eta : F_v^\times \to E^\times$ is called the *central character* of π.

PROPOSITION 4.13. *Let π be an irreducible admissible infinite dimensional representation over E of $\mathrm{GL}(2,F_v)$. Let η be the central character of π.* (1) *There exists a unique $\mathrm{GL}(2,F_v)$-invariant subspace $W(\pi,\psi)$ of $W(\psi)$ equivalent to π.* (2) *If $\phi \in W(\pi,\psi)$ then $\Lambda_\phi(t)$ is the Laurent series at $t = 0$ of a rational function, and $\tilde{\Lambda}_\phi(t)$ is the Laurent series at $t = \infty$ of a rational function.* (3) *There exists a nonzero polynomial $P \in E[t]$ such that for any $\phi \in W(\pi,\psi)$ we have $P(t)\Lambda_\phi(t) \in E[t,t^{-1}]$. There exists $\phi \in W(\pi,\psi)$ with $\Lambda_\phi(t) \neq 0$.* (4) *The quotient $\tilde{\Lambda}_\phi(t)/\Lambda_\phi(t)$ of rational functions in t does not depend on the choice of ϕ in $W(\pi,\psi)$ with $\Lambda_\phi(t) \neq 0$.* (5) *The lowest degree polynomial $P \in E[t]$ which satisfies* (3) *and $P(0) = 1$ is independent of ψ.* (6) *Put $\Gamma(\pi,\psi,t) = \tilde{\Lambda}_\phi(t)/\Lambda_\phi(t)$ and $\varepsilon(\pi,\psi,t) = \frac{\Gamma(\pi,\psi,t)L(\pi,t)}{L(\pi\otimes\eta^{-1},q_v^{-2}t^{-1})}$ where $L(\pi,t) = P(t)^{-1}$ with P of* (5). *Then $\varepsilon(\pi,\psi,t)$ has the form $c(\pi,\psi)t^{n(\pi,\psi)}$, $c(\pi,\psi)$ in $E^\times$ and $n(\pi,\psi)$ in $\mathbb{Z}$.* (7) *If $\psi_a(x)$ is $\psi(ax)$ for $a \in F_v^\times$, then $\varepsilon(\pi,\psi_a,t) = \eta(a)(q_v t)^{2\,\mathrm{val}_v(a)}\varepsilon(\pi,\psi,t)$.*

This is [JL70], Theorem 2.18. Our L and ε relate to those L_{JL}, ε_{JL} of Jacquet-Langlands by $L_{JL}(\pi,s) = L(\pi,t_v)$, $t_v = q_v^{-s}$, $\varepsilon_{JL}(\pi,\psi,s) = \varepsilon(\pi,\psi,t_v)$. Note that the proof of [JL70], which claims that $\Lambda_\phi(t)$ is a Laurent series of a meromorphic function in $\mathbb{C} - \{0\}$, shows that $\Lambda_\phi(t)$ is rational. In general, the meromorpic functions of s over p-adic and global function fields are rational functions of q^s. Every smooth finite dimensional irreducible representation of $\mathrm{GL}(2,F_v)$ is one dimensional, of the form $x \mapsto \chi(\det x)$, where $\chi : F_v^\times \to E^\times$ is a character ([JL70], Proposition 2.7).

PROPOSITION 4.14. *Let π, π' be irreducible admissible infinite dimensional representations of $\mathrm{GL}(2,F_v)$ with equal central characters. If there is a character $\psi : F_v \to E^\times$ such that for every character $\omega : F_v^\times \to E^\times$ we have $\Gamma(\pi\omega,\psi,t) = \Gamma(\pi'\omega,\psi,t)$, then $\pi \simeq \pi'$.*

For a proof see [JL70], Corollary 2.19.

The *conductor* of an irreducible admissible infinite dimensional representation π of $\mathrm{GL}(2, F_v)$ is the integer $n(\pi, \psi)$, with ψ normalized by $r(\psi) = 0$. It is well defined, as from (7) above, the integer $n(\pi, \psi)$ of (6) is not changed if ψ is replaced by $\psi_a : x \mapsto \psi(ax)$.

Proposition 4.15. *The conductor of π is the least integer n such that the representation space of π contains a nonzero vector invariant under the group $H_n = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, O_v);\ c \in \boldsymbol{\pi}_v^n O_v,\ d \in 1 + \boldsymbol{\pi}_v^n O_v\}$. For this n, $\dim_E \boldsymbol{\pi}^{H_n} = 1$.*

For a proof see Casselman, Math. Ann. 201 (1973), 301-314.

Proposition 4.16. *Let π be an irreducible admissible infinite dimensional representation, with central character η, of $\mathrm{GL}(2, F_v)$. Let $\psi : F_v \to E^\times$ be a nontrivial character. Then there exists an integer m_π such that if $\chi : F_v^\times \to E^\times$ is any character with conductor $> m_\pi$, then $L(\pi\chi, t) = 1$ and*

$$\varepsilon(\pi\chi, \psi, t) = \varepsilon(\chi, \psi, t)\varepsilon(\chi\eta, \psi, q_v t) q_v^{-r(\psi)}.$$

For a proof see [JL70], Proposition 3.8. See [JL70], Proposition 3.5, 3.6, for a proof of:

Proposition 4.17. *Let μ_1, μ_2 be characters of $F_v^\times$, and $\psi \neq 1$ a character of F_v. If $\mu_1/\mu_2 \neq |\cdot|_v^{\pm 1}$ then $L(I(\mu_1, \mu_2), t) = L(\mu_1, t)L(\mu_2, t)$ and*

$$\varepsilon(I(\mu_1, \mu_2), \psi, t) = \varepsilon(\mu_1, \psi, t)\varepsilon(\mu_2, \psi, t) q_v^{-r(\psi)}.$$

If $\mu_2/\mu_1 = |\cdot|_v$, then

$$L(\mathrm{St}(\mu_1|\cdot|_v^{-1/2}, \mu_1|\cdot|_v^{1/2}), t) = L(\mu_1|\cdot|_v^{1/2}, t),$$

$$\varepsilon(\mathrm{St}(\mu_1|\cdot|_v^{-1/2}, \mu_1|\cdot|_v^{1/2}), \psi, t) = \frac{L(\mu_1^{-1}, t^{-1})}{L(\mu_1, t)} \varepsilon(\mu_1, \psi, t)\varepsilon(\mu_1|\cdot|_v, \psi, t) q_v^{-r(\psi)}.$$

If π is a cuspidal representation of $\mathrm{GL}(2, F_v)$ then $L(\pi, t)$ is 1.

Recall that an irreducible admissible infinite dimensional representation π of $\mathrm{GL}(2, F_v)$ on a vector space V is called unramified if its space V^K of $K = \mathrm{GL}(2, O_v)$-fixed vectors is nonzero. In this case V^K is one dimensional, and $\pi = I(\mu_1, \mu_2)$ with unramified μ_1, μ_2 and $\mu_1/\mu_2 \neq |\cdot|^{\pm 1}$.

Corollary 4.18. *Let π be an unramified irreducible admissible infinite dimensional representation of $\mathrm{GL}(2, F_v)$ and $\psi \neq 1$ with $r(\psi) = 0$. Then $\varepsilon(\pi, \psi, t) = 1$.*

Proof. Here $\pi = I(\mu_1, \mu_2)$ with unramified μ_1, μ_2, so the claim follows from the last proposition and Tate's Thesis. $\square$

Let π be an admissible irreducible representation of $\mathrm{GL}(2,\mathbb{A})$ whose local components are all infinite dimensional. Put $L(\pi,t)=\prod_v L(\pi_v,t_v)$, $t_v=t^{\deg(v)}$; the infinite product converges in $E[[t]]$. For any character $\psi:\mathbb{A}/F\to E^\times, \psi\neq 1$, put $\varepsilon(\pi,\psi,t)=\prod_v \varepsilon(\pi_v,\psi_v,t_v)$; almost all factors here are 1. From (7) it follows that if the central character of π is trivial on $F^\times$, then $\varepsilon(\pi,\psi,t)$ is independent of the choice of $\psi:\mathbb{A}/F\to E^\times$. We denote it in this case by $\varepsilon(\pi,t)$.
Theorems 11.1, 11.3 of [JL70] assert:

THEOREM 4.19. *Let π be an irreducible admissible representation of* $\mathrm{GL}(2,\mathbb{A})$ *over E. Denote by $\eta:\mathbb{A}^\times\to E^\times$ its central character. Then π is cuspidal iff* (1) *η is trivial on $F^\times$*; (2) *all local components of π are infinite dimensional*; (3) *for any character $\omega:\mathbb{A}^\times/F^\times\to E^\times$, the formal series $L(\pi\omega,t)$ is a polynomial in t, and* (4) $L(\pi\omega,t)=\varepsilon(\pi\omega,t)L(\pi\eta^{-1}\omega^{-1},q^{-2}t^{-1})$.

Note that (4) makes sense due to (3). In [JL70], (3) is formulated as stating that the product $\prod_v L(\pi_v\omega_v,t_v)$ converges absolutely for sufficiently small t, and its value has an analytic continuation to a holomorphic function in $\mathbb{C}-\{0\}$. But the argument of [JL70] can be modified to lead to (3) in our case of E which is not $\mathbb{C}$, over a function field F. Note that (4) is not $\prod_v \Gamma(\pi_v\omega_v,\psi_v,t_v)=1$; indeed the product here does not converge.

PROPOSITION 4.20. *If π, π' are cuspidal representations of* $\mathrm{GL}(2,\mathbb{A})$ *and $\pi_v\simeq\pi'_v$ for almost all v, then $\pi\simeq\pi'$.*

Proof. Let S be a finite set of closed points of X with $\pi_v\simeq\pi'_v$ at $v\notin S$. Let η, η' be the central characters of π, π', and η_v, η'_v their components at v (restrictions to $F_v^\times$). By our assumption, $\eta'_v=\eta_v$ for all $v\notin S$. But the groups $F_v^\times$, $v\notin S$, generate a dense subgroup of $\mathbb{A}^\times/F^\times$. Hence $\eta'=\eta$. By the Theorem 4.19, of [JL70], above, fixing a character $\psi:\mathbb{A}/F\to E^\times$, $\psi\neq 1$, for any character $\omega:\mathbb{A}^\times/F^\times\to E^\times$ one has

$$\begin{aligned}\prod_v L(\pi_v\omega_v,t_v) &= \prod_v \varepsilon(\pi_v\omega_v,\psi_v,t_v)L(\pi_v\eta_v^{-1}\omega_v^{-1},q_v^{-2}t_v^{-1}),\\ \prod_v L(\pi'_v\omega_v,t_v) &= \prod_v \varepsilon(\pi'_v\omega_v,\psi_v,t_v)L(\pi'_v{\eta'_v}^{-1}\omega_v^{-1},q_v^{-2}t_v^{-1}).\end{aligned}$$

Since $\pi_v\simeq\pi'_v$ at all $v\notin S$, we conclude

$$\begin{aligned}\prod_{v\in S}\Gamma(\pi_v\omega_v,\psi_v,t_v) &= \prod_{v\in S}\frac{\varepsilon(\pi_v\omega_v,\psi_v,t_v)L(\pi_v\eta_v^{-1}\omega_v^{-1},q_v^{-2}t_v^{-1})}{L(\pi_v\omega_v,t_v)}\\ = \prod_{v\in S}\frac{\varepsilon(\pi'_v\omega_v,\psi_v,t_v)L(\pi'_v{\eta'_v}^{-1}\omega_v^{-1},q_v^{-2}t_v^{-1})}{L(\pi'_v\omega_v,t_v)} &= \prod_{v\in S}\Gamma(\pi'_v\omega_v,\psi_v,t_v).\end{aligned}$$

Since $\eta=\eta'$, it follows from Proposition 4.16 that for each $v\in S$ there exists $m_v>0$ such that if $\chi:F_v^\times\to E^\times$ is any character whose conductor is $\geq m_v$, then $\Gamma(\pi_v\chi, \psi_v, t) = \Gamma(\pi'_v\chi, \psi_v, t)$. Fix $v \in S$ and a character χ of $F_v^\times$. By Proposition 4.14, it suffices to show $\Gamma(\pi_v\chi, \psi_v, t) = \Gamma(\pi'_v\chi, \psi_v, t)$. For this, it suffices to choose a character $\omega : \mathbb{A}^\times/F^\times \to E^\times$ in the last displayed equation with $\omega_v = \chi$ and such that for each $u \in S - \{v\}$, the conductor of ω_u is bigger than m_u. But the group $H = F_v^\times \prod_{u\in S-\{v\}} O_u^\times$ maps isomorphically and homeomorphically onto its image in $\mathbb{A}^\times/F^\times$. Hence any character of H extends to a character of $\mathbb{A}^\times/F^\times$. $\square$

Proposition 4.21. *Let η be a character of $\mathbb{A}^\times/F^\times$, S a finite set of closed points of $X, \psi \neq 1$ a character of $\mathbb{A}/F$ with $r(\psi_u) = 0$ for all u in S. Suppose that for any closed point $v \in |X| - S$, π_v is an irreducible admissible infinite dimensional representation of $\mathrm{GL}(2, F_v)$ with central character η_v such that almost all π_v are unramified, there is no pair μ_1, μ_2 of characters of $\mathbb{A}^\times/F^\times$ with $\pi_v = \pi(\mu_{1v}, \mu_{2v})$ for almost all $v \in |X| - S$, and for any character ω of $\mathbb{A}^\times/F^\times$ which is unramified at all points of S, the formal series $\prod_{v\notin S} L(\pi_v\omega_v, t_v)$ and $\prod_{v\notin S} L(\pi_v\eta_v^{-1}\omega_v^{-1}, t_v)$ are polynomials, and there exists a number $c \in E^\times$ and integers $n_u > 0$ $(u \in S)$ such that*

$$\prod_{v\notin S} L(\pi_v\omega_v, t_v) = c \prod_{u\in S} (\omega(\boldsymbol{\pi}_u)t_u)^{n_u} \prod_{v\notin S} \varepsilon(\pi_v\omega_v, \psi_v, t_v) L(\pi_v\eta_v^{-1}\omega_v^{-1}, q_v^{-2}t_v^{-1}).$$

Then there exists a cuspidal representation π of $\mathrm{GL}(2, \mathbb{A})$ with central character η such that for every $v \in |X| - S$ the local component of π at v is π_v.

A proof is in [JL70], Theorem 11, Corollary 11.6, proof of Theorem 12.2. The representation π is unique by Proposition 4.20.

4.4 Intertwining again

We can now return to the study of the intertwining operators.

Proposition 4.22. *Let μ_1, μ_2 be characters of $F_v^\times$. Let $\psi \neq 1$ be a character of F_v. Then*

$$R(\mu_1, \mu_2, t)R(\mu_2, \mu_1, t^{-1}) = \varepsilon\left(\frac{\mu_1}{\mu_2}, \psi, q_v^{-1}t^2\right) \varepsilon\left(\frac{\mu_2}{\mu_1}, \psi, q_v^{-1}t^{-2}\right).$$

Proof. By the transformation formula for the ε-factors, the right hand side does not depend on ψ. We then choose ψ with $\ker\psi \supset O_v$ and $\ker\psi \not\supset \boldsymbol{\pi}_v^{-1}O_v$. We can rewrite the asserted equality as

$$M(\mu_1, \mu_2, t)M(\mu_2, \mu_1, t^{-1}) = \Gamma\left(\frac{\mu_2}{\mu_1}, \psi, q_v^{-1}t^2\right) \Gamma\left(\frac{\mu_2}{\mu_1}, \psi, q_v^{-1}t^{-2}\right).$$

The restriction map $I(\mu_1, \mu_2) \to I(\mu_1/\mu_2)$, where

$$I(\mu) = \{f \in C^\infty(\mathrm{SL}(2, F_v)); f\left(\left(\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right) x\right) = \mu(a)|a|_v f(x)\},$$

is an isomorphism ($\mu : F_v^\times \to E^\times$ is a character). The group $\mathrm{SL}(2, F_v)$ acts transitively on $F_v^2 - \{(0,0)\}$ on the right. The stabilizer of the vector $(0,1)$ is $N(F_v)$. Then $N(F_v)\backslash \mathrm{SL}(2, F_v)$ can be identified with $F_v^2 - \{(0,0)\}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (c,d) \in F_v^2 - \{(0,0)\}$. Using this we identify $I(\mu)$ with

$$V(\mu) = \{f \in C^\infty(F_v^2 - \{(0,0)\});$$
$$f(ax) = \mu(a)^{-1}|a|_v^{-1} f(x), a \in F_v^\times, x \in F_v^2 - \{(0,0)\}\},$$

so $I(\mu_1, \mu_2)$ with $V(\mu_1/\mu_2)$. The operator $M(\mu_1, \mu_2, t)$ corresponds to the operator $\overline{M}(\mu_1/\mu_2, t^2)$ where

$$\overline{M}(\mu, s) : V(\mu\nu_s) \to V(\mu^{-1}\nu_{s^{-1}}), \quad (\overline{M}(\mu, s)f)(x) = \int_{\{y; x \wedge y = 1\}} f(y)dy.$$

Here $\wedge$ denotes the symplectic form $(a,b)\wedge(c,d) = ad - bc$ on F_v^2. The measure on the line $\ell_x = \{y \in F_v^2; x \wedge y = 1\}$ is transferred from the Haar measure on F_v via the map $F_v \to \ell_x$ given by $a \mapsto y_0 + ax$ where y_0 is a fixed point on ℓ_x. So we need to show:

$$\overline{M}(\mu, s)\overline{M}(\mu^{-1}, s^{-1}) = \Gamma(\mu, \psi, q_v^{-1}s)\Gamma(\mu^{-1}, \psi, q_v^{-1}s^{-1}).$$

For sufficiently small $s \in \mathbb{C}^\times$ define operators $A_s : C_c^\infty(F_v^2) \to V(\mu\nu_s)$ and $B_s : C_c^\infty(F_v^2) \to V(\mu^{-1}\nu_s)$ by

$$(A_s f)(x) = \int_{F_v} f(ax)\mu(a)\nu_s(a)da, \quad (B_s f)(x) = \int_{F_v} f(ax)\mu(a)^{-1}\nu_s(a)da.$$

Restriction defines an isomorphism $V(\mu\nu_s) \to V_0(\mu)$, where

$$V_0(\mu) = \{f \in C^\infty(O_v^2 - \{(0,0)\});$$
$$f(ax) = \mu(a)^{-1} f(x),\ x \in O_v^2 - \{(0,0)\},\ a \in O_v^\times\},$$

so we can identify the spaces $V(\mu\nu_s)$ as s varies.
The operators A_s and B_s, defined above for small s, depend rationally on s. Hence they can be extended to all s.
Consider the Fourier transform

$$F : C_c^\infty(F_v^2) \to C_c^\infty(F_v^2), \quad (Ff)(y) = \int_{F_v^2} f(x)\psi(x \wedge y)dx.$$

Lemma 4.23. *We have* $\overline{M}(\mu, s)A_s = \Gamma(\mu^{-1}, \psi, q_v^{-1}s^{-1})B_{s^{-1}}F$,

$$\overline{M}(\mu^{-1}, s^{-1})B_{s^{-1}} = \Gamma(\mu, \psi, q_v^{-1}s)A_s F.$$

Proof. Given $f \in C_c^\infty(F_v^2)$, $x \in F_v^2 - \{(0,0)\}$, we first show

$$\Gamma(\mu^{-1}, \psi, q_v^{-1}s^{-1})(B_{s^{-1}}Ff)(x) = (\overline{M}(\mu, s)A_s f)(x).$$

The operators F, A_s, B_s commute with the action of $\mathrm{SL}(2, F_v)$. This action is transitive on $F_v^2 - \{(0,0)\}$, so we may assume $x = (0,1)$. We compute

$$\begin{aligned}(B_{s^{-1}}Ff)((0,1)) &= \int_{F_v} (Ff)((0,a))\mu(a)^{-1}\nu_{s^{-1}}(a)da,\\ (Ff)((0,a)) &= \int_{F_v^2} f(y,z)\psi(ya)dydz = \hat{\varphi}(-a),\\ \hat{\varphi}(a) &= \int \varphi(y)\psi(-ya)dy, \qquad \varphi(y) = \int f(y,z)dz.\end{aligned}$$

Tate's functional equation (see [L], VII, section 3-4) is

$$\Gamma(\mu^{-1}, \psi, q_v^{-1}s^{-1}) \int \hat{\varphi}(a)\mu^{-1}(a)\nu_{s^{-1}}(a)da = \int \varphi(y)\mu(y)\nu_s(y)\frac{dy}{|y|}.$$

(Formally this can be deduced from the definition of the Γ-function and the inversion formula $\varphi(y) = \int \hat{\varphi}(a)\psi(ay)da$. However the left side converges for large $|s|$, while the right for small $|s|$, so one has to show both sides are rational in s).

We conclude that the left side of the equation to be shown is

$$\int \varphi(y)\mu(-y)\nu_s(y)|y|^{-1}dy = \int\int f(y,z)\mu(-y)\nu_s(y)|y|^{-1}dydz$$

while the right side is (recall: $x = (0,1)$, so $(0,1) \wedge (y,z) = -y$)

$$\int (A_s f)(-1,z)dz = \int\int f(-y, yz)\mu(y)\nu_s(y)dydz.$$

The proof of the second identity of the lemma is similar. □

The inverse Fourier transform coincides with F since the form $(x,y) \mapsto x \wedge y$ in the definition of F is skew-symmetric. Hence $F^2 = 1$, and it follows from the Lemma that

$$\overline{M}(\mu, s)\overline{M}(\mu^{-1}, s^{-1})B_{s^{-1}} = \Gamma(\mu, \psi, q_v^{-1}s)\Gamma(\mu^{-1}, \psi, q_v^{-1}s^{-1})B_{s^{-1}}.$$

However, the operator $B_{s^{-1}}$ is onto for those s where it is defined (even its restriction to $C_c^\infty(F_v^2 - \{(0,0)\})$ is onto), as $V(\mu\nu_s)$ is irreducible, so the proposition follows. □

Proposition 4.24. *For any characters μ_1, μ_2 of $\mathbb{A}^\times/F^\times$ we have*

$$M(\mu_1, \mu_2, t)M(\mu_2, \mu_1, t^{-1}) = 1.$$

Proof. From Proposition 4.21, $M(\mu_1, \mu_2, t)M(\mu_2, \mu_2, t^{-1})$ is equal to

$$q^{2-2g}m(\mu_1/\mu_2, t^2)m(\mu_2/\mu_1, t^{-2})R(\mu_1, \mu_2, t)R(\mu_2, \mu_1, t^{-1}),$$

while Proposition 4.22 implies, for any character $\psi \neq 1$ of $\mathbb{A}/F$, that

$$R(\mu_1, \mu_2, t)R(\mu_2, \mu_1, t^{-1})$$

is

$$\prod_v [\varepsilon(\mu_{1v}/\mu_{2v}, \psi_v, q_v^{-1}t_v^2)\varepsilon(\mu_{2v}/\mu_{1v}, \psi_v, q_v^{-1}t_v^{-2})]$$
$$= q^{2g-2}\varepsilon(\mu_1/\mu_2, q^{-1}t^2)\varepsilon(\mu_2/\mu_1, q^{-1}t^{-2}).$$

As $\varepsilon(\chi, t) = q^{1-g}\prod_v \varepsilon(\chi_v, \psi_v, t_v)$ satisfies the functional equation $L(\chi, t) = \varepsilon(\chi, t)L(\chi^{-1}, q^{-1}t^{-1})$, we have that

$$\varepsilon(\mu_1/\mu_2, q^{-1}t^2)\varepsilon(\mu_2/\mu_1, q^{-1}t^{-2})m(\mu_1/\mu_2, t^2)m(\mu_2/\mu_1, t^{-2}),$$

which is equal to

$$\frac{\varepsilon(\mu_1/\mu_2, q^{-1}t^2)L(\mu_2/\mu_1, t^2)}{L(\mu_1/\mu_2, q^{-1}t^2)} \cdot \frac{\varepsilon(\mu_1/\mu_2, q^{-1}t^{-2})L(\mu_1/\mu_2, t^2)}{L(\mu_2/\mu_1, q^{-1}t^{-2})}$$

is equal to 1. □

4.5 $M^2 = 1$ via Mellin transform

We shall next study the relationship between $M : C_c^\infty(Y_\alpha) \to C^\infty(Y_\alpha)$ and $M(\mu_1, \mu_2, t) : I(\mu_1\nu^t, \mu_2\nu^{-t}) \to I(\mu_2\nu^{-t}, \mu_1\nu^t)$, and conclude that $M^2 = 1$. Both are defined by the same integral formula. Here μ_1, μ_2 are characters of $\mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}}$. Put

$$\eta(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)) = \mu_1(a)\mu_2(b)|a/b|^{1/2}\nu_t(a/b), \quad \eta : A(\mathbb{A})/A(F) \cdot \alpha^{\mathbb{Z}} \to E^\times.$$

It is a character. Recall that $Y_\alpha = \alpha^{\mathbb{Z}}N(\mathbb{A})A(F)\backslash \mathrm{GL}(2, \mathbb{A})$ and $(Mf)(x) = \int_{N(\mathbb{A})} f(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx)dn$. Suppose that $f \in C_c^\infty(Y_\alpha)$, and $t \in E^\times$. Define a function $T(f, \mu_1, \mu_2, t) : \mathrm{GL}(2, \mathbb{A}) \to \mathbb{C}$ by

$$(T(f, \mu_1, \mu_2, t))(x) = \int_{\alpha^{\mathbb{Z}}A(F)\backslash A(\mathbb{A})} f(a^{-1}x)\eta(a)d^\times a.$$

Then $T(f, \mu_1, \mu_2, t) \in I(\mu_1\nu_t, \mu_2\nu_{-t})$ is called the *Mellin transform* of f. The notation T can be used also when $f \in C^\infty(Y_\alpha)$ is not compactly supported, whenever the integral converges.

Proposition 4.25. *For $\varphi \in C_c^\infty(Y_\alpha)$, characters $\mu_1, \mu_2 : \mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}} \to E^\times$ and large enough $t \in \mathbb{C}^\times$, the integral defining T converges, and $T(M\varphi, \mu_1, \mu_2, t) = M(\mu_2, \mu_1, t^{-1})T(\varphi, \mu_2, \mu_1, t^{-1})$.*

Proof. By definition,

$$(T(f, \mu_1, \mu_2, t))(x) = \iint f(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)^{-1} x)\mu_1(a)\mu_2(b)|a/b|^{1/2}\nu_t(a/b)d^\times a d^\times b.$$

Put $f = M\varphi$, so $f((\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix})^{-1} x) = |b/a| \int_{N(\mathbb{A})} \varphi((\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix})^{-1} \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx) dn$. Hence $(T(f, \mu_1, \mu_2, t))(x)$ equals

$$\begin{aligned} & \int\int\int \varphi((\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix})^{-1} \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx) \mu_1(a)\mu_2(b)|b/a|^{1/2}\nu_t(a/b) d^\times a d^\times b dn \\ = & \int_{N(\mathbb{A})} (T(\varphi, \mu_2, \mu_1, t^{-1}))(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx) dn \\ = & (M(\mu_2, \mu_1, t^{-1})T(\varphi, \mu_2, \mu_1, t^{-1}))(x). \end{aligned}$$

If t is large enough, the integral which defines $M(\mu_2, \mu_1, t^{-1})$ converges, and so is the integral which defines $T(f, \mu_1, \mu_2, t)$, which justifies the computation. □

Proposition 4.26. *If $\varphi \in C_c^\infty(Y_\alpha)$ then $M\varphi \in C^\infty(Y_\alpha)$. If $M\varphi \in C_c^\infty(Y_\alpha)$ then $M^2\varphi = \varphi$.*

Proof. Put $f = M\varphi$ and $h = Mf = M^2\varphi$ (h is defined if $f \in C_c^\infty(Y_\alpha)$). By Proposition 4.25,

$$\begin{aligned} T(h, \mu_1, \mu_2, t) &= M(\mu_2, \mu_1, t^{-1})T(f, \mu_2, \mu_1, t^{-1}), \\ T(f, \mu_2, \mu_1, t^{-1}) &= M(\mu_1, \mu_2, t)T(\varphi, \mu_1, \mu_2, t). \end{aligned}$$

The first equation holds only for large enough t, and the second only for small enough t. However, both sides of the second equality depend rationally on t (for the left side, this is true since $f = M\varphi$ is compactly supported), hence it holds for all t in $\mathbb{C}^\times$. Hence for large enough t, by Proposition 4.24 $T(h, \mu_1, \mu_2, t) = T(\varphi, \mu_1, \mu_2, t)$ for all μ_1, μ_2. This implies $h = \varphi$. □

4.6 Poles, zeroes and values of R and M

Recall that $\nu_t(x) = t^{\deg(x)}$ is a character of $\mathbb{A}^\times/F^\times$ with $\nu_t(\boldsymbol{\pi}_v) = t_v$ $(= t^{\deg(v)})$, and locally we write ν_t for the unramified character of $F_v^\times$ with $\nu_t(\boldsymbol{\pi}_v) = t$. Let μ_1, μ_2 be characters of $F_v^\times$. Recall:

$$R(\mu_1, \mu_2, t) = \frac{L(\mu_1/\mu_2, q_v^{-1}t^2)}{L(\mu_1/\mu_2, t^2)} M(\mu_1, \mu_2, t).$$

Proposition 4.27. (1) *The function $R(\mu_1, \mu_2, t)$ is regular at $t = 0$.*
It has a pole at $\tau \in \mathbb{C}^\times$ iff $\mu_2\nu_{\tau^{-1}}/\mu_1\nu_\tau = \nu$ (with $\nu(\boldsymbol{\pi}_v) = q_v^{-1}$). This pole has order 1.
The function $R(\mu_1, \mu_2, t)^{-1}$ has a pole at $\tau \in \mathbb{C}^\times$ iff $\mu_1\nu_\tau/\mu_2\nu_{\tau^{-1}} = \nu$. This pole has order 1.
(2) *Suppose $R(\mu_1, \mu_2, t)^{-1}$ has a pole at $\tau \in \mathbb{C}^\times$. Then the function $R(\mu_1, \mu_2, t)$ is regular at $t = \tau$. Put $L = \lim_{t\to\tau}(t - \tau)R(\mu_1, \mu_2, t)^{-1}$ and $Q = R(\mu_1, \mu_2, \tau)$. The operators $Q : I(\mu_1\nu_\tau, \mu_2\nu_{\tau^{-1}}) \to I(\mu_2\nu_{\tau^{-1}}, \mu_1\nu_\tau)$ and $L : I(\mu_2\nu_{\tau^{-1}}, \mu_1\nu_\tau) \to I(\mu_1\nu_\tau, \mu_2\nu_{\tau^{-1}})$ intertwine the* $\mathrm{GL}(2, F_v)$*-action. The representations of* $\mathrm{GL}(2, F_v)$ *in the spaces* $\ker Q$, $\operatorname{coker} Q$, $\operatorname{im} L$ *are isomorphic*

to the square integrable $\mathrm{St}(\mu_1\nu_\tau, \mu_2\nu_{\tau^{-1}})$*. The representations of* $\mathrm{GL}(2, F_v)$ *in the spaces* $\ker L$*,* $\operatorname{coker} L$*,* $\operatorname{im} Q$ *are isomorphic to the one dimensional* $x \mapsto \mu_2(x)(\nu\nu_{\tau^{-1}})(x) = \mu_1(x)\nu_\tau(x)$*.*
(3) *The statement* (2) *remains true with* $R(\mu_1, \mu_2, t)$ *replaced by* $R(\mu_1, \mu_2, t)^{-1}$*.*

Proof. From the first part of the proof of Proposition 4.3 it follows that

$$M(\mu_1, \mu_2, t)/L(\mu_1/\mu_2, t^2) = R(\mu_1, \mu_2, t)/L(\mu_1/\mu_2, q_v^{-1}t^2)$$

is regular. So $R(\mu_1, \mu_2, t)$ could have a pole at $t \in \mathbb{C}^\times$ only if $L(\mu_1/\mu_2, q_v^{-1}t^2)$ is ∞, that is $\mu_2\nu_{\tau^{-1}}/\mu_1\nu_\tau = \nu$ (recall: $\nu(x) = |x|$), and the order of the pole is at most 1.
A similar statement holds for $R(\mu_1, \mu_2, t)^{-1} = c(\mu_1, \mu_2)t^{n(\mu_1,\mu_2)}R(\mu_2, \mu_1, t^{-1})$. (The last equality follows from Proposition 4.22. In fact $n(\mu_1, \mu_2) = 0$, but we do not need this.) Namely $R(\mu_1, \mu_2, t)^{-1}$ has a pole at $\tau \in \mathbb{C}^\times$ iff $\mu_1\nu_\tau/\mu_2\nu_{\tau^{-1}} = \nu$. This pole has order 1.
Suppose $\mu_1\nu_\tau/\mu_2\nu_{\tau^{-1}} = \nu$. Then $\mu_2\nu_{\tau^{-1}}/\mu_1\nu_\tau \neq \nu$ so that $R(\mu_1, \mu_2, t)^{-1}$ is regular at $t = \tau$. With L, Q defined as in the proposition, it is clear they commute with the $\mathrm{GL}(2, F_v)$-action. If $L = 0$ then $Q = R(\mu_1, \mu_2, \tau)$ has no pole, in fact it is an isomorphism. If $Q = 0$ then L would be an isomorphism, as the operator $\lim_{t\to\tau} R(\mu_1, \mu_2, t)/(t - \tau)$ would be the inverse of L. However, the representations of $\mathrm{GL}(2, F_v)$ in $I(\mu_1\nu_\tau, \mu_2\nu_{\tau^{-1}})$ and $I(\mu_2\nu_{\tau^{-1}}, \mu_1\nu_\tau)$ are not equivalent, hence $L \neq 0$, $Q \neq 0$. As $L \neq 0$, the function $R(\mu_1, \mu_2, t)^{-1}$ does have a pole at $t = \tau$. From the description of the invariant subspaces of $I(\mu_1\nu_\tau, \mu_2\nu_{\tau^{-1}})$ and $I(\mu_2\nu_{\tau^{-1}}, \mu_1\nu_\tau)$ the claims in the proposition on the description of the action of $\mathrm{GL}(2, F_v)$ follow. The regularity of $R(\mu_1, \mu_2, t)$ at $t = 0$ follows from that of $L(\mu_1/\mu_2, q_v^{-1}t^2)^{-1}R(\mu_1, \mu_2, t)$. □

In conclusion, the representation of $\mathrm{GL}(2, F_v)$ in $I(\mu_1\nu_t, \mu_2\nu_{t^{-1}})$ is reducible iff $R(\mu_1, \mu_2, t)$ or $R(\mu_1, \mu_2, t)^{-1}$ has a pole at $t = \tau$. These last operators are regular at $t \in \mathbb{C}^\times$ if μ_1/μ_2 is ramified. If μ_1/μ_2 is unramified and $(\mu_1/\mu_2)(\boldsymbol{\pi}_v) = a$, then the poles of $R(\mu_1, \mu_2, t)$ are at $\pm\sqrt{q_v/a}$, and those of $R(\mu_1, \mu_2, t)^{-1}$ are at $\pm\sqrt{a/q_v}$.

Corollary 4.28. *Let* μ_1*,* μ_2 *be characters of* $\mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}}$*. If* $R(\mu_1, \mu_2, t)$ *has a pole at* $t = \tau \in \mathbb{C}^\times$*, then* $|\tau| = \sqrt{q}$*. If* $R(\mu_1, \mu_2, t)^{-1}$ *has a pole at* $t = \tau \in \mathbb{C}^\times$ *then* $|\tau| = q^{-1/2}$*.*

Indeed, a character of $\mathbb{A}^\times/F^\times$ which takes the value 1 at α is unitary, thus $|a| = 1$.

Proposition 4.29. *Let* μ_1*,* μ_2 *be characters of* $\mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}}$ *and* $\tau \in \mathbb{C}^\times$*,* $|\tau| \leq 1$*. If* $M(\mu_1, \mu_2, t)$ *has a pole at* $t = \tau$ *then* $\mu_1 = \mu_2$ *and* $\tau = \pm q^{-1/2}$*. If* $\mu_1 = \mu_2$ *is denoted* μ *and* $\tau = \pm q^{-1/2}$ *then* $M(\mu, \mu, t)$ *has an order* 1 *pole at* τ*. The image of the operator* $C = \lim_{t\to\tau}(t - \tau)M(\mu, \mu, t)$ *in this case is one dimensional and is spanned by the function* $f(x) = \mu(\det x)\nu_\tau(\det x)$ *in* $I(\mu\nu_{\tau^{-1}}, \mu\nu_\tau)$*. Further,* $M(\mu_1, \mu_2, t)$ *is regular at* $t = 0$*.*

Proof. Recall that $M(\mu_1,\mu_2,t) = q^{1-g} m(\mu_1/\mu_2,t^2)R(\mu_1,\mu_2,t)$ where $m(\mu,t) = L(\mu,t)/L(\mu,t/q)$. Let $\tau \in \mathbb{C}^\times$, $|\tau| \le 1$. By Corollary 4.28, the function $R(\mu_1,\mu_2,t)$ is regular at τ. By Proposition 4.11, the function $m(\mu_1/\mu_2,t^2)$ is not regular at τ only if $\mu_1 = \mu_2$ and $\tau = \pm q^{-1/2}$. In these cases it has a simple pole. Hence $M(\mu_1,\mu_2,t)$ is regular at $t = \tau$ $(0 < |\tau| \le 1)$ unless $\mu_1 = \mu_2$ and $\tau = \pm q^{-1/2}$ where the order of the pole is at most 1. When $\mu_1 = \mu_2 = \mu$ and $\tau = \pm q^{-1/2}$, the operator $C = \lim_{t\to\tau}(t-\tau)M(\mu,\mu,t)$ is a scalar multiple of $R(\mu,\mu,t) = \otimes_v R(\mu_v,\mu_v,\tau_v)$, $\tau_v = \tau^{\deg(v)}$.
From (1) in Proposition 4.27, the function $R(\mu_v,\mu_v,\tau_v)^{-1}$ has a pole at $t = \tau$ $(t_v = \tau_v)$. Its statement (2) implies that the image of $R(\mu_v,\mu_v,\tau_v)$ is one dimensional and $\mathrm{GL}(2,F_v)$ acts on it via the character $x \mapsto \mu_v(\det x)\nu_\tau(\det x)^{\deg v}$. This implies the proposition, except the final claim, which follows from the regularily of $R(\mu_1,\mu_2,t)$ at $t = 0$, and that of $m(\mu_1/\mu_2,t^2)$ at $t = 0$. □

Let μ_1, μ_2 be characters of $\mathbb{A}^\times/F^\times$. The operator $M(\mu_1,\mu_2,t)$ maps $I(\mu_1\nu_t,\mu_2\nu_{t^{-1}})$ into the space $I(\mu_2\nu_{t^{-1}},\mu_1\nu_t)$, which in general is different from $I(\mu_1\nu_t,\mu_2\nu_{t^{-1}})$. However, when $\mu_1 = \mu_2 = \mu$ and $t = \pm 1$, then $M(\mu_1,\mu_2,t)$ maps $I(\mu_1\nu_t,\mu_2\nu_{t^{-1}})$ to itself; $M(\mu,\mu,t)$ is regular at $t = \pm 1$. The representation of $\mathrm{GL}(2,\mathbb{A})$ in $I(\mu\nu_\tau,\mu\nu_{\tau^{-1}})$, $\tau = \pm 1$, is irreducible, and hence $M(\mu,\mu,\tau)$ is a scalar operator. Moreover, from Proposition 4.26, $M(\mu,\mu,\tau)^2 = 1$ at $\tau = \pm 1$.

Proposition 4.30. *If μ is a character of $\mathbb{A}^\times/F^\times$ and $\tau = \pm 1$, then $M(\mu,\mu,\tau) = -1$.*

Proof. In view of the relation between M and R, it suffices to verify that

$$\lim_{t\to 1}\frac{L(1,t)}{L(1,t/q)} = -q^{g-1} \qquad \text{and} \qquad R(\mu,\mu,\tau) = 1.$$

In fact, for any character ω of $F_v^\times$, $R(\omega,\omega,\tau)$ is 1 at $\tau = \pm 1$. Indeed, suppose first ω is unramified. Then there exists a function f in $I(\omega\nu_\tau,\omega\nu_\tau)$ whose restriction to $\mathrm{GL}(2,O_v)$ is 1. By the normalization of the intertwining operator (Proposition 4.3(2)), $R(\omega,\omega,\tau)f = f$. However, the representation of $\mathrm{GL}(2,F_v)$ on $I(\omega\nu_\tau,\omega\nu_\tau)$ is irreducible, so $R(\omega,\omega,\tau) = 1$ if ω is unramified. The general case reduces to the case where ω is unramified, or even $\omega = 1$, by the commutativity of the diagram

$$\begin{array}{ccc} I(\omega\nu_\tau,\omega\nu_\tau) & \xrightarrow{R(\omega,\omega,\tau)} & I(\omega\nu_\tau,\omega\nu_\tau) \\ \uparrow & & \uparrow \\ I(\nu_\tau,\nu_\tau)\otimes\omega & \xrightarrow{R(1,1,\tau)} & I(\nu_\tau,\nu_\tau)\otimes\omega \end{array}$$

To compute the limit of the ratio of L-functions, we use the functional equation $L(1,t/q) = \varepsilon(1,t/q)L(1,t^{-1})$. Then

$$\lim_{t\to 1} L(1,t)/L(1,t/q) = \varepsilon(1,1/q)^{-1}\lim_{t\to 1} L(1,t)/L(1,t^{-1}).$$

By the definition of the global ε-function and its properties (Proposition 6.1, 6.3), $\varepsilon(1,1/q) = q^{1-g}$. Since $L(1,t)$ has a pole of order one at $t = 1$, by L'Hôpital rule $\lim_{t\to 1} L(1,t)/L(1,t^{-1})$ is -1. □

4.7 Global Eisenstein approach

These proofs of $M^2 = 1$ and rationality of $M(\mu_1, \mu_2, t)$ are based on local computations (normalization of the intertwining operators by L-functions and ε-factors), and the functional equation of the L-function. The following alternative proof of these results is based on properties of the Eisenstein map.

The alternative approach of this subsection, the following subsection 4.8, and the computation of traces in subsection 5.2 are motivated by Tate [T68]. They are the newest part of this paper, which – as noted in the introduction – cries out for generalization from our context of GL(2), and for further study.

We shall use the maps $\mathrm{ht}^+ : Y_\alpha \to \mathbb{Z}$ and $\mathrm{ht} : \alpha^{\mathbb{Z}} \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}) \to \mathbb{Z}$. Both maps are proper. However, ht^+ is onto while the image of ht contains the positive integers but only finitely many negatives. So in some sense Y_α is less compact than $\alpha^{\mathbb{Z}} \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$, so the map $E : C_c^\infty(Y_\alpha) \to C_c^\infty(\alpha^{\mathbb{Z}} \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}))$ should have a big kernel. For φ in $\ker E$ we have $(1+M)\varphi = E^*E\varphi = 0$. Hence $M^2\varphi = \varphi$. Unlike M, the operator M^2 commutes with the action of $A(\mathbb{A})$ on $C_c^\infty(Y_\alpha)$ by left translation. Hence $M^2\varphi = \varphi$ not only for $\varphi \in \ker E$ but also for φ in the span of $A(\mathbb{A})$-translates of φ in $\ker E$. The number of such linear combinations is already sufficiently large to imply $M^2 = 1$. We now turn to rigorous proofs.

Proposition 4.31. *Let $M : \mathbb{C}[z,z^{-1}]^n \to \mathbb{C}((z))^n$ be a $\mathbb{C}$-linear map with $M(zu) = z^{-1}M(u)$ for all $u \in \mathbb{C}[z,z^{-1}]^n$. Let I denote the natural embedding $\mathbb{C}[z,z^{-1}]^n \hookrightarrow \mathbb{C}((z))^n$. Put $B = I + M$. Suppose there is some $k \in \mathbb{Z}$ for which the vector space $(\mathrm{Im}\, B)/B(z^k\mathbb{C}[z^{-1}]^n)$ is finite dimensional. Then there is some*

$$P(z) \in \mathrm{GL}(n,\mathbb{C}(z)) \subset \mathrm{GL}(n,\mathbb{C}((z)))$$

with $P(z^{-1}) = P(z)^{-1}$ and $(Mu)(z) = P(z)u(z^{-1})$ for all $u(z) \in \mathbb{C}[z,z^{-1}]^n$.

Proof. Denote by e_i the column in $\mathbb{C}^n$ with nonzero entry only at the ith row, where it is 1. From $M(\sum_i(\sum_j c_{ij}z^j)e_i) = \sum_i(\sum_j c_{ij}z^{-j})Me_i$, we see that $(Mu)(z) = P(z)u(z^{-1})$ where $P(z)$ is the $n\times n$ matrix with columns $Me_1,\dots,Me_n$ whose entries are in $\mathbb{C}((z))$. If u is in the kernel of $B = I+M$, then $P(z)u(z^{-1}) = -u(z)$. Since $\mathrm{Im}\, B = \cup_{m\ge 1} B(z^m\mathbb{C}[z^{-1}]^n)$ and there is some $k \ge 0$ such that $B(z^k\mathbb{C}[z^{-1}]^n)$ has finite codimension in $\mathrm{Im}\, B$, there is some ℓ with $B(z^\ell\mathbb{C}[z^{-1}]^n) = \mathrm{Im}\, B$. Then $\ker B + z^\ell\mathbb{C}[z^{-1}]^n = \mathbb{C}[z,z^{-1}]^n$. For each i $(1 \le i \le n)$, $z^{\ell+1}e_i \in \ker B + z^\ell\mathbb{C}[z^{-1}]^n$. Hence there is a matrix $W \in M(n,\mathbb{C}[z,z^{-1}])$ whose columnes are in $\ker B$ and $W - z^{\ell+1}\,\mathrm{Id} \in z^\ell M(n,\mathbb{C}[z^{-1}])$, where Id is the identity matrix. But then $W \in \mathrm{GL}(n,\mathbb{C}(z))$, and since the columns of W are in $\ker B$, we have $P(z)W(z^{-1}) = -W(z)$. Then $P(z) = -W(z)W(z^{-1})^{-1}$, and $P(z^{-1}) = -W(z^{-1})W(z)^{-1} = P(z)^{-1}$. □

Corollary 4.32. *A $\mathbb{C}$-linear map $M : \mathbb{C}[z,z^{-1}] \to \mathbb{C}[z,z^{-1}]$ which satisfies the conditions of Proposition 4.31 has $M^2 = \mathrm{Id}$.*

Recall that $Y_\alpha = \alpha^{\mathbb{Z}} A(F)N(\mathbb{A})\backslash \mathrm{GL}(2,\mathbb{A})$. Write $C_+^\infty(Y_\alpha)$ for the space of the E-valued functions f on Y_α with (1) $f(x) = 0$ if $\mathrm{ht}^+(x)$ is large enough, and (2) f is invariant under right translation by some open subgroup U of $\mathrm{GL}(2,\mathbb{A})$. Note that $C_c^\infty(Y_\alpha) \subset C_+^\infty(Y_\alpha) \subset C^\infty(Y_\alpha)$.

Proposition 4.33. *The image of $C_c^\infty(Y_\alpha)$ under M lies in $C_+^\infty(Y_\alpha)$.*

Proof. For $f \in C_c^\infty(Y_\alpha)$ there exists an integer m such that $f(x) = 0$ if $\mathrm{ht}^+(x) < -m$. We shall show that for such f, $(Mf)(x) = \int_{N(\mathbb{A})} f\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx\right) dx$ is zero if $\mathrm{ht}^+(x) > m$. It suffices to show then that for $x \in \mathrm{GL}(2,\mathbb{A})$ with $\mathrm{ht}^+(x) > m$, and any $n \in N(\mathbb{A})$, we have $\mathrm{ht}^+\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx\right) < -m$. But by Lemma 2.7 we have

$$\mathrm{ht}^+(x) + \mathrm{ht}^+\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx\right) = \mathrm{ht}^+(nx) + \mathrm{ht}^+\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx\right) \le 0.$$

□

Proposition 4.34. *Let U be an open subgroup of* $\mathrm{GL}(2,O)$*. For every integer $m \ge 1$ define*

$$W_m^U = \{\varphi \in C_c^\infty(Y_\alpha)^U;\ \varphi(x) = 0 \ \textit{ if } \ \mathrm{ht}^+(x) < m\},$$

$$Y_m^U = \{\varphi \in C_c^\infty(\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}))^U;\ \varphi(x) = 0 \ \textit{ if } \ \mathrm{ht}^+(x) < m\}.$$

Then $E(W_m^U) = Y_m^U$ for large enough m.

Proof. Put

$$Z_m^U = \{\varphi \in C_c^\infty(\alpha^{\mathbb{Z}} \cdot A(F)N(F)\backslash \mathrm{GL}(2,\mathbb{A}))^U;\ \varphi(x) = 0 \ \text{ if } \ \mathrm{ht}^+(x) < m\}.$$

Recall that

$$E = s_* r^*, \qquad s_*(x) = \sum_\gamma \psi(\gamma x), \qquad \gamma \in A(F)N(F)\backslash \mathrm{GL}(2,F).$$

It is clear that $s_*(Z_m^U) = Y_m^U$. It suffices to show that $r^*(W_m^U) = Z_m^U$ for sufficiently large m. In fact, we showed, as the first claim in the proof of Proposition 2.13, that for an open subgroup U of $\mathrm{GL}(2,\mathbb{A})$, that there is an integer m with the property that if $z \in \mathbb{A}$, $x \in \mathrm{GL}(2,\mathbb{A})$, $\mathrm{ht}^+(x) \ge m$, then there is $u \in U$, $\beta \in F$, with $\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) x = \left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right) xu$. In other words, if $x \in \mathrm{GL}(2,\mathbb{A})$ and $\mathrm{ht}^+(x)$ is large enough, then $N(\mathbb{A})x \subset N(F)xU$. □

We shall now give a different proof of Proposition 4.26.

Proposition 4.35. *If $\varphi \in C_c^\infty(Y_\alpha)$ and $M\varphi \in C_c^\infty(Y_\alpha)$ then $M^2\varphi = \varphi$.*

Proof. Let us introduce a structure of $\mathbb{C}[z,z^{-1}]$-module on $C^\infty(Y_\alpha)$ by

$$(zf)(x) = \frac{1}{\sqrt{q}} f\left(\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)x\right), \quad f \in C^\infty(Y_\alpha), \quad x \in \mathrm{GL}(2,\mathbb{A}).$$

From

$$(M\phi)\left(\left(\begin{smallmatrix}a & 0\\ 0 & b\end{smallmatrix}\right)x\right) = \left|\frac{a}{b}\right| \int_{N(\mathbb{A})} \phi\left(\left(\begin{smallmatrix}b & 0\\ 0 & a\end{smallmatrix}\right)\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right) nx\right) dn$$

it follows that $M(zf) = z^{-1}M(f)$; recall that $|\alpha| = q$, and f is invariant under α. This is the reason for introducing the factor $\sqrt{q}$. Let U be an open subgroup of $\mathrm{GL}(2,O)$. Put

$$W_c^U = C_c^\infty(Y_\alpha)^U, \qquad W_+^U = C_+^\infty(Y_\alpha)^U.$$

Both are $\mathbb{C}[z,z^{-1}]$-submodules in $C^\infty(Y_\alpha)$. Denote by W_0^U the set of functions $f \in C^\infty(Y_\alpha)^U$ such that $f(x) = 0$ if $\mathrm{ht}^+(x) \neq 0$. Then the natural map $W_0^U \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}] \to W_c^U$ is an isomorphism. In the same way we have a canonical isomorphism $W_0^U \otimes_{\mathbb{C}} \mathbb{C}((z)) \to W_+^U$. The operator

$$M : W_c = C_c^\infty(Y_\alpha) \to W_+ = C_+^\infty(Y_\alpha)$$

maps W_c^U into W_+^U. Hence it defines a map

$$M : W_0^U \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}] \to W_0^U \otimes_{\mathbb{C}} \mathbb{C}((z))$$

satisfying the first condition of Proposition 4.31.
It remains to check the second condition of that Proposition. The space W_m^U can be identified with $W_0^U \otimes_{\mathbb{C}} z^{-m}\mathbb{C}[z^{-1}]$, and then the operator $B = I + M$ is just E^*E. Thus it suffices to show that for some $m \in \mathbb{Z}$, the space $E^*E(W_c^U)/E^*E(W_m^U)$ is finite dimensional. Since $E(W_m^U) = Y_m^U$ for large m, and $\{x \in \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}); \mathrm{ht}(x) \leq m\}$ is compact mod $Z(\mathbb{A})$, it follows that the subspace

$$E(W_m^U) \subset C_c^\infty(\alpha^{\mathbb{Z}}\,\mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}))^U$$

has finite codimension. Thus M satisfies both conditions of Proposition 4.31, and our claim follows from Corollary 4.32. □

To use Proposition 4.31 to give another proof of the rationality of $M(\mu_1,\mu_2,t)$, we take a different view of the Mellin transform and the relationship between the operators M and $M(\mu_1,\mu_2,t)$. Let $I_c(\mu_1\nu_{z^{-1}},\mu_2\nu_z)$ be the space of locally constant functions $f : \mathrm{GL}(2,\mathbb{A}) \to \mathbb{C}[z,z^{-1}]$ with

$$f\left(\left(\begin{smallmatrix}a & c\\ 0 & b\end{smallmatrix}\right)x\right) = \mu_1(a)\mu_2(b)\nu_z(b/a)|a/b|^{1/2} f(x).$$

Let $I_+(\mu_1\nu_{z^{-1}},\mu_2\nu_z)$ be

$$I_c(\mu_1\nu_{z^{-1}},\mu_2\nu_z) \otimes_{\mathbb{C}[z,z^{-1}]} \mathbb{C}((z)).$$

The group $\alpha^{\mathbb{Z}} \subset \mathrm{GL}(2,\mathbb{A})$ acts trivially on these I_c and I_+. We put

$$I_c = \oplus I_c(\mu_1\nu_{z^{-1}}, \mu_2\nu_z), \qquad I_+ = \oplus I_+(\mu_1\nu_{z^{-1}}, \mu_2\nu_z),$$

where the sums range over all characters μ_1, μ_2 of $\mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}}$.

PROPOSITION 4.36. *There exists an isomorphism of $\mathbb{C}((z))$-modules $I_+ \xrightarrow{\sim} C_+^\infty(Y_\alpha)$ which is $\mathrm{GL}(2,\mathbb{A})$-equivariant and maps I_c to $C_c^\infty(Y_\alpha)$.*

Proof. Define a map $F : I_+ \to C_+^\infty(Y_\alpha)$ by mapping

$$\varphi = \{\varphi_{\mu_1,\mu_2}\} \in I_+, \qquad \varphi_{\mu_1,\mu_2} \in I_c(\mu_1\nu_{z^{-1}}, \mu_2\nu_z),$$

to

$$(F\varphi)(x) = \text{constant term of the formal series} \sum_{\mu_1,\mu_2} \varphi_{\mu_1,\mu_2}(x) \in \mathbb{C}((z)),$$

for any $x \in \mathrm{GL}(2,\mathbb{A})$. The map F is well defined, commutes with the actions of $\mathbb{C}((z))$ and $\mathrm{GL}(2,\mathbb{A})$. The inverse of F exists, as follows. If $\psi \in C_+^\infty(Y_\alpha)$ then $F^{-1}(\psi) = \{\varphi_{\mu_1,\mu_2}\}$ with $\varphi_{\mu_1,\mu_2} \in I_+(\mu_1\nu_{z^{-1}}, \mu_2\nu_z)$ given by

$$\varphi_{\mu_1,\mu_2}(x) = \int_{A(\mathbb{A})/\alpha^{\mathbb{Z}}\cdot A(F)} \psi(h^{-1}x)\eta(h)dh,$$

where

$$\eta : A(\mathbb{A}) \to \mathbb{C}((z))^\times, \quad \eta(\mathrm{diag}(a,b)) = \mu_1(a)\mu_2(b)\nu_z(a/b).$$

The last integral converges in the field $\mathbb{C}((z))$. A base of the topology is given by $z^n\mathbb{C}[[z]]$, $n > 0$. The map F maps I_c to $C_c^\infty(Y_\alpha)$. $\square$

Put $I_0 = \oplus_{\mu_1,\mu_2} I_0(\mu_1,\mu_2)$, with

$$I_0(\mu_1,\mu_2) = \{f \in C^\infty(\mathrm{GL}(2,O));\ f\left(\left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right)x\right) = \mu_1(a)\mu_2(b)f(x)\}.$$

Denote by $M(z)$ the map $I_0 \to I_0$ which takes $I_0(\mu_1,\mu_2)$ to $I_0(\mu_2,\mu_1)$ via $M(\mu_1,\mu_2,z)$. We use the isomorphism F to identify the spaces I_+ and $C_+^\infty(Y_\alpha)$, as well as I_c and $C_c^\infty(Y_\alpha)$. The natural isomorphism

$$I_c(\mu_1\nu_{z^{-1}}, \mu_2\nu_z) \xrightarrow{\sim} I_0(\mu_1,\mu_2) \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$$

and

$$I_+(\mu_1\nu_{z^{-1}}, \mu_2\nu_z) \xrightarrow{\sim} I_0(\mu_1,\mu_2) \otimes_{\mathbb{C}} \mathbb{C}((z))$$

permit us to identify I_c and $I_0 \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$ as well as I_+ and $I_0 \otimes_{\mathbb{C}} \mathbb{C}((z))$. Thus the map $M : C_c^\infty(Y_\alpha) \to C_+^\infty(Y_\alpha)$ induces an operator

$$M_0 : I_0 \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}] \to I_0 \otimes_{\mathbb{C}} \mathbb{C}((z)).$$

Proposition 4.37. *Regard the elements of $I_0 \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$ as functions of z with values in I_0 and the elements of $I_0 \otimes_{\mathbb{C}} \mathbb{C}((z))$ as formal series in z with coefficients in I_0. Then for any $u \in I_0 \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$ one has $(M_0 u)(z) = M(z)u(z^{-1})$, $M(z)$ is viewed as a formal series in z.*

Proof. Write ι for the automorphism of $\mathbb{C}[z, z^{-1}]$ which maps z to z^{-1}. Given a function $f : \mathrm{GL}(2, \mathbb{A}) \to \mathbb{C}((z))$, denote by f_0 the function $\mathrm{GL}(2, \mathbb{A}) \to \mathbb{C}$ such that $f_0(x)$ is the constant term of $f(x)$.
Define an operator

$$M'' : I_0 \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}] \to I_0 \otimes_{\mathbb{C}} \mathbb{C}((z)) \qquad \text{by} \qquad (M''u)(z) = M(z)u(z^{-1}).$$

We claim that $M_0 = M''$. Consider M'' as a map $I_c \to I_+$. We have to show that for every $f \in I_c$, we have $FM''f = MFf$, for the isomorphism $F : I_+ \xrightarrow{\sim} C_+^\infty(Y_\alpha)$. As I_c is the sum over μ_1, μ_2 of $I_c(\mu_1\nu_{z^{-1}}, \mu_2\nu_z)$, it suffices to consider f in one of these summands.
For $x \in \mathrm{GL}(2, \mathbb{A})$, we have $(M''f)(x) = \int_{N(\mathbb{A})} \iota f\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx\right) dn$. Then

$$(FM''f)(x) = (M''f)_0(x) = \int_{N(\mathbb{A})} f_0\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx\right) dn$$

$$(MFf)(x) = \int_{N(\mathbb{A})} Ff\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx\right) dn = \int_{N(\mathbb{A})} f_0\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx\right) dn$$

are equal, as required. □

4.8 A second proof of the rationality of $M(\mu_1, \mu_2, t)$ and of the functional equation $M(\mu_1, \mu_2, t)M(\mu_2, \mu_1, t^{-1}) = 1$

Let U, W^U, A be as in the proof of Proposition 4.35. Then $W^U = \oplus_{\mu_1,\mu_2} W^U_{\mu_1,\mu_2}$, where $W^U_{\mu_1,\mu_2}$ is the space of functions $f \in W^U$ with

$$f\left(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) x\right) = \mu_1(a)^{-1}\mu_2(b)^{-1} f(x)$$

whenever $\deg(a) = \deg(b) = 0$. The natural maps $I_0(\mu_2, \mu_1)^U \xrightarrow{\sim} W^U_{\mu_1,\mu_2}$ permit one to identify W^U and the space I_0^U. The map

$$M : W^U \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}] \to W^U \otimes_{\mathbb{C}} \mathbb{C}((z))$$

is induced by the operator

$$M_0 : I_0 \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}] \to I_0 \otimes_{\mathbb{C}} \mathbb{C}((z)).$$

The proof of Proposition 4.35 implies that the operator M satisfies the conditions of Proposition 4.31. Then M is given by a formula of the form $(Mu)(z) = P(z)u(z^{-1})$, where $P(z)$ is an automorphism of V which depends on z rationally, and $P(z^{-1}) = P(z)^{-1}$. From Proposition 4.37 it follows that

$P(z)$ is just the restriction of $M(z)$ to $I_0^U \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$. The group U may be arbitrarily small. Hence $M(z)$ is a rational function of z, and $M(z)M(z^{-1}) = 1$. Hence for any characters μ_1, μ_2, of $\alpha^{\mathbb{Z}} \cdot F^\times \backslash \mathbb{A}^\times$, the operator $M(\mu_1, \mu_2, z)$ depends rationally on z, and

$$M(\mu_1, \mu_2, z)M(\mu_1, \mu_2, z^{-1}) = 1.$$

The same is true for any characters μ_1, μ_2 of $\mathbb{A}^\times/F^\times$, which are not necessarily trivial at α. To see this, it suffices to use the identities $M(\mu_1\nu_t, \mu_2\nu_t, z) = M(\mu_1, \mu_2, z)$ and $M(\mu_1\nu_t, \mu_2\nu_{t^{-1}}, z) = M(\mu_1, \mu_2, tz)$. □

5 Proof of the trace formula

5.1 The geometric part

Our aim is to compute the trace $\operatorname{tr} r_0(f)$, where $f \in C_c^\infty(\mathrm{GL}(2, \mathbb{A}))$ and r_0 is the representation of $\mathrm{GL}(2, \mathbb{A})$ by right translation on the space $A_{0,\alpha}$ of cusp forms invariant under α. Recall that the space $A_{c,\alpha}$ of α-invariant automorphic forms is equal to the direct sum of $A_{0,\alpha}$ and $A_{E,\alpha} = \mathrm{Im}(E : C_c^\infty(Y_\alpha) \to A_{c,\alpha})$. The corresponding representations of $\mathrm{GL}(2, \mathbb{A})$ are denoted by r and r_E. Had r been admissible, we would have had $\operatorname{tr} r_0(f) = \operatorname{tr} r(f) - \operatorname{tr} r_E(f)$, and the computation of $\operatorname{tr} r_0(f)$ would have reduced to that of $\operatorname{tr} r(f)$ and $\operatorname{tr} r_E(f)$. But r and r_E are not admissible, so $\operatorname{tr} r(f)$ and $\operatorname{tr} r_E(f)$ make no sense.
Suppose f is right invariant under the open subgroup U of $\mathrm{GL}(2, O)$. Denote by A_0^U, A_c^U, A_E^U the spaces of U-invariant vectors in $A_{0,\alpha}$, $A_{c,\alpha}$, $A_{E,\alpha}$. Since $\mathrm{Im}\, r_0(f) \subset A_0^U$, we have $\operatorname{tr} r_0(f) = \operatorname{tr} r_0^U(f)$, where $r_0^U(f)$ is the restriction of $r_0(f)$ to A_0^U.
Denote by χ_m the characteristic function of the set

$$\{x \in \alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2, F) \backslash \mathrm{GL}(2, \mathbb{A});\ \mathrm{ht}(x) < m\}, \quad m > 0.$$

Denote by θ_m the operator of multiplication by χ_m on $A_{c,\alpha}$.

Proposition 5.1. (1) *For any $m > 0$, $\dim \theta_m(A_c^U) < \infty$.*
(2) *If $m >> 1$ then (a) θ_m acts as the identity on A_0^U, and (b) $\theta_m(A_E^U) \subset A_E^U$.*

Proof. (1) The support of χ_m is compact mod $Z(\mathbb{A})$, the quotient by the open U is then finite. (2a) A_0^U is finite dimensional, consisting of compactly supported forms. (2b) By (2a), $(1-\theta_m)A_E^U = (1-\theta_m)A_c^U$. This lies in A_E^U as U-invariant cusp forms are uniformly compactly supported. Hence $\theta_m(A_E^U) \subset A_E^U$. □

Denote by $r^U(f)$ and $r_E^U(f)$ the restrictions of $r(f)$ to A_c^U and A_E^U. For m such that $\theta_m(A_E^U) \subset A_E^U$, denote the restriction of θ_m to A_E^U again by θ_m. Then for $m >> 1$,

$$\operatorname{tr} r_0(f) = \operatorname{tr} r_0^U(f) = \operatorname{tr}(\theta_m r^U(f)) - \operatorname{tr}(\theta_m r_E^U(f)) = \operatorname{tr}(\theta_m r(f)) - \operatorname{tr}(\theta_m r_E^U(f)).$$

We then proceed to compute $\operatorname{tr}(\theta_m r(f))$ and $\operatorname{tr}(\theta_m r_E^U(f))$.

Proposition 5.2. *There exist $c_f \in E$ and $\alpha_m \in E$ with $\lim_{n\to\infty} \alpha_m = 0$, and*

$$\operatorname{tr}(\theta_m r(f)) = \sum_{1\leq i\leq 4} S_i(f) + c_f(m - \frac{1}{2}) + \alpha_m.$$

Proof. The map $\theta_m r(f) : A_{c,\alpha} \to A_{c,\alpha}$ is an integral operator with kernel $\chi_m(y)K_f(x,y)$, where $K_f(x,y) = \sum_{\gamma\in\alpha^{\mathbb{Z}}\cdot \mathrm{GL}(2,F)} f(x^{-1}\gamma y)$. Then

$$\operatorname{tr}(\theta_m r(f)) = \int_{\alpha^{\mathbb{Z}}\cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})} \chi_m(x)K_f(x,x)dx.$$

Lemma 5.3. *There exists $m_f > 0$ such that if $x \in \mathrm{GL}(2,\mathbb{A})$, $\gamma \in \alpha^{\mathbb{Z}}\,\mathrm{GL}(2,F)$, $\mathrm{ht}^+(x) > m_f$, $f(x^{-1}\gamma x) \neq 0$, then $\gamma \in \alpha^{\mathbb{Z}}A(F)N(F)$.*

Proof. We have $\gamma x = xy$, y in $\operatorname{supp}(f)$. Since $\mathrm{ht}^+(x) + \mathrm{ht}^+(\delta x) \leq 0$ for $\delta \in \mathrm{GL}(2,F) - B(F)$, we have that $\mathrm{ht}^+(x) > 0$. If in addition we had $\mathrm{ht}^+(xy) > 0$, we would conclude that $\gamma \in \alpha^{\mathbb{Z}}B(F)$. The number $m_f = -\min\{\mathrm{ht}^+(z);\ z \in \mathrm{GL}(2,O)\cdot \operatorname{supp}(f)\}$ then has the property that $\mathrm{ht}^+(x) > m_f$, $y \in \operatorname{supp}(f)$, implies $\mathrm{ht}^+(xy) = \mathrm{ht}^+(x) + \mathrm{ht}^+(ky) > 0$, where $x = bk$ and $ky = b'k'$ so that $xy = bb'k$ (b, $b' \in B(\mathbb{A})$; k, $k' \in \mathrm{GL}(2,\mathbb{A})$). □

Denote by ξ_m the characteristic function of the set $\{x \in \mathrm{GL}(2,\mathbb{A});\ \mathrm{ht}^+(x) \geq m\}$, by $A'(F)$ the set of nonscalar diagonal matrices, and by Ell the set of elliptic matrices in $\mathrm{GL}(2,F)$, namely those whose eigenvalues are not in F. Put $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Lemma 5.4. *If m is big enough, then $\chi_m(y)K_f(x,x)$ is the sum of*

$$T_{1,m}(x) = \chi_m(x) \sum_{\gamma\in\alpha^{\mathbb{Z}}\cdot F^\times} f(\gamma), \qquad T_{2,m}(x) = \sum_{\gamma\in\alpha^{\mathbb{Z}}\cdot \mathrm{Ell}} f(x^{-1}\gamma x),$$

$$T_{3,m}(x) = \frac{1}{2} \sum_{\gamma\in\alpha^{\mathbb{Z}}\cdot A'(F)} \sum_{\delta\in A(F)\backslash \mathrm{GL}(2,F)} f(x^{-1}\delta^{-1}\gamma\delta x)\cdot(1 - \xi_m(\delta x) - \xi_m(w\delta x)),$$

$$T_{4,m}(x) = \sum_{a\in\alpha^{\mathbb{Z}}\cdot F^\times} \sum_{\delta\in F^\times N(F)\backslash \mathrm{GL}(2,F)} f(x^{-1}\delta^{-1}\left(\begin{smallmatrix} a & a \\ 0 & a \end{smallmatrix}\right)\delta x)\cdot(1 - \xi_m(\delta x)).$$

Proof. $T_{1,m}(x)$ is the contribution of the elements $\gamma \in \alpha^{\mathbb{Z}}\cdot F^\times$ in $\chi_m(x)K_f(x,x)$. We claim that the contribution of the elements $\gamma \in \alpha^{\mathbb{Z}}\cdot \mathrm{Ell}$ in $\chi_m(x)K_f(x,x)$ is $T_{2,m}(x)$. To show this, we need to see that if $x \in \mathrm{GL}(2,\mathbb{A})$, $\gamma \in \alpha^{\mathbb{Z}}\cdot \mathrm{Ell}$ and $\Phi(x^{-1}\gamma x) \neq 0$, then $\mathrm{ht}^+(x) < m$. Indeed, if $\mathrm{ht}(x) \geq m$ then there is some $\delta \in \mathrm{GL}(2,F)$ with $\mathrm{ht}^+(\delta x) \geq m$. Lemma 5.3 then implies that $\delta\gamma\delta^{-1} \in \alpha^{\mathbb{Z}}A(F)N(F)$, contradicting $\gamma \in \alpha^{\mathbb{Z}}\cdot \mathrm{Ell}$.

Denote by $T'_{3,m}(x)$ the contribution into $\chi_m(x)K_f(x,x)$ of the elements γ of the form $\alpha^j\gamma$, $j \in \mathbb{Z}$, $\gamma \in \mathrm{GL}(2,F)$ with distinct eigenvalues in F. By $T'_{4,m}(x)$

we denote the contribution of the elements $\alpha^j\gamma$, $j \in \mathbb{Z}$, $\gamma \in \mathrm{GL}(2,F)$, $\gamma \notin F^\times$ but the eigenvalues of γ are equal. We have

$$T'_{3,m}(x) = \frac{1}{2}\chi_m(x) \sum_{\gamma\in\alpha^{\mathbb{Z}}\cdot A'(F)} \sum_{\delta\in A(F)\backslash \mathrm{GL}(2,F)} f(x^{-1}\delta^{-1}\gamma\delta x).$$

The factor $\frac{1}{2}$ appears since $\mathrm{diag}(b,a)$ is conjugate to $\mathrm{diag}(a,b)$. To show that $T'_{3,m}(x) = T_{3,m}(x)$ it suffices to show that when $f(x^{-1}\delta^{-1}\gamma\delta x) \neq 0$,

$$\chi_m(x) = 1 - \xi_m(\delta x) - \xi_m(w\delta x),$$

namely if $\mathrm{ht}(x) \geq m$ then either $\mathrm{ht}^+(\delta x) \geq m$ or $\mathrm{ht}^+(w\delta x) \geq m$. So if $\mathrm{ht}(x) \geq m$, then there is some $\eta \in \mathrm{GL}(2,F)$ with $\mathrm{ht}^+(\eta x) \geq m$. By Lemma 5.3, $\eta\delta^{-1}\gamma\delta\eta^{-1} \in \alpha^{\mathbb{Z}}A(F)N(F)$, but this implies that $\eta\delta^{-1} \in A(F)N(F)$ or $\eta\delta^{-1}w \in A(F)N(F)$. Correspondingly, $\mathrm{ht}^+(\delta x) = \mathrm{ht}^+(\eta x) \geq m$ or $\mathrm{ht}^+(w\delta x) = \mathrm{ht}^+(\eta x) \geq m$, but both inequalities cannot hold simultaneously if $m > 0$.
Now

$$T'_{4,m}(x) = \chi_m(x) \sum_{a\in\alpha^{\mathbb{Z}}\cdot F^\times} \sum_{\delta\in F^\times N(F)\backslash \mathrm{GL}(2,F)} f(x^{-1}\delta^{-1} \left(\begin{smallmatrix} a & a \\ 0 & a\end{smallmatrix}\right) \delta x).$$

To show that this equals $T_{4,m}(x)$ we need to check that when

$$f(x^{-1}\delta^{-1} \left(\begin{smallmatrix} a & a \\ 0 & a\end{smallmatrix}\right) \delta x) \neq 0$$

and $\mathrm{ht}(x) \geq m$, then $\mathrm{ht}^+(\delta x) \geq m$. Suppose then that $\mathrm{ht}^+(\eta x) \geq m$ for $\eta \in \mathrm{GL}(2,F)$. Then by Lemma 5.3 we have

$$\eta\delta^{-1} \left(\begin{smallmatrix} a & a \\ 0 & a\end{smallmatrix}\right) \delta\eta^{-1} \in \alpha^{\mathbb{Z}}A(F)N(F).$$

Hence $\eta\delta^{-1} \in A(F)N(F)$, so that $\mathrm{ht}^+(\delta x) = \mathrm{ht}^+(\eta x) \geq m$. $\square$

We conclude that $\mathrm{tr}\,\theta_m r(f) = \sum_{1\leq i\leq 4} t_{i,m}$ with

$$t_{i,m} = \int_{\alpha^{\mathbb{Z}}\cdot\mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})} T_{i,m}(x)dx.$$

To prove the proposition it suffices to show that $t_{i,m} = S_i(f) + c_i(2m-1) + \beta_m$ for all i ($1 \leq i \leq 4$), where c_i does not depend on m and $\lim \beta_m = 0$. It is clear that $t_{1,m} \to S_1(f)$ as $m \to \infty$. As $T_{2,m}(x)$ is independent of m, $t_{2,m} = S_2(f)$. Now

$$\begin{aligned} t_{3,m} &= \frac{1}{2} \sum_{\gamma\in\alpha^{\mathbb{Z}}\cdot A'(F)} \int_{\alpha^{\mathbb{Z}}(A(F)\backslash \mathrm{GL}(2,\mathbb{A})} f(x^{-1}\gamma x)(1 - \xi_m(x) - \xi_m(wx))dx \\ &= \frac{1}{2} \sum_{\gamma\in\alpha^{\mathbb{Z}}\cdot A'(F)} \int_{A(\mathbb{A})\backslash \mathrm{GL}(2,\mathbb{A})} f(x^{-1}\gamma x)s(x)dx \end{aligned}$$

where

$$s(x) = \int_{\alpha^{\mathbb{Z}} A(F)\backslash A(\mathbb{A})} [1 - \xi_m(yx) - \xi_m(wyx)]dy$$

$$= \mathrm{vol}\{y \in \alpha^{\mathbb{Z}} A(F)\backslash A(\mathbb{A});\ \mathrm{ht}^+(yx) < n,\ \mathrm{ht}^+(wyx) < n\}.$$

Note that for $y \in A(\mathbb{A})$, $\mathrm{ht}^+(yx) = \mathrm{ht}^+(y) + \mathrm{ht}^+(x)$ and $\mathrm{ht}^+(wyx) = \mathrm{ht}^+(wx) - \mathrm{ht}^+(y)$. Hence

$$s(x) = |\{y \in A(\mathbb{A})/\alpha^{\mathbb{Z}} \cdot A(F);\ \mathrm{ht}^+(wx) - m < \mathrm{ht}^+(y) < m - \mathrm{ht}^+(x)\}|.$$

This is the number of integers between $\mathrm{ht}^+(wx) - m$ and $m - \mathrm{ht}^+(x)$. So $s(x) = 2m - 1 - \mathrm{ht}^+(x) - \mathrm{ht}^+(wx)$.

Lemma 5.5. *We have* $\mathrm{ht}^+(x) + \mathrm{ht}^+(wx) = -2r(x)$, *where if* $x = a\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right)k$, $a \in A(\mathbb{A})$, $k \in \mathrm{GL}(2, O)$ *and* $y \in \mathbb{A}$, *we put* $r(x) = \sum_v \max(0, \log_q |y_v|_v)$.

Proof. Note that y is determined up to a change $y \mapsto by + c$, $b \in O^\times$, $c \in O$, so $r(x)$ is well defined. The asserted relation does not change if x is replaced by axk, $a \in A(\mathbb{A})$, $k \in \mathrm{GL}(2, O)$, so we may assume $x = \left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) \in N(\mathbb{A})$. Then $\mathrm{ht}^+(x) = 0$, and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -\frac{1}{y} & 1 \\ 0 & y \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ \frac{1}{y} & 1 \end{smallmatrix}\right)$ implies that $\mathrm{ht}^+(wx) = -2r(x)$. □

Lemma 5.5 implies that

$$t_{3,m} = S_3(f) + (m - \frac{1}{2}) \sum_{\gamma \in \alpha^{\mathbb{Z}} \cdot A'(F)} \int_{A(\mathbb{A})\backslash \mathrm{GL}(2,\mathbb{A})} f(x^{-1}\gamma x)dx.$$

Next

$$t_{4,m} = \sum_{a \in \alpha^{\mathbb{Z}} \cdot F^\times} \int_{\alpha^{\mathbb{Z}} F^\times N(F)\backslash \mathrm{GL}(2,\mathbb{A})} f\left(x^{-1}\left(\begin{smallmatrix} a & a \\ 0 & a \end{smallmatrix}\right)x\right)(1 - \xi_m(x))dx$$

$$= \sum_{a \in \alpha^{\mathbb{Z}} \cdot F^\times} \int_{\{x \in \alpha^{\mathbb{Z}} F^\times N(F)\backslash \mathrm{GL}(2,\mathbb{A});\, \mathrm{ht}^+(x) < m\}} f\left(x^{-1}\left(\begin{smallmatrix} a & a \\ 0 & a \end{smallmatrix}\right)x\right)dx.$$

Recall that $\theta_{a,f}(t) = \int_{\alpha^{\mathbb{Z}} F^\times N(F)\backslash \mathrm{GL}(2,\mathbb{A})} f\left(x^{-1}\left(\begin{smallmatrix} a & a \\ 0 & a \end{smallmatrix}\right)x\right)t^{\mathrm{ht}^+(x)}dx$ is a Laurent series at $t = 0$ of a rational function of t with $\zeta_F(q^{-1}t)^{-1}\theta_{a,f}(t) \in \mathbb{C}[t, t^{-1}]$. Suppose $\theta_{a,f}(t) = \sum_k u_k(a)t^k$. Then $t_{4,m} = \sum_{a \in \alpha^{\mathbb{Z}} \cdot F^\times} \sum_{k<m} u_k(a)$. Since $\zeta_F(q^{-1}t)$ has a simple pole at $t = 1$, we have that $\theta_{a,f}(t) = \frac{\rho(a)}{1-t} + \overline{\theta}_{a,f}(t)$, with $\overline{\theta}_{a,f}(t)$ without poles on $0 < |t| \leq 1$. Then

$$\tilde{\theta}_{a,f}(t) = \frac{1}{2}(\theta_{a,f}(t) + \theta_{a,f}(t^{-1})) = \frac{1}{2}(\overline{\theta}_{a,f}(t) + \overline{\theta}_{a,f}(t^{-1})) + \frac{1}{2}\rho(a),$$

$$\tilde{\theta}_{a,f}(1) = \overline{\theta}_{a,f}(1) + \frac{1}{2}\rho(a) = \frac{1}{2}\rho(a) + \sum_k (u_k(a) - \rho(a))$$

$$= \lim_{m\to\infty}[\sum_{k<m} u_k(a) - (m-\frac{1}{2})\rho(a)].$$

Then

$$t_{4,m} = \sum_{a\in\alpha^{\mathbb{Z}}\cdot F^\times} \tilde{\theta}_{a,f}(1) + (m-\frac{1}{2})\rho(a) + \beta_m, \qquad \beta_m \to 0 \text{ as } m\to\infty,$$

and $S_4(f) = \sum_{a\in\alpha^{\mathbb{Z}}\cdot F^\times} \tilde{\theta}_{a,f}(1)$. Proposition 5.2 follows. □

Note that β_m is 0 for sufficiently large m, as will be seen below.

5.2 The Eisenstein contribution

Next we turn to computing $\mathrm{tr}(\theta_m r_E^U(f))$ for large m. Put $W_c^U = C_c^\infty(Y_\alpha)^U$, $W_M^U = (1+M)W_c^U$.

Proposition 5.6. *The operator E^* maps A_E^U isomorphically onto W_M^U.*

Proof. As $A_E^U = E(W_c^U)$ and $E^*E = 1+M$, it suffices to show that $\ker E^*E = \ker E$. For $\varphi \in \ker E^*E$ we have $(E\varphi, E\varphi) = (E^*E\varphi,\varphi) = 0$, hence $E\varphi = 0$. □

Definition 1. Denote by W_m^U the space of f in W_c^U with $f(x)=0$ if $\mathrm{ht}^+(x) < m$. Denote by ξ_m also the operator $W_m^U \to W_m^U$ of multiplication by the characteristic function of the set $\{x\in Y_\alpha;\ \mathrm{ht}^+(x)\geq m\}$. [If $m>0$ then ξ_m is a left inverse to the operator $1+M : W_m^U \to W_M^U$. Indeed, if f is in W_m^U, then $(Mf)(x) = 0$ already when $\mathrm{ht}^+(x) > -m$ since $\mathrm{ht}^+(wnx)+\mathrm{ht}^+(nx)<0$ implies $\mathrm{ht}^+(wnx)<m$ and so $f(wnx)=0$.] Hence $\pi^m = (1+M)\xi_m : W_M^U \to W_M^U$ satisfies $\pi^m\pi^m = \pi^m$, for $m>0$. Put $\pi_m = 1-\pi^m$.

Proposition 5.7. *For sufficiently large m, E^* intertwines θ_m with π_m, thus $\pi_m E^* = E^*\theta_m$, namely the diagram*

$$\begin{array}{ccc} A_E^U & \xrightarrow{E^*} & W_M^U \\ \theta_m\downarrow & & \downarrow\pi_m \\ A_E^U & \xrightarrow{E^*} & W_M^U \end{array}$$

is commutative.

Proof. Suppose $f\in A_E^U$ and $(1-\theta_m)f=0$. Then $f(x)=0$ for x with $\mathrm{ht}(x)\geq m$. As $\xi_m(x)\neq 0$ only on x with $\mathrm{ht}^+(x)\geq m$, we have $0=(1+M)\xi_m E^* f = (1-\pi_m)E^*f$, the last equality as $1-\pi_m = \pi^m = (1+M)\xi_m$. For such f we have $E^*\theta_m f = E^*f$ and $\pi_m E^* f = E^* f$.
If $f\in A_E^U$ and $\theta_m f = 0$, then by Proposition 4.34 there is $\varphi\in W_m^U$ with $f = E\varphi$. Then

$$\pi_m E^* f = \pi_m E^* E\varphi = \pi_m(1+M)\varphi = \pi_m(1+M)\xi_m\varphi = \pi_m\pi^m\varphi = 0,$$

hence $E^*\theta_m f = \pi_m E^* f$ for such f.
Any $f\in A_E^U$ can be written as $f=f_1+f_2$, $f_1 = (1-\theta_m)f$, $f_2 = \theta_m f$, thus $\theta_m f_1 = 0$ and $(1-\theta_m)f_2 = 0$. □

Definition 2. Recall that $Y_\alpha = \alpha^{\mathbb{Z}} A(F)N(\mathbb{A})\backslash \mathrm{GL}(2,\mathbb{A})$. Denote by σ_c, σ_+, σ_M the representations of $\mathrm{GL}(2,\mathbb{A})$ in the spaces

$$W_c = C_c^\infty(Y_\alpha), \qquad W_+ = C_+^\infty(Y_\alpha), \qquad W_M = (1+M)C_c^\infty(Y_\alpha).$$

Consider $\sigma_c(f)$, $\sigma_+(f)$, $\sigma_M(f)$ as operators in the spaces W_c^U, W_+^U, W_M^U.

Corollary 5.8. *We have* $\mathrm{tr}(\theta_m \cdot r_E^U(f)) = \mathrm{tr}(\pi_m \cdot \sigma_M(f))$.

Proof. The operator E^* yields an isomorphism of $A_E^U = E(W_c^U)$ with W_M^U intertwining θ_m with π_m. □

In the proof of Proposition 4.35 we introduced a structure of $\mathbb{C}[z,z^{-1}]$-module on W_c^U and W_+^U, as well as isomorphisms $W_c^U \simeq W_0^U \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$ and $W_+^U \simeq W_0^U \otimes_{\mathbb{C}} \mathbb{C}((z))$, where $W_0^U = \{f \in W_c^U;\ f(x) = 0 \text{ if } \mathrm{ht}^+(x) \neq 0\}$. Under these isomorphisms, the operator $M : W_c^U \to W_+^U$ corresponds to the operator

$$M : W_0^U \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}] \to W_0^U \otimes_{\mathbb{C}} \mathbb{C}((z)),$$

which satisfies the conditions of Proposition 4.31, hence has the form

$$(Mu)(z) = P(z)u(z^{-1}) \qquad \text{for} \qquad u \in W_0^U \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$$

which is viewed as a function of z with values in W_0^U. Here $P(z)$ is a rational function in z with values in $\mathrm{Aut}\, W_0^U$, and $P(z^{-1}) = P(z)^{-1}$.
Now $\sigma_c(f)$ is an endomorphism of W_c^U as a $\mathbb{C}[z,z^{-1}]$-module. The corresponding endomorphism of the module $W_0^U \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$ is determined by a function $B(z)$ in $\mathrm{End}(W_0^U) \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$. The endomorphism of $W_0^U \otimes_{\mathbb{C}} \mathbb{C}((z))$ corresponding to the operator $\sigma_+(f)$ is determined by the same function $B(z)$. The relation

$$M\sigma_c(f) = \sigma_+(f)M \qquad \text{becomes} \qquad P(z)B(z^{-1})u(z^{-1}) = B(z)P(z)u(z^{-1})$$

for any $u \in W_0^U \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$, thus $B(z^{-1}) = P(z)^{-1}B(z)P(z)$.

Definition 3. Under the isomorphism $W_+^U \simeq W_0^U \otimes_{\mathbb{C}} \mathbb{C}((z))$, the subspace $W_M^U = (1+M)W_c^U$ is mapped onto the subspace L consisting of all rational functions of the form $u(z) + P(z)u(z^{-1})$, with $u \in W_0^U \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$. Put

$$L_m = L \cap (W_0^U \otimes_{\mathbb{C}} z^{-m+1}\mathbb{C}[[z]]).$$

Denote by L^m the set of rational functions of the form

$$u(z) + P(z)u(z^{-1}) \qquad \text{with} \qquad u \in W_0^U \otimes_{\mathbb{C}} z^{-m}\mathbb{C}[z^{-1}].$$

For sufficiently large m we have $L = L_m \oplus L^m$. Under the isomorphism $W_M^U \xrightarrow{\sim} L$, the operator $\pi_m : W_M^U \to W_M^U$ corresponds to the idempotent operator $L \to L$ with kernel L^m and image L_m. This projection will also be denoted by π_m. Thus $\mathrm{tr}(\pi_m\sigma_M(f)) = \mathrm{tr}(\pi_m B)$, where $B : L \to L$ is the operator of

multiplication by $B(z)$. On the left, π_m is an operator on W_M^U, on the right, on L.
Fix Q_1, $Q_2 \in M(k, \mathbb{C}[z, z^{-1}])$, $k \geq 1$, such that $\det Q_i \neq 0$. Suppose the function $Q_2(z)^{-1}Q_1(z)$ is regular at $z = \infty$, thus $Q_1(z) \in Q_2(z)M(k, \mathbb{C}[[z^{-1}]])$, and the function $Q_1(z)^{-1}Q_2(z)$ is regular at $z = 0$, thus $Q_2(z) \in Q_1(z)M(k, \mathbb{C}[[z]])$. Put $R = \mathbb{C}[z, z^{-1}]^k$. For $m \geq 1$, put

$$R_m = R \cap z^{1-m}Q_1(z)\mathbb{C}[[z]]^k \cap z^{m-1}Q_2(z)\mathbb{C}[[z^{-1}]]^k.$$

Also put

$$R_-^m = z^{-m}Q_1(z)\mathbb{C}[z^{-1}]^k \qquad R_+^m = z^m Q_2(z)\mathbb{C}[z]^k.$$

Then $\dim R_m$ is finite.

Proposition 5.9. *We have* $R = R_-^m \oplus R_m \oplus R_+^m$,

$$R_m \oplus R_+^m = R \cap z^{1-m}Q_1(z)\mathbb{C}[[z]]^k$$

and

$$R_m \oplus R_-^m = R \cap z^{m-1}Q_2(z)\mathbb{C}[[z^{-1}]]^k.$$

Proof. The natural map $\varphi : R_-^m \to X_- = \mathbb{C}((z))^k / z^{1-m}Q_1(z)\mathbb{C}[[z]]^k$ is an isomorphism (note that $\mathbb{C}((z))/z^{1-m}\mathbb{C}[[z]] \simeq z^{-m}\mathbb{C}[z^{-1}]$ and $Q_1(z)$ is invertible in $\mathrm{GL}(k, \mathbb{C}((z)))$. The natural map

$$\psi : R_+^m \to X_+ = \mathbb{C}((z^{-1}))^k / z^{m-1}Q_2(z)\mathbb{C}[[z^{-1}]]^k$$

is then too. The natural map $f : R/R_m \to X_- \oplus X_+$ is injective (by definition of R_m as the intersection of R and the denominators of X_-, X_+) and the composition of the natural map $R_+^m \oplus R_-^m \to R/R_m$ with f is $\varphi \oplus \psi$. □

Definition 4. (1) Denote by $\mathrm{pr}_m : R \to R$ the projection with kernel $R_+^m \oplus R_-^m$ and image R_m. (2) If $A(z)$ is a matrix in $M(k, \mathbb{C}[z, z^{-1}])$, denote by $A[z]$ also the corresponding automorphism of $R = \mathbb{C}[z, z^{-1}]^k$. Denote by A_0 the constant term of $A(z)$.

Proposition 5.10. *The trace* $\mathrm{tr}(\mathrm{pr}_m \cdot A[z])$ *is equal to*

$$(2m-1)\,\mathrm{tr}\, A_0 - \mathrm{res}_{z=0}\, \mathrm{tr}\, A(z)Q_1'(z)Q_1(z)^{-1}dz - \mathrm{res}_{z=\infty}\, \mathrm{tr}\, A(z)Q_2'(z)Q_2(z)^{-1}dz.$$

Proof. Define a projection $\mathrm{pr}_+^m : R \to R$ with image R_+^m and kernel $R_-^m + R_m$, and a projection $\mathrm{pr}_-^m : R \to R$ with image R_-^m and kernel $R_+^m + R_m$. Analogously to the decomposition $R = R_-^m \oplus R_m \oplus R_+^m$, consider the decomposition

$$R = z^{-m}\mathbb{C}[z^{-1}]^k \oplus (z^{1-m}\mathbb{C}[z]^k \cap z^{m-1}\mathbb{C}[z^{-1}]^k) \oplus z^m\mathbb{C}[z]^k,$$

namely the case where $Q_1 = 1 = Q_2$. Denote the associated projections by p_-^m, p_m, p_+^m. Since the space $z^{-m}\mathbb{C}[z^{-1}]^k / R_-^m \cap z^{-m}\mathbb{C}[z^{-1}]^k$ is finite dimensional, the operator $\mathrm{pr}_+^m - p_+^m$ has finite rank, and the operator $\mathrm{pr}_-^m - p_-^m$ has finite rank since $z^m\mathbb{C}[z]^k / R_+^m \cap z^m\mathbb{C}[z]^k$ is finite dimensional.

LEMMA 5.11. *We have*

$$\operatorname{tr}(\operatorname{pr}_-^m \cdot A[z] - p_-^m \cdot A[z]) = \operatorname{res}_{z=0} \operatorname{tr} A(z) Q_1'(z) Q_1(z)^{-1} dz,$$

as well as

$$\operatorname{tr}(\operatorname{pr}_+^m \cdot A[z] - p_+^m \cdot A[z]) = \operatorname{res}_{z=\infty} \operatorname{tr} A(z) Q_2'(z) Q_2(z)^{-1} dz.$$

Proof. Denote by $\operatorname{Pr}_-^m : \mathbb{C}((z))^k \to \mathbb{C}((z))^k$ the projection with image $z^{-m} Q_1(z) \mathbb{C}[z^{-1}]^k$ and kernel $z^{1-m} Q_1(z) \mathbb{C}[[z]]^k$. Denote by $P_-^m : \mathbb{C}((z))^k \to \mathbb{C}((z))^k$ the projection with image $z^{-m}\mathbb{C}[z^{-1}]^k$ and kernel $z^{1-m}\mathbb{C}[[z]]^k$ (thus the case of $Q_1 = 1$). Denote by $A((z))$ the endomorphism of $\mathbb{C}((z))^k$ which is defined by multiplication by $A(z)$. Then $\operatorname{Pr}_-^m = Q_1((z)) \cdot P_-^m \cdot Q_1((z))^{-1}$. Now

$$\operatorname{Im}(\operatorname{Pr}_-^m \cdot A((z)) - P_-^m \cdot A((z))) \subset \mathbb{C}[z, z^{-1}]^k,$$

and the restriction of the operator

$$\operatorname{Pr}_-^m \cdot A((z)) - P_-^m \cdot A((z)) \qquad \text{to} \qquad \mathbb{C}[z, z^{-1}]^k \qquad (\subset \mathbb{C}((z))^k)$$

is equal to $\operatorname{pr}_-^m \cdot A[z] - p_-^m \cdot A[z]$. Hence

$$\operatorname{tr}(\operatorname{pr}_-^m \cdot A[z] - p_-^m \cdot A[z]) = \operatorname{tr}(\operatorname{Pr}_-^m \cdot A((z)) - P_-^m \cdot A((z)))$$

$$= \operatorname{tr}(Q_1((z)) \cdot P_-^m \cdot Q_1((z))^{-1} \cdot A((z)) - P_-^m \cdot A((z)))$$

$$= \operatorname{tr}(Q_1((z)) \cdot P_-^m \cdot C((z)) - P_-^m \cdot Q_1((z)) \cdot C((z))), \qquad C(z) = Q_1(z)^{-1} A(z).$$

As $\operatorname{tr} A(z) Q_1'(z) Q_1(z)^{-1} = \operatorname{tr} C(z) Q_1'(z)$, to prove the first claim of the lemma it suffices to show that

$$\operatorname{tr}(Q_1((z)) \cdot P_-^m \cdot C((z)) - P_-^m \cdot Q_1((z)) C((z))) = \operatorname{res}_{z=0} \operatorname{tr} C(z) Q_1'(z) dz$$

for any $Q_1(z) \in M(k, \mathbb{C}[z, z^{-1}])$, $C(z) \in M(k, \mathbb{C}((z)))$. By linearity, it suffices to show this when the matrices $Q_1(z)$ and $C(z)$ have a single nonzero entry. Thus we may assume $k = 1$, and that $Q_1(z) = z^b$. Thus we need to verify that for any formal power series $c(z) = \sum_d c_d z^d$ in $\mathbb{C}((z))$, we have

$$\operatorname{tr}[(((z^b)) \cdot P_-^m - P_-^m \cdot ((z^b)))c((z))] = bc_{-b},$$

where the operations here are in $\mathbb{C}((z))$. The left side is equal to

$$\operatorname{tr}[(((z^b)) \cdot P_-^m \cdot ((z^{-b})) - P_-^m) \cdot ((z^b)) c((z))] = \operatorname{tr}[(P_-^{m-b} - P_-^m) \cdot ((z^b)) c((z))]$$

$$= \operatorname{tr} \begin{pmatrix} c_{-b} & c_{-b+1} & \cdots & c_{-1} \\ c_{-b-1} & c_{-b} & \cdots & c_{-2} \\ \vdots & \vdots & \dots & \vdots \\ c_{1-2b} & c_{2-2b} & \cdots & c_{-b} \end{pmatrix} = bc_{-b}.$$

The second claim of the lemma is similarly proven. $\square$

As $\mathrm{pr}_m - p_m = (1 - \mathrm{pr}_-^m - \mathrm{pr}_+^m) - (1 - p_-^m - p_+^m) = (p_-^m - \mathrm{pr}_-^m) + (p_+^m - \mathrm{pr}_+^m)$, Lemma 5.11 implies that $\mathrm{tr}(\mathrm{pr}_m \cdot A[z] - p_m \cdot A[z])$

$$= -\operatorname{res}_{z=0} \mathrm{tr}[A(z)Q_1'(z)Q_1(z)^{-1}dz] - \operatorname{res}_{z=\infty} \mathrm{tr}[A(z)Q_2'(z)A_2(z)^{-1}dz].$$

Since $\mathrm{tr}(p_m \cdot A[z]) = (2m-1)\,\mathrm{tr}\,A_0$, the proposition follows. $\square$

Proposition 5.12. *Let $\iota : \mathbb{C}[z, z^{-1}]^k \to \mathbb{C}[z, z^{-1}]^k$ be the involution $(\iota u)(z) = u(z^{-1})$. For sufficiently large m we have $2\,\mathrm{tr}(\iota \cdot \mathrm{pr}_m \cdot A[z]) = \mathrm{tr}\,A(1) + \mathrm{tr}\,A(-1)$.*

Proof. Write $A(z) = \sum_k A_k z^k$, $A_k \in M(k, \mathbb{C})$. Then $\mathrm{tr}(\iota \cdot p_m \cdot A[z]) = \sum_{|i|<m} \mathrm{tr}\,A_{2i}$. If m is big enough the right side here is equal to $\frac{1}{2}(\mathrm{tr}\,A(1) + \mathrm{tr}\,A(-1))$. It remains to show that $\mathrm{tr}(\iota \cdot \mathrm{pr}_m \cdot A[z]) = \mathrm{tr}(\iota \cdot p_m \cdot A[z])$ for large enough m. As $\mathrm{pr}_m - p_m = p_+^m - \mathrm{pr}_+^m + (p_-^m - \mathrm{pr}_-^m)$, it suffices to show that for large enough m

$$\mathrm{tr}(\iota \cdot (p_+^m - \mathrm{pr}_+^m) \cdot A[z]) = 0 = \mathrm{tr}(\iota \cdot (p_-^m - \mathrm{pr}_-^m) \cdot A[z]).$$

Note that $\mathrm{pr}_+^m = [z^m]\,\mathrm{pr}_+^0[z^{-m}]$ and $p_+^m = [z^m]p_+^0[z^{-m}]$, where as usual $[z^m]$ here means the operator of multiplication by z^m. The operators pr_+^m and p_+^m were defined only for $m > 0$, but the definition extends to $m = 0$ so that the two relations above hold. Now

$$\mathrm{tr}(\iota \cdot (p_+^m - \mathrm{pr}_+^m) \cdot A[z]) = \mathrm{tr}(\iota \cdot [z^m](p_+^0 - \mathrm{pr}_+^0)[z^{-m}] \cdot A[z])$$
$$= \mathrm{tr}([z^{-m}]\iota \cdot (p_+^0 - \mathrm{pr}_+^0)[z^{-m}] \cdot A[z]) = \mathrm{tr}(\iota \cdot (p_+^0 - \mathrm{pr}_+^0)[z^{-m}] \cdot A[z][z^{-m}])$$
$$= \mathrm{tr}(\iota \cdot (p_+^0 - \mathrm{pr}_+^0)[z^{-2m}] \cdot A[z]).$$

Recall that $\dim V$ is finite, where $V = \mathrm{im}[\iota(p_+^0 - \mathrm{pr}_+^0)]$. If m is big enough then

$$[z^{-2m}] \cdot A[z]V \subset z^{-1}\mathbb{C}[z^{-1}]^k \cap z^{-1}Q_2(z)\mathbb{C}[[z^{-1}]]^k \subset \ker p_+^0 \cap \ker \mathrm{pr}_+^0 .$$

Hence $\mathrm{tr}(\iota \cdot (p_+^0 - \mathrm{pr}_+^0)[z^{-2m}] \cdot A[z])$ is zero. Hence $\mathrm{tr}(\iota(p_+^m - \mathrm{pr}_+^m)A[z])$ is zero. The proof of $\mathrm{tr}(\iota(p_-^m - \mathrm{pr}_-^m)A[z]) = 0$ for large m is analogous. $\square$

Definition 5. Fix $P \in \mathrm{GL}(k, \mathbb{C}(z))$ such that $P(z)$ is regular at $z = 0$ and $P(z)^{-1}$ is regular at $z = \infty$. Put $S = \mathbb{C}[z, z^{-1}]^k + P \cdot \mathbb{C}[z, z^{-1}]^k$,

$$S_m = S \cap z^{1-m}\mathbb{C}[[z]]^k \cap z^{m-1}P \cdot \mathbb{C}[[z]]^k, \qquad S^m = z^{-m}\mathbb{C}[z^{-1}]^k + z^m P \cdot \mathbb{C}[z]^k.$$

Fix B in $M(k, \mathbb{C}[z, z^{-1}])$ such that $P^{-1}BP$ lies in $M(k, \mathbb{C}[z, z^{-1}])$. Then $BS \subset S$. We denote by $[B]$ or $B[z]$ the operator $S \to S$ of multiplication by $B(z)$.

Proposition 5.13. *We have $S = S_m \oplus S^m$. Denote by $\mathrm{ps}_m : S \to S$ the projection with image S_m and kernel S^m. Then*

$$\begin{aligned} &\mathrm{tr}(\mathrm{ps}_m \cdot [B]) = (2m-1)\,\mathrm{tr}\,B_0 \\ &- \operatorname{res}_{z=\infty} \mathrm{tr}[B(z)P'(z)P(z)^{-1}]dz + \mathrm{tr}([B];\, S/\mathbb{C}[z, z^{-1}]^k). \end{aligned}$$

Here B_0 is the constant term of $B = B(z)$, and $\mathrm{tr}([B]; S/\mathbb{C}[z, z^{-1}]^k)$ denotes the trace of the endomorphism of $S/\mathbb{C}[z, z^{-1}]^k$ induced by multiplication by $B(z)$.

Proof. The space S is a k-dimensional free $\mathbb{C}[z,z^{-1}]$-submodule of $\mathbb{C}(z)^k$. Hence there exists a matrix D in $\mathrm{GL}(k,\mathbb{C}(z))$ such that $S = D\cdot\mathbb{C}[z,z^{-1}]^k$. Since S contains $\mathbb{C}[z,z^{-1}]^k$, D^{-1} lies in $M(k,\mathbb{C}[z,z^{-1}])$. Since S contains $P\cdot\mathbb{C}[z,z^{-1}]^k$ we deduce that $D^{-1}P \in M(k,\mathbb{C}[z,z^{-1}])$. Put $Q_1 = D^{-1}$, $Q_2 = D^{-1}P$. The function $Q_1(z)^{-1}Q_2(z) = P(z)$ is regular at $z=0$. The function $Q_2(z)^{-1}Q_1(z)$ is regular at $z=\infty$. Under the isomorphism $S\tilde{\to}\mathbb{C}[z,z^{-1}]^k$, $u\mapsto D^{-1}u$, the subspaces S_m and S^m correspond to the subspaces R_m and R^m of Proposition 5.9. The multiplication $[B] : S\to S$ corresponds to $[A] : \mathbb{C}[z,z^{-1}]^k \to \mathbb{C}[z,z^{-1}]^k$, $A = D^{-1}BD$. Then Proposition 5.10 implies the first part of the proposition, as well as the equality

$$\operatorname{tr}(\mathrm{ps}_m \cdot B[z]) = (2m-1)\operatorname{tr}A_0 - \operatorname{res}_{z=0}\operatorname{tr}A(z)Q_1'(z)Q_1(z)^{-1}dz$$
$$- \operatorname{res}_{z=\infty}\operatorname{tr}A(z)Q_2'(z)Q_2(z)^{-1}dz.$$

Here A_0 is the constant term of $A(z)$. We have

$$\operatorname{tr}(AQ_1'Q_1^{-1}) = -\operatorname{tr}(D^{-1}BD') = -\operatorname{tr}(BD'D^{-1}),$$

$$\operatorname{tr}(AQ_2'Q_2^{-1}) = -\operatorname{tr}(D^{-1}BP'P^{-1}D - D^{-1}BD') = \operatorname{tr}(BP'P^{-1}) - \operatorname{tr}(BD'D^{-1}).$$

As $A = D^{-1}BD$, we have $\operatorname{tr}A = \operatorname{tr}B$, and $\operatorname{tr}A_0 = \operatorname{tr}B_0$. Hence

$$\begin{aligned}\operatorname{tr}(\mathrm{ps}_m \cdot B[z]) = (2m-1)\operatorname{tr}B_0 - \operatorname{res}_{z=\infty}\operatorname{tr}B(z)P'(z)P(z)^{-1}dz&\\ +\operatorname{res}_{z=0}\operatorname{tr}B(z)D'(z)D(z)^{-1}dz + \operatorname{res}_{z=\infty}\operatorname{tr}B(z)D'(z)D(z)^{-1}dz&\\ +(2m-1)\operatorname{tr}B_0 - \operatorname{res}_{z=\infty}\operatorname{tr}B(z)P'(z)P(z)^{-1}dz&\\ -\sum_{\zeta\in\mathbb{C}^\times}\operatorname{res}_{z=\zeta}\operatorname{tr}B(z)D'(z)D(z)^{-1}dz.&\end{aligned}$$

Lemma 5.14. *Suppose $T\in\mathrm{GL}(k,\mathbb{C}((z)))$, $C\in M(k,\mathbb{C}[[z]])$ and $T^{-1}CT\in M(k,\mathbb{C}[[z]])$. Then $\operatorname{res}_{z=0}\operatorname{tr}C(z)T'(z)T(z)^{-1} = a-b$, where a denotes the trace of the operator multiplication by C in the space $(\mathbb{C}[[z]]^k + T\mathbb{C}[[z]]^k)/T\mathbb{C}[[z]]^k$, while b denotes the trace of multiplication by C in the space*

$$(\mathbb{C}[[z]]^k + T\mathbb{C}[[z]]^k)/\mathbb{C}[[z]]^k.$$

Proof. Both sides of the asserted equality do not change if (T,C) is replaced by (UTV, UCU^{-1}) where $U, V\in\mathrm{GL}(k,\mathbb{C}[[z]])$. We may then assume that T is a diagonal matrix, hence that $k=1$. When $k=1$ both sides of the asserted relation are simply $mC(0)$, where m is the multiplicity of zero of $T(z)$ at $z=0$. ∎

It follows from the lemma that $-\operatorname{res}_{z=\zeta}\operatorname{tr}(B(z)D'(z)D(z)^{-1})dz$ is just the trace of the operator of multiplication by $B(z)$ on the ζ component of the module $S/\mathbb{C}[z,z^{-1}]^k$. This, and the equality just before the lemma, implies the proposition. ∎

Suppose we have $P(z^{-1}) = P(z)^{-1}$. Replace the assumption $P(z)^{-1}B(z)P(z) \in M(k, \mathbb{C}[z, z^{-1}])$ in Proposition 5.13 with the stronger assumption

$$P(z)^{-1}B(z)P(z) = B(z^{-1}).$$

Recall that L is the space of all rational functions of the form $u(z)+P(z)u(z^{-1})$ with $u \in \mathbb{C}[z, z^{-1}]^m$. In view of the stronger assumption, L is invariant under multiplication by B.

Definition 6. Denote by B_L the operator of multiplication by B on L. Put $L_m = L \cap z^{1-m}\mathbb{C}[[z]]^k$. Denote by L^m the set of rational functions of the form $u(z) + P(z)u(z^{-1})$ with $u(z) \in z^{-m}\mathbb{C}[z^{-1}]^k$.

Proposition 5.15. *The space L_m is finite dimensional, and $L = L_m \oplus L^m$. Denote by $\pi_m : L \to L$ the projection with image L_m and kernel L^m. Suppose the function $P(z)$ is regular at $z = \pm 1$. Then for large enough m we have that* $\operatorname{tr}(\pi_m B_L)$ *equals*

$$(m - \frac{1}{2}) \operatorname{tr} B_0 - \frac{1}{2} \operatorname{res}_{z=\infty} \operatorname{tr}(B(z)P'(z)P(z)^{-1})dz$$

$$+\frac{c}{2} + \frac{1}{4}[\operatorname{tr}(B(1)P(1)) + \operatorname{tr}(B(-1)P(-1))].$$

Here B_0 is the constant term of $B(z)$, and c is the trace of the operator of multiplication by $B(z)$ in the space $(\mathbb{C}[z, z^{-1}]^k + P(z)\mathbb{C}[z, z^{-1}]^k)/\mathbb{C}[z, z^{-1}]^k$.

Proof. Let S, S_m, S^m, ps_m, B be as in Proposition 5.13. From $P(z^{-1}) = P(z)^{-1}$ it follows that if $u \in S$ then $\tilde{u}$, given by $\tilde{u}(z) = P(z)u(z^{-1})$, is also in S. Define $\tau : S \to S$ by $\tau(u) = \tilde{u}$. Then $\tau^2 = 1$, $L = \{u \in S;\, \tau(u) = u\}$, $L_m = S_m \cap L$, $L^m = S^m \cap L$, and

$$\operatorname{tr}(\pi_m B_L) = \frac{1}{2} \operatorname{tr}(\mathrm{ps}_m \cdot B[z]) + \frac{1}{2} \operatorname{tr}(\tau \cdot \mathrm{ps}_m \cdot B[z]).$$

The finite dimensionality of S_m and Proposition 5.13 then imply that the space L_m is finite dimensional, and $L = L_m \oplus L^m$. To deduce the last claim of the proposition from Proposition 5.13, it remains to show that

$$\operatorname{tr}(\tau \cdot \mathrm{ps}_m \cdot [B]) = \frac{1}{2}(\operatorname{tr}(B(1)P(1)) + \operatorname{tr}(B(-1)P(-1)))$$

for large enough m.

Let D, Q_1, Q_2 be as in Proposition 5.13. Then under the isomorphism $S \tilde{\to} \mathbb{C}[z, z^{-1}]^k$, $u \mapsto D^{-1}u$, the operator $\mathrm{ps}_m : S \to S$ translates into the operator pr_m (of Proposition 5.9), and multiplication by $B : S \to S$ translates into multiplication by

$$A = D^{-1}BD, \qquad \mathbb{C}[z, z^{-1}]^k \to \mathbb{C}[z, z^{-1}]^k.$$

The map $\tau : S \to S$ translates into

$[C]\iota : \mathbb{C}[z, z^{-1}]^k \to \mathbb{C}[z, z^{-1}]^k$, $(\iota u)(z) = u(z^{-1})$, $C(z) = D(z)^{-1}P(z)D(z^{-1})$.

Hence

$$\operatorname{tr}(\tau \cdot \mathrm{ps}_m \cdot B[z]) = \operatorname{tr}(C[z]\iota \cdot \mathrm{pr}_m \cdot A[z]) = \operatorname{tr}(\iota \, \mathrm{pr}_m \, A[z]C[z]),$$

which – by Proposition 5.9 – is

$$\frac{1}{2}(\operatorname{tr} A(1)C(1) + \operatorname{tr} A(-1)C(-1)) = \frac{1}{2}\operatorname{tr}(B(1)P(1) + \operatorname{tr} B(-1)P(-1));$$

note that $D(z)$ is regular at $z = \pm 1$, since so is $P(z)$. □

If $F \in M(k, \mathbb{C})$ and $Y \subset \mathbb{C}^k$ is an F-invariant subspace, write $\operatorname{tr}(F, Y)$ for the trace of F on Y.

Proposition 5.16. *Fix $P(z) \in \mathrm{GL}(k, \mathbb{C}(z))$ with $P(z^{-1}) = P(z)^{-1}$. Suppose that the function $P(z)$ is regular on $|z| = 1$ and at $z = 0$, and that it has order 1 at all its poles $\zeta_1, \dots, \zeta_s$ inside $\{z \in \mathbb{C};\ 0 < |z| < 1\}$. Denote by Y_i the image of the operator $\lim_{z \to \zeta_i}(z - \zeta_i)P(z)$ acting on $\mathbb{C}^k$. Fix $B(z) \in M(k, \mathbb{C}[z, z^{-1}])$ and suppose $B_1(z) = P(z)^{-1}B(z)P(z) \in M(k, \mathbb{C}[z, z^{-1}])$. Then*

$$\operatorname{tr}(\mathrm{ps}_m \cdot [B]) = (2m - 1)\operatorname{tr} B_0 + \frac{1}{2\pi i}\int_{|z|=1} \operatorname{tr} B(z)P'(z)P(z)^{-1}dz$$

$$+ \sum_{1 \le i \le s} \operatorname{tr}(B(\zeta_i) + B_1(\zeta_i^{-1}), Y_i),$$

with B_0 being the constant term of $B(z)$.
If in addition $B_1(z) = B(z^{-1})$ then

$$\operatorname{tr}(\pi_m B_L) = (m - \frac{1}{2})\operatorname{tr} B_0 + \frac{1}{4\pi i}\int_{|z|=1} \operatorname{tr} B(z)P'(z)P(z)^{-1}dz$$

$$+ \sum_{1 \le i \le s} \operatorname{tr}(B(\zeta_i), Y_i) + \frac{1}{4}[\operatorname{tr}(B(1)P(1)) + \operatorname{tr}(B(-1)P(-1))].$$

Note that the subspace Y_i of $\mathbb{C}^k$ is invariant under $B(\zeta_i)$ and $B_1(\zeta_i^{-1})$.

Proof. In view of Propositions 5.13 and 5.15 it suffices to verify that

$$\frac{1}{2\pi i}\oint_{|z|=1} \operatorname{tr} B(z)P'(z)P(z)^{-1}dz + \sum_{1 \le i \le s} \operatorname{tr}(B(\zeta_i) + B_1(\zeta_i^{-1}), Y_i)$$

$$= \operatorname{tr}([B], S/\mathbb{C}[z, z^{-1}]^k) - \operatorname{res}_{z=\infty} \operatorname{tr} B(z)P'(z)P(z)^{-1}dz,$$

where

$$S = \mathbb{C}[z, z^{-1}]^k + P(z)\mathbb{C}[z, z^{-1}]^k.$$

For any $\zeta \neq 0$ in $\mathbb{C}$ denote by M_ζ and N_ζ the ζ-components of the $\mathbb{C}[z,z^{-1}]$-modules $S/\mathbb{C}[z,z^{-1}]^k$ and $S/P(z)\mathbb{C}[z,z^{-1}]^k$, respectively. From Cauchy's formula and Lemma 5.14, it follows that

$$\frac{1}{2\pi i}\oint_{|z|=1} \operatorname{tr} B(z)P'(z)P(z)^{-1}dz = \sum_{1<|\zeta|<\infty} \operatorname{tr}([B], M_\zeta)$$

$$-\sum_{1<|\zeta|<\infty} \operatorname{tr}([B], N_\zeta) - \operatorname{res}_{z=\infty} \operatorname{tr}(B(z)P'(z)P(z)^{-1})dz.$$

On the other hand, $\operatorname{tr}([B], S/\mathbb{C}[z,z^{-1}]^k) = \sum_{\zeta\in\mathbb{C}^\times} \operatorname{tr}([B], M_\zeta)$. Hence the required identity follows from

$$\sum_{0<|\zeta|<1} \operatorname{tr}([B], M_\zeta) = \sum_{1\le i\le s} \operatorname{tr}(B(\zeta_i), Y_i),$$

$$\sum_{1<|\zeta|<\infty} \operatorname{tr}([B], N_\zeta) = \sum_{1\le i\le s} \operatorname{tr}(B_1(\zeta_i^{-1}), Y_i).$$

If $P(z)$ is regular at ζ then $M_\zeta = 0$. At each ζ_i, $P(z)$ has a pole of order one. Hence there exists isomorphisms $M_{\zeta_i}\tilde{\to}Y_i$ which translate the operator $[B] : M_{\zeta_i} \to M_{\zeta_i}$ to the operator of multiplication by $B(\zeta_i)$ on Y_i. This implies the first identity.
For the second identity, for any $\zeta \in \mathbb{C}^\times$, denote by $\overline{N}_\zeta$ the ζ-component of the module $(\mathbb{C}[z,z^{-1}]^k + P(z)^{-1}\mathbb{C}[z,z^{-1}]^k)/\mathbb{C}[z,z^{-1}]^k$. Multiplication by $P(z)^{-1}$ induces an isomorphism $N_\zeta\tilde{\to}\overline{N}_\zeta$. Under this isomorphism, multiplication by $B : N_\zeta \to N_\zeta$ translates into multiplication by $B_1 : \overline{N}_\zeta \to \overline{N}_\zeta$, hence $\operatorname{tr}([B], N_\zeta) = \operatorname{tr}([B_1], \overline{N}_\zeta)$. From $P(z)^{-1} = P(z^{-1})$ we deduce that $\overline{N}_\zeta = 0$ if $P(z)$ is regular at $z = \zeta^{-1}$, and that $\operatorname{tr}([B_1], \overline{N}_{\zeta_i^{-1}}) = \operatorname{tr}(B_1(\zeta_i^{-1}), Y_i)$. This implies the second identity, hence the proposition. □

5.3 Spectral terms

To deduce the trace formula from Proposition 5.16, we use properties of the function $M(\mu_1, \mu_2, t)$.
Recall that we have the projection $\pi_m : L \to L$ with kernel L^m and image L_m, and B_L denotes the operator of multiplication by $B(z)$ on L. The operator $P(z)$ is the restriction to the subspace of U-invariant vectors of the operator M on the space $I_0 = \oplus I_0(\mu_1, \mu_2)$ (μ_1, μ_2 range over the characters of $\mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}}$), which maps $I_0(\mu_1, \mu_2)$ to $I_0(\mu_2, \mu_1)$ via $M(\mu_1, \mu_2, z)$.

Proposition 5.17. *There exists $a_f \in \mathbb{C}$ such that for sufficiently large m,*

$$\operatorname{tr}(\pi_m B_L) = (m - \frac{1}{2})a_f - \sum_{5\le i\le 8} S_i(f).$$

Proof. By Proposition 4.29 the function $P(z)$ has two poles in the domain $|z| \leq 1$, namely at $z = \pm q^{-1/2}$, each of order 1. We have $P(z^{-1}) = P(z)^{-1}$ and $P(z)^{-1}B(z)P(z) = B(z^{-1})$. Hence the final claim of Proposition 5.16 applies and implies that for large enough m,

$$\operatorname{tr}(\pi_m[B]) = (m-\frac{1}{2}) \operatorname{tr} B_0 + \frac{1}{4\pi i}\oint_{|z|=1} \operatorname{tr} B(z)P'(z)P(z)^{-1}dz + \operatorname{tr}(B(q^{-1/2}), Y_+)$$

$$+ \operatorname{tr}(B(-q^{-1/2}), Y_-) + \frac{1}{4}[\operatorname{tr}(B(1)P(1)) + \operatorname{tr}(B(-1)P(-1))].$$

Here B_0 is the constant term of $B(z)$ and the image of the operator $\lim_{z\to\pm q^{-1/2}}(z \mp q^{-1/2})P(z)$ is denoted by $Y_\pm$. The proposition follows once we show that

$$\oint_{|z|=1} \operatorname{tr} B(z)P'(z)P(z)^{-1}dz = -4\pi i(S_5(f) + S_6(f)), \tag{1}$$

$$\operatorname{tr}(B(q^{-1/2}), Y_+) + \operatorname{tr}(B(-q^{-1/2}), Y_-) = -S_8(f), \tag{2}$$

$$\operatorname{tr}(B(1)P(1)) + \operatorname{tr}(B(-1)P(-1)) = -4S_7(f). \tag{3}$$

Denote by $r(z)$ the representation of $\mathrm{GL}(2,\mathbb{A})$ by right translation in $I(z) = \otimes_{\mu_1,\mu_2} I(\mu_1\nu_{z^{-1}}, \mu_2\nu_z)$. Here μ_1, μ_2 are characters of $\mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}}$. Let $r(z,f)$ be the convolution operator defined by $r(z)$ and the compactly supported function f in $C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$. Identify, as usual, $I(z)$ to the space I_0, and consider $r(z,f)$ as an operator in I_0. From Proposition 4.36, $B(z)$ coincides with the restriction of $r(z,f)$ to I_0^U. Also, $P(z)$ coincides with the restriction of $M(z)$ to I_0^U. Hence the integral on the left of (1) equals

$$\oint_{|z|=1} \operatorname{tr} r(z,f)M'(z)M(z)^{-1}dz$$

$$= \sum_{\mu_1,\mu_2} \oint_{|z|=1} \operatorname{tr} I(\mu_2\nu_{z^{-1}}, \mu_1\nu_z, f)M'(\mu_1,\mu_2,z)M(\mu_1,\mu_2,z)^{-1}dz$$

$$= \sum_{\mu_1,\mu_2} \oint_{|z|=1} \operatorname{tr} M(\mu_1,\mu_2,z)^{-1}I(\mu_2\nu_{z^{-1}}, \mu_1\nu_z, f)M'(\mu_1,\mu_2,z)dz$$

$$= \sum_{\mu_1,\mu_2} \oint_{|z|=1} \operatorname{tr} I(\mu_1\nu_z, \mu_2\nu_{z^{-1}}, f)M(\mu_1,\mu_2,z)^{-1}M'(\mu_1,\mu_2,z)dz.$$

Then (1) follows from Proposition 4.9.

For (2), it follows from Proposition 4.29 that $Y_+ = L^U$, with $L = \oplus L_\mu$, $L_\mu \subset I(\mu,\mu)$ being generated by the function $x \mapsto \mu(x)$. The operator $r(q^{-1/2}, f)$ acts in L_μ as the operator of multiplication by $\int_{\mathrm{GL}(2,\mathbb{A})} f(x)\mu(\det x)dx$. Hence

$$\operatorname{tr}(B(q^{-1/2}), Y_+) = \operatorname{tr}(r(q^{-1/2}, f), L) = \sum_\mu \int_{\mathrm{GL}(2,\mathbb{A})} f(x)\mu(\det x)dx,$$

where μ ranges over the set of characters of $\mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}}$. Similarly

$$\operatorname{tr}(B(q^{-1/2}), Y_-) = \operatorname{tr}(r(-q^{-1/2}, f), L)$$

$$= \sum_\mu \int_{\mathrm{GL}(2,\mathbb{A})} f(x)\mu(\det x)\nu_{-1}(\det x)dx.$$

Every character of $\mathbb{A}^\times$ which is trivial on $F^\times \cdot \alpha^{2\mathbb{Z}}$ is either trivial on $F^\times \cdot \alpha^{\mathbb{Z}}$ or its product with ν_{-1} is, so (2) follows.
For (3) note that

$$\operatorname{tr} B(1)P(1) = \operatorname{tr} r(1,f)M(1) = \sum_\mu \operatorname{tr} I(\mu,\mu,f)M(\mu,\mu,1) = -\sum_\mu \operatorname{tr} I(\mu,\mu,f)$$

by Proposition 4.30. Similarly $\operatorname{tr} B(-1)P(-1) = -\sum_\mu \operatorname{tr} I(\mu\nu_{-1}, \mu\nu_{-1}, f)$. □

This completes the proof of the trace formula.

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The Perturbed Maxwell Operator as Pseudodifferential Operator

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Abstract. As a first step to deriving effective dynamics and ray optics, we prove that the perturbed periodic Maxwell operator in $d = 3$ can be seen as a pseudodifferential operator. This necessitates a better understanding of the periodic Maxwell operator $\mathbf{M}_0$. In particular, we characterize the behavior of $\mathbf{M}_0$ and the physical initial states at small crystal momenta k and small frequencies. Among other things, we prove that generically the band spectrum is symmetric with respect to inversions at $k = 0$ and that there are exactly 4 ground state bands with approximately linear dispersion near $k = 0$.

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1 Introduction

Photonic crystals are to the transport of light (electromagnetic waves) what crystalline solids are to the transport of electrons [JJWM08]. Progress in the manufacturing techniques have allowed physicists to engineer photonic crystals with specific properties – which in turn has stimulated even more theoretical studies. One topic which has seen relatively little attention, though, is the derivation of *effective dynamics* in perturbed photonic crystals for states from a narrow range of intermediate frequencies (e. g. [OMN06, RH08, APR12, EG13]). Mathematically rigorous results are even more scarce: apart from [MP96] concerning only the unperturbed case, the

only rigorous work covering *second*-order perturbations is by Allaire, Palombaro and Rauch [APR12]. Hence, the correct form of the subleading-order terms has not yet been established – rigorously or non-rigorously.
This paucity of results motivated the two authors to apply a perturbation scheme developed by Panati, Spohn and Teufel [PST03b, PST03a], *space-adiabatic perturbation theory*, to derive effective dynamics and ray optics equations for adiabatically perturbed Maxwell operators. Among other things, we settle the important question about the correct form of the next-to-leading order terms in the ray optics equations; these terms are necessary to explain topological effects in photonic crystals. The current paper is a preliminary, but necessary step to implement space-adiabatic perturbation theory [DL13]: we establish that the Maxwell operator can be seen as a *semiclassical pseudodifferential operator* (ΨDO) with band structure defined over the cotangent bundle over the Brillouin torus.
This is not just the content of an innocent lemma, it turns out there are quite a few technical and conceptual hurdles to overcome. To mention but one, we need a better understanding of the band structure of the periodic Maxwell operator. Despite the body of work on periodic Maxwell operators (see e. g. [Kuc01] for a review), proofs of rather fundamental results are either scattered throughout the literature or, in some cases, seem to have not been published at all.

Before we expound on this point in more detail, let us recall the L^2-theory of electromagnetism first established in [BS87]. The two dynamical equations

$$\partial_t \mathbf{E} = +\varepsilon^{-1} \nabla_x \times \mathbf{H}, \qquad \partial_t \mathbf{H} = -\mu^{-1} \nabla_x \times \mathbf{E}, \tag{1}$$

can be recast as a time-dependent Schrödinger equation

$$\mathrm{i}\partial_t \Psi = \mathbf{M}_w \Psi \tag{2}$$

where $\Psi = (\mathbf{E}, \mathbf{H})$ consists of the electric field $\mathbf{E} = (E_1, E_2, E_3)$ and the magnetic field $\mathbf{H} = (H_1, H_2, H_3)$, and

$$\mathbf{M}_w := \begin{pmatrix} 0 & +\mathrm{i}\,\varepsilon^{-1}\,\nabla_x^{\times} \\ -\mathrm{i}\,\mu^{-1}\,\nabla_x^{\times} & 0 \end{pmatrix} \tag{3}$$

is the *Maxwell operator*. Here we used $\nabla_x^{\times}$ as shorthand for the curl (cf. Appendix A). The second set of Maxwell equation which imposes the absence of sources,

$$\nabla_x \cdot \varepsilon \mathbf{E} = 0, \qquad \nabla_x \cdot \mu \mathbf{H} = 0, \tag{4}$$

enter as a constraint on the initial conditions for equation (2) or, equivalently, one can restrict the domain to the physical states of $\mathbf{M}_w$ (see Section 2.1). We shall always make the following assumptions on the material weights $w = (\varepsilon, \mu)$:

Assumption 1.1 (Material weights) *Assume* $\varepsilon, \mu \in L^\infty\big(\mathbb{R}^3, \mathrm{Mat}_{\mathbb{C}}(3)\big)$ *are hermitian-matrix-valued functions which are bounded away from* 0 *and* $+\infty$*, i. e.* $0 < c\,\mathrm{id}_{\mathbb{R}^3} \leq \varepsilon, \mu \leq C\,\mathrm{id}_{\mathbb{R}^3}$ *for some* $0 < c \leq C < \infty$*. We say the material weights* (ε, μ) *are* real *iff their entries are all real-valued functions.*

These assumptions are rather natural in the setting we are interested in: First of all, asking for boundedness of ε and μ only instead of continuity is necessary to include the most common cases, because many photonic crystals are made by alternating two different materials, e. g. a dielectric and air, in a periodic fashion. The selfadjointness of the multiplication operator defined by the *electric permittivity tensor* $\varepsilon^* = \varepsilon$ and the *magnetic permeability tensor* $\mu^* = \mu$ ensure that the medium neither absorbs nor amplifies electromagnetic waves. The positivity of ε and μ excludes the case of metamaterials with negative refraction indices (see e. g. [SPV$^+$00]); moreover, combined with the boundedness away from 0 and $+\infty$, it implies that ε^{-1} and μ^{-1} exist as bounded operators which again satisfy Assumption 1.1. Lastly, our assumptions also include the interesting case of *gyrotropic photonic crystals* where the offdiagonal entries of $\varepsilon = \varepsilon^*$ and $\mu = \mu^*$ are complex-valued functions.
Under these assumptions, we can proceed with a rigorous definition of the Maxwell operator (3): it can be conveniently factored into

$$\mathbf{M}_w = W\,\mathbf{Rot}\,. \tag{5}$$

where the first term is the bounded operator involving the weights

$$W(\hat{x}) := \begin{pmatrix} \varepsilon^{-1}(\hat{x}) & 0 \\ 0 & \mu^{-1}(\hat{x}) \end{pmatrix} \tag{6}$$

and the free Maxwell operator

$$\mathbf{Rot} := \begin{pmatrix} 0 & +\mathrm{i}\,\nabla_x^\times \\ -\mathrm{i}\,\nabla_x^\times & 0 \end{pmatrix} = \begin{pmatrix} 0 & +\mathrm{i}\,\mathbf{curl} \\ -\mathrm{i}\,\mathbf{curl} & 0 \end{pmatrix}\,. \tag{7}$$

$\mathbf{Rot}$ equipped with the domain $\mathfrak{D} := \mathfrak{D}(\mathbf{Rot}) \subset L^2(\mathbb{R}^3, \mathbb{C}^6)$ is selfadjoint (see Appendix A for a precise characterization of $\mathfrak{D}$). For reasons that will be clear in the following, we refer to (5) as the *physical representation* of the Maxwell operator. From the representation (5) one gets two immediate consequences: first, $\mathfrak{D}(\mathbf{M}_w) = \mathfrak{D}$ since W is bounded and second, $\mathbf{M}_w$ is *not* self-adjoint on $L^2(\mathbb{R}^3, \mathbb{C}^6)$. In order to cure the lack of selfadjointness one introduces the *weighted* scalar product

$$\big\langle \Psi, \Phi \big\rangle_w := \big\langle \Psi, W^{-1} \Phi \big\rangle_{L^2(\mathbb{R}^3, \mathbb{C}^6)} = \big\langle W^{-1} \Psi, \Phi \big\rangle_{L^2(\mathbb{R}^3, \mathbb{C}^6)}\,. \tag{8}$$

on the Banach space $L^2(\mathbb{R}^3, \mathbb{C}^6)$, and we will denote this Hilbert space with $\mathfrak{H}_w$. Then, one can show that the Maxwell operator $\mathbf{M}_w$ is self-adjoint on $\mathfrak{D} \subset \mathfrak{H}_w$ (cf. Theorem 2.1). Only with respect to the correctly weighted scalar product, the evolutionary semigroup $\mathrm{e}^{-\mathrm{i}t\mathbf{M}_w}$ is unitary – which physically corresponds to conservation of field energy $\mathcal{E}\big(\mathbf{E}(t),\mathbf{H}(t)\big) = \mathcal{E}(\mathbf{E},\mathbf{H})$,

$$\begin{aligned}\mathcal{E}(\mathbf{E},\mathbf{H}) &= \frac{1}{2}\int_{\mathbb{R}^3}\mathrm{d}x\,\mathbf{E}(x)\cdot\varepsilon(x)\mathbf{E}(x) + \frac{1}{2}\int_{\mathbb{R}^3}\mathrm{d}x\,\mathbf{H}(x)\cdot\mu(x)\mathbf{H}(x)\\ &= \frac{1}{2}\,\big\|(\mathbf{E},\mathbf{H})\big\|_w^2\,.\end{aligned}$$

Periodic Maxwell operators describe photonic crystals; here, the material weights ε and μ are periodic with respect to some lattice Γ. As the analog of periodic Schrödinger operators, one can use Bloch-Floquet theory to analyze the properties of $\mathbf{M}_w$ (cf. Section 3). Hence, many properties of photonic crystals mimic those of crystalline solids (both physically and mathematically). However, the rapidly increasing interest for photonic crystals resides in the fact that, as they are artificially created by patterning several materials, they can be engineered to have certain desired properties. To name one example, one of the early successes was to design a *photonic semiconductor* with a band gap in the frequency spectrum [JJ00, JJWM08]. Such a "semiconductor for light" is of great interest to the quantum optics community (e. g. [Yab93]).
Since perfectly periodic media are only a mathematical abstraction, one is led to study more realistic models of photonic crystals. One well-explored possibility is to include effects of disorder by interpreting ε and μ as random variables and leads to the "Anderson localization of light" (see e. g. [Joh91, FK96b, FK97] and references therein). We will concern ourselves with another class of perturbations where the perfectly periodic weights ε and μ are modulated slowly,

$$\varepsilon_\lambda(x) := \frac{\varepsilon(x)}{\tau_\varepsilon(\lambda x)^2}\,, \qquad \mu_\lambda(x) := \frac{\mu(x)}{\tau_\mu(\lambda x)^2}\,. \tag{9}$$

The perturbation parameter $\lambda \ll 1$ quantifies the separation of spatial scales on which (ε,μ) and the scalar *modulation functions* $(\tau_\varepsilon,\tau_\mu)$ vary. The latter are assumed to verify the following

ASSUMPTION 1.2 (MODULATION FUNCTIONS) *Suppose $\tau_\varepsilon,\tau_\mu \in \mathcal{C}^\infty_{\mathrm{b}}(\mathbb{R}^3)$ are bounded away from 0 and $+\infty$ as well as $\tau_\varepsilon(0) = 1$ and $\tau_\mu(0) = 1$.*

To shorten the notation, we define $\mathbf{M}_\lambda := \mathbf{M}_{(\varepsilon_\lambda,\mu_\lambda)}$ and $\mathfrak{H}_\lambda := \mathfrak{H}_{(\varepsilon_\lambda,\mu_\lambda)}$.

As mentioned in the very beginning our goal is to rigorously derive both, the effective "quantum-like" and "semiclassical" dynamics for perturbed Maxwell operators $\mathbf{M}_\lambda$ in the adiabatic limit $\lambda \ll 1$ [DL13]. Apart from ray optics, we will derive *effective light dynamics* $\mathrm{e}^{-\mathrm{i}t\mathbf{M}_{\mathrm{eff}}}$ which approximate the full light dynamics $\mathrm{e}^{-\mathrm{i}t\mathbf{M}_\lambda}$ for initial states supported in a narrow range of frequencies,

$$\Big\|\big(\mathrm{e}^{-\mathrm{i}t\mathbf{M}_\lambda} - \mathrm{e}^{-\mathrm{i}t\mathbf{M}_{\mathrm{eff}}}\big)\,\mathbf{\Pi}_\lambda\Big\|_{\mathfrak{H}_\lambda} = \mathcal{O}(\lambda^\infty)\,. \tag{10}$$

$\mathbf{\Pi}_\lambda$ is the projection on the superadiabatic subspace associated with a narrow range of frequencies and, up to a unitary transformation, the effective operator $\mathbf{M}_{\mathrm{eff}}$ can be constructed order-by-order in λ as the Weyl quantization $\mathfrak{Op}_\lambda(\mathcal{M}_{\mathrm{eff}})$ of a semiclassical symbol; in case additional assumptions are placed on the frequency bands, the leading-order terms are given by

$$\mathcal{M}_{\mathrm{eff}}(r,k) = \sum_{n\in\mathcal{I}} \tau_\varepsilon(r)\,\tau_\mu(r)\,\omega_n(k)\,|\chi_n\rangle\langle\chi_n| + \mathcal{O}(\lambda)\,.$$

Here, the ω_n are the Bloch frequency band functions and χ_n denotes a fixed orthonormal basis in the reference space [DL13, Theorem 3.1]. As usual one can also prove that the subleading-order terms of $\mathcal{M}_{\mathrm{eff}}(r,k)$ contain geometric quantities such as the Berry connection.
Similarly, the superadiabatic projection $\mathbf{\Pi}_\lambda$ is also constructed on the level of symbols in terms of $\boldsymbol{\mathcal{M}}_\lambda$, the symbol of the Maxwell operator, and hence, proving that the Maxwell operator is a ΨDO associated to a semiclassical symbol is the first order of business.

Theorem 1.3 *Suppose Assumptions 3.1 on the material weights (ε,μ) and 1.2 on the modulation functions $(\tau_\varepsilon,\tau_\mu)$ are satisfied. Then the Maxwell operator (in Zak representation) $\mathbf{M}_\lambda^{\mathcal{Z}} = \mathfrak{Op}_\lambda(\boldsymbol{\mathcal{M}}_\lambda)$ is the pseudodifferential operator associated to*

$$\begin{aligned}\boldsymbol{\mathcal{M}}_\lambda(r,k) = &\begin{pmatrix}\tau_\varepsilon^2(r) & 0\\ 0 & \tau_\mu^2(r)\end{pmatrix}\mathbf{M}_0(k) + \\ &+\lambda\, W\begin{pmatrix}0 & -\mathrm{i}\,\tau_\varepsilon(r)\,\big(\nabla_r\tau_\varepsilon\big)^\times(r)\\ +\mathrm{i}\,\tau_\mu(r)\,\big(\nabla_r\tau_\mu\big)^\times(r) & 0\end{pmatrix}\end{aligned} \tag{11}$$

where

$$\begin{aligned}\mathbf{M}_0(k) &:= W\,\mathbf{Rot}(k)\\ &:= \begin{pmatrix}\varepsilon^{-1}(\hat{y}) & 0\\ 0 & \mu^{-1}(\hat{y})\end{pmatrix}\begin{pmatrix}0 & -(-\mathrm{i}\nabla_y + k)^\times\\ +(-\mathrm{i}\nabla_y+k)^\times & 0\end{pmatrix}\end{aligned}$$

is the periodic Maxwell operator acting on the fiber at k defined in terms of the weight operator W and the free Maxwell operator $\mathbf{Rot}(k)$. The function $\boldsymbol{\mathcal{M}}_\lambda \in AS^1_{1,\mathrm{eq}}\big(\mathcal{B}\big(\mathfrak{d}, L^2(\mathbb{T}^3,\mathbb{C}^6)\big)\big)$ is an equivariant semiclassical operator-valued symbol in the sense of Definition 4.1.

For the precise definitions and the proof, we refer to Section 4.

Despite the similarities to the case of the Bloch electron [PST03a], applying space-adiabatic perturbation theory to photonic crystals required us to solve numerous technical and conceptual problems. In addition to defining pseudodifferential operators on weighted L^2-spaces, one other major difficulty is to make $\mathcal{O}(\lambda^n)$ estimates in norm, because the norm *also* depends on λ (see e. g. equation (10)). Such estimates are crucial when one wants to make sense

of perturbation expansions of operators. This conceptual problem is solved by introducing a λ-independent auxiliary representation (cf. Section 2.2).
However, the biggest obstacle to control the symbol $\boldsymbol{\mathcal{M}}_\lambda$ is to gain a better understanding of the *periodic* Maxwell operator $\mathbf{M}_0(k)$ and its band structure. In particular, pseudodifferential theory requires us to understand the *pointwise* behavior of $\mathbf{M}_0(k)$ and associated objects. Even though $k \mapsto \mathbf{M}_0(k)$ is linear and defined on a k-independent domain, and thus trivially analytic, the splitting of the *fiber* Hilbert space $\mathfrak{h}_0 = J_0(k) \oplus_\perp G_0(k)$ into physical and unphysical states is not even *continuous* at $k = 0$. Here, $\mathfrak{h}_0$ is defined as the Banach space $L^2(\mathbb{T}^3, \mathbb{C}^6)$ equipped with a scalar product analogous to (8), and elements of $J_0(k)$ satisfy the source-free condition on the fiber space. We characterize how this discontinuity enters into the band structure of $\mathbf{M}_0(k)$, and show that it is connected to the *ground state bands*, i. e. those frequency bands which go to 0 linearly as $k \to 0$. The precise band structure of $\mathbf{M}_0^{\mathcal{Z}} = \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \, \mathbf{M}_0(k)$ is studied in great detail in Section 3.3 where the following result is proven:

Theorem 1.4 (The band picture of $\mathbf{M}_0^{\mathcal{Z}}$) *Suppose ε and μ satisfy Assumption 3.1.*

(i) For each $n \in \mathbb{Z}$, the band functions $\mathbb{R}^3 \ni k \mapsto \omega_n(k)$ are continuous, analytic away from band crossings and Γ^*-periodic.

(ii) If the weights (ε, μ) are real, then for all $n \in \mathbb{Z}$, there exists $j \in \mathbb{Z}$ such that $\omega_n(k) = -\omega_j(-k)$ holds for all $k \in \mathbb{R}^3$.

(iii) $\mathbf{M}_0^{\mathcal{Z}}$ has 4 ground state bands indexed by the set $\mathcal{I}_{\mathrm{gs}}$ which are characterized as follows:

(1) $\omega_n(k) = 0 \Leftrightarrow n \in \mathcal{I}_{\mathrm{gs}}$ and $k = 0$.

(2) $\omega_n(k) = \pm c_n(\underline{k}) \, |k| + o(|k|)$ holds for $n \in \mathcal{I}_{\mathrm{gs}}$ where the $c_n(\underline{k})$ are the positive eigenvalues of the matrix (36) for the unit vector $\underline{k} := \frac{k}{|k|}$.

The content of Theorem 1.4 is sketched in Figure 1. Among other things, we prove that the ground state bands of the Maxwell operator always have a doubly degenerate conical intersection at $k = 0$ and $\omega = 0$.

The remainder of the paper is dedicated to explaining and proving Theorem 1.3 and Theorem 1.4: In Section 2, we give some basic facts on the Maxwell operator. Section 3 is devoted to the study of the properties of the periodic operator $\mathbf{M}_0^{\mathcal{Z}}$ with a particular attention to the analysis of the band picture. Finally, in Section 4 where discuss pseudodifferential theory on weighted Hilbert spaces and finish the proof of Theorem 1.3. For the benefit of the reader, we have included some auxiliary results in Appendix A.

Before we proceed, let us collect some conventions and introduce notation used throughout the remainder of the paper.

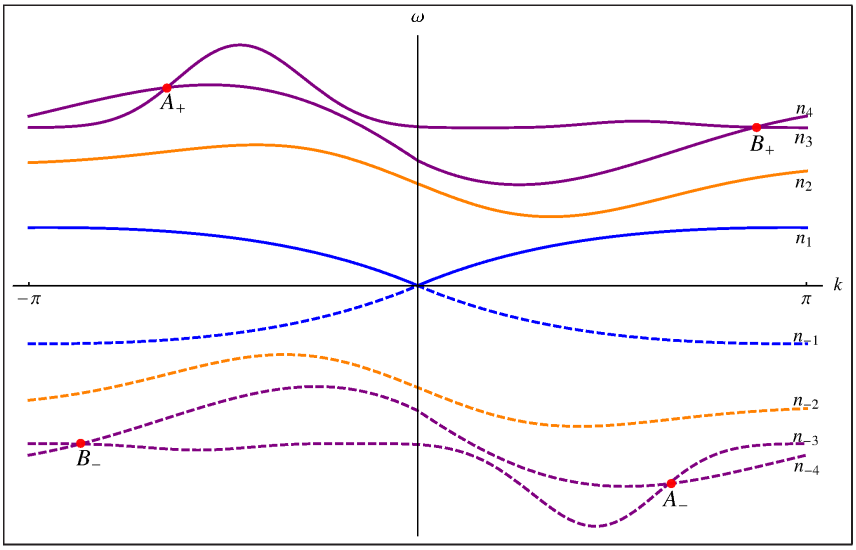

Figure 1: A sketch of a typical band spectrum of $\mathbf{M}_0(k)|_{J_0(k)}$. The $2+2$ ground state bands with linear dispersion around $k = 0$ are blue. Positive frequency bands are drawn using solid lines while the lines for the symmetrically-related negative frequency bands are in the same color, but dashed.

1.1 Notation and remarks

The Maxwell operator is naturally defined on *weighted* L^2-spaces $\mathfrak{H}_w$ where the scalar product is weighted by the tensors $w = (\varepsilon, \mu)$ according to the prescription (8). We will use capital greek letters such as Ψ and Φ to denote elements of $\mathfrak{H}_w$ and small greek letters with the appropriate index to indicate they are the electric (first three) or the magnetic (last three) component[1] , for instance $\Psi = (\psi^E, \psi^H)$ and $\Phi = (\phi^E, \phi^H)$. Componentwise the scalar product (8) reads

$$\langle \Psi, \Phi \rangle_w := \int_{\mathbb{R}^3} \mathrm{d}x \; \psi^E(x) \cdot \varepsilon(x) \phi^E(x) + \int_{\mathbb{R}^3} \mathrm{d}x \, \psi^H(x) \cdot \mu(x) \phi^H(x) \, . \tag{12}$$

Let us point out that with this convention the complex conjugation is implicit in the scalar product like $a \cdot b := \sum_{j=1}^N \overline{a_j} \, b_j$ on $\mathbb{C}^N$. Equation (12) leads to the natural (orthogonal) splitting

$$\mathfrak{H}_w := L^2_\varepsilon(\mathbb{R}^3, \mathbb{C}^3) \oplus_\perp L^2_\mu(\mathbb{R}^3, \mathbb{C}^3) \, ,$$

[1] Note that even though physical electromagnetic fields are real-valued, we assume $\Psi \in \mathfrak{H}_w$ takes values in the complex vector space $\mathbb{C}^6$, and hence our distinction in notation to the physical fields $(\mathbf{E}, \mathbf{H})$. It turns out to be crucial in the analysis of photonic crystals to admit complex solutions.

where $L^2_\varepsilon(\mathbb{R}^3, \mathbb{C}^3)$ is the Banach space $L^2(\mathbb{R}^3, \mathbb{C}^3)$ with the scalar product twisted by the tensor ε and similarly for μ.
Even though the Hilbert space structure of $\mathfrak{H}_w$ depends crucially on the weights $w = (\varepsilon, \mu)$, the Assumption 1.1 implies the equivalence of the norm $\|\cdot\|_w$ with the usual $L^2(\mathbb{R}^3, \mathbb{C}^6)$-norm $\|\cdot\|$. This means that $\mathfrak{H}_w$ agrees with the usual $L^2(\mathbb{R}^3, \mathbb{C}^6)$ as Banach spaces. For many arguments in this paper, only the Banach space structure of $\mathfrak{H}_w$ is important, and thus, whenever convenient, we will use the canonical identification of $\mathfrak{H}_w \simeq L^2(\mathbb{R}^3, \mathbb{C}^6)$. In particular, any closed operator $\mathbf{T}$ on $\mathfrak{H}_w$ can also be seen as a closed operator on $L^2(\mathbb{R}^3, \mathbb{C}^6)$ which we denote with the same symbol. We will use the same notation for weighted L^2-spaces over $\mathbb{T}^3$: for instance, the Hilbert space

$$\mathfrak{h}_0 := L^2_\varepsilon(\mathbb{T}^3, \mathbb{C}^3) \oplus_\perp L^2_\mu(\mathbb{T}^3, \mathbb{C}^3)$$

is defined as the Banach space $L^2(\mathbb{T}^3, \mathbb{C}^6)$ equipped with a scalar product analogous to equation (12).
Let us turn to conventions regarding operators: Suppose $A : \mathfrak{D}_0(A) \subseteq \mathfrak{B}_1 \longrightarrow \mathfrak{B}_2$ is a possibly unbounded linear operator between the Banach spaces $\mathfrak{B}_1$ and $\mathfrak{B}_2$ defined on the dense domain $\mathfrak{D}_0(A)$. The operator A is called *closable* if and only if for every $\{\psi_n\} \subset \mathfrak{D}_0(A)$ such that $\psi_n \to 0$, then also $A\psi_n \to 0$. The *closure* of the operator A (still denoted with the same symbol) is the extension of A to $\mathfrak{D}(A) := \overline{\mathfrak{D}_0(A)}^{\|\cdot\|_A}$ with respect to the *graph norm*

$$\|\psi\|_A := \sqrt{\|\psi\|^2_{\mathfrak{B}_1} + \|A\psi\|^2_{\mathfrak{B}_2}}. \tag{13}$$

When $\mathfrak{D}_0(A) = \mathfrak{D}(A)$, the operator A is said to be *closed*. A *core* $\mathfrak{C}$ of a closed operator is any subset of $\mathfrak{D}(A)$ which is dense with respect to $\|\cdot\|_A$. Given any closed operator $A : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ between Banach spaces, the kernel (or null space) and range of A are defined as

$$\begin{aligned} \ker A &:= \bigl\{\psi \in \mathcal{B}_1 \mid A\psi = 0\bigr\} \subset \mathfrak{D}(A) \subseteq \mathfrak{B}_1, \\ \operatorname{ran}_0 A &:= \bigl\{A\psi \;\mid\; \psi \in \mathfrak{D}(A)\bigr\} \subseteq \mathfrak{B}_2 \end{aligned}$$

While $\ker A$ is automatically a closed subspace of $\mathfrak{B}_1$, in general $\operatorname{ran}_0 A$ is not. For this reason, we need to introduce its closure $\operatorname{ran} A := \overline{\operatorname{ran}_0 A}^{\|\cdot\|_{\mathfrak{B}_2}}$.
Other properties, most notably selfadjointness, crucially depend on the scalar product. Whenever the Hilbert structure of $\mathfrak{H}_w$ is important, we will make this explicit either in the text or in notation. To give one example, we distinguish between the *direct* sum $J \oplus G$ and the *orthogonal* sum $J \oplus_\perp G$ of vector spaces.
We found it convenient to use the shorthand $v^\times \psi := v \times \psi$ to associate the antisymmetric matrix

$$v^\times = \begin{pmatrix} 0 & -v_3 & +v_2 \\ +v_3 & 0 & -v_1 \\ -v_2 & +v_1 & 0 \end{pmatrix} \tag{14}$$

to any vectorial quantity $v = (v_1, v_2, v_3)$.

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2 The perturbed Maxwell operator

We will use this section to recall standard facts on the Maxwell operator [BS87, Kuc01] and introduce the main definitions and notions. This initial part is completed by a compendium of classical results in vector field analysis sketched in Appendix A.

2.1 General properties of the Maxwell operator

In order to identify the domain $\mathfrak{D}(\mathbf{M}_w)$ explicitly we start with the free case $\mathbf{M}_{w=(1,1)} = \mathbf{Rot}$ which is reviewed in detail in Appendix A.5. Assumption 1.1 on $w = (\varepsilon, \mu)$ implies that $\mathfrak{H}_w \simeq L^2(\mathbb{R}^3, \mathbb{C}^6)$ agree as Banach spaces and that W defines a bounded operator with bounded inverse. Moreover, $\mathbf{Rot}|_{\mathcal{C}_c^\infty}$ is a densely defined operators on $\mathfrak{H}_w$ and $\mathbf{Rot}$ is its unique closed extension defined on the domain $\mathfrak{D} := \mathfrak{D}(\mathbf{Rot})$ (cf. eq. (59)). Since, the graph norms $\|\cdot\|_{\mathbf{M}_w}$ and $\|\cdot\|_{\mathbf{Rot}}$ are equivalent, this immediately implies

$$\mathfrak{D}(\mathbf{M}_w) = \mathfrak{D} = \big(\ker \mathbf{Div} \cap H^1(\mathbb{R}^3, \mathbb{C}^6)\big) \oplus \operatorname{ran} \mathbf{Grad}, \tag{15}$$

because $\mathbf{M}_w|_{\mathcal{C}_c^\infty} = W\,\mathbf{Rot}|_{\mathcal{C}_c^\infty}$ is closable and its *unique* closure is the product of the bounded operator W and $(\mathbf{Rot}, \mathfrak{D})$.
The weighted scalar products (8) also implies $\mathbf{M}_w$ is not only closed but also symmetric, and thus, selfadjoint: for all $\Psi, \Phi \in \mathfrak{D}$, we have

$$\begin{aligned}\big\langle \Psi, \mathbf{M}_w \Phi \big\rangle_w &= \big\langle \Psi, W^{-1}\, W\, \mathbf{Rot}\, \Phi \big\rangle_{L^2(\mathbb{R}^3,\mathbb{C}^6)} = \big\langle \mathbf{Rot}\, \Psi, \Phi \big\rangle_{L^2(\mathbb{R}^3,\mathbb{C}^6)} \\ &= \big\langle W^{-1}\, W\, \mathbf{Rot}\, \Psi, \Phi \big\rangle_{L^2(\mathbb{R}^3,\mathbb{C}^6)} = \big\langle \mathbf{M}_w \Psi, \Phi \big\rangle_w .\end{aligned}$$

The weights in the scalar products imply that the Helmholtz-Hodge-Weyl-Leray decomposition of the domain (15) is no longer orthogonal with respect to $\langle \cdot\,, \cdot \rangle_w$. However, Theorem A.1 readily generalizes to the case with weights and yields an orthogonal splitting

$$\mathfrak{H}_w = \mathbf{J}_w \oplus_\perp \mathbf{G} \tag{16}$$

where we identify the *physical* (or *transversal*) subspace

$$\mathbf{J}_w = \ker\big(\mathbf{Div}\, W^{-1}\big) = \big\{ \Psi \in \mathfrak{H}_w \;\mid\; \mathbf{Div}\big(W^{-1}\Psi\big) = 0 \big\} = W\,\mathbf{J} \tag{17}$$

and the *unphysical* (or *longitudinal*) subspace

$$\mathbf{G} = \mathrm{ran}\,\mathbf{Grad} = \big\{ \Psi = \mathbf{Grad}\,\varphi \in \mathfrak{H}_w \;\mid\; \varphi \in L^2_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^2) \big\} = \ker \mathbf{Rot}. \tag{18}$$

We also call $\mathbf{G}$ the space of *zero modes*, because $\mathbf{G} = \ker \mathbf{Rot}$ coincides with $\ker \mathbf{M}_w$ as W has a bounded inverse. From the first equation of (8) we conclude that $\mathbf{J}_w = \mathbf{G}^{\perp_w}$ is the $\langle \cdot\,, \cdot \rangle_w$-orthogonal complement to $\mathbf{G}$. We will denote the orthogonal projections onto $\mathbf{J}_w$ and $\mathbf{G}$ with $\mathbf{P}_w$ and $\mathbf{Q}_w$. For later reference, we summarize these facts into a

Theorem 2.1 ([BS87]) *Suppose Assumption 1.1 on ε and μ is satisfied.*

*(i) The Maxwell operator $\mathbf{M}_w$ equipped with the (ε, μ)-*independent *domain*

$$\mathfrak{D} = \big(\mathfrak{D} \cap H^1(\mathbb{R}^3, \mathbb{C}^6)\big) \oplus \mathrm{ran}\,\mathbf{Grad} = \big(\ker \mathbf{Div} \cap H^1(\mathbb{R}^3, \mathbb{C}^6)\big) \oplus \mathbf{G}$$

defines a selfadjoint operator on $\mathfrak{H}_w$, and $H^1(\mathbb{R}^3, \mathbb{C}^6)$ and $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3, \mathbb{C}^6)$ are cores.

*(ii) The Maxwell operator $\mathbf{M}_w = \mathbf{M}_w|_{\mathbf{J}_w} \oplus_\perp 0|_{\mathbf{G}}$ is block diagonal with respect to the (ε, μ)-*dependent *orthogonal decomposition of $\mathfrak{H}_w = \mathbf{J}_w \oplus_\perp \mathbf{G}$. In this decomposition, the domain splits into*

$$\mathfrak{D} = \big(\mathfrak{D} \cap \mathbf{J}_w\big) \oplus_\perp \mathbf{G}\,.$$

(iii) The restrictions of $\mathbf{M}_w$ to $\mathbf{J}_w$ or $\mathbf{G}$ again define selfadjoint operators, and thus, the dynamics $\mathrm{e}^{-\mathrm{i}t\mathbf{M}_w}$ leave $\mathbf{J}_w$ and $\mathbf{G}$ invariant.

With the exception of the explicit computation of the domain, all of this is contained in [BS87, Lemma 2.2].
We have mentioned the significance of admitting *complex* vector fields in the introduction (cf. Footnote 1), and the question arises whether we can construct solutions by evolving $\Psi \in \mathfrak{H}_w$ in time and then taking real and imaginary part of $\Psi(t) = \mathrm{e}^{-\mathrm{i}t\mathbf{M}_w}\Psi$. This question will be crucial as to why usually one needs to consider "counter-propagating waves" whose frequencies $\pm\omega$ differ by a sign. So let $(C\Psi)(x) := \overline{\Psi(x)}$, $\Psi \in L^2(\mathbb{R}^3, \mathbb{C}^N)$, be component-wise complex conjugation; for simplicity, we shall always use the same symbol independently of $N \in \mathbb{N}$. *If ε and μ are real,* then the weights commute with C, and

$$\big(C\mathbf{M}_w C\Psi\big)^E = C\big(+\mathrm{i}\,\varepsilon^{-1}(\hat{x})\,\nabla^\times_x\big)C\psi^H = -\mathrm{i}\,\varepsilon^{-1}(\hat{x})\,\nabla_x \times \psi^H$$

as well as an analogous computation for the other component of $\mathbf{M}_w\Psi$ imply

$$C\,\mathbf{M}_w\,C = -\mathbf{M}_w\,. \tag{19}$$

Consequently, the spectra for Maxwell operators with real weights are symmetric with respect to reflections at 0; the same holds for all spectral components.

Theorem 2.2 *Suppose Assumption 1.1 on the weights ε and μ is satisfied, and assume in addition that they are real. Then equation* (19) *holds and thus the spectra* $\sigma(\mathbf{M}_w) = -\sigma(\mathbf{M}_w)$ *and* $\sigma_\sharp(\mathbf{M}_w) = -\sigma_\sharp(\mathbf{M}_w)$*,* $\sharp = \mathrm{pp}, \mathrm{ac}, \mathrm{sc}$*, are symmetric with respect to reflections about the origin* $0 \in \mathbb{R}$*.*

In case ε and μ have non-trivial complex offdiagonal entries, the weights no longer commute with complex conjugation, and (19) as well as the above theorem do not hold.

Remark 2.3 Symmetries of type (19), i. e. *anti*-unitary operators which map $\mathbf{M}_w$ onto $-\mathbf{M}_w$, are known in the physics literature as *particle-hole symmetries* or *PH symmetries* for short [AZ97, SRFL08]. However, as many physicists and mathematicians consider the second-order equation $\partial_t^2 \Psi = -\mathbf{M}_w^2 \Psi$ because it is block-diagonal, the PH symmetry for $\mathbf{M}_w$ is replaced by a *time-reversal symmetry* for the second-order equation. Ordinary Schrödinger operators $H = -\Delta_x + V$ on the other hand possess time-reversal symmetry, $C\,H\,C = H$. Discrete symmetries which square to $\pm\mathrm{id}$ have been classified systematically for topological insulators (cf. Table II in [SRFL08]); the presence of the PH symmetry means that $\mathbf{M}_w$ is in *symmetry class D* (provided there are no other symmetries). According to general results on the topological classification of band insulators (aka periodic operators), one expects that D-type operators in dimension $d = 2$ admit protected states parametrized by $\mathbb{Z}$-valued topological invariants (cf. Table I in [SRFL08]). This suggests there is an analog of the quantum Hall effect in 2-dimensional photonic crystals [RH08]. In contrast, for topological invariants to exist in $d = 3$, additional symmetries appear to be necessary (e. g. $\varepsilon = \mu$ or ε and μ have a common center of inversion); the presence of PH symmetry alone seems to prevent the formation of topologically protected states. Certainly, a direct proof for the Maxwell operator establishing the existence ($d = 2$) or absence ($d = 3$) of topological invariants would be an interesting avenue to explore.

2.2 Slow modulation of the Maxwell operator

One of the key differences between Maxwell and Schrödinger operators is that perturbations are *multiplicative* rather than *additive*. Given material weights ε and μ (which verify Assumption 1.1), we define their slow modulations $(\varepsilon_\lambda, \mu_\lambda)$ to be of the form (9). Assumption 1.2 for the modulation functions $(\tau_\varepsilon, \tau_\mu)$ ensures that also $(\varepsilon_\lambda, \mu_\lambda)$ satisfy Assumption 1.1 because they are again bounded away from 0 and $+\infty$.
We denote the λ-dependence of the weights with $w(\lambda) = (\varepsilon_\lambda, \mu_\lambda)$ and define shorthand notation for the λ-dependent family of Hilbert spaces, projections and Maxwell operators by setting

$$\begin{aligned} &\mathfrak{H}_\lambda := \mathfrak{H}_{w(\lambda)}\,, && \mathbf{J}_\lambda := \mathbf{J}_{w(\lambda)} && && \text{(spaces)} \\ &\mathbf{M}_\lambda := \mathbf{M}_{w(\lambda)}\,, && \mathbf{P}_\lambda := \mathbf{P}_{w(\lambda)}\,, && \mathbf{Q}_\lambda := \mathbf{Q}_{w(\lambda)} && \text{(operators)}\,. \end{aligned}$$

Similarly, we will denote the scalar product and norm of $\mathfrak{H}_\lambda$ by $\langle \cdot \,, \cdot \rangle_\lambda$ and $\lVert \cdot \rVert_\lambda$.

To compare these operators for different values of λ, we will represent them on a *common, λ-independent* Hilbert space: the scaling operator

$$S(\lambda\hat{x}) : \mathfrak{H}_\lambda \longrightarrow \mathfrak{H}_0, \quad S(\lambda\hat{x}) = \begin{pmatrix} \tau_\varepsilon^{-1}(\lambda\hat{x}) & 0 \\ 0 & \tau_\mu^{-1}(\lambda\hat{x}) \end{pmatrix}, \tag{20}$$

is a unitary since it is surjective and preserves scalar products. The Maxwell operator in this new representation can be calculated explicitly: for instance, the upper-right matrix element of $\mathbf{M}_\lambda$ transforms to

$$\begin{aligned} &\tau_\varepsilon^{-1}(\lambda\hat{x}) \left(-\tau_\varepsilon^2(\lambda\hat{x})\, \varepsilon^{-1}(\hat{x})\, (-\mathrm{i}\nabla_x)^\times\right) \tau_\mu(\lambda\hat{x}) = \\ &\qquad = -\tau_\varepsilon(\lambda\hat{x})\, \tau_\mu(\lambda\hat{x}) \Big(\varepsilon^{-1}(\hat{x})\, (-\mathrm{i}\nabla_x)^\times + \lambda\, \varepsilon^{-1}(\hat{x}) \big(-\mathrm{i}\nabla_x \ln \tau_\mu\big)^\times (\lambda\hat{x})\Big), \end{aligned}$$

and if we introduce the functions $\tau(\lambda x) := \tau_\varepsilon(\lambda x)\, \tau_\mu(\lambda x)$ and

$$\Upsilon(\lambda x) := \begin{pmatrix} 0 & +\mathrm{i}\big(\nabla_x \ln \tau_\mu\big)^\times(\lambda x) \\ -\mathrm{i}\big(\nabla_x \ln \tau_\varepsilon\big)^\times(\lambda x) & 0 \end{pmatrix},$$

we can write the Maxwell operator as

$$\begin{aligned} M_\lambda &:= S(\lambda\hat{x})\, \mathbf{M}_\lambda\, S(\lambda\hat{x})^{-1} = M_0 + \lambda\, M_1 \\ &= \tau(\lambda\hat{x})\, \mathbf{M}_0 + \lambda\, \tau(\lambda\hat{x})\, W\, \Upsilon(\lambda\hat{x}). \end{aligned} \tag{21}$$

As a product of bounded multiplication operators, M_1 is an element of $\mathcal{B}(\mathfrak{H}_0)$. The regularity of τ_ε and τ_μ also ensures the domain is preserved.

Lemma 2.4 *$S(\lambda\hat{x})$ maps $\mathfrak{D}$ bijectively onto itself.*

This means all of the operators, $\mathbf{M}_0$, $\mathbf{M}_\lambda$ and M_λ, have the same λ-independent domain $\mathfrak{D}$ and cores (e. g. $H^1(\mathbb{R}^3, \mathbb{C}^6)$) – even though the splitting of the domain into physical and unphysical subspaces depends on λ. We denote the invariant subspaces

$$J_\lambda := S(\lambda\hat{x})\, \mathbf{J}_\lambda, \qquad G_\lambda := S(\lambda\hat{x})\, \mathbf{G}$$

of M_λ with regular letters instead of bold letters, and in the same vein, the corresponding projections are

$$P_\lambda := S(\lambda\hat{x})\, \mathbf{P}_\lambda\, S(\lambda\hat{x})^{-1}, \qquad Q_\lambda := S(\lambda\hat{x})\, \mathbf{Q}_\lambda\, S(\lambda\hat{x})^{-1}.$$

For $\lambda = 0$, the λ-independent representation coincides with the physical representation since $S(\lambda\hat{x})|_{\lambda=0} = \mathrm{id}_{\mathfrak{H}_0}$ reduces to the identity by Assumption 1.2, and we have $J_0 = \mathbf{J}_0$ and $G_0 = \mathbf{G}$ for the subspaces, as well as $P_0 = \mathbf{P}_0$ and $Q_0 = \mathbf{Q}_0$ for the corresponding projections.
The unitarity of $S(\lambda\hat{x})$ and Theorem 2.1 imply $\mathfrak{H}_0 = J_\lambda \oplus_\perp G_\lambda$ is a λ-dependent decomposition of $\mathfrak{H}_0$ into $\langle \cdot\,, \cdot \rangle_0$-orthogonal subspaces which are invariant under the dynamics $\mathrm{e}^{-\mathrm{i}tM_\lambda}$.

3 Properties of the periodic Maxwell operator

Photonic crystals are materials where the unperturbed material weights (ε, μ) are periodic with respect to a lattice

$$\Gamma := \mathrm{span}_{\mathbb{Z}}\{e_1, e_2, e_3\} \cong \mathbb{Z}^3 \,,$$

and henceforth, we shall always make the following

Assumption 3.1 (Photonic crystal) *Suppose that ε and μ are Γ-periodic and satisfy Assumption 1.1.*

The lattice periodicity suggests we borrow the language of crystalline solids [GP03]: we can decompose vectors $x = y + \gamma$ in real space $\mathbb{R}^3 \cong \mathbb{W} \times \Gamma$ into a component y which lies in the so-called Wigner–Seitz cell $\mathbb{W}$ and a lattice vector $\gamma \in \Gamma$. Whenever convenient we will identify this fundamental cell $\mathbb{W}$ with the 3-dimensional torus $\mathbb{T}^3$.
Given a lattice Γ, then there is a canonical way to decompose momentum space $\mathbb{R}^3 \cong \mathbb{B} \times \Gamma^*$: here, the *dual lattice* $\Gamma^* = \mathrm{span}_{\mathbb{Z}}\{e_1^*, e_2^*, e_3^*\}$ is generated by the family of vectors which are defined through the relations $e_j \cdot e_n^* = 2\pi\, \delta_{jn}$, $j, n = 1, 2, 3$. The standard choice of fundamental cell

$$\mathbb{B} := \Big\{ \textstyle\sum_{j=1}^3 \alpha_j \, e_j^* \in \mathbb{R}^3 \;\big|\; \alpha_1, \alpha_2, \alpha_3 \in [-1/2, +1/2) \Big\}$$

is called (first) Brillouin zone, and elements $k \in \mathbb{B}$ are known as *crystal momentum.*

3.1 The Zak transform

The lattice-periodicity of ε and μ sugests to use a Fourier basis: for any $\mathbb{C}^N$-valued Schwartz function $\Psi \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^N)$ we define the *Zak transform* [Zak68] evaluated at $k \in \mathbb{R}^3$ and $y \in \mathbb{R}^3$ as

$$(\mathcal{Z}\Psi)(k, y) := \sum_{\gamma \in \Gamma} \mathrm{e}^{-\mathrm{i}k \cdot (y+\gamma)} \, \Psi(y + \gamma) \,. \tag{22}$$

The Zak transform is a variant of the Bloch-Floquet transform with the following periodicity properties:

$$\begin{aligned} (\mathcal{Z}\Psi)(k, y - \gamma) &= (\mathcal{Z}\Psi)(k, y) && \gamma \in \Gamma \\ (\mathcal{Z}\Psi)(k - \gamma^*, y) &= \mathrm{e}^{+\mathrm{i}\gamma^* \cdot y} (\mathcal{Z}\Psi)(k, y) && \gamma^* \in \Gamma^* \end{aligned}$$

In other words, $\mathcal{Z}\Psi$ is a Γ-periodic function in y and periodic up to a phase in k. The Schwartz functions are dense in $\mathfrak{H}_0$, so

$$\mathcal{Z} : \mathfrak{H}_0 \longrightarrow L^2_{\mathrm{eq}}(\mathbb{R}^3, \mathfrak{h}_0) \cong L^2(\mathbb{B}) \otimes \mathfrak{h}_0$$

extends to a unitary map between $\mathfrak{H}_0$ and the L^2-space of equivariant functions in k with values in $\mathfrak{h}_0 := L^2_{\varepsilon}(\mathbb{T}^3, \mathbb{C}^3) \oplus_{\perp} L^2_{\mu}(\mathbb{T}^3, \mathbb{C}^3)$,

$$L^2_{\mathrm{eq}}(\mathbb{R}^3, \mathfrak{h}_0) := \Bigl\{ \Psi \in L^2_{\mathrm{loc}}(\mathbb{R}^3, \mathfrak{h}_0) \bigm\vert \Psi(k - \gamma^*) = \mathrm{e}^{+\mathrm{i}\gamma^* \cdot \hat{y}} \Psi(k) \text{ a. e. } \forall \gamma^* \in \Gamma^* \Bigr\} , \tag{23}$$

which is equipped with the scalar product

$$\bigl\langle \Psi, \Phi \bigr\rangle_{\mathrm{eq}} := \int_{\mathbb{B}} \mathrm{d}k \, \bigl\langle \Psi(k), \Phi(k) \bigr\rangle_{\mathfrak{h}_0}$$

where

$$\begin{aligned} \bigl\langle \Psi(k), \Phi(k) \bigr\rangle_{\mathfrak{h}_0} := \int_{\mathbb{T}^3} \mathrm{d}y \; \psi^E(k,y) \cdot \varepsilon(y) \phi^E(k,y) \, + \\ + \int_{\mathbb{T}^3} \mathrm{d}y \; \psi^H(k,y) \cdot \mu(y) \phi^H(k,y) \, . \end{aligned}$$

Due to the (quasi-)periodicity of Zak transformed functions, they are uniquely determined by the values they take on $\mathbb{B} \times \mathbb{T}^3$.
To see how the Maxwell operator transforms when conjugating it with $\mathcal{Z}$, we compute the Zak representation of its building block operators positions $\hat{x}$ and momentum $-\mathrm{i}\nabla_x$ (which are equipped with the obvious domains):

$$\mathcal{Z} \, \hat{x} \, \mathcal{Z}^{-1} = \mathrm{i}\nabla_k \tag{24}$$

$$\mathcal{Z} \, (-\mathrm{i}\nabla_x) \, \mathcal{Z}^{-1} = \mathrm{id}_{L^2(\mathbb{B})} \otimes (-\mathrm{i}\nabla_y) + \hat{k} \otimes \mathrm{id}_{\mathfrak{h}_0} \equiv -\mathrm{i}\nabla_y + \hat{k} \tag{25}$$

The common domains of the components $\mathrm{i}\partial_{k_j}$ and $-\mathrm{i}\partial_{y_j} + \hat{k}_j$ Zak transform to $L^2_{\mathrm{eq}}(\mathbb{R}^3, \mathfrak{h}_0) \cap H^1_{\mathrm{loc}}\bigl(\mathbb{R}^3, \mathfrak{h}_0\bigr)$ and

$$\mathcal{Z} H^1(\mathbb{R}^3, \mathbb{C}^6) = L^2_{\mathrm{eq}}\bigl(\mathbb{R}^3, H^1(\mathbb{T}^3, \mathbb{C}^6)\bigr) \cong L^2(\mathbb{B}) \otimes H^1(\mathbb{T}^3, \mathbb{C}^6) \, . \tag{26}$$

Note that the position operator in Zak representation does not factor, unless we consider Γ-periodic functions ε,

$$\mathcal{Z} \, \varepsilon(\hat{x}) \, \mathcal{Z}^{-1} = \mathrm{id}_{L^2(\mathbb{B})} \otimes \varepsilon(\hat{y}) \equiv \varepsilon(\hat{y}) \, . \tag{27}$$

Operators $\mathbf{A} : \mathfrak{D}(A) \longrightarrow \mathfrak{H}_0$ which commute with lattice translations, e. g. operators of the form (25), (27) or the periodic Maxwell operator, fiber in k,

$$\mathbf{A}^{\mathcal{Z}} = \mathcal{Z} \, \mathbf{A} \, \mathcal{Z}^{-1} = \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \, \mathbf{A}(k) \, ,$$

and the fiber operators at $k \in \mathbb{R}^3$ and $k - \gamma^*$, $\gamma^* \in \Gamma^*$, are unitarily equivalent,

$$\mathbf{A}(k - \gamma^*) = \mathrm{e}^{+\mathrm{i}\gamma^* \cdot \hat{y}} \, \mathbf{A}(k) \, \mathrm{e}^{-\mathrm{i}\gamma^* \cdot \hat{y}} \, , \tag{28}$$

Operator-valued functions $k \mapsto \mathbf{A}(k)$ which satisfy (28) are called *equivariant*. It is for this reason that it suffices to consider all objects only for $k \in \mathbb{B}$ and extend them by equivariance if necessary.

3.2 Analytic decomposition of the fiber Hilbert space

Clearly, Q_0 and P_0 also commute with lattice translations, and thus, the Zak transform yields a fiber decomposition into

$$Q_0^{\mathcal{Z}} := \mathcal{Z}\, Q_0\, \mathcal{Z}^{-1} = \int_{\mathbb{B}}^{\oplus} \mathrm{d}k\, Q_0(k)\,, \qquad P_0^{\mathcal{Z}} := \mathcal{Z}\, P_0\, \mathcal{Z}^{-1} = \int_{\mathbb{B}}^{\oplus} \mathrm{d}k\, P_0(k)\,.$$

These fibrations also identify physical and unphysical subspaces of the fiber Hilbert space

$$\mathfrak{h}_0 = J_0(k) \oplus_{\perp} G_0(k)$$

for each $k \in \mathbb{B}$ where $G_0(k) = \operatorname{ran} Q_0(k)$ and $J_0(k) = \operatorname{ran} P_0(k)$. A priori, all we know is that this fibration is *measurable* in k. However, we are interested in the *analyticity* properties of the fiber projections. Figotin and Kuchment have recognized that $k \mapsto Q_0(k)$ and thus also $k \mapsto P_0(k)$ are not analytic at $k \in \Gamma^*$ [FK96a]. The purpose of this section is to define *regularized* projections $k \mapsto Q_0^{\mathrm{reg}}(k)$ and $k \mapsto P_0^{\mathrm{reg}}(k)$ which are analytic on *all* of $\mathbb{R}^3$. These regularized projections enter crucially in the proof on the existence of ground state bands (Theorem 1.4 (iii)).

Lemma 3.2 *(i) The orthogonal projections $k \mapsto Q_0(k)$ and $k \mapsto P_0(k)$ onto unphysical and physical subspace are analytic on $\mathbb{R}^3 \setminus \Gamma^*$.*

(ii) The regularized orthogonal projections $k \mapsto Q_0^{\mathrm{reg}}(k)$ and $k \mapsto P_0^{\mathrm{reg}}(k)$ are analytic on all *of $\mathbb{R}^3$. Moreover, $P_0^{\mathrm{reg}}(\gamma^*) = P_0(\gamma^*)$ and $Q_0^{\mathrm{reg}}(\gamma^*) = Q_0(\gamma^*)$ holds for all $\gamma^* \in \Gamma^*$.*

(iii) $\dim\big(G_0(k) \cap J_0^{\mathrm{reg}}(k)\big) = 2$ for all $k \in \mathbb{R}^3 \setminus \Gamma^*$

Essentially, the idea for the definition of $Q_0^{\mathrm{reg}}(k)$ is already contained in the proofs of Lemma 51 and Corollary 52 of [FK96a], so we will briefly sketch the construction of $Q_0(k)$ and then proceed to define $Q_0^{\mathrm{reg}}(k)$.

Assume from now on that $k \in \mathbb{B}$. The idea is to use the fact that $G_0(k) := \operatorname{ran}_0 \mathbf{Grad}(k)$ and define an auxiliary projection $\widetilde{Q}_0(k) = \mathbf{Grad}(k)\, T(k)$ with range $G_0(k)$ as the product of the operator

$$\mathbf{Grad}(k) = (\nabla_y + \mathrm{i}k, \nabla_y + \mathrm{i}k) : H^1(\mathbb{T}^3, \mathbb{C}^2) \longrightarrow \mathfrak{h}_0.$$

which depends analytically on $k \in \mathbb{R}^3$ and its left-inverse $T(k)$. Such a left-inverse exists if and only if $\mathbf{Grad}(k)$ is injective, and *if* it exists, it is also bounded [FK96a, p. 52] and analytic in k [ZKKP75, Theorem 4.4]. Note that the closedness of $\operatorname{ran}_0 \mathbf{Grad}(k) = \mathbf{Grad}(k)\, H^1(\mathbb{T}^3, \mathbb{C}^2)$ for $k \neq 0$ follows from the boundedness of $T(k)$.

One can check that for $k \neq 0$, the operator $\mathbf{Grad}(k)$ is injective while for $k = 0$ there are zero modes,

$$Z(\mathbb{T}^3, \mathbb{C}^2) := \left\{ y \mapsto \begin{pmatrix} \beta^E \\ \beta^H \end{pmatrix} \;\middle|\; \begin{pmatrix} \beta^E \\ \beta^H \end{pmatrix} \in \mathbb{C}^2 \right\} = \ker \mathbf{Grad}(0)\,.$$

Consequently, the projection $\widetilde{Q}_0(k) = \mathbf{Grad}(k)\, T(k)$ can only be defined in this fashion for $k \neq 0$, and there is a point of non-analyticity at $k = 0$, because $\mathrm{ran}\, \mathbf{Grad}(0)$ is "smaller" by two dimensions than $G_0(k)$, $k \neq 0$.
Even though $\widetilde{Q}_0(k)$ need not be an orthogonal projection (the proofs in [All67] and [ZKKP75] only make reference to the Banach algebra structure), these arguments show that $G_0(k) = \mathrm{ran}\, Q_0(k) = \mathrm{ran}\, \widetilde{Q}_0(k)$ depends analytically on k away from Γ^*. Thus, the unique *orthogonal* projection $Q_0(k)$ onto $G_0(k)$ necessarily also depends analytically on $k \in \mathbb{R}^3 \setminus \Gamma^*$.
The behavior of $\mathbf{Grad}(k)$ at $k = 0$ suggests to define the *regularized* unphysical space as

$$G_0^{\mathrm{reg}}(k) := \mathrm{ran}_0\, \mathbf{Grad}(k)|_{H^1_{\mathrm{reg}}}$$

where the closed subspace

$$\begin{aligned} H^1_{\mathrm{reg}}(\mathbb{T}^3, \mathbb{C}^2) &:= \Big\{ \varphi = (\varphi^E, \varphi^H) \in H^1(\mathbb{T}^3, \mathbb{C}^2) \;\big|\; \big\langle 1, \varphi^\sharp \big\rangle_{L^2(\mathbb{T}^3)} = 0, \; \sharp = E, H \Big\} \\ &= Z(\mathbb{T}^3, \mathbb{C}^2)^\perp \cap H^1(\mathbb{T}^3, \mathbb{C}^2) \end{aligned}$$

consists of all H^1-functions orthogonal to the constant functions. Now $\mathbf{Grad}(k)|_{H^1_{\mathrm{reg}}}$ *is* injective for all $k \in \mathbb{B}$, and by modifying the estimates on [FK96a, p. 52] we deduce there exists an *analytic* bounded left-inverse $T_{\mathrm{reg}}(k)$ for all $k \in \mathbb{B}$. Hence, the composition

$$k \mapsto \widetilde{Q_0^{\mathrm{reg}}}(k) := \mathbf{Grad}(k)|_{H^1_{\mathrm{reg}}}\, T_{\mathrm{reg}}(k)$$

defines a projection onto $G_0^{\mathrm{reg}}(k)$ that depends analytically on k for all of $\mathbb{B}$, including $k = 0$; again, the boundedness of $T_{\mathrm{reg}}(k)$ implies $G_0^{\mathrm{reg}}(k)$ is a closed subset of $\mathfrak{h}_0$. By the same arguments as above, the uniquely determined *orthogonal* projection $Q_0^{\mathrm{reg}}(k)$ onto $G_0^{\mathrm{reg}}(k)$ inherits the analyticity of $\widetilde{Q_0^{\mathrm{reg}}}(k)$ [Kat95, Theorem 6.35]. At $k = 0$, this regularized projection coincides with the usual one, $Q_0^{\mathrm{reg}}(0) = Q_0(0)$, as their ranges

$$G_0^{\mathrm{reg}}(0) = \mathrm{ran}\, \mathbf{Grad}(0)|_{H^1_{\mathrm{reg}}} = \mathrm{ran}\, \mathbf{Grad}(0) = G_0(0) \tag{29}$$

are the same (this also proves that $G_0(0)$ is closed). Moreover, $k \mapsto Q_0^{\mathrm{reg}}(k)$ has a unique extension by equivariance (cf. (28)) to all of $\mathbb{R}^3$.
Now the analyticity of the orthogonal projection

$$P_0^{\mathrm{reg}}(k) := \mathrm{id}_{\mathfrak{h}_0} - Q_0^{\mathrm{reg}}(k)$$

onto the $\langle \cdot\,, \cdot \rangle_{\mathfrak{h}_0}$-orthogonal complement

$$J_0^{\mathrm{reg}}(k) := G_0^{\mathrm{reg}}(k)^\perp$$

follows from the analyticity of $k \mapsto Q_0^{\mathrm{reg}}(k)$.

Before we prove (iii), it is instructive to juxtapose the decomposition $\mathfrak{h}_0 = J_0(k) \oplus_\perp G_0(k)$ with the regularized decomposition

$$\mathfrak{h}_0 = J_0^{\mathrm{reg}}(k) \oplus_\perp G_0^{\mathrm{reg}}(k)$$

for the special case $\mathbf{M}_0 = \mathbf{Rot}$, i. e. $\varepsilon = 1 = \mu$. The difference between the two is how the constant functions $y \mapsto (\alpha^E, \alpha^H) \in \mathbb{C}^6$, are distributed amongst them: for $k \neq 0$ only *certain* constant functions belong to $J_0(k)$,

$$y \mapsto \begin{pmatrix} \alpha^E \\ \alpha^H \end{pmatrix} \in J_0(k) \quad \Longleftrightarrow \quad \mathbf{Div}(k) \begin{pmatrix} \alpha^E \\ \alpha^H \end{pmatrix} = -\mathrm{i} \begin{pmatrix} k \cdot \alpha^E \\ k \cdot \alpha^H \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

while for $k = 0$ *all* constant functions are elements of $J_0(0)$ and the physical subspace "grows" by 2 dimensions at the expense of $G_0(0)$. In contrast, the regularized physical subspace $J_0^{\mathrm{reg}}(k)$ contains all constant functions for *all* values of k. We will now extend these arguments to the case of non-trivial weights (ε, μ).

PROOF (LEMMA 3.2) We have already shown (i) and (ii) in the text preceding the lemma and it remains to prove (iii). Without loss of generality, we restrict ourselves to $k \in \mathbb{B}$. First of all, we note that the unphysical subspace

$$G_0(k) = \left\{ \sum_{\gamma^* \in \Gamma^*} \begin{pmatrix} \beta^E(\gamma^*)\,(\gamma^* + k) \\ \beta^H(\gamma^*)\,(\gamma^* + k) \end{pmatrix} \mathrm{e}^{+\mathrm{i}\gamma^* \cdot y} \;\middle|\; \left\{ \left| \beta^\sharp(\gamma^*)\,\gamma^* \right| \right\}_{\gamma^* \in \Gamma^*} \in \ell^2(\Gamma^*),\ \sharp = E, H \right\}$$

and the *regularized* unphysical subspace

$$G_0^{\mathrm{reg}}(k) = \left\{ \sum_{\gamma^* \in \Gamma^* \setminus \{0\}} \begin{pmatrix} \beta^E(\gamma^*)\,(\gamma^* + k) \\ \beta^H(\gamma^*)\,(\gamma^* + k) \end{pmatrix} \mathrm{e}^{+\mathrm{i}\gamma^* \cdot y} \;\middle|\; \left\{ \left| \beta^\sharp(\gamma^*)\,\gamma^* \right| \right\}_{\gamma^* \in \Gamma^*} \in \ell^2(\Gamma^*),\ \sharp = E, H \right\}. \tag{30}$$

coincide for $k = 0$, and we immediately deduce

$$\dim\bigl(G_0(0) \cap J_0^{\mathrm{reg}}(0)\bigr) = \dim\bigl(G_0(0) \cap J_0(0)\bigr) = 0\,.$$

Hence, we assume from now on $k \in \mathbb{B} \setminus \{0\}$. That means, we can write the intersection as the regularized projection applied to a two-dimensional subspace,

$$G_0(k) \cap J_0^{\mathrm{reg}}(k) = P_0^{\mathrm{reg}}(k) \left\{ y \mapsto \begin{pmatrix} \beta^E\, k \\ \beta^H\, k \end{pmatrix} \;\middle|\; \beta^E, \beta^H \in \mathbb{C} \right\}.$$

The image is again two-dimensional: if we write any $\Psi = \Psi_Q \oplus_\perp \Psi_P \in \mathfrak{h}_0$ as the sum of $\Psi_Q \in G_0^{\mathrm{reg}}(k)$ and $\Psi_P \in J_0^{\mathrm{reg}}(k)$, then in view of equation (30) the $\gamma^* = 0$ Fourier coefficient of $\psi_Q = Q_0^{\mathrm{reg}}(k)\Psi$ necessarily has to vanish, $\widehat{\psi}_Q(0) = 0$. Thus, $\widehat{\psi}_P(0) = \widehat{\psi}(0)$ follows, and the map

$$\mathbb{C}^2 \ni \begin{pmatrix} \beta^E \\ \beta^H \end{pmatrix} \mapsto P_0^{\mathrm{reg}}(k) \begin{pmatrix} \beta^E \, k \\ \beta^H \, k \end{pmatrix} \in J_0^{\mathrm{reg}}(k)$$

is injective. That means $\dim\big(G_0(k) \cap J_0^{\mathrm{reg}}(k)\big) = 2$ for $k \in \mathbb{R}^3 \setminus \Gamma^*$. □

3.3 Analyticity properties of the fiber Maxwell operator

The Zak transform fibers the periodic Maxwell operator in crystal momentum,

$$\mathbf{M}_0^{\mathcal{Z}} := \mathcal{Z}\, \mathbf{M}_0 \, \mathcal{Z}^{-1} = \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \, \mathbf{M}_0(k) \, . \tag{31}$$

Each of the fiber operators

$$\mathbf{M}_0(k) = W \, \mathbf{Rot}(k) = \begin{pmatrix} 0 & -\varepsilon^{-1} \, (-\mathrm{i}\nabla_y + k)^{\times} \\ +\mu^{-1} \, (-\mathrm{i}\nabla_y + k)^{\times} & 0 \end{pmatrix} ,$$

acts on a *potentially k-dependent* subspace $\mathfrak{d}(k)$ of $\mathfrak{h}_0$, and has a splitting into physical and unphysical part, $\mathbf{M}_0(k) = \mathbf{M}_0(k)|_{J_0(k)} \oplus 0|_{G_0(k)}$. In any case, the selfadjointness of $\mathbf{M}_0$ on $\mathfrak{D}$ implies the selfadjointness of each fiber operator $\mathbf{M}_0(k)$ on $\mathfrak{D}(k)$. Since the domain of each fiber operator $\mathbf{M}_0(k)$ may depend on k, it is not obvious whether $k \mapsto \mathbf{M}_0(k)$ is analytic in k even though the operator prescription is linear.

Proposition 3.3 (Analyticity) *Suppose Assumption 3.1 on ε and μ holds.*

(i) The domain of selfadjointness

$$\mathfrak{d} = \big(\ker \mathbf{Div}(k) \cap H^1(\mathbb{T}^3, \mathbb{C}^6)\big) \oplus \operatorname{ran} \mathbf{Grad}(k) \tag{32}$$

of $\mathbf{M}_0(k)$ is independent of k.

(ii) The map $\mathbb{R}^3 \ni k \mapsto \mathbf{M}_0(k) \in \mathcal{B}(\mathfrak{d}, \mathfrak{h}_0)$ is analytic.

Proof (i) Since $H^1(\mathbb{R}^3, \mathbb{C}^6)$ is a core for $\mathbf{M}_0$ (Theorem 2.1 (i)) and (26), we know that $H^1(\mathbb{T}^3, \mathbb{C}^6)$ is a common core of $\mathbf{M}_0(k)$ for all values of k. Moreover, combining equations (59) and (26) with the fact that **Div** and **Grad** also fiber in k yields the decomposition of $\mathfrak{d}$ as a k-dependent direct sum as given by (32).

The difference of the two fiber operators restricted to $H^1(\mathbb{T}^3, \mathbb{C}^6)$ extends to a bounded operator on all of $\mathfrak{h}_0$,

$$\begin{aligned} \mathbf{M}_0(k)|_{H^1} - \mathbf{M}_0(k')|_{H^1} &= W \begin{pmatrix} 0 & -(k-k')^{\times} \\ +(k-k')^{\times} & 0 \end{pmatrix} \\ &=: \sum_{j=1}^{3} (k_j - k'_j) \, \mathbf{A}_j =: (k-k') \cdot \mathbf{A} \, . \end{aligned} \tag{33}$$

Using $\left\|k \cdot \mathbf{A}\right\|_{\mathcal{B}(\mathfrak{h}_0)} = |k| \, \|W\|_{\mathcal{B}(\mathfrak{h}_0)}$, it is straightforward to see that these graph norms of $\mathbf{M}_0(k)$ and $\mathbf{M}_0(0)$ are equivalent on $H^1(\mathbb{T}^3, \mathbb{C}^6)$,

$$\big(1 + |k| \; \|W\|\big)^{-1} \, \|\Psi\|_{\mathbf{M}_0(0)} \leq \|\Psi\|_{\mathbf{M}_0(k)} \leq \big(1 + |k| \; \|W\|\big) \, \|\Psi\|_{\mathbf{M}_0(0)} \, .$$

The equivalence of the graph norms now implies that the domains, seen as completions of $H^1(\mathbb{T}^3, \mathbb{C}^6)$ with respect to these graph norms, are independent of k,

$$\mathfrak{d}(k) = \overline{H^1(\mathbb{T}^3, \mathbb{C}^6)}^{\|\cdot\|_{\mathbf{M}_0(k)}} = \overline{H^1(\mathbb{T}^3, \mathbb{C}^6)}^{\|\cdot\|_{\mathbf{M}_0(0)}} = \mathfrak{d}(0) \, .$$

(ii) By (i) the domain $\mathfrak{d}$ of each $\mathbf{M}_0(k)$ is independent of k, and thus the analyticity of the linear polynomial $k \mapsto \mathbf{M}_0(k)$ is trivial. □

The fibration of $\mathbf{M}_0^{\mathcal{Z}}$ can be used to extract a great deal of information on the spectra of $\mathbf{M}_0$ and $\mathbf{M}_0(k)$:

Theorem 3.4 (Spectral properties) *Suppose Assumption 3.1 on ε and μ is satisfied. Then for any $k \in \mathbb{R}^3$ the following holds true:*

(i) $\sigma\big(\mathbf{M}_0(k)|_{G_0(k)}\big) = \sigma_{\mathrm{ess}}\big(\mathbf{M}_0(k)|_{G_0(k)}\big) = \sigma_{\mathrm{pp}}\big(\mathbf{M}_0(k)|_{G_0(k)}\big) = \{0\}$

(ii) $\sigma\big(\mathbf{M}_0(k)|_{J_0(k)}\big) = \sigma_{\mathrm{disc}}\big(\mathbf{M}_0(k)|_{J_0(k)}\big)$

(iii) $\sigma\big(\mathbf{M}_0(k)|_{J_0^{\mathrm{reg}}(k)}\big) = \sigma_{\mathrm{disc}}\big(\mathbf{M}_0(k)|_{J_0^{\mathrm{reg}}(k)}\big) = \sigma\big(\mathbf{M}_0(k)\big)$

(iv) $\sigma(\mathbf{M}_0) = \bigcup_{k \in \mathbb{B}} \sigma\big(\mathbf{M}_0(k)\big) = \bigcup_{k \in \mathbb{R}^3} \sigma\big(\mathbf{M}_0(k)\big)$

(v) $\sigma(\mathbf{M}_0) = \sigma_{\mathrm{ac}}(\mathbf{M}_0) \cup \sigma_{\mathrm{pp}}(\mathbf{M}_0)$

Proof (i) For any $\varphi \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^3, \mathbb{C}^2)$, the vector $\mathbf{Grad}(\varphi) \in G_0$ is an element of the unphysical subspace, and thus we have found an eigenvector to 0,

$$\mathbf{M}_0(k)(\mathcal{Z}\Psi)(k) = \big(\mathcal{Z}\mathbf{M}_0\Psi\big)(k) = 0.$$

This means we have found a countably infinite family of eigenvectors, and we have shown (i).

(ii) According to Lemma A.4, $\big(\mathbf{Rot}(k)|_{J_{\mathbf{Rot}}(k)} - z\big)^{-1}$ is compact for all $k \in \mathbb{R}^3$ where $J_{\mathbf{Rot}}(k) = \ker \mathbf{Div}(k)$ is the physical subspace for the free Maxwell operator. Because we can write $\big(\mathbf{M}_0(k)|_{J_0(k)} - z\big)^{-1}$ as a product of bounded operators and $\big(\mathbf{Rot}(k)|_{J_{\mathbf{Rot}}(k)} - z\big)^{-1}$ [SEK+05, equation (4.23)], the resolvent of $\mathbf{M}_0(k)|_{J_0(k)}$ is also compact. Thus, the spectrum of $\mathbf{M}_0(k)|_{J_0(k)}$ is purely discrete.

(iii) This follows from (ii) and the observation that by Lemma 3.2 (iii), $J_0(k)$ and $J_0^{\mathrm{reg}}(k)$ differ by an at most 2-dimensional subspace $J_0^{\mathrm{reg}}(k) \cap G_0(k)$.

(iv) The proof is analogous to that of [FK96a, Corollary 57].

(v) From (iv) we know that $\sigma(\mathbf{M}_0)$ can be written as the union of the spectra of the fiber operators $\mathbf{M}_0(k)$. Because these spectra $\sigma\big(\mathbf{M}_0(k)\big) = \big\{\omega_n(k)\big\}_{n\in\mathbb{Z}}$ in turn can be expressed in terms of *piecewise analytic* frequency band functions $k \mapsto \omega_n(k)$, $n \in \mathbb{Z}$ (cf. Theorem 1.4 (i)), $\sigma_{\mathrm{sc}}(\mathbf{M}_0)$ must be empty. □

Remark 3.5 (Absolute continuity of $\sigma\big(\mathbf{M}_0|_{J_0}\big)$) Unlike in the case of periodic Schrödinger operators, it has not yet been proven that the spectrum of $\mathbf{M}_0|_{J_0}$ is purely absolutely continuous. To show $\sigma\big(\mathbf{M}_0|_{J_0}\big) = \sigma_{\mathrm{ac}}\big(\mathbf{M}_0|_{J_0}\big)$, all of the known proofs reduce the Maxwell operator to a possibly non-selfadjoint Schrödinger-type operator with magnetic field, and these transformations involve derivatives of ε and μ [Mor00, Sus00, KL01]. Hence, one needs additional regularity assumptions on ε and μ; the best currently known are $\varepsilon, \mu \in \mathcal{C}^1(\mathbb{R}^3)$ [KL01, Section 7.4]. This means, even though it is widely expected that the spectrum is always purely absolutely continuous, flat bands (apart from $\omega \equiv 0$) currently cannot be excluded unless we make additional regularity assumptions on ε and μ.

So far, most spectral and analytic properties mirror of $\mathbf{M}_0^{\mathcal{Z}}$ those of periodic Schrödinger operators, but there are two important differences: (i) $\mathbf{M}_0$ is not bounded from below and (ii) in case of real weights the PH symmetry of the spectrum (cf. Theorem 2.2) implies a symmetry for the frequency band spectrum (cf. Figure 1).
The first item in conjunction with the non-analyticity of $J_0(k)$ at $k \in \Gamma^*$ potentially complicates the labeling of frequency bands. For simplicity, we solve this *using* the band picture proven in Theorem 1.4: first of all, we know there exists an infinitely degenerate flat band $\omega_0(k) = 0$ associated to the unphysical states (cf. Theorem 3.4 (i)). Moreover, it is easy to prove that 0 is an eigenvalue of $\mathbf{M}_0(k)|_{J_0(k)}$ if and only if $k \in \Gamma^*$. Away from $k \in \Gamma^*$, we repeat *non-zero* eigenvalues $\omega_j(k)$ of $\mathbf{M}_0(k)$ according to their multiplicity, arrange them in non-increasing order and label positive (negative) eigenvalues with positive (negative) integers, i. e. away from $k \in \Gamma^*$ we set

$$\ldots \leq \omega_{-2}(k) \leq \omega_{-1}(k) < \omega_0(k) = 0 < \omega_1(k) \leq \omega_2(k) \leq \ldots$$

Moreover, due to the analyticity of $k \mapsto \mathbf{M}_0(k)$, the eigenvalues depend on k in a continuous fashion, and we extend this labeling by continuity to $k \in \Gamma^*$. This procedure yields a family $\big\{k \mapsto \omega_n(k)\big\}_{n\in\mathbb{Z}}$ of Γ^*-periodic functions.
Two types of bands are special: beside the zero mode band $\omega_0(k) = 0$ which is due to states in $G_0(k)$, the *ground state bands* are those of lowest frequency in absolute value:

Definition 3.6 (Ground state bands) *We call a frequency band $k \mapsto \omega_n(k)$ of $\mathbf{M}_0^{\mathcal{Z}}$ a ground state band if and only if* *(i)* $\lim_{k \to 0} \omega_n(k) = 0$ *and*

(ii) ω_n *is not identically* 0 *in a neighborhood of* $k = 0$.

Moreover, we define $\mathcal{I}_{\mathrm{gs}} \subset \mathbb{Z}$ *to be the set of ground state band indices.*

The ground state bands can be recovered from the space of zero modes

$$\mathrm{GS} := \ker \mathbf{M}_0(0) \cap J_0(0)\,.$$

using analytic continuation, and hence, also the use of $P_0^{\mathrm{reg}}(k)$ instead of $P_0(k)$ even though they coincide at $k = 0$.

LEMMA 3.7 (GROUND STATE EIGENFUNCTIONS AT $k = 0$) *Suppose Assumption 3.1 holds true. Then* $\mathrm{GS} = P_0^{\mathrm{reg}}(0)\big\{y \mapsto a \mid a \in \mathbb{C}^6\big\}$ *is six-dimensional and any of its elements can be uniquely written as*

$$\Psi_a(y) := \big(P_0^{\mathrm{reg}}(0)a\big)(y) = \sum_{\gamma^* \in \Gamma} \widehat{\Psi}_a(\gamma^*)\,\mathrm{e}^{+\mathrm{i}\gamma^* \cdot y}$$

for some $a \in \mathbb{C}^6$*. The Fourier coefficients* $\widehat{\Psi}_a(\gamma^*) = \big(\widehat{\psi}_a^E(\gamma^*), \widehat{\psi}_a^H(\gamma^*)\big)$ *satisfy the following relations:*

$$\begin{aligned} &\widehat{\Psi}_a(0) = a \in \mathbb{C}^6 && \\ &\widehat{\psi}_a^\sharp(\gamma^*) \propto \gamma^* && \forall \gamma^* \in \Gamma^* \setminus \{0\},\ \sharp = E, H \end{aligned} \tag{34}$$

PROOF First of all, seeing as W is bounded with bounded inverse, $\ker \mathbf{M}_0(k) = \ker \mathbf{Rot}(k)$. A simple computation (cf. Lemma A.4) yields that any $\Psi \in \ker \mathbf{Rot}(0)$ is of the form

$$\Psi = a + \Psi_G$$

for some $a \in \mathbb{C}^6$ and $\Psi_G \in G_0^{\mathrm{reg}}(0) = G_0(0)$. Applying $P_0^{\mathrm{reg}}(0)$ to both sides yields $P_0^{\mathrm{reg}}(0)\Psi = \Psi_a$ and consequently, $\dim \mathrm{GS} \leq 6$.
From $\big[\mathbf{M}_0(k), P_0^{\mathrm{reg}}(k)\big] = 0$ we deduce $P_0^{\mathrm{reg}}(0)\big\{y \mapsto a \mid a \in \mathbb{C}^6\big\} \subseteq \mathrm{GS}$. Moreover, in view of (30), $y \mapsto a \in \mathbb{C}^6$ is an element of $G_0^{\mathrm{reg}}(0) = G_0(0)$ if and only if $a = 0$. Hence, $a \mapsto \Psi_a$ is injective and

$$\dim P_0^{\mathrm{reg}}(0)\big\{y \mapsto a \mid a \in \mathbb{C}^6\big\} = \dim \mathrm{GS} = 6\,.$$

Finally, $P_0^{\mathrm{reg}}(0)\big(\Psi_a - a\big) = 0$ means $\Psi_a - a \in G_0^{\mathrm{reg}}(0)$, and thus using equation (30) once more, we deduce $\widehat{\Psi}_a(\gamma^*) \propto \gamma^*$ and $\widehat{\Psi}_a(0) = a$. $\square$

We now proceed to the proof of Theorem 1.4 which establishes the frequency band picture for periodic Maxwell operators (cf. Figure 1).

Proof (of Theorem 1.4) (i) Since $\mathbf{M}_0(k)$ is isospectral to its restriction $\mathbf{M}_0(k)|_{J_0^{\mathrm{reg}}(k)}$, let us consider the latter. First of all, $k \mapsto \omega_0(k) = 0$ is trivially analytic, we may assume $n \neq 0$. Thus, the analyticity away from band crossings follows from the purely discrete nature of the spectrum of $\mathbf{M}_0(k)|_{J_0^{\mathrm{reg}}(k)}$ (Theorem 3.4 (iii)), the analyticity of $k \mapsto \mathbf{M}_0(k)$ (Proposition 3.3 (ii)) and $k \mapsto P_0^{\mathrm{reg}}(k)$ (Lemma 3.2) combined with standard perturbation theory in the sense of Kato [Kat95].

The Γ^*-periodicity of $k \mapsto \omega_n(k)$ is deduced from the equivariance of $k \mapsto \mathbf{M}_0(k)$.

(ii) Now assume in addition that ε and μ are real. For $n = 0$, we trivially find $\omega_0(k) = 0 = -\omega_0(-k)$. So from now on, suppose $n \in \mathbb{Z} \setminus \{0\}$.

One can check that upon Zak transform, the PH operator (complex conjugation) $C^{\mathcal{Z}} := \mathcal{Z} C \mathcal{Z}^{-1}$ acts on elements of $\Psi \in L^2_{\mathrm{eq}}(\mathbb{B}, \mathfrak{h}_0)$ as $\big(C^{\mathcal{Z}} \Psi\big)(k) = \overline{\Psi(-k)}$. Combined with $C^{\mathcal{Z}} \, \mathbf{M}_0^{\mathcal{Z}} = -\mathbf{M}_0^{\mathcal{Z}} \, C^{\mathcal{Z}}$ which follows from equation (19) since ε and μ are real, a straight-forward calculation shows that if $u_n(k)$ is an eigenfunction to $\omega_n(k)$, then $\big(C^{\mathcal{Z}} u_n\big)(k)$ is an eigenfunction to $-\omega_n(-k)$, and we have shown (ii).

(iii) To show (1), we will prove

$$0 \in \sigma\big(\mathbf{M}_0(k)|_{J_0(k)}\big) \iff 0 \in \sigma\big(\mathbf{Rot}(k)|_{J_{\mathbf{Rot}}(k)}\big) \tag{35}$$

first where $J_{\mathbf{Rot}}(k) = \ker \mathbf{Div}(k)$ is the physical subspace of the free Maxwell operator, and since the spectrum of $\mathbf{Rot}$,

$$\sigma\big(\mathbf{Rot}(k)|_{J_{\mathbf{Rot}}(k)}\big) = \bigcup_{\gamma^* \in \Gamma^*} \big\{\pm|k + \gamma^*|\big\},$$

is known explicitly (cf. Lemma A.4), this will prove $0 \in \sigma\big(\mathbf{M}_0(k)|_{J_0(k)}\big)$ if and only if $k \in \Gamma^*$. Hence, combined with Definition 3.6 this implies (1).

First of all, since the spectra $\sigma\big(\mathbf{M}_0(k)|_{J_0(k)}\big)$ are discrete for any $k \in \mathbb{B}$ (Theorem 3.4 (ii)), we only need to consider the existence of eigenvectors. As the inverse of W is bounded, the equations $\mathbf{M}_0(k)\Psi = 0$ and $\mathbf{Rot}(k)\Psi = 0$ are equivalent on the domain $\mathfrak{d}$. We will now show that the existence of $\Psi_{\mathbf{M}_0} \in J_0(k) \cap \mathfrak{d}$ to $\mathbf{M}_0(k)\Psi_{\mathbf{M}_0} = 0$ is equivalent to the existence of a $\Psi_{\mathbf{Rot}} \in \ker \mathbf{Div}(k)$ which satisfies $\mathbf{Rot}(k)\Psi_{\mathbf{Rot}} = 0$.

Assume there exists an eigenvector $\Psi_{\mathbf{M}_0} \in J_0(k) \cap \mathfrak{d}$. Then by the direct decomposition of the domain $\mathfrak{D} = \ker \mathbf{Div}(k) \oplus \operatorname{ran} \mathbf{Grad}(k)$ implies we can uniquely write

$$\Psi_{\mathbf{M}_0} = \Psi_{\mathbf{Rot}} + \Psi_G$$

as the sum of $\Psi_{\mathbf{Rot}} \in \ker \mathbf{Div}(k)$ and $\Psi_G \in G_0(k)$. Because the intersection $J_0(k) \cap G_0(k) = \{0\}$ is trivial, we know $\Psi_{\mathbf{Rot}} \neq 0$. Hence, $\Psi_{\mathbf{Rot}}$ is an eigenvector of $\mathbf{Rot}(k)$,

$$\mathbf{Rot}(k)\Psi_{\mathbf{Rot}} = \mathbf{Rot}(k)\big(\Psi_{\mathbf{M}_0} - \Psi_G\big) = 0.$$

The converse statement is shown analogously and we have proven (35).

Now we turn to (2): let us define $N := |\mathcal{I}_{\mathrm{gs}}|$. By (ii), N needs to be even. Due to (29), we may replace the physical subspace $J_0(0)$ with its regularized version $J_0^{\mathrm{reg}}(0)$, and the six-dimensional space GS from Lemma 3.7 can also be defined in terms of $J_0^{\mathrm{reg}}(0)$. Thus, we already know $N \leq \dim \mathrm{GS} = 6$. Moreover, since $\dim\big(G_0(k) \cap J_0^{\mathrm{reg}}(k)\big) = 2$ (Lemma 3.2 (iii)) and $Q_0(k) J_0^{\mathrm{reg}}(k) \subset G_0(k)$, the operator $\mathbf{M}_0(k)|_{J_0^{\mathrm{reg}}(k)}$ has a two-fold degenerate flat band $k \mapsto 0$ and we conclude that in fact, $N \leq 4$.

To show $N = 4$ and property (2), we use standard analytic perturbation theory in the sense of Kato around the eigenvalue 0: We have proven in (i) that all band functions are continuous, and thus if $\omega_n(0) = 0$ there exists a neighborhood V of $k = 0$ and a $\delta > 0$ such that $|\omega_n(k)| < \delta$ holds on V. Let us pick an orthonormal basis $\big\{\Psi_1, \ldots, \Psi_6\big\}$ of GS; according to Lemma 3.7, each of these Ψ_j is associated to a coefficient $a_{(j)} = \big(a^E_{(j)}, a^H_{(j)}\big) \in \mathbb{C}^6$, $j = 1, \ldots, 6$ via (34). Then $\mathbf{M}_0(0)\Psi_j = 0$ and [Kat95, equation (2.40)] imply the ground state band functions $\{\omega_n(k)\}_{n \in \mathcal{I}_{\mathrm{gs}}}$ are approximately equal to the non-zero eigenvalues of the k-dependent matrix

$$k \cdot A := \Big(\big\langle \Psi_l, k \cdot \mathbf{A}\Psi_j \big\rangle_{\mathfrak{h}_0}\Big)_{1 \leq l,j \leq 6} \tag{36}$$

where $k \cdot \mathbf{A} = \mathbf{M}_0(k) - \mathbf{M}_0(0)$ is explicitly given in equation (33) and $k \cdot A := \sum_{j=1}^3 k_j \, A_j$ involves the implicitly defined matrices A_j. For $a, b \in \mathbb{C}^6$, we can directly compute the scalar product:

$$\begin{aligned}
\big\langle \Psi_a, k \cdot \mathbf{A}\Psi_b \big\rangle_{\mathfrak{h}_0} &= \left\langle \begin{pmatrix} \psi_a^E \\ \psi_a^H \end{pmatrix}, W \begin{pmatrix} -k \times \psi_b^H \\ +k \times \psi_b^E \end{pmatrix} \right\rangle_{\mathfrak{h}_0} \\
&= k \cdot \int_{\mathbb{T}^3} \mathrm{d}y \left(\overline{\psi_a^E(y)} \times \psi_b^H(y) - \overline{\psi_a^H(y)} \times \psi_b^E(y) \right) \\
&= k \cdot \left(\overline{a^E} \times b^H - \overline{a^H} \times b^E \right) && (37) \\
&= \left\langle \begin{pmatrix} a^E \\ a^H \end{pmatrix}, \begin{pmatrix} 0 & -k^{\times} \\ +k^{\times} & 0 \end{pmatrix} \begin{pmatrix} b^E \\ b^H \end{pmatrix} \right\rangle_{\mathbb{C}^6} && (38)
\end{aligned}$$

To arrive at the last line, we plug in the ansatz (34) for the ground state function, use the orthogonality of the plane waves with respect to the standard scalar product on $L^2(\mathbb{T}^3)$ and exploit $\gamma^* \times \gamma^* = 0$.

Now let us define the invertible 6×6 matrix $\Lambda := \big(a_{(1)} \mid \cdots \mid a_{(6)}\big)$ which maps the canonical basis $\big\{v_{(1)}, \ldots, v_{(6)}\big\} \subset \mathbb{C}^6$ onto $\big\{a_{(1)}, \ldots, a_{(6)}\big\}$. Then we can express the matrix elements of $k \cdot A$ in terms of Λ:

$$\begin{aligned} \big\langle v_{(j)}, k \cdot A \, v_{(n)} \big\rangle_{\mathbb{C}^6} &:= \big(k \cdot A \big)_{jn} \\ &= \left\langle a_{(j)}, \begin{pmatrix} 0 & -k^{\times} \\ +k^{\times} & 0 \end{pmatrix} a_{(n)} \right\rangle_{\mathbb{C}^6} \\ &= \left\langle v_{(j)}, \Lambda^* \begin{pmatrix} 0 & -k^{\times} \\ +k^{\times} & 0 \end{pmatrix} \Lambda v_{(n)} \right\rangle_{\mathbb{C}^6} \end{aligned} \tag{39}$$

In view of equation (37), the matrix elements possess an $SO(3)$ symmetry: if we define the action of $R \in SO(3)$ on $a \in \mathbb{C}^6$ by setting $Ra := \big(Ra^E, Ra^H \big)$, then equation (37) in conjunction with $R(v \times w) = Rv \times Rw$, $v, w \in \mathbb{C}^3$, yields

$$\big\langle \Psi_a, k \cdot \mathbf{A} \Psi_b \big\rangle_{\mathfrak{h}_0} = \big\langle \Psi_{Ra}, (Rk) \cdot \mathbf{A} \Psi_{Rb} \big\rangle_{\mathfrak{h}_0}. \tag{40}$$

Combining this symmetry with equation (38), we get

$$\begin{aligned} \big(k \cdot A \big)_{jn} &= \left\langle R \, \Lambda v_{(j)}, \begin{pmatrix} 0 & +(Rk)^{\times} \\ -(Rk)^{\times} & 0 \end{pmatrix} R \, \Lambda v_{(n)} \right\rangle_{\mathbb{C}^6} \\ &= \left\langle v_{(j)}, \big(\Lambda^{-1} \, R \, \Lambda \big)^* \, \Lambda^* \begin{pmatrix} 0 & +(Rk)^{\times} \\ -(Rk)^{\times} & 0 \end{pmatrix} \Lambda \, \big(\Lambda^{-1} \, R \, \Lambda \big) v_{(n)} \right\rangle_{\mathbb{C}^6} \end{aligned}$$

or, put more succinctly after replacing R with R^{-1} and k with Rk,

$$(Rk) \cdot A = \big(\Lambda^{-1} \, R^{-1} \, \Lambda \big)^* \, \big(k \cdot A \big) \, \big(\Lambda^{-1} \, R^{-1} \, \Lambda \big).$$

As the matrix $\Lambda^{-1} \, R^{-1} \, \Lambda$ is invertible, we deduce

$$\mathrm{rank}\big(k \cdot A \big) = \mathrm{rank}\big((Rk) \cdot A \big) = \mathrm{rank}\big(\lambda \, k \cdot A \big) \tag{41}$$

holds for all $R \in SO(3)$ and $\lambda \in \mathbb{C} \setminus \{0\}$, i. e. the rank of the matrix $k \cdot A$ is independent of $k \neq 0$. In particular, it means that if $0 \in \sigma\big(k_0 \cdot A \big)$ for some special $k_0 \neq 0$, then 0 is an eigenvalue of *all* matrices $k \cdot A$.

Now we will reduce this problem of 6×6 matrices to a problem of 3×3 matrices: first of all, *any* basis $\big\{ v_{(j)} \big\}_{j=1}^6$ of $\mathbb{C}^6$ gives rise to a basis $\big\{ \Psi_{v_{(j)}} \big\}_{j=1}^6$ of GS. In particular, if we take $\big\{ v_{(j)} \big\}_{j=1}^6$ to be the canonical basis of $\mathbb{C}^6$, we can apply the Gram-Schmidt procedure to $\big\{ \Psi_{v_{(j)}} \big\}_{j=1}^6$ and obtain a $\langle \cdot \, , \cdot \rangle_{\mathfrak{h}_0}$-*orthonormal* basis $\big\{ \Psi_{a_{(1)}} \big\}_{j=1}^6$ of GS with coefficients $a_{(j)} = \big(a^E_{(j)}, a^H_{(j)} \big) \in \mathbb{C}^6$. Due to the block structure of W^{-1} that is also inherited by $\big\langle \Phi, \Psi \big\rangle_{\mathfrak{h}_0} = \big\langle \Phi, W^{-1} \Psi \big\rangle_{L^2(\mathbb{T}^3, \mathbb{C}^6)}$, the fact that $v^H_{(1)} = v^H_{(2)} = v^H_{(3)} = 0$ and $v^E_{(4)} = v^E_{(5)} = v^E_{(6)} = 0$ forces also the corresponding coefficients of the orthonormalized vectors to be 0,

$$a^H_{(1)} = a^H_{(2)} = a^H_{(3)} = 0 \, , \qquad a^E_{(4)} = a^E_{(5)} = a^E_{(6)} = 0 \, .$$

Moreover, $\{a^E_{(1)}, a^E_{(2)}, a^E_{(3)}\}$ and $\{a^H_{(4)}, a^H_{(5)}, a^H_{(6)}\}$ are two sets of linearly independent vectors in $\mathbb{C}^3$ with $a^E_{(1)}, a^H_{(4)} \propto (1,0,0)$.

Thus, using equation (37), one sees that the symmetric matrix $k \cdot A$ is purely block-offdiagonal and can be written in term of three 3×3 matrices $B = (B_1, B_2, B_3)$ as

$$k \cdot A =: \begin{pmatrix} 0 & k \cdot B \\ \bigl(k \cdot B\bigr)^* & 0 \end{pmatrix} . \tag{42}$$

The block structure implies that

$$\operatorname{rank}\bigl(k \cdot A\bigr) = \operatorname{rank}\bigl(k \cdot B\bigr) + \operatorname{rank}\bigl(k \cdot B\bigr)^* = 2 \operatorname{rank}\bigl(k \cdot B\bigr) . \tag{43}$$

Then in order to conclude that $\operatorname{rank}\bigl(k \cdot A\bigr) = 4$, we only need to show that $\operatorname{rank}\bigl(k \cdot B\bigr) = 2$. Since the result is independent of k, we pick $k_0 = (1,0,0)$ and use the basis obtained after Gram-Schmidt orthonormalizing $\{\Psi_{v_{(1)}}, \ldots, \Psi_{v_{(6)}}\}$. Then $a_{(1)} \propto v_{(1)}$ and $a_{(4)} \propto v_{(4)}$ are non-trivial scalar multiples of $v_{(1)}$ and $v_{(4)}$, and consequently, one obtains again from (37)

$$k_0 \cdot B = \begin{pmatrix} 0 & 0 \\ 0 & k_0 \cdot \tilde{B} \end{pmatrix}$$

where the 2×2 matrix

$$k_0 \cdot \tilde{B} = \begin{pmatrix} k_0 \cdot \bigl(\overline{a^E_{(2)}} \times a^H_{(5)}\bigr) & k_0 \cdot \bigl(\overline{a^E_{(2)}} \times a^H_{(6)}\bigr) \\ k_0 \cdot \bigl(\overline{a^E_{(3)}} \times a^H_{(5)}\bigr) & k_0 \cdot \bigl(\overline{a^E_{(3)}} \times a^H_{(6)}\bigr) \end{pmatrix}$$

has full rank, because $k_0 = v^E_{(1)} = v^H_{(4)} \propto a^E_{(1)}, a^H_{(4)}$ implies

$$\begin{aligned} \det\bigl(k_0 \cdot \tilde{B}\bigr) &= \overline{\Bigl(k_0 \cdot \bigl(a^E_{(2)} \times a^E_{(3)}\bigr)\Bigr)} \Bigl(k_0 \cdot \bigl(a^H_{(5)} \times a^H_{(6)}\bigr)\Bigr) \\ &\propto \overline{\det\Bigl(a^E_{(1)} \bigm| a^E_{(2)} \bigm| a^E_{(3)}\Bigr)} \det\Bigl(a^H_{(4)} \bigm| a^H_{(5)} \bigm| a^H_{(6)}\Bigr) \neq 0 . \end{aligned}$$

Hence, piecing together $\operatorname{rank}\bigl(k_0 \cdot B\bigr) = 2$ with equations (41) and (43) yields that the degeneracy of the ground state bands is 4. □

3.4 Comparison to existing literature

Even though most of the results in this section are neither new nor surprising, we still feel they fill a void in the literature: To the best of our knowledge, it is the first time the most important fundamental properties of the fiber Maxwell operator $\mathbf{M}_0(k)$ are all proven rigorously in one place. Many of these are scattered throughout the literature, e. g. various authors have proven the discrete nature of the spectrum of $\mathbf{M}_0(k)$ [FK97, Mor00, SEK+05] or have shown the non-analyticity of $k \mapsto P_0(k)$ at $k = 0$ [FK96a]. Certainly there is no dearth of literature on the subject (see also [Kuc01, JJWM08] and references therein). However, most of these results are piecemeal: Some of them are contained in publications which do not really focus on the periodic Maxwell operator, but random Maxwell operators ([FK96b, FK97], for instance). Other publications do not study $\mathbf{M}_0$ but rather operators associated to $\mathbf{M}_0^2$: since $\mathbf{M}_0^2$ is block-diagonal, it suffices to study a second-order equation for either $\mathbf{E}$ or $\mathbf{B}$, see e. g. [FK96a, FK97]. In the two-dimensional case, this leads to a *scalar* equation where the right-hand side is a second-order operator [FK96a].

Nevertheless, one result is new, namely Theorem 1.4 (iii): even though the presence of ground state bands is heuristically well-understood, we provide rather simple and straight-forward proof. The $k \to 0$ limit is related in spirit to the *homogenization limit* where the wavelength of the electromagnetic wave is large compared to the lattice spacing (see e. g. [Sus04, Sus05, SEK+05, BS07, APR12] and references therein). On the one hand, many homogenization techniques yield much farther-reaching results, most notably effective equations for the dynamics (e. g. [BS07, Theorem 2.1]) while Theorem 1.4 (iii) only makes a statement about the behavior of the ground state frequency bands. On the other hand, compared to, say, [BS07, Theorem 2.1] or [SEK+05, Theorem 6.2], computing the dispersion of the ground state bands for small k seems much easier in our approach: given ε and μ, the problem reduces to orthonormalizing 2×3 vectors numerically and solving an eigenvalue problem for an explicitly given 3×3 matrix $|k \cdot B|$ defined through (42) with one known eigenvalue (namely 0). Moreover, a proof of the fact that there are 4 ground state bands also appears to be new, e. g. in a recent publication this was stated as [SEK+05, Conjecture 1]. Proving this fact, however, required a better insight into the nature of the singularity of $k \mapsto P_0(k)$ at $k = 0$ and necessitated the introduction of a regularized projection P_0^{reg}.

4 $\mathbf{M}_\lambda^{\mathcal{Z}}$ and $M_\lambda^{\mathcal{Z}}$ as ΨDOs

After expounding the properties of the periodic Maxwell operator, we proceed to the proof of Theorem 1.3. The essential ingredient is a suitable interpretation of the usual Weyl quantization rule

$$\mathfrak{Op}_\lambda(f) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{d}r' \int_{\mathbb{R}^3} \mathrm{d}k' \, (\mathcal{F}_\sigma f)(r', k') \, \mathrm{e}^{-\mathrm{i}(k' \cdot (\mathrm{i}\lambda\nabla_k) - r' \cdot \hat{k})} \tag{44}$$

where

$$(\mathcal{F}_\sigma f)(r', k') := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{d}r' \int_{\mathbb{R}^3} \mathrm{d}k' \, \mathrm{e}^{+\mathrm{i}(k' \cdot r - r' \cdot k)} \, f(r, k)$$

is the symplectic Fourier transform. The idea is to combine the point of view from [Teu03, Appendix B] and [DL11, Section 2.2] with the fact that most results of standard pseudodifferential theory depend only on the *Banach* structure of the spaces involved and not on the Hilbert structure.

First of all, equation (44) defines a ΨDO for a large class of scalar [Fol89, Hö79, Kg81, Tay81] and vector-valued functions [Luk72, Lev90]. For instance, if f is a Hörmander symbol or order $m \in \mathbb{R}$ and type $\rho \in [0,1]$ taking values in the Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$,

$$f \in S^m_\rho(\mathcal{B}) := \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^6, \mathcal{B}) \;\middle|\; \forall \alpha, \beta \in \mathbb{N}_0^3 : \|f\|_{m,\alpha\beta} < \infty \right\}, \tag{45}$$

where the seminorms $\left\{ \|\cdot\|_{m,\alpha\beta} \right\}_{\alpha,\beta \in \mathbb{N}_0^3}$ are defined by

$$\|f\|_{m,\alpha\beta} := \sup_{(r,k) \in \mathbb{R}^6} \left(\sqrt{1+k^2}^{\,-m+|\beta|\rho} \left\| \partial_r^\alpha \partial_k^\beta f(r,k) \right\|_{\mathcal{B}} \right),$$

then (44) is defined as an oscillatory integral [Hö71]. The vector-valuedness of f usually does not create any technical difficulties, most standard results readily extend to vector-valued symbols, e. g. Caldéron-Vaillancourt-type theorems and the composition of Hörmander-type symbols (see e. g. [Luk72, GMS91, MS09] and [Teu03, Appendix A]).

In our applications $\mathcal{B} = \mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_2)$ will always be some Banach space of bounded operators between the Hilbert spaces $\mathfrak{h}_1$ and $\mathfrak{h}_2$ whose elements are L^2-functions on the torus, e. g. $L^2(\mathbb{T}^3, \mathbb{C}^N)$, $\mathfrak{h}_0$ or $\mathfrak{d}$. As explained in [DL11, Section 2.2.1], when compared to the pseudodifferential calculus associated to $(-\mathrm{i}\lambda\nabla_x, \hat{x})$, equation (44) can be seen as an equivalent representation of the same underlying Moyal algebra [GBV88a, GBV88b]. Hence, the usual formulas and results apply, and we may use standard Hörmander classes instead of the less common weighted Hörmander classes as in [PST03a].

4.1 Equivariant ΨDOs

The relevant Hilbert spaces, $\mathcal{Z}\mathfrak{H}_\lambda$ and $\mathcal{Z}\mathfrak{H}_0$, coincide with $L^2_{\mathrm{eq}}\big(\mathbb{R}^3, L^2(\mathbb{T}^3, \mathbb{C}^6)\big)$ as Banach spaces, and we are in the same framework as in [Teu03, Appendix B] and [DL11, Section 2.2.2]. The building block operators are macroscopic position $\mathrm{i}\lambda\nabla_k$ and crystal momentum $\hat{k}$ whose domains are dense in $L^2_{\mathrm{eq}}(\mathbb{R}^3, \mathfrak{h}_0)$ (cf. Section 3.1).

Operators which fiber-decompose in Zak representation have the equivariance property (28), and thus $\mathbf{M}_0^{\mathcal{Z}} : L^2_{\mathrm{eq}}(\mathbb{R}^3, \mathfrak{d}) \longrightarrow L^2_{\mathrm{eq}}(\mathbb{R}^3, \mathfrak{h}_0)$ defines a selfadjoint operator between Hilbert spaces of equivariant functions, for instance. This motivates the following

Definition 4.1 (Semiclassical symbols) *Assume $\mathfrak{h}_j$, $j = 1, 2$, are Hilbert spaces consisting of functions on $\mathbb{T}^3$. A map $f : [0, \lambda_0) \longrightarrow S^m_{\rho,\mathrm{eq}}\big(\mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_2)\big)$, $\lambda \mapsto f_\lambda$, is called a semiclassical equivariant symbol of order $m \in \mathbb{R}$ and weight $\rho \in [0,1]$, that is $f \in AS^m_{\rho,\mathrm{eq}}\big(\mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_2)\big)$, if and only if*

(i) $f_\lambda(r, k - \gamma^*) = \mathrm{e}^{-\mathrm{i}\gamma^* \cdot \hat{y}} \, f_\lambda(r,k) \, \mathrm{e}^{+\mathrm{i}\gamma^* \cdot \hat{y}}$ holds $\forall\ (r,k) \in \mathbb{R}^6$, $\gamma^* \in \Gamma^*$ and

(ii) there exists a sequence $\{f_n\}_{n \in \mathbb{N}_0}$*,* $f_n \in S^{m-n\rho}_{\rho}$*, such that for all* $N \in \mathbb{N}_0$

$$\lambda^{-N} \left(f_\lambda - \sum_{n=0}^{N-1} \lambda^n \, f_n \right) \in S^{m-N\rho}_{\rho}\big(\mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_2)\big)$$

holds true uniformly in λ *in the sense that for any* $N \in \mathbb{N}_0$ *and* $\alpha, \beta \in \mathbb{N}_0^3$*, there exist constants* $C_{N\alpha\beta} > 0$ *so that the estimate*

$$\left\| f_\lambda - \sum_{n=0}^{N-1} \lambda^n \, f_n \right\|_{m,\alpha\beta} \leq C_{N\alpha\beta} \, \lambda^N$$

is satisfied for all $\lambda \in [0, \lambda_0)$*.*

Since $S^m_\rho\big(\mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_2)\big)$ and $S^m_{\rho,\mathrm{eq}}\big(\mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_2)\big)$ are contained in the Moyal algebra [GBV88a, Section III], the associated ΨDOs extend from continuous maps between vector-valued Schwartz functions to continuous maps between vector-valued tempered distributions,

$$\begin{aligned} \mathfrak{Op}_\lambda \left(S^m_\rho\big(\mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_2)\big)\right) &\subset \mathfrak{Op}_\lambda \left(S^m_{\rho,\mathrm{eq}}\big(\mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_2)\big)\right) \\ &\subset \mathcal{L}\left(\mathcal{S}(\mathbb{R}^3, \mathfrak{h}_1), \mathcal{S}(\mathbb{R}^3, \mathfrak{h}_2)\right) \cap \mathcal{L}\left(\mathcal{S}'(\mathbb{R}^3, \mathfrak{h}_1), \mathcal{S}'(\mathbb{R}^3, \mathfrak{h}_2)\right). \end{aligned}$$

Furthermore, one can easily check that equivariant ΨDOs also preserve equivariance on the level of tempered distributions: let us define translations and multiplication with the phase $\mathrm{e}^{+\mathrm{i}\gamma^* \cdot \hat{y}}$ on $\mathcal{S}'(\mathbb{R}^3, \mathfrak{h}_j)$, $j = 1, 2$, by duality, i. e. we set

$$\big(L_{\gamma^*} F, \varphi\big)_{\mathcal{S}} := \big(T, \varphi(\cdot + \gamma^*)\big)_{\mathcal{S}}, \qquad \big(\mathrm{e}^{-\mathrm{i}\gamma^* \cdot \hat{y}} F, \varphi\big)_{\mathcal{S}} := \big(T, \mathrm{e}^{+\mathrm{i}\gamma^* \cdot \hat{y}} \varphi\big)_{\mathcal{S}},$$

for all $\gamma^* \in \Gamma^* \subset \mathbb{R}^3$. The set of equivariant tempered distributions $\mathcal{S}'_{\mathrm{eq}}(\mathbb{R}^3, \mathfrak{h}_j)$, $j = 1, 2$, is comprised of those tempered distributions which satisfy

$$L_{\gamma^*} F = \mathrm{e}^{-\mathrm{i}\gamma^* \cdot \hat{y}} F.$$

Then [Teu03, Proposition B.3] states that

$$\mathfrak{Op}_\lambda(f) : \mathcal{S}'_{\mathrm{eq}}(\mathbb{R}^3, \mathfrak{h}_1) \longrightarrow \mathcal{S}'_{\mathrm{eq}}(\mathbb{R}^3, \mathfrak{h}_2)$$

holds for all $f \in S^m_{\rho,\mathrm{eq}}\big(\mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_2)\big)$. Consequently, the inclusion $L^2_{\mathrm{eq}}(\mathbb{R}^3, \mathfrak{h}_j) \subset \mathcal{S}'_{\mathrm{eq}}(\mathbb{R}^3, \mathfrak{h}_j)$ and the standard Caldéron-Vaillancourt theorem imply [Teu03, Proposition B.5]

$$\mathfrak{Op}_\lambda \left(S^0_{\rho,\mathrm{eq}}\big(\mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_2)\big)\right) \subset \mathcal{B}\left(L^2_{\mathrm{eq}}(\mathbb{R}^3, \mathfrak{h}_1), \, L^2_{\mathrm{eq}}(\mathbb{R}^3, \mathfrak{h}_2)\right).$$

Similarly, the Moyal product $\sharp$ which is implicitly defined through

$$\mathfrak{Op}_\lambda(f \sharp g) := \mathfrak{Op}_\lambda(f) \, \mathfrak{Op}_\lambda(g)$$

extends as a bilinear, continuous map which respects equivariance,

$$\sharp : S^{m_1}_{\rho,\mathrm{eq}}\big(\mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_2)\big) \times S^{m_2}_{\rho,\mathrm{eq}}\big(\mathcal{B}(\mathfrak{h}_2, \mathfrak{h}_3)\big) \longrightarrow S^{m_1+m_2}_{\rho,\mathrm{eq}}\big(\mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_3)\big). \tag{46}$$

4.2 Extension to weighted L^2-spaces

We have seen that certain equivariant operator-valued functions define bounded ΨDOs mapping between Hilbert spaces of equivariant L^2-functions. The fact $\mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_2)$ only depends on the Banach space structure of $\mathfrak{h}_1$ and $\mathfrak{h}_2$ immediately implies

$$\mathcal{B}\big(\mathcal{Z}\mathfrak{D}\,,\,\mathcal{Z}\mathfrak{H}_\lambda\big) = \mathcal{B}\Big(L^2_{\mathrm{eq}}(\mathbb{R}^3, \mathfrak{d})\,,\, L^2_{\mathrm{eq}}\big(\mathbb{R}^3, L^2(\mathbb{T}^3, \mathbb{C}^6)\big)\Big),$$

for instance, and hence any $f \in S^0_{\rho,\mathrm{eq}}\big(\mathcal{B}\big(L^2(\mathbb{T}^3, \mathbb{C}^6)\big)\big)$ uniquely defines a ΨDO

$$\mathfrak{Op}_\lambda(f) : \mathcal{Z}\mathfrak{H}_\lambda \longrightarrow \mathcal{Z}\mathfrak{H}_\lambda. \tag{47}$$

One only needs to be careful about taking adjoints: the adjoint operator crucially depends on the scalar product (see e. g. the discussion of selfadjointness of $\mathbf{M}_w$ in Section 2.1), but in our applications, properties such as selfadjointness are checked “by hand”.

4.3 Proof of Theorem 1.3

Assumption 3.1 on the material weights ε and μ as well as Assumption 1.2 placed on the modulation functions imply $\mathfrak{H}_\lambda$ and $\mathfrak{H}_0$ coincide with $L^2(\mathbb{R}^3, \mathbb{C}^6)$ as Banach spaces. Similarly, we have $\mathfrak{h}_0 = L^2(\mathbb{T}^3, \mathbb{C}^6)$ on the level of Banach spaces. This means, $\mathcal{Z}\mathfrak{H}_\lambda$ and $\mathcal{Z}\mathfrak{H}_0$ agree with $L^2_{\mathrm{eq}}\big(\mathbb{R}^3, L^2(\mathbb{T}^3, \mathbb{C}^6)\big)$ as normed vector spaces.
Seeing as we can write $\mathbf{M}^{\mathcal{Z}}_\lambda = S(\mathrm{i}\lambda\nabla_k)^{-2}\,\mathbf{M}^{\mathcal{Z}}_0$, Theorem 1.3 follows from the following

Lemma 4.2 *Under the assumptions of Theorem 1.3, the following two operators are semiclassical pseudodifferential operators:*

(i) $S(\mathrm{i}\lambda\nabla_k)^{\pm 1} = \mathfrak{Op}_\lambda\big(S^{\pm 1}\big)$ *where* $S, S^{-1} \in S^0_{1,\mathrm{eq}}\left(\mathcal{B}\big(L^2(\mathbb{T}^3, \mathbb{C}^6)\big)\right)$

(ii) $\mathbf{M}^{\mathcal{Z}}_0 = \mathfrak{Op}_\lambda\big(\mathbf{M}_0(\,\cdot\,)\big)$ *where* $\mathbf{M}_0(\,\cdot\,) \in S^1_{1,\mathrm{eq}}\left(\mathcal{B}\big(\mathfrak{d}, L^2(\mathbb{T}^3, \mathbb{C}^6)\big)\right)$

Proof (i) The matrix $S(r)$ is block-diagonal with respect to $L^2(\mathbb{T}^3, \mathbb{C}^6) \cong L^2(\mathbb{T}^3, \mathbb{C}^3) \oplus L^2(\mathbb{T}^3, \mathbb{C}^3)$ and each block is proportional to the identity in $L^2(\mathbb{T}^3, \mathbb{C}^3)$. Due to the assumption on the modulation functions, we conclude

$$S \in \mathcal{C}^\infty_{\mathrm{b}}\left(\mathbb{R}^3, \mathcal{B}\big(L^2(\mathbb{T}^3, \mathbb{C}^6)\big)\right) \subset S^0_1\left(\mathcal{B}\big(L^2(\mathbb{T}^3, \mathbb{C}^6)\big)\right).$$

Equivariance is trivial, because $S(\mathrm{i}\lambda\nabla_k)$ commutes with $\mathrm{e}^{-\mathrm{i}\gamma^*\cdot\hat{y}}$ and hence

$$S(r) = \mathrm{e}^{+\mathrm{i}\gamma^*\cdot\hat{y}}\, S(r)\, \mathrm{e}^{-\mathrm{i}\gamma^*\cdot\hat{y}}$$

holds. Lastly, S^{-1} has the same properties as S since τ_ε^{-1} and τ_μ^{-1} also satisfy Assumption 1.2. This concludes the proof of (i).

(ii) By Proposition 3.3, the map $k \mapsto \mathbf{M}_0(k)$ is linear (the domain is independent of k), and thus $S^1_1\big(\mathcal{B}\big(\mathfrak{d}, L^2(\mathbb{T}^3,\mathbb{C}^6)\big)\big)$. Equivariance follows from equation (28), and thus we have shown (ii). □

Seeing as $\mathbf{M}_0(\,\cdot\,)$ is linear, the asymptotic expansion of $\sharp$ terminates after two terms and the symbols of the Maxwell operators in the physical representation can be computed from

$$\mathbf{M}^{\mathcal{Z}} = \mathfrak{Op}_\lambda\big(S^{-2}\sharp\mathbf{M}_0(\,\cdot\,)\big) =: \mathfrak{Op}_\lambda(\boldsymbol{\mathcal{M}}_\lambda).$$

That $\boldsymbol{\mathcal{M}}_\lambda$ is an element of $AS^1_{1,\mathrm{eq}}\big(\mathcal{B}\big(\mathfrak{d}, L^2(\mathbb{T}^3,\mathbb{C}^6)\big)\big)$ is implied by the composition properties of equivariant symbols (46) and the preceding Lemma. This concludes the proof of Theorem 1.3.

Consequently, also the Maxwell operator in the auxiliary representation is a semiclassical ΨDO,

$$M^{\mathcal{Z}}_\lambda = \mathfrak{Op}_\lambda\big(S\sharp\boldsymbol{\mathcal{M}}_\lambda\sharp S^{-1}\big) = \mathfrak{Op}_\lambda\big(S^{-1}\sharp\mathbf{M}_0(\,\cdot\,)\sharp S^{-1}\big) =: \mathfrak{Op}_\lambda(\mathcal{M}_\lambda)\,,$$

whose semiclassical symbol $\mathcal{M}_\lambda$ is in the same symbol class.

Corrolary 4.3 *Under the assumptions of Theorem 1.3, the Maxwell operator $M^{\mathcal{Z}}_\lambda = \mathfrak{Op}_\lambda(\mathcal{M}_\lambda)$ in the rescaled representation is the semiclassical pseudodifferential operator associated to*

$$\mathcal{M}_\lambda(r,k) = \tau(r)\,\mathbf{M}_0(k) - \lambda\,\tau(r)\,W\begin{pmatrix} 0 & \frac{\mathrm{i}}{2}\left(\nabla_r \ln {}^{\tau_\varepsilon}\!/_{\tau_\mu}\right)^{\times}(r) \\ \frac{\mathrm{i}}{2}\left(\nabla_r \ln {}^{\tau_\varepsilon}\!/_{\tau_\mu}\right)^{\times}(r) & 0 \end{pmatrix}$$

where $\tau(r) := \tau_\varepsilon(r)\,\tau_\mu(r)$. The function $\mathcal{M}_\lambda \in AS^1_{1,\mathrm{eq}}\big(\mathcal{B}\big(\mathfrak{d}, L^2(\mathbb{T}^3,\mathbb{C}^6)\big)\big)$ is an equivariant semiclassical symbol in the sense of Definition 4.1.

A The **curl** operator and the **Rot** operator

The aim of this Appendix is to clarify the meaning of the relation $\mathfrak{D}(\mathbf{Rot}) = \mathfrak{D}(\mathbf{curl}) \oplus \mathfrak{D}(\mathbf{curl})$ used in Section 2.1 in order to define the domain of the Maxwell operator. So to conclude our arguments from Section 2.1, we give a brief overview on the theory of the operators $\mathbf{curl} := \nabla^\times_x$ and $\mathbf{Rot}$. Many works have been devoted to the rigorous study of $\mathbf{curl}$ on $L^2(\Omega,\mathbb{C}^3)$ where $\Omega \subseteq \mathbb{R}^3$ can be a bounded [YG90, ABDG98, HKT12] or unbounded domain [Pic98] whose boundary satisfies various regularity properties. A lot of related results are contained in standard texts on the Navier-Stokes equation [DL72, FT78, GR86, Gal11]. In this Appendix, we enumerate some elementary results for the special case $\Omega = \mathbb{R}^3$. The crucial result is the so-called *Helmholtz-Hodge-Weyl-Leray decomposition* which leads to a decomposition of any $\psi \in L^2(\mathbb{R}^3,\mathbb{C}^3)$ into divergence and rotation-free component.

A.1 The gradient operator

The gradient operator is initially defined on the smooth functions with compact support by

$$\nabla_x : \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3) \longrightarrow \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3, \mathbb{C}^3), \qquad \nabla_x \varphi := \begin{pmatrix} \partial_{x_1}\varphi \\ \partial_{x_2}\varphi \\ \partial_{x_3}\varphi \end{pmatrix}. \tag{48}$$

The operator ∇_x is closable (any component ∂_{x_j} is anti-symmetric) and its closure, still denoted with ∇_x, has domain $\mathfrak{D}(\nabla_x) = H^1(\mathbb{R}^3)$ and trivial null space, $\ker \nabla_x = \{0\}$.

A.2 The divergence operator

The second operator of relevance, the divergence

$$\mathbf{div} : \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3, \mathbb{C}^3) \longrightarrow \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3), \qquad \mathbf{div}\, \psi := \sum_{j=1}^{3} \partial_{x_j} \psi_j, \tag{49}$$

is also closable and its closure, still denoted with **div**, has domain [Tem01, Section 1.2 and Theorem 1.1]

$$\mathfrak{D}(\mathbf{div}) := \overline{\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3, \mathbb{C}^3)}^{\|\cdot\|_{\mathbf{div}}} = \bigl\{ \psi \in L^2(\mathbb{R}^3, \mathbb{C}^3) \ \mid \ \mathbf{div}\, \psi \in L^2(\mathbb{R}^3) \bigr\}.$$

A relevant result is the *Stokes formula* [Tem01, Theorem 1.2], i. e. we have

$$X_\psi(\varphi) := \bigl\langle \psi, \nabla_x \varphi \bigr\rangle_{L^2(\mathbb{R}^3, \mathbb{C}^3)} + \bigl\langle \mathbf{div}\, \psi, \varphi \bigr\rangle_{L^2(\mathbb{R}^3)} = 0$$

for all $\psi \in \mathfrak{D}(\mathbf{div})$ and $\varphi \in H^1(\mathbb{R}^3)$. This follows mainly from the Cauchy-Schwarz inequality $\bigl| X_\psi(\phi) \bigr| \leqslant 2 \, \|\psi\|_{\mathbf{div}} \, \|\phi\|_{\nabla_x}$. The above relation shows that **div** is the *adjoint* of $-\nabla_x$ and vice versa (cf. [Pic98]). In this sense $\mathfrak{D}(\mathbf{div})$ can be seen as the space of vector fields with weak divergence.

A.3 The rotor operator

Lastly, the

$$\mathbf{curl} : \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3, \mathbb{C}^3) \longrightarrow \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3, \mathbb{C}^3), \qquad \mathbf{curl}\, \psi := \begin{pmatrix} \partial_{x_2}\psi_3 - \partial_{x_3}\psi_2 \\ \partial_{x_3}\psi_1 - \partial_{x_1}\psi_3 \\ \partial_{x_1}\psi_2 - \partial_{x_2}\psi_1 \end{pmatrix} \tag{50}$$

is essentially selfadjoint, and thus, uniquely extends to a selfadjoint operator whose domain

$$\mathfrak{D}(\mathbf{curl}) := \overline{\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3, \mathbb{C}^3)}^{\|\cdot\|_{\mathbf{curl}}} = \bigl\{ \psi \in L^2(\mathbb{R}^3, \mathbb{C}^3) \ \mid \ \mathbf{curl}\, \psi \in L^2(\mathbb{R}^3, \mathbb{C}^3) \bigr\} \tag{51}$$

is the closure of the core with respect to the graph norm. The characterization of $\mathfrak{D}(\mathbf{curl})$ by the second equality in (51) is proven in a slightly more general context in [DL72, Chapter 7, Lemma 4.1] (cf. also [ABDG98, Definition 2.2] and [Urb01]). By showing that the deficiency indices of **curl** are both 0, i. e. $\mathbf{curl}\,\psi = \pm\mathrm{i}\,\psi$ has no non-trivial solutions, one deduces **curl** is indeed selfadjoint (cf. [CK57, Pic98]). A very interesting fact relates the domains of **curl** and **div**, and the space $H^1(\mathbb{R}^3,\mathbb{C}^3)$: Theorem 2.5 of [ABDG98] states

$$\mathfrak{D}(\mathbf{curl}) \cap \mathfrak{D}(\mathbf{div}) = H^1(\mathbb{R}^3,\mathbb{C}^3) \tag{52}$$

which follows from the identity

$$\|\psi\|^2_{H^1(\mathbb{R}^3,\mathbb{C}^3)} = \|\psi\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^3)} + \|\mathbf{curl}\,\psi\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^3)} + \|\mathbf{div}\,\psi\|^2_{L^2(\mathbb{R}^3)} . \tag{53}$$

This decomposition of the $H^1(\mathbb{R}^3,\mathbb{C}^3)$-norm follows from integration by parts and the identity

$$\big(\mathbf{curl}\big)^2 = \nabla_x\,\mathbf{div} - \Delta_x$$

on $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3,\mathbb{C}^3)$, and a simple density argument. Note that (52) implies $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3,\mathbb{C}^3)$ and $H^1(\mathbb{R}^3,\mathbb{C}^3)$ are cores for both, **div** and **curl**.

A.4 The Helmholtz-Hodge-Weyl-Leray decomposition

For a more precise characterization of the domain $\mathfrak{D}(\mathbf{curl})$ we need the *Helmholtz-Hodge-Weyl-Leray decomposition* (see [Tem01, Chapter I, Section 1.4], [FT78, Section 1.1] and [Gal11, Section III.1]). Let us introduce the subspaces

$$\mathbf{C}_\sigma := \big\{\psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3,\mathbb{C}^3) \;\mid\; \mathbf{div}\,\psi = 0\big\}, \qquad \mathbf{J} := \overline{\mathbf{C}_\sigma}^{\|\cdot\|_{L^2(\mathbb{R}^3,\mathbb{C}^3)}}.$$

Theorem A.1 (Helmholtz-Hodge-Weyl-Leray decomposition) *The space $L^2(\mathbb{R}^3,\mathbb{C}^3)$ admits the following orthogonal decomposition*

$$L^2(\mathbb{R}^3,\mathbb{C}^3) = \mathbf{J} \oplus_\perp \mathbf{G} \tag{54}$$

where $\mathbf{J} \subset \mathfrak{D}(\mathbf{div})$ is defined by

$$\mathbf{J} = \big\{\psi \in L^2(\mathbb{R}^3,\mathbb{C}^3) \;\mid\; \mathbf{div}\,\psi = 0\big\} = \ker \mathbf{div} \tag{55}$$

and

$$\mathbf{G} := \big\{\psi \in L^2(\mathbb{R}^3,\mathbb{C}^3) \;\mid\; \psi = \nabla_x\varphi,\ \varphi \in L^2_{\mathrm{loc}}(\mathbb{R}^3)\big\} = \mathrm{ran}\,\nabla_x . \tag{56}$$

Moreover, one has also the following characterization:

$$\mathbf{J} = \ker \mathbf{div} = \mathrm{ran}\,\mathbf{curl}, \qquad \mathbf{G} = \ker \mathbf{curl} = \mathrm{ran}\,\nabla_x . \tag{57}$$

Proof (Sketch) Equation (55) is proven in [Tem01, Chapter I, Theorem 1.4, eq. (1.34)]. The inclusion $\mathbf{J} \subset \mathfrak{D}(\mathbf{div})$ follows from the observation that the norms $\|\cdot\|_{L^2(\mathbb{R}^3,\mathbb{C}^3)}$ and $\|\cdot\|_{\mathbf{div}}$ coincide on $\mathbf{C}_\sigma$.
The definition of $\mathbf{G}$ as gradient fields (first equality) has been shown in [Tem01, Chapter I, Theorem 1.4, eq. (1.33) and Remark 1.5]. The closedness of $\mathbf{G}$, and thus, the second equality is discussed in the proof of [Pic98, Lemma 2.5]. (According to our choice of convention in Section 1.1, $\operatorname{ran} \nabla_x$ is the closure of $\operatorname{ran}_0 \nabla_x = \nabla_x H^1(\mathbb{R}^3)$, and for an example of $\varphi \in L^2_{\mathrm{loc}}(\mathbb{R}^3) \setminus H^1(\mathbb{R}^3)$ such that $\nabla_x \varphi \in L^2(\mathbb{R}^3, \mathbb{C}^3)$ we refer to [Gal11, Note 2, pg. 156].)
The proofs of the two remaining equalities in (57) can be found in [Pic98, Theorem 1.1].
We remark that in case of the vector fields on all of $\mathbb{R}^3$, the space of harmonic vector fields $H_N := \ker \mathbf{div} \cap \ker \mathbf{curl} = \{0\}$ is the trivial vector space, because $\Delta\psi = 0$ has no non-trivial solutions on $L^2(\mathbb{R}^3, \mathbb{C}^3)$. This concludes the proof of (54). □

Remark A.2 According to the standard nomenclature $\mathbf{J}$ is known as the space of the *solenoidal* or *transversal* vector fields while $\mathbf{G}$ is the space of the *irrotational* or *longitudinal* vector fields. The orthogonal projection $\mathbf{P} : L^2(\mathbb{R}^3, \mathbb{C}^3) \longrightarrow \mathbf{J}$ is called *Leray projection.* The identification $\mathbf{J} = \operatorname{ran} \mathbf{curl}$ implies that $\mathbf{curl} : \mathbf{J} \longrightarrow \mathbf{J}$ and this is enough for $[\mathbf{P}, \mathbf{curl}] = 0$.

Theorem A.1 has two immediate consequences: The first is the *Helmholtz splitting*, meaning each $\psi \in L^2(\mathbb{R}^3, \mathbb{C}^3)$ can be uniquely decomposed into a *stream field* $\phi \in \mathfrak{D}(\mathbf{curl})$ and the gradient of a *potential function* $\varphi \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$,

$$\psi = \mathbf{curl}\, \phi + \nabla_x \varphi,$$

where $\mathbf{curl}\, \phi$ and $\nabla_x \varphi$ are mutually orthogonal. The second is the content of the following

Corrolary A.3 (Domain of $\mathbf{curl}$) *The domain $\mathfrak{D}(\mathbf{curl})$ of the operator $\mathbf{curl}$ admits the following splitting*

$$\begin{aligned} \mathfrak{D}(\mathbf{curl}) &= \big(\mathbf{J} \cap \mathfrak{D}(\mathbf{curl})\big) \oplus_\perp \mathbf{G} \\ &= \big(\mathbf{J} \cap H^1(\mathbb{R}^3, \mathbb{C}^3)\big) \oplus_\perp \mathbf{G} \\ &= \big(\ker \mathbf{div} \cap H^1(\mathbb{R}^3, \mathbb{C}^3)\big) \oplus_\perp \ker \mathbf{curl} \\ &= \big(\ker \mathbf{div} \cap H^1(\mathbb{R}^3, \mathbb{C}^3)\big) \oplus_\perp \operatorname{ran} \nabla_x. \end{aligned} \tag{58}$$

Proof Theorem A.1 implies $\mathfrak{D}(\mathbf{curl}) = \big(\mathbf{J} \cap \mathfrak{D}(\mathbf{curl})\big) \oplus_\perp \mathbf{G}$ since $\mathbf{G} \subset \mathfrak{D}(\mathbf{curl})$. Moreover, relation (52) and $\mathbf{J} = \ker \mathbf{div}$ lead to $\mathbf{J} \cap \mathfrak{D}(\mathbf{curl}) = \big(\mathbf{J} \cap \mathfrak{D}(\mathbf{div})\big) \cap \mathfrak{D}(\mathbf{curl}) = \mathbf{J} \cap H^1(\mathbb{R}^3, \mathbb{C}^3)$. □

A.5 The Rot operator

The block structure displayed in equation (7) implies $\mathbf{Rot}$ defines a selfadjoint operator on $\mathfrak{D}(\mathbf{Rot}) = \mathfrak{D}(\mathbf{curl}) \oplus_\perp \mathfrak{D}(\mathbf{curl})$ where $\mathfrak{D}(\mathbf{curl})$ is the domain of the rotation operator **curl** as given in Corollary A.3. The splitting (58) of $\mathfrak{D}(\mathbf{curl})$ carries over to **Rot**, namely

$$\mathfrak{D} := \mathfrak{D}(\mathbf{Rot}) = \big(\ker \mathbf{Div} \cap H^1(\mathbb{R}^3, \mathbb{C}^6)\big) \oplus_\perp \operatorname{ran} \mathbf{Grad}, \tag{59}$$

where $\mathbf{Div} := \mathbf{div} \oplus \mathbf{div}$ and $\mathbf{Grad} := \nabla_x \oplus \nabla_x$ consist of two copies of **div** and ∇_x which are defined as in Appendix A, and $\operatorname{ran} \mathbf{Grad}$ is the closure of $\operatorname{ran}_0 \mathbf{Grad}$.
The splitting of the domain (59) is motivated by the orthogonal decomposition of

$$L^2(\mathbb{R}^3, \mathbb{C}^6) = \mathbf{J} \oplus_\perp \mathbf{G} := \ker \mathbf{Div} \oplus_\perp \operatorname{ran} \mathbf{Grad} = \operatorname{ran} \mathbf{Rot} \oplus_\perp \ker \mathbf{Rot}$$

into transversal and longitudinal vector fields provided by the Helmholtz-Hodge-Weyl-Leray theorem (cf. Section A.4); it extends the unique splitting

$$\Psi = \mathbf{Rot}\, \Phi + \mathbf{Grad}\, \varphi, \qquad \Phi \in L^2(\mathbb{R}^3, \mathbb{C}^6),\ \varphi \in L^2_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^2),$$

from $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3, \mathbb{C}^6)$ to all of $L^2(\mathbb{R}^3, \mathbb{C}^6)$. Note that the vectors $\mathbf{Rot}\, \Phi$ and $\mathbf{Grad}\, \varphi$ are orthogonal with respect to the scalar product $\langle \cdot , \cdot \rangle_{L^2(\mathbb{R}^3, \mathbb{C}^6)}$, and thus there exist orthogonal projections **P** and **Q** onto **J** and **G**. Moreover, Remark A.2 implies $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3, \mathbb{C}^6)$ and $H^1(\mathbb{R}^3, \mathbb{C}^6)$ are cores of **Rot**.
The free Maxwell operator $\mathbf{Rot} \cong \int_{\mathbb{B}}^{\oplus} \mathrm{d}k\, \mathbf{Rot}(k)$ is periodic with respect to any lattice, and thus we can use the Zak transform to fiber decompose it. The eigenvectors to any eigenvalue of $\mathbf{Rot}(k)$ can be explicitly constructed in terms of plane waves.

LEMMA A.4 (BAND SPECTRUM OF $\mathbf{Rot}^{\mathcal{Z}}$)

(i) $\sigma\big(\mathbf{Rot}(k)\big) = \{0\} \cup \bigcup_{\gamma^* \in \Gamma^*} \big\{ \pm |\gamma^* + k| \big\}$

(ii) There exists a k-dependent family of linearly independent vectors

$$\big\{ u_{j \pm \gamma^*}(k) \ \big|\ \gamma^* \in \Gamma^*,\ j = 1, 2, 3 \big\}$$

which spans all of $L^2(\mathbb{T}^3, \mathbb{C}^6)$ and has the following properties:

(1) The $u_{j \pm \gamma^*}(k)$ are eigenfunctions to $\mathbf{Rot}(k)$ with eigenvalues $\pm|\gamma^* + k|$ or 0 for all $k \in \mathbb{R}^3$.

(2) Away from $\Gamma^* \subset \mathbb{R}^3$, all maps $k \mapsto u_{j \pm \gamma^*}(k) \in L^2(\mathbb{T}^3, \mathbb{C}^6)$ are locally analytic on a small neighborhood which can be chosen to be independent of j and γ^*.

(3) Near $\gamma^*_0 \in \Gamma^*$, only those $u_{j \pm \gamma^*}(k)$ are locally analytic on a common neighborhood for which $\gamma^* \neq -\gamma^*_0$ holds.

Proof We begin by analyzing the original operator $\mathbf{Rot} = \mathbf{curl} \otimes \sigma_2$ which can be factorized into an operator acting on $L^2(\mathbb{R}^3, \mathbb{C}^3)$ and a 2×2 matrix. The Pauli matrix σ_2 has eigenvalues ± 1 and eigenvectors $w_\pm$. **curl** fibers in ξ after applying the usual Fourier transform $\mathcal{F} : L^2(\mathbb{R}^3, \mathbb{C}^3) \longrightarrow L^2(\mathbb{R}^3, \mathbb{C}^3)$,

$$\mathcal{F} \, \nabla_x^\times \, \mathcal{F}^{-1} = \int_{\mathbb{R}^3}^{\oplus} \mathrm{d}\xi \, (\mathrm{i}\xi)^\times =: \int_{\mathbb{R}^3}^{\oplus} \mathrm{d}\xi \, \mathbf{curl}(\xi),$$

and $\mathbf{curl}(\xi) = \mathrm{i}\xi^\times$ (see equation (5)) can be diagonalized explicitly: it has eigenvalues $\{0, \pm |\xi|\}$. Moreover, it can be seen that the eigenvectors $v_j(\xi)$, $j = 1, 2, 3$, are analytic away from $\xi = 0$. For $\xi \neq 0$, we set $v_1(\xi)$, $v_2(\xi)$ and $v_3(\xi)$ to be the eigenvectors to $+|\xi|$, $-|\xi|$ and 0, respectively. At $\xi = 0$ neither the eigenvalues $\pm |\xi|$ nor the eigenvectors are analytic.
Now to the proof of the Lemma: For $j = 1, 2, 3$ let us set

$$u_{j \pm \gamma^*}(k) := \mathrm{e}^{+\mathrm{i}\gamma^* \cdot y} \, v_j(\gamma^* + k) \otimes w_\pm$$

where $v_j(\gamma^* + k)$ is defined as in the preceding paragraph for $\xi = \gamma^* + k$. The exponential functions $\{\mathrm{e}^{+\mathrm{i}\gamma^* \cdot y}\}_{\gamma^* \in \Gamma^*}$ and the $\{v_j(\xi) \otimes w_\pm\}_{j=1,2,3}$ form a basis of $L^2(\mathbb{T}^3)$ and $\mathbb{C}^3 \otimes \mathbb{C}^2 \cong \mathbb{C}^6$, respectively, and hence, the set of all $u_{j \pm \gamma^*}$ forms a basis of $L^2(\mathbb{T}^3, \mathbb{C}^6)$. Moreover, these vectors are eigenfunctions to $\mathbf{Rot}(k)$ with eigenvalues $\pm|\gamma^* + k|$ ($j = 1, 2$) or 0 ($j = 3$), and thus we have shown (i), $\sigma\big(\mathbf{Rot}(k)\big) = \{0\} \cup \bigcup_{\gamma^* \in \Gamma^*} \big\{\pm|\gamma^* + k|\big\}$, and (ii) (1).
If $k_0 \in \mathbb{R}^3 \setminus \Gamma^*$, then

$$|\gamma^* + k| \geq \mathrm{dist}\big(k_0, \Gamma^*\big) > 0$$

is bounded from below which implies the eigenvectors $u_{j \pm \gamma^*}$ are analytic in some neighborhood of k_0. These vectors $v_j(\gamma^* + k)$, $j = 1, 2, 3$, are analytic on an open ball around k_0 with radius $\mathrm{dist}\big(k_0, \Gamma^*\big)$, proving (ii) (2).
If, on the other hand, $k_0 = \gamma_0^* \in \Gamma^*$, then the basis involves the vector

$$u_{j \pm -\gamma_0^*}(\gamma_0^*) = \mathrm{e}^{-\mathrm{i}\gamma_0^* \cdot y} \, v_j(0) \otimes w_\pm$$

which cannot be extended analytically to a neighborhood of $k_0 = \gamma_0^*$, thus proving (ii) (3). □

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Exact Criterion for Global Existence and Blow Up to a Degenerate Keller-Segel System

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Abstract. A degenerate Keller-Segel system with diffusion exponent m with $\frac{2n}{n+2} < m < 2 - \frac{2}{n}$ in multi dimension is studied. An exact criterion for global existence and blow up of solution is obtained. The estimates on $L^{\frac{2n}{n+2}}$ norm of the solution play important roles in our analysis. These estimates are closely related to the optimal constant in the Hardy- Littlewood- Sobolev inequality. In the case of initial free energy less than a universal constant which depends on the inverse of total mass, there exists a constant such that if the $L^{\frac{2n}{n+2}}$ norm of initial data is less than this constant, then the weak solution exists globally; if the $L^{\frac{2n}{n+2}}$ norm of initial data is larger than the same constant, then the solution must blow-up in finite time. Our result shows that the total mass, which plays the deterministic role in two dimension case, might not be an appropriate criterion for existence and blow up discussion in multi-dimension, while the $L^{\frac{2n}{n+2}}$ norm of the initial data and the relation between initial free energy and initial mass are more important.

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1. Introduction

In this article, we will study a degenerate Keller-Segel system for $n \geq 3$ dimension:

$$
\begin{cases} \rho_t = \Delta \rho^m - \operatorname{div}(\rho \nabla c), & x \in \mathbb{R}^n,\, t \geq 0, \\ -\Delta c = \rho, & x \in \mathbb{R}^n,\ t \geq 0, \\ \rho(x,0) = \rho_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}
$$

where diffusion exponent $m \in (\frac{2n}{n+2}, 2 - \frac{2}{n})$, $\rho(x,t)$ represents the density of bacteria and $c(x,t)$ represents the chemical substance concentration. Mass conservation of the system implies $\|\rho(\cdot,t)\|_{L^1} = \|\rho_0(\cdot)\|_{L^1} = M_0$.

Keller-Segel system with linear diffusion was proposed by Patlak [16] and Keller-Segel [13, 14]. It is used to describe the collective motion of cells or the evolution of the density of bacteria. This model plays important roles in the study of chemotaxis in mathematical biology. Since 1980, Keller-Segel system was widely studied in the literature. From the work by Childress [7], we known that the behavior of this model strongly depends on the space dimension, the readers can refer to two surveys given by Horstmann [11, 12].

Recently, many mathematicians are interested in finding the criterion for global existence and blow up of solution to Keller-Segel type systems. In particular, the 2-dimensional case has been well studied. It is well known that 8π is the critical mass of 2-dimensional Keller-Segel system [5, 10, 17]. More precisely, if the initial mass $M_0 < 8\pi$, then there exists global weak solution; if $M_0 > 8\pi$, then the solution blows up in finite time; The more delicate case $M_0 = 8\pi$ was studied in [2, 4].

In dimension $n \geq 3$, one has to use nonlinear diffusion to balance the non-local aggregation effect. A natural question is to find a criterion for initial data to separate the global existence and finite time blow up to degenerate Keller-Segel system (1.1) with diffusion exponent $m > 1$.

There were two critical diffusion exponents of (1.1) which have been studied recently. One is that $m^* = 2 - \frac{2}{n}$, which came from the scaling invariance of the total mass. The following results were obtained in [18, 19]. If $m > m^*$, the solution exists globally for any initial data; if $1 < m \leq m^*$, both global existence and blow-up can happen for some initial data. Later on, Blanchet-Carrillo-Laurencot in [3] studied the degenerate system with diffusion exponent $m = m^*$, a critical mass was given there. Another critical exponent of (1.1), $m_c = \frac{2n}{n+2}$ was given in [6], which came from the conformal invariance of the free energy. The authors in [6] showed that L^{m_c} norm of a family of positive stationary solution can be viewed as the criterion for the global existence and blow up of solutions.

In this paper we are interested in finding a criterion to classify the initial data to get either global existence or blow up of the solution. Our analysis will work for all the diffusion exponents m such that $\frac{2n}{n+2} = m_c < m < m^* = 2 - \frac{2}{n}$.

There are two very important quantities of system (1.1). One is the total mass which is time independent,

$$\int_{\mathbb{R}^n} \rho(x,t)dx = \int_{\mathbb{R}^n} \rho_0(x)dx = M_0,$$

the other is the free energy

$$\mathcal{F}(\rho) = \frac{1}{m-1}\int_{\mathbb{R}^n} \rho^m(x,t)dx - \frac{1}{2}\int_{\mathbb{R}^n} \rho(x,t)c(x,t)dx,$$

which decays in time due to the following entropy-entropy production relation

$$\frac{d}{dt}\mathcal{F}(\rho(\cdot,t)) + \int_{\mathbb{R}^n} \rho\Big|\nabla(\frac{m}{m-1}\rho^{m-1} - c)\Big|^2 dx = 0.$$

The main result of this paper is

THEOREM 1.1. *Assume that the initial density $\rho_0 \in L^1_+(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)$ and $\mathcal{F}(\rho_0) < \mathcal{F}^*$, the following holds,*

(1) *If $\|\rho_0\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} < (s^*)^{\frac{n-2}{2n(m-1)}}$, then (1.1) has a global weak solution, i.e. for all $T > 0$ and some $1 < r, s \le 2$, there is a function $\rho(x,t)$ with*

$$\rho \in L^\infty(0,+\infty; L^1_+(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)),$$
$$\nabla\rho \in L^2(0,T; L^r(\mathbb{R}^n)),\ \partial_t\rho \in L^2(0,T; W^{-1,s}_{\mathrm{loc}}(\mathbb{R}^n)),$$

such that it satisfies (1.1) in the sense of distribution.

(2) *If $\|\rho_0\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} > (s^*)^{\frac{n-2}{2n(m-1)}}$ and ρ_0 has finite second moment, $\rho(x,t)$ is a solution of (1.1), then there exists a $T^* > 0$ such that*

$$\lim_{t\to T^*} \|\rho(\cdot,t)\|_{L^m(\mathbb{R}^n)} = +\infty. \tag{1.2}$$

Here $\mathcal{F}^*$ and s^* are universal constants given by

$$\mathcal{F}^* = \frac{2-\frac{2}{n}-m}{(m-1)(1-\frac{2}{n})}\Big(\frac{2n^2\alpha(n)}{C(n)}\Big)^{\frac{n(m-1)}{2n-2-mn}} M_0^{\frac{2n-m(n+2)}{2n-2-mn}} > 0, \tag{1.3}$$

$$s^* = \Big(\frac{2n^2\alpha(n)M_0^{\frac{2n-m(n+2)}{n-2}}}{C(n)}\Big)^{\frac{n(m-1)}{2n-2-mn}} > 0, \tag{1.4}$$

where M_0 is the initial mass $\|\rho_0\|_{L^1(\mathbb{R}^n)}$, $\alpha(n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ is the volume of the unit ball of $\mathbb{R}^n$ and $C(n)$ is the best constant of the Hardy-Littlewood-Sobolev inequality, see (1.9).

Remark 1.1. We remark here that under the condition $\mathcal{F}(\rho_0) < \mathcal{F}^*$, $L^{\frac{2n}{n+2}}$ norm of the initial data can not be $(s^*)^{\frac{n-2}{2n(m-1)}}$, which can be easily checked by using the decomposition of the free energy. Thus the classification of the initial data in Theorem 1.1 is complete.

Remark 1.2. The result does not hold for $m = m^* = 2 - \frac{2}{n}$, thus there is no contradiction with the result by Blanchet *et al.* in [3], where a critical mass was obtained.

Remark 1.3. The conditions $\rho_0 \in L^1_+(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)$ and $\|\rho_0\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} < (s^*)^{\frac{n-2}{2n(m-1)}}$ for the existence result imply that the initial free energy is positive, i.e. $\mathcal{F}(\rho_0) > 0$, which can be easily checked by direct computations. Conversely, if the initial free energy is negative, i.e. $\mathcal{F}(\rho_0) < 0$ and $\rho_0 \in L^1_+(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)$, then $\|\rho_0\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} > (s^*)^{\frac{n-2}{2n(m-1)}}$. Therefore, our result on the blow-up of solutions allows more initial data than those in the work by Sugiyama. Thus the blow up result improves her work with $\gamma = 0$. (In [18], Y. Sugiyama proved that if the initial free energy is negative and $\rho_0 \in L^1_+(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)$, then the solution to the degenerate Keller-Segel with Bessel potential blows up in finite time.) In fact, Theorem 1.1 gives an exact classification of the initial data so that the solution either exists globally or blow-up in finite time. More precisely, it is the constant $(s^*)^{\frac{n-2}{2n(m-1)}}$, where s^* is stated in (1.4), which classifies the initial data in $L^{\frac{2n}{n+2}}$ norm.

Remark 1.4. The exponents of M_0 in (1.3) and (1.4) are both negative due to the fact that $\frac{2n}{n+2} < m < 2 - \frac{2}{n}$. The assumption $\mathcal{F}(\rho_0) < \mathcal{F}^*$ in Theorem 1.1 gives a relation between the initial mass and the initial free energy, i.e.

$$\mathcal{F}(\rho_0) M_0^{\frac{m(n+2)-2n}{2n-2-mn}} < \frac{2-\frac{2}{n}-m}{(m-1)(1-\frac{2}{n})} \Big(\frac{2n^2\alpha(n)}{C(n)}\Big)^{\frac{n(m-1)}{2n-2-mn}}. \tag{1.5}$$

As a conclusion, Theorem 1.1 implies that *the initial mass itself might not be an important quantity in the existence and blow up analysis in multi-dimension.* More precisely, no matter how small the initial mass is, the solution can still blow up in case that $\|\rho_0\|_{L^{\frac{2n}{n+2}}} > (s^*)^{\frac{n-2}{2n(m-1)}}$. No matter how large the initial mass is, there still exists a global weak solution if $\|\rho_0\|_{L^{\frac{2n}{n+2}}} < (s^*)^{\frac{n-2}{2n(m-1)}}$. The similar fact that the initial mass is not a relevant quantity for blow-up in the multi-dimensional Keller-Segel model is known in the literature, such as in [9] where (1.1) with $m = 1$ was considered. Moreover, we can find a consistent phenomenon with this result in parabolic-parabolic model, such as in [20, 8]. In [8], the norm of $\|\rho_0\|_{L^{\frac{n}{2}}}$ was used to discuss existence and blow-up. The author in [20] studied the case with smooth bounded domain with homogeneous Neumann boundary conditions, they obtained the existence result for small initial data in L^q, $q > \frac{n}{2}$ and if the domain is ball, there is always an unbounded solution developed from initial data with arbitrary small mass.

Example 1. *For given $\varepsilon_0 > 0$ arbitrarily small, let the initial data be*

$$\rho_0(x) = \begin{cases} \varepsilon_0 \frac{K^n}{\alpha(n)}, & |x| \le \frac{1}{K}, \\ 0, & |x| > \frac{1}{K}, \end{cases}$$

where K to be determined later. Then

$$\|\rho_0\|_{L^1} = \varepsilon_0, \quad \|\rho_0\|_{L^{\frac{2n}{n+2}}} = \varepsilon_0 \Big(\frac{K^n}{\alpha(n)}\Big)^{\frac{n-2}{2n}} \quad \textit{and} \quad \int_{\mathbb{R}^n} |x|^2 \rho_0 dx < \infty.$$

Now we can choose K large such that

$$\|\rho_0\|_{L^{\frac{2n}{n+2}}} > (s^*)^{\frac{n-2}{2n(m-1)}}, \tag{1.6}$$

and

$$\mathcal{F}(\rho_0) M_0^{\frac{m(n+2)-2n}{2n-2-mn}} < \frac{2-\frac{2}{n}-m}{(m-1)(1-\frac{2}{n})}\Big(\frac{2n^2\alpha(n)}{C(n)}\Big)^{\frac{n(m-1)}{2n-2-mn}}. \tag{1.7}$$

Therefore according to our result in theorem 1.1, the solution must blow up in finite time.

We will give a detailed calculation of this example in the Appendix.

Similarly, we can find some initial data with large initial mass such that the solution exist globally.

It should also be mentioned that the constants appeared in the main result have close relation to the critical Hardy-Littlewood-Sobolev inequality. For completeness, we cite this result from [15].

Proposition 1.1 (H.-L.-S. inequality). *Let $\rho \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)$, then*

$$\int\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\rho(x)\rho(y)}{|x-y|^{n-2}} dxdy \le C(n)\|\rho\|^2_{L^{\frac{2n}{n+2}}}, \tag{1.8}$$

where

$$C(n) = \pi^{(n-2)/2}\frac{1}{\Gamma(n/2+1)}\left\{\frac{\Gamma(n/2)}{\Gamma(n)}\right\}^{-2/n}. \tag{1.9}$$

Moreover, the equality holds if and only if $\rho(x) = AU_{\lambda,x_0}$, for some constant A and parameters $\lambda > 0$, $x_0 \in \mathbb{R}^n$, where

$$U_{\lambda,x_0} = 2^{\frac{n+2}{4}} n^{\frac{n+2}{2}} \left(\frac{\lambda}{\lambda^2 + |x-x_0|^2}\right)^{\frac{n+2}{2}}. \tag{1.10}$$

This family of radially symmetric functions (1.10) is also a class of stationary solution of the degenerate system (1.1) with diffusion exponent $m = m_c = \frac{2n}{n+2}$. The readers are referred to [6] for the relations among stationary solution, the Hardy-Littlewood-Sobolev inequality and conformal invariance of the free energy. A direct scaling analysis tells us that $L^{\frac{2n}{n+2}}$ norm of U_{λ,x_0} is a universal constant independent of the parameters λ and x_0.

We can separate the free energy into two parts by using the Hardy-Littlewood-Sobolev inequality (1.8), namely,

$$\begin{aligned} \mathcal{F}(\rho) \quad &= \quad \frac{1}{m-1}\int_{\mathbb{R}^n} \rho^m(x,t)dx - \frac{C(n)}{2(n-2)n\alpha(n)}\|\rho\|^2_{L^{\frac{2n}{n+2}}} \\ &+ \quad \frac{C(n)}{2(n-2)n\alpha(n)}\|\rho\|^2_{L^{\frac{2n}{n+2}}} - \frac{1}{2(n-2)n\alpha(n)}\int\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\rho(x,t)\rho(y,t)}{|x-y|^{n-2}}dxdy \\ &=: \quad \mathcal{F}_1(\rho) + \mathcal{F}_2(\rho). \end{aligned}$$

Proposition 1.1 says that $\mathcal{F}_2(\rho) \ge 0$.

Since the first part of the free energy is concave in $L^{\frac{2n}{n+2}}$ norm of the solution, it is not difficult to get *a priori* estimates, which shows that in the cases of supercritical and subcritical initial data, the quantity $\|\rho\|_{L^{\frac{2n}{n+2}}}$ can be bounded from below or from above separately. More precisely, if the initial free energy $\mathcal{F}(\rho_0) < \mathcal{F}^*$, then the following estimates hold

(1) If $\|\rho_0\|_{\frac{2n}{n+2}} < (s^*)^{\frac{n-2}{2n(m-1)}}$, then there exists a constant $\mu_1 < 1$ such that

$$\|\rho(\cdot,t)\|_{\frac{2n}{n+2}} < (\mu_1 s^*)^{\frac{n-2}{2n(m-1)}}, \text{ for all } t > 0.$$

(2) If $\|\rho_0\|_{\frac{2n}{n+2}} > (s^*)^{\frac{n-2}{2n(m-1)}}$, then there exists a constant $\mu_2 > 1$ such that

$$\|\rho(\cdot,t)\|_{\frac{2n}{n+2}} > (\mu_2 s^*)^{\frac{n-2}{2n(m-1)}}, \text{ for all } t > 0.$$

We will give the proof of the first fact for the regularized solution in the Lemma 2.1 in section 2, and show that the second is true in Lemma 3.1 in section 3.

This paper is arranged as follows. In section 2, we will give the proof of the global existence of weak solution. After introducing the regularized problem, a uniform estimate for the $L^{\frac{2n}{n+2}}$ norm of the regularized solution by using decomposition of the free energy is obtained. Based on this estimate, further estimates, including the spacial and time derivatives, are derived. Then the global existence follows from standard compactness arguments with the help of Aubin's lemma. In section 3, with supercritical initial data, it is shown that any solution will blow-up in finite time by studying the time derivative of second moment.

2. Existence of weak solution

We follow the same way on the construction of the regularized problem as in [3, 18, 19], namely,

(2.1)
$$\begin{cases} \partial_t \rho_\varepsilon = \Delta[(\rho_\varepsilon+\varepsilon)^m - \varepsilon^m] - \operatorname{div}((\rho_\varepsilon+\varepsilon)\nabla c_\varepsilon), & x \in \mathbb{R}^n,\ t \geq 0, \\ -\Delta c_\varepsilon = J_\varepsilon * \rho_\varepsilon, & x \in \mathbb{R}^n,\ t \geq 0, \\ \rho(x,0) = \rho_{0\varepsilon}(x), & x \in \mathbb{R}^n \end{cases}$$

for $\varepsilon > 0$, $J_\varepsilon(x) = \frac{1}{\varepsilon^n} J(\frac{x}{\varepsilon})$, $J(x) = \frac{1}{\alpha(n)}(1+|x|^2)^{-(n+2)/2}$ satisfying $\displaystyle\int_{\mathbb{R}^n} J_\varepsilon(x)dx = 1$. A simple computation derives

$$c_\varepsilon = \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^n} \frac{1}{(|x-y|^2+\varepsilon^2)^{\frac{n-2}{2}}} \rho_\varepsilon(y)dy.$$

The initial data $\rho_{0\varepsilon}$ is the regularization of the function ρ_0, it satisfies that there exists a positive constant δ such that for all $0 < \varepsilon < \delta$,

$$\rho_{0\varepsilon} > 0, \qquad \rho_{0\varepsilon} \in L^r(\mathbb{R}^n),\ r \geq 1, \qquad \|\rho_{0\varepsilon}\|_{L^1} = \|\rho_0\|_{L^1} = M_0,$$

Moreover, as $\varepsilon \to 0$,

$$\text{if } \rho_0 \in L^p \text{ for some } p > 1, \text{ then } \|\rho_{0\varepsilon} - \rho_0\|_{L^p} \to 0, \text{ as } \varepsilon \to 0,$$
$$\int_{\mathbb{R}^n} |x|^2 \rho_{0\varepsilon} dx \to \int_{\mathbb{R}^n} |x|^2 \rho_0 dx, \qquad \mathcal{F}_\varepsilon(\rho_{0\varepsilon}) \to \mathcal{F}(\rho_0),$$

where $\mathcal{F}_\varepsilon(\rho_{0\varepsilon})$ is the initial regularized entropy, see (2.2).

The classical parabolic theory implies that the above regularized problem has a global smooth non-negative solution ρ_ε for $t > 0$ if the initial data is non-negative. Notice that the solution of the regularized problem (2.1) still conserves the mass.

We will mainly focus on the estimates of the regularized solutions in this section. After getting $L^{\frac{2n}{n+2}}$ estimate with the help of the free energy, we obtain the uniform L^p estimates by using standard method. Furthermore, the uniform estimates for space and time derivatives will be derived carefully. With all these uniform estimates, a standard compactness argument as in [6, 1] by using Aubin's lemma will give the global existence.

From now on, we will present the uniform estimates in five steps and will skip the compactness arguments.

Step 1. Free energy of the regularized problem

The free energy on the regularized solution ρ_ε is

$$\mathcal{F}_\varepsilon(\rho_\varepsilon) = \frac{1}{m-1} \int_{\mathbb{R}^n} ((\rho_\varepsilon + \varepsilon)^m - \varepsilon^m) dx - \frac{1}{2} \int_{\mathbb{R}^n} \rho_\varepsilon c_\varepsilon dx. \tag{2.2}$$

Or, the free energy has an equivalent form in the following

$$\begin{aligned} \mathcal{F}_\varepsilon(\rho_\varepsilon) &= \frac{1}{m-1} \int_{\mathbb{R}^n} ((\rho_\varepsilon + \varepsilon)^m - \varepsilon^m) dx \\ &\quad - \frac{1}{2(n-2)n\alpha(n)} \int\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\rho_\varepsilon(x,t)\rho_\varepsilon(y,t)}{(|x-y|^2 + \varepsilon^2)^{\frac{n-2}{2}}} dxdy. \end{aligned} \tag{2.3}$$

It is easy to check that $\mathcal{F}_\varepsilon(\rho_\varepsilon)$ is non-increasing in time. In fact, the system (2.1) has the gradient flow structure

$$\rho_{\varepsilon t} = \operatorname{div}\left((\rho_\varepsilon + \varepsilon)\nabla\left(\frac{m}{m-1}(\rho_\varepsilon + \varepsilon)^{m-1} - c_\varepsilon\right)\right). \tag{2.4}$$

Now by taking $\frac{m}{m-1}\left((\rho_\varepsilon + \varepsilon)^{m-1} - \varepsilon^{m-1}\right) - c_\varepsilon$ as a test function, we have the following entropy-entropy production relation

$$\frac{d}{dt}\mathcal{F}_\varepsilon(\rho_\varepsilon(\cdot,t)) + \int_{\mathbb{R}^n} (\rho_\varepsilon + \varepsilon)\left|\nabla\left(\frac{m}{m-1}(\rho_\varepsilon + \varepsilon)^{m-1} - c_\varepsilon\right)\right|^2 dx = 0.$$

The monotone decreasing property of the free energy follows immediately by the non-negativity of the entropy production.

Next, we separate the free energy into two parts by using the Hardy-Littlewood-Sobolev inequality (1.8), i.e.,

$$\begin{aligned}
\mathcal{F}_\varepsilon(\rho_\varepsilon) &= \frac{1}{m-1}\int_{\mathbb{R}^n}((\rho_\varepsilon+\varepsilon)^m-\varepsilon^m)\,dx - \frac{C(n)}{2(n-2)n\alpha(n)}\|\rho_\varepsilon\|^2_{L^{\frac{2n}{n+2}}} \\
&+\frac{C(n)}{2(n-2)n\alpha(n)}\|\rho_\varepsilon\|^2_{L^{\frac{2n}{n+2}}} - \frac{1}{2(n-2)n\alpha(n)}\int\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\rho_\varepsilon(x,t)\rho_\varepsilon(y,t)}{(|x-y|^2+\varepsilon^2)^{\frac{n-2}{2}}}dxdy \\
&\geq \frac{1}{m-1}\int_{\mathbb{R}^n}\rho_\varepsilon^m dx - \frac{C(n)}{2(n-2)n\alpha(n)}\|\rho_\varepsilon\|^2_{L^{\frac{2n}{n+2}}} \\
&+\frac{C(n)}{2(n-2)n\alpha(n)}\|\rho_\varepsilon\|^2_{L^{\frac{2n}{n+2}}} - \frac{1}{2(n-2)n\alpha(n)}\int\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\rho_\varepsilon(x,t)\rho_\varepsilon(y,t)}{|x-y|^{n-2}}dxdy \\
&=: \mathcal{F}_1(\rho_\varepsilon)+\mathcal{F}_2(\rho_\varepsilon).
\end{aligned}$$

Proposition 1.1 shows that the second part of the free energy is non-negative, i.e. $\mathcal{F}_2(\rho_\varepsilon)\geq 0$.

Due to $m>\dfrac{2n}{n+2}$, interpolation shows that

$$\|\rho_\varepsilon\|_{L^{\frac{2n}{n+2}}} \leq \|\rho_\varepsilon\|^{1-\theta}_{L^1}\|\rho_\varepsilon\|^{\theta}_{L^m}, \quad \theta=\frac{m(n-2)}{2n(m-1)}. \tag{2.5}$$

Thus the first part of the free energy is

$$\begin{aligned}
\mathcal{F}_1(\rho_\varepsilon) &= \frac{1}{m-1}\int_{\mathbb{R}^n}\rho_\varepsilon^m(x,t)dx - \frac{C(n)}{2(n-2)n\alpha(n)}\|\rho_\varepsilon\|^2_{L^{\frac{2n}{n+2}}} \\
&\geq \frac{1}{m-1}\|\rho_\varepsilon\|_{L^1}^{\frac{(\theta-1)m}{\theta}}\|\rho_\varepsilon\|^{\frac{m}{\theta}}_{L^{\frac{2n}{n+2}}} - \frac{C(n)}{2(n-2)n\alpha(n)}\|\rho_\varepsilon\|^2_{L^{\frac{2n}{n+2}}} \qquad (2.6)\\
&\geq \frac{1}{m-1}M_0^{\frac{2n-m(n+2)}{n-2}}\|\rho_\varepsilon\|^{\frac{2n(m-1)}{n-2}}_{L^{\frac{2n}{n+2}}} - \frac{C(n)}{2(n-2)n\alpha(n)}\|\rho_\varepsilon\|^2_{L^{\frac{2n}{n+2}}}.
\end{aligned}$$

According to the previous analysis, let

$$f(s)=\frac{1}{m-1}M_0^{\frac{2n-m(n+2)}{n-2}}s-\frac{C(n)}{2(n-2)n\alpha(n)}s^{\frac{n-2}{n(m-1)}}.$$

We now have a lower bound of the first part of free energy, i.e. $f\Big(\|\rho_\varepsilon\|^{\frac{2n(m-1)}{n-2}}_{L^{\frac{2n}{n+2}}}\Big)\leq \mathcal{F}_1(\rho_\varepsilon)$.

Step 2. Uniform $L^{\frac{2n}{n+2}}$ norm estimate of the regularized solution.

The following lemma shows that for subcritical initial data, the quantity $\|\rho_\varepsilon\|_{L^{\frac{2n}{n+2}}}$ can be bounded.

Lemma 2.1. *If the initial free energy $\mathcal{F}_\varepsilon(\rho_{0\varepsilon})<\mathcal{F}^*:=f(s^*)$, $\|\rho_{0\varepsilon}\|_{\frac{2n}{n+2}}<(s^*)^{\frac{n-2}{2n(m-1)}}$, let $\rho_\varepsilon(x,t)$ be a solution of problem (2.1), then there exists a constant $\mu_1<1$ such that*

$$\|\rho_\varepsilon(\cdot,t)\|_{\frac{2n}{n+2}}<(\mu_1 s^*)^{\frac{n-2}{2n(m-1)}},\ \textit{for all } t>0,$$

where s^* is the maximum point of $f(s)$:

$$s^* = \Big(\frac{2n^2\alpha(n)M_0^{\frac{2n-m(n+2)}{n-2}}}{C(n)}\Big)^{\frac{n(m-1)}{2n-2-mn}}. \tag{2.7}$$

Proof. Notice that $1 < m < 2 - \frac{2}{n}$ implies $\frac{n-2}{n(m-1)} > 1$, we know that $f(s)$ is a strictly concave function in $0 < s < \infty$. Directly calculation shows that

$$f'(s) = \frac{1}{m-1}M_0^{\frac{2n-m(n+2)}{n-2}} - \frac{C(n)}{2(n-2)n\alpha(n)}\frac{n-2}{n(m-1)}s^{\frac{2n-2-mn}{n(m-1)}}.$$

As a consequence, s^* is a unique maximum point of $f(s)$. Therefore the important property of f is that $f(s)$ is monotone increasing for $0 < s < s^*$, while $f(s)$ is monotone decreasing for $s > s^*$.

In the case that initial free energy $\mathcal{F}_\varepsilon(\rho_{0\varepsilon}) < f(s^*)$, we can make it even smaller, i.e. there is a $\delta < 1$ such that $\mathcal{F}_\varepsilon(\rho_{0\varepsilon}) < \delta f(s^*)$.

Combining all the facts we know, including the interpolation, the Hardy-Littlewood-Sobolev inequality and the monotonicity of free energy, we have

$$f\Big(\|\rho_\varepsilon\|_{L^{\frac{2n}{n+2}}}^{\frac{2n(m-1)}{n-2}}\Big) \le \mathcal{F}_1(\rho_\varepsilon) \le \mathcal{F}_\varepsilon(\rho_\varepsilon) \le \mathcal{F}_\varepsilon(\rho_{0\varepsilon}) < \delta f(s^*). \tag{2.8}$$

If initially $\|\rho_{0\varepsilon}\|_{L^{\frac{2n}{n+2}}}^{\frac{2n(m-1)}{n-2}} < s^*$, due to the fact that $f(s)$ is increasing in $0 < s < s^*$, there exists a $\mu_1 < 1$ such that $\|\rho_\varepsilon\|_{L^{\frac{2n}{n+2}}}^{\frac{2n(m-1)}{n-2}} < \mu_1 s^*$. □

Step 3. Uniform L^p $(1 < p < n)$ estimates of the regularized solution.

Under the assumption of $\rho_{0\varepsilon} \in L^p(\mathbb{R}^n)$ with $1 < p < n$, we will give the estimate on $\|\rho_\varepsilon\|_{L^p}$, and as a byproduct, also the uniform estimates on space derivatives $\nabla\rho_\varepsilon^{\frac{m+p-1}{2}}$ and ∇c_ε.

Lemma 2.2. *Assume $\rho_{0\varepsilon} \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $\|\rho_{0\varepsilon}\|_{L^{\frac{2n}{n+2}}} < (s^*)^{\frac{n-2}{2n(m-1)}}$ and $\mathcal{F}_\varepsilon(\rho_{0\varepsilon}) < \mathcal{F}^* := f(s^*)$, ρ_ε is a smooth solution of the regularized problem* (2.1)*, then*

$$\|\rho_\varepsilon\|_{L^\infty(0,T;L^p(\mathbb{R}^n)\cap L^{p+1}(0,T;L^{p+1}(\mathbb{R}^n)))} \le C, \ \ \|\nabla\rho_\varepsilon^{\frac{m+p-1}{2}}\|_{L^2(0,T;L^2(\mathbb{R}^n))} \le C, \tag{2.9}$$

moreover, for $1 < p < n$, it holds

$$\|\nabla c_\varepsilon\|_{L^\infty(0,T;L^s(\mathbb{R}^n))} \le C, \quad s \in (\frac{n}{n-1}, \frac{np}{n-p}]. \tag{2.10}$$

Proof. Multiplying the first equation of (2.1) by $p\rho_\varepsilon^{p-1}$ with $p>1$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho_\varepsilon^p dx\\
=\;& -pm(p-1)\int_{\mathbb{R}^n}(\rho_\varepsilon+\varepsilon)^{m-1}\rho_\varepsilon^{p-2}|\nabla\rho_\varepsilon|^2dx\\
&+(p-1)\int_{\mathbb{R}^n}\nabla\rho_\varepsilon^p\cdot\nabla c_\varepsilon dx+\varepsilon p\int_{\mathbb{R}^n}\nabla\rho_\varepsilon^{p-1}\cdot\nabla c_\varepsilon dx\\
\le\;& -pm(p-1)\int_{\mathbb{R}^n}\rho_\varepsilon^{p+m-3}|\nabla\rho_\varepsilon|^2dx+(p-1)\int_{\mathbb{R}^n}\rho_\varepsilon^{p+1}dx+p\varepsilon\int_{\mathbb{R}^n}\rho_\varepsilon^p dx\\
=\;& -\frac{4pm(p-1)}{(m+p-1)^2}\int_{\mathbb{R}^n}|\nabla\rho_\varepsilon^{\frac{m+p-1}{2}}|^2dx+(p-1)\int_{\mathbb{R}^n}\rho_\varepsilon^{p+1}+p\varepsilon\int_{\mathbb{R}^n}\rho_\varepsilon^p dx.
\end{aligned}$$

Now we will focus on the estimate on $\displaystyle\int_{\mathbb{R}^n}\rho_\varepsilon^{p+1}$.

$$\begin{aligned}
\int_{\mathbb{R}^n}\rho_\varepsilon^{p+1} &= \left\|\rho_\varepsilon^{\frac{m+p-1}{2}}\right\|_{L^{\frac{2(p+1)}{m+p-1}}}^{\frac{2(p+1)}{m+p-1}}\\
&\le G^{\frac{2(p+1)}{m+p-1}}\left\|\nabla\rho_\varepsilon^{\frac{m+p-1}{2}}\right\|_{L^2}^{\alpha\frac{2(p+1)}{m+p-1}}\cdot\left\|\rho_\varepsilon^{\frac{m+p-1}{2}}\right\|_{L^r}^{(1-\alpha)\frac{2(p+1)}{m+p-1}},
\end{aligned}$$

where G is the constant from Gagliardo-Nirenberg-Sobolev ineqality,

$$\frac{m+p-1}{2}r=\frac{2n}{n+2},\qquad \frac{m+p-1}{2(p+1)}=\frac{\alpha(n-2)}{2n}+\frac{1-\alpha}{r},$$

and

$$\alpha=\frac{\frac{m+p-1}{2}\left(\frac{n+2}{2n}-\frac{1}{p+1}\right)}{\frac{(n+2)(m+p-1)-2(n-2)}{4n}}.$$

In the next, we will use notation

$$\nu:=\alpha\frac{2(p+1)}{m+p-1}=\frac{2(n+2)(p+1)-4n}{(n+2)(m+p-1)-2(n-2)}<2$$

in the case of $m>\dfrac{2n}{n+2}$. Thus by Young's inequality, we get

$$\begin{aligned}
\int_{\mathbb{R}^n}\rho_\varepsilon^{p+1} &\le G^{\frac{2(p+1)}{m+p-1}}\left\|\nabla\rho_\varepsilon^{\frac{m+p-1}{2}}\right\|_{L^2}^{\nu}\|\rho_\varepsilon\|_{L^{\frac{2n}{n+2}}}^{(1-\alpha)(p+1)}\\
&\le G^{\frac{2(p+1)}{m+p-1}}\left(\epsilon\left\|\nabla\rho_\varepsilon^{\frac{m+p-1}{2}}\right\|_{L^2}^2+C(\epsilon)\|\rho_\varepsilon\|_{L^{\frac{2n}{n+2}}}^{\frac{2(1-\alpha)(p+1)}{2-\nu}}\right). \qquad (2.11)
\end{aligned}$$

Now we can choose ϵ such that

$$(p-1)G^{\frac{2(p+1)}{m+p-1}}\epsilon=\frac{2pm(p-1)}{(m+p-1)^2}.$$

By using the boundedness of $\|\rho_\varepsilon\|_{L^{\frac{2n}{n+2}}}$ from Lemma 2.1, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n}\rho_\varepsilon^p dx &+ \frac{2pm(p-1)}{(m+p-1)^2}\int_{\mathbb{R}^n}|\nabla\rho_\varepsilon^{\frac{m+p-1}{2}}|^2 dx\\ &\le p\varepsilon\int_{\mathbb{R}^n}\rho_\varepsilon^p dx + C(M_0,p,n).\end{aligned}\tag{2.12}$$

Gronwall's inequality implies that $\rho_\varepsilon \in L^\infty(0,T;L^p(\mathbb{R}^n))$. Therefore we have the uniform estimate by integrating (2.12) in t, for any fixed $T>0$,

$$\sup_{0\le t\le T}\int_{\mathbb{R}^n}\rho_\varepsilon^p(x,t)dx + \frac{2pm(p-1)}{(m+p-1)^2}\int_0^T\int_{\mathbb{R}^n}|\nabla\rho_\varepsilon^{\frac{m+p-1}{2}}|^2 dxdt \le C(M_0,p,n,T).$$

Moreover combining this estimate with (2.11), it is easy to see that $\rho_\varepsilon \in L^{p+1}(0,T;L^{p+1}(\mathbb{R}^n))$. The estimate for ∇c_ε in (2.10) can be directly obtained from the weak Young inequality. $\square$

Remark 2.1. The above lemma gives a general L^p estimate. In particular, we can take $p=m$ and get the estimate $\rho_\varepsilon \in L^\infty(0,T;L^m(\mathbb{R}^n)) \cap L^{m+1}(0,T;L^{m+1}(\mathbb{R}^n))$ which will be used later.

Remark 2.2. The fact that $m > \frac{2n}{n+2}$ is very important in the above proof. It makes the use of Young's inequality successful (see (2.11)), which is impossible in the case $m=\frac{2n}{n+2}$, $\nu=2$.

Step 4. Uniform estimates for the space derivatives

The estimate on space derivative of ρ_ε is important in order to use Aubin's lemma for compactness arguments. We will use the L^p estimate when $p=m$.

Lemma 2.3. *Assume $p=m$ and the assumptions of Lemma 2.1 hold, then*

$$\|\nabla\rho_\varepsilon\|_{L^2(0,T;L^{\frac{2m}{3-m}}(\mathbb{R}^n))} \le C, \quad \text{in the case of } m<\frac{3}{2},\tag{2.13}$$

$$\|\nabla\rho_\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^n))} \le C, \quad \text{in the case of } m\ge\frac{3}{2}.\tag{2.14}$$

Proof. In the case of $m<\frac{3}{2}$, using (2.9), it holds for $p=m$ that

$$\|\rho_\varepsilon\|_{L^\infty(0,T;L^m(\mathbb{R}^n))} \le C, \quad \|\nabla\rho_\varepsilon^{m-\frac{1}{2}}\|_{L^2(0,T;L^2(\mathbb{R}^n))} \le C.\tag{2.15}$$

We can use the expression

$$\nabla\rho_\varepsilon = \frac{2}{2m-1}\rho_\varepsilon^{\frac{3}{2}-m}\nabla\rho_\varepsilon^{m-\frac{1}{2}},$$

then the Hölder inequality and (2.15) imply (2.13).

In the case of $m\ge\frac{3}{2}$, taking ρ_ε^{2-m} as test function in (1.1), we have

$$\begin{aligned}&\frac{1}{3-m}\frac{d}{dt}\int_{\mathbb{R}^n}\rho_\varepsilon^{3-m}dx + m(2-m)\int_{\mathbb{R}^n}|\nabla\rho_\varepsilon|^2 dx\\ &\le \frac{2-m}{3-m}\int_{\mathbb{R}^n}\rho_\varepsilon^{4-m}dx + \varepsilon\int_{\mathbb{R}^n}\rho_\varepsilon^{3-m}dx.\end{aligned}$$

Next we only need to estimate $\int_{\mathbb{R}^n}\rho_\varepsilon^{4-m}dx$ by $\|\rho_\varepsilon\|_{L^m}$ and $\|\nabla\rho_\varepsilon^{m-\frac{1}{2}}\|_{L^2(0,T;L^2(\mathbb{R}^n))}$. By the Gagliardo-Nirenberg-Sobolev inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\rho_\varepsilon^{4-m}dx=\|\rho_\varepsilon^{m-1/2}\|_{L^{\frac{4-m}{m-1/2}}}^{\frac{4-m}{m-1/2}} &\le C\|\nabla\rho_\varepsilon^{m-1/2}\|_{L^2}^{\theta\frac{4-m}{m-1/2}}\|\rho_\varepsilon^{m-1/2}\|_{L^{\frac{m}{m-1/2}}}^{(1-\theta)\frac{4-m}{m-1/2}}\\ &= C\|\nabla\rho_\varepsilon^{m-1/2}\|_{L^2}^{\theta\frac{4-m}{m-1/2}}\|\rho_\varepsilon\|_{L^m}^{(1-\theta)(4-m)},\end{aligned}\tag{2.16}$$

where $0<\theta=\frac{2(2-m)(m-1/2)}{m(4-m)\left(\frac{m-1/2}{m}-\frac{n-2}{2n}\right)}<1$. Thus it remains to show if $m\ge\frac{3}{2}$ and $\frac{2n}{n+2}<m<2-\frac{2}{n}$, it holds that

$$\theta\frac{4-m}{m-1/2}=\frac{2(2-m)}{m-\frac{1}{2}-\frac{m(n-2)}{2n}}\le 2.\tag{2.17}$$

Actually, (2.17) is equivalent to $m\ge\frac{5n}{3n+2}$, which can be obtained from the following two facts.

- When $n\ge 6$, since $\frac{2n}{n+2}\ge\frac{5n}{3n+2}$, we have $m>\frac{5n}{3n+2}$;
- When $n<6$, since $\frac{3}{2}>\frac{5n}{3n+2}$, we have $m>\frac{5n}{3n+2}$.

Now by integrating (2.16) in time, we have

$$\int_0^T\int_{\mathbb{R}^n}\rho_\varepsilon^{4-m}dxdt\le C\left(\|\rho_\varepsilon\|_{L^\infty(0,T;L^m(\mathbb{R}^n))},\|\nabla\rho_\varepsilon^{m-1/2}\|_{L^2(0,T;L^2(\mathbb{R}^n))},T\right).$$

Therefore,

$$\begin{aligned}&\frac{1}{(3-m)}\int_{\mathbb{R}^n}\rho_\varepsilon^{3-m}dx+m(2-m)\int_0^T\int_{\mathbb{R}^n}|\nabla\rho_\varepsilon|^2dxdt\\ \le\;&\frac{1}{(3-m)}\|\rho_{0\varepsilon}\|_{L^{3-m}}^{3-m}+C\le C\left(\|\rho_{0\varepsilon}\|_{L^m},\|\rho_{0\varepsilon}\|_{L^1}\right)+C,\end{aligned}$$

where we have used the fact that $3-m\le m$. So, (2.14) holds. □

STEP 5. Uniform estimate for the time derivative.

This subsection will give another important fact in order to use Aubins lemma, i.e. the estimate of the time derivative of ρ_ε.

LEMMA 2.4. *Assume $p=m$ and the assumptions of Lemma 2.1 hold, then*

$$\|\partial_t\rho_\varepsilon\|_{L^2(0,T;W^{-1,s}_{\mathrm{loc}}(\mathbb{R}^n))}\le C,\quad s=\min\{\frac{2m}{m+1},\frac{nm(m+1)}{nm+(n-m)(m+1)}\}>1.$$

Proof. By using the weak formulation of the equation, we know the estimate for time derivative $\partial_t\rho_\varepsilon$ can be obtained directly from the estimates on $\nabla(\rho_\varepsilon+\varepsilon)^m$ and $(\rho_\varepsilon+\varepsilon)\cdot\nabla c_\varepsilon$. We will prove the following facts,

$$\begin{aligned}&\|\nabla(\rho_\varepsilon+\varepsilon)^m\|_{L^2(0,T;L^{\frac{2m}{m+1}}(\mathbb{R}^n))}\le C,\\ &\|(\rho_\varepsilon+\varepsilon)\cdot\nabla c_\varepsilon\|_{L^{m+1}(0,T;L^{\frac{nm(m+1)}{nm+(n-m)(m+1)}}(\mathbb{R}^n))}\le C.\end{aligned}$$

In fact,

$$\begin{aligned} |\nabla(\rho_\varepsilon+\varepsilon)^m| &= m|(\rho_\varepsilon+\varepsilon)^{m-1}|\cdot|\nabla\rho_\varepsilon| \\ &\le m|(\rho_\varepsilon^{m-1}+\varepsilon^{m-1})|\cdot|\nabla\rho_\varepsilon| \le |\nabla\rho_\varepsilon^m| + m\varepsilon^{m-1}|\nabla\rho_\varepsilon|. \end{aligned} \tag{2.18}$$

By writing

$$|\nabla\rho_\varepsilon^m| = \big|\frac{2m}{2m-1}\rho_\varepsilon^{1/2}\nabla\rho_\varepsilon^{m-1/2}\big|,$$

the Hölder inequality and lemma 2.2, we have

$$\int_{\mathbb{R}^n}|\nabla\rho_\varepsilon^m|^{\frac{2m}{m+1}} \le C\Big(\int_{\mathbb{R}^n}\rho_\varepsilon^m\Big)^{\frac{1}{m+1}}\Big(\int_{\mathbb{R}^n}|\nabla\rho_\varepsilon^{m-1/2}|^2\Big)^{\frac{m}{m+1}}.$$

Therefore,

$$\int_0^T\|\nabla\rho_\varepsilon^m\|^2_{L^{\frac{2m}{m+1}}} \le \int_0^T\|\rho_\varepsilon\|_{L^m}\|\nabla\rho_\varepsilon^{m-1/2}\|^2_{L^2}dt \le C,$$

i.e.,

$$\|\nabla\rho_\varepsilon^m\|_{L^2(0,T;L^{\frac{2m}{m+1}}(\mathbb{R}^n))} \le C. \tag{2.19}$$

By Lemma 2.3, since $\frac{2m}{m+1} < \min\{2, \frac{2m}{3-m}\}$ and (2.19), we know that

$$\nabla(\rho_\varepsilon+\varepsilon)^m \in L^2(0,T;L^{\frac{2m}{m+1}}_{\text{loc}}(\mathbb{R}^n)).$$

As a direct consequence of Lemma 2.2, we have

$$\|\rho_\varepsilon\cdot\nabla c_\varepsilon\|_{L^{m+1}(0,T;L^{\frac{nm(m+1)}{nm+(n-m)(m+1)}}(\mathbb{R}^n))} \le C, \tag{2.20}$$

where $\frac{nm(m+1)}{nm+(n-m)(m+1)} > 1$ due to $\frac{2n}{n+2} < m < 2-\frac{2}{n}$. By Lemma 2.2 with (2.20) and noticing $\frac{nm(m+1)}{nm+(n-m)(m+1)} \in (\frac{n}{n-1}, \frac{mn}{n-m}]$, we get

$$\|(\rho_\varepsilon+\varepsilon)\cdot\nabla c_\varepsilon\|_{L^{m+1}(0,T;L^{\frac{nm(m+1)}{nm+(n-m)(m+1)}}(\mathbb{R}^n))} \le C.$$

□

3. Blow up of the solution

In this section, we will discuss the blow-up of the solution when $\|\rho_0\|_{L^{\frac{2n}{n+2}}} > (s^*)^{\frac{n-2}{2n(m-1)}}$ and $\mathcal{F}(\rho_0) < \mathcal{F}^* := f(s^*)$. Before we prove the result of blow-up, we need to give a key lemma that shows in the cases of subcritical initial data, the quantity $\|\rho\|_{L^{\frac{2n}{n+2}}}$ can be bounded from below.

3.1. Lower bound of $\|\rho\|_{L^{\frac{2n}{n+2}}}$.

Similar to the decomposition of free energy of the regularized problem, we can separate the free energy into two parts by using the Hardy-Littlewood-Sobolev inequality (1.8)

$$\begin{aligned}\mathcal{F}(\rho) &= \frac{1}{m-1}\int_{\mathbb{R}^n}\rho^m(x,t)dx - \frac{C(n)}{2(n-2)n\alpha(n)}\|\rho\|^2_{L^{\frac{2n}{n+2}}}\\ &+ \frac{C(n)}{2(n-2)n\alpha(n)}\|\rho\|^2_{L^{\frac{2n}{n+2}}} - \frac{1}{2(n-2)n\alpha(n)}\int\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\rho(x,t)\rho(y,t)}{|x-y|^{n-2}}dxdy\\ &=: \mathcal{F}_1(\rho)+\mathcal{F}_2(\rho).\end{aligned}$$

Proposition 1.1 says that that $\mathcal{F}_2(\rho)\geq 0$.

Due to $m>\dfrac{2n}{n+2}$, interpolation tells us

$$\|\rho\|_{L^{\frac{2n}{n+2}}}\leq \|\rho\|^{1-\theta}_{L^1}\|\rho\|^{\theta}_{L^m},\quad \theta=\frac{m(n-2)}{2n(m-1)}.$$

Thus the first part of the free energy is

$$\mathcal{F}_1(\rho)\geq \frac{1}{m-1}M_0^{\frac{2n-m(n+2)}{n-2}}\|\rho\|^{\frac{2n(m-1)}{n-2}}_{L^{\frac{2n}{n+2}}} - \frac{C(n)}{2(n-2)n\alpha(n)}\|\rho\|^2_{L^{\frac{2n}{n+2}}}.$$

According to the previous analysis, let

$$f(s)=\frac{1}{m-1}M_0^{\frac{2n-m(n+2)}{n-2}}s-\frac{C(n)}{2(n-2)n\alpha(n)}s^{\frac{n-2}{n(m-1)}}.$$

We now have a lower bound of the first part of free energy, i.e. $f\Big(\|\rho\|^{\frac{2n(m-1)}{n-2}}_{L^{\frac{2n}{n+2}}}\Big)\leq \mathcal{F}_1(\rho)$.

Lemma 3.1. *If the initial free energy $\mathcal{F}(\rho_0)<\mathcal{F}^*:=f(s^*)$ and $\|\rho_0\|_{L^{\frac{2n}{n+2}}}>(s^*)^{\frac{n-2}{2n(m-1)}}$, let $\rho(x,t)$ be a solution of problem* (1.1)*, then there exists a constant $\mu_2>1$ such that*

$$\|\rho(\cdot,t)\|_{\frac{2n}{n+2}}>(\mu_2 s^*)^{\frac{n-2}{2n(m-1)}},\ \textit{for all } t>0,$$

where s^* is the maximum point of $f(s)$:

$$s^*=\Big(\frac{2n^2\alpha(n)M_0^{\frac{2n-m(n+2)}{n-2}}}{C(n)}\Big)^{\frac{n(m-1)}{2n-2-mn}}.$$

Proof. Notice that $1<m<2-\frac{2}{n}$ implies $\frac{n-2}{n(m-1)}>1$, we know that $f(s)$ is a strictly concave function in $0<s<\infty$. Directly calculation shows that

$$f'(s)=\frac{1}{m-1}M_0^{\frac{2n-m(n+2)}{n-2}}-\frac{C(n)}{2(n-2)n\alpha(n)}\frac{n-2}{n(m-1)}s^{\frac{2n-2-mn}{n(m-1)}}.$$

As a consequence, s^* is a unique maximum point of $f(s)$. Therefore the important property of f is that $f(s)$ is monotone increasing for $0<s<s^*$, while $f(s)$ is monotone decreasing for $s>s^*$.

In the case that initial free energy $\mathcal{F}(\rho_0) < f(s^*)$, we can make it even smaller, i.e. there is a $\delta < 1$ such that $\mathcal{F}(\rho_0) < \delta f(s^*)$.

Combining all the facts we know, including the interpolation, the Hardy-Littlewood-Sobolev inequality and the monotonicity of free energy, we have

$$f\Big(\|\rho\|_{L^{\frac{2n}{n+2}}}^{\frac{2n(m-1)}{n-2}}\Big) \le \mathcal{F}_1(\rho) \le \mathcal{F}(\rho) \le \mathcal{F}(\rho_0) < \delta f(s^*).$$

If initially $\|\rho_0\|_{L^{\frac{2n}{n+2}}}^{\frac{2n(m-1)}{n-2}} > s^*$, due to the fact that $f(s)$ is increasing in $s > s^*$, there exists a $\mu_2 > 1$ such that $\|\rho\|_{L^{\frac{2n}{n+2}}}^{\frac{2n(m-1)}{n-2}} > \mu_2 s^*$. □

3.2. TIME DERIVATIVE OF SECOND MOMENT.
In this subsection, we will focus on studying the time evolution of the second moment. The following lemma is obtained from Lemma 3.1.

LEMMA 3.2. *If* $\mathcal{F}(\rho_0) < \mathcal{F}^* := f(s^*)$ *and* $\|\rho_0\|_{\frac{2n}{n+2}} > (s^*)^{\frac{n-2}{2n(m-1)}}$, ρ *is a solution of* (1.1), *then*

$$\frac{dm_2(t)}{dt} < 0. \tag{3.1}$$

Proof. By direct calculation, we have

$$\frac{dm_2(t)}{dt} = \Big(2n - \frac{2(n-2)}{m-1}\Big)\int_{\mathbb{R}^n} \rho^m dx + 2(n-2)\mathcal{F}(\rho).$$

The restriction on $m < 2 - \frac{2}{n}$ gives that $2n - \frac{2(n-2)}{m-1} < 0$. Then by using the interpolation inequality, the decreasing properties of free energy and Lemma 3.1 with $\mu_2 > 1$, we have

$$\begin{aligned}
\frac{dm_2(t)}{dt} &\le \Big(2n - \frac{2(n-2)}{m-1}\Big) M_0^{\frac{(\theta-1)m}{\theta}} \|\rho\|_{L^{\frac{2n}{n+2}}}^{\frac{m}{\theta}} + 2(n-2)\mathcal{F}(\rho_0) \\
&< \Big(2n - \frac{2(n-2)}{m-1}\Big) M_0^{\frac{(\theta-1)m}{\theta}} \mu_2 s^* + 2(n-2) f(s^*) \\
&= \Big(2n - \frac{2(n-2)}{m-1}\Big) M_0^{\frac{(\theta-1)m}{\theta}} (\mu_2 - 1)s^* + \Big(2n - \frac{2(n-2)}{m-1}\Big) M_0^{\frac{(\theta-1)m}{\theta}} s^* \\
&\qquad + 2(n-2)\Big(\frac{1}{m-1} M_0^{\frac{(\theta-1)m}{\theta}} s^* - \frac{C(n)}{2(n-2)n\alpha(n)}(s^*)^{\frac{2\theta}{m}}\Big) \\
&= \Big(2n - \frac{2(n-2)}{m-1}\Big) M_0^{\frac{(\theta-1)m}{\theta}} (\mu_2 - 1)s^* + 2n M_0^{\frac{(\theta-1)m}{\theta}} s^* - \frac{C(n)}{n\alpha(n)}(s^*)^{\frac{2\theta}{m}} \\
&= \Big(2n - \frac{2(n-2)}{m-1}\Big) M_0^{\frac{(\theta-1)m}{\theta}} (\mu_2 - 1)s^* < 0.
\end{aligned}$$

where the last second equation follows from the definition of s^*. □

3.3. THE PROOF ON THE BLOW-UP RESULT IN THEOREM 1.1.
From Lemma 3.2, we know that there exists a finite time T such that

$$\lim_{t\to T} m_2(t) = 0.$$

The relation between the second moment and L^m norm of ρ can be obtained by using Hölder's inequality, $\forall R > 0$, we have

$$\int_{\mathbb{R}^n} \rho(x)dx \le \int_{B_R} \rho(x)dx + \int_{B_R^c} \rho(x)dx \le CR^{n(m-1)/m}\|\rho\|_{L^m} + \frac{1}{R^2}m_2(t).$$

Now by choosing $R = (\frac{m_2(t)}{C\|\rho\|_{L^m}})^{\frac{m}{(m-1)n+2m}}$, we have

$$\|\rho\|_{L^1} \le C\|\rho\|_{L^m}^{\frac{2m}{(m-1)n+2m}} m_2(t)^{\frac{n(m-1)}{(m-1)n+2m}}.$$

Consequently, there exists $T^* \le T$ such that $\lim_{t\to T^*} \|\rho\|_{L^m} = \infty$.

APPENDIX

In Example 1, we gave an initial data of the system with small mass and showed that the solution must blow up in finite time according to the main result of this paper. Here in this appendix, we will give a detailed calculation for the quantities appeared in Example 1 to make sure that the assumptions in theorem 1.1 satisfied.

For given $\varepsilon_0 > 0$ small, let the initial data be

$$\rho_0(x) = \begin{cases} \varepsilon_0 \frac{K^n}{\alpha(n)}, & |x| \le \frac{1}{K}, \\ 0, & |x| > \frac{1}{K}, \end{cases} \tag{3.2}$$

where $\alpha(n)$ is the volume of n dimensional unit ball, and K will be determined later.

First of all, since $\|\rho_0\|_{L^{\frac{2n}{n+2}}} = \varepsilon_0\Big(\frac{K^n}{\alpha(n)}\Big)^{\frac{n-2}{2n}}$, to prove (1.6), i.e. $\|\rho_0\|_{L^{\frac{2n}{n+2}}} > (s^*)^{\frac{n-2}{2n(m-1)}}$, it is necessary to show

$$\varepsilon_0^{1+\frac{m(n+2)-2n}{2(2n-2-mn)}} K^{\frac{n-2}{2}} > (\alpha(n))^{\frac{n-2}{2n}} \Big(\frac{2n^2\alpha(n)}{C(n)}\Big)^{\frac{n-2}{2(2n-2-mn)}}. \tag{3.3}$$

Notice that $n > 2$, there exists a constant $K_1 > 0$ such that for all $K > K_1$, the formula (3.3) is true.

The corresponding initial free energy is

$$
\begin{aligned}
\mathcal{F}(\rho_0) &= \frac{1}{m-1}\int_{\mathbb{R}^n}\rho_0^m dx - \frac{1}{2(n-2)n\alpha(n)}\int\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\rho_0(x)\rho_0(y)}{|x-y|^{n-2}}dxdy \\
&= \frac{1}{m-1}\int_{|x|\le\frac{1}{K}}\varepsilon_0^m\Big(\frac{K^n}{\alpha(n)}\Big)^m dx - \frac{1}{2(n-2)n\alpha(n)}\int_{|x|\le\frac{1}{K}}\int_{|y|\le\frac{1}{K}}\frac{\big(\varepsilon_0\frac{K^n}{\alpha(n)}\big)^2}{|x-y|^{n-2}}dxdy \\
&\le \frac{\varepsilon_0^m}{m-1}K^{n(m-1)}(\alpha(n))^{1-m} - \frac{1}{2(n-2)n\alpha(n)}\int_{|x|\le\frac{1}{K}}\int_{|y|\le\frac{1}{K}}\frac{\big(\varepsilon_0\frac{K^n}{\alpha(n)}\big)^2}{(|x|+|y|)^{n-2}}dxdy \\
&\le \frac{\varepsilon_0^m}{m-1}K^{n(m-1)}(\alpha(n))^{1-m} - \frac{1}{2(n-2)n\alpha(n)}\int_{|x|\le\frac{1}{K}}\int_{|y|\le\frac{1}{K}}\frac{\big(\varepsilon_0\frac{K^n}{\alpha(n)}\big)^2}{(\frac{2}{K})^{n-2}}dxdy \\
&= \frac{\varepsilon_0^m}{m-1}K^{n(m-1)}(\alpha(n))^{1-m} - \frac{2^{2-n}}{2(n-2)n\alpha(n)}\varepsilon_0^2K^{n-2}.
\end{aligned}
$$

To show that (1.7) is true, it is necessary to show that

$$
\begin{aligned}
\varepsilon_0^{m+\frac{m(n+2)-2n}{2n-2-mn}}K^{n(m-1)}(\alpha(n))^{1-m} \quad &< \quad \frac{(m-1)2^{2-n}}{2(n-2)n\alpha(n)}\varepsilon_0^{2+\frac{m(n+2)-2n}{2n-2-mn}}K^{n-2} \\
&\quad + \frac{2-\frac{2}{n}-m}{1-\frac{2}{n}}\Big(\frac{2n^2\alpha(n)}{C(n)}\Big)^{\frac{n(m-1)}{2n-2-mn}}.
\end{aligned}
\tag{3.4}
$$

Notice that $m < 2-\frac{2}{n}$ implies $n(m-1) < n-2$. Thus there exists a constant $K_2 > 0$ such that when $K > K_2$, (3.4) holds.

Hence taking $K_0 = \max\{K_1, K_2\}$, we know that when $K > K_0$, the initial data satisfies blow-up condition in Theorem 1.1.

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E_n-Regularity Implies E_{n-1}-Regularity

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Abstract. Vorst and Dayton-Weibel proved that K_n-regularity implies K_{n-1}-regularity. In this article we generalize this result from (commutative) rings to differential graded categories and from algebraic K-theory to any functor which is Morita invariant, continuous, and localizing. Moreover, we show that regularity is preserved under taking desuspensions, fibers of morphisms, direct factors, and arbitrary direct sums. As an application, we prove that the above implication also holds for schemes. Along the way, we extend Bass' fundamental theorem to this broader setting and establish a Nisnevich descent result which is of independent interest.

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[1]

1. Introduction

Let $n \in \mathbb{Z}$. Following Bass [1, §XII], a (commutative) ring R is called K_n-*regular* if $K_n(R) \simeq K_n(R[t_1, \ldots, t_m])$ for all $m \geq 1$. The following implication

$$R \text{ is } K_n\text{-regular} \Rightarrow R \text{ is } K_{n-1}\text{-regular} \tag{1.1}$$

was proved by Vorst [29, Cor. 2.1] for $n \geq 1$ and latter by Dayton-Weibel [9, Cor. 4.4] for $n \leq 0$. It is then natural to ask the following:

Question: Does implication (1.1) *holds more generally ?*

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Statement of results. A *differential graded (=dg) category* $\mathcal{A}$, over a base commutative ring k, is a category enriched over complexes of k-modules; see §2. Every (dg) k-algebra A gives naturally rise to a dg category $\underline{A}$ with a single object and (dg) k-algebra of endomorphisms A. Another source of examples is provided by k-schemes since, as explained in [7, Example 5.5], the derived category of perfect complexes of every quasi-compact separated k-scheme X admits a canonical dg enhancement $\mathsf{perf}(X)$.

A functor $E : \mathsf{dgcat} \to \mathcal{M}$ defined on the category of (small) dg categories and with values in a stable Quillen model category (see [12, §7][18]) is called:

(i) *Morita invariant* if it sends Morita equivalences (see §2) to weak equivalences;

(ii) *Continuous* if it preserves filtered (homotopy) colimits;

(iii) *Localizing* if it sends short exact sequences of dg categories (see [13, §4.6]) to distinguished triangles

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0 \quad \mapsto \quad E(\mathcal{A}) \to E(\mathcal{B}) \to E(\mathcal{C}) \stackrel{\partial}{\to} \Sigma E(\mathcal{A})$$

in the triangulated homotopy category $\mathsf{Ho}(\mathcal{M})$.

Thanks to the work of Thomason-Trobaugh, Schlichting, Keller, Blumberg-Mandell and others (see [3, 15, 16, 20, 22, 28]), examples of functors satisfying the above conditions (i)-(iii) include (nonconnective) algebraic K-theory (K), Hochschild homology, cyclic homology (and its variants), topological Hochschild homology, *etc*. As proved in *loc. cit.*, when applied to $\underline{A}$ (resp. to $\mathsf{perf}(X)$) these functors reduce to the classical invariants of (dg) k-algebras (resp. of k-schemes). Making use of the language of Grothendieck derivators, the universal functor with respect to the above conditions (i)-(iii) was constructed in [21, §10]

$$U : \mathsf{dgcat} \to \mathrm{Mot}\,; \tag{1.2}$$

in *loc. cit.* U was denoted by $\mathcal{U}_l$ and Mot by $\mathcal{M}^{\mathsf{loc}}_{\mathsf{dg}}$. Any other functor $E : \mathsf{dgcat} \to \mathcal{M}$ satisfying the above conditions (i)-(iii) factors through U via a triangulated functor $\overline{E} : \mathsf{Ho}(\mathrm{Mot}) \to \mathsf{Ho}(\mathcal{M})$; see Proposition 2.1. Because of this universal property, which is reminiscent from motives, $\mathsf{Ho}(\mathrm{Mot})$ is called the triangulated category of noncommutative motives; consult the survey article [24]. Moreover, as proved in [6, Thm. 7.6][21, Thm. 15.10], $U(\underline{k})$ is a compact object and for every dg category $\mathcal{A}$ we have the isomorphisms

$$\mathrm{Hom}_{\mathsf{Ho}(\mathrm{Mot})}(\Sigma^n U(\underline{k}), U(\mathcal{A})) \simeq K_n(\mathcal{A}) \qquad n \in \mathbb{Z}\,. \tag{1.3}$$

Given a dg category $\mathcal{A}$, an integer n, a functor $E : \mathsf{dgcat} \to \mathcal{M}$, and an object $b \in \mathsf{Ho}(\mathcal{M})$, let us write $E^b_n(\mathcal{A})$ for the abelian group $\mathrm{Hom}_{\mathsf{Ho}(\mathcal{M})}(\Sigma^n b, E(\mathcal{A}))$. For instance, when $\mathcal{A} = \underline{A}$, $E = U$ and $b = U(\underline{k})$, $E^b_n(\mathcal{A})$ identifies, thanks to (1.3), with the n^{th} algebraic K-theory group $K_n(A)$ of A. Following Bass, a dg category $\mathcal{A}$ is called E^b_n*-regular* if $E^b_n(\mathcal{A}) \simeq E^b_n(\mathcal{A}[t_1, \ldots, t_m])$ for all $m \geq 1$, where $\mathcal{A}[t_1, \ldots, t_m] := \mathcal{A} \otimes \underline{k[t_1, \ldots, t_m]}$. Our main result, which answers affirmatively the above question, is the following:

Theorem 1.4. *Let $\mathcal{A}$ be a dg category, n an integer, $E : \mathsf{dgcat} \to \mathcal{M}$ a functor satisfying the above conditions (i)-(iii), and b a compact object of $\mathsf{Ho}(\mathcal{M})$. Under these notations and assumptions, the following implication holds:*

$$\mathcal{A} \text{ is } E_n^b\text{-regular} \Rightarrow \mathcal{A} \text{ is } E_{n-1}^b\text{-regular}. \tag{1.5}$$

Note that Theorem 1.4 uncovers in a direct and elegant way the three key conceptual properties (= Morita invariance, continuity, and localization) that underlie Vorst and Dayton-Weibel's implication (1.1). Along its proof, we have generalized Bass' fundamental theorem and introduced a Nisnevich descent result; see Theorems 3.1 and 4.2. These results are of independent interest. The above implication (1.5) shows us that regularity is preserved when n is replaced by $n-1$. The same holds in the following five cases:

Theorem 1.6. *Let $\mathcal{A}, n, E, b$ be as in Theorem 1.4.*

(i) *Given an integer $i > 0$, we have: $\mathcal{A}$ is E_n^b-regular $\Rightarrow$ $\mathcal{A}$ is $E_n^{\Sigma^{-i}b}$-regular.*
(ii) *Given a distinguished triangle $c \to c' \to c'' \to \Sigma c$ of compact objects in $\mathsf{Ho}(\mathcal{M})$, we have:*

$$\mathcal{A} \text{ is } E_n^{c'}\text{-regular and } E_n^{c''}\text{-regular} \;\Rightarrow\; \mathcal{A} \text{ is } E_n^{c}\text{-regular}. \tag{1.7}$$

(iii) *Given a direct factor d of b, we have: $\mathcal{A}$ is E_n^b-regular $\Rightarrow$ $\mathcal{A}$ is E_n^d-regular.*
(iv) *Given a family of objects $\{c_i\}_{i\in I}$ in $\mathsf{Ho}(\mathcal{M})$, we have: $\mathcal{A}$ is $E_n^{c_i}$-regular for every $i \in I$ $\Rightarrow$ $\mathcal{A}$ is $E_n^{\oplus_{i\in I}c_i}$-regular.*
(v) *Consider the k-algebra Γ of those $\mathbb{N}\times\mathbb{N}$-matrices M which satisfy the following two conditions: (1) the set $\{M_{ij}\,|\,i,j\in\mathbb{N}\}$ is finite; (2) there exists a natural number n_M such that each row and column has at most n_M non-zero entries. Let σ be the quotient of Γ by the two-sided ideal consisting of those matrices with finitely many non-zero entries. Under these notations, we have: $\mathcal{A}$ is E_n^b-regular $\Rightarrow$ $\sigma(\mathcal{A}) := \mathcal{A}\otimes\underline{\sigma}$ is E_{n+1}^b-regular.*

In items (iii)-(iv) the assumptions of Theorem 1.4 are not necessary.

Roughly speaking, item (v) shows us that the converse of implication (1.5) also holds as long as on the right-hand-side one tensors $\mathcal{A}$ with σ. Let us denote by $\langle\Sigma^n b|^{\natural,\oplus}$ the smallest subcategory of $\mathsf{Ho}(\mathcal{M})$ which contains the object $\Sigma^n b$ and which is stable under taking desuspensions, fibers of morphisms, direct factors, and arbitrary direct sums. Thanks to the above items (i)-(iv) we have:

$$\mathcal{A} \text{ is } E_n^b\text{-regular} \Rightarrow \mathcal{A} \text{ is } E_n^c\text{-regular} \qquad \forall\, c\in\langle\Sigma^n b|^{\natural,\oplus}. \tag{1.8}$$

Moreover, in the particular case where $\mathcal{A}$ is E_n^b-regular for every $n\in\mathbb{Z}$ one can replace $\langle\Sigma^n b|^{\natural,\oplus}$ in the above implication (1.8) by the smallest thick localizing (=stable under arbitrary direct sums) triangulated subcategory $\langle b\rangle^{\natural,\oplus}$ of $\mathsf{Ho}(\mathcal{M})$ which contains b. When $E = U$ and $b = U(\underline{k})$, (1.8) reduces to

$$\mathcal{A} \text{ is } K_n\text{-regular} \Rightarrow \mathcal{A} \text{ is } U_n^c\text{-regular} \qquad \forall\, c\in\langle\Sigma^n U(\underline{k})|^{\natural,\oplus} \tag{1.9}$$

and that in the particular case where $\mathcal{A}$ is K_n-regular for every $n \in \mathbb{Z}$ (*e.g.* $\mathcal{A} = \underline{A}$ with A a noetherian regular k-algebra) one can replace $\langle \Sigma^n U(\underline{k})|^{\natural,\oplus}$ by the triangulated category $\langle U(\underline{k})\rangle^{\natural,\oplus}$. Here is one example of the above implication (1.9):

Proposition 1.10. *Consider the following distinguished triangle in* $\mathsf{Ho}(\mathrm{Mot})$

$$\mathrm{fib}(l) \longrightarrow U(\underline{k}) \xrightarrow{\cdot l} U(\underline{k}) \longrightarrow \Sigma \mathrm{fib}(l)\,,$$

where $l \geq 2$ is an integer and $\cdot l$ stands for the l-fold multiple of the identity morphism. Under these notations, $U_n^{\mathrm{fib}(l)}(\mathcal{A})$ identifies with Browder-Karoubi [5] mod-l algebraic K-theory $K_n(\mathcal{A};\mathbb{Z}/l)$. Consequently, the above implication (1.9) with $c := \mathrm{fib}(l)$ reduces to: $\mathcal{A}$ is K_n-regular $\Rightarrow$ $\mathcal{A}$ is $K_n(-;\mathbb{Z}/l)$-regular.

Remark 1.11. In the particular case where $\mathcal{A}$ is a k-algebra $\underline{A}$ such that $1/l \in A$, Weibel proved in [30, 31, 32] that A is $K_n(-;\mathbb{Z}/l)$-regular for every $n \in \mathbb{Z}$.

Intuitively speaking, Proposition 1.10 shows us that mod-l algebraic K-theory is the simplest replacement of algebraic K-theory (using fibers of morphisms) for which regularity is preserved. Many other replacements, preserving regularity, can be obtained by combining the above implication (1.9) with the description (1.3) of the Hom-sets of the category of noncommutative motives. Following Bass, a (quasi-compact separated) k-scheme X is called *K_n-regular* if $K_n(X) \simeq K_n(X \times \mathbb{A}^m)$ for all $m \geq 1$, where $\mathbb{A}^1$ stands for the affine line. As mentioned above, all the invariants of X can be recovered from its derived dg category of perfect complexes $\mathsf{perf}(X)$. Hence, let us define $E_n^b(X)$ to be the abelian group $E_n^b(\mathsf{perf}(X))$ and call a k-scheme X *E_n^b-regular* if $E_n^b(X) \simeq E_n^b(X \times \mathbb{A}^m)$ for all $m \geq 1$. Making use of Theorems 1.4 and 1.6 and of Proposition 1.10 one then obtains the following result:

Theorem 1.12. *Let X be a quasi-compact separated k-scheme, n an integer, $E : \mathsf{dgcat} \to \mathcal{M}$ a functor satisfying the above conditions (i)-(iii), and b a compact object of $\mathsf{Ho}(\mathcal{M})$. Under these notations and assumptions, the following implications hold:*

$$X \text{ is } E_n^b\text{-regular} \Rightarrow X \text{ is } E_{n-1}^b\text{-regular} \tag{1.13}$$

$$X \text{ is } E_n^b\text{-regular} \Rightarrow X \text{ is } E_n^c\text{-regular} \qquad \forall\, c \in \langle \Sigma^n b|^{\natural,\oplus} \tag{1.14}$$

$$X \text{ is } K_n(-;\mathbb{Z}/l^\nu)\text{-regular} \Rightarrow X \text{ is } K_{n-1}(-;\mathbb{Z}/l^\nu)\text{-regular}\,, \tag{1.15}$$

where in (1.15) l^ν is a prime power; see Thomason-Trobaugh [28, §9.3].

Remark 1.16. As in the above Remark 1.11, Weibel proved that in the particular case where $1/l \in \mathcal{O}_X$ the k-scheme X is $K_n(-;\mathbb{Z}/l^\nu)$-regular for every $n \in \mathbb{Z}$.

When $E = U$ and $b = U(\underline{k})$, (1.13) reduces to K_n-regularity $\Rightarrow$ K_{n-1}-regularity. Chuck Weibel kindly informed the author that this latter implication was proved (in a totally different way) by Cortiñas-Haesemeyer-Walker-Weibel [8, Cor. 4.4] in the particular case where k is a field of characteristic zero. To the best of the author's knowledge all the remaining cases (with k an arbitrary commutative ring) are new in the literature. On the other hand, (1.14) reduces to the implication

$$X \text{ is } K_n\text{-regular} \Rightarrow X \text{ is } U_n^c\text{-regular} \qquad \forall\, c \in \langle \Sigma^n U(\underline{k})|^{\natural,\oplus}\,.$$

Moreover, in the particular case where X is K_n-regular for every $n \in \mathbb{Z}$ (*e.g.* X a regular k-scheme) one can replace $\langle \Sigma^n U(\underline{k})|^{\natural,\oplus}$ by the triangulated category $\langle U(\underline{k})\rangle^{\natural,\oplus}$. Finally, to the best of the author's knowledge, implication (1.15) is also new in the literature.

Remark 1.17. Theorem 1.4 admits a "cohomological" analogue. Given a dg category $\mathcal{A}$, an integer n, a functor $E : \mathsf{dgcat} \to \mathcal{M}$, and an object $b \in \mathsf{Ho}(\mathcal{M})$, let us write $E_b^{-n}(\mathcal{A})$ for the abelian group $\mathrm{Hom}_{\mathsf{Ho}(\mathcal{M})}(E(\mathcal{A}), \Sigma^n b)$. The dg category $\mathcal{A}$ is called E_b^{-n}*-regular* if $E_b^{-n}(\mathcal{A}) \simeq E_b^{-n}(\mathcal{A}[t_1, \ldots, t_m])$ for all $m \geq 1$. Under these notations, the following implication

$$\mathcal{A} \text{ is } E_b^{-n}\text{-regular} \Rightarrow \mathcal{A} \text{ is } E_b^{-n+1}\text{-regular} \tag{1.18}$$

holds for every functor E which satisfies the above conditions (i)-(iii). Moreover, and in contrast with implication (1.5), it is *not* necessary to assume that b is a compact object of $\mathsf{Ho}(\mathcal{M})$. The proof of (1.18) is similar to the proof of (1.5). First replace $NE_n^b(\mathcal{A})$ by the cokernel $CE_b^{-n}(\mathcal{A})$ of the group homomorphism

$$E_b^{-n}(\mathrm{id} \otimes (t=0)) : E_b^{-n}(\mathcal{A}) \longrightarrow E_b^{-n}(\mathcal{A}[t])\,,$$

then replace (5.2) by the group isomorphism $\lim CE_b^{-n}(\mathcal{B}[x]) \simeq CE_b^{-n}(\mathcal{B}[x, x^{-1}])$, and finally use the new key fact that the contravariant functor $\mathrm{Hom}_{\mathsf{Ho}(\mathcal{M})}(-, \Sigma^n b)$ sends colimits to limits.
Theorem 1.12 also admits a "cohomological" analogue. In items (i)-(iv) replace $E_n^?$ by $E_?^{-n}$ and in item (v) replace the above implication by: $\mathcal{A}$ is E_b^{-n}-regular $\Rightarrow \sigma(\mathcal{A})$ is E_b^{-n-1}-regular. As a consequence we obtain:

$$\mathcal{A} \text{ is } E_b^{-n}\text{-regular} \Rightarrow \mathcal{A} \text{ is } E_c^{-n}\text{-regular} \qquad \forall\, c \in \langle \Sigma^n b|^{\natural,\oplus}\,.$$

In the particular case where $\mathcal{A}$ is E_b^{-n}-regular for every $n \in \mathbb{Z}$ we can furthermore replace $\langle \Sigma^n b|^{\natural,\oplus}$ by the thick localizing triangulated category $\langle b\rangle^{\natural,\oplus}$.

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2. Preliminaries

Dg categories. Let k be a base commutative ring and $\mathcal{C}(k)$ the category of complexes of k-modules. A *differential graded (=dg) category* $\mathcal{A}$ is a category enriched over $\mathcal{C}(k)$ (morphism sets $\mathcal{A}(x, y)$ are complexes) in such a way that composition fulfills the Leibniz rule: $d(f \circ g) = d(f) \circ g + (-1)^{\deg(f)} f \circ d(g)$. A *dg functor* $\mathcal{A} \to \mathcal{B}$ is a functor enriched over $\mathcal{C}(k)$; consult Keller's ICM

survey [13]. In what follows we will write dgcat for the category of (small) dg categories and dg functors.
A dg functor $\mathcal{A} \to \mathcal{B}$ is called a *Morita equivalence* if the restriction functor induces an equivalence $\mathcal{D}(\mathcal{B}) \xrightarrow{\sim} \mathcal{D}(\mathcal{A})$ on derived categories; see [13, §3]. The localization of dgcat with respect to the class of Morita equivalences will be denoted by $\mathsf{Ho}(\mathsf{dgcat})$. Note that every Morita invariant functor $E : \mathsf{dgcat} \to \mathcal{M}$ descends uniquely to $\mathsf{Ho}(\mathsf{dgcat})$.
The tensor product of k-algebras extends naturally to dg categories, giving rise to a symmetric monoidal structure $-\otimes-$ on dgcat with $\otimes$-unit the dg category $\underline{k}$. As explained in [13, §4.2], this tensor product descends to a derived tensor product $-\otimes^{\mathbb{L}}-$ on $\mathsf{Ho}(\mathsf{dgcat})$. Finally, recall that a dg category $\mathcal{A}$ is called *k-flat* if for any two objects x and y the functor $\mathcal{A}(x,y) \otimes - : \mathcal{C}(k) \to \mathcal{C}(k)$ preserves quasi-isomorphisms. In this particular case the derived tensor product $\mathcal{A} \otimes^{\mathbb{L}} \mathcal{B}$ agrees with the classical one $\mathcal{A} \otimes \mathcal{B}$.

SCHEMES. Throughout this article all schemes will be quasi-compact and separated. By a k-scheme X we mean a scheme X over $\mathrm{spec}(k)$. Given a dg category $\mathcal{A}$ and a k-scheme X, we will often write $\mathcal{A} \otimes^{\mathbb{L}} X$ instead of $\mathcal{A} \otimes^{\mathbb{L}} \mathsf{perf}(X)$. When $X = \mathrm{spec}(C)$ is affine we will furthermore replace $\mathcal{A} \otimes^{\mathbb{L}} \mathrm{spec}(C)$ by $\mathcal{A} \otimes^{\mathbb{L}} \underline{C}$.

NONCOMMUTATIVE MOTIVES.

PROPOSITION 2.1. *Given a functor $E : \mathsf{dgcat} \to \mathcal{M}$ which satisfies the above conditions (i)-(iii), there exists a triangulated functor $\overline{E} : \mathsf{Ho}(\mathrm{Mot}) \to \mathsf{Ho}(\mathcal{M})$ such that $\overline{E} \circ U = E$.*

Proof. The category dgcat carries a (cofibrantly generated) Quillen model category whose weak equivalences are precisely the Morita equivalences; see [26, Thm. 5.3]. Hence, it gives rise to a well-defined Grothendieck derivator $\mathsf{HO}(\mathsf{dgcat})$; consult [7, Appendix A] for the notion of derivator. Since by hypothesis $\mathcal{M}$ is stable and the functor E satisfies conditions (i)-(iii), we then obtain a well-defined localizing invariant of dg categories $\mathsf{HO}(E) : \mathsf{HO}(\mathsf{dgcat}) \to \mathsf{HO}(\mathcal{M})$ in the sense of [21, Notation 15.5]. Thanks to the universal property of [21, Thm. 10.5] this localizing invariant of dg categories factors (uniquely) through $\mathsf{HO}(\mathrm{Mot})$ via an homotopy colimit preserving morphism of derivators $\mathsf{HO}(\mathrm{Mot}) \to \mathsf{HO}(\mathcal{M})$. By passing to the underlying homotopy categories of this latter morphism of derivators we hence obtain the searched triangulated functor $\overline{E} : \mathsf{Ho}(\mathrm{Mot}) \to \mathsf{Ho}(\mathcal{M})$ which verifies $\overline{E} \circ U = E$. □

3. NISNEVICH DESCENT

In this section we prove the following Nisnevich descent result, which is of independent interest. Its Corollary 3.4 will play a key role in the next section.

Theorem 3.1. *(Nisnevich descent) Consider the following (distinguished) square of k-schemes*

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "j"'] & X , \end{tikzcd} \tag{3.2}$$

where j is an open immersion and p is an étale morphism inducing an isomorphism of reduced k-schemes $p^{-1}(X-U)_{\mathrm{red}} \simeq (X-U)_{\mathrm{red}}$. Then, given a dg category $\mathcal{A}$ and a Morita invariant localizing functor $E : \mathsf{dgcat} \to \mathcal{M}$, one obtains a homotopy (co)cartesian square

$$\begin{tikzcd} E(\mathcal{A} \otimes^{\mathbb{L}} X) \arrow[r, "E(\mathrm{id} \otimes^{\mathbb{L}} j^*)"] \arrow[d, "E(\mathrm{id} \otimes^{\mathbb{L}} p^*)"'] \arrow[dr, phantom, "\square"] & E(\mathcal{A} \otimes^{\mathbb{L}} U) \arrow[d] \\ E(\mathcal{A} \otimes^{\mathbb{L}} V) \arrow[r] & E(\mathcal{A} \otimes^{\mathbb{L}} (U \times_X V)) \end{tikzcd} \tag{3.3}$$

in the homotopy category $\mathsf{Ho}(\mathcal{M})$; see [19, Def. 1.4.1].

Proof. Consider the following commutative diagram in $\mathsf{Ho}(\mathsf{dgcat})$

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{perf}(X)_Z \arrow[r] \arrow[d, "\sim"'] & \mathsf{perf}(X) \arrow[r, "j^*"] \arrow[d, "p^*"'] & \mathsf{perf}(U) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathsf{perf}(V)_{Z'} \arrow[r] & \mathsf{perf}(V) \arrow[r] & \mathsf{perf}(U \times_X V) \arrow[r] & 0 , \end{tikzcd}$$

where Z (resp. Z') is the closed set $X - U$ (resp. $p^{-1}(X-U)$) and $\mathsf{perf}(X)_Z$ (resp. $\mathsf{perf}(V)_{Z'}$) the dg category of those perfect complexes of $\mathcal{O}_X$-modules (resp. of $\mathcal{O}_V$-modules) that are supported on Z (resp. on Z'). As explained by Thomason-Trobaugh in [28, §5], both rows are short exact sequences of dg categories; see also [13, §4.6]. Furthermore, as proved in [28, Thm. 2.6.3], the induced dg functor $\mathsf{perf}(X)_Z \xrightarrow{\sim} \mathsf{perf}(V)_{Z'}$ is a Morita equivalence and hence an isomorphism in $\mathsf{Ho}(\mathsf{dgcat})$. Following Drinfeld [10, Prop. 1.6.3], the functor $\mathcal{A} \otimes^{\mathbb{L}} - : \mathsf{Ho}(\mathsf{dgcat}) \to \mathsf{Ho}(\mathsf{dgcat})$ preserves short exact sequences of dg categories. As a consequence, we obtain the commutative diagram in $\mathsf{Ho}(\mathsf{dgcat})$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A} \otimes^{\mathbb{L}} \mathsf{perf}(X)_Z \arrow[r] \arrow[d, "\sim"'] & \mathcal{A} \otimes^{\mathbb{L}} \mathsf{perf}(X) \arrow[r, "\mathrm{id} \otimes^{\mathbb{L}} j^*"] \arrow[d, "\mathrm{id} \otimes^{\mathbb{L}} p^*"'] & \mathcal{A} \otimes^{\mathbb{L}} \mathsf{perf}(U) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{A} \otimes^{\mathbb{L}} \mathsf{perf}(V)_{Z'} \arrow[r] & \mathcal{A} \otimes^{\mathbb{L}} \mathsf{perf}(V) \arrow[r] & \mathcal{A} \otimes^{\mathbb{L}} \mathsf{perf}(U \times_X V) \arrow[r] & 0 , \end{tikzcd}$$

where both rows are short exact sequences of dg categories. Recall that by hypothesis E sends (in a functorial way) short exact sequences of dg categories to distinguished triangles. Consequently, by applying E to the preceding commutative diagram we obtain the following morphism between distinguished triangles:

$$\begin{tikzcd}
E(\mathcal{A}\otimes^{\mathbb{L}}\mathsf{perf}(X)_Z) \arrow[r] \arrow[d, "\sim"'] & E(\mathcal{A}\otimes^{\mathbb{L}} X) \arrow[r, "E(\mathrm{id}\otimes^{\mathbb{L}} j^*)"] \arrow[d, "E(\mathrm{id}\otimes^{\mathbb{L}} p^*)"'] & E(\mathcal{A}\otimes^{\mathbb{L}} U) \arrow[r, "\partial"] \arrow[d] & \Sigma E(\mathcal{A}\otimes^{\mathbb{L}}\mathsf{perf}(X)_Z) \arrow[d, "\sim"] \\
E(\mathcal{A}\otimes^{\mathbb{L}}\mathsf{perf}(V)_{Z'}) \arrow[r] & E(\mathcal{A}\otimes^{\mathbb{L}} V) \arrow[r] & E(\mathcal{A}\otimes^{\mathbb{L}}(U\times_X V)) \arrow[r, "\partial"'] & \Sigma E(\mathcal{A}\otimes^{\mathbb{L}}\mathsf{perf}(V)_{Z'}).
\end{tikzcd}$$

Since the outer left and right vertical maps are isomorphisms we conclude that the middle square (which agrees with the above square (3.3)) is homotopy (co)cartesian. This achieves the proof. □

Corollary 3.4. *(Mayer-Vietoris for open covers) Let X be a k-scheme which is covered by two Zariski open subschemes $U, V \subset X$. Then, given a dg category $\mathcal{A}$ and a Morita invariant localizing functor $E : \mathsf{dgcat} \to \mathcal{M}$, one obtains a Mayer-Vietoris triangle*

$$E(\mathcal{A}\otimes^{\mathbb{L}} X) \to E(\mathcal{A}\otimes^{\mathbb{L}} U)\oplus E(\mathcal{A}\otimes^{\mathbb{L}} V) \stackrel{\pm}{\to} E(\mathcal{A}\otimes^{\mathbb{L}}(U\cap V)) \stackrel{\partial}{\to} \Sigma E(\mathcal{A}\otimes^{\mathbb{L}} X)\,.$$

Proof. This follows from the fact that when the morphism p in the square (3.2) is an open immersion, $U\times_X V$ identifies with $U\cap V$; recall also from [19, §1.4] that every homotopy (co)cartesian square has an associated distinguished "Mayer-Vietoris" triangle. □

4. Generalized fundamental theorem

The following theorem was proved by Bass [1, §XII-§7-8] for $n \leq 0$ and by Quillen [11] for $n \geq 1$.

Theorem 4.1. *(Bass' fundamental theorem) Let R be a ring and n an integer. Then, we have the following exact sequence of abelian groups*

$$0 \to K_n(R) \stackrel{\Delta}{\to} K_n(R[x])\oplus K_n(R[1/x]) \stackrel{\pm}{\to} K_n(R[x,1/x]) \stackrel{\partial_n}{\to} K_{n-1}(R)\to 0\,.$$

In this section we generalize it as follows:

Theorem 4.2. *(Generalized fundamental theorem) Let $\mathcal{A}$ be a dg category, n an integer, $E : \mathsf{dgcat}\to\mathcal{M}$ a Morita invariant localizing functor, and b and object of $\mathcal{M}$. Then, we have the following exact sequence of abelian groups*

(4.3)
$$0 \to E_n^b(\mathcal{A}) \stackrel{\Delta}{\to} E_n^b(\mathcal{A}[x])\oplus E_n^b(\mathcal{A}[1/x]) \stackrel{\pm}{\to} E_n^b(\mathcal{A}[x,1/x]) \stackrel{\partial_n}{\to} E_{n-1}^b(\mathcal{A})\to 0\,.$$

Remark 4.4. A version of (4.3) for k-schemes can be found in Remark 8.6.

Proof. Let $\mathbb{P}^1$ be the projective line over $\mathrm{spec}(k)$ and $i : \mathrm{spec}(k[x]) \subset \mathbb{P}^1$ and $j : \mathrm{spec}(k[1/x]) \subset \mathbb{P}^1$ its standard Zariski open cover. Since $\mathrm{spec}(k[x]) \cap \mathrm{spec}(k[1/x]) = \mathrm{spec}(k[x,1/x])$, one obtains from Corollary 3.4 the following distinguished triangle

(4.5)
$$E(\mathcal{A}\otimes^{\mathbb{L}}\mathbb{P}^1) \stackrel{(E(\mathrm{id}\otimes^{\mathbb{L}} i^*),E(\mathrm{id}\otimes^{\mathbb{L}} j^*))}{\longrightarrow} E(\mathcal{A}[x])\oplus E(\mathcal{A}[1/x]) \stackrel{\pm}{\to} E(\mathcal{A}[x,1/x]) \stackrel{\partial}{\to} \Sigma E(\mathcal{A}\otimes^{\mathbb{L}}\mathbb{P}^1).$$

Note that since $k[x]$, $k[1/x]$ and $k[x,1/x]$ are all k-flat algebras, the derived tensor product agrees with the classical one. Let us now study the object

$E(\mathcal{A} \otimes^{\mathbb{L}} \mathbb{P}^1)$. As explained by Thomason in [27, §2.5-2.7], we have two fully faithful dg functors

$$\begin{aligned} \iota_0 : \mathsf{perf}(\mathrm{pt}) \to \mathsf{perf}(\mathbb{P}^1) \qquad & \mathcal{O}_{\mathrm{pt}} \mapsto \mathcal{O}_{\mathbb{P}^1}(0) \\ \iota_{-1} : \mathsf{perf}(\mathrm{pt}) \to \mathsf{perf}(\mathbb{P}^1) \qquad & \mathcal{O}_{\mathrm{pt}} \mapsto \mathcal{O}_{\mathbb{P}^1}(-1)\,. \end{aligned}$$

Moreover, ι_{-1} induces a Morita equivalence between $\mathsf{perf}(\mathrm{pt})$ and Drinfeld's dg quotient $\mathsf{perf}(\mathbb{P}^1)/\iota_0(\mathsf{perf}(\mathrm{pt}))$ (see [13, §4.4]). Following [21, §13], we obtain then a well-defined *split* short exact sequence of dg categories

$$0 \longrightarrow \mathsf{perf}(\mathrm{pt}) \underset{\iota_0}{\overset{r}{\leftrightarrows}} \mathsf{perf}(\mathbb{P}^1) \underset{s}{\overset{\iota_{-1}}{\leftrightarrows}} \mathsf{perf}(\mathrm{pt}) \longrightarrow 0\,, \tag{4.6}$$

where r is the right adjoint of ι_0, $r \circ \iota_0 = \mathrm{id}$, ι_{-1} is right adjoint of s, and $\iota_{-1} \circ s = \mathrm{id}$. As explained in the proof of Theorem 3.1, the functor $\mathcal{A} \otimes^{\mathbb{L}} - : \mathsf{Ho}(\mathsf{dgcat}) \to \mathsf{Ho}(\mathsf{dgcat})$ preserves split short exact sequences of dg categories. Moreover, every localizing functor sends split short exact sequences to split distinguished triangles, *i.e.* to direct sums in $\mathsf{Ho}(\mathcal{M})$. Therefore, by first applying $\mathcal{A} \otimes^{\mathbb{L}} -$ to (4.6) and then the functor E we obtain the following isomorphism

$$(E(\mathrm{id} \otimes^{\mathbb{L}} \iota_0), E(\mathrm{id} \otimes^{\mathbb{L}} \iota_{-1})) : E(\mathcal{A} \otimes \underline{k}) \oplus E(\mathcal{A} \otimes \underline{k}) \xrightarrow{\sim} E(\mathcal{A} \otimes^{\mathbb{L}} \mathbb{P}^1)\,. \tag{4.7}$$

Recall that the line bundles $\mathcal{O}_{\mathbb{P}^1}(0)$ and $\mathcal{O}_{\mathbb{P}^1}(-1)$ become isomorphic when restricted to $\mathrm{spec}(k[x])$ and $\mathrm{spec}(k[1/x])$. Hence, we have the commutative diagrams

$$\begin{aligned} &\mathsf{perf}(\mathrm{pt}) \underset{\iota_{-1}}{\overset{\iota_0}{\rightrightarrows}} \mathsf{perf}(\mathbb{P}^1) \xrightarrow{\;i^*\;} \mathsf{perf}(\mathrm{spec}(k[x])) \\ &\mathsf{perf}(\mathrm{pt}) \underset{\iota_{-1}}{\overset{\iota_0}{\rightrightarrows}} \mathsf{perf}(\mathbb{P}^1) \xrightarrow{\;j^*\;} \mathsf{perf}(\mathrm{spec}(k[1/x])) \end{aligned}$$

and consequently we obtain the equalities:

$$E(\mathrm{id} \otimes^{\mathbb{L}} i^*) \circ E(\mathrm{id} \otimes^{\mathbb{L}} \iota_0) = E(\mathrm{id} \otimes^{\mathbb{L}} i^*) \circ E(\mathrm{id} \otimes^{\mathbb{L}} \iota_{-1}) \tag{4.8}$$

$$E(\mathrm{id} \otimes^{\mathbb{L}} j^*) \circ E(\mathrm{id} \otimes^{\mathbb{L}} \iota_0) = E(\mathrm{id} \otimes^{\mathbb{L}} j^*) \circ E(\mathrm{id} \otimes^{\mathbb{L}} \iota_{-1})\,. \tag{4.9}$$

Now, apply Lemma 4.11 to isomorphism (4.7) and then compose the result with $(E(\mathrm{id} \otimes^{\mathbb{L}} i^*), E(\mathrm{id} \otimes^{\mathbb{L}} j^*))$. Thanks to (4.8)-(4.9), we obtain a morphism

$$\Psi : E(\mathcal{A} \otimes \underline{k}) \oplus E(\mathcal{A} \otimes \underline{k}) \longrightarrow E(\mathcal{A}[x]) \oplus E(\mathcal{A}[1/x]) \tag{4.10}$$

which is zero on the second component and

$$\big(E(\mathrm{id} \otimes i^*) \circ E(\mathrm{id} \otimes \iota_0), E(\mathrm{id} \otimes j^*) \circ E(\mathrm{id} \otimes \iota_0)\big)$$

on the first component; note once again that since k, $k[x]$ and $k[1/x]$ are k-flat the derived tensor product agrees with the classical one. Making use of (4.10), the above distinguished triangle (4.5) identifies with

$$E(\mathcal{A}) \oplus E(\mathcal{A}) \xrightarrow{\Psi} E(\mathcal{A}[x]) \oplus E(\mathcal{A}[1/x]) \xrightarrow{\pm} E(\mathcal{A}[x, 1/x]) \xrightarrow{\partial} \Sigma E(\mathcal{A}) \oplus \Sigma E(\mathcal{A})\,.$$

By applying to it the functor $\mathrm{Hom}_{\mathsf{Ho}(\mathcal{M})}(\Sigma^n b, -)$ we obtain then a long exact sequence

$$\cdots \to E_n^b(\mathcal{A}) \oplus E_n^b(\mathcal{A}) \xrightarrow{\Psi_n} E_n^b(\mathcal{A}[x]) \oplus E_n^b(\mathcal{A}[1/x]) \xrightarrow{\pm} E_n^b(\mathcal{A}[x,1/x]) \longrightarrow$$
$$\stackrel{\partial_n}{\to} E_{n-1}^b(\mathcal{A}) \oplus E_{n-1}^b(\mathcal{A}) \stackrel{\Psi_{n-1}}{\to} E_{n-1}^b(\mathcal{A}[x]) \oplus E_{n-1}^b(\mathcal{A}[1/x]) \stackrel{\pm}{\to} E_{n-1}^b(\mathcal{A}[x,1/x]) \to \cdots$$

As explained above, Ψ_n is zero when restricted to the second component. Moreover, since the inclusions $k \subset k[x]$ and $k \subset k[1/x]$ admits canonical retractions, Ψ_n is injective when restricted to the first component. This implies that the image of ∂_n is precisely the second component of the direct sum. As a consequence, the above long exact sequence breaks up into the exact sequences (4.3). This achieves the proof. □

Lemma 4.11. *If $(f,g) : A \oplus A \xrightarrow{\sim} B$ is an isomorphism in an additive category, then $(f, f-g) : A \oplus A \xrightarrow{\sim} B$ is also an isomorphism.*

Proof. Since (f,g) is an isomorphism, there exist maps $i, h : B \to A$ such that $fi + gh = \mathrm{id}$, $if = \mathrm{id}$, $hf = 0$, $ig = 0$, and $hg = \mathrm{id}$. Using these equalities one observes that $(i+h, -h) : B \xrightarrow{\sim} A \oplus A$ is the inverse of $(f, f-g)$. □

Notation 4.12. Given a dg category $\mathcal{A}$, let us denote by $NE_n^b(\mathcal{A})$ the kernel of the surjective group homomorphism

$$E_n^b(\mathrm{id} \otimes (t=0)) : E_n^b(\mathcal{A}[t]) \longrightarrow E_n^b(\mathcal{A}) \,. \tag{4.13}$$

Note that the inclusion $k \subset k[t]$ gives rise to a direct sum decomposition $E_n^b(\mathcal{A}[t]) \simeq NE_n^b(\mathcal{A}) \oplus E_n^b(\mathcal{A})$. Note also that by induction on m, $\mathcal{A}$ is E_n^b-regular if and only if $NE_n^b(\mathcal{A}[t_1, \ldots, t_m]) = 0$ for all $m \geq 0$.

Corollary 4.14. *Under the notations and assumptions of Theorem 4.2, we have the following exact sequence of abelian groups*

$$0 \to NE_n^b(\mathcal{A}) \stackrel{\Delta}{\to} NE_n^b(\mathcal{A}[x]) \oplus NE_n^b(\mathcal{A}[1/x]) \stackrel{\pm}{\to} NE_n^b(\mathcal{A}[x,1/x]) \stackrel{\partial_n}{\to} NE_{n-1}^b(\mathcal{A}) \to 0 \,.$$

Proof. This follows automatically from the naturality of (4.3). □

5. Proof of Theorem 1.4

Consider the following "substitution" k-algebra homomorphism

$$k[x][t] \longrightarrow k[x][t] \qquad p(x,t) \mapsto p(x, xt) \,. \tag{5.1}$$

Given a dg category $\mathcal{B}$, let us denote by $\mathrm{colim}\, NE_n^b(\mathcal{B}[x])$ the direct limit of the following diagram of abelian groups

$$NE_n^b(\mathcal{B}[x]) \xrightarrow{NE_n^b(\mathrm{id}\otimes(5.1))} NE_n^b(\mathcal{B}[x]) \xrightarrow{NE_n^b(\mathrm{id}\otimes(5.1))} NE_n^b(\mathcal{B}[x]) \xrightarrow{NE_n^b(\mathrm{id}\otimes(5.1))} \cdots$$

We start by proving that we have a group isomorphism

$$\mathrm{colim}\, NE_n^b(\mathcal{B}[x]) \simeq NE_n^b(\mathcal{B}[x, x^{-1}]) \,. \tag{5.2}$$

Consider first the commutative diagram

$$
\begin{tikzcd}
k[x][t] \arrow[r, "(5.1)"] \arrow[d, "(t=0)"'] & k[x][t] \arrow[r, "(5.1)"] \arrow[d, "(t=0)"'] & k[x][t] \arrow[r, "(5.1)"] \arrow[d, "(t=0)"'] & \cdots \\
k[x] \arrow[r, equal] & k[x] \arrow[r, equal] & k[x] \arrow[r, equal] & \cdots
\end{tikzcd}
\tag{5.3}
$$

Note that the colimit of the lower row is $k[x]$ while the colimit of the upper row is the k-algebra $R := k[x] + tk[x, 1/x][t] \subset k[x, 1/x][t]$. By first tensoring (5.3) with $\mathcal{B}$ and then applying the functor E_n^b we obtain the commutative diagram (5.4)

$$
\begin{tikzcd}
NE_n^b(\mathcal{B}[x]) \arrow[r, "NE_n^b(\mathrm{id}\otimes(5.1))"] \arrow[d] & NE_n^b(\mathcal{B}[x]) \arrow[r, "NE_n^b(\mathrm{id}\otimes(5.1))"] \arrow[d] & NE_n^b(\mathcal{B}[x]) \arrow[r, "NE_n^b(\mathrm{id}\otimes(5.1))"] \arrow[d] & \cdots \\
E_n^b(\mathcal{B}[x][t]) \arrow[r, "E_n^b(\mathrm{id}\otimes(5.1))"] \arrow[d, "(4.13)"'] & E_n^b(\mathcal{B}[x][t]) \arrow[r, "E_n^b(\mathrm{id}\otimes(5.1))"] \arrow[d, "(4.13)"'] & E_n^b(\mathcal{B}[x][t]) \arrow[r, "E_n^b(\mathrm{id}\otimes(5.1))"] \arrow[d, "(4.13)"'] & \cdots \\
E_n^b(\mathcal{B}[x]) \arrow[r, equal] & E_n^b(\mathcal{B}[x]) \arrow[r, equal] & E_n^b(\mathcal{B}[x]) \arrow[r, equal] & \cdots
\end{tikzcd}
$$

Recall from Notation (4.12) that each column is a (split) short exact sequence of abelian groups. The colimit of the lower row is clearly $E_n^b(\mathcal{B}[x])$. Since the functors $\mathcal{B} \otimes - : \mathsf{dgcat} \to \mathsf{dgcat}$ and $E : \mathsf{dgcat} \to \mathcal{M}$ preserve filtered (homotopy) colimits and b is a compact object of $\mathsf{Ho}(\mathcal{M})$, the colimit of the middle row identifies with $E_n^b(\mathcal{B} \otimes \underline{R})$. Hence, from diagram (5.4) one obtains the isomorphism

$$
\operatorname{colim} NE_n^b(\mathcal{B}[x]) \simeq \operatorname{Ker}\big(E_n^b(\mathcal{B} \otimes \underline{R}) \stackrel{(4.13)}{\longrightarrow} E_n^b(\mathcal{B}[x])\big)\,. \tag{5.5}
$$

Now, consider the k-algebras R and $k[x]$ endowed with the sets of left denominators $S_1 := \{x^n\}_{n \geq 0} \subset R$ and $S_2 := \{x^n\}_{n \geq 0} \subset k[x]$. The k-algebra homomorphism

$$
R = k[x] + tk[x, 1/x][t] \longrightarrow k[x] \qquad t \mapsto 0 \tag{5.6}
$$

identifies S_1 with S_2 and moreover induces a quasi-isomorphism

$$
\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, "(5.6)"'] & R[S_1^{-1}] = k[x, 1/x][t] \arrow[r] \arrow[d, "(5.6)"'] & 0 \\
0 \arrow[r] & k[x] \arrow[r] & k[x][S_2^{-1}] = k[x, 1/x] \arrow[r] & 0\,.
\end{tikzcd}
$$

As a consequence, since R and $k[x]$ are clearly k-flat algebras, conditions a) and b) of [14, §4.2] are satisfied. In *loc. cit.* Keller also assumes that the base ring k is coherent and of finite dimensional global dimension. However, these extra assumptions are only used to prove the localization theorem for model categories; see [14, §5-6]. We obtain then a commutative diagram in $\mathsf{Ho}(\mathsf{dgcat})$

$$\begin{tikzcd} 0 \arrow[r] & \underline{A_1} \arrow[r] \arrow[d, "\sim"'] & \mathsf{perf}(R) \arrow[r] \arrow[d] & \mathsf{perf}(k[x,x^{-1}][t]) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \underline{A_2} \arrow[r] & \mathsf{perf}(k[x]) \arrow[r] & \mathsf{perf}(k[x,x^{-1}]) \arrow[r] & 0\,, \end{tikzcd} \tag{5.7}$$

where moreover each row is a short exact sequence of dg categories and the left vertical map is a quasi-isomorphism (and hence a Morita equivalence) of dg k-algebras; consult [14, §4.3] for further details. By first tensoring (5.7) with $\mathcal{B}$ and then applying the functor E we obtain (as in the proof of Theorem 3.1) a homotopy (co)cartesian square

$$\begin{tikzcd} E(\mathcal{B}\otimes \underline{R}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E(\mathcal{B}[x,1/x][t]) \arrow[d] \\ E(\mathcal{B}[x]) \arrow[r] & E(\mathcal{B}[x,1/x])\,. \end{tikzcd} \tag{5.8}$$

Note that since R, $k[x]$, $k[x,1/x]$, and $k[x,1/x][t]$ are all k-flat algebras, the derived tensor product agrees with the classical one. Note also that the natural inclusions $k[x]\subset R$ and $k[x,1/x]\subset k[x,1/x][t]$ give rise to sections of the vertical maps. As a consequence, since (5.8) is homotopy (co)cartesian, we obtain an induced isomorphism

$$\mathrm{Ker}\big(E_n^b(\mathcal{B}\otimes\underline{R}) \overset{(4.13)}{\to} E_n^b(\mathcal{B}[x])\big) \xrightarrow{\sim} \mathrm{Ker}\big(E_n^b(\mathcal{B}[x,1/x][t]) \overset{(4.13)}{\to} E_n^b(\mathcal{B}[x,1/x])\big)\,.$$

Since the right-hand-side is by definition $NE_n^b(\mathcal{B}[x,1/x])$ the searched isomorphism (5.2) follows now from isomorphism (5.5).

We are now ready to conclude the proof. As explained in Notation 4.12, a dg category $\mathcal{A}$ is E_n^b-regular if and only if $NE_n^b(\mathcal{A}[t_1,\ldots,t_m])=0$ for any all $m\geq 0$. Since $\mathcal{A}$ is E_n^b-regular we hence have $NE_n^b(\mathcal{A}[t_1,\ldots,t_m])=0$ for all $m\geq 0$. Using isomorphism (5.2) (with $\mathcal{B}=\mathcal{A}[t_1,\ldots t_{m-1}]$) we conclude that

$$\mathrm{colim}\, NE_n^b(\mathcal{A}[t_1,\ldots,t_{m-1}][x]) \simeq NE_n^b(\mathcal{A}[t_1,\ldots,t_{m-1}][x,1/x]) = 0\,.$$

The exact sequence of Corollary 4.14 (with $\mathcal{A}=\mathcal{A}[t_1,\ldots,t_{m-1}]$) implies that $NE_{n-1}^b(\mathcal{A}[t_1,\ldots,t_{m-1}])=0$. Since this holds for every $m\geq 0$, we conclude finally that $\mathcal{A}$ is E_{n-1}^b-regular. This concludes the proof of Theorem 1.4.

6. Proof of Theorem 1.6

Item (i) follows from the combination of implication (1.5) with the equalities

$$E_n^{\Sigma^{-i}b}(\mathcal{A}) := \mathrm{Hom}_{\mathsf{Ho}(\mathcal{M})}(\Sigma^n(\Sigma^{-i}b), E(\mathcal{A})) = \mathrm{Hom}_{\mathsf{Ho}(\mathcal{M})}(\Sigma^{n-i}b, E(\mathcal{A})) =: E_{n-i}^b(\mathcal{A})\,.$$

In what concerns item (ii), note that by applying the bifunctor $\mathrm{Hom}_{\mathsf{Ho}(\mathcal{M})}(-,-)$ to the sequence

$$\Sigma^{n-1}c' \to \Sigma^{n-1}c'' \to \Sigma^n c \to \Sigma^n c' \to \Sigma^n c''$$

in the first variable and to the morphism $E(\mathcal{A}) \to E(\mathcal{A}[t_1,\ldots,t_m])$ in the second variable, one obtains the following commutative diagram

$$\begin{tikzcd}
E_n^{c''}(\mathcal{A}) \arrow[r] \arrow[d] & E_n^{c''}(\mathcal{A}[t_1,\ldots,t_m]) \arrow[d] \\
E_n^{c'}(\mathcal{A}) \arrow[r] \arrow[d] & E_n^{c'}(\mathcal{A}[t_1,\ldots,t_m]) \arrow[d] \\
E_n^{c}(\mathcal{A}) \arrow[r] \arrow[d] & E_n^{c}(\mathcal{A}[t_1,\ldots,t_m]) \arrow[d] \\
E_{n-1}^{c''}(\mathcal{A}) \arrow[r] \arrow[d] & E_{n-1}^{c''}(\mathcal{A}[t_1,\ldots,t_m]) \arrow[d] \\
E_{n-1}^{c'}(\mathcal{A}) \arrow[r] & E_{n-1}^{c'}(\mathcal{A}[t_1,\ldots,t_m]) \,,
\end{tikzcd}$$

where each column is exact. Since by hypothesis $\mathcal{A}$ is $E_n^{c'}$-regular and $E_n^{c''}$-regular the two top horizontal morphisms are isomorphisms. Using implication (1.5) we conclude that the two bottom horizontal morphisms are also isomorphisms. Using the 5-lemma one then concludes that the horizontal middle morphism is an isomorphism. This implies that $\mathcal{A}$ is E_n^c-regular.
Let us now prove item (iii). Since by hypothesis d is a direct factor of b, there exist morphisms $d \to b$ and $b \to d$ such that the composition $d \to b \to d$ equals the identity of d. This data gives naturally rise to the following commutative diagram

$$\begin{tikzcd}
E_n^{d}(\mathcal{A}) \arrow[r] \arrow[d] & E_n^{b}(\mathcal{A}) \arrow[r] \arrow[d] & E_n^{d}(\mathcal{A}) \arrow[d] \\
E_n^{d}(\mathcal{A}[t_1,\ldots,t_m]) \arrow[r] & E_n^{b}(\mathcal{A}[t_1,\ldots,t_m]) \arrow[r] & E_n^{d}(\mathcal{A}[t_1,\ldots,t_m]) \,,
\end{tikzcd} \tag{6.1}$$

where both horizontal compositions are the identity. By assumption, $\mathcal{A}$ is E_n^b-regular and so the vertical middle morphism in (6.1) is an isomorphism. From the commutativity of (6.1) and the fact that isomorphisms are stable under retractions, one concludes that the vertical left-hand-side (or right-hand-side) morphism is also an isomorphism. This implies that $\mathcal{A}$ is E_n^d-regular.
Item (iv) follow from the combination of implication (1.5) with the equalities

$$E_n^{\oplus_{i\in I}c_i}(\mathcal{A}) := \mathrm{Hom}(\Sigma^n(\oplus_{i\in I}c_i), E(\mathcal{A})) = \prod_{i\in I}\mathrm{Hom}(\Sigma^n c_i, E(\mathcal{A})) =: \prod_{i\in I} E_n^{c_i}(\mathcal{A})\,,$$

where we have removed the subscripts of Hom in order to simplify the exposition. Let us now prove item (v). As explained in [23, Thm. 1.2], we have a canonical isomorphism $U(\sigma(\mathcal{A})) \xrightarrow{\sim} \Sigma U(\mathcal{A})$ in $\mathsf{Ho}(\mathrm{Mot})$; in *loc. cit.* $\sigma(\mathcal{A})$ was denoted by $\Sigma(\mathcal{A})$ and U by $\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}$. Hence, by applying the triangulated functor $\overline{E}$ of Proposition 2.1 to the square below (6.6) (with $\mathcal{B} := \mathcal{A}[t_1,\ldots,t_m]$), one obtains the square

$$
\begin{tikzcd}
E(\sigma(\mathcal{A})) \arrow[r, "\sim"] \arrow[d] & \Sigma E(\mathcal{A}) \arrow[d] \\
E(\sigma(\mathcal{A}[t_1, \ldots, t_m])) \arrow[r, "\sim"'] & \Sigma E(\mathcal{A}[t_1, \ldots, t_m])
\end{tikzcd}
\tag{6.2}
$$

in the homotopy category $\mathsf{Ho}(\mathcal{M})$. Since by construction $\sigma(\mathcal{A}[t_1, \ldots, t_m])$ and $\sigma(\mathcal{A})[t_1, \ldots, t_m]$ are canonically isomorphic, (6.2) gives rise to the following commutative diagram

(6.3)

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Ho}(\mathcal{M})}(\Sigma^{n+1}b, E(\sigma(\mathcal{A}))) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}_{\mathsf{Ho}(\mathcal{M})}(\Sigma^{n+1}b, \Sigma E(\mathcal{A})) \arrow[d] \\
\mathrm{Hom}_{\mathsf{Ho}(\mathcal{M})}(\Sigma^{n+1}b, E(\sigma(\mathcal{A})[t_1, \ldots, t_m])) \arrow[r, "\sim"'] & \mathrm{Hom}_{\mathsf{Ho}(\mathcal{M})}(\Sigma^{n}b, \Sigma E(\mathcal{A}[t_1, \ldots, t_m]))\,.
\end{tikzcd}
$$

Moreover, using the fact that $\Sigma^{-1}(-)$ is an autoequivalence of $\mathsf{Ho}(\mathcal{M})$, we have

(6.4)

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Ho}(\mathcal{M})}(\Sigma^{n+1}b, \Sigma E(\mathcal{A})) \arrow[r, "\Sigma^{-1}(-)", "\sim"'] \arrow[d] & \mathrm{Hom}_{\mathsf{Ho}(\mathcal{M})}(\Sigma^{n}b, E(\mathcal{A})) \arrow[d] \\
\mathrm{Hom}_{\mathsf{Ho}(\mathcal{M})}(\Sigma^{n+1}b, \Sigma E(\mathcal{A}[t_1, \ldots, t_m])) \arrow[r, "\sim", "\Sigma^{-1}(-)"'] & \mathrm{Hom}_{\mathsf{Ho}(\mathcal{M})}(\Sigma^{n}b, E(\mathcal{A}[t_1, \ldots, t_m]))\,.
\end{tikzcd}
$$

Now, recall that by hypothesis $\mathcal{A}$ is E^b_n-regular. Hence, the vertical right-hand-side morphism in (6.4) is an isomorphism. Consequently, by combining (6.3)-(6.4), we conclude that the vertical left-hand-side morphism in (6.3), *i.e.* $E^b_{n+1}(\sigma(\mathcal{A})) \to E^b_{n+1}(\sigma(\mathcal{A})[t_1, \ldots, t_m])$ is an isomorphism. This implies that $\sigma(\mathcal{A})$ is E^b_{n+1}-regular and so the proof is finished.

Lemma 6.5. *Given a dg functor $F : \mathcal{A} \to \mathcal{B}$, we have a commutative diagram*

$$
\begin{tikzcd}
U(\sigma(\mathcal{A})) \arrow[r, "\sim"] \arrow[d, "U(\sigma(F))"'] & \Sigma U(\mathcal{A}) \arrow[d, "\Sigma U(F)"] \\
U(\sigma(\mathcal{B})) \arrow[r, "\sim"'] & \Sigma U(\mathcal{B})
\end{tikzcd}
\tag{6.6}
$$

in the homotopy category $\mathsf{Ho}(\mathrm{Mot})$.

Proof. Thanks to [23, Prop. 4.9], we have the diagram in $\mathsf{Ho}(\mathsf{dgcat})$

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{A} \otimes \underline{k} \arrow[r] \arrow[d, "F\otimes \mathrm{id}"'] & \mathcal{A} \otimes \underline{\Gamma} \arrow[r] \arrow[d, "F\otimes \mathrm{id}"'] & \mathcal{A} \otimes \underline{\sigma} \arrow[r] \arrow[d, "F\otimes \mathrm{id}"'] & 0 \\
0 \arrow[r] & \mathcal{B} \otimes \underline{k} \arrow[r] & \mathcal{B} \otimes \underline{\Gamma} \arrow[r] & \mathcal{B} \otimes \underline{\sigma} \arrow[r] & 0\,,
\end{tikzcd}
\tag{6.7}
$$

where both rows are short exact sequences of dg categories. Consequently, by applying the functor U to (6.7) we obtain the following morphism between distinguished triangles:

$$
\text{(6.8)}\qquad
\begin{tikzcd}
U(\mathcal{A}) \arrow[r] \arrow[d, "U(F)"'] & U(\mathcal{A}\otimes\underline{\Gamma}) \arrow[r] \arrow[d] & U(\sigma(\mathcal{A})) \arrow[r, "\partial"] \arrow[d, "U(\sigma(F))"'] & \Sigma E(\mathcal{A}) \arrow[d, "\Sigma U(F)"] \\
U(\mathcal{B}) \arrow[r] & U(\mathcal{B}\otimes\underline{\Gamma}) \arrow[r] & U(\sigma(\mathcal{B})) \arrow[r, "\partial"'] & \Sigma E(\mathcal{B})\,.
\end{tikzcd}
$$

As explained in [23, §6], $U(\mathcal{A}\otimes\underline{\Gamma})$ and $U(\mathcal{B}\otimes\underline{\Gamma})$ are isomorphic to zero in $\mathsf{Ho}(\mathrm{Mot})$. Hence, the connecting morphisms ∂ are isomorphisms and so the searched commutative square (6.6) is the right-hand-side square in (6.8). This achieves the proof. □

7. Proof of Proposition 1.10

Consider the following distinguished triangle in $\mathsf{Ho}(\mathrm{Mot})$

$$U(\underline{k}) \xrightarrow{\cdot l} U(\underline{k}) \longrightarrow U(\underline{k})/l \longrightarrow \Sigma U(\underline{k})\,.$$

As proved in [25, Prop. 2.12], one has the following isomorphisms

$$\mathrm{Hom}_{\mathsf{Ho}(\mathrm{Mot})}(\Sigma^n(U(\underline{k})/l), U(\mathcal{A})) \simeq K_{n+1}(\mathcal{A};\mathbb{Z}/l) \qquad n\in\mathbb{Z}\,.$$

In *loc. cit.* the author worked with $k=\mathbb{Z}$ and with the additive version of Mot where localization is replaced by additivity; however, the arguments are exactly the same. The proof follows now from the fact that $U(\underline{k})/l \simeq \Sigma\mathrm{fib}(l)$ and from the definition $U_n^{\mathrm{fib}(l)}(\mathcal{A}) := \mathrm{Hom}_{\mathsf{Ho}(\mathrm{Mot})}(\Sigma^n\mathrm{fib}(l), U(\mathcal{A}))$.

8. Proof of Theorem 1.12

Since by hypothesis X is E_n^b-regular the isomorphism $E_n^b(X)\simeq E_n^b(X\times\mathbb{A}^m)$ holds for all $m\geq 1$. By applying Proposition 8.2 below to X and to the k-flat k-scheme $Y=\mathbb{A}^m$ we obtain moreover the following isomorphisms

$$\text{(8.1)}\quad E_n^b(X\times\mathbb{A}^m) \overset{(8.3)}{\simeq} E_n^b(\mathsf{perf}(X)\otimes\mathsf{perf}(\mathbb{A}^m)) \simeq E_n^b(\mathsf{perf}(X)[t_1,\ldots,t_m])\,.$$

Note that since $\mathbb{A}^m=\mathrm{spec}(k[t_1,\ldots,t_m])$ is an affine k-flat algebra the derived tensor product agrees with the classical one. By combining (8.1) with the isomorphism $E_n^b(X)\simeq E_n^b(X\times\mathbb{A}^m)$ we conclude then that the dg category $\mathsf{perf}(X)$ is E_n^b-regular. By Theorem 1.4 it is also E_{n-1}^b-regular. Hence, using again the above isomorphisms (8.1) (with n replaced by $n-1$) one concludes that the isomorphism $E_{n-1}^b(X)\simeq E_{n-1}^b(X\times\mathbb{A}^m)$ holds for all $m\geq 1$, *i.e.* that X is E_{n-1}^b-regular. This proves implication (1.13). Implication (1.14) follows automatically from the combination of the above isomorphism (8.1) with implication (1.8). Finally, implication (1.15) follows from the combination of Proposition 1.10 with implication (1.13) and with [25, Example 2.13]. This achieves the proof.

Proposition 8.2. *Let X and Y be two quasi-compact separated k-schemes with Y k-flat, n an integer, $E:\mathsf{dgcat}\to\mathcal{M}$ a Morita invariant localizing functor,* *and b an object of $\mathcal{M}$. Under these notations and assumptions, we have a canonical isomorphism*

$$E_n^b(-\boxtimes^{\mathbb{L}}-):E_n^b(\mathsf{perf}(X)\otimes^{\mathbb{L}}\mathsf{perf}(Y))\xrightarrow{\sim}E_n^b(\mathsf{perf}(X\times Y))\,. \tag{8.3}$$

Proof. The proof will consist on showing that the canonical maps

$$E(-\boxtimes^{\mathbb{L}}-)_{Z,W}:E(\mathsf{perf}(Z)\otimes^{\mathbb{L}}\mathsf{perf}(W))\longrightarrow E(\mathsf{perf}(Z\times W))\,, \tag{8.4}$$

parametrized by the pairs (Z,W) of quasi-compact separated k-schemes with W k-flat, are isomorphisms. The above isomorphism (8.3) will follow then from (8.4) (with $Z:=X$ and $W:=Y$) by applying the functor $\mathrm{Hom}_{\mathsf{Ho}(\mathcal{M})}(\Sigma^n b,-)$. Let us denote by Sch the category of quasi-compact separated k-schemes and by $\mathrm{Sch}_{\mathsf{flat}}$ the full subcategory of k-flat schemes. Note that we have two well-defined contravariant bifunctors

$$E(\mathsf{perf}(-)\otimes^{\mathbb{L}}\mathsf{perf}(-))\qquad E(\mathsf{perf}(-\times-))$$

from $\mathrm{Sch}\times\mathrm{Sch}_{\mathsf{flat}}$ to $\mathsf{Ho}(\mathcal{M})$. Moreover, the above canonical maps (8.4) give rise to a natural transformation of bifunctors

$$E(\mathsf{perf}(-)\otimes^{\mathbb{L}}\mathsf{perf}(-))\Rightarrow E(\mathsf{perf}(-\times-))\,. \tag{8.5}$$

Our goal is then to show that (8.5) is an isomorphism when evaluated at any pair $(Z,W)\in\mathrm{Sch}\times\mathrm{Sch}_{\mathsf{flat}}$. Let us start by fixing W. Thanks to Theorem 3.1 (applied to $\mathcal{A}=\mathsf{perf}(W)$) one observes that the functor $E(\mathsf{perf}(-)\otimes^{\mathbb{L}}\mathsf{perf}(W))$ satisfies Nisnevich descent and hence by Corollary 3.4 Zariski descent. In what concerns $E(\mathsf{perf}(-\times W))$ note first that by applying the functor $-\times W$ to (3.2) one still obtains a (distinguished) square of k-schemes. Therefore, Theorem 3.1 (applied to $\mathcal{A}=\underline{k}$) allows us to conclude that $E(\mathsf{perf}(-\times W))$ satisfies also Nisnevich descent.
Now, by the reduction principle of Bondal and Van den Bergh (see [4, Prop. 3.3.1]) the above natural transformation (8.5) is an isomorphism when evaluated at the pairs (Z,W), with W fixed, if and only if it is an isomorphism when evaluated at the pairs $(\mathrm{spec}(C),W)$, with C a commutative k-algebra. By fixing Z and making the same argument one concludes also from the reduction principle that (8.5) is an isomorphism when evaluated at the pairs (Z,W), with Z fixed, if and only if it is an isomorphism when evaluated at the pairs $(Z,\mathrm{spec}(D))$, with D a k-flat commutative k-algebra. In conclusion it suffices to show that (8.5) is an isomorphism when evaluated at the pairs $(\mathrm{spec}(C),\mathrm{spec}(D))$. Note that in this particular case we have the following canonical Morita equivalences

$$\mathsf{perf}(\mathrm{spec}(C))\simeq\underline{C}\quad\mathsf{perf}(\mathrm{spec}(D))\simeq\underline{D}\quad\mathsf{perf}(\mathrm{spec}(C)\times\mathrm{spec}(D))\simeq\underline{C}\otimes\underline{D}\,.$$

Moreover, since the k-algebra D is k-flat, the derived tensor product $\underline{C}\otimes^{\mathbb{L}}\underline{D}$ agrees with the classical one $\underline{C}\otimes\underline{D}$. By applying the functor E to this latter isomorphism one obtains the evaluation $E(\underline{C}\otimes^{\mathbb{L}}\underline{D})\simeq E(\underline{C}\otimes\underline{D})$ of the above natural transformation (8.5) at the pair $(\mathrm{spec}(C),\mathrm{spec}(D))$. This concludes the proof of Proposition 8.2. □

Remark 8.6. Given a quasi-compact separated k-scheme X, let

$$X[x] := X \times \mathbb{A}^1 \quad X[1/x] := X \times \mathrm{spec}(k[1/x]) \quad X[x, 1/x] := X \times \mathrm{spec}(k[x, 1/x])\,.$$

Making use of Proposition 8.2 and the k-flatness of $k[x]$, $k[1/x]$ and $k[x, 1/x]$, one observes that Theorem 4.2 applied to $\mathcal{A} = \mathsf{perf}(X)$ reduces to the following exact sequence of abelian groups

$$0 \to E_n^b(X) \stackrel{\Delta}{\to} E_n^b(X[x]) \oplus E_n^b(X[1/x]) \stackrel{\pm}{\to} E_n^b(X[x, 1/x]) \stackrel{\partial_n}{\to} E_{n-1}^b(X) \to 0\,.$$

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An Equivariant Lefschetz Fixed-Point Formula for Correspondences

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Abstract. We compute the trace of an endomorphism in equivariant bivariant K-theory for a compact group G in several ways: geometrically using geometric correspondences, algebraically using localisation, and as a Hattori–Stallings trace. This results in an equivariant version of the classical Lefschetz fixed-point theorem, which applies to arbitrary equivariant correspondences, not just maps.

We dedicate this article to Tamaz Kandelaki, who was a coauthor in an earlier version of this article, and passed away in 2012. We will remember him for his warm character and his perseverance in doing mathematics in difficult circumstances.

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1 Introduction

Here we continue a series of articles by the last two authors about Euler characteristics and Lefschetz invariants in equivariant bivariant K-theory. These invariants were introduced in [11, 13–16]. The goal is to compute Lefschetz invariants explicitly in a way that generalises the Lefschetz–Hopf fixed-point formula.

Let X be a smooth compact manifold and $f\colon X \to X$ a self-map with simple isolated fixed points. The Lefschetz–Hopf fixed-point formula identifies

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1. the sum over the fixed points of f, where each fixed point contributes ± 1 depending on its index;

2. the supertrace of the $\mathbb{Q}$-linear, grading-preserving map on $\mathrm{K}^*(X) \otimes \mathbb{Q}$ induced by f.

It makes no difference in (2) whether we use rational cohomology or K-theory because the Chern character is an isomorphism between them.

We will generalise this result in two ways. First, we allow a compact group G to act on X and get elements of the representation ring $\mathrm{R}(G)$ instead of numbers. Secondly, we replace self-maps by self-correspondences in the sense of [15]. Sections 2 and 3 generalise the invariants (1) and (2) respectively to this setting. The invariant of Section 2 is local and geometric and generalises (1) above; the formulas in Sections 3 and 4 are global and homological and generalise (2) (in two different ways.) The equality of the geometric and homological invariants is our generalisation of the Lefschetz fixed-point theorem.

A first step is to interpret the invariants (1) or (2) in a category-theoretic way in terms of the trace of an endomorphism of a dualisable object in a symmetric monoidal category.

Let $\mathcal{C}$ be a symmetric monoidal category with tensor product $\otimes$ and tensor unit $\mathbb{1}$. An object A of $\mathcal{C}$ is called *dualisable* if there is an object A^*, called its *dual*, and a natural isomorphism

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(B, A^* \otimes C)$$

for all objects B and C of $\mathcal{C}$. Such duality isomorphisms exist if and only if there are two morphisms $\eta\colon \mathbb{1} \to A \otimes A^*$ and $\varepsilon\colon A^* \otimes A \to \mathbb{1}$, called unit and counit of the duality, that satisfy two appropriate conditions. Let $f\colon A \to A$ be an endomorphism in $\mathcal{C}$. Then the *trace* of f is the composite endomorphism

$$\mathbb{1} \xrightarrow{\eta} A \otimes A^* \xrightarrow{\mathsf{sym}} A^* \otimes A \xrightarrow{\mathrm{id}_{A^*} \otimes f} A^* \otimes A \xrightarrow{\varepsilon} \mathbb{1},$$

where sym denotes the symmetry (or braiding) isomorphism. In this article we also call the trace the *Lefschetz index* of the morphism. This is justified by the following example.

Let $\mathcal{C}$ be the Kasparov category KK with its usual tensor product structure, $A = \mathrm{C}(X)$ for a smooth compact manifold X, and $\hat{f} \in \mathrm{KK}_0(A, A)$ for some morphism. We may construct a dual A^* from the tangent bundle or the stable normal bundle of X. In the case of a smooth self-map of X, and assuming a certain transversality condition, the trace of the morphism $\hat{f}$ induced by the self-map equals the invariant (1), that is, equals the number of fixed-points of the map, counted with appropriate signs. This is checked by direct computation in Kasparov theory, see [13] for more general results.

This paper springs in part from the reference [13]. A similar invariant to the Lefschetz index was introduced there, called the *Lefschetz class* (of the morphism). The Lefschetz class for an equivariant Kasparov endomorphism of X was defined as an equivariant K-homology class for X. The Lefschetz *index*, that is, the categorical trace, discussed above, is the Atiyah–Singer index of the Lefschetz class of [13].

The main goal of this article is to give a global, homological formula for the Lefschetz index generalising the invariant (2) for a non-equivariant self-map. The formulation and proof of our homological formula works best for Hodgkin Lie groups. A more complicated form applies to all compact groups. The article [13] also provides two formulas for the equivariant Lefschetz class whose equality generalises that of the invariants (1) and (2), but the methods there are completely different.

The other main contribution of this article is to compute the geometric expression for the Lefschetz index in the category $\widehat{\mathrm{KK}}^G$ of geometric correspondences introduced in [15]. This simplifies the computation in Kasparov's analytic theory in [13] and also gives a more general result, since we can work with general smooth correspondences rather than just maps. Furthermore, using an idea of Baum and Block in [4], we give a recipe for composing two smooth equivariant correspondences under a weakening of the usual transversality assumption (of [6]). This technique is important for computing the Lefschetz index in the case of continuous group actions, where transversality is sometimes difficult to achieve, and in particular, aids in describing equivariant Euler characteristics in our framework.

Section 2 contains our geometric formula for the Lefschetz index of an equivariant self-correspondence. Why is there a nice geometric formula for the Lefschetz index of a self-map in Kasparov theory? A good explanation is that Connes and Skandalis [6] describe KK-theory for commutative C*-algebras geometrically, including the Kasparov product; furthermore, the unit and counit of the KK-duality for smooth manifolds have a simple form in this geometric variant of KK. An equivariant version of the theory in [6] is developed in [15]. In Section 2, we also recall some basic results about the geometric KK-theory introduced in [15]. If X is a smooth compact G-manifold for a compact group G, then $\mathrm{KK}^G_*(\mathrm{C}(X),\mathrm{C}(X))$ is isomorphic to the geometrically defined group $\widehat{\mathrm{KK}}^G_*(X,X)$. Its elements are smooth *correspondences*

$$X \xleftarrow{b} (M,\xi) \xrightarrow{f} X \tag{1.1}$$

consisting of a smooth G-map b, a K_G-oriented smooth G-map f, and $\xi \in \mathrm{K}^*_G(M)$. Theorem 2.18 computes the categorical trace, or Lefschetz index, of such a correspondence under suitable assumptions on b and f.

Assume first that X has no boundary and that b and f are transverse; equivalently, for all $m \in M$ with $f(m) = b(m)$ the linear map $Db - Df\colon \mathrm{T}_m M \to \mathrm{T}_{f(m)}X$ is surjective. Then

$$Q := \{m \in M \mid b(m) = f(m)\} \tag{1.2}$$

is naturally a K_G-oriented smooth manifold. We show that the Lefschetz index is the G-index of the Dirac operator on Q twisted by $\xi|_Q \in \mathrm{K}^*_G(Q)$ (Theorem 2.18). More generally, suppose that the coincidence space Q as defined above is merely assumed to be a smooth submanifold of M, and that $x \in \mathrm{T}X$ and $Df(\xi) = Db(\xi)$ implies that $\xi \in \mathrm{T}Q$. Then we say that f and b *intersect smoothly*. For example, the identity correspondence, where f and b are the identity maps on X, does not satisfy the above transversality hypothesis, but f and b clearly intersect smoothly. In the case of a smooth intersection, the cokernels of the map $Df - Db$ form a vector bundle on Q which we call the *excess intersection bundle* η. This bundle measures the failure of transversality of f and b. Let η be K_G-oriented. Then $\mathrm{T}Q$ also inherits a canonical K_G-orientation. The restriction of the Thom class of η to the zero section gives a class $e(\eta) \in \mathrm{K}^*_G(Q)$.

Then Theorem 2.18 asserts that the Lefschetz index of the correspondence (1.1) with smoothly intersecting f and b is the index of the Dirac operator on the coincidence manifold Q twisted by $\xi \otimes e(\eta)$. This is the main result of Section 2.

In Section 3 we generalise the global homological formula involved in the classical Lefschetz fixed-point theorem, to the equivalent situation. This involves completely different ideas. The basic idea to use Künneth and Universal Coefficient theorems for such a formula already appears in [9]. In the equivariant case, these theorems become much more complicated, however. The new idea that we need here is to first localise KK^G and compute the Lefschetz index in the localisation.

In the introduction, we only state our result in the simpler case of a Hodgkin Lie group G. Then $\mathrm{R}(G)$ is an integral domain and thus embeds into its field of fractions F. For any G-C^*-algebra A, $\mathrm{K}^G_*(A)$ is a $\mathbb{Z}/2$-graded $\mathrm{R}(G)$-module. Thus $\mathrm{K}^G_*(A;F) := \mathrm{K}^G_*(A) \otimes_{\mathrm{R}(G)} F$ becomes a $\mathbb{Z}/2$-graded F-vector space. Assume that A is dualisable and belongs to the bootstrap class in KK^G. Then $\mathrm{K}^G_*(A;F)$ is finite-dimensional, so that the map on $\mathrm{K}^G_*(A;F)$ induced by an endomorphism $\varphi \in \mathrm{KK}^G_0(A,A)$ has a well-defined (super)trace in F. Theorem 3.4 asserts that this supertrace belongs to $\mathrm{R}(G) \subseteq F$ and is equal to the Lefschetz index of φ. In particular, this applies if $A = \mathrm{C}(X)$ for a compact G-manifold.

The results of Sections 2 and 3 together thus prove the following:

Theorem 1.1. *Let G be a Hodgkin Lie group, let F be the field of fractions of $\mathrm{R}(G)$. Let X be a closed G-manifold. Let $X \xleftarrow{b} (M,\xi) \xrightarrow{f} X$ be a smooth G-equivariant correspondence from X to X with $\xi \in \mathrm{K}_G^{\dim M - \dim X}(X)$; it represents a class $\varphi \in \widehat{\mathrm{KK}}{}^G_0(X,X)$. Assume that b and f intersect smoothly with K_G-oriented excess intersection bundle η. Equip $Q := \{m \in M \mid b(m) = f(m)\}$ with its induced K_G-orientation.*

Then the $\mathrm{R}(G)$-valued index of the Dirac operator on Q twisted by $\xi|_Q \otimes e(\eta)$ is equal to the supertrace of the F-linear map on $\mathrm{K}^*_G(X) \otimes_{\mathrm{R}(G)} F$ induced by φ.

If G is a connected Lie group, then there is a finite covering $\hat{G} \twoheadrightarrow G$ that is a Hodgkin Lie group. We may turn G-actions into $\hat{G}$-actions using the projection map, and get a symmetric monoidal functor $\mathrm{KK}^G \to \mathrm{KK}^{\hat{G}}$. Since the map $\mathrm{R}(G) \to \mathrm{R}(\hat{G})$ is clearly injective, we may compute the Lefschetz index of $\varphi \in \mathrm{KK}_0^G(A, A)$ by computing instead the Lefschetz index of the image of φ in $\mathrm{KK}_0^{\hat{G}}(A, A)$. By the result mentioned above, this uses the induced map on $\mathrm{K}_*^{\hat{G}}(A) \otimes_{\mathrm{R}(\hat{G})} \hat{F}$, where $\hat{F}$ is the field of fractions of $\mathrm{R}(\hat{G})$. Thus we get a satisfactory trace formula for all connected Lie groups. But the result may be quite different from the trace of the induced map on $\mathrm{K}_*^G(A) \otimes_{\mathrm{R}(G)} F$.

If G is not connected, then the total ring of fractions of G is a product of finitely many fields. Its factors correspond to conjugacy classes of Cartan subgroups in G. Each Cartan subgroup $H \subseteq G$ corresponds to a minimal prime ideal $\mathfrak{p}_H$ in $\mathrm{R}(G)$. The quotient $\mathrm{R}(G)/\mathfrak{p}_H$ is an integral domain and embeds into a field of fractions F_H. We show that the map $\mathrm{R}(G) \to F_H$ maps the Lefschetz index of φ to the supertrace of $\mathrm{K}_*^H(\varphi; F_H)$ (Theorem 3.23). It is crucial to use H-equivariant K-theory here. The very simple counterexample 3.7 shows that there may be two elements $\varphi_1, \varphi_2 \in \mathrm{KK}_0^G(A, A)$ with different Lefschetz index but inducing the same map on $\mathrm{K}_*^G(A)$.

Thus the generalisation of Theorem 1.1 to disconnected G identifies the image of the index of the Dirac operator on Q twisted by $\xi|_Q \otimes e(\eta)$ under the canonical map $\mathrm{R}(G) \to F_H$ with the supertrace of the F_H-linear map on $\mathrm{K}^*_G(X) \otimes_{\mathrm{R}(G)} F_H$ induced by φ, for each Cartan subgroup H.

The trace formulas in Section 3 require the algebra A on which we compute the trace to be dualisable and to belong to an appropriate bootstrap class, namely, the class of all G-C^*-algebras that are KK^G-equivalent to a type I G-C^*-algebra. This is strictly larger than the class of G-C^*-algebras that are KK^G-equivalent to a commutative one, already if G is the circle group (see [10]). We describe the bootstrap class as being generated by so-called elementary G-C^*-algebras in Section 3.1. This list of generators is rather long, but for the purpose of the trace computations, we may localise KK^G at the multiplicatively closed subset of non-zero divisors in $\mathrm{R}(G)$. The image of the bootstrap class in this localisation has a very simple structure, which is described in Section 3.2. The homological formula for the Lefschetz index follows easily from this description of the localised bootstrap category.

In Section 4, we give a variant of the global homological formula for the trace for a Hodgkin Lie group G. Given a commutative ring R and an R-module M with a projective resolution of finite type, we may define a Hattori–Stallings trace for endomorphisms of M by lifting the endomorphism to a finite type projective resolution and using the standard trace for endomorphisms of finitely generated projective resolutions. This defines the trace of the $\mathrm{R}(G)$-module homomorphism $\mathrm{K}_*^G(\varphi)\colon \mathrm{K}_*^G(A) \to \mathrm{K}_*^G(A)$ in $\mathrm{R}(G)$ without passing through a field of fractions.

2 Lefschetz indices in geometric bivariant K-theory

The category $\widehat{\mathrm{KK}}^G$ introduced in [15] provides a geometric analogue of Kasparov theory. We first recall some basic facts about this category and duality in bivariant K-theory from [14–16] and then compute Lefschetz indices in it as intersection products. Later we are going to compare this with other formulas for Lefschetz indices. We also prove an excess intersection formula to compute the composition of geometric correspondences under a weaker assumption than transversality. This formula goes back to Baum and Block [4].

All results in this section extend to the case where G is a proper Lie groupoid with enough G-vector bundles in the sense of [14, Definition 3.1] because the theory in [14–16] is already developed in this generality. For the sake of concreteness, we limit our treatment here to compact Lie groups acting on smooth manifolds.

The results in this section work both for real and complex K-theory. For concreteness, we assume in our notation that we are dealing with the complex case. In the real case, K must be replaced by KO throughout. In particular, K_G-orientations (that is, G-equivariant Spin^c-structures) must be replaced by KO^G-orientations (that is, G-equivariant Spin structures). In some examples, we use the isomorphisms $\widehat{\mathrm{KK}}^G_{2n}(\mathrm{pt},\mathrm{pt}) = \mathrm{R}(G)$ and $\widehat{\mathrm{KK}}^G_{2n+1}(\mathrm{pt},\mathrm{pt}) = 0$ for all $n \in \mathbb{Z}$. Here $\mathrm{R}(G)$ denotes the representation ring of G. Of course, this is true only in complex K-theory.

2.1 Geometric bivariant K-theory

Like Kasparov theory, geometric bivariant K-theory yields a category $\widehat{\mathrm{KK}}^G$. Its objects are (Hausdorff) locally compact G-spaces; arrows from X to Y are *geometric correspondences* from X to Y in the sense of [15, Definition 2.3]. These consist of

- M: a G-space;
- b: a G-map (that is, a continuous G-equivariant map) $b\colon M \to X$;
- ξ: a G-equivariant K-theory class on M with X-compact support (where we view M as a space over X via the map b); we write $\xi \in \mathrm{RK}^*_{G,X}(M)$;
- f: a K_G-oriented normally non-singular G-map $f\colon M \to Y$.

Equivariant K-theory with X-compact support and equivariant vector bundles are defined in [12, Definitions 2.5 and 2.6]. If b is a proper map, in particular if M is compact, then $\mathrm{RK}^*_{G,X}(M)$ is the ordinary G-equivariant (compactly supported) K-theory $\mathrm{K}^*_G(M)$ of M.

A K_G*-oriented normally non-singular map* from M to Y consists of

- V: a K_G-oriented G-vector bundle on M,
- E: a K_G-oriented finite-dimensional linear G-representation, giving rise to a trivial K_G-oriented G-vector bundle $Y \times E$ on Y,
- $\hat{f}$: a G-equivariant homeomorphism from the total space of V to an open subset in the total space of $Y \times E$, $\hat{f}\colon V \hookrightarrow Y \times E$.

We will not distinguish between a vector bundle and its total space in our notation.

A normally non-singular map $f = (V, E, \hat{f})$ has an *underlying map*

$$M \rightarrowtail V \xhookrightarrow{\hat{f}} Y \times E \twoheadrightarrow Y,$$

where the first map is the zero section of the vector bundle V and the third map is the coordinate projection. This map is called its "trace" in [14], but we avoid this name here because we use "trace" in a different sense. The *degree* of f is $d = \dim V - \dim E$. A wrong-way element $f_! \in \mathrm{KK}^G_d(\mathrm{C}_0(M), \mathrm{C}_0(Y))$ induced by f is defined in [14, Section 5.3]).

Our geometric correspondences are variants of those introduced by Alain Connes and Georges Skandalis in [6]. The changes in the definition avoid technical problems with the usual definition in the equivariant case.

The ($\mathbb{Z}/2$-graded) geometric KK-group $\widehat{\mathrm{KK}}^G_*(X, Y)$ is defined as the quotient of the set of geometric correspondences from X to Y by an appropriate equivalence relation, generated by bordism, Thom modification, and equivalence of normally non-singular maps. Bordism includes homotopies for the maps b and f by [15, Lemma 2.12]. We will use this several times below. The Thom modification allows to replace the space M by the total space of a K_G-oriented vector bundle on M. In particular, we could take the K_G-oriented vector bundle from the normally non-singular map f. This results in an equivalent normally non-singular map where $f\colon M \to Y$ is a *special submersion*, that is, an open embedding followed by a coordinate projection $Y \times E \twoheadrightarrow Y$ for some linear G-representation E. Correspondences with this property are called *special*.

The composition in $\widehat{\mathrm{KK}}^G$ is defined as an intersection product (see Section 2.2) if the map $f\colon M \to Y$ is such a special submersion. This turns $\widehat{\mathrm{KK}}^G$ into a category; the identity map on X is the correspondence with $f = b = \mathrm{id}_X$ and $\xi = 1$. The product of G-spaces provides a symmetric monoidal structure in $\widehat{\mathrm{KK}}^G$ (see [15, Theorem 2.27]).

There is an additive, grading-preserving, symmetric monoidal functor

$$\widehat{\mathrm{KK}}^G_*(X, Y) \to \mathrm{KK}^G_*(\mathrm{C}_0(X), \mathrm{C}_0(Y)).$$

This is an isomorphism if X is *normally non-singular* by [15, Corollary 4.3], that is, if there is a normally non-singular map $X \to \mathrm{pt}$. This means that there is a G-vector bundle V over X whose total space is G-equivariantly homeomorphic to an open G-invariant subset of some linear G-space. In particular, by Mostow's Embedding Theorem smooth G-manifolds of finite orbit type are normally non-singular (see [14, Theorem 3.22]).

Stable K_G-orientations play an important technical role in our trace formulas and should therefore be treated with care. A K_G*-orientation* on a G-vector bundle V is, by definition, a G-equivariant complex spinor bundle for V. (This is equivalent to a reduction of the structure group to Spin^c.) Given such

K_G-orientations on V_1 and V_2, we get an induced K_G-orientation on $V_1 \oplus V_2$; conversely, K_G-orientations on $V_1 \oplus V_2$ and V_1 induce one on V_2.

Let $\xi \in \mathrm{RK}^0_G(M)$ be represented by the formal difference $[V_1] - [V_2]$ of two G-vector bundles. A *stable* K_G*-orientation* on ξ means that we are given another G-vector bundle V_3 and K_G-orientations on both $V_1 \oplus V_3$ and $V_2 \oplus V_3$. Since $\xi = [V_1 \oplus V_3] - [V_2 \oplus V_3]$, this implies that ξ is a formal difference of two K_G-oriented G-vector bundles. Conversely, assume that $\xi = [W_1] - [W_2]$ with two K_G-oriented G-vector bundles; then there are G-vector bundles V_3 and W_3 such that $V_i \oplus V_3 \cong W_i \oplus W_3$ for $i = 1, 2$; since W_3 is a direct summand in a K_G-oriented G-vector bundle, we may enlarge V_3 and W_3 so that W_3 itself is K_G-oriented. Then $V_i \oplus V_3 \cong W_i \oplus W_3$ for $i = 1, 2$ inherit K_G-orientations. Roughly speaking, stably K_G-oriented K-theory classes are equivalent to formal differences of K_G-oriented G-vector bundles.

A K_G-orientation on a normally non-singular map $f = (V, E, \hat{f})$ from M to Y means that both V and E are K_G-oriented. Since "lifting" allows us to replace E by $E \oplus E'$ and V by $V \oplus (M \times E')$, we may assume without loss of generality that E is already K_G-oriented. Thus a K_G-orientation on f becomes equivalent to one on V. But the chosen K_G-orientation on E remains part of the data: changing it changes the K_G-orientation on f. By [14, Lemma 5.13], all essential information is contained in a K_G-orientation on the formal difference $[V] - [M \times E] \in \mathrm{RK}^0_G(M)$, which we call the *stable normal bundle* of the normally non-singular map f. If $[V] - [M \times E]$ is K_G-oriented, then we may find a G-vector bundle V_3 such that $V \oplus V_3$ and $(M \times E) \oplus V_3$ are K_G-oriented. Since $(M \times E) \oplus V_3$ is a direct summand in a K_G-oriented trivial G-vector bundle, we may assume without loss of generality that V_3 itself is trivial, $V_3 = M \times E'$, and that already $E \oplus E'$ is K_G-oriented. Lifting f along E' then gives a normally non-singular map $(V \oplus (M \times E'), E \oplus E', \hat{f} \times \mathrm{id}_{E'})$, where both $V \oplus (M \times E')$ and $E \oplus E'$ are K_G-oriented. Thus a K_G-orientation on f is equivalent to a stable K_G-orientation on the stable normal bundle of f.

Lemma 2.1. *If $f = (V, E, \hat{f})$ is a smooth normally non-singular map with underlying map $\bar{f}\colon M \to Y$, then its stable normal bundle is equal to $\bar{f}^*[\mathrm{T}Y] - [\mathrm{T}M] \in \mathrm{RK}^0_G(M)$.*

Proof. The tangent bundles of the total spaces of V and $Y \times E$ are $\mathrm{T}M \oplus V$ and $\mathrm{T}Y \oplus E$, respectively. Since $\hat{f}$ is an open embedding, $\hat{f}^*(\mathrm{T}Y \oplus E) \cong \mathrm{T}M \oplus V$. This implies $\bar{f}^*(\mathrm{T}Y) \oplus (M \times E) \cong \mathrm{T}M \oplus V$. Thus $[V] - [M \times E] = \bar{f}^*[\mathrm{T}Y] - [\mathrm{T}M]$. ∎

This lemma also shows that the stable normal bundle of f and hence the orientability assumption depend only on the equivalence class of f.

Another equivalent way to describe stable K_G-orientations is the following. Suppose we are already given a G-vector bundle V on Y such that $\mathrm{T}Y \oplus V$ is K_G-oriented. Then a stable K_G-orientation on f is equivalent to one on

$$[\bar{f}^*V \oplus \mathrm{T}M] = \bar{f}^*[\mathrm{T}Y \oplus V] - (\bar{f}^*[\mathrm{T}Y] - [\mathrm{T}M]),$$

which is equivalent to a K_G-orientation on $\bar{f}^*V \oplus \mathrm{T}M$ in the usual sense.
If X and Y are smooth G-manifolds (without boundary), we may require the maps b and $\hat{f}$ and the vector bundles V and E to be smooth. This leads to a smooth variant of $\widehat{\mathrm{KK}}^G$. This variant is isomorphic to the one defined above by [15, Theorem 4.8] provided X is of finite orbit type and hence normally non-singular.
Working in the smooth setting has two advantages.
First, assuming M to be of finite orbit type, [14, Theorem 3.22] shows that any smooth G-map $f\colon M \to Y$ lifts to a smooth normally non-singular map that is unique up to equivalence. Thus we may replace normally non-singular maps by smooth maps in the usual sense in the definition of a geometric correspondence. Moreover, $Nf = f^*[\mathrm{T}Y] - [\mathrm{T}M]$, so f is K_G-oriented if and only if there are K_G-oriented G-vector bundles V_1 and V_2 over M with $f^*[\mathrm{T}Y] \oplus V_1 \cong \mathrm{T}M \oplus V_2$ (compare [14, Corollary 5.15]).
Secondly, in the smooth setting there is a particularly elegant way of composing correspondences when they satisfy a suitable transversality condition, see [15, Corollary 2.39]. This description of the composition is due to Connes and Skandalis [6].

2.2 Composition of geometric correspondences

By [15, Theorem 2.38], a smooth normally non-singular map lifting $f\colon M_1 \to Y$ and a smooth map $b\colon M_2 \to Y$ are *transverse* if

$$D_{m_1} f(\mathrm{T}_{m_1} M_1) + D_{m_2} b(\mathrm{T}_{m_2} M_2) = \mathrm{T}_y Y$$

for all $m_1 \in M_1$, $m_2 \in M_2$ with $y := f(m_1) = b(m_2)$. Equivalently, the map

$$Df - Db\colon \mathrm{pr}_1^*(\mathrm{T}M_1) \oplus \mathrm{pr}_2^*(\mathrm{T}M_2) \to (f \circ \mathrm{pr}_1)^*(\mathrm{T}Y)$$

is surjective; this is a bundle map of vector bundles over

$$M_1 \times_Y M_2 := \{(m_1, m_2) \mid f(m_1) = b(m_2)\},$$

where $\mathrm{pr}_1\colon M_1 \times_Y M_2 \to M_1$ and $\mathrm{pr}_2\colon M_1 \times_Y M_2 \to M_2$ denote the restrictions to $M_1 \times_Y M_2$ of the coordinate projections. (We shall always use this notation for restrictions of coordinate projections.)
A commuting square diagram of smooth manifolds is called *Cartesian* if it is isomorphic (as a diagram) to a square

$$\begin{tikzcd}
M_1 \times_Y M_2 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & M_2 \arrow[d, "f"] \\
M_1 \arrow[r, "b"] & Y
\end{tikzcd}$$

where f and b are transverse smooth maps in the sense above; then $M_1 \times_Y M_2$ is again a smooth manifold and pr_1 and pr_2 are smooth maps.

The tangent bundles of these four manifolds are related by an exact sequence

$$0 \to \mathrm{T}(M_1 \times_Y M_2) \xrightarrow{(D\mathrm{pr}_1, D\mathrm{pr}_2)} \mathrm{pr}_1^*(\mathrm{T}M_1) \oplus \mathrm{pr}_2^*(\mathrm{T}M_2) \xrightarrow{Df - Db} (f \circ \mathrm{pr}_1)^* \mathrm{T}Y \to 0. \tag{2.1}$$

That is, $\mathrm{T}(M_1 \times_Y M_2)$ is the sub-bundle of $\mathrm{pr}_1^*(\mathrm{T}M_1) \oplus \mathrm{pr}_2^*(\mathrm{T}M_2)$ consisting of those vectors $(m_1, \xi, m_2, \eta) \in \mathrm{T}M_1 \oplus \mathrm{T}M_2$ (where $f(m_1) = b(m_2)$) with $D_{m_1} f(\xi) = D_{m_2} b(\eta)$. We may denote this bundle briefly by $\mathrm{T}M_1 \oplus_{\mathrm{T}Y} \mathrm{T}M_2$. Furthermore, from (2.1),

$$\mathrm{T}(M_1 \times_Y M_2) - \mathrm{pr}_2^*(\mathrm{T}M_2) = \mathrm{pr}_1^*(\mathrm{T}M_1 - f^*(\mathrm{T}Y)) \tag{2.2}$$

as stable G-vector bundles. Thus a stable K_G-orientation for $\mathrm{T}M_1 - f^*(\mathrm{T}Y)$ may be pulled back to one for $\mathrm{T}(M_1 \times_Y M_2) - \mathrm{pr}_2^*(\mathrm{T}M_2)$. More succinctly, a K_G-orientation for the map f induces one for pr_2.
Now consider two composable smooth correspondences

$$\begin{tikzcd}
& M_1 \arrow[dl, "b_1"'] \arrow[dr, "f_1"] & & M_2 \arrow[dl, "b_2"'] \arrow[dr, "f_2"] & \\
X & & Y & & Z,
\end{tikzcd} \tag{2.3}$$

with K-theory classes $\xi_1 \in \mathrm{RK}^G_{*,X}(M_1)$ and $\xi_2 \in \mathrm{RK}^G_{*,Y}(M_2)$. We assume that the pair of smooth maps (f_1, b_2) is transverse. Then there is an essentially unique commuting diagram

$$\begin{tikzcd}
& & M_1 \times_Y M_2 \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & & \\
& M_1 \arrow[dl, "b_1"'] \arrow[dr, "f_1"] & & M_2 \arrow[dl, "b_2"'] \arrow[dr, "f_2"] & \\
X & & Y & & Z,
\end{tikzcd} \tag{2.4}$$

where the square is Cartesian. We briefly call such a diagram an *intersection diagram* for the two given correspondences.
By the discussion above, the map pr_2 inherits a K_G-orientation from f_1, so that the map $f := f_2 \circ \mathrm{pr}_2$ is also K_G-oriented. Let $M := M_1 \times_Y M_2$ and $b := b_1 \circ \mathrm{pr}_1$. The product $\xi := \mathrm{pr}_1^*(\xi_1) \otimes \mathrm{pr}_2^*(\xi_2)$ belongs to $\mathrm{RK}^G_{*,X}(M)$, that is, it has X-compact support with respect to the map $b\colon M \to X$. Thus we get a G-equivariant correspondence (M, b, f, ξ) from X to Y. The assertion of [15, Corollary 2.39] – following [6] – is that this represents the composition of the two given correspondences. It is called their *intersection product*.

Example 2.2. Consider the diagonal embedding $\delta\colon X \to X \times X$ and the graph embedding $\bar{f}\colon X \to X \times X$, $x \mapsto (x, f(x))$, for a smooth map $f\colon X \to X$. These two maps are transverse if and only if f has simple fixed points. If this is the case, then the intersection space is the set of fixed points of f. If, say, $f = \mathrm{id}_X$, then δ and $\bar{f}$ are not transverse.

To define the composition also in the non-transverse case, a Thom modification is used in [15] to achieve transversality (see [15, Theorem 2.32]). Take two composable (smooth) correspondences as in (2.3), and let $f_1 = (V_1, E_1, \hat{f}_1)$ as a normally non-singular map. By a Thom modification, the geometric correspondence $X \xleftarrow{b_1} (M_1, \xi) \xrightarrow{f_1} Y$ is equivalent to

$$X \xleftarrow{b_1 \circ \pi_{V_1}} (V_1, \tau_{V_1} \otimes \pi_{V_1}^* \xi) \xrightarrow{\pi_{E_1} \circ \hat{f}_1} Y, \tag{2.5}$$

where $\pi_{V_1}\colon V_1 \to M_1$ and $\pi_{E_1}\colon Y \times E \to Y$ are the bundle projections, and $\tau_{V_1} \in \mathrm{RK}^*_{G,M_1}(V_1)$ is the Thom class of V_1. We write $\otimes$ for the multiplication of K-theory classes. The support of such a product is the intersection of the supports of the factors. Hence the support of $\tau_{V_1} \otimes \pi_{V_1}^* \xi$ is an X-compact subset of V_1.

The forward map $V_1 \to Y$ in (2.5) is a special submersion and, in particular, a submersion. As such it is transverse to any other map $b_2\colon M_2 \to Y$. Hence after the Thom modification we may compute the composition of correspondences as an intersection product of the correspondence (2.5) with the correspondence $Y \xleftarrow{b_2} M_2 \xrightarrow{f_2} Y$. This yields

$$X \xleftarrow{b_1 \circ \pi_{V_1} \circ \mathrm{pr}_1} \big(V_1 \times_Y M_2, \mathrm{pr}_{V_1}^*(\tau_{V_1} \otimes \pi_{V_1}^*(\xi))\big) \xrightarrow{f_2 \circ \mathrm{pr}_2} Z, \tag{2.6}$$

where

$$V_1 \times_Y M_2 := \{(x, v, m_2) \in V_1 \times M_2 \mid (\pi_{E_1} \circ \hat{f}_1)(x, v) = b_2(m_2)\}$$

and $\mathrm{pr}_1\colon V_1 \times_Y M_2 \to V_1$ and $\mathrm{pr}_2\colon V_1 \times_Y M_2 \to M_2$ are the coordinate projections. The intersection space $V_1 \times_Y M_2$ is a smooth manifold with tangent bundle

$$\mathrm{T}V_1 \oplus_{\mathrm{T}Y} \mathrm{T}M_2 := \mathrm{pr}_1^*(\mathrm{T}V_1) \oplus_{(\pi_{E_1} \circ \hat{f}_1)^*(\mathrm{T}Y)} \mathrm{pr}_2^*(\mathrm{T}M_2),$$

and the map pr_2 is a submersion with fibres tangent to E_1. Thus it is K_G-oriented.

This recipe to define the composition product for all geometric correspondences is introduced in [15]. It is shown there that it is equivalent to the intersection product if f_1 and b_2 are transverse. But the space $V_1 \times_Y M_2$ has high dimension, making it inefficient to compute with this formula. And we are usually given only the underlying map $f_1\colon M_1 \to Y$, not its factorisation as a normally non-singular map – and the latter is difficult to compute. We will weaken the transversality requirement in Section 2.5. The more general condition still applies, say, if $f_1 = b_2$. This is particularly useful for computing Euler characteristics.

2.3 Duality and the Lefschetz index

Duality plays a crucial role in [15] in order to compare the geometric and analytic models of equivariant Kasparov theory. Duality is also used in [16, Definition 4.26] to construct a Lefschetz map

$$\mathcal{L}\colon \mathrm{KK}^G_*\big(\mathrm{C}(X), \mathrm{C}(X)\big) \to \mathrm{KK}^G_*(\mathrm{C}(X), \mathbb{C}), \tag{2.7}$$

for a compact smooth G-manifold X. We may compose $\mathcal{L}$ with the index map $\mathrm{KK}^G_*(\mathrm{C}(X), \mathbb{C}) \to \mathrm{KK}^G_*(\mathbb{C}, \mathbb{C}) \cong \mathrm{R}(G)$ to get a Lefschetz index $\text{L-ind}(f) \in \mathrm{R}(G)$ for any $f \in \mathrm{KK}^G_*\big(\mathrm{C}(X), \mathrm{C}(X)\big)$. This is the invariant we will be studying in this paper.

This Lefschetz map $\mathcal{L}$ is a special case of a very general construction. Let $\mathcal{C}$ be a symmetric monoidal category. Let A be a dualisable object of $\mathcal{C}$ with a dual A^*. Let $\eta\colon \mathbb{1} \to A \otimes A^*$ and $\varepsilon\colon A^* \otimes A \to \mathbb{1}$ be the unit and counit of the duality. Being unit and counit of a duality means that they satisfy the zigzag equations: the composition

$$A \xrightarrow{\eta\otimes\mathrm{id}_A} A \otimes A^* \otimes A \xrightarrow{\mathrm{id}_A\otimes\varepsilon} A \tag{2.8}$$

is equal to the identity $\mathrm{id}_A\colon A \to A$, and similarly for the composition

$$A^* \xrightarrow{\mathrm{id}_{A^*}\otimes\eta} A^* \otimes A \otimes A^* \xrightarrow{\varepsilon\otimes\mathrm{id}_{A^*}} A^*. \tag{2.9}$$

If $\mathcal{C}$ is $\mathbb{Z}$-graded, then we may allow dualities to shift degrees. Then some signs are necessary in the zigzag equations, see [16, Theorem 5.5].

Given a multiplication map $m\colon A \otimes A \to A$, we define the *Lefschetz map*

$$\mathcal{L}\colon \mathcal{C}(A, A) \to \mathcal{C}(A, \mathbb{1})$$

by sending an endomorphism $f\colon A \to A$ to the composite morphism

$$A \cong A\otimes\mathbb{1} \xrightarrow{\mathrm{id}_A\otimes\eta} A\otimes A\otimes A^* \xrightarrow{m\otimes\mathrm{id}_{A^*}} A\otimes A^* \xrightarrow{f\otimes\mathrm{id}_{A^*}} A\otimes A^* \xrightarrow{\mathsf{sym}} A^*\otimes A \xrightarrow{\varepsilon} \mathbb{1}.$$

This depends only on m and f, not on the choices of the dual, unit and counit. For $f = \mathrm{id}_A$ we get the *higher Euler characteristic* of A in $\mathcal{C}(A, \mathbb{1})$.

While the geometric computations below give the Lefschetz map as defined above, the global homological computations in Sections 3 and 4 only apply to the following coarser invariant:

Definition 2.3. The *Lefschetz index* $\text{L-ind}(f)$ (or *trace* $\mathrm{tr}(f)$) of an endomorphism $f\colon A \to A$ is the composite

$$\mathbb{1} \xrightarrow{\eta} A \otimes A^* \xrightarrow{\mathsf{sym}} A^* \otimes A \xrightarrow{\mathrm{id}_{A^*}\otimes f} A^* \otimes A \xrightarrow{\varepsilon} \mathbb{1}, \tag{2.10}$$

where sym denotes the symmetry isomorphism. The Lefschetz index of id_A is called the *Euler characteristic* of A.

If A is a unital algebra object in $\mathcal{C}$ with multiplication $m\colon A\otimes A\to A$ and unit $u\colon \mathbb{1}\to A$, then $\text{L-ind}(f)=\mathcal{L}(f)\circ u$. In particular, the Euler characteristic is the composite of the higher Euler characteristic with u.

In this section, we work in $\mathcal{C}=\widehat{\mathrm{KK}}^G$ for a compact group G with $\mathbb{1}=\mathrm{pt}$ and $\otimes=\times$. In Section 3, we work in the related analytic category $\mathcal{C}=\mathrm{KK}^G$ with $\mathbb{1}=\mathbb{C}$ and the usual tensor product.

We will show below that any compact smooth G-manifold X is dualisable in $\widehat{\mathrm{KK}}^G$. The multiplication $m\colon X\times X\to X$ and unit $u\colon \mathrm{pt}\to X$ are given by the geometric correspondences

$$X\times X \xleftarrow{\Delta} X \xrightarrow[=]{\mathrm{id}_X} X,\qquad \mathrm{pt}\leftarrow X \xrightarrow[=]{\mathrm{id}_X} X$$

with $\Delta(x)=(x,x)$; these induce the multiplication *-homomorphism

$$m\colon \mathrm{C}(X\times X)\cong \mathrm{C}(X)\otimes \mathrm{C}(X)\to \mathrm{C}(X)$$

and the embedding $\mathbb{C}\to\mathrm{C}(X)$ of constant functions. Composing with u corresponds to taking the *index* of a K-homology class.

Remark 2.4. In [11, 13, 16] Lefschetz maps are also studied for non-compact spaces X, equipped with group actions of possibly non-compact groups. A non-compact G-manifold X is usually not dualisable in $\widehat{\mathrm{KK}}^G$, and even if it were, the Lefschetz map that we would get from this duality would not be the one studied in [11, 13, 16].

2.4 Duality for smooth compact manifolds

We are going to show that compact smooth G-manifolds are dualisable in the equivariant correspondence theory $\widehat{\mathrm{KK}}^G$. This was already proved in [15], but since we need to know the unit and counit to compute Lefschetz indices, we recall the proof in detail. It is of some interest to treat duality for smooth manifolds with boundary because any finite CW-complex is homotopy equivalent to a manifold with boundary.

In case X has a boundary ∂X, let $\mathring{X}:=X\setminus\partial X$ denote its interior and let $\iota\colon\mathring{X}\to X$ denote the inclusion map. The boundary $\partial X\subseteq X$ admits a G-equivariant collar, that is, the embedding $\partial X\to X$ extends to a G-equivariant diffeomorphism from $\partial X\times[0,1)$ onto an open neighbourhood of ∂X in X (see also [16, Lemma 7.6] for this standard result). This collar neighbourhood together with a smooth map $[0,1)\to(0,1)$ that is the identity near 1 provides a smooth G-equivariant map $\rho\colon X\to\mathring{X}$ that is inverse to ι up to smooth G-homotopy. Furthermore, we may assume that ρ is a diffeomorphism onto its image.

If X has no boundary, then $\mathring{X}=X$, $\iota=\mathrm{id}$, and $\rho=\mathrm{id}$.

The results about smooth normally non-singular maps in [14] extend to smooth manifolds with boundary if we add suitable assumptions about the behaviour near the boundary. We mention one result of this type and a counterexample.

Proposition 2.5. *Let X and Y be smooth G-manifolds with X of finite orbit type and let $f\colon X \to Y$ be a smooth map with $f(\partial X) \subseteq \partial Y$ and f transverse to ∂Y. Then f lifts to a normally non-singular map, and any two such normally non-singular liftings of f are equivalent.*

Proof. Since X has finite orbit type, we may smoothly embed X into a finite-dimensional linear G-representation E. Our assumptions ensure that the resulting map $X \to Y \times E$ is a smooth embedding between G-manifolds with boundary in the sense of [14, Definition 3.17] and hence has a tubular neighbourhood by [14, Theorem 3.18]. This provides a normally non-singular map $X \to Y$ lifting f. The uniqueness up to equivalence is proved as in the proof of [14, Theorem 4.36]. □

Example 2.6. The inclusion map $\{0\} \to [0,1)$ is a smooth map between manifolds with boundary, but it does not lift to a smooth normally non-singular map.

Let X be a smooth compact G-manifold. Since X has finite orbit type, it embeds into some linear G-representation E. We may choose this G-representation to be K_G-oriented and even-dimensional by a further stabilisation. Let $\mathrm{N}X \twoheadrightarrow X$ be the *normal bundle* for such an embedding $X \to E$. Thus $\mathrm{T}X \oplus \mathrm{N}X \cong X \times E$ is G-equivariantly isomorphic to a K_G-oriented trivial G-vector bundle.

Theorem 2.7. *Let X be a smooth compact G-manifold, possibly with boundary. Then X is dualisable in $\widehat{\mathrm{KK}}^G_*$ with dual $\mathrm{N}\mathring{X}$, and the unit and counit for the duality are the geometric correspondences*

$$\mathrm{pt} \leftarrow X \xrightarrow{(\mathrm{id},\zeta\rho)} X \times \mathrm{N}\mathring{X}, \qquad \mathrm{N}\mathring{X} \times X \xleftarrow{(\mathrm{id},\iota\pi)} \mathrm{N}\mathring{X} \to \mathrm{pt},$$

where $\zeta\colon \mathring{X} \to \mathrm{N}\mathring{X}$ is the zero section, $\rho\colon X \to \mathring{X}$ is some G-equivariant collar retraction, $\pi\colon \mathrm{N}\mathring{X} \to \mathring{X}$ is the bundle projection, and $\iota\colon \mathring{X} \to X$ the identical inclusion. The K*-theory classes on the space in the middle are the trivial rank-one vector bundles for both correspondences.*

Proof. First we must check that the purported unit and counit above are indeed geometric correspondences; this contains describing the K_G-orientations on the forward maps, which is part of the data of the geometric correspondences.

The maps $X \to \mathrm{pt}$ and $\mathrm{N}\mathring{X} \to \mathrm{N}\mathring{X} \times X$ above are proper. Hence there is no support restriction for the K-theory class on the middle space, and the trivial rank-one vector bundle is allowed.

By the Tubular Neighbourhood Theorem, the normal bundle $\mathrm{N}\mathring{X}$ of the embedding $\mathring{X} \to E$ is diffeomorphic to an open subset of E. This gives a canonical isomorphism between the tangent bundle of $\mathrm{N}\mathring{X}$ and E. We choose this isomorphism and the given K_G-orientation on the linear G-representation E to K_G-orient $\mathrm{N}\mathring{X}$ and thus the projection $\mathrm{N}\mathring{X} \to \mathrm{pt}$. With this K_G-orientation, the counit $\mathrm{N}\mathring{X} \times X \xleftarrow{(\mathrm{id},\iota\pi)} \mathrm{N}\mathring{X} \to \mathrm{pt}$ is a G-equivariant geometric correspondence – even a special one in the sense of [15].

We identify the tangent bundle of $X \times \mathrm{N}\mathring{X}$ with $\mathrm{T}X \times \mathrm{T}\mathring{X} \oplus \mathrm{N}\mathring{X}$ in the obvious way. The normal bundle of the embedding $(\mathrm{id}, \zeta\rho)\colon X \to X \times \mathrm{N}\mathring{X}$ is isomorphic to the quotient of $\mathrm{T}X \oplus \rho^*(\mathrm{T}\mathring{X}) \oplus \rho^*(\mathrm{N}\mathring{X})$ by the relation $(\xi, D\rho(\xi), 0) \sim 0$ for $\xi \in \mathrm{T}X$. We identify this with $\mathrm{T}X \oplus \mathrm{N}X \cong X \times E$ by $(\xi_1, \xi_2, \eta) \mapsto (D\rho^{-1}(\xi_2) - \xi_1, D\rho^{-1}(\eta))$ for $\xi_1 \in \mathrm{T}_xX$, $\xi_2 \in \mathrm{T}_{\rho(x)}X$, $\eta \in \rho^*(\mathrm{N}\mathring{X})_x = \mathrm{N}_{\rho(x)}X$. With this K_G-orientation on $(\mathrm{id}, \zeta\rho)$, the unit above is a G-equivariant geometric correspondence. A boundary of X, if present, causes no problems here. The same goes for the computations below: although the results in [15] are formulated for smooth manifolds without boundary, they continue to hold in the cases we need.

We establish the duality isomorphism by checking the zigzag equations as in [16, Theorem 5.5]. This amounts to composing geometric correspondences. In the case at hand, the correspondences we want to compose are transverse, so that they may be composed by intersections as in Section 2.2. Actually, we are dealing with manifolds with boundary, but the argument goes through nevertheless. We write down the diagrams together with the relevant Cartesian square.

The intersection diagram for the first zigzag equation is

$$\begin{tikzcd}
& & X \arrow[dl, "(\mathrm{id},\iota\rho)"'] \arrow[dr, "(\mathrm{id},\zeta\rho)"] & & \\
& X \times X \arrow[dl, "\mathrm{pr}_2"'] \arrow[dr, "(\mathrm{id},\zeta\rho)\times\mathrm{id}"] & & X \times \mathrm{N}\mathring{X} \arrow[dl, "\mathrm{id}\times(\mathrm{id},\iota\pi)"'] \arrow[dr, "\mathrm{pr}_1"] & \\
X & & X \times \mathrm{N}\mathring{X} \times X & & X.
\end{tikzcd} \tag{2.11}$$

The square is Cartesian because $(x, y, z, (w, \nu)) \in X^3 \times \mathrm{N}\mathring{X}$ satisfies

$$(x, (\rho(x), 0), y) = (z, (w, \nu), w)$$

if and only if $y = \rho(x)$, $z = x$, $w = \rho(x)$, and $\nu = 0$ for some $x \in X$. The K_G-orientation on the map $(\mathrm{id}, \zeta\rho)$ described above is chosen such that the composite map $f := \mathrm{pr}_1 \circ (\mathrm{id}, \zeta\rho) = \mathrm{id}$ carries the standard K_G-orientation. The map $b := \mathrm{pr}_2 \circ (\mathrm{id}, \iota\rho) = \iota\rho$ is properly homotopic to the identity map. Hence the composition above gives the identity map on X as required.

The intersection diagram for the second zigzag equation is

$$\begin{tikzcd}
& & \mathrm{N}\mathring{X} \arrow[dl, "(\mathrm{id},\iota\pi)"'] \arrow[dr, "(\mathrm{id},\zeta\rho\pi)"] & & \\
& \mathrm{N}\mathring{X}\times X \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{id}\times(\mathrm{id},\zeta\rho)"] & & \mathrm{N}\mathring{X}\times\mathrm{N}\mathring{X} \arrow[dl, "(\mathrm{id},\iota\pi)\times\mathrm{id}"'] \arrow[dr, "\mathrm{pr}_2"] & \\
\mathrm{N}\mathring{X} & & \mathrm{N}\mathring{X}\times X\times\mathrm{N}\mathring{X} & & \mathrm{N}\mathring{X}
\end{tikzcd} \tag{2.12}$$

because $((x,\nu),y,(w,\mu),(z,\kappa)) \in \mathrm{N}\mathring{X}\times X\times(\mathrm{N}\mathring{X})^2$ satisfy

$$((x,\nu),y,(\rho(y),0)) = ((w,\mu),w,(z,\kappa))$$

if and only if $(w,\mu)=(x,\nu)$, $y=x$, $z=\rho(x)$, $\kappa=0$ for some $(x,\nu)\in\mathrm{N}\mathring{X}$. The map $(\mathrm{id},\zeta\rho\pi)$ is smoothly homotopic to the diagonal embedding $\delta\colon \mathrm{N}\mathring{X}\to\mathrm{N}\mathring{X}\times\mathrm{N}\mathring{X}$. Replacing $(\mathrm{id},\zeta\rho\pi)$ by δ gives an equivalent geometric correspondence. The K_G-orientation on the normal bundle of $(\mathrm{id},\zeta\rho\pi)$ that comes with the composition product is transformed by this homotopy to the K_G-orientation on the normal bundle of the diagonal embedding that we get by identifying the latter with the pull-back of E by mapping

$$(\xi_1,\eta_1,\xi_2,\eta_2)\in\mathrm{T}_{(x,\zeta,x,\zeta)}(\mathrm{N}\mathring{X}\times\mathrm{N}\mathring{X})\cong\mathrm{T}_x\mathring{X}\oplus\mathrm{N}_x\mathring{X}\times\mathrm{T}_x\mathring{X}\times\mathrm{N}_x\mathring{X}\cong E_x\times E_x$$

to $(\xi_2-\xi_1,\eta_2-\eta_1)\in E_x$. Since E has even dimension, changing this to $(\xi_1-\xi_2,\eta_1-\eta_2)$ does not change the K_G-orientation. Hence the induced K_G-orientation on the fibres of $D\mathrm{pr}_2$ is the same one that we used to K_G-orient pr_2. The induced K_G-orientation on $\mathrm{pr}_2\circ\delta=\mathrm{id}$ is the standard one. Thus the composition in (2.12) is the identity on $\mathrm{N}\mathring{X}$. ☐

Corollary 2.8. *Let X be a compact smooth G-manifold and let Y be any locally compact G-space. Then every element of $\widehat{\mathrm{KK}}^G_*(X,Y)$ is represented by a geometric correspondence of the form*

$$X \xleftarrow{\iota\circ\pi\circ\mathrm{pr}_1} \mathrm{N}\mathring{X}\times Y \xrightarrow{\mathrm{pr}_2} Y,\qquad \xi\in\mathrm{K}^*_G(\mathrm{N}\mathring{X}\times Y),$$

and two such correspondences for $\xi_1,\xi_2\in\mathrm{K}^*_G(\mathrm{N}\mathring{X}\times Y)$ give the same element of $\widehat{\mathrm{KK}}^G_*(X,Y)$ if and only if $\xi_1=\xi_2$. Here $\mathrm{pr}_1\colon\mathrm{N}\mathring{X}\times Y\to\mathrm{N}\mathring{X}$ and $\mathrm{pr}_2\colon\mathrm{N}\mathring{X}\times Y\to Y$ are the coordinate projections and $\iota\circ\pi\colon\mathrm{N}\mathring{X}\to\mathring{X}\subseteq X$ is as above.

Proof. Duality provides a canonical isomorphism

$$\mathrm{K}^*_G(\mathrm{N}\mathring{X}\times Y)\cong\widehat{\mathrm{KK}}^G_*(\mathrm{pt},\mathrm{N}\mathring{X}\times Y)\cong\widehat{\mathrm{KK}}^G_*(X,Y).$$

It maps $\xi \in \mathrm{K}^*_G(\mathrm{N}\mathring{X} \times Y)$ to the composition of correspondences described by the following intersection diagram:

$$\begin{tikzcd}
& & \mathrm{N}\mathring{X} \times Y \arrow[dl, "(\iota\pi,\mathrm{id})\times\mathrm{id}"'] \arrow[dr, "id"] & & \\
& X \times \mathrm{N}\mathring{X} \times Y \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "id"] & & \mathrm{N}\mathring{X} \times Y \arrow[dl, "(\iota\pi,\mathrm{id})\times\mathrm{id}"'] \arrow[dr, "\mathrm{pr}_2"] & \\
X & & X \times \mathrm{N}\mathring{X} \times Y & & Y,
\end{tikzcd}$$

with the K-theory class ξ on $\mathrm{N}\mathring{X} \times Y$. Hence it involves the maps $\iota\pi\colon \mathrm{N}\mathring{X} \times Y \to X$ and $\mathrm{pr}_2\colon \mathrm{N}\mathring{X} \times Y \to Y$. □

If X is, in addition, K_G-oriented, then the Thom isomorphism provides an isomorphism $\mathrm{N}\mathring{X} \cong \mathring{X}$ in $\widehat{\mathrm{KK}}{}^G_*$ (which has odd parity if the dimension of X is odd). A variant of Corollary 2.8 yields a duality isomorphism

$$\mathrm{K}^{*+\dim(X)}_G(\mathring{X} \times Y) \cong \widehat{\mathrm{KK}}{}^G_*(X, Y),$$

which maps $\xi \in \mathrm{K}^*_G(\mathring{X} \times Y)$ to the geometric correspondence

$$X \xleftarrow{\iota\circ\mathrm{pr}_1} \mathring{X} \times Y \xrightarrow{\mathrm{pr}_2} Y, \qquad \xi \in \mathrm{K}^*_G(\mathring{X} \times Y).$$

Hence any element of $\widehat{\mathrm{KK}}{}^G_*(X, Y)$ is represented by a correspondence of this form.
If X is K_G-oriented and has no boundary, this becomes

$$X \xleftarrow{\mathrm{pr}_1} X \times Y \xrightarrow{\mathrm{pr}_2} Y, \qquad \xi \in \mathrm{K}^*_G(X \times Y).$$

These standard forms for correspondences are less useful than one may hope at first because their intersection products are no longer in this standard form.

2.5 More on composition of geometric correspondences

With our geometric formulas for the unit and counit of the duality, we could now compute Lefschetz indices geometrically, assuming the necessary intersections are transverse. While this works well, say, for self-maps with regular nondegenerate fixed points, it fails badly for the identity correspondence, whose Lefschetz index is the Euler characteristic. Building on work of Baum and Block [4], we now describe the composition as a modified intersection product under a much weaker assumption than transversality that still covers the computation of Euler characteristics.

DEFINITION 2.9. We say that the smooth maps $f_1\colon M_1 \to Y$ and $b_2\colon M_2 \to Y$ *intersect smoothly* if

$$M := M_1 \times_Y M_2$$

is a smooth submanifold of $M_1 \times M_2$ and any $(\xi_1, \xi_2) \in \mathrm{T}M_1 \times \mathrm{T}M_2$ with $Df_1(\xi_1) = Db_2(\xi_2) \in \mathrm{T}Y$ is tangent to M.

If f_1 and b_2 intersect smoothly, then we define the *excess intersection bundle* $\eta(f_1, b_2)$ on M as the cokernel of the vector bundle map

$$(Df_1, -Db_2)\colon \mathrm{pr}_1^*(\mathrm{T}M_1) \oplus \mathrm{pr}_2^*(\mathrm{T}M_2) \to f^*(\mathrm{T}Y), \tag{2.13}$$

where $f := f_1 \circ \mathrm{pr}_1 = b_2 \circ \mathrm{pr}_2\colon M \to Y$.

If the maps f_1 and b_2 are G-equivariant with respect to a compact group G, then the excess intersection bundle is a G-vector bundle.

We call the square

$$\begin{tikzcd} M \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & M_2 \arrow[d, "f_1"] \\ M_1 \arrow[r, "b_2"] & Y \end{tikzcd}$$

η-Cartesian if f_1 and b_2 intersect smoothly with excess intersection bundle η.

If M is a smooth submanifold of $M_1 \times M_2$, then $\mathrm{T}M \subseteq \mathrm{T}(M_1 \times M_2)$; and if $(\xi_1, \xi_2) \in \mathrm{T}(M_1 \times M_2)$ is tangent to M, then $Df_1(\xi_1) = Db_2(\xi_2)$ in $\mathrm{T}Y$. These pairs (ξ_1, ξ_2) form a subspace of $\mathrm{T}(M_1 \times M_2)|_M = \mathrm{pr}_1^*\mathrm{T}M_1 \oplus \mathrm{pr}_2^*\mathrm{T}M_2$, which in general need not be a vector bundle, that is, its rank need not be locally constant. The smooth intersection assumption forces it to be a subbundle: the kernel of the map in (2.13). Hence the excess intersection bundle is a vector bundle over M, and there is the following exact sequence of vector bundles over M:

$$0 \to \mathrm{T}M \to \mathrm{pr}_1^*(\mathrm{T}M_1) \oplus \mathrm{pr}_2^*(\mathrm{T}M_2) \xrightarrow{(Df_1, -Db_2)} (f_1 \circ \mathrm{pr}_1)^*(\mathrm{T}Y) \to \eta \to 0. \tag{2.14}$$

Example 2.10. Let $M_1 = M_2 = X$ and let $f_1 = b_2 = i\colon X \to Y$ be an injective immersion. Then $M_1 \times_Y M_2 \cong X$ is the diagonal in $M_1 \times M_2 = X^2$, which is a smooth submanifold. Furthermore, if $(\xi_1, \xi_2) \in \mathrm{T}M_1 \times \mathrm{T}M_2$ satisfy $Df_1(\xi_1) = Db_2(\xi_2)$, then $\xi_1 = \xi_2$ because $Di\colon \mathrm{T}M \to \mathrm{T}Y$ is assumed injective. Hence M_1 and M_2 intersect smoothly, and the excess intersection bundle is the normal bundle of the immersion i.

Example 2.11. Let M_1 and M_2 be two circles embedded in $Y = \mathbb{R}^2$. The four possible configurations are illustrated in Figure 1.

1. The circles meet in two points. Then $M = \{p_1, p_2\}$ and the intersection is transverse.
2. The two circles are disjoint. Then $M = \emptyset$ and the intersection is transverse.
3. The two circles are identical. Then $M = M_1 = M_2$. The intersection is not transverse, but smooth by Example 2.10; the excess intersection bundle is the normal bundle of the circle, which is trivial of rank 1.

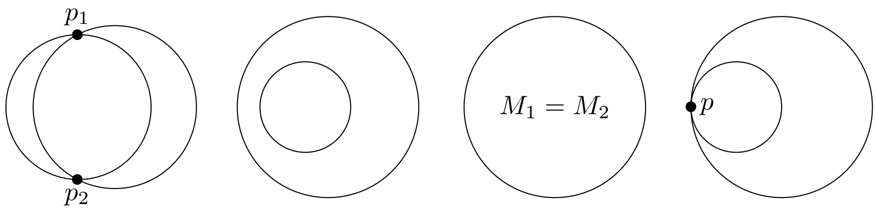

Figure 1: Four possible configurations of two circles in the plane

4. The two circles touch in one point. Then $M := M_1 \times_Y M_2 = \{p\}$, so that the tangent bundle of M is zero-dimensional. But $\mathrm{T}_p M_1 \cap \mathrm{T}_p M_2$ is one-dimensional because $\mathrm{T}_p M_1 = \mathrm{T}_p M_2$. Hence the embeddings do *not* intersect smoothly.

Remark 2.12. The maps $f\colon M_1 \to Y$ and $b\colon M_2 \to Y$ intersect smoothly if and only if $f \times b\colon M_1 \times M_2 \to Y \times Y$ and the diagonal embedding $Y \to Y \times Y$ intersect smoothly; both pairs of maps have the same excess intersection bundle. Thus we may always normalise intersections to the case where one map is a diagonal embedding and thus an embedding.

Example 2.13. Let η be a K_G-oriented vector bundle over X. Let $M_1 = M_2 = X$, $Y = \eta$, and let $f_1 = b_2 = \zeta\colon Y \to \eta$ be the zero section of η. This is a special case of Example 2.10. The maps f_1 and b_2 intersect smoothly with excess intersection bundle η.
In this example it is easy to compose the geometric correspondences $X = X \to \eta$ and $\eta \leftarrow X = X$. A Thom modification of the first one along the K_G-oriented vector bundle η gives the special correspondence

$$X \leftarrow (\eta, \tau_\eta) = \eta,$$

where $\tau_\eta \in \mathrm{RK}^*_{G,X}(\eta)$ is the Thom class of η. The intersection product of this with $\eta \leftarrow X = X$ is $X = (X, \zeta^*(\tau_\eta)) = X$, that is, it is the class in $\widehat{\mathrm{KK}}^G_*(X, X)$ of $\zeta^*(\tau_\eta) \in \mathrm{RK}^*_G(X)$. This K-theory class is the restriction of τ_η to the zero section of η. By the construction of the Thom class, it is the K-theory class of the spinor bundle of η.

DEFINITION 2.14. Let η be a K_G-oriented G-vector bundle over a G-space X. Let $\zeta\colon X \to \eta$ be the zero section and let $\tau_\eta \in \mathrm{RK}^*_{G,X}(\eta)$ be the Thom class. The *Euler class* of η is $\zeta^*(\tau_\eta)$, the restriction of τ_η to the zero section.

By definition, the Euler class is the composition of the correspondences $\mathrm{pt} \leftarrow X \to \eta$ and $\eta \leftarrow X = X$ involving the zero section $\zeta\colon X \to \eta$ in both cases.

Example 2.15. Assume that there is a G-equivariant section $s\colon X \to \eta$ of η with isolated simple zeros; that is, s and ζ are transverse. The linear homotopy connects s to the zero section and hence gives an equivalent correspondence

$\eta \xleftarrow{s} X = X$. Since s and ζ are transverse by assumption, the composition is $X \leftarrow Z \rightarrow X$, where Z is the zero set of s and the maps $Z \rightarrow X$ are the inclusion map, suitably K_G-oriented.

Example 2.16. Let $M_1 = S^1$, $M_2 = S^2$, $Y = \mathbb{R}^3$, $b_2 \colon M_2 \to \mathbb{R}^3$ be the standard embedding of the 2-sphere in $\mathbb{R}^3$, and let $f_1 \colon M_1 \to M_2 \to \mathbb{R}^3$ be the embedding corresponding to the equator of the circle. Then $M_1 \times_Y M_2 = M_1 \times_{M_2} M_2 = M_1$, embedded diagonally into $M_1 \times M_1 \subset M_1 \times M_2$. This is a case of smooth intersection. The excess intersection bundle is the restriction to the equator of the normal bundle of the embedding b_2. This is isomorphic to the rank-one trivial bundle on S^2. Hence the Euler class $e(\eta)$ is zero in this case.

Theorem 2.17. *Let*

$$X \xleftarrow{b_1} (M_1, \xi_1) \xrightarrow{f_1} Y \xleftarrow{b_2} (M_2, \xi_2) \xrightarrow{f_2} Z \tag{2.15}$$

be a pair of G-equivariant correspondences as in (2.3)*. Assume that b_2 and f_1 intersect smoothly and with a K_G-oriented excess intersection bundle η. Then the composition of* (2.15) *is represented by the G-equivariant correspondence*

$$X \xleftarrow{b_1 \circ \mathrm{pr}_1} \big(M_1 \times_Y M_2, e(\eta) \otimes \mathrm{pr}_1^*(\xi_1) \otimes \mathrm{pr}_2^*(\xi_2)\big) \xrightarrow{f_2 \circ \mathrm{pr}_2} Z, \tag{2.16}$$

where $e(\eta)$ is the Euler class and the projection $\mathrm{pr}_2 \colon M_1 \times_Y M_2 \to M_2$ carries the K_G-orientation induced by the K_G-orientations on f_1 and η (explained below).

In the above situation of smooth intersection, we call the diagram (2.4) an *η-intersection diagram.* It still computes the composition, but we need the Euler class of the excess intersection bundle η to compensate the lack of transversality.

We describe the canonical K_G-orientation of $\mathrm{pr}_2 \colon M_1 \times_Y M_2 \to M_2$. The excess intersection bundle η is defined so as to give an exact sequence of vector bundles (2.14). From this it follows that

$$[\eta] = (f_1 \circ \mathrm{pr}_1)^*[\mathrm{T}Y] + \mathrm{T}M - \mathrm{pr}_1^*[\mathrm{T}M_1] - \mathrm{pr}_2^*[\mathrm{T}M_2].$$

On the other hand, the stable normal bundle $N\mathrm{pr}_2$ of pr_2 is equal to $\mathrm{pr}_2^*[\mathrm{T}M_2] - [\mathrm{T}M]$. Hence

$$[\eta] = \mathrm{pr}_1^*\big(f_1^*[\mathrm{T}Y] - [\mathrm{T}M_1]\big) - N\mathrm{pr}_2.$$

A K_G-orientation on f_1 means a stable K_G-orientation on $Nf_1 = f_1^*[\mathrm{T}Y] - [\mathrm{T}M_1]$. If such an orientation is given, it pulls back to one on $\mathrm{pr}_1^*\big(f_1^*[\mathrm{T}Y] - [\mathrm{T}M_1]\big)$, and then (stable) K_G-orientations on $[\eta]$ and on $N\mathrm{pr}_2$ are in 1-to-1-correspondence. In particular, a K_G-orientation on the bundle η induces one on the normal bundle of pr_2. This induced K_G-orientation on pr_2 is used in (2.16). ([14, Lemma 5.13] justifies working with K_G-orientations on stable normal bundles.)

Proof of Theorem 2.17. Lift f_1 to a G-equivariant smooth normally non-singular map $(V_1, E_1, \hat{f}_1)$. The composition of (2.15) is defined in [15, Section 2.5] as the intersection product

$$X \xleftarrow{b_1\circ\pi_{V_1}\circ\mathrm{pr}_{V_1}} V_1 \times_Y M_2 \xrightarrow{f_2\circ\mathrm{pr}_2} Z \tag{2.17}$$

with K-theory datum $\mathrm{pr}_{V_1}^*(\tau_{V_1}) \otimes \pi_{V_1}^*(\xi_1) \otimes \mathrm{pr}_2^*(\xi_2) \in \mathrm{RK}^*_{G,X}(V_1 \times_Y M_2)$. We define the manifold $V_1 \times_Y M_2$ using the (transverse) maps $\pi_{E_1} \circ \hat{f}_1 \colon V_1 \to Y$ and $b_2 \colon M_2 \to Y$. We must compare this with the correspondence in the statement of the theorem.

We have a commuting square of embeddings of smooth manifolds

$$\begin{tikzcd} M_1 \times_Y M_2 \arrow[r, "\iota_0"] \arrow[d, "\zeta_0"] & M_1 \times M_2 \arrow[d, "\zeta_1"] \\ V_1 \times_Y M_2 \arrow[r, "\iota_1"] & V_1 \times M_2 \end{tikzcd} \tag{2.18}$$

where the vertical maps are induced by the zero section $M_1 \to V_1$ and the horizontal ones are the obvious inclusion maps. The map ζ_0 is a smooth embedding because the other three maps in the square are so.

Let $N\iota_0$ and $\nu := N\zeta_0$ denote the normal bundles of the maps ι_0 and ζ_0 in (2.18). The normal bundle of ι_1 is isomorphic to the pull-back of $\mathrm{T}Y$ because $V_1 \to Y$ is submersive. Since $M_1 \times M_2 \to V_1 \times M_2$ is the zero section of the pull back of the vector bundle V_1 to $M_1 \times M_2$, the normal bundle of ζ_1 is isomorphic to $\mathrm{pr}_1^*(V_1)$. Recall that $M := M_1 \times_Y M_2$. We get a diagram of vector bundles over M:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathrm{T}M \arrow[r, "D\iota_0"] \arrow[d, "D\zeta_0"] & \mathrm{T}(M_1 \times M_2)|_M \arrow[r] \arrow[d, "D\zeta_1"] & N\iota_0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{T}(V_1 \times_Y M_2)|_M \arrow[r, "D\iota_1"] \arrow[d] & \mathrm{T}(V_1 \times M_2)|_M \arrow[r] \arrow[d] & f^*(\mathrm{T}Y) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \nu \arrow[r] \arrow[d] & \mathrm{pr}_1^*(V_1) \arrow[r, dotted] \arrow[d] & \eta \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

The first two rows and the first two columns are exact by definition or by our description of the normal bundles of ζ_1 and ι_1. The third row is exact with the excess intersection bundle η by (2.14). Hence the dotted arrow exists and makes the third row exact. Since extensions of G-vector bundles always split, we get

$$\nu \oplus \eta \cong \mathrm{pr}_1^*(V_1).$$

Since η and V_1 are K_G-oriented, the bundle ν inherits a K_G-orientation.
We apply Thom modification with the K_G-oriented G-vector bundle ν to the correspondence in (2.16). This gives the geometric correspondence

$$X \xleftarrow{b_1\circ \mathrm{pr}_1\circ \pi_\nu} \nu \xrightarrow{f_2\circ \mathrm{pr}_2\circ \pi_\nu} Z \tag{2.19}$$

with K-theory datum

$$\xi := \tau_\nu \otimes \pi_\nu^*\bigl(e(\eta)\otimes \mathrm{pr}_1^*(\xi_1)\otimes \mathrm{pr}_2^*(\xi_2)\bigr) \in \mathrm{RK}^*_{G,X}(\nu).$$

The Tubular Neighbourhood Theorem gives a G-equivariant open embedding $\hat{\zeta}_0\colon \nu \to V_1 \times_Y M_2$ onto some G-invariant open neighbourhood of M (see [14, Theorem 3.18]).
We may find an open G-invariant neighbourhood U of the zero section in V_1 such that $U \times_Y M_2 \subseteq V_1 \times_Y M_2$ is contained in the image of $\hat{\zeta}_0$ and relatively M-compact. We may choose the Thom class $\tau_{V_1} \in \mathrm{K}_G^{\dim V_1}(V_1)$ to be supported in U. Hence we may assume that $\mathrm{pr}_1^*(\tau_{V_1})$, the pull-back of τ_{V_1} along the coordinate projection $\mathrm{pr}_1^*\colon V_1 \times_Y M_2 \to V_1$, is supported inside a relatively M-compact subset of $\hat{\zeta}_0(\nu)$.
Then [15, Example 2.14] provides a bordism between the cycle in (2.17) and

$$X \xleftarrow{b_1\circ\pi_{V_1}\circ \mathrm{pr}_1\circ\hat{\zeta}_0} \nu \xrightarrow{f_2\circ \mathrm{pr}_2\circ\hat{\zeta}_0} Z, \tag{2.20}$$

with K-theory class $\mathrm{pr}_1^*(\tau_{V_1}) \otimes \hat{\zeta}_0^*\mathrm{pr}_1^*\pi_{V_1}^*(\xi_1) \otimes \hat{\zeta}_0^*\mathrm{pr}_2^*(\xi_2)$.
Let $s_t\colon \nu \to \nu$ be the scalar multiplication by $t \in [0,1]$. Composition with s_t is a G-equivariant homotopy

$$\pi_{V_1}\mathrm{pr}_1\hat{\zeta}_0 \sim \mathrm{pr}_1\pi_\nu\colon \nu \to M_1, \qquad \mathrm{pr}_2\hat{\zeta}_0 \sim \mathrm{pr}_2\pi_\nu\colon \nu \to M_2.$$

Hence $s_t^*(\hat{\zeta}_0^*\mathrm{pr}_1^*\pi_{V_1}^*(\xi_1) \otimes \hat{\zeta}_0^*\mathrm{pr}_2^*(\xi_2))$ is a G-equivariant homotopy

$$\hat{\zeta}_0^*\mathrm{pr}_1^*\pi_{V_1}^*(\xi_1) \otimes \hat{\zeta}_0^*\mathrm{pr}_2^*(\xi_2) \sim \pi_\nu^*\bigl(\mathrm{pr}_1^*(\xi_1)\otimes \mathrm{pr}_2^*(\xi_2)\bigr).$$

When we tensor with $\mathrm{pr}_1^*(\tau_{V_1})$, this homotopy has X-compact support because the support of $\mathrm{pr}_1^*(\tau_{V_1})$ is relatively M-compact.
This gives a homotopy of geometric correspondences between (2.17) and the variant of (2.19) with K-theory datum

$$\mathrm{pr}_1^*(\tau_{V_1}) \otimes \pi_\nu^*\mathrm{pr}_1^*(\xi_1) \otimes \pi_\nu^*\mathrm{pr}_2^*(\xi_2);$$

the relative M-compactness of the support of $\mathrm{pr}_{V_1}^*(\tau_{V_1})$ ensures that the homotopy of K_G-cycles implicit here has X-compact support. (We use [15, Lemma

2.12] here, but the statement of the lemma is unclear about the necessary compatibility between the homotopy and the support of ξ.)
The K-theory class $\mathrm{pr}_1^*(\tau_{V_1})$ in this formula is the restriction of the Thom class for the vector bundle $\mathrm{pr}_1^*(V_1)$ over M to ν. Since $\mathrm{pr}_1^*(V_1) \cong \nu \oplus \eta$ and the Thom isomorphism for a direct sum bundle is the composition of the Thom isomorphisms for the factors, the Thom class of $\mathrm{pr}_1^*(V_1)$ is $\mathrm{pr}_1^*(\tau_{V_1}) = \tau_\nu \otimes \tau_\eta$. Restricting this to the subbundle ν gives $\tau_\nu \otimes \pi_\nu^*(e(\eta))$. Hence the K-theory classes that come from (2.17) and (2.19) are equal. This finishes the proof. □

2.6 The geometric Lefschetz index formula

In this section we compute Lefschetz indices in the symmetric monoidal category $\widehat{\mathrm{KK}}^G$ for smooth G-manifolds with boundary. Our computation is geometric and uses the intersection theory of equivariant correspondences discussed in Sections 2.2 and 2.5.
Let X be a smooth compact G-manifold, possibly with boundary. Let $\mathring{X}$ be its interior. Let

$$X \xleftarrow{b} M \xrightarrow{f} X, \qquad \xi \in \mathrm{RK}^*_{G,X}(M) \tag{2.21}$$

be a K_G-oriented smooth geometric correspondence from X to itself, with M of finite orbit type to ensure that $f\colon M \to X$ lifts to an essentially unique normally non-singular map. Since X is compact, $\mathrm{RK}^*_{G,X}(M) = \mathrm{K}^*_G(M)$ is the usual K-theory with compact support. The K_G-orientation for (2.21) means a K_G-orientation on the stable normal bundle of f. This is equivalent to giving a G-vector bundle V over X and K_G-orientations on $\mathrm{T}M \oplus f^*(V)$ and $\mathrm{T}X \oplus V$. If X has a boundary, then the requirements for a smooth correspondence are that M be a smooth manifold with boundary of finite orbit type, such that $f(\partial M) \subseteq \partial X$ and f is transverse to ∂X. This ensures that f has an essentially unique lift to a normally non-singular map from M to X by Proposition 2.5. Recall the map $\rho\colon X \to \mathring{X}$, which is shrinking the collar around ∂X.

Theorem 2.18. *Let $\alpha \in \widehat{\mathrm{KK}}{}_i^G(X, X)$ be represented by a K_G-oriented smooth geometric correspondence as in (2.21). Assume that $(\rho b, f)\colon M \to X \times X$ and the diagonal embedding $X \to X \times X$ intersect smoothly with a K_G-oriented excess intersection bundle η. Then $Q_{\rho b,f} := \{m \in M \mid \rho b(m) = f(m)\}$ is a smooth manifold without boundary. For a certain canonical K_G-orientation on $Q_{\rho b,f}$, $\mathcal{L}(\alpha) \in \widehat{\mathrm{KK}}{}_i^G(X, \mathrm{pt})$ is represented by the geometric correspondence $X \leftarrow Q_{\rho b,f} \to \mathrm{pt}$ with K-theory class $\xi|_{Q_{\rho b,f}} \otimes e(\eta)$ on $Q_{\rho b,f}$; here the map $Q_{\rho b,f} \to X$ is given by $m \mapsto \rho b(m) = f(m)$.*
The Lefschetz index of α in $\widehat{\mathrm{KK}}{}_i^G(\mathrm{pt}, \mathrm{pt})$ is represented by the geometric correspondence $\mathrm{pt} \leftarrow Q_{\rho b,f} \to \mathrm{pt}$ with K_G-theory class $\xi|_{Q_{\rho b,f}} \otimes e(\eta)$ on $Q_{\rho b,f}$.
The Lefschetz index of α is the index of the Dirac operator on $Q_{\rho b,f}$ with coefficients in $\xi|_{Q_{\rho b,f}} \otimes e(\eta)$.

Proof. We abbreviate $Q := Q_{\rho b,f}$ throughout the proof. We have $Q \subseteq \mathring{M}$ because $\rho b(M) \subseteq \rho(X) \subseteq \mathring{X}$ and $f(\partial M) \subseteq \partial X$. The intersection $\mathring{M} \times_{\mathring{X} \times \mathring{X}} \mathring{X}$

$$
\begin{tikzcd}
 & & & & Q_{\rho b,f} \arrow[dl, "j"'] \arrow[ddrr, "\zeta f j"] & & & \\
 & & & M \arrow[dll, "\Delta b"'] \arrow[dr, "(\mathrm{id},\zeta\rho b)"] & & & & \\
 & X\times X \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{id}\times(\mathrm{id},\zeta\rho)"] & & & M\times \mathrm{N}\mathring{X} \arrow[dll, "(\Delta b)\times \mathrm{id}"'] \arrow[dr, "f\times\mathrm{id}"] & & \mathrm{N}\mathring{X} \arrow[dl, "(\mathrm{id},\iota\pi)"'] \arrow[dr] & \\
X & & X\times X\times \mathrm{N}\mathring{X} & & & X\times \mathrm{N}\mathring{X}\cong \mathrm{N}\mathring{X}\times X & & \mathrm{pt}
\end{tikzcd}
$$

Figure 2: The intersection diagram for the computation of $\mathcal{L}(\alpha)$ in the proof of Theorem 2.18. Here $j\colon Q_{\rho b,f}\to M$ denotes the inclusion map; ζ the zero section $X\to \mathrm{N}X$ or $\mathring{X}\to\mathrm{N}\mathring{X}$; $\pi\colon \mathrm{N}\mathring{X}\to\mathring{X}$ the bundle projection; $\iota\colon\mathring{X}\to X$ the inclusion; $\Delta\colon X\to X\times X$ the diagonal embedding; $\mathrm{pr}_1\colon X\times X\to X$ the projection onto the first factor.

is Q and hence a smooth submanifold of $\mathring{M}$.

We compute $\mathcal{L}(\alpha)$ using the dual of X constructed in Theorem 2.7. This involves a G-vector bundle $\mathrm{N}X$ such that $\mathrm{T}X\oplus\mathrm{N}X\cong X\times E$ for a K_G-oriented G-vector space E.

With the unit and counit from Theorem 2.7, $\mathcal{L}(\alpha)$ becomes the composition of the three geometric correspondences in the bottom zigzag in Figure 2; here we already composed α with the multiplication correspondence, which simply composes b with Δ.

We first consider the small left square. Computing its intersection space naively gives M, which is a manifold with boundary. We would hope that this square is Cartesian. But $X\times X$ is only a manifold with corners if X has a boundary, and we we did not discuss smooth correspondences in this generality. Hence we check directly that the composition of the correspondences from X to $X\times X\times\mathrm{N}\mathring{X}$ and on to $M\times\mathrm{N}\mathring{X}$ is represented by $X\leftarrow M\to M\times\mathrm{N}\mathring{X}$.

The manifold $\mathrm{N}\mathring{X}$ is an open subset of E by construction. Hence the map

$$\mathrm{id}\times(\mathrm{id},\zeta\rho b)\colon X\times X\to X\times X\times\mathrm{N}\mathring{X}$$

extends to an open embedding

$$\psi\colon X\times X\times E\to X\times X\times\mathrm{N}\mathring{X},\qquad (x_1,x_2,e)\mapsto\bigl(x_1,x_2,\zeta\rho(x_2)+h_{x_2}(\|e\|^2)\cdot e\bigr),$$

where $h_{x_2}\colon\mathbb{R}_+\to\mathbb{R}_+$ is a diffeomorphism onto a bounded interval $[0,t)$ depending smoothly and G-invariantly on x_2, such that the t-ball in E around

$\zeta\rho(x_2) \in \mathrm{N}\mathring{X}$ is contained in $\mathrm{N}\mathring{X}$.
The map ψ gives a special correspondence

$$X \xleftarrow{\mathrm{pr}_1 \circ \pi_E} X \times X \times E \xrightarrow{\psi} X \times X \times \mathrm{N}\mathring{X}$$

with K-theory class the pull-back of the Thom class of E. This is equivalent to the given correspondence from X to $X \times X \times \mathrm{N}\mathring{X}$ because of a Thom modification for the trivial vector bundle E and a homotopy. In particular, the K_G-orientation of $\mathrm{id} \times (\mathrm{id}, \zeta\rho)$ that is implicit here is the one that we get from the K_G-orientation in the proof of Theorem 2.7.
For a special correspondence, the intersection always gives the composition product. Here we get the space

$$\big\{(x_1, x_2, e, m, y, \mu) \in X \times X \times E \times M \times \mathrm{N}\mathring{X} \mid \\ (x_1, x_2, \zeta\rho(x_2) + h_{x_2}(\|e\|^2) \cdot e)) = (b(m), b(m), y, \mu)\big\}.$$

That is, $x_1 = x_2 = b(m)$, $(y, \mu) = \rho b(m) + h_{b(m)}(\|e\|^2) \cdot e)$. Since $m \in M$ and $e \in E$ may be arbitrary and determine the other variables, we may identify this space with $M \times E$.
In the same way, we may replace

$$X \xleftarrow{b} M \xrightarrow{(\mathrm{id}, \zeta\rho b)} M \times \mathrm{N}\mathring{X} \tag{2.22}$$

by an equivalent special correspondence with space $M \times E$ in the middle. This gives exactly the composition computed above. Hence (2.22) also represents the composition of the correspondences from X to $M \times \mathrm{N}\mathring{X}$ in Figure 2.
Composing further with $f \times \mathrm{id}$ simply composes K_G-oriented normally non-singular maps. Since we are now in the world of manifolds with boundary, we may identify smooth maps and smooth normally non-singular maps. The large right square contains the G-maps

$$(f, \zeta\rho b) = (f \times \mathrm{id}) \circ (\mathrm{id}, \zeta\rho b) \colon M \to X \times \mathrm{N}\mathring{X}, \\ (\iota\pi, \mathrm{id}) \colon \mathrm{N}\mathring{X} \to X \times \mathrm{N}\mathring{X}.$$

The pull-back contains those $(m, x, \mu) \in M \times \mathrm{N}\mathring{X}$ with $(f(m), \rho b(x), 0) = (x, x, \mu)$ in $X \times \mathrm{N}\mathring{X}$. This is equivalent to $x = f(m) = \rho b(m)$ and $\mu = 0$, so that the pull-back is Q. Since all vectors tangent to the fibres of $\mathrm{N}\mathring{X}$ are in the image of $D(\iota\pi, \mathrm{id})$, the intersection is smooth and the excess intersection bundle is the same bundle η as for $(f, \rho b) \colon \mathring{M} \to \mathring{X} \times \mathring{X}$ and $\delta \colon \mathring{X} \to \mathring{X} \times \mathring{X}$. Hence the right square is η-Cartesian.
Theorem 2.17 shows that $\mathcal{L}(\alpha)$ is represented by a correspondence of the form $X \xleftarrow{bj} Q \to \mathrm{pt}$, with a suitable class in $\mathrm{K}^*_G(Q)$ and a suitable K_G-orientation on the map $Q \to \mathrm{pt}$ or, equivalently, the manifold Q. Here we may replace bj by the properly homotopic map $\rho bj = fj$. It remains to describe the K-theory and orientation data.

First, the given K-theory class ξ on M is pulled back to $\xi \otimes 1$ on $M \times \mathrm{N}\mathring{X}$ when we take the exterior product with $\mathrm{N}\mathring{X}$. In the intersection product, this is pulled back to M along $(\mathrm{id}, \zeta\rho b)$, giving ξ again, and then to Q along j, giving the restriction of ξ to $Q \subseteq M$. The unit and counit have 1 as its K-theory datum. Thus the Lefschetz index has $\xi|_Q \otimes e(\eta) \in \mathrm{K}^*_G(Q)$ as its K-theory datum by Theorem 2.17.

The given K_G-orientations on E, f and η induce K_G-orientations on all maps in Figure 2 that point to the right. This is the K_G-orientation on the map $Q \to \mathrm{pt}$ that we need. We describe it in greater detail after the proof of the theorem.

The K_G-orientation on the map $Q \to \mathrm{pt}$ is equivalent to a G-equivariant Spin^c-structure on Q. The isomorphism

$$\widehat{\mathrm{KK}}^G_*(\mathrm{pt}, \mathrm{pt}) \to \widehat{\mathrm{KK}}^G_*(\mathrm{C}(\mathrm{pt}), \mathrm{C}(\mathrm{pt}))$$

described in [15, Theorem 4.2] maps the geometric correspondence just described to the index of the Dirac operator on Q for the chosen Spin^c-structure twisted by $\xi|_Q \otimes e(\eta)$. This gives the last assertion of the theorem. □

Since the K_G-orientation on $Q_{\rho b, f}$ is necessary for computations, we describe it more explicitly now. We still use the notation from the previous proof.

We are given K_G-orientations on E, f and η. The K_G-orientation on f is equivalent to one on the G-vector bundle $\mathrm{T}M \oplus f^*(\mathrm{N}X)$ over M because

$$\mathrm{T}X \oplus \mathrm{N}X \cong X \times E$$

is a K_G-oriented G-vector bundle on X.

We already discussed during the proof of the theorem that $\mathrm{id} \times (\mathrm{id}, \zeta\rho)$ and $(\mathrm{id}, \zeta\rho)$ are normally non-singular embeddings with normal bundle E; this gives the correct K_G-orientation for these maps as well.

A K_G-orientation on the map $(f, \zeta\rho b)\colon M \to X \times \mathrm{N}\mathring{X}$ is equivalent to one for $\mathrm{T}M \oplus f^*(\mathrm{N}X)$ because the bundle $\mathrm{T}(X \times \mathrm{N}\mathring{X}) \oplus \mathrm{pr}_1^*(\mathrm{N}X)$ over $X \times \mathrm{N}\mathring{X}$ is isomorphic to the trivial bundle with fibre $E \oplus E$ and $(f, \zeta\rho b)^*\mathrm{pr}_1^*(\mathrm{N}X) = f^*(\mathrm{N}X)$. We are already given such a K_G-orientation from the K_G-orientation of f.

Lemma 2.19. *The given K_G-orientation on $\mathrm{T}M \oplus f^*(\mathrm{N}X)$ is also the one that we get by inducing K_G-orientations on $(\mathrm{id}, \zeta\rho b)$ from $(\mathrm{id}, \zeta\rho)$ and on $f \times \mathrm{id}$ from f and then composing.*

Proof. The K_G-orientation of f induces one for $f \times \mathrm{id}$, which is equivalent to a K_G-orientation for

$$\mathrm{T}(M \times \mathrm{N}\mathring{X}) \oplus (f\mathrm{pr}_1)^*(\mathrm{N}X) \cong \big(\mathrm{T}M \oplus f^*(\mathrm{N}X)\big) \times (\mathrm{N}\mathring{X} \times E).$$

This K_G-orientation is exactly the direct sum orientation from $\mathrm{T}M \oplus f^*(\mathrm{N}X)$ and E; no sign appears in changing the order because E has even dimension.

The map $h = (\mathrm{id}, \zeta\rho)$ is a smooth embedding with normal bundle E. Hence we get an extension of vector bundles

$$\mathrm{T}M \oplus f^*(\mathrm{N}X) \rightarrowtail h^*\big(\mathrm{T}(M \times \mathrm{N}\mathring{X}) \oplus (f\mathrm{pr}_1)^*(\mathrm{N}X)\big) \twoheadrightarrow E.$$

The given K_G-orientations on $\mathrm{T}M \oplus f^*(\mathrm{N}X)$ and E induce one on the vector bundle in the middle. This is the same one as the pull-back of the one constructed above. This means that the K_G-orientation on $\mathrm{T}M \oplus f^*(\mathrm{N}X)$ induced by h is the given one. $\square$

Equation (2.14) provides the following exact sequence of vector bundles over Q:

$$\begin{aligned} 0 \to \mathrm{T}Q \xrightarrow{Dj, D(\zeta f j)} j^*(\mathrm{T}M) \oplus (\zeta f j)^*\mathrm{T}(\mathrm{N}\mathring{X}) \\ \xrightarrow{D(f,\zeta\rho b), -D(\iota\pi,\mathrm{id})} (f, \zeta\rho b)^*\mathrm{T}(X \times \mathrm{N}\mathring{X}) \to \eta \to 0. \end{aligned}$$

Since $-D(\iota\pi, \mathrm{id})$ is injective, we may divide out $\mathrm{T}(\mathrm{N}\mathring{X})$ and its image to get the simpler short exact sequence

$$0 \to \mathrm{T}Q \xrightarrow{Dj} j^*\mathrm{T}M \xrightarrow{Df - D(\rho b)} f^*\mathrm{T}X \to \eta \to 0.$$

Then we add the identity map on $j^*f^*(\mathrm{N}X)$ to get

$$0 \to \mathrm{T}Q \xrightarrow{(Dj,0)} j^*(\mathrm{T}M \oplus f^*\mathrm{N}X) \xrightarrow{(Df - D(\rho b), \mathrm{id})} f^*(\mathrm{T}X \oplus \mathrm{N}X) \to \eta \to 0. \tag{2.23}$$

In the last long exact sequence, the vector bundles $j^*(\mathrm{T}M \oplus f^*\mathrm{N}X)$, $f^*(\mathrm{T}X \oplus \mathrm{N}X) \cong Q \times E$ and η carry K_G-orientations. These together induce one on $\mathrm{T}Q$. This is the K_G-orientation that appears in Theorem 2.18.

Of course, the resulting geometric cycle should not depend on the auxiliary choice of a K_G-orientation on η. Indeed, if we change it, then we change both $e(\eta)$ and the K_G-orientation on $\mathrm{T}Q$, and these changes cancel each other.

We now consider some examples of Theorem 2.18.

2.6.1 Self-maps transverse to the identity map

Let X be a compact G-manifold with boundary and let $b\colon X \to X$ be a smooth G-map that is transverse to the identity map. Thus b has only finitely many isolated fixed points and $1 - D_x b\colon \mathrm{T}_x X \to \mathrm{T}_x X$ is invertible for all fixed points x of b. We turn b into a geometric correspondence α from X to itself by taking $M = X$, $f = \mathrm{id}$ (with standard K_G-orientation) and $\xi = 1$.

Since b has only finitely many fixed points, we may choose the collar neighbourhood so small that all fixed points that do not lie on ∂X lie outside the collar neighbourhood, and such that the fixed points of ρb are precisely the fixed points of b not on the boundary of X. Hence $\rho b = b$ near all fixed points. Then ρb is also transverse to the diagonal map and Theorem 2.18 applies. The intersection space in Theorem 2.18 is

$$Q = Q_{\rho b,\mathrm{id}} = \{x \in X \mid \rho b(x) = x\} = \{x \in \mathring{X} \mid b(x) = x\},$$

the set of fixed points of b in $\mathring{X}$. The K-theory class on Q is 1 because $\xi = 1$ and the intersection is transverse. More precisely, the bundle η is zero-dimensional, and we may give it a trivial K_G-orientation for which $e(\eta) = 1$.

Although Q is discrete, the K_G-orientation of the map $Q \to \mathrm{pt}$ is important extra information: it provides the signs that appear in the familiar Lefschetz fixed-point formula. Equation (2.23) simplifies to

$$0 \to \mathrm{T}Q \to (\mathrm{T}X \oplus \mathrm{N}X)|_Q \xrightarrow{(\mathrm{id}-Db,\mathrm{id})} (\mathrm{T}X \oplus \mathrm{N}X)|_Q \to 0.$$

We left out η because it is zero-dimensional and carries the trivial K_G-orientation to ensure that $e(\eta) = 1$. The bundle $\mathrm{T}Q$ is also zero-dimensional. But a zero-dimensional bundle has non-trivial K_G-orientations. The Clifford algebra bundle of a zero-dimensional bundle is the trivial, trivially graded one-dimensional bundle spanned by the unit section. Thus an irreducible Clifford module (spinor bundle) for it is the same as a $\mathbb{Z}/2$-graded G-equivariant complex line bundle.

Let S be the spinor bundle associated to the given K_G-orientation on $\mathrm{T}X \oplus \mathrm{N}X \cong E$. The exact sequence (2.23) says that the K_G-orientation of Q is the $\mathbb{Z}/2$-graded G-equivariant complex line bundle ℓ such that $(\mathrm{id}-Db)^*(S|_Q)\otimes\ell \cong S|_Q$ as Clifford modules. This uniquely determines ℓ. Thus ℓ measures whether Db changes orientation or not. This is exactly the *sign* of the G-equivariant vector bundle automorphism $1 - Db$ on $\mathrm{T}X|_Q$, which is studied in detail in [13]. In particular, it is shown in [13] that ℓ is the complexification of a $\mathbb{Z}/2$-graded G-equivariant *real* line bundle. The $\mathbb{Z}/2$-grading gives one sign for each G-orbit in Q, namely, the index of $\mathrm{id} - Db_x$. In addition, the sign gives a real character $G_x \to \{-1, +1\}$ for each orbit, where G_x denotes the stabiliser of a point in the orbit.

Twisting the K_G-orientation by a line bundle over Q has the same effect as taking the trivial K_G-orientation and putting this line bundle on Q. Thus $\mathcal{L}(\alpha)$ is represented by the geometric correspondence

$$X \leftarrow (Q, \mathrm{sign}(1 - Db|_Q)) \to \mathrm{pt}$$

with the trivial K_G-orientation on the map $Q \to \mathrm{pt}$.

The Lefschetz index of α is the index of the Dirac operator on Q with coefficients in the line bundle $\mathrm{sign}(1-Db)|_Q$; this is simply the $\mathbb{Z}/2$-graded G-representation on the space of sections of $\mathrm{sign}(1 - Db)|_Q$, which is a certain finite-dimensional $\mathbb{Z}/2$-graded, real G-representation.

If the group G is trivial, then the Lefschetz index is a number and $\mathrm{sign}(1 - Db)$ is the family of $\mathrm{sign}(1 - D_x b) \in \{\pm 1\}$ for $x \in Q$. If X is connected, then all maps $X \leftarrow \mathrm{pt}$ give the same element in $\widehat{\mathrm{KK}}$. Thus $\mathcal{L}(\alpha)$ is $\text{L-ind}(\alpha)$ times the point evaluation class $[X \leftarrow \mathrm{pt} = \mathrm{pt}]$, and $\text{L-ind}(\alpha)$ is the sum of the indices of all fixed points of b in $\mathring{X}$.

2.6.2 Euler characteristics

Now let $\xi \in \mathrm{K}^*_G(X)$ and consider the correspondence with $M = X$, $b = f = \mathrm{id}$, and the above class ξ. We want to compute the Lefschetz index of the geometric correspondence α associated to ξ. In particular, for $\xi = 1$ we get the Lefschetz index of the identity element in $\widehat{\mathrm{KK}}^G_0(X, X)$, which is the Euler characteristic of X.

We only compute the Lefschetz index of $\xi \in \mathrm{K}^*_G(X)$ for X with trivial boundary. Then the map ρ in Theorem 2.18 is the identity map, and id_X intersects itself smoothly. The intersection space is $Q = X$, embedded diagonally into $X \times X$. The excess intersection bundle η is $\mathrm{T}X$. To apply Theorem 2.18, we also assume that X is K_G-oriented. Then $\mathcal{L}(\alpha)$ is represented by the geometric correspondence

$$X \xleftarrow{\mathrm{id}_X} (X, \xi \otimes e(\mathrm{T}X)) \to \mathrm{pt}.$$

Here $e(\mathrm{T}X)$ and the map $X \to \mathrm{pt}$ both use the same K_G-orientation on X. The Lefschetz index of α is represented by

$$\mathrm{pt} \leftarrow (X, \xi \otimes e(\mathrm{T}X)) \to \mathrm{pt}.$$

By Theorem 2.18, this is the index of the Dirac operator of X with coefficients in $\xi \otimes e(\mathrm{T}X)$.

Twisting the Dirac operator by $e(\mathrm{T}X)$ gives the de Rham operator: this is the operator $d + d^*$ on differential forms with usual $\mathbb{Z}/2$-grading, so that its index is the Euler characteristic of X. Thus (the analytic version of) $\mathcal{L}(\alpha)$ is the class in $\mathrm{KK}^G_0(\mathrm{C}(X), \mathbb{C})$ of the de Rham operator with coefficients in ξ. This was proved already in [11] by computations in Kasparov's analytic KK-theory. Now we have a purely geometric proof of this fact, at least if X is K_G-oriented. Theorem 2.18 no longer works for X without K_G-orientation because there is no K_G-orientation on the excess intersection bundle. A way around this restriction would be to use twisted K-theory throughout. We shall not pursue this here, however.

We can now clarify the relationship between the Euler class $e(\mathrm{T}X) \in \mathrm{K}^{\dim(X)}_G(X)$ and the higher Euler characteristic $\mathrm{Eul}_X \in \mathrm{KK}^G_0(\mathrm{C}(X), \mathbb{C})$ introduced already in [11]. Since we assume X K_G-oriented and without boundary, there is a duality isomorphism $\mathrm{K}^{\dim(X)}_G(X) \cong \mathrm{K}^G_0(X) = \mathrm{KK}^G_0(\mathrm{C}(X), \mathbb{C})$. This duality isomorphism maps $e(\mathrm{T}X)$ to Eul_X.

2.6.3 Self-maps without transversality

Let X be a compact G-manifold and let $b\colon X \to X$ be a smooth G-map. We want to compute the Lefschetz map on the geometric correspondence

$$X \xleftarrow{b} X \xrightarrow{\mathrm{id}_X} X$$

with K_G-theory class 1 on X.

If b is transverse to the identity map, then this is done already in Section 2.6.1. The case $b = \mathrm{id}_X$ is done already in Section 2.6.2. Now we assume that b and id_X intersect smoothly. We also assume that b has no fixed points on the boundary; then we may choose the collar neighbourhood of ∂X to contain no fixed points of b, so that $\rho(x) = x$ in a neighbourhood of the fixed point subset of b. Furthermore, all fixed points of ρb are already fixed points of b.

That b and id_X intersect smoothly and away from ∂X means that

$$Q := \{x \in X \mid b(x) = x\} = \{x \in X \mid \rho b(x) = x\}$$

is a smooth submanifold of $\mathring{X}$ and that there is an exact sequence of G-vector bundles over Q:

$$0 \to \mathrm{T}Q \to \mathrm{T}X|_Q \xrightarrow{1-D(\rho b)} \mathrm{T}X|_Q \to \eta \to 0,$$

where η is the excess intersection bundle.

Remark 2.20. The maps b and id_X always intersect smoothly if $b\colon X \to X$ is isometric with respect to a Riemannian metric on X; the reason is that if Db fixes a vector (x, ξ) at a fixed point of b, then b fixes the entire geodesic through x in direction ξ.

The vector bundles $\mathrm{T}Q$ and η are the kernel and cokernel of the vector bundle endomorphism $1 - D(\rho b)$ on $\mathrm{T}X|_Q$. Since both are vector bundles, $1 - D(\rho b)$ has locally constant rank. We may split

$$\mathrm{T}X|_Q \cong \ker(\mathrm{id} - D(\rho b)) \oplus \mathrm{im}(\mathrm{id} - D(\rho b)) = \mathrm{T}Q \oplus \mathrm{im}(\mathrm{id} - D(\rho b)),$$
$$\mathrm{T}X|_Q \cong \mathrm{coker}(\mathrm{id} - D(\rho b)) \oplus \mathrm{coim}(\mathrm{id} - D(\rho b)) = \eta \oplus \mathrm{coim}(\mathrm{id} - D(\rho b)).$$

Since $\mathrm{im}(\varphi) \cong \mathrm{coim}(\varphi)$ for any vector bundle homomorphism, it follows that η and $\mathrm{T}Q$ are stably isomorphic as G-vector bundles. Thus K_G-orientations on one of them translate to K_G-orientations on the other.

Remark 2.21. Given two stably isomorphic vector bundles, there is always a vector bundle endomorphism with these two as kernel and cokernel. Hence we cannot expect η and $\mathrm{T}Q$ to be isomorphic.

Corollary 2.22. *Let X be a compact G-manifold. Let $b\colon X \to X$ be a smooth G-map without fixed points on ∂X, such that b and id_X intersect smoothly. Let the fixed point submanifold Q of b be K_G-oriented, and equip the excess intersection bundle with the induced K_G-orientation. Then the Lefschetz index of the geometric correspondence*

$$X \xleftarrow{b} X \xrightarrow{\mathrm{id}_X} X$$

with K_G-theory class 1 on X is the index of the Dirac operator on Q twisted by $e(\eta)$.

The Lefschetz map sends the correspondence above to

$$X \xleftarrow{bj} Q \to \mathrm{pt}$$

with K-theory class $e(\eta)$ on Q.

2.6.4 Trace computation for standard correspondences

By Corollary 2.8, any element of $\widehat{\mathrm{KK}}^G_*(X,X)$ is represented by a correspondence of the form

$$X \xleftarrow{\iota\circ\pi\circ\mathrm{pr}_1} \mathrm{N}\mathring{X} \times X \xrightarrow{\mathrm{pr}_2} X$$

for a unique $\xi \in \mathrm{K}^*_G(\mathrm{N}\mathring{X} \times X)$. We may view this as a standard form for an element in $\widehat{\mathrm{KK}}^G_*(X,X)$.

The map $(\rho\circ\iota\circ\pi\circ\mathrm{pr}_1, \mathrm{pr}_2) = (\rho\circ\pi)\times\mathrm{id}\colon \mathrm{N}\mathring{X}\times X \to X\times X$ is a submersion and hence transverse to the diagonal. Thus Theorem 2.18 applies. The space $Q_{\rho b,f}$ is the graph of $\rho\pi\colon \mathrm{N}\mathring{X} \to X$. Thus the Lefschetz map gives the geometric correspondence

$$X \leftarrow \mathrm{N}\mathring{X} \to \mathrm{pt}, \qquad \xi|_{\mathrm{N}\mathring{X}} \in \mathrm{K}^*_G(\mathrm{N}\mathring{X}),$$

where we embed $\mathrm{N}\mathring{X} \to \mathrm{N}\mathring{X} \times X$ via $(\mathrm{id}, \rho\pi)$ and use the canonical K_G-orientation on $\mathrm{N}\mathring{X}$. The Lefschetz index in $\widehat{\mathrm{KK}}^G_*(\mathrm{pt},\mathrm{pt}) \cong \mathrm{K}^*_G(\mathrm{pt})$ is computed analytically as the G-equivariant index of the Dirac operator on $\mathrm{N}\mathring{X}$ twisted by $\xi|_{\mathrm{N}\mathring{X}}$.

2.6.5 Trace computation for another standard form

Assume now that X has no boundary and is K_G-oriented. As we remarked at the end of Section 2.4, any element of $\widehat{\mathrm{KK}}^G_*(X,X)$ is represented by a correspondence

$$X \xleftarrow{\mathrm{pr}_1} X\times X \xrightarrow{\mathrm{pr}_2} X, \qquad \xi \in \mathrm{K}^*_G(X\times X).$$

The same computation as in Section 2.6.4 shows that the Lefschetz map sends this to

$$X = X \to \mathrm{pt}, \qquad \xi|_X \in \mathrm{K}^*_G(X),$$

where $\xi|_X$ is for the diagonal embedding $X \to X\times X$. Analytically, this is the K_G-homology class of the Dirac operator on X with coefficients $\xi|_X$.

2.6.6 Homogeneous correspondences

We call a self-correspondence $X \xleftarrow{b} M \xrightarrow{f} X$ *homogeneous* if X and M are homogeneous G-spaces. That is, $X := G/H$ and $M := G/L$ for closed subgroups $H, L \subseteq G$. Then there are elements $t_b, t_f \in G$ with $b(gL) := gt_bH$, $f(gL) := gt_fH$; we need $L \subseteq t_bHt_b^{-1} \cap t_fHt_f^{-1}$ for this to be well-defined. Since $G/L \cong G/t_f^{-1}Lt_f$ by $gL \mapsto gLt_f$, any homogeneous correspondence is isomorphic to one with $t_f = 1$, so that $L \subseteq H$. We assume this from now on and abbreviate $t = t_b$.

Since M and X are compact, the relevant K-theory group $\mathrm{RK}^*_{G,X}(M)$ for a homogeneous correspondence is just $\mathrm{K}^*_G(M)$. The induction isomorphism gives $\mathrm{RK}^*_{G,X}(M) = \mathrm{K}^*_G(G/L) \cong \mathrm{K}^*_L(\mathrm{pt})$.

A K_G-orientation for $f\colon G/L \to G/H$ is equivalent to a K^H-orientation for the projection map $H/L \to \mathrm{pt}$ because f is obtained from this H-map by induction. Thus we must assume a K^H-orientation on H/L. Equivalently, the representation of L on $\mathrm{T}_{1L}(H/L)$ factors through Spin^c. This tangent space is the quotient $\mathfrak{h}/\mathfrak{l}$, where $\mathfrak{h}$ and $\mathfrak{l}$ denote the Lie algebras of H and L, respectively. Let $L' := H \cap tHt^{-1}$. Then $L \subseteq L'$ and both maps $f, b\colon G/L \to G/H$ factor through the quotient map $p\colon G/L \to G/L'$. The geometric correspondence

$$G/H \xleftarrow{b} G/L \xrightarrow{f} G/H, \qquad \xi \in \mathrm{K}^*_G(G/L)$$

is equivalent to the geometric correspondence

$$G/H \xleftarrow{b'} G/L' \xrightarrow{f'} G/H, \qquad \xi' \in \mathrm{K}^*_G(G/L')$$

with $\xi' := p_!(\xi)$ and $b'(gL') = gtH$ and $f'(gL') = gH$. (To construct the equivalence, we first need a normally non-singular map lifting p; then we apply vector bundle modifications on the domain and target of p to replace p by an open embedding; finally, for an open embedding we may construct a bordism as in [15, Example 2.14].)

Thus we may further normalise a homogeneous geometric self-correspondence to one with $L = H \cap tHt^{-1}$.

Now we compute the Lefschetz map for a such a normalised homogeneous self-correspondence.

First let $t \notin H$. Then the image of the map $(f, b)\colon G/L \to G/H \times G/H$ does not intersect the diagonal. Hence (f, b) is transverse to the diagonal and the coincidence space $Q_{b,f}$ is empty. Thus the Lefschetz map vanishes on a homogeneous correspondence with $t \notin H$ by Theorem 2.18.

Now let $t \in H$. Then $b = f\colon G/L \to G/H$ is the canonical projection map. Our normalisation condition yields $L = H$ and $b = f = \mathrm{id}$ in this case; that is, our geometric correspondence is the class in $\widehat{\mathrm{KK}}^G_*(G/H, G/H)$ of some $\xi \in \mathrm{K}^*_G(G/H)$. Thus we have a special case of the Euler characteristic computation in Section 2.6.2. The Lefschetz map gives the class of the geometric correspondence

$$G/H \xleftarrow[=]{\mathrm{id}} (G/H, e(\mathrm{T}G/H) \otimes \xi) \to \mathrm{pt},$$

provided G/H is K_G-oriented. The Lefschetz index is the index of the de Rham operator with coefficients in ξ.

When we identify $\mathrm{K}^*_G(G/H) \cong \mathrm{K}^*_H(\mathrm{pt})$, the Lefschetz index becomes a map

$$\mathrm{K}^*_H(\mathrm{pt}) \to \mathrm{K}^*_G(\mathrm{pt}).$$

In complex K-theory, this is a map $\mathrm{R}(H) \to \mathrm{R}(G)$. Graeme Segal studied this map in [28, Section 2], where it was denoted by $i_!$.

For instance, assume G to be connected and let $H = L$ be its maximal torus. Let $t \in W := N_G H/H$, the Weyl group of G. Assume that we are working with complex K-theory, so that $\mathrm{K}^*_G(G/H) \cong \mathrm{K}^*_H(\mathrm{pt}) \cong \mathrm{R}(H)$. The Weyl group W acts on G/H by right translations; these are G-equivariant maps.

Taking the correspondences $X \xleftarrow{w^{-1}} X = X$, this gives a representation $W \to \widehat{\mathrm{KK}}{}_0^G(G/H, G/H)$. We also map $\mathrm{R}(H) \cong \mathrm{K}_G^0(G/H) \to \widehat{\mathrm{KK}}{}_0^G(G/H, G/H)$ using the correspondences $X = (X, \xi) = X$. These representations of W and $\mathrm{R}(H)$ are a covariant pair of representations with respect to the canonical action of W on $\mathrm{R}(H)$ induced by the automorphisms $h \mapsto whw^{-1}$ of H for $w \in W$. Hence we map

$$\mathrm{R}(H) \rtimes W \to \widehat{\mathrm{KK}}{}_0^G(G/H, G/H).$$

The Lefschetz index $\mathrm{R}(H) \rtimes W \to \mathrm{R}(G)$ maps $a \cdot t \mapsto 0$ for $t \in W \setminus \{1\}$ and $a \cdot 1 \mapsto \mathrm{ind}_G \Lambda_a$, where Λ_a means the de Rham operator on G/H twisted by a.

2.7 Fixed points submanifolds for torus actions

As another application of our excess intersection formula, we reprove a result that is used in a recent article by Block and Higson [5] to reformulate the Weyl Character Formula in KK-theory.

Block and Higson also develop a more geometric framework for equivariant KK-theory for a compact group. For two locally compact G-spaces X and Y, they identify $\mathrm{KK}_*^G(X, Y)$ with the group of continuous natural transformations $\Phi_Z \colon \mathrm{K}_G^*(X \times Z) \to \mathrm{K}_G^*(Y \times Z)$ for all compact G-spaces Z; here continuity means that each Φ_Z is a $\mathrm{K}_G^*(Z)$-module homomorphism. The Kasparov product then becomes the composition of natural transformations. This reduces Kasparov's equivariant KK-theory to equivariant K-theory.

The theory $\widehat{\mathrm{KK}}{}^G$ does more: it contains the knowledge that all such natural transformations come from geometric correspondences, when geometric correspondences give the same natural transformation, and how to compose geometric correspondences. Thus we get a more concrete KK-theory.

Theorem 2.23. *Let T be a compact torus and let X be a smooth, K_T-oriented T-manifold with boundary. Let $e(\mathrm{T}X) \in \mathrm{K}_T^0(X)$ be the Euler class of X for the chosen K_T-orientation. Let $F \subseteq X$ be the fixed-point subset of the T-action on X and let $j \colon F \to X$ be the inclusion map. Then F is again a smooth K-oriented manifold with boundary, with trivial T-action, so that the inclusion map j is K_T-oriented. Let $e(\mathrm{T}F) \in \mathrm{K}^0(F) \subseteq \mathrm{K}_T^0(F)$ be the Euler class of F. The two geometric correspondences*

$$X \xleftarrow{\mathrm{id}_X} (X, e(\mathrm{T}X)) \xrightarrow{\mathrm{id}_X} X,$$
$$X \xleftarrow{j} (F, e(\mathrm{T}F)) \xrightarrow{j} X$$

represent the same element in $\widehat{\mathrm{KK}}{}_0^T(X, X)$.

This is a generalisation of [5, Lemma 3.1]. We allow $\mathrm{Spin^c}$-manifolds instead of complex manifolds. For a $\mathrm{Spin^c}$-structure coming from a complex structure, the Euler class is $[\Lambda^* \mathrm{T}^* X] \in \mathrm{K}_T^0(X)$, which appears in [5]. The following proof is a translation of the proof in [5] into the category $\widehat{\mathrm{KK}}{}^G$.

Proof. The first geometric correspondence above, involving the Euler class of X, is represented by the composition of geometric correspondences

$$X \xleftarrow[=]{\mathrm{id}_X} X \xrightarrow{\zeta} \mathrm{T}X \xleftarrow{\zeta} X \xrightarrow[=]{\mathrm{id}_X} X$$

by Example 2.13; here ζ denotes the zero section, which is K_T-oriented using the given K_T-orientation on the T-vector bundle $\mathrm{T}X$.

Choose a generic element ξ in the Lie algebra of T, that is, the one-parameter group $\exp(s\xi)$, $s \in \mathbb{R}$, is dense in T. Let $\alpha_t\colon X \to X$ denote the action of $t \in T$ on X. The action of T maps ξ to a vector field $\alpha_\xi\colon X \to \mathrm{T}X$. There is a homotopy of geometric correspondences

$$X \xleftarrow[=]{\mathrm{id}_X} X \xrightarrow{\alpha_{s\xi}} \mathrm{T}X \xleftarrow{\zeta} X \xrightarrow[=]{\mathrm{id}_X} X$$

for $s \in [0,1]$. For $t = 0$ we get the composition above, involving $e(\mathrm{T}X)$. We claim that for $s = 1$, the two correspondences intersect smoothly and that the intersection product is the second geometric correspondence in the theorem, involving F and its Euler class.

First we show that the fixed-point submanifold F is a closed submanifold. Equip X with a T-invariant Riemannian metric. Let $x \in F$, that is, $\alpha_t(x) = x$ for all $t \in T$. Split T_xX into

$$V = \{v \in \mathrm{T}_xX \mid D\alpha_t(x,v) = (x,v) \text{ for all } t \in T\}$$

and its orthogonal complement $V^\perp$. Since the metric is T-invariant,

$$\alpha_t(\exp(x,v)) = \exp(D\alpha_t(x,v))$$

for all $v \in \mathrm{T}_xX$. Since the exponential mapping restricts to a diffeomorphism between a neighbourhood of 0 in T_xX and a neighbourhood of x in X, we have $\exp(x,v) \in F$ if $v \in V$, and the converse holds for v in a suitable neighbourhood of 0. Thus we get a closed submanifold chart for F near x with $\mathrm{T}_xF = V$. Hence F is a closed submanifold with

$$\mathrm{T}F = \{(x,v) \in \mathrm{T}X \mid D\alpha_t(x,v) = (x,v) \text{ for all } t \in T\}.$$

Since ξ is generic, $\alpha_\xi(x) = 0$ in T_xX if and only if $x \in F$. Thus F is the coincidence space of the pair of maps $\zeta, \alpha_\xi\colon X \to \mathrm{T}X$. Let $x \in F$ and let $v_1, v_2 \in \mathrm{T}_xX$ satisfy $D\zeta(x,v_1) = D\alpha_\xi(x,v_2)$. Then $v_1 = v_2$ by taking the horizontal components; and the vertical component of $D\alpha_\xi(x,v_2)$ vanishes, which means that $D\alpha_{\exp(s\xi)}(x,v_2) = (x,v_2)$ for all $s \in \mathbb{R}$. Hence $v_2 \in \mathrm{T}_xF$. This proves that ζ and α_ξ intersect smoothly. The excess intersection bundle is the cokernel of $D\alpha_{\exp(s\xi)} - \mathrm{id}$; since the action of T is by isometries, $D\alpha_{\exp(s\xi)} - \mathrm{id}$ is normal in each fibre, so that its image and kernel are orthogonal complements. Hence the cokernel is canonically isomorphic to the kernel of $D\alpha_{\exp(s\xi)} - \mathrm{id}$. Thus the excess intersection bundle is canonically isomorphic to $\mathrm{T}F$.

Hence Theorem 2.17 gives the geometric correspondence $X \xleftarrow{j} (F, e(\mathrm{T}F)) \xrightarrow{j} X$ as the composition, as desired. ∎

3 The homological Lefschetz index of a Kasparov morphism

The example in Section 2.6.1 shows in what sense the geometric Lefschetz index computations in Section 2 generalise the local fixed-point formula for the Lefschetz index of a self-map. Now we turn to generalisations of the global homological formula for the Lefschetz index.

The classical Lefschetz fixed-point formula for a self-map $f\colon X \to X$ contains the (super)trace of the map on the cohomology of X with rational coefficients induced by f. We take rational coefficients in order to get vector spaces over a field, where there is a good notion of trace for endomorphisms. By the Chern character, we may as well take $\mathrm{K}^*(X) \otimes \mathbb{Q}$ instead of rational cohomology. It is checked in [9] that the Lefschetz index of $f \in \mathrm{KK}_0(A, A)$ for a dualisable C^*-algebra A in the bootstrap class is equal to the supertrace of the map on $\mathrm{K}_*(A) \otimes \mathbb{Q}$ induced by f.

We are going to generalise this result to the equivariant situation for a compact Lie group G. We assume that we are working with complex C^*-algebras, so that $\widehat{\mathrm{KK}}{}^G_*(\mathrm{pt}, \mathrm{pt}) = \mathrm{KK}^G_*(\mathbb{C}, \mathbb{C})$ vanishes in odd degrees and is the representation ring $\mathrm{R}(G)$ in even degrees. Our methods do not apply to the torsion invariants in $\mathrm{KK}^G_d(\mathbb{R}, \mathbb{R})$ for $d \neq 0$ in the real case because we (implicitly) tensor everything with $\mathbb{Q}$ to simplify the Lefschetz index.

Furthermore, we work in KK^G instead of $\widehat{\mathrm{KK}}{}^G$ in this section because the category KK^G is triangulated, unlike $\widehat{\mathrm{KK}}{}^G$. We explain in Remark 3.11 why $\widehat{\mathrm{KK}}{}^G$ is not triangulated; the triangulated structure on KK^G is introduced in [21].

Let $S \subseteq \mathrm{R}(G)$ be the set of all elements that are not zero divisors. This is a saturated, multiplicatively closed subset; even more, it is the largest multiplicatively closed subset for which the canonical map $\mathrm{R}(G) \to S^{-1}\mathrm{R}(G)$ to the ring of fractions is injective (see [1, Exercise 9 on p. 44]). The localisation $S^{-1}\mathrm{R}(G)$ is also called the *total ring of fractions* of $\mathrm{R}(G)$.

Since KK^G is symmetric monoidal with unit $\mathbb{1} = \mathbb{C}$ and $\mathrm{R}(G) = \mathrm{KK}^G_0(\mathbb{C}, \mathbb{C})$, the category KK^G is $\mathrm{R}(G)$-linear. Hence we may localise it at S as in [17]. The resulting category $\mathcal{T} := S^{-1}\mathrm{KK}^G$ has the same objects as KK^G and arrows

$$\mathcal{T}_*(A, B) := S^{-1}\mathrm{KK}^G_*(A, B) = S^{-1}\mathrm{R}(G) \otimes_{\mathrm{R}(G)} \mathrm{KK}^G_*(A, B).$$

The category $\mathcal{T}$ is $S^{-1}\mathrm{R}(G)$-linear. There is an obvious functor $\natural\colon \mathrm{KK}^G \to \mathcal{T}$. If A is a separable G-C^*-algebra, then

$$\mathcal{T}_*(\mathbb{C}, A) = S^{-1}\mathrm{KK}^G_*(\mathbb{C}, A) \cong S^{-1}\mathrm{R}(G) \otimes_{\mathrm{R}(G)} \mathrm{K}^G_*(A),$$

where we use the usual $\mathrm{R}(G)$-module structure on $\mathrm{K}^G_*(A) \cong \mathrm{KK}^G_*(\mathbb{C}, A)$.

There is a unique symmetric monoidal structure on $\mathcal{T}$ for which $\natural$ is a strict symmetric monoidal functor: simply extend the exterior tensor product on KK^G $S^{-1}\mathrm{R}(G)$-linearly. Hence if A is dualisable in KK^G, then its image in $\mathcal{T}$ is dualisable as well, and

$$\natural(\operatorname{tr} f) = \operatorname{tr}(\natural f) \qquad \text{for all } f \in \mathrm{KK}^G_*(A, A).$$

The crucial point for us is that $\natural\operatorname{tr}(f) = \operatorname{tr}(\natural f)$ uniquely determines $\operatorname{tr} f$ because the map

$$\mathrm{R}(G) \cong \mathrm{KK}_0^G(\mathbb{1},\mathbb{1}) \xrightarrow{\natural} \mathcal{T}_0(\mathbb{1},\mathbb{1}) \cong S^{-1}\,\mathrm{R}(G)$$

is injective. Thus it suffices to compute Lefschetz indices in $\mathcal{T}$. This may be easier because $\mathcal{T}$ has more isomorphisms and thus fewer isomorphism classes of objects. Furthermore, the endomorphism ring of the unit $\mathcal{T}_*(\mathbb{1},\mathbb{1}) = S^{-1}\,\mathrm{R}(G)$ has a rather simple structure:

Lemma 3.1. *The ring $S^{-1}\,\mathrm{R}(G)$ is a product of finitely many fields.*

Proof. Let $G/\operatorname{Ad} G$ be the space of conjugacy classes in G and let $\mathrm{C}(G/\operatorname{Ad} G)$ be the algebra of continuous functions on $G/\operatorname{Ad} G$. Taking characters provides a ring homomorphism $\chi\colon \mathrm{R}(G) \to \mathrm{C}(G/\operatorname{Ad} G)$, which is well-known to be injective. Hence $\mathrm{R}(G)$ is torsion-free as an Abelian group and has no nilpotent elements. Since G is a compact Lie group, $\mathrm{R}(G)$ is a finitely generated commutative ring by [28, Corollary 3.3]. Thus $\mathrm{R}(G)$ is Noetherian and reduced. This implies that its total ring of fractions is a finite product of fields (see [18, Exercise 6.5]). ☐

The fields in this product decomposition correspond bijectively to minimal prime ideals in $\mathrm{R}(G)$. By [28, Proposition 3.7.iii], these correspond bijectively to cyclic subgroups of G/G^0, where G^0 denotes the connected component of the identity element. In particular, $S^{-1}\,\mathrm{R}(G)$ is a field if and only if G is connected.

Example 3.2. Let G be a connected compact Lie group. Let T be a maximal torus in G and let W be the Weyl group, $W := N_G(T)/T$. Highest weight theory provides an isomorphism $\mathrm{R}(G) \cong \mathrm{R}(T)^W$. Here $\mathrm{R}(T)$ is a ring of integral Laurent polynomials in r variables, where r is the rank of T. Since elements of $\mathbb{N}_{\ge 1}$ are not zero divisors in $\mathrm{R}(G)$, the total ring of fractions of $\mathrm{R}(G)$ is equal to the total ring of fractions of $\mathrm{R}(G) \otimes \mathbb{Q}$. The latter is the $\mathbb{Q}$-algebra of W-invariant elements in $\mathbb{Q}[x_1, \dots, x_r, (x_1 \cdots x_r)^{-1}]$. This is the algebra of polynomial functions on the algebraic $\mathbb{Q}$-variety $(\mathbb{Q}^\times)^r$, and the W-invariants give the algebra of polynomials on the quotient variety $(\mathbb{Q}^\times)^r/W$. This variety is connected, so that the total ring of fractions $S^{-1}\,\mathrm{R}(G)$ in this case is the field of rational functions on the algebraic $\mathbb{Q}$-variety $(\mathbb{Q}^\times)^r/W$.

Now we can define an equivariant analogue of the trace of the map on $\mathrm{K}_*(A)\otimes\mathbb{Q}$ induced by $f \in \mathrm{KK}_0(A,A)$:

Definition 3.3. Let $S^{-1}\,\mathrm{R}(G) = \prod_{i=1}^n F_i$ with fields F_i. A module over $S^{-1}\,\mathrm{R}(G)$ is a product $\prod_{i=1}^n V_i$, where each V_i is an F_i-vector space. In particular, if A is a G-C*-algebra, then $\mathcal{T}_*(\mathbb{C},A) = S^{-1}\mathrm{K}_*^G(A) = \prod_{i=1}^n \mathrm{K}_{*,i}^G(A)$ for certain $\mathbb{Z}/2$-graded F_i-vector spaces $\mathrm{K}_{*,i}^G(A)$. An endomorphism $f \in \mathcal{T}_0(A,A)$ induces grading-preserving endomorphisms $\mathrm{K}_{*,i}^G(f)\colon \mathrm{K}_{*,i}^G(A) \to \mathrm{K}_{*,i}^G(A)$.

If the vector spaces $\mathrm{K}^G_{*,i}(A)$ are all finite-dimensional, then the (super)trace of $\mathrm{K}^G_{*,i}(f)$ is defined to be $\operatorname{tr}\mathrm{K}^G_{0,i}(f) - \operatorname{tr}\mathrm{K}^G_{1,i}(f) \in F_i$, and

$$\operatorname{tr} S^{-1}\mathrm{K}^G_*(f) := (\operatorname{tr}\mathrm{K}^G_{*,i}(f))_{i=1}^n \in \prod_{i=1}^n F_i = S^{-1}\,\mathrm{R}(G).$$

We will see below that dualisability for objects in appropriate bootstrap classes already implies that $\mathrm{K}^G_*(A)$ is a finitely generated $\mathrm{R}(G)$-module, and then each $\mathrm{K}^G_{*,i}(A)$ must be a finite-dimensional F_i-vector space.

Theorem 3.4. *Let A belong to the thick subcategory of KK^G generated by $\mathbb{C}$ and let $f \in \mathrm{KK}^G_0(A, A)$. Then A is dualisable in KK^G, so that $\operatorname{tr} f$ is defined, and*

$$\natural(\operatorname{tr} f) = \operatorname{tr} S^{-1}\mathrm{K}^G_*(f) \in S^{-1}\,\mathrm{R}(G).$$

Thick subcategories are defined in [26, Definition 2.1.6]. The thick subcategory generated by $\mathbb{C}$ is, of course, the smallest thick subcategory that contains the object $\mathbb{C}$. We denote the thick subcategory generated by a set A of objects or a single object by $\langle A\rangle$.

As we remarked above, $\natural(\operatorname{tr} f)$ uniquely determines $\operatorname{tr} f \in \mathrm{R}(G)$ because the canonical embedding $\natural\colon \mathrm{R}(G) \to S^{-1}\,\mathrm{R}(G)$ is injective.

We will prove Theorem 3.4 in Section 3.3.

How restrictive is the assumption that X should belong to the thick subcategory of KK^G generated by $\mathbb{C}$? The answer depends on the group G.

We consider the two extreme cases: *Hodgkin Lie groups* and finite groups.

A Hodgkin Lie group is, by definition, a connected Lie group with simply connected fundamental group; they are the groups to which the Universal Coefficient Theorem and the Künneth Theorem in [27] apply.

Theorem 3.5. *Let G be a compact connected Lie group with torsion-free fundamental group. Then a G-C^*-algebra A belongs to the thick subcategory generated by $\mathbb{C}$ if and only if*

- *A, without the G-action, belongs to the bootstrap category in* KK*, and*
- *A is dualisable.*

We postpone the proof of this theorem until after the proof of Proposition 3.13, which generalises part of this theorem to arbitrary compact Lie groups.

The first condition in Theorem 3.5 is automatic for commutative C^*-algebras because the non-equivariant bootstrap category is the class of all separable C^*-algebras that are KK-equivalent to a commutative separable C^*-algebra. Hence Theorem 3.5 verifies the assumptions needed for Theorem 3.4 if $A = \mathrm{C}_0(X)$ and $\mathrm{C}_0(X)$ is dualisable in KK^G; the latter is necessary for the Lefschetz index to be defined, anyway.

In particular, let X be a compact smooth G-manifold with boundary, for a Hodgkin Lie group G. Then X is dualisable in $\widehat{\mathrm{KK}}^G$ by Theorem 2.7, and hence $\mathrm{C}(X)$ is dualisable in KK^G because the functor $\widehat{\mathrm{KK}}^G \to \mathrm{KK}^G$ is symmetric monoidal. Furthermore, $\widehat{\mathrm{KK}}^G_*(X,X) \cong \mathrm{KK}^G_*(\mathrm{C}(X),\mathrm{C}(X))$ in this case, so that any endomorphism $f \in \mathrm{KK}^G_0(\mathrm{C}(X),\mathrm{C}(X))$ comes from some self-correspondence in $\widehat{\mathrm{KK}}^G_0(X,X)$. We get the following generalisation of the Lefschetz fixed-point formula:

Corollary 3.6. *Let G be a Hodgkin Lie group, X a smooth compact G-manifold, possibly with boundary, and $f \in \widehat{\mathrm{KK}}^G_0(X,X)$. Then $\mathrm{tr}(f) \in \mathrm{R}(G) \subseteq S^{-1}\mathrm{R}(G)$ is equal to the supertrace of $S^{-1}\mathrm{K}^*_G(f)$, acting on the $S^{-1}\mathrm{R}(G)$-vector space $S^{-1}\mathrm{K}^*_G(X)$.*

Notice that $S^{-1}\mathrm{R}(G)$ for a Hodgkin Lie group is a field, not just a product of fields.

In particular, Corollary 3.6 for the trivial group gives the Lefschetz index formula in [9].

Whereas Theorem 3.4 yields quite satisfactory results for Hodgkin Lie groups, its scope for a finite group G is quite limited:

Example 3.7. For $G = \mathbb{Z}/2$ there is a locally compact G-space X with $\mathrm{K}^*_G(X) = 0$ but $\mathrm{K}^*(X) \neq 0$. Equivalently, $\mathrm{KK}^G_*(\mathbb{C}, \mathrm{C}_0(X)) = 0$ and $\mathrm{KK}^G_*(\mathrm{C}(G), \mathrm{C}_0(X)) \neq 0$. This shows that $\mathrm{C}(G)$ does not belong to $\langle\mathbb{C}\rangle$.

Worse, the Lefschetz index formula in Theorem 3.4 is false for endomorphisms of $\mathrm{C}(G)$. We have $\widehat{\mathrm{KK}}^G_*(G,G) \cong \mathbb{Z}[G]$, spanned by the classes of the translation maps $G \to G$, $x \mapsto x\cdot g$, for $g \in G$, and these are homogeneous correspondences as in Section 2.6.6.

Translation by $g = 1$ is the identity map, and its Lefschetz index is the class of the regular representation of G in $\mathrm{R}(G)$. For $g \neq 1$, the Lefschetz index is zero because the fixed point subset is empty. However, $\mathrm{K}^*_G(G) = \mathrm{K}^*(\mathrm{pt}) = \mathbb{Z}[0]$ and all translation maps induce the identity map on $\mathrm{K}^*_G(G)$. Thus the induced map on $\mathrm{K}^*_G(G)$ is not enough information to compute the Lefschetz index of an endomorphism of G in $\widehat{\mathrm{KK}}^G$.

3.1 The equivariant bootstrap category

A reasonable Lefschetz index formula should apply at least to KK^G-endomorphisms of $\mathrm{C}(X)$ for all smooth compact G-manifolds and thus, in particular, for finite G-sets X. Example 3.7 shows that Theorem 3.4 fails on such a larger category. This leads us to improve the Lefschetz index formula. First we discuss the class of G-C^*-algebras where we expect it to hold.

We are going to describe an equivariant analogue of the bootstrap class in KK^G. Our class is larger than the class of C^*-algebras that are KK^G-equivalent to a commutative C^*-algebra. The latter subcategory is too small because it is not thick. The thick (or localising) subcategory of KK^G generated by commutative C^*-algebras is a better choice, but such a definition is not very intrinsic. We will choose an even larger subcategory of KK^G because it is not more difficult to treat and has a nicer characterisation.

The category KK^G only has countable coproducts because we need C^*-algebras to be separable. Hence the standard notions of compact objects and localising subcategories have to be modified so that they only involve countable coproducts. As in [7, Definition 2.1], we speak of *compact*$_{\aleph_1}$ objects, *localising*$_{\aleph_1}$ subcategories, and *compactly*$_{\aleph_1}$ generated subcategories.

Definition 3.8. Call a G-C^*-algebra A *elementary* if it is of the form $\mathrm{Ind}_H^G \mathbb{M}_n\mathbb{C} = \mathrm{C}(G, \mathbb{M}_n\mathbb{C})^H$ for some closed subgroup $H \subseteq G$ and some action of H on $\mathbb{M}_n\mathbb{C}$ by automorphisms; the superscript H means the fixed points for the diagonal action of H.

Definition 3.9. Let $\mathcal{B}^G \subseteq \mathrm{KK}^G$ be the localising$_{\aleph_1}$ subcategory generated by all elementary G-C^*-algebras. We call $\mathcal{B}^G$ the *G-equivariant bootstrap category*.

An action of H on $\mathbb{M}_n\mathbb{C}$ comes from a projective representation of H on $\mathbb{C}^n$. Such a projective representation is a representation of an extension of H by the circle group. The extension is classified by a cohomology class in $\mathrm{H}^2(H, \mathrm{U}(1))$. Two actions on $\mathbb{M}_n\mathbb{C}$ are H-equivariantly Morita equivalent if and only if they belong to the same class in $\mathrm{H}^2(H, \mathrm{U}(1))$. The G-C^*-algebras $\mathrm{Ind}_H^G \mathbb{M}_n\mathbb{C}$ for actions of H on $\mathbb{M}_n\mathbb{C}$ with different cohomology classes need not be KK^G-equivalent.

Theorem 3.10. *A G-C^*-algebra belongs to the localising$_{\aleph_1}$ subcategory generated by the elementary G-C^*-algebras if and only if it is KK^G-equivalent to a G-action on a type I C^*-algebra.*

Proof. It is already shown in [27, Theorem 2.8] that all G-actions on type I C^*-algebras belong to the localising$_{\aleph_1}$ subcategory generated by the elementary G-C^*-algebras. By definition, localising$_{\aleph_1}$ subcategories are closed under KK^G-equivalence. Elementary G-C^*-algebras are type I C^*-algebras, even continuous trace C^*-algebras. To finish the proof we must show that the G-C^*-algebras that are KK^G-equivalent to type I G-C^*-algebras form a localising$_{\aleph_1}$ subcategory of KK^G.

Let $\mathcal{T}_1 \subseteq \mathrm{KK}^G$ be the full subcategory of type I, separable G-C^*-algebras. If $A \in \mathcal{T}_1$, then $\mathrm{C}_0(\mathbb{R}, A) \in \mathcal{T}_1$, so that $\mathcal{T}_1$ is closed under suspension and desuspension. Let $A, B \in \mathcal{T}_1$ and $f \in \mathrm{KK}_0^G(A, B)$. We have $\mathrm{KK}_0^G(A, B) \cong \mathrm{KK}_1^G(A, \mathrm{C}_0(\mathbb{R}, B))$, and cycles for the latter group correspond to (equivariantly) semisplit extensions of G-C^*-algebras

$$\mathrm{C}_0(\mathbb{R}, B) \otimes \mathcal{K} \rightarrowtail D \twoheadrightarrow A$$

with $\mathcal{K} := \mathcal{K}(L^2(G \times \mathbb{N}))$. Since B and A are type I, so are $\mathrm{C}_0(\mathbb{R}, B) \otimes \mathcal{K}$ and D because the property of being type I is inherited by extensions. The semisplit extension above provides an exact triangle isomorphic to

$$B[-1] \to D \to A \xrightarrow{f} B.$$

Thus there is an exact triangle containing f with all three entries in $\mathcal{T}_1$. Furthermore, countable direct sums of type I C^*-algebras are again type I. This implies that the G-C^*-algebras KK^G-equivalent to one in $\mathcal{T}_1$ form a localising$_{\aleph_1}$ subcategory of KK^G. □

Remark 3.11. In the non-equivariant case, any C^*-algebra in the bootstrap class is KK-equivalent to a commutative one. This criterion fails already for $G = \mathrm{U}(1)$, as shown by a counterexample in [10]. Since the bootstrap class is the smallest localising subcategory containing $\mathbb{C}$, it follows that the commutative C^*-algebras do not form a localising subcategory. Thus $\widehat{\mathrm{KK}}^G$ is not triangulated: it lacks cones for some maps.
In this case, the equivariant bootstrap class is already generated by $\mathbb{C}$ and contains all $\mathrm{U}(1)$-actions on C^*-algebras in the non-equivariant bootstrap category. It is shown in [10] that the $\mathrm{U}(1)$-equivariant K-theory of a suitable Cuntz–Krieger algebra with its natural gauge action cannot arise from any $\mathrm{U}(1)$-action on a locally compact space.

Corollary 3.12. *The restriction and induction functors* $\mathrm{KK}^G \to \mathrm{KK}^H$ *and* $\mathrm{KK}^H \to \mathrm{KK}^G$ *for a closed subgroup H in a compact Lie group G restrict to functors between the bootstrap classes in* KK^G *and* KK^H.

Proof. Restriction does not change the underlying C^*-algebra and thus preserves the property of being type I. Induction maps elementary H-C^*-algebras to elementary G-C^*-algebras, is triangulated, and commutes with direct sums. Hence it maps $\mathcal{B}^H$ to $\mathcal{B}^G$. □

Proposition 3.13. *An object of $\mathcal{B}^G$ is compact$_{\aleph_1}$ if and only if it is dualisable, if and only if it belongs to the thick subcategory of $\mathcal{B}^G$ (or of KK^G) generated by the elementary G-C^*-algebras.*

Proof. The tensor unit $\mathbb{C}$ is compact$_{\aleph_1}$ because $\mathrm{KK}^G_*(\mathbb{C}, A) \cong \mathrm{K}^G_*(A) \cong \mathrm{K}_*(G \ltimes A)$ is countable for all G-C^*-algebras A, and the functors $A \mapsto G \ltimes A$ and K_* are well-known to commute with coproducts. Furthermore, the tensor product in KK^G commutes with coproducts in both variables.
Using this, we show that dualisable objects of $\mathcal{B}^G$ are compact$_{\aleph_1}$. If A is dualisable with dual A^*, then $\mathrm{KK}^G(A, B) \cong \mathrm{KK}^G(\mathbb{C}, A^* \otimes B)$, and since $\mathbb{C}$ is compact$_{\aleph_1}$ and $\otimes$ commutes with countable direct sums, it follows that A is compact$_{\aleph_1}$.
It follows from [8, Corollary 2.2] that elementary G-C^*-algebras are dualisable and hence compact$_{\aleph_1}$. A compact group has only at most countably many compact subgroups by Lemma 3.14 below; and any of them has at most finitely many projective representations. Hence the set of elementary G-C^*-algebras is at most countable. Therefore, $\mathcal{B}^G$ is compactly$_{\aleph_1}$ generated in the sense of [7, Definition 2.1]. By [7, Corollary 2.4] an object of $\mathcal{B}^G$ is compact$_{\aleph_1}$ if and only if it belongs to the thick subcategory generated by the elementary G-C^*-algebras.
The Brown Representability Theorem [7, Corollary 2.2] shows that for every compact$_{\aleph_1}$ object A of $\mathcal{B}^G$ there is a functor $\mathrm{Hom}(A, \sqcup)$ from $\mathcal{B}^G$ to $\mathcal{B}^G$ such

that
$$\mathrm{KK}^G(A \otimes B, D) \cong \mathrm{KK}^G(B, \mathrm{Hom}(A, D))$$
for all B, D in $\mathcal{B}^G$. Using exactness properties of the internal Hom functor in the first variable, we then show that the class of dualisable objects in $\mathcal{B}^G$ is thick (see [7, Section 2.3]). Thus all objects of the thick subcategory generated by the elementary G-C*-algebras are dualisable. □

The following lemma is well-known, see [25].

Lemma 3.14. *A compact Lie group has at most countably many conjugacy classes of closed subgroups.*

Proof. Let H be a closed subgroup of a compact Lie group G. By the Mostow Embedding Theorem, G/H embeds into a linear representation of G, that is, H is a stabiliser of a point in some linear representation of G. Up to isomorphism, there are only countably many linear representations of G. Each linear representation has finite orbit type, that is, it admits only finitely many different conjugacy classes of stabilisers. Hence there are altogether at most countably many conjugacy classes of closed subgroups in G. □

Proof of Theorem 3.5. Let G be a Hodgkin Lie group. The main result of [23] says that A belongs to the localising subcategory of KK^G generated by $\mathbb{C}$ if and only if $A \rtimes G$ belongs to the non-equivariant bootstrap category (this is special for Hodgkin Lie groups). Since this covers all elementary G-C*-algebras, we conclude that the localising subcategory generated by $\mathbb{C}$ contains $\mathcal{B}^G$ and is, therefore, equal to $\mathcal{B}^G$.
The same argument as in the proof of Proposition 3.13 shows that the following are equivalent for an object A of $\mathcal{B}^G$:

- A is dualisable;
- A is compact$_{\aleph_1}$;
- A belongs to the thick subcategory generated by $\mathbb{C}$.

This finishes the proof of Theorem 3.5. □

So far we always used the bootstrap class, which is the domain where a Universal Coefficient Theorem holds. The next proposition is a side remark showing that we may also use the domain where a Künneth formula holds.

Definition 3.15. An object $A \in \mathrm{KK}^G$ satisfies the Künneth formula if $\mathrm{K}_*^G(A\otimes B) = 0$ for all B that satisfy $\mathrm{K}_*^G(C \otimes B) = 0$ for all elementary G-C*-algebras C.

By results of [20, 24], the assumption in Definition 3.15 is necessary and sufficient for a certain natural spectral sequence that computes $\mathrm{K}_*^G(A \otimes B)$ from $\mathrm{KK}_*^G(C, A)$ and $\mathrm{KK}_*^G(C, B)$ for elementary C to converge for all B; we have no need to describe this spectral sequence.

Proposition 3.16. *Let $A \in \mathrm{KK}^G$ be dualisable with dual A^*. If A or A^* satisfies a Künneth formula, then both A and A^* belong to $\mathcal{B}^G$, and vice versa.*

Proof. Since $\mathcal{B}^G$ is generated by the elementary G-C*-algebras, $\mathrm{KK}^G_*(C,B)=0$ for all elementary G-C*-algebras C if and only if $\mathrm{KK}^G_*(C,B)=0$ for all $C \in \mathcal{B}^G$. Any elementary G-C*-algebra C is dualisable with a dual in $\mathcal{B}^G$. Hence $\mathrm{K}^G_*(C\otimes B) \cong \mathrm{KK}^G_*(C^*,B) = 0$ for elementary C if $\mathrm{KK}^G_*(C',B)=0$ for all elementary G-C*-algebras C'; conversely $\mathrm{KK}^G_*(C,B) \cong \mathrm{K}^G_*(C^*\otimes B)=0$ for elementary C if $\mathrm{K}^G_*(C'\otimes B)=0$ for all elementary G-C*-algebras C'. Let us denote the class of G-C*-algebras with these equivalent properties $\mathcal{B}^{G,\perp}$.
It follows from [20, Theorem 3.16] that $(\mathcal{B}^G, \mathcal{B}^{G,\perp})$ is a complementary pair of localising subcategories. In particular, if $\mathrm{KK}^G_*(A,B)=0$ for all $B \in \mathcal{B}^{G,\perp}$, then $A \in \mathcal{B}^G$.
Now assume, say, that A satisfies a Künneth formula. Then $\mathrm{KK}^G_*(A^*,B) \cong \mathrm{K}^G_*(A\otimes B)=0$ for all $B\in\mathcal{B}^{G,\perp}$. Thus $A^*\in\mathcal{B}^G$. Then $\mathrm{K}^G_*(A^*\otimes B)=0$ for all $B \in \mathcal{B}^{G,\perp}$ because the class of C with $\mathrm{K}^G_*(C\otimes B)=0$ is localising and contains all elementary C if $B\in\mathcal{B}^{G,\perp}$. As above, this implies $(A^*)^* = A \in \mathcal{B}^G$. ☐

The proof of Theorem 3.5 above used that, for a Hodgkin Lie group, $\mathcal{B}^G$ is already generated by $\mathbb{C}$. For more general groups, we also expect that fewer generators suffice to generate $\mathcal{B}^G$. But we only need and only prove a result about topologically cyclic groups here.
A locally compact group G is called *topologically cyclic* if there is an element $g \in G$ that generates a dense subgroup of G. A topologically cyclic group is necessarily Abelian. We are interested in topologically cyclic, compact Lie groups here. A compact Lie group is topologically cyclic if and only if it is isomorphic to $\mathbb{T}^r \times F$ for some $r \geq 0$ and some finite cyclic group F (possibly the trivial group), where $\mathbb{T} = \mathbb{R}/\mathbb{Z} \cong \mathrm{U}(1)$. Here we use that any extension $\mathbb{T}^r \rightarrowtail E \twoheadrightarrow F$ for a finite cyclic group F splits. This also implies that any projective representation of a finite cyclic groups is a representation.

Theorem 3.17. *Let G be a topologically cyclic, compact Lie group. Then the bootstrap class $\mathcal{B}^G \subseteq \mathrm{KK}^G$ is already generated by the finitely many G-C*-algebras $\mathrm{C}(G/H)$ for all open subgroups $H \subseteq G$.*
Furthermore, an object of $\mathcal{B}^G$ is compact$_{\aleph_1}$ if and only if it is dualisable if and only if it belongs to the thick subcategory generated by $\mathrm{C}(G/H)$ for open subgroups $H \subseteq G$.

Proof. The second statement about compact$_{\aleph_1}$ objects in $\mathcal{B}^G$ follows from the first one and [7, Corollary 2.4], compare the proof of Proposition 3.13. Thus it suffices to prove that the objects $\mathrm{C}(G/H)$ for open subgroups already generate $\mathcal{B}^G$. For this, we use an isomorphism $G \cong \mathbb{T}^r \times F$ for some $r \geq 0$ and some finite cyclic subgroup F.
Let us first consider the special case $r = 0$, that is, G is a finite cyclic group. In this case, any subgroup of G is open and again cyclic. We observed above that cyclic groups have no non-trivial projective representations. Thus any elementary G-C*-algebra is Morita equivalent to $\mathrm{C}(G/H)$ for some open subgroup H

in G. Hence the assertion of the theorem is just the definition of $\mathcal{B}^G$ in this case.

If F is trivial, then the assertion follows from Theorem 3.5. Now we consider the general case where both F and $\mathbb{T}^r$ are non-trivial.

The Pontryagin dual $\hat{G}$ of G is isomorphic to the discrete group $\mathbb{Z}^r \times F$. If A is a G-C*-algebra, then $G \ltimes A$ carries a canonical action of $\hat{G}$ called the dual action. Similarly, $\hat{G} \ltimes A$ for a $\hat{G}$-C*-algebra A carries a canonical dual action of G. This provides functors $\mathrm{KK}^G \to \mathrm{KK}^{\hat{G}}$ and $\mathrm{KK}^{\hat{G}} \to \mathrm{KK}^G$. Baaj–Skandalis duality says that they are inverse to each other up to natural equivalence (see [2, Section 6]). Since both functors are triangulated, this is an equivalence of triangulated categories.

If A is type I, then so is $G \ltimes A$. Hence all objects in $\mathcal{B}^{\hat{G}} \subseteq \mathrm{KK}^{\hat{G}}$ are $\mathrm{KK}^{\hat{G}}$-equivalent to a $\hat{G}$-action on a type I C*-algebra by Theorem 3.10.

The group $\hat{G}$ is Abelian and hence satisfies a very strong form of the Baum–Connes conjecture: it has a dual Dirac morphism and $\gamma = 1$ in the sense of [22, Definition 8.1]. From this it follows that any $\hat{G}$-C*-algebra A belongs to the localising subcategory of $\mathrm{KK}^{\hat{G}}$ that is generated by $\mathrm{Ind}_{\hat{H}}^{\hat{G}} A$ for finite subgroups $\hat{H} \subseteq \hat{G}$ (this is shown as in the proof of [22, Theorem 9.3]).

The finite subgroups in $\mathbb{Z}^r \times \hat{F}$ are exactly the subgroups of $\hat{F}$, of course. Since we have induction in stages, we may assume $\hat{H} = \hat{F}$. Thus the subcategory of type I $\hat{G}$-C*-algebras is already generated by $\mathrm{Ind}_{\hat{F}}^{\hat{G}} A$ for type I $\hat{F}$-C*-algebras A. Since $\hat{F}$ is a finite cyclic group, the discussion above shows that the category of type I $\hat{F}$-C*-algebras A is already generated by $\mathrm{C}_0(\hat{F}/\hat{H})$ for subgroups $\hat{H} \subseteq \hat{F}$. Thus $\mathcal{B}^{\hat{G}}$ is generated by the $\hat{G}$-C*-algebras $\mathrm{Ind}_{\hat{F}}^{\hat{G}} \mathrm{C}_0(\hat{F}/\hat{H}) \cong \mathrm{C}_0(\hat{G}/\hat{H})$. The finite subgroups $\hat{H} \subseteq \hat{G}$ are exactly the orthogonal complements of (finite-index) open subgroups $H \subseteq G$.

Now $G \ltimes \mathrm{C}_0(G/H)$ is Morita equivalent to $\mathrm{C}^*(H) \cong \mathrm{C}_0(\hat{G}/\hat{H})$ for any open subgroup $H \subseteq G$, where $\hat{H} \subseteq \hat{G}$ denotes the orthogonal complement of H in $\hat{G}$. The dual action on $\mathrm{C}_0(\hat{G}/\hat{H})$ comes from the translation action of $\hat{G}$. Thus the G- and $\hat{G}$-C*-algebras $\mathrm{C}_0(G/H)$ and $\mathrm{C}_0(\hat{G}/\hat{H})$ correspond to each other via Baaj–Skandalis duality. We conclude that the G-C*-algebras $\mathrm{C}_0(G/H)$ for open subgroups $H \subseteq G$ generate $\mathcal{B}^G$. □

Let G be topologically cyclic, say, $G \cong \mathbb{T}^r \times \mathbb{Z}/k$ for some $r \geq 0$, $k \geq 1$. Then open subgroups of G correspond to subgroups of $\mathbb{Z}/k$ and thus to divisors d of k. The representation ring of G is

$$\mathrm{R}(G) \cong \mathrm{R}(\mathbb{T}^r) \otimes \mathrm{R}(\mathbb{Z}/k) \cong \mathbb{Z}[x_1, \dots, x_r, (x_1 \cdots x_r)^{-1}] \otimes \mathbb{Z}[t]/(t^k - 1). \tag{3.1}$$

Let

$$t^k - 1 = \prod_{d \mid k} \Phi_d(t)$$

be the decomposition into cyclotomic polynomials. Each factor Φ_d generates a minimal prime ideal of $\mathrm{R}(G)$, and these are all minimal prime ideals of $\mathrm{R}(G)$.

The localisation at (Φ_d) gives the field $\mathbb{Q}(\theta_d)(x_1, \dots, x_r)$ of rational functions in r variables over the cyclotomic field $\mathbb{Q}(\theta_d)$, and the product of these localisations is the total ring of fractions of $\mathrm{R}(G)$,

$$S^{-1}\,\mathrm{R}(G) = \prod_{d|k} \mathbb{Q}(\theta_d)(x_1, \dots, x_r).$$

(Compare Lemma 3.1.)

Lemma 3.18. *Let $H \subsetneq G$ be a proper open subgroup. The canonical map*

$$\mathrm{R}(G) \to \mathrm{KK}_0^G(\mathrm{C}(G/H), \mathrm{C}(G/H))$$

from the exterior product in KK^G factors through the restriction map $\mathrm{R}(G) \to \mathrm{R}(H)$. The image of $\mathrm{C}(G/H)$ in the localisation of KK^G at the prime ideal (Φ_k) vanishes.

Proof. The exterior product of the identity map on $\mathrm{C}(G/H)$ and $\xi \in \mathrm{R}(G) \cong \mathrm{KK}_0^G(\mathbb{C}, \mathbb{C})$ is given by the geometric correspondence $G/H = G/H = G/H$ with the class $p^*(\xi) \in \mathrm{K}_G^0(G/H)$, where $p\colon G/H \to \mathrm{pt}$ is the constant map. Now identify $\mathrm{K}_G^0(G/H) \cong \mathrm{K}_H^0(\mathrm{pt}) \cong \mathrm{R}(H)$ and p^* with the restricton map $\mathrm{R}(G) \to \mathrm{R}(H)$ to get the first statement.
We have $H \cong \mathbb{T}^r \times \mathbb{Z}/d$ embedded via $(x, j) \mapsto (x, jk/d)$ into $G \cong \mathbb{T}^r \times \mathbb{Z}/k$. If $H \neq G$, then $d \neq k$. The restriction map $\mathrm{R}(G) \to \mathrm{R}(H)$ annihilates the polynomial $(t^k - 1)/\Phi_k = \prod_{d|k, d\neq k} \Phi_d$. This polynomial does not belong to the prime ideal (Φ_k) and hence becomes invertible in the localisation of $\mathrm{R}(G)$ at (Φ_k). Since an invertible endomorphism can only be zero on the zero object, $\mathrm{C}(G/H)$ becomes zero in the localisation of KK^G at (Φ_k). □

3.2 Localisation of the bootstrap class

Proposition 3.19. *Let $G \cong \mathbb{T}^r \times \mathbb{Z}/k$ be topologically cyclic. Let $\mathcal{B}_d^G$ be the thick subcategory of dualisable objects in the bootstrap class $\mathcal{B}^G \subseteq \mathrm{KK}^G$. Any object in the localisation of $\mathcal{B}_d^G$ at the prime ideal (Φ_k) in $\mathrm{R}(G)$ is isomorphic to a finite direct sum of suspensions of $\mathbb{C}$.*

Proof. By Theorem 3.17 an object of $\mathcal{B}^G$ is dualisable if and only if it belongs to the thick subcategory generated by $\mathrm{C}(G/H)$ for open subgroups $H \subseteq G$. Lemma 3.18 shows that all of them except $\mathbb{C} = \mathrm{C}(G/G)$ become zero when we localise at (Φ_k). Hence the image of $\mathcal{B}_d^G$ in the localisation is contained in the thick subcategory generated by $\mathbb{C}$. We must show that the objects isomorphic to a direct sum of suspensions of $\mathbb{C}$ already form a thick subcategory in the localisation of KK^G at (Φ_k).
The graded endomorphism ring of $\mathbb{C}$ in this localisation is

$$\mathrm{KK}_*^G(\mathbb{C}, \mathbb{C}) \otimes_{\mathrm{R}(G)} \mathrm{R}(G)_{(\Phi_k)} \cong \mathbb{Q}(\theta_k)(x_1, \dots, x_r)[\beta, \beta^{-1}]$$

with β of degree two generating Bott periodicity. It is crucial that $\mathrm{KK}^G_*(\mathbb{C},\mathbb{C}) \cong F[\beta,\beta^{-1}]$ for a field $F := \mathbb{Q}(\theta_k)(x_1,\dots,x_r)$. The following argument only uses this fact.

We map a finite direct sum $A = \bigoplus_{i\in I}\mathbb{C}[\varepsilon_i]$ of suspensions of $\mathbb{C}$ to the $\mathbb{Z}/2$-graded F-vector spaces $V(A)$ with basis I and generators of degree ε_i. For two such direct sums, $\mathrm{KK}^G_0(A,B)$ is isomorphic to the space of grading-preserving F-linear maps $V(A)\to V(B)$ because this clearly holds for a single summand.

Now let $f\in \mathrm{KK}^G_0(A,B)$ and consider the associated linear map $V(f)\colon V(A)\to V(B)$. Choose a basis for the kernel of $V(f)$ of homogeneous elements and extend it to a homogeneous basis for $V(A)$, and extend the resulting basis for the image of $V(f)$ to a homogeneous basis of $V(B)$. This provides isomorphisms $V(A)\cong V_0\oplus V_1$, $V(B)\cong W_1\oplus W_2$ such that $f|_{V_0}=0$, $f(V_1)=W_1$ and $f|_{V_1}\colon V_1\to W_1$ is an isomorphism. The chosen bases describe how to lift the $\mathbb{Z}/2$-graded vector spaces V_i and W_i to direct sums of suspensions of $\mathbb{C}$. Thus the map f is equivalent to a direct sum of three maps $f_0\oplus f_1\oplus f_2$ with $f_0\colon A_0\to 0$ mapping to the zero object, f_1 invertible, and $f_2\colon 0\to B_2$ with domain the zero object. The mapping cone of f_0 is the suspension of A_0, the cone of f_2 is B_2, and the cone of f_1 is zero. Hence the cone is again a direct sum of suspensions of $\mathbb{C}$. Furthermore, any idempotent endomorphism has a range object.

Thus the direct sums of suspensions of $\mathbb{C}$ already form an idempotent complete triangulated category. As a consequence, any object in the thick subcategory generated by $\mathbb{C}$ is isomorphic to a direct sum of copies of $\mathbb{C}$. ☐

Proposition 3.20. *Let G be a Hodgkin Lie group. Let $\mathcal{B}^G_d$ be the thick subcategory of dualisable objects in the bootstrap class $\mathcal{B}^G\subseteq \mathrm{KK}^G$. Any object in the localisation of $\mathcal{B}^G_d$ at S is isomorphic to a finite direct sum of suspensions of $\mathbb{C}$.*

Proof. Theorem 3.5 shows that $\mathcal{B}^G_d$ is the thick subcategory of KK^G generated by $\mathbb{C}$. The localisation $F := S^{-1}\mathrm{R}(G)$ is a field because G is connected, and the graded endomorphism ring of $\mathbb{C}$ in the localisation of KK^G at S is $F[\beta,\beta^{-1}]$ with β the generator of Bott periodicity. Now the argument is finished as in the proof of Proposition 3.19. ☐

Remark 3.21. The localisations above use the groups $\mathrm{KK}^G(A,B)\otimes_{\mathrm{R}(G)} S^{-1}\mathrm{R}(G)$ for some multiplicatively closed subset $S\subseteq \mathrm{End}(\mathbb{1}) = R(G)$, following [17]. A drawback of this localisation is that the canonical functor $\mathrm{KK}^G\to S^{-1}\mathrm{KK}^G$ does not commute with (countable) coproducts. This is why Propositions 3.19 and 3.20 are formulated only for $\mathcal{B}^G_d$ and not for all of $\mathcal{B}^G$.

Another way to localise $\mathcal{B}^G$ at S is described in [7, Theorem 2.33]. Both localisations agree on $\mathcal{B}^G_d$ by [7, Theorem 2.33.h]. The construction in [7] has the advantage that the canonical functor from KK^G to this localisation commutes with $\mathrm{small}_{\aleph_1}$ (that is, countable) coproducts. Hence analogues of Propositions

3.19 and 3.20 hold for the whole bootstrap category $\mathcal{B}^G$, with small$_{\aleph_1}$ coproducts of suspensions of $\mathbb{C}$ instead of finite direct sums of suspensions of $\mathbb{C}$.

3.3 The Lefschetz index computation using localisation

Now we have all the tools available to formulate and prove a Lefschetz index formula for general compact Lie groups. We first prove Theorem 3.4, which deals with endomorphisms of objects in the thick subcategory generated by $\mathbb{C}$. Then we formulate and prove the general Lefschetz index formula.

Proof of Theorem 3.4. Since A belongs to the thick subcategory generated by $\mathbb{C}$, it is dualisable in KK^G by Proposition 3.13. Hence $\mathrm{tr}(f) \in \mathrm{R}(G)$ is defined for $f \in \mathrm{KK}_0^G(A, A)$.
The image of $\mathrm{tr}(f)$ in $S^{-1}\,\mathrm{R}(G)$ is the Lefschetz index of the image of f in the localisation of KK^G at S. The localisation $S^{-1}\,\mathrm{R}(G)$ is a product of fields. It is more convenient to compute each component separately. This means that we localise at larger multiplicatively closed subsets $\bar{S}$ such that $\bar{S}^{-1}\,\mathrm{R}(G)$ is one of the factors of $S^{-1}\,\mathrm{R}(G)$. In this localisation, the endomorphisms of $\mathbb{C}$ form a field again, not a product of fields. If our trace formula holds for all these localisations, it also holds for $S^{-1}\,\mathrm{R}(G)$.
Since the endomorphisms of $\mathbb{C}$ form a field, the same argument as in the proof of Proposition 3.19 show that, in this localisation, A is isomorphic to a finite sum of copies of suspensions of $\mathbb{C}$. Write $A \cong \bigoplus_{i=1}^n A_i$ with $A_i \cong \mathbb{C}[\varepsilon_i]$ in $S^{-1}\mathrm{KK}^G$ for some $\varepsilon_i \in \mathbb{Z}/2$. Then f becomes a matrix (f_{ij}) with $f_{ij} \in S^{-1}\mathrm{KK}_0^G(A_j, A_i)$. The dual of $A_i \cong \mathbb{C}[\varepsilon_i]$ is $A_i^* \cong \mathbb{C}[\varepsilon_i] \cong A_i$, and the unit and counit of adjunction $\mathbb{C} \leftrightarrows \mathbb{C}[\varepsilon_i] \otimes \mathbb{C}[\varepsilon_i]$ are the canonical isomorphism and its inverse with sign $(-1)^{\varepsilon_i}$, respectively; the sign is necessary because the exterior product is *graded* commutative. Hence the dual of A is isomorphic to A, with unit and counit

$$\mathbb{C} \leftrightarrows A \otimes A \cong \bigoplus_{i,j=1}^{n} A_i \otimes A_j$$

the sum of the canonical isomorphisms $\mathbb{C} \leftrightarrows A_i \oplus A_i$, up to signs, and the zero maps $\mathbb{C} \leftrightarrows A_i \oplus A_j$ for $i \neq j$. Thus the Lefschetz index of f is the sum $\sum_{i=1}^n (-1)^{\varepsilon_i} f_{ii}[\varepsilon_i]$ as an element in $S^{-1}\mathrm{KK}_0^G(\mathbb{C}, \mathbb{C})$. This is exactly the supertrace of f acting on $S^{-1}\mathrm{K}_*^G(A) \cong \bigoplus_{i=1}^n S^{-1}\,\mathrm{R}(G)[\varepsilon_i]$. ☐

Let G be a general compact Lie group. Let C_G denote the set of conjugacy classes of Cartan subgroups of G in the sense of [28, Definition 1.1]. Such subgroups correspond bijectively to conjugacy classes of cyclic subgroups in the finite group G/G^0, where G^0 denotes the connected component of the identity element in G. Thus C_G is a non-empty, finite set, and it has a single element if and only if G is connected.
The support of a prime ideal $\mathfrak{p}$ in $\mathrm{R}(G)$ is defind in [28] as the smallest subgroup H such that $\mathfrak{p}$ comes from a prime ideal in $\mathrm{R}(H)$ via the restriction map $\mathrm{R}(G) \to \mathrm{R}(H)$. Given any Cartan subgroup H, there is a unique minimal

prime ideal with support H, and this gives a bijection between C_G and the set of minimal prime ideals in $\mathrm{R}(G)$ (see [28, Proposition 3.7]).

More precisely, if $H \subseteq G$ is a Cartan subgroup, then H is topologically cyclic and hence $H \cong \mathbb{T}^r \times \mathbb{Z}/k$ for some $r \geq 0$, $k \geq 1$. We described a prime ideal (Φ_k) in $\mathrm{R}(H)$ before Lemma 3.18, and its preimage in $\mathrm{R}(G)$ is a minimal prime ideal $\mathfrak{p}_H$ in $\mathrm{R}(G)$.

The total ring of fractions $S^{-1}\,\mathrm{R}(G)$ is a product of fields by Lemma 3.1. We can make this more explicit:

$$S^{-1}\,\mathrm{R}(G) \cong \prod_{H \in C_G} F(\mathrm{R}(G)/\mathfrak{p}_H),$$

where $F(\sqcup)$ denotes the field of fractions for an integral domain.

Definition 3.22. Let A be dualisable in $\mathcal{B}^G \subseteq \mathrm{KK}^G$, let $\varphi \in \mathrm{KK}_0^G(A,A)$, and let $H \in C_G$. Let $F := F(\mathrm{R}(G)/\mathfrak{p}_H)$ and let $\mathrm{K}_H(A) := \mathrm{K}_*^H(A) \otimes_{\mathrm{R}(H)} F$, considered as a $\mathbb{Z}/2$-graded F-vector space. Let $\mathrm{K}_H(\varphi)$ be the grading-preserving F-linear endomorphism of $\mathrm{K}_H(A)$ induced by φ.

Theorem 3.23. *Let A be dualisable in $\mathcal{B}^G \subseteq \mathrm{KK}^G$, let $\varphi \in \mathrm{KK}_0^G(A,A)$, and let $H \in C_G$. Then the image of $\mathrm{tr}(\varphi)$ in $F(\mathrm{R}(G)/\mathfrak{p}_H)$ is the supertrace of $\mathrm{K}_H(\varphi)$.*

Proof. The map $\mathrm{R}(G) \to F(\mathrm{R}(G)/\mathfrak{p}_H)$ factors through the restriction homomorphism $\mathrm{R}(G) \to \mathrm{R}(H)$ because $\mathfrak{p}_H$ is supported in H. Restricting the group action to H maps the bootstrap category in KK^G into the bootstrap category in KK^H by Corollary 3.12, and commutes with taking Lefschetz indices because restriction is a tensor functor. Hence we may replace G by H and take $\varphi \in \mathrm{KK}_0^H(A,A)$ throughout.

Since H is topologically cyclic, Proposition 3.19 applies. It shows that in the localisation of KK^H at $\mathfrak{p}_H$, any dualisable object in $\mathcal{B}^G$ becomes isomorphic to a finite direct sum of suspensions of $\mathbb{C}$. Now the argument continues as in the proof of Theorem 3.4 above. ☐

4 Hattori–Stallings traces

Before we found the above approach through localisation, we developed a different trace formula where, in the case of a Hodgkin Lie group, the trace is identified with the Hattori–Stallings trace of the $\mathrm{R}(G)$-module map $\mathrm{K}_*^G(f)$ on $\mathrm{K}_*^G(A)$. We briefly sketch this alternative formula here, although the localisation approach above seems much more useful for computations. The Hattori–Stallings trace has the advantage that it obviously belongs to $\mathrm{R}(G)$.

We work in the general setting of a tensor triangulated category $(\mathcal{T}, \otimes, \mathbb{1})$. We assume that $\mathcal{T}$ satisfies additivity of traces, that is:

Assumption 4.1. *Let $A \to B \to C \to A[1]$ be an exact triangle in $\mathcal{T}$ and assume that A and B are dualisable. Assume also that the left square in the* *following diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f_A"] & B \arrow[r] \arrow[d, "f_B"] & C \arrow[r] \arrow[d, dotted, "f_C"] & A[1] \arrow[d, "f_A[1]"] \\ A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1] \end{tikzcd}$$

commutes. Then C is dualisable and there is an arrow $f_C\colon C \to C$ such that the whole diagram commutes and $\mathrm{tr}(f_C) - \mathrm{tr}(f_B) + \mathrm{tr}(f_A) = 0$.

Additivity of traces holds in the bootstrap category $\mathcal{B}^G \subseteq \mathrm{KK}^G$. The quickest way to check this is the localisation formula for the trace in Theorem 3.23. It shows that $\mathcal{B}^G$ satisfies even more: $\mathrm{tr}(f_C) - \mathrm{tr}(f_B) + \mathrm{tr}(f_A) = 0$ holds for *any* arrow f_C that makes the diagram commute.

There are several more direct ways to verify additivity of traces, but all require significant work which we do not want to get into here. The axioms worked out by J. Peter May in [19] are lengthy and therefore rather unpleasant to check by hand. In a previous manuscript we embedded the localising subcategory of KK^G generated by $\mathbb{C}$ into a category of module spectra. Since additivity is known for categories of module spectra, this implies the required additivity result at least for this smaller subcategory. Another way would be to show that additivity of traces follows from the derivator axioms and to embed KK^G into a triangulated derivator.

In the following, we will just assume additivity of traces and use it to compute the trace. Let

$$R := \mathcal{T}_*(\mathbb{1}, \mathbb{1}) = \bigoplus_{n\in\mathbb{Z}} \mathcal{T}_n(\mathbb{1}, \mathbb{1})$$

be the graded endomorphism ring of the tensor unit. It is graded-commutative provided $\mathcal{T}$ satisfies some very basic compatibility axioms; see [29] for details. If A is any object of $\mathcal{T}$, then $M(A) := \mathcal{T}_*(\mathbb{1}, A) = \bigoplus_{n\in\mathbb{Z}} \mathcal{T}_n(\mathbb{1}, A)$ is an R-module in a canonical way, and an endomorphism $f \in \mathcal{T}_n(A, A)$ yields a degree-n endomorphism $M(f)$ of $M(A)$. We will prove in Theorem 4.2 below that, under some assumptions, the trace of f equals the Hattori–Stallings trace of $M(f)$ and, in particular, depends only on $M(f)$.

Before we can state our theorem, we must define the Hattori–Stallings trace for endomorphisms of graded modules over graded rings. This is well-known for ungraded rings (see [3]). The grading causes some notational overhead. Let R be a (unital) graded-commutative graded ring. A finitely generated free R-module is a direct sum of copies of $R[n]$, where $R[n]$ denotes R with degree shifted by n, that is $R[n]_i = R_{n+i}$. Let $F\colon P \to P$ be a module endomorphism of such a free module, let us assume that F is homogeneous of degree d. We use an isomorphism

$$P \cong \bigoplus_{i=1}^{r} R[n_i] \tag{4.1}$$

to rewrite F as a matrix $(f_{ij})_{1\le i,j\le r}$, where $f_{ij}\colon R[n_j]\to R[n_i]$ are R-module homomorphisms of degree d. The entry f_{ij} is given by right multiplication by some element of R of degree $n_i - n_j + d$. The (super)trace $\operatorname{tr} F$ is defined as

$$\operatorname{tr} F := \sum_{i=1}^{r} (-1)^{n_i} \operatorname{tr} f_{ii};$$

this is an element of R of degree d.

It is straightforward to check that $\operatorname{tr} F$ is well-defined, that is, independent of the choice of the isomorphism in (4.1). Here we use that the degree-zero part of R is central in R (otherwise, we still get a well-defined element in the commutator quotient $R_d/[R_d, R_0]$). Furthermore, if we shift the grading on P by n, then the trace is multiplied by the sign $(-1)^n$ – it is a supertrace.

If P is a finitely generated projective graded R-module, then $P\oplus Q$ is finitely generated and free for some Q, and for an endomorphism F of P we let

$$\operatorname{tr} F := \operatorname{tr}(F\oplus 0\colon P\oplus Q\to P\oplus Q).$$

This does not depend on the choice of Q.

A *finite projective resolution* of a graded R-module M is a resolution

$$\cdots \to P_\ell \xrightarrow{d_\ell} P_{\ell-1} \xrightarrow{d_{\ell-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \tag{4.2}$$

of finite length by finitely generated projective graded R-modules P_j. *We assume that the maps d_j have degree one* (or at least odd degree). Assume that M has such a resolution and let $f\colon M\to M$ be a module homomorphism. Lift f to a chain map $f_j\colon P_j\to P_j$, $j=0,\dots,\ell$. We define the *Hattori–Stallings trace* of f as

$$\operatorname{tr}(f) = \sum_{j=0}^{\ell} \operatorname{tr}(f_j).$$

It may be shown that this trace does not depend on the choice of resolution. It is important for this that we choose d_j of degree one. Since shifting the degree by one alters the sign of the trace of an endomorphism, the sum in the definition of the trace becomes an *alternating* sum when we change conventions to have even-degree boundary maps d_j. Still the trace changes sign when we shift the degree of M.

Theorem 4.2. *Let $F\in\mathcal{T}(A,A)$ be an endomorphism of some object A of $\mathcal{T}$. Assume that A belongs to the localising subcategory of $\mathcal{T}$ generated by $\mathbb{1}$. If the graded R-module $M(A) := \mathcal{T}_*(\mathbb{1}, A)$ has a finite projective resolution, then A is dualisable in $\mathcal{T}$ and the trace of F is equal to the Hattori–Stallings trace of the induced module endomorphism $\mathcal{T}_*(\mathbb{1}, f)$ of $M(A)$.*

Proof. Our main tool is the phantom tower over A, which is constructed in [20]. We recall some details of this construction.

Let $M^\perp$ be the functor from finitely generated projective R-modules to $\mathcal{T}$ defined by the adjointness property $\mathcal{T}(M^\perp(P), B) \cong \mathcal{T}(P, M(B))$ for all $B \in \mathcal{T}$. The functor $M^\perp$ maps the free rank-one module R to $\mathbb{1}$, is additive, and commutes with suspensions; this determines $M^\perp$ on objects. Since $R = \mathcal{T}_*(\mathbb{1}, \mathbb{1})$, $\mathcal{T}_*(M^\perp(P_1), M^\perp(P_2))$ is isomorphic (as a graded Abelian group) to the space of R-module homomorphisms $P_1 \to P_2$. Furthermore, we have canonical isomorphisms $M\bigl(M^\perp(P)\bigr) \cong P$ for all finitely generated projective R-modules P. By assumption, $M(A)$ has a finite projective resolution as in (4.2). Using $M^\perp$, we lift it to a chain complex in $\mathcal{T}$, with entries $\hat{P}_j := M^\perp(P_j)$ and boundary maps $\hat{d}_j := M^\perp(d_j)$ for $j \geq 1$. The map $\hat{d}_0 \colon \hat{P}_0 \to A$ is the pre-image of d_0 under the adjointness isomorphism $\mathcal{T}(M^\perp(P), B) \cong \mathcal{T}(P, M(B))$. We get back the resolution of modules by applying M to the chain complex $(\hat{P}_j, \hat{d}_j)$.

Next, it is shown in [20] that we may embed this chain complex into a diagram

$$\begin{tikzcd}
A = N_0 \arrow[rr, "\iota_0^1"] & & N_1 \arrow[rr, "\iota_1^2"] \arrow[dl, rightsquigarrow, "\varepsilon_0"'] & & N_2 \arrow[rr, "\iota_2^3"] \arrow[dl, rightsquigarrow, "\varepsilon_1"'] & & N_3 \arrow[rr] \arrow[dl, rightsquigarrow, "\varepsilon_2"'] & & \cdots \arrow[dl, rightsquigarrow] & \\
& \hat{P}_0 \arrow[ul, "\hat{d}_0 = \pi_0"] & & \hat{P}_1 \arrow[ul, "\pi_1"'] \arrow[ll, rightsquigarrow, "\hat{d}_1"] & & \hat{P}_2 \arrow[ul, "\pi_2"'] \arrow[ll, rightsquigarrow, "\hat{d}_2"] & & \hat{P}_3 \arrow[ul, "\pi_3"'] \arrow[ll, rightsquigarrow, "\hat{d}_3"] & & \cdots \arrow[ll, rightsquigarrow]
\end{tikzcd} \tag{4.3}$$

where the wriggly lines are maps of degree one; the triangles involving $\hat{d}_j$ commute; and the other triangles are exact. This diagram is called the *phantom tower* in [20].

Since $\hat{P}_j = 0$ for $j > \ell$, the maps ι_j^{j+1} are invertible for $j > \ell$. Furthermore, a crucial property of the phantom tower is that these maps ι_j^{j+1} are *phantom maps*, that is, they induce the zero map on $\mathcal{T}_*(\mathbb{1}, \sqcup)$. Together, these facts imply that $M(N_j) = 0$ for $j > \ell$. Since we assumed $\mathbb{1}$ to be a generator of $\mathcal{T}$, this further implies $N_j = 0$ for $j > \ell$. Therefore, $A \in \langle \mathbb{1} \rangle$, so that A is dualisable as claimed.

Next we recursively extend the endomorphism F of $A = N_0$ to an endomorphism of the phantom tower. We start with $F_0 = F \colon N_0 \to N_0$. Assume $F_j \colon N_j \to N_j$ has been constructed. As in [20], we may then lift F_j to a map $\hat{F}_j \colon \hat{P}_j \to \hat{P}_j$ such that the square

$$\begin{tikzcd}
\hat{P}_j \arrow[r, "\pi_j"] \arrow[d, "\hat{F}_j"] & N_j \arrow[d, "F_j"] \\
\hat{P}_j \arrow[r, "\pi_j"] & N_j
\end{tikzcd}$$

commutes. Now we apply additivity of traces (Assumption 4.1) to construct an endomorphism $F_{j+1} \colon N_{j+1} \to N_{j+1}$ such that $(\hat{F}_j, F_j, F_{j+1})$ is a triangle morphism and $\operatorname{tr}(F_j) = \operatorname{tr}(\hat{F}_j) + \operatorname{tr}(F_{j+1})$. Then we repeat the recursion step with F_{j+1} and thus construct a sequence of maps F_j. We get

$$\operatorname{tr}(F) = \operatorname{tr}(F_0) = \operatorname{tr}(\hat{F}_0) + \operatorname{tr}(F_1) = \cdots = \operatorname{tr}(\hat{F}_0) + \cdots + \operatorname{tr}(\hat{F}_\ell) + \operatorname{tr}(F_{\ell+1}).$$

Since $N_{\ell+1} = 0$, we may leave out the last term.
Finally, it remains to observe that the trace of $\hat{F}_j$ as an endomorphism of $\hat{P}_j$ agrees with the trace of the induced map on the projective module P_j. Since both traces are additive with respect to direct sums of maps, the case of general finitely generated projective modules reduces first to free modules and then to free modules of rank one. Both traces change by a sign if we suspend or desuspend once, hence we reduce to the case of endomorphisms of $\mathbb{1}$, which is trivial. Hence the computation above does indeed yield the Hattori–Stallings trace of $M(A)$ as asserted. □

Remark 4.3. Note that if a module has a finite projective resolution, then it must be finitely generated. Conversely, if the graded ring R is *coherent* and *regular*, then any finitely generated module has a finite projective resolution. (Regular means that every finitely generated module has a finite *length* projective resolution; coherent means that every finitely generated homogeneous ideal is finitely presented – for instance, this holds if R is (graded) Noetherian; coherence implies that any finitely generated graded module has a resolution by finitely generated projectives.)
Moreover, if R is coherent then the finitely presented R-modules form an abelian category, and this implies (by an easy induction on the triangular length of A) that for every $A \in \langle \mathbb{1} \rangle = (\langle \mathbb{1} \rangle_{\mathrm{loc}})_d$ the module $M(A)$ is finitely presented and thus *a fortiori* finitely generated. If R is also regular, each such $M(A)$ has a finite projective resolution.
In conclusion: if R is regular and coherent, an object $A \in \langle \mathbb{1} \rangle_{\mathrm{loc}}$ is dualisable if and only if the graded R-module $M(A)$ has a finite projective resolution.

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An Application of Hermitian K-Theory: Sums-of-Squares Formulas

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Abstract. By using Hermitian K-theory, we improve D. Dugger and D. Isaksen's condition (some powers of 2 dividing some binomial coefficients) for the existence of sums-of-squares formulas.

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1 Introduction

A sums-of-squares formula of type $[r, s, n]$ over a field F of characteristic $\neq 2$ (with strictly positive integers r, s and n) is a formula

$$\Big(\sum_{i=1}^{r} x_i^2\Big) \cdot \Big(\sum_{i=1}^{s} y_i^2\Big) = \Big(\sum_{i=1}^{n} z_i^2\Big) \in F[x_1, \ldots, x_r, y_1, \ldots, y_s] \tag{1}$$

where $z_i = z_i(X, Y)$ for each $i \in \{1, \ldots, n\}$ is a bilinear form in X and Y (with coefficients in F), i.e. $z_i \in F[x_1, \ldots, x_r, y_1, \ldots, y_s]$ is homogeneous of degree 2 and F-linear in X and Y. Here, $X = (x_1, \ldots, x_r)$ and $Y = (y_1, \ldots, y_s)$ are coordinate systems. To be specific, $z_i = \sum_{k,j} c_{kj}^{(i)} x_k y_j$ for $c_{kj}^{(i)} \in F$. An old problem of Adolf Hurwitz concerns the existence of sums-of-squares formulas. Historical remarks can be found in [18] and [20]. For any $m \in \mathbb{Z}_{>0}$, we let $\varphi(m)$ denote the cardinality of the set $\{l \in \mathbb{Z} : 0 < l \leq m \text{ and } l \equiv 0, 1, 2 \text{ or } 4 \pmod 8\}$. The aim of this paper is to introduce the following result.

Theorem 1.1. *If a sums-of-squares formula of type* $[r, s, n]$ *exists over a field* F *of characteristic* $\neq 2$*, then* $2^{\varphi(s-1)-i+1}$ *divides* $\binom{n}{i}$ *for* $n - r < i \leq \varphi(s-1)$.

The proof of Theorem 1.1 over $\mathbb{R}$ was provided by [2] and [21]. It involves computations of topological KO-theory of real projective spaces and γ^i-operations. The statement of Theorem 1.1 over $\mathbb{R}$ can be extended to any field of characteristic 0 by an algebraic remark of T.Y. Lam and K.Y. Lam, cf. Theorem 3.3 [18]. By using algebraic K-theory, D. Dugger and D. Isaksen prove a similar result over an arbitrary field of characteristic $\neq 2$, where $\varphi(s-1)$ in the above theorem is replaced by $\lfloor \frac{s-1}{2} \rfloor$, cf. Theorem 1.1 [7]. They actually conjectured the above statement. Since $\varphi(s-1) \geq \lfloor \frac{s-1}{2} \rfloor$, our main theorem generalizes theirs. One may wish to look at the following table.

n	1	2	3	4	5	6	7	8	9	$\cdots$
$\varphi(n)$	1	2	2	3	3	3	3	4	5	$\cdots$
$\lfloor \frac{n}{2} \rfloor$	0	1	1	2	2	3	3	4	4	$\cdots$

Example 1.1. Consider the triplet $[15, 10, 16]$ which does not exist over F by the above theorem. Neither Hopf's condition [8] nor the weaker condition in [7] can give the non-existence of $[15, 10, 16]$.

Remark 1.1. The necessary condition of our main theorem does not imply the existence of $[r, s, n]$. To illustrate, $[3, 5, 5]$ does not exist over the field F by the Hurwitz-Radon theorem. However, it satisfies the necessary condition.

Remark 1.2. The algebraic K-theory analog (cf. Theorem 1.1 [7]) of our main theorem works even if the assumption 'if a sums-of-squares formula of type $[r, s, n]$ exists over F' is replaced by 'if a nonsingular bilinear map of size $[r, s, n]$ exists over F'. The statement with the latter assumption is 'stronger'. However, this is not the case under our proof, since we will use the sums-of-squares identity (1).

Remark 1.3. The triplet $[r, s, n]$ is independent of the base fields whenever $r \leq 4$ and whenever $s \geq n-2$ (cf. Corollary 14.21 [20]), so that the main theorem is true. There is a bold conjecture which states that the existence of $[r, s, n]$ is independent of the base field F (of characteristic $\neq 2$), cf. Conjecture 3.8 [18] or Conjecture 14.22 [20]. Our main theorem and Dugger-Isaksen's Hopf condition (cf. [8]) suggest this conjecture to some extent. However, as Shapiro points out in Chapter 14 [20], there is indeed very little evidence to support this conjecture.

In [22], it is shown that the Grothendieck-Witt group of a complex cellular variety is isomorphic to the KO-theory of its set of $\mathbb{C}$-rational points with analytic topology. The set of $\mathbb{C}$-rational points of a deleted quadric is homotopy equivalent to the real projective space of the same dimension, cf. Lemma 6.3 [15]. Moreover, the computation of topological KO-theory of a real projective space is well-known, cf. Theorem 7.4 [1]. We therefore have motivations to work on the Grothendieck-Witt group of a deleted quadric and on the γ^i-operations. The proof of our main theorem requires the computation of Grothendieck-Witt group of a deleted quadric which will be explored in Section 3.

2 Terminology, notation and remark

Let $(\mathcal{E}, *, \eta)$ be a $\mathbb{Z}[\frac{1}{2}]$-linear exact category with duality. For $i \in \mathbb{Z}$, Walter's Grothendieck-Witt groups $GW^i(\mathcal{E}, *, \eta)$ are defined in Section 4.3 [16]. The triplet $(\mathrm{Vect}(X), \mathcal{H}om(\ \ , \mathcal{L}), \mathrm{can})$ (notation in Example 2.3 [16]) is an exact category with duality. If X is any $\mathbb{Z}[\frac{1}{2}]$-scheme, then we define

$$GW^i(X, \mathcal{L}) := GW^i(\mathrm{Vect}(X), \mathcal{H}om(\ \ , \mathcal{L}), \mathrm{can}).$$

By the symbols $GW^i(X)$, we mean the groups $GW^i(X, \mathcal{O})$. Note that $GW^0(X)$ is just Knebusch's $L(X)$ which is defined in [14]. The notation in [3] is used for the Witt theory. For KO-theory and comparison maps, we refer to [22].

Definition 2.1. Let T be a scheme. For us, a smooth T-variety X is called *T-cellular* if it has a filtration by closed subvarieties

$$X = Z_0 \supset Z_1 \supset \cdots \supset Z_N = \emptyset$$

such that $Z_{k-1} - Z_k \cong \mathbb{A}_T^{n_k}$ for each k.

In this paper, the following notations are introduced for convenience:

F	—	a field of characteristic $\neq 2$;
K	—	an algebraically closed field of characteristic $\neq 2$;
V	—	the ring of Witt vectors over K;
L	—	the field of fractions of V;
X_F	—	the base-change scheme $X \times_{\mathbb{Z}[\frac{1}{2}]} F$ for any $\mathbb{Z}[\frac{1}{2}]$-scheme X;
S	—	the polynomial ring $F[y_1, \ldots, y_s]$;
$\mathbb{P}^{s-1}$	—	the scheme Proj $\mathbb{Z}[\frac{1}{2}][y_1, \ldots, y_s]$;
q_s	—	the quadratic polynomial $q_s(y) = y_1^2 + \ldots + y_s^2$;
$V_+(q_s)$	—	the closed subscheme of $\mathbb{P}^{s-1}$ defined by q_s;
$D_+(q_s)$	—	the open subscheme $\mathbb{P}^{s-1} - V_+(q_s)$ of $\mathbb{P}^{s-1}$;
ξ	—	the line bundle $\mathcal{O}(-1)$ of $\mathbb{P}_F^{s-1}$ restricted to $D_+(q_s)_F$;
R	—	the ring of elements of total degrees 0 in S_{q_s};
P	—	the R-module of elements of total degrees -1 in S_{q_s};
Q_n	—	the $\mathbb{Z}[\frac{1}{2}]$-scheme defined by $\sum_{i=0}^{n/2} x_i y_i = 0$ in $\mathbb{P}^{n+1}$, if $n > 0$ is even; $\sum_{i=0}^{(n-1)/2} x_i y_i + c^2 = 0$ in $\mathbb{P}^{n+1}$, if $n > 0$ is odd;
DQ_{n+1}	—	the open subscheme $\mathbb{P}^{n+1} - Q_n$ of $\mathbb{P}^{n+1}$.

Remark 2.1. (i) Let E be a field containing $\sqrt{-1}$ and of characteristic $\neq 2$. Note that $(Q_{s-2})_E$ is isomorphic to the projective variety $V_+(q_s)_E$, cf. Lemma 2.2 [8]. This map induces an isomorphism $i_E : (DQ_{s-1})_E \to D_+(q_s)_E$.
(ii) Observe that V is a complete DVR with the quotient field K, cf. Chapter II [17]. Also, note that the fraction field L of V has characteristic 0, cf. *loc. cit.*.
(iii) The scheme $D_+(q_s)_F$ is affine over the base field F, since $D_+(q_s)_F$ and Spec R are isomorphic, cf. the proof of Proposition 2.2 [7].

3 Proof of Theorem 1.1

Lemma 3.1. *If a sums-of-squares formula of type $[r, s, n]$ exists over F, then there exist a non-degenerate bilinear form $\sigma : \xi \times \xi \to \mathcal{O}$ on $D_+(q_s)_F$ and a bilinear space ζ on $D_+(q_s)_F$ of rank $n - r$ such that*

$$r[\xi, \sigma] + [\zeta] = n \in GW^0(D_+(q_s)_F)$$

where n is the trivial bilinear space of the rank n.

Proof. The K-theory analog has been proved, cf. Proposition 2.2 [7]. It is clear that the group $GW^0(D_+(q_s)_F)$ is isomorphic to $GW_0(R)$ by Remark 2.1 (iii). If the equation (1) exits, we are able to construct a graded S-module homomorphism $(S(-1))^r \to S^n$ by $f = (f_1, \dots, f_r) \mapsto (z_1(f, Y), \dots, z_n(f, Y))$ where $Y = (y_1, \dots, y_s)$ is the coordinate system introduced in Section 1. This map induces a homomorphism $\alpha : P^r \to R^n$ of R-modules by localizing it at q_s.

The isomorphism $P \otimes_R P \to R, f \otimes g \mapsto (fg) \cdot q_s$ gives a non-degenerate bilinear form $\sigma : P \times P \to R$. Let $\langle -, - \rangle_{R^n}$ be the unit bilinear form over R^n. Let $f = (f_1, \dots, f_r), g = (g_1, \dots, g_r) \in P^r$. We claim that $\langle \alpha(f), \alpha(g) \rangle_{R^n}$ equals $\sum_{i=1}^r \sigma(f_i, g_i)$. It is enough to show that $\langle \alpha(f), \alpha(f) \rangle_{R^n} = \sum_{i=1}^r \sigma(f_i, f_i)$. Note that $\langle \alpha(f), \alpha(f) \rangle_{R^n} = z_1(f, Y)^2 + \dots + z_n(f, Y)^2$. By the existence of the triplet $[r, s, n]$, we obtain $z_1(f, Y)^2 + \dots + z_n(f, Y)^2 = (f_1^2 + \dots + f_r^2) q_s = \sum_{i=1}^r \sigma(f_i, f_i)$. Note that $(P^r, \sum_{i=1}^r \sigma)$ is non-degenerate. It follows that α is injective and $(P^r, \sum_{i=1}^r \sigma)$ can be viewed as a non-degenerate subspace of $(R^n, \langle -, - \rangle_{R^n})$ via α. Define ζ to be its orthogonal complement $(P^r)^\perp$ with the unit form $\langle -, - \rangle_{R_n}$ restricting to $(P^r)^\perp$. By a basic fact of quadratic form theory, ζ is non-degenerate and $\zeta \perp (P^r, \sum_{i=1}^r \sigma) \cong (R^n, \langle -, - \rangle_{R^n})$. □

Theorem 3.1. *Let ν denote the element $[\xi, \sigma] - 1$ in the ring $GW^0(D_+(q_s)_K)$. Then, the ring $GW^0(D_+(q_s)_K)$ is isomorphic to*

$$\mathbb{Z}[\nu]/(\nu^2 + 2\nu, 2^{\varphi(s-1)}\nu)$$

where $\varphi(s-1)$ is the number defined in Section 1. Therefore, for any rational point $\varsigma : \mathrm{Spec}\ K \to D_+(q_s)_K$, the reduced Grothendieck-Witt ring

$$\widetilde{GW}^0(D_+(q_s)_K) := \ker \left(\varsigma^* : GW^0(D_+(q_s)_K) \to GW^0(\mathrm{Spec}\ K) \cong \mathbb{Z} \right)$$

is isomorphic to $\mathbb{Z}/2^{\varphi(s-1)}$.

Theorem 3.1 will be proved in the next section.

Proof of Theorem 1.1. It is enough to show this theorem over the algebraic closure $\bar{F}$ of F. Indeed, if $[r, s, n]$ exists over F, then it also exists over $\bar{F}$. In order to apply the standard trick (cf. the proof of Theorem 1.3 [7]), we have to take care of γ^i-operations on $GW^0(D_+(q_s)_{\bar{F}})$. To be specific, this standard trick can not be applied without the list of three properties (cf. Properties (i)-(iii) in *loc. cit.*) of γ^i-operations and their generating power series $\gamma_t = 1+\sum_{i>0}\gamma^i t^i$ on $GW^0(D_+(q_s)_{\bar{F}})$. Due to the lack of reference, we will develop γ^i-operations on $K(\mathrm{Bil}(X))$ and prove these three properties (see Appendix A). It is enough for our purpose because $GW^0(X)$ is just $K(\mathrm{Bil}(X))$ if X is affine (see Remark A.1), and the scheme $D_+(q_s)_{\bar{F}}$ is affine by Remark 2.1 (iii). Hence, together with Lemma 3.1, we are allowed to apply the standard trick. One checks that details are the same as in the proof of Theorem 1.3 [7] by replacing K-theory analogs with GW-theory and $\lfloor \frac{s-1}{2} \rfloor$ with $\varphi(s-1)$. Combining with a reformulation of powers of 2 dividing correspondent binomial coefficients (cf. Section 1.2 [7]), we are done. $\square$

4 Proof of Theorem 3.1

4.1 Rigidity and Hermitian K-theory of cellular varieties

By Remark 2.1 (ii), there is always an inclusion map $\overline{\mathbb{Q}} \to \overline{L}$ where $\overline{\mathbb{Q}}$ (resp. $\overline{L}$) is the algebraic closure of $\mathbb{Q}$ (resp. $\overline{L}$). Consider the following diagram (2).

$$\begin{array}{ccccc} K & \longleftarrow & V & \longrightarrow & \overline{L} \\ & & & & \uparrow \\ \mathbb{C} & \longleftarrow & \overline{\mathbb{Q}} & & \end{array} \qquad \begin{array}{ccccc} W^i(K) & \xleftarrow[\cong]{\beta^i} & W^i(V) & \xrightarrow[\cong]{\alpha^i} & W^i(\overline{L}) \\ & & & & \cong\uparrow \chi^i \\ & & W^i(\mathbb{C}) & \xleftarrow[\eta^i]{\cong} & W^i(\overline{\mathbb{Q}}) \end{array} \tag{2}$$

On the right-hand side of the diagram (2), the maps of Witt groups are all induced by the correspondent ring maps of the left-hand side for a fixed $i \in \mathbb{Z}$. All these Witt groups are trivial if $i \not\equiv 0 \pmod 4$, cf. Theorem 5.6 [5]. Note that β^0 is an isomorphism by Satz 3.3 [13]. It is also clear that $W^0(K)$ is isomorphic to $\mathbb{Z}/2$ and that all the maps on the right-hand side of the diagram (2) preserve multiplicative identities for $i = 0$. Since Witt groups are four periodicity in shifting, we obtain

Lemma 4.1. *The map $\eta^i \circ (\chi^i)^{-1} \circ \alpha^i \circ (\beta^i)^{-1}$ yields an isomorphism from $W^i(K)$ to $W^i(\mathbb{C})$. Moreover, by Karoubi induction (cf. Section 3 [6]), the left-hand side of the digram (2) gives an isomorphism $GW^i(K) \to GW^i(\mathbb{C})$ of Grothendieck-Witt groups.* $\square$

Lemma 4.2. *Let X be a smooth $\mathbb{Z}[\frac{1}{2}]$-cellular variety. Let $f : A \to B$ be a map of regular local rings of finite Krull dimensions with $1/2$. Suppose that the map $W^i(A) \to W^i(B)$ induced by f is an isomorphism for each i, then f gives an isomorphism of Witt groups (resp. Grothendieck-Witt groups)*

$$W^i(X_A, \mathcal{L}_A) \to W^i(X_B, \mathcal{L}_B) \text{ (resp. } GW^i(X_A, \mathcal{L}_A) \to GW^i(X_B, \mathcal{L}_B))$$

for each i and any line bundle $\mathcal{L}$ over X.

Proof. We may use $W^i(X,\mathcal{L})_*$ to simplify the notation $W^i(X_*,\mathcal{L}_*)$. We wish to prove the Witt theory case by induction on cells. Firstly, note that the pullback maps $W^i(A)\to W^i(\mathbb{A}^n_A)$ and $W^i(B)\to W^i(\mathbb{A}^n_B)$ are isomorphisms by homotopy invariance, cf. Theorem 3.1 [4]. It follows that

$$W^i(\mathbb{A}^n_A)\cong W^i(\mathbb{A}^n_B).$$

Let $X=Z_0\supset Z_1\supset\cdots\supset Z_N=\emptyset$ be the filtration such that

$$Z_{k-1}-Z_k\cong\mathbb{A}^{n_k}=:C_k.$$

In general, the closed subvarieties Z_k may not be smooth. However, let U_k be the open subvariety $X-Z_k$ for each $0\le k\le N$. Every U_k is smooth in X. There is another filtration $X=U_N\supset U_{N-1}\supset\cdots\supset U_0=\emptyset$ with $U_k-U_{k-1}=Z_{k-1}-Z_k\cong C_k$ closed in U_k of codimension d_k. Consider the following commutative diagram of localization sequences.

$$\begin{array}{ccccccccc}
W^{i-1}(U_{k-1})_A & \longrightarrow & W^i_{C_k}(U_k,\mathcal{L})_A & \longrightarrow & W^i(U_k,\mathcal{L})_A & \longrightarrow & W^i(U_{k-1})_A & \longrightarrow & W^{i+1}_{C_k}(U_k,\mathcal{L})_A\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
W^{i-1}(U_{k-1})_B & \longrightarrow & W^i_{C_k}(U_k,\mathcal{L})_B & \longrightarrow & W^i(U_k,\mathcal{L})_B & \longrightarrow & W^i(U_{k-1})_B & \longrightarrow & W^{i+1}_{C_k}(U_k,\mathcal{L})_B
\end{array}$$

Here, $W^i_{C_k}(U_k,\mathcal{L})$ means the $\mathcal{L}$-twisted ith-Witt group of U_k with supports on C_k. Note that any line bundle over $(C_k)_A$ is trivial, since

$$\mathrm{Pic}(\mathbb{A}^n_A)\cong\mathrm{Pic}(A)=0\ (A\text{ is regular local and so it is a UFD}).$$

By the dévissage theorem (cf. [10]), we deduce that

$$W^i_{C_k}(U_k,\mathcal{L})_A\cong W^i_{C_k}(U_k,\mathcal{L})_B\text{ for all }i.$$

Moreover, by induction hypothesis,

$$W^i(U_{k-1})_A\cong W^i(U_{k-1})_B\text{ for all }i.$$

Applying the 5-lemma, one sees that the middle vertical map is an isomorphism. Since the K-theory analog of this theorem is also true by induction on cells, the GW-theory cases follow by Karoubi induction, cf. Section 3 [6]. □

COROLLARY 4.1. *The Witt group (resp. the Grothendieck-Witt group)*

$$W^i(X,\mathcal{L})_K\ \textit{(resp. } GW^i(X,\mathcal{L})_K)$$

is isomorphic to

$$W^i(X,\mathcal{L})_{\mathbb{C}}\ \textit{(resp. } GW^i(X,\mathcal{L})_{\mathbb{C}})$$

for each i and any line bundle $\mathcal{L}$ over X.

4.2 Comparison maps and rank one bilinear spaces

If X is a smooth variety over $\mathbb{C}$, we let $X(\mathbb{C})$ be the set of $\mathbb{C}$-rational points of X with analytic topology. One can define comparison maps (cf. Section 2 [22])

$$\begin{array}{rccl} k^0: & K_0(X) & \to & K^0(X(\mathbb{C})) \\ gw^0: & GW^0(X) & \to & KO^0(X(\mathbb{C})) \\ w^0: & W^0(X) & \to & \frac{KO}{K}^0(X(\mathbb{C})) \end{array} \tag{3}$$

where $\frac{KO}{K}^0(X(\mathbb{C}))$ means the cokernel of the realification map from $K^0(X)$ to $KO^0(X(\mathbb{C}))$. Let $GW^0_{\mathrm{top}}(X(\mathbb{C}))$ be the Grothendieck-Witt group of complex bilinear spaces over $X(\mathbb{C})$. The map gw^0 consists of the composition of the following two maps

$$f: GW^0(X) \to GW^0_{\mathrm{top}}(X(\mathbb{C})) \qquad g: GW^0_{\mathrm{top}}(X(\mathbb{C})) \to KO^0(X(\mathbb{C}))$$

where the map f takes a class $[M,\phi]$ on X to the class $[M(\mathbb{C}),\phi(\mathbb{C})]$ on $X(\mathbb{C})$. The map g sends a class $[N,\epsilon]$ on $X(\mathbb{C})$ to the class represented by the underlying real vector bundle $\mathfrak{R}(N,\epsilon)$ such that $\mathfrak{R}(N,\epsilon)\otimes_{\mathbb{R}}\mathbb{C} = N$ and that $\epsilon|_{\mathfrak{R}(N,\epsilon)}$ is real and positive definite, cf. Lemma 1.3 [22]. Let $Q(X)$ (resp. $Q_{\mathrm{top}}(X)$) denote the group of isometry (resp. isomorphism) classes of rank one bilinear spaces (resp. rank one complex bilinear spaces) over X (resp. $X(\mathbb{C})$) with the group law defined by the tensor product. There are maps of sets

$$Q(X) \to GW^0(X), [\mathcal{L},\phi] \mapsto [\mathcal{L},\phi] \qquad Q_{\mathrm{top}}(X(\mathbb{C})) \to GW^0_{\mathrm{top}}(X(\mathbb{C})), [L,\epsilon] \mapsto [L,\epsilon].$$

Let $\mathrm{Pic}_{\mathbb{R}}(X(\mathbb{C}))$ be the group of isomorphism classes of rank one real vector bundles over $X(\mathbb{C})$.

Lemma 4.3. *The following diagram is commutative*

$$\begin{array}{ccccc} GW^0(X) & \xrightarrow{f} & GW^0_{\mathrm{top}}(X(\mathbb{C})) & \xrightarrow{g} & KO^0(X(\mathbb{C})) \\ \uparrow & & u\uparrow & & v\uparrow \\ Q(X) & \xrightarrow{\tilde{f}} & Q_{\mathrm{top}}(X(\mathbb{C})) & \xrightarrow{\tilde{g}} & \mathrm{Pic}_{\mathbb{R}}(X(\mathbb{C})) \end{array}$$

where $\tilde{f}([\mathcal{L},\phi])$ (resp. $\tilde{g}([L,\epsilon])$) is defined as $[\mathcal{L}(\mathbb{C}),\phi(\mathbb{C})]$ (resp. $[\mathfrak{R}(L,\epsilon)]$).

Proof. The square on the left-hand side is obviously commutative. It remains to show that the right-hand side square is commutative. Check that the map $\tilde{g}$ is well-defined. Note that, for each couple of complex bilinear spaces (L',ϵ') and (L,ϵ) on $X(\mathbb{C})$, if $\mathfrak{R}(L',\epsilon')$ is isomorphic to $\mathfrak{R}(L,\epsilon)$, then (L',ϵ') is isometric to (L,ϵ). Besides, the map $\tilde{g}$ has image in $\mathrm{Pic}_{\mathbb{R}}(X(\mathbb{C}))$. To see this, suppose $\tilde{g}([L,\epsilon]) = [\mathfrak{R}(L,\epsilon)]$ is not in $\mathrm{Pic}_{\mathbb{R}}(X(\mathbb{C}))$ for some $[L,\epsilon] \in Q_{\mathrm{top}}(X(\mathbb{C}))$. It follows that $X(\mathbb{C})$ has a point with an open neighborhood U such that $\mathfrak{R}(L,\epsilon)|_U$ is isomorphic to $U\times\mathbb{R}^n$ with $n\neq 1$. Then, $L|_U$ is isomorphic to $U\times\mathbb{C}^n$ $(n\neq 1)$, since $\mathfrak{R}(L,\epsilon)\otimes_{\mathbb{R}}\mathbb{C}\cong L$. This contradicts the assumption that the bundle L has rank one. Then, it is clear that $g\circ u = v\circ\tilde{g}$. $\square$

4.3 Comparison maps and cellular varieties

Let $\mathcal{H}(\mathbb{C})$ (resp. $\mathcal{SH}(\mathbb{C})$) be the unstable $\mathbb{A}^1$-homotopy category (resp. the stable $\mathbb{A}^1$-homotopy category) over $\mathbb{C}$. Let $\mathcal{H}_\bullet(\mathbb{C})$ be the pointed version of $\mathcal{H}(\mathbb{C})$. There are objects in $\mathcal{H}_\bullet(\mathbb{C})$:

S^1_s	$-$	the constant sheaf represented by $\Delta^1/\partial\Delta^1$ pointed canonically;
S^1_t	$-$	the sheaf represented by $\mathbb{A}^1 - \{0\}$ pointed by 1;
T	$-$	the sheaf represented by the projective line $\mathbb{P}^1$ pointed by ∞.

Set $S^{p,q} = (S^1_s)^{\wedge(p-q)} \wedge (S^1_t)^{\wedge q}$ with $p \geq q \geq 0$. Then, $S^{2,1}$ and T are $\mathbb{A}^1$-weakly equivalent. See Section 3.2 [9] for details and Section 1.4 [22] for discussion. One may take these objects to $\mathcal{SH}(\mathbb{C})$. The category $\mathcal{SH}(\mathbb{C})$ is triangulated with translation functor $S^{1,0} \wedge -$. Set $\widetilde{KO}^{p,q}(\mathcal{X}) := [\Sigma^\infty \mathcal{X}, S^{p,q} \wedge KO]$ and $KO^{p,q}(X) := [\Sigma^\infty X_+, S^{p,q} \wedge KO]$ where $\mathcal{X} \in \mathcal{H}_\bullet(\mathbb{C})$ and $X \in \mathcal{H}(\mathbb{C})$. The object $KO \in \mathcal{SH}(\mathbb{C})$ is the geometric model of Hermitian K-theory in the $\mathbb{A}^1$-homotopy theory defined by Schlichting and Tripathi (See Section 1.5 [22]). Moreover, there are isomorphisms $GW^q(X) \cong KO^{2q,q}(X)$ and $W^q(X) \cong KO^{2q-1,q-1}(X)$. One defines comparison maps (cf. Section 2 [22])

$$\begin{array}{rlcl} \tilde{k}_h^{p,q}(\mathcal{X}): & \widetilde{KO}^{p,q}(\mathcal{X}) & \to & \widetilde{KO}^{p}(\mathcal{X}(\mathbb{C})) \\ k_h^{p,q}(X): & KO^{p,q}(X) & \to & KO^p(X(\mathbb{C})). \end{array}$$

In particular, when X is a complex smooth variety, we have

$$\begin{array}{rlccl} gw^q = & k_h^{2q,q}: & GW^q(X) & \to & KO^{2q}(X(\mathbb{C})) \\ w^{q+1} = & k_h^{2q+1,q}: & W^{q+1}(X) & \to & KO^{2q+1}(X(\mathbb{C})). \end{array}$$

Theorem 4.1. *Let X be a complex smooth cellular variety. Assume further that Z is cellular and closed in X, and let $U := X - Z$. Then, the map $k_h^{2q,q}(U)$ is an isomorphism and the map $k_h^{2q+1,q}(U)$ is injective.*

Proof. When $Z = \emptyset$, this theorem is a special case of Theorem 2.6 [22]. We slightly modify the proof of Theorem 2.6 [22] to show this theorem by induction on cells. Let $Z = Z_N \supset Z_{N-1} \supset \cdots \supset Z_0 = \emptyset$ be the filtration such that

$$Z_{k+1} - Z_k \cong \mathbb{A}^{n_k} =: C_k.$$

Set $U_k := X - Z_k$ for each $0 \leq k \leq N$. Note that there is another filtration $X = U_0 \supset U_1 \supset \cdots \supset U_N = U$ with $U_k - U_{k+1} = Z_{k+1} - Z_k \cong C_k$ closed in U_k. Then, the normal bundle $\mathcal{N}_{U_k/C_k}$ of U_k in C_k is trivial. Hence, $\mathrm{Thom}(\mathcal{N}_{U_k/C_k})$ and $S^{2d,d}$ are $\mathbb{A}^1$-weakly equivalent, where d is the codimension of C_k in U_k, cf. Proposition 2.17 (page 112) [9]. We can therefore deduce the commutative ladder diagram in Figure 1 (page 486) [22]. Assume by induction, the theorem is true for U_k, and we want to prove it for U_{k+1}. It is known that $\tilde{k}_h^{2q,q}(S^{2d,d})$ and $\tilde{k}_h^{2q+1,q}(S^{2d,d})$ are isomorphisms and that $\tilde{k}_h^{2q+2,q}(S^{2d,d})$ is injective, cf. the proof of Theorem 2.6 [22]. The results follow by the 5-lemma. $\square$

4.4 Grothendieck-Witt group of a deleted quadric

In this subsection, we simply write $X = D_+(q_s), Q = Q_{s-2}$ and $DQ = DQ_{s-1}$. Note that Q is smooth and closed in $\mathbb{P}^{s-1}$ of codimension 1. The normal bundle $\mathcal{N}$ of Q in $\mathbb{P}^{s-1}$ is isomorphic to $\mathcal{O}_Q(2)$.

Theorem 4.2. *The comparison map* $gw^q : GW^q(DQ_{\mathbb{C}}) \to KO^{2q}(DQ(\mathbb{C}))$ *is an isomorphism for each* $q \in \mathbb{Z}$.

Proof. This theorem is a consequence of Theorem 4.1. □

Lemma 4.4. *The group* $GW^0(DQ_{\mathbb{C}})$ *is isomorphic to* $GW^0(DQ_K)$.

Proof. Applying Corollary 4.1 and the dévissage theorem, we observe that the vertical maps of W and GW-groups in the following commutative diagram are all isomorphisms

$$\begin{array}{ccccccccc}
GW^0_{Q_K}(\mathbb{P}^{s-1}_K) & \longrightarrow & GW^0(\mathbb{P}^{s-1}_K) & \longrightarrow & GW^0(DQ_K) & \longrightarrow & W^1_{Q_K}(\mathbb{P}^{s-1}_K) & \longrightarrow & W^1(\mathbb{P}^{s-1}_K) \\
\downarrow & & \downarrow & & \Omega\downarrow & & \downarrow & & \downarrow \\
GW^0_{Q_{\mathbb{C}}}(\mathbb{P}^{s-1}_{\mathbb{C}}) & \longrightarrow & GW^0(\mathbb{P}^{s-1}_{\mathbb{C}}) & \longrightarrow & GW^0(DQ_{\mathbb{C}}) & \longrightarrow & W^1_{Q_{\mathbb{C}}}(\mathbb{P}^{s-1}_{\mathbb{C}}) & \longrightarrow & W^1(\mathbb{P}^{s-1}_{\mathbb{C}})
\end{array}$$

where all vertical maps are induced from the left-hand side of the diagram (2) (use the 5-lemma to see the middle map Ω is an isomorphism). □

Recall the isomorphism of varieties $i_K : DQ_K \to X_K$ in Remark 2.1 (i). Note that $i_{\mathbb{C}} : DQ_{\mathbb{C}} \to X_{\mathbb{C}}$ gives a homeomorphism $i_{(\mathbb{C})} : DQ(\mathbb{C}) \to X(\mathbb{C})$ by taking $\mathbb{C}$-rational points. Besides, let $\upsilon : \mathbb{R}P^{s-1} \to X(\mathbb{C})$ be the natural embedding. The space $\mathbb{R}P^{s-1}$ is a deformation retract of the space $X(\mathbb{C})$ in the category of real spaces, cf. Lemma 6.3 [15]. These maps that induce isomorphisms in KO-theory or GW-theory are described in the diagram (4).

$$\begin{array}{ccc|c|ccc}
\multicolumn{3}{c}{\text{Hermitian } K\text{-theory}} & & \multicolumn{3}{c}{\text{Topological } KO\text{-theory}} \\
 & & GW^0(DQ_{\mathbb{C}}) & \xrightarrow{gw^0} & KO^0(DQ(\mathbb{C})) & & \\
 & & \Omega\uparrow & & \downarrow\ i^*_{(\mathbb{C})} & & \\
GW^0(X_K) & \xrightarrow{i^*_K} & GW^0(DQ_K) & & KO^0(X(\mathbb{C})) & \xrightarrow{\upsilon^*} & KO^0(\mathbb{R}P^{s-1})
\end{array} \qquad (4)$$

Proof of Theorem 3.1. Let ξ_{top} denote the tautological line bundle over $\mathbb{R}P^{s-1}$. Recall that there is an isomorphism of rings

$$KO^0(\mathbb{R}P^{s-1}) \cong \mathbb{Z}[\nu_{\text{top}}]/(\nu_{\text{top}}^2 + 2\nu_{\text{top}}, 2^{\varphi(s-1)}\nu_{\text{top}})$$

where ν_{top} represents the class $[\xi_{\text{top}}]-1$, cf. Section 7 [1] or Chapter IV [12]. Note that $\text{Pic}_{\mathbb{R}}(\mathbb{R}P^{s-1})$ is isomorphic to $\mathbb{Z}/2$. Let $\vartheta : GW^0(X_K) \to KO^0(\mathbb{R}P^{s-1})$ be the composition of maps in the diagram (4). We have known ϑ is an isomorphism. Therefore, to prove Theorem 3.1, we only need to show $\vartheta(\nu) = \nu_{\text{top}}$. To achieve this, we give the following lemma.

LEMMA 4.5. *The group $Q(X_K)$ (cf. Section 4.2) is isomorphic to $\mathbb{Z}/2$.*

Proof. There is an exact sequence (cf. Chapter IV.1 (page 229) [14])

$$1 \longrightarrow \mathcal{O}(X_K)^*/\mathcal{O}(X_K)^{2*} \longrightarrow Q(X_K) \xrightarrow{F} {}_2\mathrm{Pic}(X_K) \longrightarrow 1$$

where ${}_2\mathrm{Pic}(X_K)$ means the subgroup of elements of order ≤ 2 in $\mathrm{Pic}(X_K)$ and where F is the forgetful map. Note that ${}_2\mathrm{Pic}(X_K) \cong \mathbb{Z}/2$, cf. [19]. In addition, observe that $\mathcal{O}(X_K)^* \cong R^* = K^*$ and that the group K^*/K^{2*} is trivial. It follows that the forgetful map F is an isomorphism. In fact, it sends the non-trivial element $[\xi, \sigma]$ (in Lemma 3.1) to the non-trivial element $[\xi]$. □

Proof of Theorem 3.1 (Continued). In light of Lemma 4.3, there is a map

$$\tilde{\vartheta} : Q(X_K) \to \mathrm{Pic}_{\mathbb{R}}(\mathbb{R}P^{s-1})$$

(obtained in an obvious way) such that the following diagram is commutative

$$\begin{tikzcd}
GW^0(X_K) \arrow[r, "\vartheta"] & KO^0(\mathbb{R}P^{s-1}) \\
\mathbb{Z}/2 \cong Q(X_K) \arrow[u, "i"] \arrow[r, "\tilde{\vartheta}"] & \mathrm{Pic}_{\mathbb{R}}(\mathbb{R}P^{s-1}) \cong \mathbb{Z}/2. \arrow[u, "j"]
\end{tikzcd}$$

The map i is injective (Note that $[\xi]$ and 1 are distinct elements in $K_0(X_K)$ by its computation in Proposition 2.4 [7]). The map j is injective by the computation of $KO^0(\mathbb{R}P^{s-1})$. Then, we see that $\tilde{\vartheta}$ is bijective and must send $[\xi, \sigma]$ to $[\xi_{\mathrm{top}}]$. Therefore, $\vartheta([\xi, \sigma]) = [\xi_{\mathrm{top}}]$, so that $\vartheta(\nu) = \nu_{\mathrm{top}}$. □

A OPERATIONS ON THE GROTHENDIECK-WITT GROUP

The γ^i-operations on GW^0 of an affine scheme are analogous to those on the topological KO-theory which have been explained in Section 1 and 2 in [2]. For readers' convenience, details have been added.

Let $\mathrm{Bil}(X)$ be the set of isometry classes of bilinear spaces over a scheme X. The orthogonal sum and the tensor product of bilinear spaces over the scheme X make $\mathrm{Bil}(X)$ a semi-ring with a zero and a multiplicative identity. Then, by taking the associated Grothendieck ring $K(\mathrm{Bil}(X))$, we have a homomorphism of the underlying semi-rings

$$\iota : \mathrm{Bil}(X) \to K(\mathrm{Bil}(X))$$

satisfying the universal property (see Chapter I.4 (page 137) [14] for details).

REMARK A.1. For an affine scheme X, the ring $GW^0(X)$ is identified with $K(\mathrm{Bil}(X))$, cf. Chapter I.4 Proposition 1 (page 138) [14].

Definition A.1 (Chapter IV.3 (page 235) [14]). Let $(\mathcal{F}, \phi)$ be a bilinear space over a scheme X. Let i be a strictly positive integer. The *i-th exterior power of* $(\mathcal{F}, \phi)$, denoted by $\Lambda^i(\mathcal{F}, \phi)$, is the symmetric bilinear space $(\Lambda^i\mathcal{F}, \Lambda^i\phi)$ over X, where $\Lambda^i\mathcal{F}$ is the i-th exterior power of the locally free sheaf $\mathcal{F}$ and where

$$\Lambda^i\phi : \Lambda^i\mathcal{F} \times_X \Lambda^i\mathcal{F} \to \mathcal{O}_X$$

is a morphism of sheaves consisting of a symmetric bilinear form

$$\Lambda^i\phi(U) : \Lambda^i\mathcal{F}(U) \times \Lambda^i\mathcal{F}(U) \to \mathcal{O}_X(U)$$

defined by

$$\Lambda^i\phi(U)(x_1 \wedge \cdots \wedge x_i, y_1 \wedge \cdots \wedge y_i) = \det([\phi(U)(x_j, y_k)]_{i\times i})$$

for each open subscheme U of X. The exterior power $\Lambda^0(\mathcal{F}, \phi)$ for every bilinear space $(\mathcal{F}, \phi)$ (over X) is defined as $1 = (\mathcal{O}, \mathrm{id})$.

Lemma A.1. *Let $(\mathcal{F}, \phi), (\mathcal{G}, \psi)$ be bilinear spaces over X. Then, we have that*

(a) $\Lambda^1(\mathcal{F}, \phi) = (\mathcal{F}, \phi)$;

(b) $\Lambda^k((\mathcal{F}, \phi) \oplus (\mathcal{G}, \psi)) \cong \bigoplus_{r+s=k} \Lambda^r(\mathcal{F}, \phi) \otimes \Lambda^s(\mathcal{G}, \psi)$;

(c) *If $(\mathcal{F}, \phi)$ is of constant rank $\Theta \geq 1$, $\Lambda^i(\mathcal{F}, \phi) = 0$ whenever $i \geq \Theta$.*

Proof. (a) and (c) are clear. For (b), it is enough to show that the canonical isomorphism of locally free sheaves

$$\varrho : \bigoplus_{r+s=k} \Lambda^r\mathcal{F} \otimes \Lambda^s\mathcal{G} \to \Lambda^k(\mathcal{F} \oplus \mathcal{G})$$

respects the symmetric bilinear forms. This may be checked locally. Let U be an affine open subset of the scheme X. One may choose elements

$$x^{(t)} = x_{1,t} \wedge \cdots \wedge x_{r,t} \in \Lambda^r\mathcal{F}(U) \text{ and } y^{(t)} = y_{1,t} \wedge \cdots \wedge y_{s,t} \in \Lambda^s\mathcal{G}(U)$$

for $t \in \{1, 2\}$. Let $a_{i,j} := \phi(U)(x_{i,1}, x_{j,2})$ and $b_{k,l} := \psi(U)(y_{k,1}, y_{l,2})$. We have matrices $A = [a_{i,j}]_{r\times r}$ and $B = [b_{k,l}]_{s\times s}$. On the one hand, we get that

$$\Lambda^r\phi(U) \otimes \Lambda^s\psi(U)(x^{(1)} \otimes y^{(1)}, x^{(2)} \otimes y^{(2)}) = \det(A) \times \det(B). \tag{5}$$

On the other hand, set

$$u^{(t)} := \varrho(U)(x^{(t)} \otimes y^{(t)}) \in \Lambda^{r+s}(\mathcal{F}(U) \oplus \mathcal{G}(U))$$

for $t \in \{1, 2\}$. Consider the elements

$$(x_{j,t}, 0), (0, y_{k,t}) \in \mathcal{F}(U) \oplus \mathcal{G}(U)$$

for $1 \leq j \leq r, 1 \leq k \leq s$ and $t \in \{1, 2\}$. It is clear that

$$u^{(t)} = (x_{1,t}, 0) \wedge \cdots \wedge (x_{r,t}, 0) \wedge (0, y_{1,t}) \wedge \cdots \wedge (0, y_{s,t})$$

for $t \in \{1,2\}$. Then, we deduce that

$$\Lambda^{r+s}(\phi(U) \oplus \psi(U))(u^{(1)}, u^{(2)}) = \det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{6}$$

Note (5) = (6). The result follows. □

Let $A(X)$ denote the group $1 + tK(\mathrm{Bil}(X))[[t]]$ of formal power series with constant term 1 (under multiplication). Consider a map

$$\Lambda_t : \mathrm{Bil}(X) \to A(X), [\mathcal{F}, \phi] \mapsto 1 + \sum_{i \geq 1} \Lambda^i([\mathcal{F}, \phi])t^i.$$

If $I : (\mathcal{F}, \phi) \to (\mathcal{G}, \psi)$ is an isometry of bilinear spaces, so is the natural map

$$\Lambda^i I : \Lambda^i(\mathcal{F}, \phi) \to \Lambda^i(\mathcal{G}, \psi).$$

Then, the map Λ_t is well-defined. Furthermore, Lemma A.1 (b) implies that Λ_t is a homomorphism of the underlying monoids. By the universal property of K-theory, we can lift Λ_t to a homomorphism of groups

$$\lambda_t : K(\mathrm{Bil}(X)) \to A(X)$$

such that $\lambda_t \circ \iota = \Lambda_t$. Taking coefficients of λ_t, we get operators (not homomorphisms in general)

$$\lambda^i : K(\mathrm{Bil}(X)) \to K(\mathrm{Bil}(X)).$$

Set $\gamma_t = \lambda_{t/(1-t)}$ and write $\gamma_t = 1 + \sum_{i \geq 1} \gamma^i t^i$. Again, we obtain operators

$$\gamma^i : K(\mathrm{Bil}(X)) \to K(\mathrm{Bil}(X)).$$

Explicitly, we deduce

$$\sum_{i \geq 0} \gamma^i t^i = \sum_{i \geq 0} \lambda^i t^i (1-t)^{-i} = 1 + \sum_{i \geq 1} (\sum_{i \geq s \geq 1} \lambda^s \binom{i-1}{s-1}) t^i.$$

Hence, the γ^i are certain $\mathbb{Z}$-linear combinations of the λ^s. By definition, the map γ_t is a homomorphism of groups. Hence, for all $x, y \in K(\mathrm{Bil}(X))$, we have

Corollary A.1. (a) *$\gamma_t(x+y) = \gamma_t(x)\gamma_t(y)$;*

(b) *$\gamma_t([\eta]-1) = 1 + t([\eta]-1)$ where η is a bilinear space of rank 1 over X;*

(c) *If $(\mathcal{F}, \phi) \in \mathrm{Bil}(X)$ is of constant rank $\Theta \geq 1$, $\gamma^i((\mathcal{F}, \phi) - \Theta) = 0$ if $i \geq \Theta$.*

Proof. (a) is proved. For (b), we deduce

$$\gamma_t([\eta]-1) = \frac{\gamma_t([\eta])}{\gamma_t(1)} = \frac{\lambda_{t/(1-t)}([\eta])}{\lambda_{t/(1-t)}(1)} = \frac{1 + [\eta]t/(1-t)}{(1-t)^{-1}} = 1 + t([\eta]-1).$$

For (c), see the proof of Lemma 2.1 [2]. □

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A New Approach to the Geometric Satake Equivalence

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Abstract. I give another proof of the geometric Satake equivalence from I. Mirković and K. Vilonen [16] over a separably closed field. Over a not necessarily separably closed field, I obtain a canonical construction of the Galois form of the full L-group.

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1 Introduction

Connected reductive groups over separably closed fields are classified by their root data. These come in pairs: to every root datum, there is associated its dual root datum and vice versa. Hence, to every connected reductive group G, there is associated its dual group $\hat{G}$. Following Drinfeld's geometric interpretation of Langlands' philosophy, Mirković and Vilonen [16] show that the representation theory of $\hat{G}$ is encoded in the geometry of an ind-scheme canonically associated to G as follows.

Let G be a connected reductive group over an arbitrary field F. The *loop group* LG is the fpqc-sheaf associated with group functor on the category of F-algebras

$$LG : R \longmapsto G(R(\!(t)\!)).$$

The *positive loop group* L^+G is the fpqc-sheaf associated with the group functor

$$L^+G : R \longmapsto G(R[\![t]\!]).$$

Then $L^+G \subset LG$ is a subgroup functor, and the fpqc-quotient $\mathrm{Gr}_G = LG/L^+G$ is called the *affine Grassmannian*. It is representable by an ind-projective ind-scheme (= inductive limit of projective schemes). Now fix a prime $\ell \neq \mathrm{char}(F)$, and consider the category $P_{L^+G}(\mathrm{Gr}_G)$ of L^+G-equivariant ℓ-adic perverse sheaves on Gr_G. This is a $\bar{\mathbb{Q}}_\ell$-linear abelian category.

First assume that F is separably closed. Then the simple objects in $P_{L^+G}(\mathrm{Gr}_G)$ are as follows. Fix $T \subset B \subset G$ a maximal torus contained in a Borel. For every cocharacter μ, denote by

$$\overline{\mathcal{O}}_\mu \overset{\text{def}}{=} \overline{L^+G \cdot t^\mu}$$

the reduced L^+G-orbit closure of $t^\mu \in T(F(\!(t)\!))$ inside Gr_G. Then $\overline{\mathcal{O}}_\mu$ is a projective variety over F. Let IC_μ be the intersection complex of $\overline{\mathcal{O}}_\mu$. The simple objects of $P_{L^+G}(\mathrm{Gr}_G)$ are the IC_μ's where μ ranges over the set of dominant cocharacters $X_+^\vee$. Furthermore, the category $P_{L^+G}(\mathrm{Gr}_G)$ is equipped with an inner product: to every $\mathcal{A}_1, \mathcal{A}_2 \in P_{L^+G}(\mathrm{Gr}_G)$, there is associated a perverse sheaf $\mathcal{A}_1 \star \mathcal{A}_2 \in P_{L^+G}(\mathrm{Gr}_G)$ called the *convolution product* of $\mathcal{A}_1$ and $\mathcal{A}_2$ (cf. §3 below). Denote by

$$\omega(\text{-}) \overset{\text{def}}{=} \bigoplus_{i \in \mathbb{Z}} R^i\Gamma(\mathrm{Gr}_G, \text{-}) : P_{L^+G}(\mathrm{Gr}_G) \longrightarrow \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}$$

the global cohomology functor with values in the category of finite dimensional $\bar{\mathbb{Q}}_\ell$-vector spaces. Fix a pinning of G, and let $\hat{G}$ be the Langlands dual group over $\bar{\mathbb{Q}}_\ell$, i.e. the reductive group over $\bar{\mathbb{Q}}_\ell$ whose root datum is dual to the root datum of G. Let $\hat{T}$ be the dual torus, i.e. the $\bar{\mathbb{Q}}_\ell$-torus with $X^*(\hat{T}) = X_*(T)$.

Theorem 1.1. (i) *The pair $(P_{L^+G}(\mathrm{Gr}_G), \star)$ admits a unique symmetric monoidal structure such that the functor ω is symmetric monoidal.*

(ii) *The functor ω is a faithful exact tensor functor, and induces via the Tannakian formalism an equivalence of tensor categories*

$$\begin{aligned}(P_{L^+G}(\mathrm{Gr}_G), \star) &\xrightarrow{\simeq} (\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G}), \otimes)\\ \mathcal{A} &\longmapsto \omega(\mathcal{A}),\end{aligned}$$

which is uniquely determined up to inner automorphisms by $\hat{T}$ by the property that $\omega(IC_\mu)$ is the irreducible representation of highest weight μ.

In the case $F = \mathbb{C}$, this reduces to the theorem of Mirković and Vilonen [16] for coefficient fields of characteristic 0. The drawback of our method is the restriction to $\bar{\mathbb{Q}}_\ell$-coefficients. Mirkovic and Vilonen are able to establish a geometric Satake equivalence with coefficients in any Noetherian ring of finite global dimension (in the analytic topology). I give a proof of the theorem over any separably closed field F using ℓ-adic perverse sheaves. My proof is different from the one of Mirković and Vilonen. It proceeds in two main steps as follows. In the first step I show that the pair $(P_{L^+G}(\mathrm{Gr}_G), \star)$ is a symmetric monoidal category. This relies on the *Beilinson-Drinfeld Grassmannians* [2] and the comparison of the convolution product with the *fusion product* via Beilinson's construction of the nearby cycles functor. Here the fact that the convolution of two perverse sheaves is perverse is deduced from the fact that nearby cycles preserve perversity. The method is based on ideas of Gaitsgory [7] which were extended by Reich [19]. The constructions in this first step are essentially known, my purpose was to give a coherent account of these results.

The second step is the identification of the group of tensor automorphisms $\underline{\mathrm{Aut}}^\star(\omega)$ with the reductive group $\hat{G}$. I use a theorem of Kazhdan, Larsen and Varshavsky [10] which states that the root datum of a split reductive group can be reconstructed from the Grothendieck semiring of its algebraic representations. The reconstruction of the root datum relies on the PRV-conjecture proven by Kumar [11]. I prove the following geometric analogue of the PRV-conjecture.

Theorem 1.2 (Geometric analogue of the PRV-Conjecture). *Denote by $W = W(G,T)$ the Weyl group. Let $\mu_1, \ldots, \mu_n \in X_+^\vee$ be dominant coweights. Then, for every $\lambda \in X_+^\vee$ of the form $\lambda = \nu_1 + \ldots + \nu_k$ with $\nu_i \in W\mu_i$ for $i = 1, \ldots, k$, the perverse sheaf* IC_λ *appears as a direct summand in the convolution product $IC_{\mu_1} \star \ldots \star IC_{\mu_n}$.*

Using this theorem and the method in [10], I show that the Grothendieck semirings of $P_{L^+G}(\mathrm{Gr}_G)$ and $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G})$ are isomorphic. Hence, the root data of $\underline{\mathrm{Aut}}^\star(\omega)$ and $\hat{G}$ are the same. This shows that $\underline{\mathrm{Aut}}^\star(\omega) \simeq \hat{G}$ uniquely up to inner automorphisms by $\hat{T}$.
If F is not neccessarily separably closed, we are able to apply Galois descent to reconstruct the full L-group. Fix a separable closure $\bar{F}$ of F, and denote by $\Gamma = \mathrm{Gal}(\bar{F}/F)$ the absolute Galois group. Let ${}^LG = \hat{G}(\bar{\mathbb{Q}}_\ell) \rtimes \Gamma$ be the Galois form of the full L-group with respect to some pinning.

Theorem 1.3. *The functor $\mathcal{A} \mapsto \omega(\mathcal{A}_{\bar{F}})$ induces an equivalence of abelian tensor categories*

$$(P_{L^+G}(\mathrm{Gr}_G), \star) \;\simeq\; (\mathrm{Rep}^c_{\bar{\mathbb{Q}}_\ell}({}^LG), \otimes),$$

where $\mathrm{Rep}^c_{\bar{\mathbb{Q}}_\ell}({}^LG)$ is the full subcategory of the category of finite dimensional continuous ℓ-adic representations of ${}^LG$ such that the restriction to $\hat{G}(\bar{\mathbb{Q}}_\ell)$ is algebraic.

Theorem 1.3 may be seen as an extension of Theorem A.12 in my joint work with Zhu [20]. In [*loc. cit.*] we consider the category $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}({}^LG)$ of algebraic representations of ${}^LG$ regarded as a pro-algebraic group over $\bar{\mathbb{Q}}_\ell$. Then $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}({}^LG)$ is a full subcategory of $\mathrm{Rep}^c_{\bar{\mathbb{Q}}_\ell}({}^LG)$, and we identify the corresponding subcategory of $P_{L^+G}(\mathrm{Gr}_G)$ explicitly.
My method of proof here is similiar to the method used in [20]. Besides some general Tannakian formalism, the key ingredient is the identification of the Γ-action on $\hat{G}$ obtained via the geometric Satake equivalence over $\bar{F}$. It differs from the usual action by a twist with the cyclotomic character, cf. Proposition 6.6 below.
The structure of the paper is as follows. In §2 we introduce the Satake category $P_{L^+G}(\mathrm{Gr}_G)$. Appendix A supplements the definition of $P_{L^+G}(\mathrm{Gr}_G)$ and explains some basic facts on perverse sheaves on ind-schemes as used in the paper. In §3-§4 we clarify the tensor structure of the tuple $(P_{L^+G}(\mathrm{Gr}_G), \star)$,

and show that it is neutralized Tannakian with fiber functor ω. Section 5 is devoted to the identification of the dual group. This section is supplemented by Appendix B on the reconstruction of root data from the Grothendieck semiring of algebraic representations. The reader who is just interested in the case of an algebraically closed ground field may assume F to be algebraically closed throughout §2-§5. The last section §6 is concerned with Galois descent and the reconstruction of the full L-group.

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2 The Satake Category

Let G a connected reductive group over any field F. The *loop group* LG is the fpqc-sheaf associated with the group functor on the category of F-algebras

$$LG : R \longmapsto G(R(\!(t)\!)).$$

The *positive loop group* L^+G is the fpqc-sheaf associated with the group functor

$$L^+G : R \longmapsto G(R[\![t]\!]).$$

Then $L^+G \subset LG$ is a subgroup functor, and the fpqc-quotient $\mathrm{Gr}_G = LG/L^+G$ is called the *affine Grassmannian* (associated to G over F).

Lemma 2.1. *The affine Grassmannian* Gr_G *is representable by an ind-projective strict ind-scheme over* F*. It represents the functor which assigns to every* F*-algebra* R *the set of isomorphism classes of pairs* $(\mathcal{F}, \beta)$*, where* $\mathcal{F}$ *is a* G*-torsor over* $\mathrm{Spec}(R[\![t]\!])$ *and* β *a trivialization of* $\mathcal{F}[\frac{1}{t}]$ *over* $\mathrm{Spec}(R(\!(t)\!))$.

We postpone the proof of Lemma 2.1 to Section 3.1 below. For every $i \geq 0$, let G_i denote i-th jet group, given for any F-algebra R by $G_i : R \mapsto G(R[t]/t^{i+1})$. Then G_i is representable by a smooth connected affine group scheme over F and, as fpqc-sheaves,

$$L^+G \simeq \varprojlim_i G_i.$$

In particular, if G is non trivial, then L^+G is not of finite type over F. The positive loop group L^+G operates on Gr_G and, for every orbit $\mathcal{O}$, the L^+G-action factors through G_i for some i. Let $\overline{\mathcal{O}}$ denote the reduced closure of $\mathcal{O}$ in Gr_G, a projective L^+G-stable subvariety. This presents the reduced locus as the direct limit of L^+G-stable subvarieties

$$(\mathrm{Gr}_G)_{\mathrm{red}} = \varinjlim_{\mathcal{O}} \overline{\mathcal{O}},$$

where the transition maps are closed immersions.

Fix a prime $\ell \neq \mathrm{char}(F)$, and denote by $\mathbb{Q}_\ell$ the field of ℓ-adic numbers with algebraic closure $\bar{\mathbb{Q}}_\ell$. For any separated scheme T of finite type over F, we consider the bounded derived category $D_c^b(T, \bar{\mathbb{Q}}_\ell)$ of constructible ℓ-adic complexes on T, and its abelian full subcategory $P(T)$ of ℓ-adic perverse sheaves. If H is a connected smooth affine group scheme acting on T, then let $P_H(T)$ be the abelian subcategory of $P(T)$ of H-equivariant objects with H-equivariant morphisms. We refer to Appendix A for an explanation of these concepts.
The category of ℓ-adic perverse sheaves $P(\mathrm{Gr}_G)$ on the affine Grassmannian is the direct limit

$$P(\mathrm{Gr}_G) \overset{\text{def}}{=} \varinjlim_{\mathcal{O}} P(\overline{\mathcal{O}}),$$

which is well-defined, since all transition maps are closed immersions, cf. Appendix A.

Definition 2.2. The *Satake category* is the category of L^+G-equivariant ℓ-adic perverse sheaves on the affine Grassmannian Gr_G

$$P_{L^+G}(\mathrm{Gr}_G) \overset{\text{def}}{=} \varinjlim_{\mathcal{O}} P_{L^+G}(\overline{\mathcal{O}}),$$

where $\mathcal{O}$ ranges over the L^+G-orbits.

The Satake category $P_{L^+G}(\mathrm{Gr}_G)$ is an abelian $\bar{\mathbb{Q}}_\ell$-linear category, cf. Appendix A.

3 The Convolution Product

We are going to equip the category $P_{L^+G}(\mathrm{Gr}_G)$ with a tensor structure. Let

$$\text{-} \star \text{-} : P(\mathrm{Gr}_G) \times P_{L^+G}(\mathrm{Gr}_G) \longrightarrow D_c^b(\mathrm{Gr}_G, \bar{\mathbb{Q}}_\ell)$$

be the convolution product with values in the derived category. We recall its definition [17, §2]. Consider the following diagram of ind-schemes

$$\mathrm{Gr}_G \times \mathrm{Gr}_G \xleftarrow{p} LG \times \mathrm{Gr}_G \xrightarrow{q} LG \times^{L^+G} \mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G. \tag{3.1}$$

Here p (resp. q) is a right L^+G-torsor with respect to the L^+G-action on the left factor (resp. the diagonal action).The LG-action on Gr_G factors through q, giving rise to the morphism m.
For perverse sheaves $\mathcal{A}_1, \mathcal{A}_2$ on Gr_G, their box product $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ is a perverse sheaf on $\mathrm{Gr}_G \times \mathrm{Gr}_G$. If $\mathcal{A}_2$ is L^+G-equivariant, then there is a unique perverse sheaf $\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2$ on $LG \times^{L^+G} \mathrm{Gr}_G$ such that there is an isomorphism equivariant for the diagonal L^+G-action[1]

$$p^*(\mathcal{A}_1 \boxtimes \mathcal{A}_2) \simeq q^*(\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2).$$

Then the convolution is defined as $\mathcal{A}_1 \star \mathcal{A}_2 \overset{\text{def}}{=} m_*(\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2)$.

[1]Though LG is not of ind-finite type, we use Lemma 3.20 below to define $\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2$.

Theorem 3.1. (i) *For perverse sheaves $\mathcal{A}_1, \mathcal{A}_2$ on Gr_G with $\mathcal{A}_2$ being L^+G-equivariant, their convolution $\mathcal{A}_1 \star \mathcal{A}_2$ is a perverse sheaf. If $\mathcal{A}_1$ is also L^+G-equivariant, then $\mathcal{A}_1 \star \mathcal{A}_2$ is L^+G-equivariant.*

(ii) *Let $\bar{F}$ be a separable closure of F. The convolution product is a bifunctor*

$$\text{-} \star \text{-} : P_{L^+G}(\mathrm{Gr}_G) \times P_{L^+G}(\mathrm{Gr}_G) \longrightarrow P_{L^+G}(\mathrm{Gr}_G),$$

and $(P_{L^+G}(\mathrm{Gr}_G), \star)$ has a unique structure of a symmetric monoidal category such that the cohomology functor with values in finite dimensional $\bar{\mathbb{Q}}_\ell$-vector spaces

$$\bigoplus_{i\in\mathbb{Z}} R^i\Gamma(\mathrm{Gr}_{G,\bar{F}}, (\text{-})_{\bar{F}}) \colon P_{L^+G}(\mathrm{Gr}_G) \longrightarrow \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}$$

is symmetric monoidal.

Part (i) is due to Lusztig [12] and Gaitsgory [7]. Part (ii) is based on methods due to Reich [19]. Both parts of Theorem 3.1 are proved simultaneously in Subsection 3.3 below using universally locally acyclic perverse sheaves (cf. Subsection 3.2 below) and a global version of diagram (3.1) which we introduce in the next subsection.

3.1 Beilinson-Drinfeld Grassmannians

Let X a smooth geometrically connected curve over F. For any F-algebra R, let $X_R = X \times \mathrm{Spec}(R)$. Denote by Σ the moduli space of relative effective Cartier divisors on X, i.e. the fppf-sheaf associated with the functor on the category of F-algebras

$$R \longmapsto \{D \subset X_R \text{ relative effective Cartier divisor}\}.$$

Lemma 3.2. *The $fppf$-sheaf Σ is represented by the disjoint union of fppf-quotients $\coprod_{n\geq 1} X^n/S_n$, where the symmetric group S_n acts on X^n by permuting its coordinates.*

□

Definition 3.3. The *Beilinson-Drinfeld Grassmannian (associated to G and X)* is the functor $\mathcal{G}r = \mathcal{G}r_{G,X}$ on the category of F-algebras which assings to every R the set of isomorphism classes of triples $(D, \mathcal{F}, \beta)$ with

$$\begin{cases} D \in \Sigma(R) \text{ a relative effective Cartier divisor;} \\ \mathcal{F} \text{ a } G\text{-torsor on } X_R; \\ \beta : \mathcal{F}|_{X_R \backslash D} \overset{\simeq}{\to} \mathcal{F}_0|_{X_R\backslash D} \text{ a trivialisation,} \end{cases}$$

where $\mathcal{F}_0$ denotes the trivial G-torsor. The projection $\mathcal{G}r \to \Sigma$, $(D, \mathcal{F}, \beta) \mapsto D$ is a morphism of functors.

Lemma 3.4. *The Beilinson-Drinfeld Grassmannian $\mathcal{G}r = \mathcal{G}r_{G,X}$ associated to a reductive group G and a smooth curve X is representable by an ind-proper strict ind-scheme over Σ.*

Proof. This is proven in [7, Appendix A.5.]. We sketch the argument. If $G = \mathrm{GL}_n$, consider the functor $\mathcal{G}r_{(m)}$ parametrizing

$$J \subset \mathcal{O}^n_{X_R}(-m\cdot D)/\mathcal{O}^n_{X_R}(m\cdot D),$$

where J is a coherent $\mathcal{O}_{X_R}$-submodule such that $\mathcal{O}_{X_R}(-m\cdot D)/J$ is flat over R. By the theory of Hilbert schemes, the functor $\mathcal{G}r_{(m)}$ is representable by a proper scheme over Σ. For $m_1 < m_2$, there are closed immersions $\mathcal{G}r_{(m_1)} \hookrightarrow \mathcal{G}r_{(m_2)}$. Then as fpqc-sheaves

$$\varinjlim_m \mathcal{G}r_{(m)} \xrightarrow{\simeq} \mathcal{G}r.$$

For general reductive G, choose an embedding $G \hookrightarrow \mathrm{GL}_n$. Then the fppf-quotient GL_n/G is affine, and the natural morphism $\mathcal{G}r_G \to \mathcal{G}r_{\mathrm{GL}_n}$ is a closed immersion. The ind-scheme structure of $\mathcal{G}r_G$ does not depend on the choosen embedding $G \hookrightarrow \mathrm{GL}_n$. This proves the lemma. □

Now we define a global version of the loop group. For every $D \in \Sigma(R)$, the formal completion of X_R along D is a formal affine scheme. We denote by $\hat{\mathcal{O}}_{X,D}$ its underlying R-algebra. Let $\hat{D} = \mathrm{Spec}(\hat{\mathcal{O}}_{X,D})$ be the associated affine scheme over R. Then D is a closed subscheme of $\hat{D}$, and we set $\hat{D}^o = \hat{D}\backslash D$. The *global loop group* is the fpqc-sheaf associated with the group functor on the category of F-algebras

$$\mathcal{L}G : R \mapsto \{(s, D) \mid D \in \Sigma(R),\ s \in G(\hat{D}^o)\}.$$

The *global positive loop group* is the fpqc-sheaf associated with the group functor

$$\mathcal{L}^+G : R \mapsto \{(s, D) \mid D \in \Sigma(R),\ s \in G(\hat{D})\}.$$

Then $\mathcal{L}^+G \subset \mathcal{L}G$ is a subgroup functor over Σ.

Lemma 3.5. (i) *The global loop group $\mathcal{L}G$ is representable by an ind-group scheme over Σ. It represents the functor on the category of F-algebras which assigns to every R the set of isomorphism classes of quadruples $(D, \mathcal{F}, \beta, \sigma)$, where $D \in \Sigma(R)$, $\mathcal{F}$ is a G-torsor on X_R, $\beta : \mathcal{F} \xrightarrow{\simeq} \mathcal{F}_0$ is a trivialisation over $X_R\backslash D$ and $\sigma : \mathcal{F}_0 \xrightarrow{\simeq} \mathcal{F}|_{\hat{D}}$ is a trivialisation over $\hat{D}$.*

(ii) *The global positive loop group $\mathcal{L}^+G$ is representable by an affine group scheme over Σ with geometrically connected fibers.*

(iii) *The projection $\mathcal{L}G \to \mathcal{G}r_G$, $(D, \mathcal{F}, \beta, \sigma) \to (D, \mathcal{F}, \beta)$ is a right $\mathcal{L}^+G$-torsor, and induces an isomorphism of fpqc-sheaves over Σ*

$$\mathcal{L}G/\mathcal{L}^+G \xrightarrow{\simeq} \mathcal{G}r_G.$$

Proof. We reduce to the case that X is affine. Note that fppf-locally on R every $D \in \Sigma(R)$ is of the form $V(f)$. Then the moduli description in (i) follows from the descent lemma of Beauville-Laszlo [1] (cf. [14, Proposition 3.8]). The ind-representability follows from part (ii) and (iii). This proves (i).
For any $D \in \Sigma(R)$ denote by $D^{(i)}$ its i-th infinitesimal neighbourhood in X_R. Then $D^{(i)}$ is finite over R, and the Weil restriction $\mathrm{Res}_{D^{(i)}/R}(G)$ is representable by a smooth affine group scheme with geometrically connected fibers. For $i \leq j$, there are affine transition maps $\mathrm{Res}_{D^{(j)}/R}(G) \to \mathrm{Res}_{D^{(i)}/R}(G)$ with geometrically connected fibers. Hence, $\varprojlim_i \mathrm{Res}_{D^{(i)}/R}(G)$ is an affine scheme, and the canonical map

$$\mathcal{L}^+G \times_{\Sigma,D} \mathrm{Spec}(R) \longrightarrow \varprojlim_i \mathrm{Res}_{D^{(i)}/R}(G)$$

is an isomorphism of fpqc-sheaves. This proves (ii).
To prove (iii), the crucial point is that after a faithfully flat extension $R \to R'$ a G-torsor $\mathcal{F}$ on $\hat{D}$ admits a global section. Indeed, $\mathcal{F}$ admits a R'-section which extends to $\hat{D}_{R'}$ by smoothness and Grothendieck's algebraization theorem. This finishes (iii). $\square$

Remark 3.6. The connection with the affine Grassmannian Gr_G is as follows. Lemma 3.2 identifies X with a connected component of Σ. Choose a point $x \in X(F)$ considered as an element $D_x \in \Sigma(F)$. Then $\hat{D}_x \simeq \mathrm{Spec}(F[\![t]\!])$, where t is a local parameter of X in x. Under this identification, there are isomorphisms of fpqc-sheaves

$$\begin{aligned} \mathcal{L}G_x &\simeq LG \\ \mathcal{L}^+G_x &\simeq L^+G \\ \mathcal{G}r_{G,x} &\simeq \mathrm{Gr}_G. \end{aligned}$$

Using the theory of Hilbert schemes, the proof of Lemma 3.4 also implies that $\mathrm{Gr}_{\mathrm{GL}_n}$, and hence Gr_G is ind-projective. This proves Lemma 2.1 above.

By Lemma 3.5 (iii), the global positive loop groop $\mathcal{L}^+G$ acts on $\mathcal{G}r$ from the left. For $D \in \Sigma(R)$ and $(D, \mathcal{F}, \beta) \in \mathcal{G}r_G(R)$, denote the action by

$$((g, D), (\mathcal{F}, \beta, D)) \longmapsto (g\mathcal{F}, g\beta, D).$$

Corollary 3.7. *The $\mathcal{L}^+G$-orbits on $\mathcal{G}r$ are of finite type and smooth over Σ.*

Proof. Let $D \in \Sigma(R)$. It is enough to prove that the action of

$$\mathcal{L}^+G \times_{\Sigma,D} \mathrm{Spec}(R) \;\simeq\; \varprojlim_i \mathrm{Res}_{D^{(i)}/R}(G)$$

on $\mathcal{G}r \times_{\Sigma,D} \mathrm{Spec}(R)$ factors over $\mathrm{Res}_{D^{(i)}/R}(G)$ for some $i >> 0$. Choose a faithful representation $\rho : G \to \mathrm{GL}_n$, and consider the corresponding closed immersion $\mathcal{G}r_G \to \mathcal{G}r_{\mathrm{GL}_n}$. This reduces us to the case $G = \mathrm{GL}_n$. In this case, the $\mathcal{G}r_{(m)}$'s (cf. proof of Lemma 3.4) are $\mathcal{L}^+\mathrm{GL}_n$ stable, and it is easy to see that the action on $\mathcal{G}r_{(m)}$ factors through $\mathrm{Res}_{D^{(2m)}/R}(\mathrm{GL}_n)$. This proves the corollary. □

Now we globalize the convolution morphism m from diagram (3.1) above. The moduli space Σ of relative effective Cartier divisors has a natural monoid structure

$$\begin{aligned} \text{-} \cup \text{-} : \Sigma \times \Sigma &\longrightarrow \Sigma \\ (D_1, D_2) &\longmapsto D_1 \cup D_2. \end{aligned}$$

The key definition is the following.

DEFINITION 3.8. For $k \geq 1$, the *k-fold convolution Grassmannian* $\tilde{\mathcal{G}r}_k$ is the functor on the category of F-algebras which associates to every R the set of isomorphism classes of tuples $((D_i, \mathcal{F}_i, \beta_i)_{i=1,\dots,k})$ with

$$\begin{cases} D_i \in \Sigma(R) \text{ relative effective Cartier divisors, } i = 1, \dots, k; \\ \mathcal{F}_i \text{ are } G\text{-torsors on } X_R; \\ \beta_i : \mathcal{F}_i|_{X_R \backslash D_i} \overset{\simeq}{\to} \mathcal{F}_{i-1}|_{X_R \backslash D_i} \text{ isomorphisms, } i = 1, \dots, k, \end{cases}$$

where $\mathcal{F}_0$ is the trivial G-torsor. The projection $\tilde{\mathcal{G}r}_k \to \Sigma^k$, $((D_i, \mathcal{F}_i, \beta_i)_{i=1,\dots,k}) \mapsto ((D_i)_{i=1,\dots,k})$ is a morphism of functors.

LEMMA 3.9. *For $k \geq 1$, the k-fold convolution Grassmannian $\tilde{\mathcal{G}r}_k$ is representable by a strict ind-scheme which is ind-proper over Σ^k.*

Proof. The lemma follows by induction on k. If $k = 1$, then $\tilde{\mathcal{G}r}_k = \mathcal{G}r$. For $k > 1$, consider the projection

$$\begin{aligned} p: \tilde{\mathcal{G}r}_k &\longrightarrow \tilde{\mathcal{G}r}_{k-1} \times \Sigma \\ ((D_i, \mathcal{F}_i, \beta_i)_{i=1,\dots,k}) &\longmapsto ((D_i, \mathcal{F}_i, \beta_i)_{i=1,\dots,k-1}, D_k). \end{aligned}$$

Then the fiber over a R-point $((D_i, \mathcal{F}_i, \beta_i)_{i=1,\dots,k-1}, D_k)$ is

$$p^{-1}(((D_i, \mathcal{F}_i, \beta_i)_{i=1,\dots,k-1}, D_k)) \simeq \mathcal{F}_{k-1} \times^G (\mathcal{G}r \times_{X_R} D_k),$$

which is ind-proper. This proves the lemma. □

For $k \geq 1$, there is the *k-fold global convolution morphism*

$$\begin{aligned} m_k : \tilde{\mathcal{G}r}_k &\longrightarrow \mathcal{G}r \\ ((D_i, \mathcal{F}_i, \beta_i)_{i=1,\dots,k}) &\longmapsto (D, \mathcal{F}_k, \beta_1|_{X_R \backslash D} \circ \ldots \circ \beta_k|_{X_R \backslash D}), \end{aligned}$$

where $D = D_1 \cup \ldots \cup D_k$. This yields a commutative diagram of ind-schemes

$$\begin{array}{ccc} \tilde{\mathcal{G}r}_k & \xrightarrow{m_k} & \mathcal{G}r \\ \downarrow & & \downarrow \\ \Sigma^k & \xrightarrow{\cup} & \Sigma, \end{array}$$

i.e., regarding $\tilde{\mathcal{G}r}_k$ as a Σ-scheme via $\Sigma^k \to \Sigma$, $(D_i)_i \mapsto \cup_i D_i$, the morphism m_k is a morphism of Σ-ind-schemes. The global positive loop group $\mathcal{L}^+G$ acts on $\tilde{\mathcal{G}r}_k$ over Σ as follows: let $(D_i, \mathcal{F}_i, \beta_i)_i \in \tilde{\mathcal{G}r}_k(R)$ and $g \in G(\hat{D})$ with $D = \cup_i D_i$. Then the action is defined as

$$((g, D), (D_i, \mathcal{F}_i, \beta_i)_i) \longmapsto (D_i, g\mathcal{F}_i, g\beta_i g^{-1})_i.$$

Corollary 3.10. *The morphism $m_k : \tilde{\mathcal{G}r}_k \to \mathcal{G}r$ is a $\mathcal{L}^+G$-equivariant morphism of ind-proper strict ind-schemes over Σ.*

Proof. The $\mathcal{L}^+G$-equivariance is immediate from the definition of the action. Note that $\Sigma^k \xrightarrow{\cup} \Sigma$ is finite, and hence $\tilde{\mathcal{G}r}_k$ is an ind-proper strict ind-scheme over Σ. This proves the corollary. □

Now we explain the global analogue of the L^+G-torsors p and q from (3.1). For $k \geq 1$, let $\tilde{\mathcal{L}}G_k$ be the functor on the category of F-algebras which associates to every R the set of isomorphism classes of tuples $((D_i, \mathcal{F}_i, \beta_i)_{i=1,\dots,k}, (\sigma_i)_{i=2,\dots,k})$ with

$$\begin{cases} D_i \in \Sigma(R),\ i = 1, \dots, k; \\ \mathcal{F}_i \text{ are } G\text{-torsors on } X_R; \\ \beta_i : \mathcal{F}_i|_{X_R \setminus D_i} \xrightarrow{\cong} \mathcal{F}_0|_{X_R \setminus D_i} \text{ trivialisations, } i = 1, \dots, k; \\ \sigma_i : \mathcal{F}_0|_{\hat{D}_i} \xrightarrow{\cong} \mathcal{F}_{i-1}|_{\hat{D}_i},\ i = 2, \dots, k, \end{cases}$$

where $\mathcal{F}_0$ is the trivial G-torsor. There are two natural projections over Σ^k. Let

$$\mathcal{L}^+G_\Sigma^{k-1} = \Sigma^k \times_{\Sigma^{k-1}} \mathcal{L}^+G^{k-1}.$$

The first projection is given by

$$\begin{aligned} p_k : \tilde{\mathcal{L}}G_k &\longrightarrow \mathcal{G}r^k \\ ((D_i, \mathcal{F}_i, \beta_i)_{i=1,\dots,k}, (\sigma_i)_{i=2,\dots,k}) &\longmapsto ((D_i, \mathcal{F}_i, \beta_i)_{i=1,\dots,k}). \end{aligned}$$

Then p_k is a right $\mathcal{L}^+G_\Sigma^{k-1}$-torsor for the action on the σ_i's. The second projection is given by

$$\begin{aligned} q_k : \tilde{\mathcal{L}}G_k &\longrightarrow \tilde{\mathcal{G}r}_k \\ ((D_i, \mathcal{F}_i, \beta_i)_{i=1,\dots,k}, (\sigma_i)_{i=2,\dots,k}) &\longmapsto ((D_i, \mathcal{F}'_i, \beta'_i)_{i=1,\dots,k}), \end{aligned}$$

where $\mathcal{F}'_1 = \mathcal{F}_1$ and for $i \geq 2$, the G-torsor $\mathcal{F}'_i$ is defined successively by gluing $\mathcal{F}_i|_{X_R \setminus D_i}$ to $\mathcal{F}'_{i-1}|_{\hat{D}_i}$ along $\sigma_i|_{\hat{D}_i^o} \circ \beta_i|_{\hat{D}_i^o}$. Then q_k is a right $\mathcal{L}^+G_\Sigma^{k-1}$-torsor for the action given by

$$\begin{aligned} (((D_i, \mathcal{F}_i, \beta_i)_{i\geq 1}, (\sigma_i)_{i\geq 2}),&(D_1, (D_i, g_i)_{i\geq 2})) \longmapsto \\ &((D_1, \mathcal{F}_1, \beta_1), (D_i, g_i^{-1}\mathcal{F}_i, g_i^{-1}\beta_i)_{i\geq 2}, (\sigma_i g_i)_{i\geq 2}). \end{aligned}$$

In the following, we consider ind-schemes over Σ^k as ind-schemes over Σ via $\Sigma^k \to \Sigma$.

Definition 3.11. For every $k \geq 1$, the *k-fold global convolution diagram* is the diagram of ind-schemes over Σ

$$\mathcal{G}r^k \xleftarrow{p_k} \tilde{\mathcal{L}}G_k \xrightarrow{q_k} \tilde{\mathcal{G}r}_k \xrightarrow{m_k} \mathcal{G}r.$$

Remark 3.12. Fix $x \in X(F)$, and choose a local coordinate t at x. Taking the fiber over $\mathrm{diag}(\{x\}) \in X^k(F)$ in the k-fold global convolution diagram, then

$$\mathrm{Gr}_G^k \longleftarrow LG^{k-1} \times \mathrm{Gr}_G \longrightarrow \underbrace{LG \times^{L^+G} \ldots \times^{L^+G} \mathrm{Gr}_G}_{k\text{-times}} \longrightarrow \mathrm{Gr}_G.$$

For $k = 2$, we recover diagram (3.1).

3.2 Universal Local Acyclicity

The notion of universal local acyclicity (ULA) is used in Reich's thesis [19], cf. also the paper [3] by Braverman and Gaitsgory. We recall the definition. Let S be a smooth geometrically connected scheme over F, and $f : T \to S$ a separated morphism of finite type. For complexes $\mathcal{A}_T \in D_c^b(T, \bar{\mathbb{Q}}_\ell)$, $\mathcal{A}_S \in D_c^b(S, \bar{\mathbb{Q}}_\ell)$, there is a natural morphism

$$\mathcal{A}_T \otimes f^*\mathcal{A}_S \;\longrightarrow\; (\mathcal{A}_T \overset{!}{\otimes} f^!\mathcal{A}_S)[2\dim(S)], \tag{3.2}$$

where $\mathcal{A} \overset{!}{\otimes} \mathcal{B} \overset{\text{def}}{=} \mathbb{D}(\mathbb{D}\mathcal{A} \otimes \mathbb{D}\mathcal{B})$ for $\mathcal{A}, \mathcal{B} \in D_c^b(T, \bar{\mathbb{Q}}_\ell)$. The morphism (3.2) is constructed as follows. Let $\Gamma_f : T \to T \times S$ be the graph of f. The projection formula gives a map

$$\Gamma_{f,!}(\Gamma_f^*(\mathcal{A}_T \boxtimes \mathcal{A}_S) \otimes \Gamma_f^! \bar{\mathbb{Q}}_\ell) \;\simeq\; (\mathcal{A}_T \boxtimes \mathcal{A}_S) \otimes \Gamma_{f,!}\Gamma_f^! \bar{\mathbb{Q}}_\ell \;\longrightarrow\; \mathcal{A}_T \boxtimes \mathcal{A}_S,$$

and by adjunction a map $\Gamma_f^*(\mathcal{A}_T \boxtimes \mathcal{A}_S) \otimes \Gamma_f^! \bar{\mathbb{Q}}_\ell \to \Gamma_f^!(\mathcal{A}_T \boxtimes \mathcal{A}_S)$. Note that

$$\Gamma_f^*(\mathcal{A}_T \boxtimes \mathcal{A}_S) \;\simeq\; \mathcal{A}_T \otimes f^*\mathcal{A}_S \qquad \text{and} \qquad \Gamma_f^!(\mathcal{A}_T \boxtimes \mathcal{A}_S) \;\simeq\; \mathcal{A}_T \overset{!}{\otimes} f^!\mathcal{A}_S,$$

using $\mathbb{D}(\mathcal{A}_T \boxtimes \mathcal{A}_S) \simeq \mathbb{D}\mathcal{A}_T \boxtimes \mathbb{D}\mathcal{A}_S$. Since S is smooth, Γ_f is a regular embedding, and thus $\Gamma_f^! \bar{\mathbb{Q}}_\ell \simeq \bar{\mathbb{Q}}_\ell[-2\dim(S)]$. This gives after shifting by $[2\dim(S)]$ the map (3.2).

Definition 3.13. (i) A complex $\mathcal{A}_T \in D_c^b(T, \bar{\mathbb{Q}}_\ell)$ is called *locally acyclic with respect to f* (f-LA) if (3.2) is an isomorphism for all $\mathcal{A}_S \in D_c^b(S, \bar{\mathbb{Q}}_\ell)$.
(ii) A complex $\mathcal{A}_T \in D_c^b(T, \bar{\mathbb{Q}}_\ell)$ is called *universally locally acyclic with respect to f* (f-ULA) if $f_{S'}^*\mathcal{A}_T$ is $f_{S'}$-LA for all $f_{S'} = f \times_S S'$ with $S' \to S$ smooth, S' geometrically connected.

Remark 3.14. (i) If f is smooth, then the trivial complex $\mathcal{A}_T = \bar{\mathbb{Q}}_\ell$ is f-ULA.
(ii) If $S = \mathrm{Spec}(F)$ is a point, then every complex $\mathcal{A}_T \in D_c^b(T, \bar{\mathbb{Q}}_\ell)$ is f-ULA.
(iii) The ULA property is local in the smooth topology on T.

Lemma 3.15. *Let $g : T \to T'$ be a proper morphism of S-schemes of finite type. For every ULA complex $\mathcal{A}_T \in D^b_c(T, \bar{\mathbb{Q}}_\ell)$, the push forward $g_*\mathcal{A}_T$ is ULA.*

Proof. For any morphism of finite type $g : T \to T'$ and any two complexes $\mathcal{A}_T$, $\mathcal{A}_{T'}$, we have the projection formulas

$$g_!(\mathcal{A}_T \otimes g^*\mathcal{A}_{T'}) \;\simeq\; g_!\mathcal{A}_T \otimes \mathcal{A}_{T'} \qquad \text{and} \qquad g_*(\mathcal{A}_T \overset{!}{\otimes} g^!\mathcal{A}_{T'}) \;\simeq\; g_*\mathcal{A}_T \overset{!}{\otimes} \mathcal{A}_{T'}.$$

If g is proper, then $g_* = g_!$, and the lemma follows from an application of the projection formulas and proper base change. □

Theorem 3.16 ([19]). *Let $D \subset S$ be a smooth Cartier divisor, and consider a cartesian diagram of morphisms of finite type*

$$\begin{array}{ccccc} E & \xrightarrow{\;i\;} & T & \xleftarrow{\;j\;} & U \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ D & \longrightarrow & S & \longleftarrow & S\backslash D. \end{array}$$

Let $\mathcal{A}$ be a f-ULA complex on T such that $\mathcal{A}|_U$ is perverse. Then:
(i) *There is a functorial isomorphism*

$$i^*[-1]\mathcal{A} \;\simeq\; i^![1]\mathcal{A},$$

and both complexes $i^*[-1]\mathcal{A}$, $i^![1]\mathcal{A}$ are perverse. Furthermore, the complex $\mathcal{A}$ is perverse and is the middle perverse extension $\mathcal{A} \simeq j_{!*}(\mathcal{A}|_U)$.
(ii) *The complex $i^*[-1]\mathcal{A}$ is $f|_E$-ULA.*

□

Remark 3.17. The proof of Theorem 3.16 uses Beilinson's construction of the unipotent part of the tame nearby cycles as follows. Suppose the Cartier divisor D is principal, this gives a morphism $\varphi : S \to \mathbb{A}^1_F$ such that $\varphi^{-1}(\{0\}) = S\backslash D$. Let $g = \varphi \circ f$ be the composition. Fix a separable closure $\bar{F}$ of F. In SGA VII, Deligne constructs the nearby cycles functor $\psi = \psi_g : P(U) \to P(E_{\bar{F}})$. Let ψ_{tame} be the tame nearby cycles, i.e. the invariants under the pro-p-part of $\pi_1(\mathbb{G}_{m,\bar{F}}, 1)$. Fix a topological generator T of the maximal prime-p-quotient of $\pi_1(\mathbb{G}_{m,\bar{F}}, 1)$. Then T acts on ψ_{tame}, and there is an exact triangle

$$\psi_{\mathrm{tame}} \xrightarrow{T-1} \psi_{\mathrm{tame}} \longrightarrow i^*j_* \xrightarrow{+1}$$

Under the action of $T-1$ the nearby cycles decompose as $\psi_{\mathrm{tame}} \simeq \psi^{\mathrm{u}}_{\mathrm{tame}} \oplus \psi^{\mathrm{nu}}_{\mathrm{tame}}$, where $T-1$ acts nilpotently on $\psi^{\mathrm{u}}_{\mathrm{tame}}$ and invertibly on $\psi^{\mathrm{un}}_{\mathrm{tame}}$. Let $N : \psi_{\mathrm{tame}} \to \psi_{\mathrm{tame}}(-1)$ be the logarithm of T, i.e. the unique nilpotent operator N such that $T = \exp(\bar{T}N)$ where $\bar{T}$ is the image of T under $\pi_1(\mathbb{G}_{m,\bar{F}}, 1) \twoheadrightarrow \mathbb{Z}_\ell(1)$. Then for any $a \geq 0$, Beilinson constructs a local system $\mathcal{L}_a$ on $\mathbb{G}_m$

together with a nilpotent operator N_a such that for $\mathcal{A}_U \in P(U)$ and $a \geq 0$ with $N^{a+1}(\psi^{\mathrm{u}}_{\mathrm{tame}}(\mathcal{A}_U)) = 0$ there is an isomorphism

$$(\psi^{\mathrm{u}}_{\mathrm{tame}}(\mathcal{A}_U), N) \;\simeq\; (i^*[-1]j_{!*}(\mathcal{A}_U \otimes g^*\mathcal{L}_a)_{\bar{F}}, 1 \otimes N_a).$$

Set $\Psi^{\mathrm{u}}_g(\mathcal{A}_U) \stackrel{\mathrm{def}}{=} \lim_{a\to\infty} i^*[-1]j_{!*}(\mathcal{A}_U \otimes g^*\mathcal{L}_a)$. Then $\Psi^{\mathrm{u}}_g : P(U) \to P(E)$ is a functor, and we obtain that N acts trivially on $\psi^{\mathrm{u}}_{\mathrm{tame}}(\mathcal{A}_U)$ if and only if $\Psi^{\mathrm{u}}_g(\mathcal{A}_U) = i^*[-1]j_{!*}(\mathcal{A}_U)$. In this case, Ψ^u_g is also defined for non-principal Cartier divisors by the formula $\Psi^u_g = i^*[-1] \circ j_{!*}$.
In the situation of Theorem 3.16 above Reich shows that the unipotent monodromy along E is trivial, and consequently

$$i^*[-1]\mathcal{A} \;\simeq\; \Psi^u_g \circ j^*(\mathcal{A}) \;\simeq\; i^![1]\mathcal{A}.$$

.

Corollary 3.18 ([19]). *Let $\mathcal{A}$ be a perverse sheaf on S whose support contains an open subset of S. Then the following are equivalent:*
(i) *The perverse sheaf $\mathcal{A}$ is ULA with respect to the identity $id : S \to S$.*
(ii) *The complex $\mathcal{A}[-\dim(S)]$ is a locally constant system, i.e. a lisse sheaf.*

□

We use the universal local acyclicity to show the perversity of certain complexes on the Beilinson-Drinfeld Grassmannian. For every finite index set I, there is the quotient map $X^I \to \Sigma$ onto a connected component of Σ. Set

$$\mathcal{G}r_I \stackrel{\mathrm{def}}{=} \mathcal{G}r \times_\Sigma X^I.$$

If $I = \{*\}$ has cardinality 1, we write $\mathcal{G}r_X = \mathcal{G}r_I$.

Remark 3.19. Let $X = \mathbb{A}^1_F$ with global coordinate t. Then $\mathbb{G}_a$ acts on X via translations. We construct a $\mathbb{G}_a$-action on $\mathcal{G}r$ as follows. For every $x \in \mathbb{G}_a(R)$, let a_x be the associated automorphism of X_R. If $D \in \Sigma(R)$, then we get an isomorphism $a_{-x} : a_x D \to D$. Let $(D, \mathcal{F}, \beta) \in \mathcal{G}r_G(R)$. Then the $\mathbb{G}_a$-action on $\mathcal{G}r_G \to \Sigma$ is given as

$$(D, \mathcal{F}, \beta) \;\longmapsto\; (a^*_{-x}\mathcal{F}, a^*_{-x}\beta, a_x D).$$

Let $\mathbb{G}_a$ act diagonally on X^I, then the structure morphism $\mathcal{G}r_I \to X^I$ is $\mathbb{G}_a$-equivariant. If $|I| = 1$, then by the transitivity of the $\mathbb{G}_a$-action on X, we get $\mathcal{G}r_X = \mathrm{Gr}_G \times X$. Let $p : \mathcal{G}r_X \to \mathrm{Gr}_G$ be the projection. Then for every perverse sheaf $\mathcal{A}$ on Gr_G, the complex $p^*[1]\mathcal{A}$ is a ULA perverse sheaf on $\mathcal{G}r_X$ by Remark 3.14 (ii) and the smoothness of p.

Now fix a finite index set I of cardinality $k \geq 1$. Consider the base change along $X^I \to \Sigma$ of the k-fold convolution diagram from Definition 3.11,

$$\prod_{i\in I} \mathcal{G}r_{X,i} \xleftarrow{p_I} \tilde{\mathcal{L}}G_I \xrightarrow{q_I} \tilde{\mathcal{G}}r_I \xrightarrow{m_I} \mathcal{G}r_I. \tag{3.3}$$

Now choose a total order $I = \{1, \dots, k\}$, and set $I^o = I \backslash \{1\}$. Then p_I (resp. q_I) is a $\mathcal{L}^+G_I^o$-torsor, where $\mathcal{L}^+G_I^o = X^I \times_{X^{I^o}} \mathcal{L}^+G_{I^o}$.
Let $\mathcal{L}^+G_X = \mathcal{L}^+G \times_\Sigma X$, and denote by $P_{\mathcal{L}^+G_X}(\mathcal{G}r_X)^{\mathrm{ULA}}$ the category of $\mathcal{L}^+G_X$-equivariant ULA perverse sheaves on $\mathcal{G}r_X$. For any $i \in I$, let $\mathcal{A}_{X,i} \in P(\mathcal{G}r_X)^{\mathrm{ULA}}$ such that $\mathcal{A}_{X,i}$ are $\mathcal{L}^+G_X$-equivariant for $i \geq 2$. We have the $\prod_{i\geq 2} \mathcal{L}^+G_{X,i}$-equivariant ULA perverse sheaf $\boxtimes_{i\in I}\mathcal{A}_{X,i}$ on $\prod_{i\in I} \mathcal{G}r_{X,i}$.

Lemma 3.20. *There is a unique ULA perverse sheaf $\widetilde{\boxtimes}_{i\in I}\mathcal{A}_{X,i}$ on $\tilde{\mathcal{G}r}_I$ such that there is a q_I-equivariant isomorphism*[2]

$$q_I^*(\widetilde{\boxtimes}_{i\in I}\mathcal{A}_{X,i}) \;\simeq\; p_I^*(\boxtimes_{i\in I}\mathcal{A}_{X,i}),$$

where q_I-equivariant means with respect to the action on the $\mathcal{L}^+G_I^o$-torsor $q_I : \tilde{\mathcal{L}}G_I \to \tilde{\mathcal{G}r}_I$. If $\mathcal{A}_{X,1}$ is also $\mathcal{L}^+G_X$-equivariant, then $\widetilde{\boxtimes}_{i\in I}\mathcal{A}_{X,i}$ is $\mathcal{L}^+G_I$-equivariant

Remark 3.21. The ind-scheme $\tilde{\mathcal{L}}G_I$ is not of ind-finite type. We explain how the pullback functors p_I^*, q_I^* should be understood. Let $Y_1, \dots, Y_k$ be $\mathcal{L}^+G$-equivariant closed subschemes of $\mathcal{G}r_X$ containing the supports of $\mathcal{A}_1, \dots, \mathcal{A}_k$. Choose $N >> 0$ such that the action of $\mathcal{L}^+G_X$ on each $Y_1, \dots, Y_k$ factors over the smooth affine group scheme $H_N = \mathrm{Res}_{D^{(N)}/X}(G)$, where $D^{(N)}$ is the N-th infinitesimal neighbourhoud of the universal Cartier divisor D over X. Let $K_N = \ker(\mathcal{L}^+G_X \to H_N)$, and $Y = Y_1 \times \dots Y_k$. Then the left K_N-action on each Y_i is trivial, and hence the restriction of the p_I-action resp. q_I-action on $p_I^{-1}(Y)$ to $\prod_{i\geq 2} K_N$ agree. Let $h_N : p_I^{-1}(Y) \to Y_N$ be the resulting $\prod_{i\geq 2} K_N$-torsor. By Lemma A.4 below, we get a factorization

$$\begin{tikzcd} & p_I^{-1}(Y) \arrow[dl, "p_I"'] \arrow[d, "h_N"] \arrow[dr, "q_I"] & \\ Y & Y_N \arrow[l, "p_{I,N}"] \arrow[r, "q_{I,N}"'] & q_I(p_I^{-1}(Y)), \end{tikzcd}$$

where $p_{I,N}, q_{I,N}$ are $\prod_{i\geq 2} H_N$-torsors. In particular, Y_N is a separated scheme of finite type, and we can replace p_I^* (resp. q_I^*) by $p_{I,N}^*$ (resp. $q_{I,N}^*$).

Proof of Lemma 3.20. We use the notation from Remark 3.21 above. The sheaf $p_{I;N}^*(\boxtimes_{i\in I}\mathcal{A}_{X,i})$ is $\prod_{i\geq 2} H_N$-equivariant for the $q_{I,N}$-action. Using descent along smooth torsors (cf. Lemma A.2 below), we get the perverse sheaf $\widetilde{\boxtimes}_{i\in I}\mathcal{A}_{X,i}$, which is ULA. Indeed, $p_{I;N}^*(\boxtimes_{i\in I}\mathcal{A}_{X,i})$ is ULA, and the ULA property is local in the smooth topology. Since the diagram (3.3) is $\mathcal{L}^+G_I$-equivariant, the sheaf $\widetilde{\boxtimes}_{i\in I}\mathcal{A}_{X,i}$ is $\mathcal{L}^+G_I$-equivariant, if $\mathcal{A}_{X,1}$ is $\mathcal{L}^+G_X$-equivariant. This proves the lemma. $\square$

[2]See Remark 3.21 below.

Let U_I be the open locus of pairwise distinct coordinates in X^I. There is a cartesian diagram

$$\begin{tikzcd} \mathcal{G}r_I \arrow[d] & (\mathcal{G}r_X^I)|_{U_I} \arrow[l, "j_I"'] \arrow[d] \\ X^I & U_I. \arrow[l] \end{tikzcd}$$

Proposition 3.22. *The complex $m_{I,*}(\widetilde{\boxtimes}_{i\in I}\mathcal{A}_{X,i})$ is a ULA perverse sheaf on $\mathcal{G}r_I$, and there is a unique isomorphism of perverse sheaves*

$$m_{I,*}(\widetilde{\boxtimes}_{i\in I}\mathcal{A}_{X,i}) \simeq j_{I,!*}(\boxtimes_{i\in I}\mathcal{A}_{X,i}|_{U_I}),$$

which is $\mathcal{L}^+G_I$-equivariant, if $\mathcal{A}_{X,1}$ is $\mathcal{L}^+G_X$-equivariant.

Proof. The sheaf $\widetilde{\boxtimes}_{i\in I}\mathcal{A}_{X,i}$ is by Lemma 3.20 a ULA perverse sheaf on $\tilde{\mathcal{G}r}_I$. Now the restriction of the global convolution morphism m_I to the support of $\widetilde{\boxtimes}_{i\in I}\mathcal{A}_{X,i}$ is a proper morphism, and hence $m_{I,*}(\widetilde{\boxtimes}_{i\in I}\mathcal{A}_{X,i})$ is a ULA complex by Lemma 3.15. Then $m_{I,*}(\widetilde{\boxtimes}_{i\in I}\mathcal{A}_{X,i}) \simeq j_{!*}((\boxtimes_{i\in I}\mathcal{A}_{X,i})|_{U_I})$, as follows from Theorem 3.16 (i) and the formula $u_{!*}\circ v_{!*} \simeq (u\circ v)_{!*}$ for open immersions $V \overset{v}{\hookrightarrow} U \overset{u}{\hookrightarrow} T$, because $m_I|_{U_I}$ is an isomorphism. In particular, $m_{I,*}(\widetilde{\boxtimes}_{i\in I}\mathcal{A}_{X,i})$ is perverse. Since m_I is $\mathcal{L}^+G_I$-equivariant, it follows from proper base change that $m_{I,*}(\widetilde{\boxtimes}_{i\in I}\mathcal{A}_{X,i})$ is $\mathcal{L}^+G_I$-equivariant, if $\mathcal{A}_{X,1}$ is $\mathcal{L}^+G_X$-equivariant. This proves the proposition.

□

3.3 The Symmetric Monoidal Structure

First we equip $P_{\mathcal{L}^+G_X}(\mathcal{G}r_X)^{\mathrm{ULA}}$ with a symmetric monoidal structure $*$ which allows us later to define a symmetric monoidal structure with respect to the usual convolution (3.1) of L^+G-equivariant perverse sheaves on Gr_G.
Fix I, and let U_I be the open locus of pairwise distinct coordinates in X^I. Then the diagram

$$\begin{tikzcd} \mathcal{G}r_X \arrow[r, "i_I"] \arrow[d] & \mathcal{G}r_I \arrow[d] & (\mathcal{G}r_X^I)|_{U_I} \arrow[l, "j_I"'] \arrow[d] \\ X \arrow[r, "\mathrm{diag}"] & X^I & U_I. \arrow[l] \end{tikzcd} \tag{3.4}$$

is cartesian.

Definition 3.23. Fix some total order on I. For every tuple $(\mathcal{A}_{X,i})_{i\in I}$ with $\mathcal{A}_{X,i} \in P(\mathcal{G}r_X)^{\mathrm{ULA}}$ for $i \in I$, the *I-fold fusion product* $*_{i\in I}\mathcal{A}_{X,i}$ is the complex

$$*_{i\in I}\mathcal{A}_{X,i} \overset{\mathrm{def}}{=} i_I^*[-k+1]j_{I,!*}((\boxtimes_{i\in I}\mathcal{A}_{X,i})|_{U_I}) \in D_c^b(\mathcal{G}r_X, \bar{\mathbb{Q}}_\ell),$$

where $k = |I|$.

Now let $\pi : I \to J$ be a surjection of finite index sets. For $j \in J$, let $I_j = \pi^{-1}(j)$, and denote by U_π the open locus in X^I such that the I_j-coordinates are pairwise distinct from the $I_{j'}$-coordinates for $j \neq j'$. Then the diagram

$$\begin{tikzcd} \mathcal{G}r_J \arrow[r, "i_\pi"] \arrow[d] & \mathcal{G}r_I \arrow[d] & (\prod_j \mathcal{G}r_{I_j})|_{U_\pi} \arrow[l, "j_\pi"'] \arrow[d] \\ X^J \arrow[r] & X^I & U_\pi, \arrow[l] \end{tikzcd} \tag{3.5}$$

is cartesian. The following theorem combined with Proposition 3.22 is the key to the symmetric monoidal structure:

Theorem 3.24. *Let I be a finite index set, and let $\mathcal{A}_{X,i} \in P_{\mathcal{L}^+G_X}(\mathcal{G}r_X)^{ULA}$ for $i \in I$. Let $\pi : I \to J$ be a surjection of finite index sets, and set $k_\pi = |I| - |J|$.*
(i) *As complexes*

$$i_\pi^*[-k_\pi] j_{I,!*}((\boxtimes_{i\in I}\mathcal{A}_{X,i})|_{U_I}) \simeq i_\pi^![k_\pi] j_{I,!*}((\boxtimes_{i\in I}\mathcal{A}_{X,i})|_{U_I}),$$

and both are $\mathcal{L}^+G_J$-equivariant ULA perverse sheaves on $\mathcal{G}r_J$. Hence, $*_{i\in I}\mathcal{A}_{X,i} \in P_{\mathcal{L}^+G_X}(\mathcal{G}r_X)^{ULA}$.
(ii) *There is an associativity and a commutativity constraint for the fusion product such that there is a canonical isomorphism*

$$*_{i\in I}\mathcal{A}_{X,i} \simeq *_{j\in J}(*_{i\in I_j}\mathcal{A}_{X,i}),$$

where $I_j = \pi^{-1}(j)$ for $j \in J$. In particular, $(P_{\mathcal{L}^+G_X}(\mathcal{G}r_X)^{ULA}, *)$ is symmetric monoidal.

Proof. Factor π as a chain of surjective maps $I = I_1 \to I_2 \to \ldots \to I_{k_\pi} = J$ with $|I_{i+1}| = |I_i| + 1$, and consider the corresponding chain of smooth Cartier divisors

$$X^J = X^{I_{k_\pi}} \longrightarrow \ldots \longrightarrow X^{I_2} \longrightarrow X^{I_1} = X^I.$$

By Proposition 3.22, the complex $j_{I,!*}((\boxtimes_{i\in I}\mathcal{A}_{X,i})|_{U_I})$ is ULA. Then part (i) follows inductively from Theorem 3.16 (i) and (ii). This shows (i).
Let $\tau : I \to I$ be a bijection. Then τ acts on X^I by permutation of coordinates, and diagram (3.4) becomes equivariant for this action. Then

$$\tau^* j_{I,!*}((\boxtimes_{i\in I}\mathcal{A}_{X,i})|_{U_I}) \simeq j_{I,!*}((\boxtimes_{i\in I}\mathcal{A}_{X,\tau^{-1}(i)})|_{U_I}).$$

Since the action on $\mathrm{diag}(X) \subset X^I$ is trivial, we obtain

$$\begin{aligned} i_I^* j_{I,!*}((\boxtimes_{i\in I}\mathcal{A}_{X,i})|_{U_I}) &\simeq i_I^*\tau^* j_{I,!*}((\boxtimes_{i\in I}\mathcal{A}_{X,i})|_{U_I}) \\ &\simeq i_I^* j_{I,!*}((\boxtimes_{i\in I}\mathcal{A}_{X,\tau^{-1}(i)})|_{U_I}), \end{aligned}$$

and hence $*_{i\in I}\mathcal{A}_{X,i} \simeq *_{i\in I}\mathcal{A}_{X,\tau^{-1}(i)}$. It remains to give the isomorphism defining the symmetric monoidal structure. Since $j_I = j_\pi \circ \prod_j j_{I_j}$, diagram (3.5) gives

$$(j_{I,!*}((\boxtimes_{i\in I}\mathcal{A}_{X,i})|_{U_I}))|_{U_\pi} \simeq \boxtimes_{j\in J} j_{I_j,!*}((\boxtimes_{i\in I_j}\mathcal{A}_{X,i})|_{U_{I_j}}).$$

Applying $(i_\pi|_{U_\pi})^*[k_\pi]$ and using that $U_\pi \cap X^J = U_J$, we obtain

$$(i_\pi^*[k_\pi]j_{I,!*}((\boxtimes_{i\in I}\mathcal{A}_{X,i})|_{U_I}))|_{U_J} \simeq \boxtimes_{j\in J}(*_{i\in I_j}\mathcal{A}_{X,i}).$$

But by (i), the perverse sheaf $i_\pi^*[k_\pi]j_{I,!*}((\boxtimes_{i\in I}\mathcal{A}_{X,i})|_{U_I})$ is ULA, thus

$$i_\pi^*[k_\pi]j_{I,!*}((\boxtimes_{i\in I}\mathcal{A}_{X,i})|_{U_I}) \simeq j_{J,!*}((\boxtimes_{j\in J}(*_{i\in I_j}\mathcal{A}_{X,i}))|_{U_J}),$$

and restriction along the diagonal in X^J gives the isomorphism $*_{i\in I}\mathcal{A}_{X,i} \simeq *_{j\in J}(*_{i\in I_j}\mathcal{A}_{X,i})$. This proves (ii). □

EXAMPLE 3.25. Let $G = \{e\}$ be the trivial group. Then $\mathcal{G}r_X = X$. Let $\mathrm{Loc}(X)$ be the category of ℓ-adic local systems on X. Using Corollary 3.18, we obtain an equivalence of symmetric monoidal categories

$$\mathcal{H}^0 \circ [-1] : (P(X)^{\mathrm{ULA}}, *) \xrightarrow{\simeq} (\mathrm{Loc}(X), \otimes),$$

where $\mathrm{Loc}(X)$ is endowed with the usual symmetric monoidal structure with respect to the tensor product $\otimes$.

COROLLARY 3.26. *Let $D_c^b(X, \bar{\mathbb{Q}}_\ell)^{\mathrm{ULA}}$ be the category of ULA complexes on X. Denote by $f : \mathcal{G}r_X \to X$ the structure morphism. Then the functor*

$$f_*[-1] : (P(\mathcal{G}r_X)^{\mathrm{ULA}}, *) \longrightarrow (D_c^b(X, \bar{\mathbb{Q}}_\ell), \otimes)$$

is symmetric monoidal.

Proof. If $\mathcal{A}_X \in P(\mathcal{G}r_X)^{\mathrm{ULA}}$, then $f_*\mathcal{A}_X \in D_c^b(X, \bar{\mathbb{Q}}_\ell)^{\mathrm{ULA}}$ by Lemma 3.15 and the ind-properness of f. Now apply f_* to the isomorphism in Theorem 3.24 (ii) defining the symmetric monoidal structure on $P(\mathcal{G}r_X)^{\mathrm{ULA}}$. Then by proper base change and going backwards through the arguments in the proof of Theorem 3.24 (ii), we get that $f_*[-1]$ is symmetric monoidal. □

COROLLARY 3.27. *Let $X = \mathbb{A}^1_F$. Let $p : \mathcal{G}r_X \to \mathrm{Gr}_G$ be the projection, cf. Remark 3.19.*
(i) *The functor*

$$p^*[1] : P_{L^+G}(\mathrm{Gr}_G) \longrightarrow P_{\mathcal{L}^+G_X}(\mathcal{G}r_X)^{\mathrm{ULA}}$$

embeds $P_{L^+G}(\mathrm{Gr}_G)$ as a full subcategory and is an equivalence of categories with the subcategory of $\mathbb{G}_a$-equivariant objects in $P_{\mathcal{L}^+G_X}(\mathcal{G}r_X)^{\mathrm{ULA}}$.
(ii) *For every I and $\mathcal{A}_i \in P_{L^+G}(\mathrm{Gr}_G)$, $i \in I$, there is a canonical $\mathcal{L}^+G_X$-equivariant isomorphism*

$$p^*[1](\star_{i\in I}\mathcal{A}_i) \simeq *_{i\in I}(p^*[1]\mathcal{A}_i),$$

where the product is taken with respect to some total order on I.

Proof. Under the simply transitive action of $\mathbb{G}_a$ on X, the isomorphism $\mathcal{G}r_X \simeq \mathrm{Gr}_G \times X$ is compatible with the action of L^+G under the zero section $L^+G \hookrightarrow \mathcal{L}^+G_X$. By Lemma 3.19, the complex $p^*[1]\mathcal{A}$ is a ULA perverse sheaf on $\mathcal{G}r_X$. It is obvious that the functor $p^*[1]$ is fully faithful. Denote by $i_0 : \mathrm{Gr}_G \to \mathcal{G}r_X$ the zero section. If $\mathcal{A}_X$ on $\mathcal{G}r_X$ is $\mathbb{G}_a$-equivariant, then $\mathcal{A}_X \simeq p^*[1]i_0^*[-1]\mathcal{A}_X$. This proves (i).
By Remark 3.12, the fiber over $\mathrm{diag}(\{0\}) \in X^I(F)$ of (3.3) is the usual convolution diagram (3.1). Hence, by proper base change,

$$i_0^*[-1](\ast_{i\in I}p^*[1]\mathcal{A}_i) \;\simeq\; \star_{i\in I}i_0^*[-1]p^*[1]\mathcal{A}_i \;\simeq\; \star_{i\in I}\mathcal{A}_i.$$

Since $\ast_{i\in I}p^*[1]\mathcal{A}_i$ is $\mathbb{G}_a$-equivariant, this proves (ii). □

Now we are prepared for the proof of Theorem 3.1.

Proof of Theorem 3.1. Let $X = \mathbb{A}^1_F$. For every $\mathcal{A}_1, \mathcal{A}_2 \in P(\mathrm{Gr}_G)$ with $\mathcal{A}_2$ being L^+G-equivariant, we have to prove that $\mathcal{A}_1 \star \mathcal{A}_2 \in P(\mathrm{Gr}_G)$. By Theorem 3.24 (i), the $\ast$-convolution is perverse. Then the perversity of $\mathcal{A}_1 \star \mathcal{A}_2$ follows from Corollary 3.27 (ii). Again by Corollary 3.27 (ii), the convolution $\mathcal{A}_1 \star \mathcal{A}_2$ is L^+G-equivariant, if $\mathcal{A}_1$ is L^+G-equivariant. This proves (i).
We have to equip $(P_{L^+G}(\mathrm{Gr}_G), \star)$ with a symmetric monoidal structure. By Corollary 3.27, the tuple $(P_{L^+G}(\mathrm{Gr}_G), \star)$ is a full subcategory of $(P_{\mathcal{L}^+G_X}(\mathcal{G}r_X)^{\mathrm{ULA}}, \ast)$, and the latter is symmetric monoidal by Theorem 3.24 (ii), hence so is $(P_{L^+G}(\mathrm{Gr}_G), \star)$. Since taking cohomology is only graded commutative, we need to modify the commutativity constraint of $(P_{L^+G}(\mathrm{Gr}_G), \star)$ by a sign as follows. Let $\bar{F}$ be a separable closure of F. The $L^+G_{\bar{F}}$-orbits in one connected component of $\mathrm{Gr}_{G,\bar{F}}$ are all either even or odd dimensional. Because the Galois action on $\mathrm{Gr}_{G,\bar{F}}$ commutes with the $L^+G_{\bar{F}}$-action, the connected components of Gr_G are divided into those of even or odd parity. Consider the corresponding $\mathbb{Z}/2$-grading on $P_{L^+G}(\mathrm{Gr}_G)$ given by the parity of the connected components of Gr_G. Then we equip $(P_{L^+G}(\mathrm{Gr}_G), \star)$ with the super commutativity constraint with respect to this $\mathbb{Z}/2$-grading, i.e. if $\mathcal{A}$ (resp. $\mathcal{B}$) is an L^+G-equivariant perverse sheaf supported on a connected component $X_{\mathcal{A}}$ (resp. $X_{\mathcal{B}}$) of Gr_G, then the modified commutativity constraint differs by the sign $(-1)^{p(X_{\mathcal{A}})p(X_{\mathcal{B}})}$, where $p(X) \in \mathbb{Z}/2$ denotes the parity of a connected component X of Gr_G.
Now consider the global cohomology functor

$$\omega(\text{-}) = \bigoplus_{i\in\mathbb{Z}} R^i\Gamma(\mathrm{Gr}_{G,\bar{F}}, (\text{-})_{\bar{F}}) \colon P_{L^+G}(\mathrm{Gr}_G) \longrightarrow \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}.$$

Let $f : \mathcal{G}r_X \to X$ be the structure morphism. Then the diagram

$$\begin{tikzcd} P_{\mathcal{L}^+G_{X,\bar{F}}}(\mathcal{G}r_{X,\bar{F}})^{\mathrm{ULA}} \arrow[r, "f_*[-1]"] & D_c^b(X_{\bar{F}}, \bar{\mathbb{Q}}_\ell) \arrow[d, "\oplus_{i\in\mathbb{Z}}\mathcal{H}^i \circ i_0^*"] \\ P_{L^+G}(\mathrm{Gr}_G) \arrow[u, "p^*[1] \circ (\text{-})_{\bar{F}}"] \arrow[r, "\omega"] & \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell} \end{tikzcd}$$

is commutative up to natural isomorphism. Now if $\mathcal{A}$ is a perverse sheaf supported on a connected component X of Gr_G, then by a theorem of Lusztig [12, Theorem 11c],

$$R^i\Gamma(\mathrm{Gr}_{G,\bar{F}}, \mathcal{A}_{\bar{F}}) = 0, \qquad\qquad i \not\equiv p(X) \pmod 2,$$

where $p(X) \in \mathbb{Z}/2$ denotes the parity of X. Hence, Corollary 3.26 shows that ω is symmetric monoidal with respect to the super commutativity constraint on $P_{L^+G}(\mathrm{Gr}_G)$. To prove uniqueness of the symmetric monoidal structure, it is enough to prove that ω is faithful, which follows from Lemma 4.4 below. This proves (ii). □

4 The Tannakian Structure

In this section we assume that $F = \bar{F}$ is separably closed. Let $X_+^\vee$ be a set of representatives of the L^+G-orbits on Gr_G. For $\mu \in X_+^\vee$ we denote by $\mathcal{O}_\mu$ the corresponding L^+G-orbit, and by $\overline{\mathcal{O}}_\mu$ its reduced closure with open embeddding $j^\mu : \mathcal{O}_\mu \hookrightarrow \overline{\mathcal{O}}_\mu$. We equip $X_+^\vee$ with the partial order defined as follows: for every $\lambda, \mu \in X_+^\vee$, we define $\lambda \leq \mu$ if and only if $\mathcal{O}_\lambda \subset \overline{\mathcal{O}}_\mu$.

Proposition 4.1. *The category $P_{L^+G}(\mathrm{Gr}_G)$ is semisimple with simple objects the intersection complexes*

$$\mathrm{IC}_\mu = j^\mu_{!*}\bar{\mathbb{Q}}_\ell[\dim(\mathcal{O}_\mu)], \qquad \textit{for } \mu \in X_+^\vee.$$

In particular, if ${}^pj^\mu_*$ (resp. ${}^pj^\mu_!$) denotes the perverse push forward (resp. perverse extension by zero), then $j^\mu_{!*} \simeq {}^pj^\mu_! \simeq {}^pj^\mu_*$.

Proof. For any $\mu \in X_+^\vee$, the étale fundamental group $\pi_1^{\text{ét}}(\mathcal{O}_\mu)$ is trivial. Indeed, since $\overline{\mathcal{O}}_\mu \backslash \mathcal{O}_\mu$ is of codimension at least 2 in $\overline{\mathcal{O}}_\mu$, Grothendieck's purity theorem implies that $\pi_1^{\text{ét}}(\mathcal{O}_\mu) = \pi_1^{\text{ét}}(\overline{\mathcal{O}}_\mu)$. The latter group is trivial by [SGA1, XI.1 Corollaire 1.2], because $\overline{\mathcal{O}}_\lambda$ is normal (cf. [6]), projective and rational. This shows the claim.

Since by [17, Lemme 2.3] the stabilizers of the L^+G-action are connected, any L^+G-equivariant irreducible local system supported on $\mathcal{O}_\mu$ is isomorphic to the constant sheaf $\bar{\mathbb{Q}}_\ell$. Hence, the simple objects in $P_{L^+G}(\mathrm{Gr}_G)$ are the intersection complexes IC_μ for $\mu \in X_+^\vee$.

To show semisimplicity of the Satake category, it is enough to prove

$$\mathrm{Ext}^1_{P(\mathrm{Gr}_G)}(\mathrm{IC}_\lambda, \mathrm{IC}_\mu) = \mathrm{Hom}_{D_c^b(\mathrm{Gr}_G)}(\mathrm{IC}_\lambda, \mathrm{IC}_\mu[1]) \stackrel{!}{=} 0.$$

We distinguish several cases:

Case (i): $\lambda = \mu$.

Let $\mathcal{O}_\mu \xrightarrow{j} \overline{\mathcal{O}}_\mu \xleftarrow{i} \overline{\mathcal{O}}_\mu \backslash \mathcal{O}_\mu$, and consider the exact sequence of abelian groups

$$\mathrm{Hom}(\mathrm{IC}_\mu, i_!i^!\mathrm{IC}_\mu[1]) \longrightarrow \mathrm{Hom}(\mathrm{IC}_\mu, \mathrm{IC}_\mu[1]) \longrightarrow \mathrm{Hom}(\mathrm{IC}_\mu, j_*j^*\mathrm{IC}_\mu[1]) \quad (4.1)$$

associated to the distinguished triangle $i_! i^! \mathrm{IC}_\mu \to \mathrm{IC}_\mu \to j_* j^* \mathrm{IC}_\mu$. We show that the outer groups in (4.1) are trivial. Indeed, the last group is trivial, since $j^*\mathrm{IC}_\mu = \bar{\mathbb{Q}}_\ell[\dim(\mathcal{O}_\mu)]$ gives

$$\mathrm{Hom}(\mathrm{IC}_\mu, j_* j^* \mathrm{IC}_\mu[1]) = \mathrm{Hom}(j^*\mathrm{IC}_\mu, j^*\mathrm{IC}_\mu[1]) = \mathrm{Ext}^1(\bar{\mathbb{Q}}_\ell, \bar{\mathbb{Q}}_\ell).$$

And $\mathrm{Ext}^1(\bar{\mathbb{Q}}_\ell, \bar{\mathbb{Q}}_\ell) = H^1_{\text{ét}}(\mathcal{O}_\mu, \bar{\mathbb{Q}}_\ell) = 0$, because $\mathcal{O}_\mu$ is simply connected. To show that the first group

$$\mathrm{Hom}(\mathrm{IC}_\mu, i_! i^! \mathrm{IC}_\mu[1]) = \mathrm{Hom}(i^*\mathrm{IC}_\mu, i^!\mathrm{IC}_\mu[1])$$

is trivial, note that $i^*\mathrm{IC}_\mu$ lives in perverse degrees ≤ -1 because the 0th perverse cohomology vanishes, since IC_μ is a middle perverse extension along j. Hence, the Verdier dual $D(i^*\mathrm{IC}_\mu)[1] = i^!\mathrm{IC}_\mu[1]$ lives in perverse degrees ≥ 0. This proves case (i).

Case (ii): $\lambda \neq \mu$ *and either* $\lambda \leq \mu$ *or* $\mu \leq \lambda$.

If $\lambda \leq \mu$, let $i : \overline{\mathcal{O}}_\lambda \hookrightarrow \overline{\mathcal{O}}_\mu$ be the closed embedding. Then

$$\mathrm{Hom}(i_*\mathrm{IC}_\lambda, \mathrm{IC}_\mu[1]) = \mathrm{Hom}(\mathrm{IC}_\lambda, i^!\mathrm{IC}_\mu[1]),$$

and this vanishes, since $i^!\mathrm{IC}_\mu[1]$ lives in perverse degrees ≥ 1 or equivalently, the Verdier dual $D(i^!\mathrm{IC}_\mu) = i^*\mathrm{IC}_\mu$ lives in perverse degrees ≤ -2. Indeed, by a theorem of Lusztig [12, Theorem 11c], $i^*\mathrm{IC}_\mu$ is concentrated in even perverse degrees, and the 0th perverse cohomology vanishes, since IC_μ is a middle perverse extension. If $\mu \leq \lambda$, let $i : \overline{\mathcal{O}}_\mu \hookrightarrow \overline{\mathcal{O}}_\lambda$ the closed embedding. Then

$$\mathrm{Hom}(\mathrm{IC}_\lambda, i_*\mathrm{IC}_\mu[1]) = \mathrm{Hom}(i^*\mathrm{IC}_\lambda, \mathrm{IC}_\mu[1])$$

vanishes, since $i^*\mathrm{IC}_\lambda$ lives in perverse degrees ≤ -2 as before. This proves case (ii).

Case (iii): $\lambda \not\leq \mu$ *and* $\mu \not\leq \lambda$.

We may assume that λ and μ are contained in the same connected component of Gr_G. Choose some $\nu \in X_+^\vee$ with $\lambda, \mu \leq \nu$. Consider the cartesian diagram

$$\begin{tikzcd} \overline{\mathcal{O}}_\lambda \times_{\overline{\mathcal{O}}_\nu} \overline{\mathcal{O}}_\mu \arrow[r, hook, "\iota_1"] \arrow[d, hook, "\iota_2"] & \overline{\mathcal{O}}_\mu \arrow[d, hook, "i_2"] \\ \overline{\mathcal{O}}_\lambda \arrow[r, hook, "i_1"] & \overline{\mathcal{O}}_\nu . \end{tikzcd}$$

Then adjunction gives

$$\mathrm{Hom}(i_{1,*}\mathrm{IC}_\lambda, i_{2,*}\mathrm{IC}_\mu[1]) = \mathrm{Hom}(i_2^* i_{1,*}\mathrm{IC}_\lambda, \mathrm{IC}_\mu[1]), \tag{4.2}$$

and $i_2^* i_{1,*}\mathrm{IC}_\lambda \simeq \iota_{1,*}\iota_2^*\mathrm{IC}_\lambda$ by proper base change. Hence (4.2) equals $\mathrm{Hom}(\iota_2^*\mathrm{IC}_\lambda, \iota_1^!\mathrm{IC}_\mu[1])$ which vanishes. This proves case (iii), hence the proposition. □

The affine group scheme $L^+\mathbb{G}_m$ acts on Gr_G as follows. For $x \in L^+\mathbb{G}_m(R)$, denote by v_x the automorphism of $\mathrm{Spec}(R[\![t]\!])$ induced by multiplication with x. If $\mathcal{F}$ is a G-torsor over $\mathrm{Spec}(R[\![t]\!])$, we denote by $v_x^*\mathcal{F}$ the pullback of $\mathcal{F}$ along v_x. Let $(\mathcal{F}, \beta) \in \mathrm{Gr}_G(R)$. Then the action of $L^+\mathbb{G}_m$ on Gr_G is given by

$$(\mathcal{F}, \beta) \longmapsto (v_{x^{-1}}^*\mathcal{F}, v_{x^{-1}}^*\beta),$$

and is called the *Virasoro action.*
Note that every L^+G-orbit in Gr_G is stable under $L^+\mathbb{G}_m$. The semidirect product $L^+G \rtimes L^+\mathbb{G}_m$ acts on Gr_G, and the action on each orbit factors through a smooth connected affine group scheme. Hence, we may consider the category $P_{L^+G\rtimes L^+\mathbb{G}_m}(\mathrm{Gr}_G)$ of $L^+G \rtimes L^+\mathbb{G}_m$-equivariant perverse sheaves on Gr_G.

Corollary 4.2. *The forgetful functor*

$$P_{L^+G\rtimes L^+\mathbb{G}_m}(\mathrm{Gr}_G) \longrightarrow P_{L^+G}(\mathrm{Gr}_G)$$

is an equivalence of categories. In particular, the category $P_{L^+G}(\mathrm{Gr}_G)$ does not depend on the choice of the parameter t.

Proof. By Proposition 4.1 above, every L^+G-equivariant perverse sheaf is a direct sum of intersection complexes, and these are $L^+\mathbb{G}_m$-equivariant. □

Remark 4.3. If $X = \mathbb{A}^1_F$ is the base curve, then the global affine Grassmannian $\mathcal{G}r_X$ splits as $\mathcal{G}r_X \simeq \mathrm{Gr}_G \times X$. Corollary 4.2 shows that we can work over an arbitrary curve X as follows. Let $\mathcal{X}$ be the functor on the category of F-algebras R parametrizing tuples (x, s) with

$$\begin{cases} x \in X(R) \text{ is a point;} \\ s \text{ is a continuous isomorphism of } R\text{-modules } \hat{\mathcal{O}}_{X_R,x} \overset{\simeq}{\to} R[\![t]\!], \end{cases}$$

where $\hat{\mathcal{O}}_{X_R,x}$ is the completion of the R-module $\mathcal{O}_{X_R,x}$ along the maximal ideal $\mathfrak{m}_x$ at x. The affine group scheme $L^+\mathbb{G}_m$ operates from left on $\mathcal{X}$ by $(g, (x, s)) \mapsto (x, gs)$. The projection $p : \mathcal{X} \to X, (x, s) \mapsto x$ gives $\mathcal{X}$ the structure of a $L^+\mathbb{G}_m$-torsor. Then $\mathcal{G}r_X \simeq \mathrm{Gr}_G \times^{L^+\mathbb{G}_m} \mathcal{X}$, and we get a diagram of $L^+\mathbb{G}_m$-torsors

$$\begin{tikzcd} & \mathrm{Gr}_G \times \mathcal{X} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{Gr}_G \times X & & \mathcal{G}r_X. \end{tikzcd}$$

For any $\mathcal{A} \in P_{L^+G}(\mathrm{Gr}_G)$, the perverse sheaf $\mathcal{A} \boxtimes \bar{\mathbb{Q}}_\ell[1]$ on $\mathrm{Gr}_G \times X$ is $L^+\mathbb{G}_m$-equivariant by Corollary 4.2. Hence, $p^*(\mathcal{A} \boxtimes \bar{\mathbb{Q}}_\ell[1])$ descends along q to a perverse sheaf $\mathcal{A}\tilde{\boxtimes}\bar{\mathbb{Q}}_\ell[1]$ on $\mathcal{G}r_X$.

We are going to define a fiber functor on $P_{L^+G}(\mathrm{Gr}_G)$. Denote by

$$\omega(\text{-}) = \bigoplus_{i\in\mathbb{Z}} R^i\Gamma(\mathrm{Gr}_G, \text{-}) : P_{L^+G}(\mathrm{Gr}_G) \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell} \tag{4.3}$$

the cohomology functor with values in the category of finite dimensional $\bar{\mathbb{Q}}_\ell$-vector spaces.

Lemma 4.4. *The functor* $\omega : P_{L^+G}(\mathrm{Gr}_G) \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}$ *is additive, exact and faithful.*

Proof. Additivity is immediate. Exactness follows from Proposition 4.1, since every exact sequence splits, and ω is additive. To show faithfulness, it is enough, again by Proposition 4.1, to show that the intersection cohomology of the Schubert varieties is non-zero. Indeed, we claim that the intersection cohomology of any projective variety T is non-zero. Embedding T into projective space and projecting down on hyperplanes, we obtain a generically finite morphism $\pi : T \to \mathbb{P}^n$. Using the decomposition theorem, we see that the intersection complex of $\mathbb{P}^n$ appears as a direct summand in $\pi_*\mathrm{IC}_T$. Hence, the intersection cohomology of T is non-zero. This proves the lemma. □

Corollary 4.5. *The tuple* $(P_{L^+G}(\mathrm{Gr}_G), \star)$ *is a neutralized Tannakian category with fiber functor* $\omega : P_{L^+G}(\mathrm{Gr}_G) \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}$.

Proof. We check the criterion in [5, Prop. 1.20]:
The category $(P_{L^+G}(\mathrm{Gr}_G), \star)$ is abelian $\bar{\mathbb{Q}}_\ell$-linear (cf. Appendix A below) and by Theorem 3.1 (ii) above symmetric monoidal. To prove that ω is a fiber functor, we must show that ω is an additive exact faithful tensor functor. Lemma 4.4 shows that ω is additive exact and faithful, and Theorem 3.1 (ii) shows that ω is symmetric monoidal.
It remains to show that $(P_{L^+G}(\mathrm{Gr}_G), \star)$ has a unit object and that any one dimensional object has an inverse. The unit object is the constant sheaf $\mathrm{IC}_0 = \bar{\mathbb{Q}}_\ell$ concentrated in the base point e_0. We have $\mathrm{End}(\mathrm{IC}_0) = \bar{\mathbb{Q}}_\ell$, and $\dim(\omega(\mathrm{IC}_0)) = 1$. Now, let $\mathcal{A} \in P_{L^+G}(\mathrm{Gr}_G)$ with $\dim(\omega(A)) = 1$. Then $\mathcal{A}$ is supported on a L^+G-invariant closed point $z_0 \in \mathrm{Gr}_G$. There exists z in the center of LG such that $z \cdot z_0 = e_0$ is the basepoint. If $z_0' = z \cdot e_0$, then the intersection cohomology complex $\mathcal{A}'$ supported on z_0' satisfies $\mathcal{A} \star \mathcal{A}' = \mathrm{IC}_0$. This shows the corollary. □

5 The Geometric Satake Equivalence

In this section we assume that $F = \bar{F}$ is separably closed. Denote by $H = \mathrm{Aut}^\star(\omega)$ the affine $\bar{\mathbb{Q}}_\ell$-group scheme of tensor automorphisms defined by Corollary 4.5.

Theorem 5.1. *The group scheme* H *is a connected reductive group over* $\bar{\mathbb{Q}}_\ell$ *which is dual to* G *in the sense of Langlands, i.e. if we denote by* $\hat{G}$ *the Langlands dual group with respect to some pinning of* G*, then there exists an isomorphism* $H \simeq \hat{G}$ *determined uniquely up to inner automorphisms.*

We fix some notation. Let T be a maximal split torus of G and B a Borel subgroup containing T with unipotent radical U. We denote by $\langle \text{-}, \text{-} \rangle$ the natural pairing between $X = \mathrm{Hom}(T, \mathbb{G}_m)$ and $X^\vee = \mathrm{Hom}(\mathbb{G}_m, T)$. Let $R \subset X$ be the root system associated to (G, T), and R_+ be the set of positive roots corresponding to B. Let $R^\vee \subset X^\vee$ the dual root system with the bijection $R \to R^\vee$, $\alpha \mapsto \alpha^\vee$. Denote by $R_+^\vee$ the set of positive coroots. Let W the Weyl group of (G, T). Consider the half sum of all positive roots

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha.$$

Let $Q^\vee$ (resp. $Q_+^\vee$) the subgroup (resp. submonoid) of $X^\vee$ generated by $R^\vee$ (resp. $R_+^\vee$). We denote by

$$X_+^\vee = \{\mu \in X^\vee \mid \langle \alpha, \mu \rangle \geq 0, \ \forall \alpha \in R_+\}$$

the cone of dominant cocharacters with the partial order on $X^\vee$ defined as follows: $\lambda \leq \mu$ if and only if $\mu - \lambda \in Q_+^\vee$.
Note that $(X_+^\vee, \leq)$ identifies with the partially ordered set of orbit representatives in Section 4 as follows: for every $\mu \in X_+^\vee$, let t^μ the corresponding element in $LT(F)$, and denote by $e_0 \in \mathrm{Gr}_G$ the base point. Then $\mu \mapsto t^\mu \cdot e_0$ gives the bijection of partial ordered sets, i.e. the orbit closures satisfy

$$\overline{\mathcal{O}}_\mu \ = \ \coprod_{\lambda \leq \mu} \mathcal{O}_\lambda, \qquad \text{(Cartan stratification)}$$

where $\mathcal{O}_\lambda$ denotes the L^+G-orbit of $t^\lambda \cdot e_0$ (cf. [17, §2]).
For every $\nu \in X^\vee$, consider the LU-orbit $S_\nu = LU \cdot t^\nu e_0$ inside Gr_G (cf. [17, §3]). Then S_ν is a locally closed ind-subscheme of Gr_G, and for every $\mu \in X_+^\vee$, there is a locally closed stratification

$$\overline{\mathcal{O}}_\mu \ = \ \coprod_{\nu \in X^\vee} S_\nu \cap \overline{\mathcal{O}}_\mu. \qquad \text{(Iwasawa stratification)}$$

For $\mu \in X_+^\vee$, let

$$\Omega(\mu) \ \stackrel{\text{def}}{=} \ \{\nu \in X^\vee \mid w\nu \leq \mu, \ \forall w \in W\}.$$

Proposition 5.2. *For every $\nu \in X^\vee$ and $\mu \in X_+^\vee$ the stratum $S_\nu \cap \overline{\mathcal{O}}_\mu$ is non-empty if and only if $\nu \in \Omega(\mu)$, and in this case it is pure of dimension $\langle \rho, \mu + \nu \rangle$.*

Proof. The schemes G, B, T and all the associated data are already defined over a finitely generated $\mathbb{Z}$-algebra. By generic flatness, we reduce to the case where $F = \mathbb{F}_q$ is a finite field. The proposition is proven in [8, Proof of Lemma 2.17.4], which relies on [17, Theorem 3.1]. □

For every sequence $\mu_\bullet = (\mu_1, \ldots, \mu_k)$ of dominant cocharacters, consider the projective variety over F

$$\overline{\mathcal{O}}_{\mu_\bullet} \ \stackrel{\text{def}}{=} \ p^{-1}(\overline{\mathcal{O}}_{\mu_1}) \times^{L^+G} \ldots \times^{L^+G} p^{-1}(\overline{\mathcal{O}}_{\mu_{k-1}}) \times^{L^+G} \overline{\mathcal{O}}_{\mu_k},$$

inside $LG\times^{L^+G}\ldots\times^{L^+G}\mathrm{Gr}_G$, where $p: LG\to \mathrm{Gr}_G$ denotes the quotient map. The quotient exists, by the ind-properness of Gr_G and Lemma A.4 below.
Now let $|\mu_\bullet| = \mu_1+\ldots+\mu_k$. Then the restriction $m_{\mu_\bullet} = m|_{\overline{\mathcal{O}}_{\mu_\bullet}}$ of the k-fold convolution morphism factors as

$$m_{\mu_\bullet}: \overline{\mathcal{O}}_{\mu_\bullet} \;\longrightarrow\; \overline{\mathcal{O}}_{|\mu_\bullet|},$$

and is an isomorphism over $\mathcal{O}_{|\mu_\bullet|}\subset\overline{\mathcal{O}}_{|\mu_\bullet|}$.

Corollary 5.3. *For every $\lambda\in X_+^\vee$ with $\lambda\le|\mu_\bullet|$ and $x\in\mathcal{O}_\lambda(F)$, one has*

$$\dim(m_{\mu_\bullet}^{-1}(x)) \;\le\; \langle\rho, |\mu_\bullet|-\lambda\rangle,$$

i.e. the convolution morphism is semismall.

Proof. The proof of [17, Lemme 9.3] carries over word by word, and we obtain that $\dim(m_{\mu_\bullet}^{-1}(\mathcal{O}_\lambda)) \le \langle\rho,|\mu_\bullet|+\lambda\rangle$. Since $m_{\mu_\bullet}$ is L^+G-equivariant and $\dim(\mathcal{O}_\lambda)=\langle 2\rho,\lambda\rangle$, the corollary follows. □

The convolution $\mathrm{IC}_{\mu_1}\star\ldots\star\mathrm{IC}_{\mu_n}$ is a L^+G-equivariant perverse sheaf, and by Proposition 4.1, we can write

$$\mathrm{IC}_{\mu_1}\star\ldots\star\mathrm{IC}_{\mu_n} \;\simeq\; \bigoplus_{\lambda\le|\mu_\bullet|} V_{\mu_\bullet}^\lambda\otimes\mathrm{IC}_\lambda, \tag{5.1}$$

where $V_{\mu_\bullet}^\lambda$ are finite dimensional $\bar{\mathbb{Q}}_\ell$-vector spaces.

Lemma 5.4. *For every $\lambda\in X_+^\vee$ with $\lambda\le|\mu_\bullet|$ and $x\in\mathcal{O}_\lambda(F)$, the vector space $V_{\mu_\bullet}^\lambda$ has a canonical basis indexed by the irreducible components of $m_{\mu_\bullet}^{-1}(x)$ of exact dimension $\langle\rho,|\mu_\bullet|-\lambda\rangle$.*

Proof. We follow the argument in Haines [9]. We claim that $\mathrm{IC}_{\mu_\bullet} = \mathrm{IC}_{\mu_1}\tilde{\boxtimes}\ldots\tilde{\boxtimes}\mathrm{IC}_{\mu_k}$ is the intersection complex on $\overline{\mathcal{O}}_{\mu_\bullet}$. Indeed, this can be checked locally in the smooth topology, and then easily follows from the definitions. Hence, the left hand side of (5.1) is equal to $m_{\mu_\bullet,*}(\mathrm{IC}_{\mu_\bullet})$. If $d=-\dim(\mathcal{O}_\lambda)$, then taking the d-th stalk cohomology at x in (5.1) gives by proper base change

$$R^d\Gamma(m_{\mu_\bullet}^{-1}(x),\mathrm{IC}_{\mu_\bullet}) \;\simeq\; V_{\mu_\bullet}^\lambda.$$

Since $m_{\mu_\bullet}:\overline{\mathcal{O}}_{\mu_\bullet}\to\overline{\mathcal{O}}_{|\mu_\bullet|}$ is semismall, the cohomology $R^d\Gamma(m_{\mu_\bullet}^{-1}(x),\mathrm{IC}_{\mu_\bullet})$ admits by [9, Lemma 3.2] a canonical basis indexed by the top dimensional irreducible components. This proves the lemma. □

In the following, we consider $\overline{\mathcal{O}}_{\mu_\bullet}$ as a closed projective subvariety of

$$\overline{\mathcal{O}}_{\mu_1}\times\overline{\mathcal{O}}_{\mu_1+\mu_2}\times\ldots\times\overline{\mathcal{O}}_{\mu_1+\ldots+\mu_k},$$

via $(g_1,\ldots,g_k)\mapsto(g_1,g_1g_2,\ldots,g_1\ldots g_k)$. The lemma below is the geometric analogue of the PRV-conjecture.

Lemma 5.5. *For every* $\lambda \in X_+^\vee$ *of the form* $\lambda = \nu_1 + \ldots + \nu_k$ *with* $\nu_i \in W\mu_i$ *for* $i = 1, \ldots, k$, *the perverse sheaf* IC_λ *appears as a direct summand in* $\mathrm{IC}_{\mu_1} \star \ldots \star \mathrm{IC}_{\mu_k}$.

Proof. Let $\nu = w(\nu_2 + \ldots + \nu_k)$ be the unique dominant element in the W-orbit of $\nu_2 + \ldots + \nu_k$. Then $\lambda = \nu_1 + w^{-1}\nu$. Hence, by induction, we may assume $k = 2$. By Lemma 5.4, it is enough to show that there exists $x \in \mathcal{O}_\lambda(F)$ such that $m_{\mu_\bullet}^{-1}(x)$ is of exact dimension $\langle \rho, |\mu_\bullet| - \lambda \rangle$.
Let $w \in W$ such that $w\nu_1$ is dominant, and consider $w\lambda = w\nu_1 + w\nu_2$. We denote by $S_{w\nu_\bullet} \cap \overline{\mathcal{O}}_{\mu_\bullet}$ the intersection inside $\overline{\mathcal{O}}_{\mu_1} \times \overline{\mathcal{O}}_{\mu_1+\mu_2}$

$$S_{w\nu_\bullet} \cap \overline{\mathcal{O}}_{\mu_\bullet} \overset{\text{def}}{=} (S_{w\nu_1} \times S_{w\nu_1 + w\nu_2}) \cap \overline{\mathcal{O}}_{\mu_\bullet}.$$

The convolution is then given by projection on the second factor. By [17, Lemme 9.1], we have a canonical isomorphism

$$S_{w\nu_\bullet} \cap \overline{\mathcal{O}}_{\mu_\bullet} \simeq (S_{w\nu_1} \cap \overline{\mathcal{O}}_{\mu_1}) \times (S_{w\nu_2} \cap \overline{\mathcal{O}}_{\mu_2}).$$

Let $y = (y_1, y_2)$ in $(S_{w\nu_\bullet} \cap \overline{\mathcal{O}}_{\mu_\bullet})(F)$. Since for $i = 1, 2$ the elements $w\nu_i$ are conjugate under W to μ_i, there exist by [17, Lemme 5.2] elements $u_1, u_2 \in L^+U(F)$ such that

$$\begin{aligned} y_1 &= u_1 t^{w\nu_1} \cdot e_0 \\ y_2 &= u_1 t^{w\nu_1} u_2 t^{w\nu_2} \cdot e_0. \end{aligned}$$

The dominance of $w\nu_1$ implies $t^{w\nu_1} u_2 t^{-w\nu_1} \in L^+U(F)$, and hence $Y = S_{w\nu_\bullet} \cap \overline{\mathcal{O}}_{\mu_\bullet}$ maps under the convolution morphism onto an open dense subset Y' in $S_{w\lambda} \cap \mathcal{O}_\lambda$. Denote by $h = m_{\mu_\bullet}|_Y$ the restriction to Y. Both Y, Y' are irreducible schemes (their reduced loci are isomorphic to affine space), thus by generic flatness, there exists $x \in Y'(F)$ such that

$$\dim(h^{-1}(x)) = \dim(Y) - \dim(Y') = \langle \rho, |\mu_\bullet| + w\lambda \rangle - \langle \rho, \lambda + w\lambda \rangle = \langle \rho, |\mu_\bullet| - \lambda \rangle.$$

In particular, $\dim(m_{\mu_\bullet}^{-1}(x)) \geq \langle \rho, |\mu_\bullet| - \lambda \rangle$, and hence equality by Corollary 5.3. ☐

For the proof of Theorem 5.1, we introduce a weaker partial order $\preceq$ on $X_+^\vee$ defined as follows: $\lambda \preceq \mu$ if and only if $\mu - \lambda \in \mathbb{R}_+ Q_+^\vee$. Then $\lambda \leq \mu$ if and only if $\lambda \preceq \mu$ and their images in $X^\vee / Q^\vee$ coincide (cf. Lemma B.2 below).

Proof of Theorem 5.1. We proceed in several steps:

(1) *The affine group scheme H is of finite type over $\bar{\mathbb{Q}}_\ell$.*

By [5, Proposition 2.20 (b)] this is equivalent to the existence of a tensor generator in $P_{L^+G}(\mathrm{Gr}_G)$. Now there exist $\mu_1, \ldots, \mu_k \in X_+^\vee$ which generate $X_+^\vee$ as semigroups. Then $\mathrm{IC}_{\mu_1} \oplus \ldots \oplus \mathrm{IC}_{\mu_k}$ is a tensor generator.

(2) *The affine group scheme H is connected reductive.*

For every $\mu \in X_+^\vee$ and $k \in \mathbb{N}$, the sheaf $\mathrm{IC}_{k\mu}$ is a direct summand of $\mathrm{IC}_\mu^{\star k}$, hence the scheme H is connected by [5, Corollary 2.22]. By [5, Proposition 2.23], the connected algebraic group H is reductive if and only if $P_{L^+G}(\mathrm{Gr}_G)$ is semisimple, and this is true by Proposition 4.1.

(3) *The root datum of H is dual to the root datum of G.*

Let $(X', R', \Delta', X'^\vee, R'^\vee, \Delta'^\vee)$ the based root datum of H constructed in Theorem B.1 below. By Lemma B.5 below it is enough to show that we have an isomorphism of partially ordered semigroups

$$(X_+^\vee, \leq) \xrightarrow{\simeq} (X'_+, \leq'). \tag{5.2}$$

By Proposition 4.1, the map $X_+^\vee \to X'_+$, $\mu \mapsto [\mathrm{IC}_\mu]$, where $[\mathrm{IC}_\mu]$ is the class of IC_μ in $K_0^+ P_{L^+G}(\mathrm{Gr}_G)$ is a bijection of sets.

For every $\lambda, \mu \in X_+^\vee$, we claim that $\lambda \preceq \mu$ if and only if $[\mathrm{IC}_\lambda] \preceq' [\mathrm{IC}_\mu]$. Assume $\lambda \preceq \mu$, and choose a finite subset $F \subset X_+^\vee$ satisfying Proposition B.3 (iii). Let $\mathcal{A} = \oplus_{\nu \in F} \mathrm{IC}_\nu$, and suppose IC_χ is a direct summand of $\mathrm{IC}_\lambda^{\star k}$ for some $k \in \mathbb{N}$. In particular, $\chi \leq k\lambda$ and so $\chi \in WF + \sum_{i=1}^k W\mu$. By Lemma 5.5, the sheaf IC_χ is a direct summand of $\mathrm{IC}_\mu^{\star k} \star \mathcal{A}$, which means $[\mathrm{IC}_\lambda] \preceq' [\mathrm{IC}_\mu]$. Conversely, assume $[\mathrm{IC}_\lambda] \preceq' [\mathrm{IC}_\mu]$. Using Proposition B.3 (iv) below, this translates, by looking at the support, into the following condition: there exists $\nu \in X_+^\vee$ such that $\overline{\mathcal{O}}_{k\lambda} \subset \overline{\mathcal{O}}_{k\mu+\nu}$ holds for infinitely many $k \in \mathbb{N}$. Equivalently, $k\lambda \leq k\mu + \nu$ for infinitely many $k \in \mathbb{N}$ which implies $\lambda \preceq \mu$.

For every $\lambda, \mu \in X_+^\vee$, we claim that $[\mathrm{IC}_\lambda] + [\mathrm{IC}_\mu] = [\mathrm{IC}_{\lambda+\mu}]$ in X'_+: by the proof of Theorem B.1 below, $[\mathrm{IC}_\lambda] + [\mathrm{IC}_\mu]$ is the class of the maximal element appearing in $\mathrm{IC}_\lambda \star \mathrm{IC}_\mu$. Since the partial orders $\preceq, \preceq'$ agree, this is $[\mathrm{IC}_{\lambda+\mu}]$.

It remains to show that the partial orders $\leq, \leq'$ agree. The identification $X_+^\vee = X'_+$ prolongs to $X^\vee = X'$. We claim that $Q_+^\vee = Q'_+$ under this identification and hence $Q^\vee = Q'$, which is enough by Lemma B.2 below. Let $\alpha^\vee \in Q_+^\vee$ a simple coroot, and choose some $\mu \in X_+^\vee$ with $\langle \alpha, \mu \rangle = 2$. Then $\mu + s_\alpha(\mu) = 2\mu - \alpha^\vee$ is dominant, and hence $\mathrm{IC}_{2\mu-\alpha^\vee}$ appears by Lemma 5.5 as a direct summand in $\mathrm{IC}_\mu^{\star 2}$. By Lemma B.4 this means $\alpha^\vee \in Q'_+$, and thus $Q_+^\vee \subset Q'_+$. Conversely, assume $\alpha' \in Q'_+$ has the property that there exists $\mu \in X'_+$ with $2\mu - \alpha' \in X'_+$ and $\mathrm{IC}_{2\mu-\alpha'}$ appears as a direct summand in $\mathrm{IC}_\mu^{\star 2}$. Note that every element in Q'_+ is a sum of these elements. Then $2\mu - \alpha' \leq 2\mu$, and hence $\alpha' \in Q_+^\vee$. This shows $Q'_+ \subset Q_+^\vee$ and finishes the proof of (5.2). $\square$

6 Galois Descent

Let F be any field, and G a connected reductive group defined over F. Fix a separable closure $\bar{F}$, and let $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$ be the absolute Galois group. Let $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_F)$ be the category of finite dimensional continuous ℓ-adic Galois representations. For any object defined over F, we denote by a subscript $(\text{-})_{\bar{F}}$

its base change to $\bar{F}$. Consider the functor

$$\begin{aligned}\Omega : P_{L^+G}(\mathrm{Gr}_G) &\longrightarrow \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_F)\\ \mathcal{A} &\longmapsto \bigoplus_{i\in\mathbb{Z}} R^i\Gamma(\mathrm{Gr}_{G,\bar{F}}, \mathcal{A}_{\bar{F}}).\end{aligned}$$

There are canonical isomorphisms of fpqc-sheaves $(LG)_{\bar{F}} \simeq LG_{\bar{F}}$, $(L^+G)_{\bar{F}} \simeq L^+G_{\bar{F}}$ and $\mathrm{Gr}_{G,\bar{F}} \simeq \mathrm{Gr}_{G_{\bar{F}}}$. Hence, $\Omega \simeq \omega \circ (\text{-})_{\bar{F}}$, cf. (4.3).
The absolute Galois group Γ_F operates on the Tannakian category $P_{L^+G_{\bar{F}}}(\mathrm{Gr}_{G_{\bar{F}}})$ by tensor equivalences compatible with the fiber functor ω. Hence, we may form the semidirect product ${}^LG = \mathrm{Aut}^\star(\omega)(\bar{\mathbb{Q}}_\ell) \rtimes \Gamma_F$ considered as a topological group as follows. The group $\mathrm{Aut}^\star(\omega)(\bar{\mathbb{Q}}_\ell)$ is equipped with the ℓ-adic topology, the Galois Γ_F group with the profinite topology and ${}^LG$ with the product topology. Note that Γ_F acts continuously on $\mathrm{Aut}^\star(\omega)(\bar{\mathbb{Q}}_\ell)$ by Proposition 6.6 below. Let $\mathrm{Rep}^c_{\bar{\mathbb{Q}}_\ell}({}^LG)$ be the full subcategory of the category finite dimensional continuous ℓ-adic representations of ${}^LG$ such that the restriction to $\mathrm{Aut}^\star(\omega)(\bar{\mathbb{Q}}_\ell)$ is algebraic.

THEOREM 6.1. *The functor Ω is an equivalence of abelian tensor categories*

$$\begin{aligned}\Omega : P_{L^+G}(\mathrm{Gr}_G) &\longrightarrow \mathrm{Rep}^c_{\bar{\mathbb{Q}}_\ell}({}^LG)\\ \mathcal{A} &\longmapsto \Omega(\mathcal{A}).\end{aligned}$$

The proof of Theorem 6.1 proceeds in several steps.

LEMMA 6.2. *Let H be an affine group scheme over a field k. Let $\mathrm{Rep}_k(H)$ be the category of algebraic representations of H, and let $\mathrm{Rep}_k(H(k))$ be the category of finite dimensional representations of the abstract group $H(k)$. Assume that H is reduced and that $H(k) \subset H$ is dense. Then the functor*

$$\begin{aligned}\Psi : \mathrm{Rep}_k(H) &\longrightarrow \mathrm{Rep}_k(H(k))\\ \rho &\longmapsto \rho(k)\end{aligned}$$

is a fully faithful embedding.

□

We recall some facts on the Tannakian formalism from [20]. Let $(\mathcal{C}, \otimes)$ be a neutralized Tannakian category over a field k with fiber functor v. We define a monoidal category $\mathrm{Aut}^\otimes(\mathcal{C}, v)$ as follows. Objects are pairs (σ, α), where $\sigma : \mathcal{C} \to \mathcal{C}$ is a tensor automorphism and $\alpha : v \circ \sigma \to v$ is a natural isomorphism of tensor functors. Morphisms between (σ, α) and (σ', α') are natural tensor isomorphisms between σ and σ' that are compatible with α, α' in an obvious way. The monoidal structure is given by compositions. Since v is faithful, $\mathrm{Aut}^\otimes(\mathcal{C}, v)$ is equivalent to a set, and in fact is a group.
Let $H = \mathrm{Aut}^\otimes_{\mathcal{C}}(v)$, the Tannakian group defined by $(\mathcal{C}, v)$. There is a canonical action of $\mathrm{Aut}^\otimes(\mathcal{C}, v)$ on H by automorphisms as follows. Let (σ, α) be in

$\mathrm{Aut}^{\otimes}(\mathcal{C}, v)$. Let R be a k-algebra, and let $h : v_R \to v_R$ be a R-point of H. Then $(\sigma, \alpha) \cdot h$ is the following composition

$$v_R \xrightarrow{\alpha^{-1}} v_R \circ \sigma \xrightarrow{h \circ \mathrm{id}} v_R \circ \sigma \xrightarrow{\alpha} v_R.$$

Let Γ be an abstract group. Then an action of Γ on $(\mathcal{C}, v)$ is by definition a group homomorphism $\mathrm{act} : \Gamma \to \mathrm{Aut}^{\otimes}(\mathcal{C}, v)$.
Assume that Γ acts on $(\mathcal{C}, v)$. Then we define $\mathcal{C}^{\Gamma}$, the category of Γ-equivariant objects in $\mathcal{C}$ as follows. Objects are $(X, \{c_\gamma\}_{\gamma \in \Gamma})$, where X is an object in $\mathcal{C}$ and $c_\gamma : \mathrm{act}_\gamma(X) \simeq X$ is an isomorphism, satisfying the natural cocycle condition, i.e. $c_{\gamma'\gamma} = c_{\gamma'} \circ \mathrm{act}_{\gamma'}(c_\gamma)$. The morphisms between $(X, \{c_\gamma\}_{\gamma \in \Gamma})$ and $(X', \{c'_\gamma\}_{\gamma \in \Gamma})$ are morphisms between X and X', compatible with c_γ, c'_γ in an obvious way.

Lemma 6.3. *Let Γ be a group acting on $(\mathcal{C}, v)$.*

(i) *The category $\mathcal{C}^{\Gamma}$ is an abelian tensor category.*

(ii) *Assume that H is reduced and that k is algebraically closed. The functor v is an equivalence of abelian tensor categories*

$$\mathcal{C}^{\Gamma} \simeq \mathrm{Rep}_k^o(H(k) \rtimes \Gamma)$$

where $\mathrm{Rep}_k^o(H(k) \rtimes \Gamma)$ is the full subcategory of finite dimensional representations of the abstract group $H(k) \rtimes \Gamma$ such that the restriction to $H(k)$ is algebraic.

Remark 6.4. In fact, the category $\mathcal{C}^{\Gamma}$ is neutralized Tannakian with fiber functor v. If Γ is finite, then $\mathrm{Aut}^{\otimes}_{\mathcal{C}^{\Gamma}}(v) \simeq H \rtimes \Gamma$. However, if Γ is not finite, then $\mathrm{Aut}^{\otimes}_{\mathcal{C}^{\Gamma}}(v)$ is in general not $H \rtimes \Gamma$, where the latter is regarded as an affine group scheme.

Proof of Lemma 6.3. The monoidal structure on $\mathcal{C}^{\Gamma}$ is defined as

$$(X, \{c_\gamma\}_{\gamma \in \Gamma}) \otimes (X', \{c'_\gamma\}_{\gamma \in \Gamma}) = (X'', \{c''_\gamma\}_{\gamma \in \Gamma}),$$

where $X'' = X \otimes X'$ and $c''_\gamma : \mathrm{act}_\gamma(X'') \to X''$ is the composition

$$\mathrm{act}_\gamma(X \otimes X') \simeq \mathrm{act}_\gamma(X) \otimes \mathrm{act}_\gamma(X') \xrightarrow{c_\gamma \otimes c'_\gamma} X \otimes X'.$$

This gives $\mathcal{C}^{\Gamma}$ the structure of an abelian tensor category.
Now assume that H is reduced and that k is algebraically closed. It is enough to show that as tensor categories

$$\Psi : \mathrm{Rep}_k(H)^{\Gamma} \xrightarrow{\simeq} \mathrm{Rep}_k^o(H(k) \rtimes \Gamma)$$

compatible with the forgetful functors. Let $((V, \rho), \{c_\gamma\}_{\gamma \in \Gamma}) \in \mathrm{Rep}_k(H)^{\Gamma}$. Then we define $(V, \rho_\Gamma) \in \mathrm{Rep}_k^o(H(k) \rtimes \Gamma)$ by

$$(h, \gamma) \longmapsto \rho(h) \circ \alpha_h(V) \circ v \circ c_\gamma^{-1} \in \mathrm{GL}(V),$$

where $\alpha_h : v \circ \sigma_h \simeq v$ is induced by the action of Γ as above. Using the cocycle relation, one checks that this is indeed a representation. By Lemma 6.2, the natural map

$$\mathrm{Hom}_H(\rho, \rho') \longrightarrow \mathrm{Hom}_{H(k)}(\rho(k), \rho'(k))$$

is bijective. Taking Γ-invariants shows that the functor Ψ is fully faithful. Essential surjectivity is obvious. $\square$

Now we specialize to the case $(\mathcal{C}, \otimes) = (P_{L^+G_{\bar{F}}}(\mathrm{Gr}_{G,\bar{F}}), \star)$ with fiber functor $v = \omega$. Then the absolute Galois group $\Gamma = \Gamma_{\bar{F}}$ acts on this Tannakian category (cf. Appendix A.1).

Proof of Theorem 6.1. The functor Ω is fully faithful.

Let $P_{L^+G_{\bar{F}}}(\mathrm{Gr}_{G,\bar{F}})^{\Gamma,c}$ be the full subcategory of $P_{L^+G_{\bar{F}}}(\mathrm{Gr}_{G,\bar{F}})^{\Gamma}$ consisting of perverse sheaves together with a continuous descent datum (cf. Appendix A.1). By Lemma A.6, the functor $\mathcal{A} \mapsto \mathcal{A}_{\bar{F}}$ is an equivalence of abelian categories $P_{L^+G}(\mathrm{Gr}_G) \simeq P_{L^+G_{\bar{F}}}(\mathrm{Gr}_{G,\bar{F}})^{\Gamma,c}$. Hence, we get a commutative diagram

$$\begin{array}{ccc} P_{L^+G_{\bar{F}}}(\mathrm{Gr}_{G,\bar{F}})^{\Gamma} & \xrightarrow{\ \omega\ } & \mathrm{Rep}^{o}_{\bar{\mathbb{Q}}_\ell}({}^LG) \\ {\scriptstyle \mathcal{A} \mapsto \mathcal{A}_{\bar{F}}}\uparrow & & \uparrow \\ P_{L^+G}(\mathrm{Gr}_G) & \xrightarrow{\ \Omega\ } & \mathrm{Rep}^{c}_{\bar{\mathbb{Q}}_\ell}({}^LG), \end{array}$$

where ω is an equivalence of categories by Lemma 6.3 (ii), and where the vertical arrows are fully faithful. Hence, Ω is fully faithful.

The functor Ω is essentially surjective.

Let ρ be in $\mathrm{Rep}^{c}_{\bar{\mathbb{Q}}_\ell}({}^LG)$. Without loss of generality, we assume that ρ is indecomposable. Let $H = \mathrm{Aut}^*(\omega)$. By Proposition 4.1, the restriction $\rho|_H$ is semisimple. Denote by A the set of isotypic components of $\rho|_H$. Then Γ_F operates transitively on A, and for every $a \in A$ its stabilizer in Γ_F is the absolute Galois group Γ_E for some finite separable extension E/F. By Galois descent along finite extensions, we may assume that $E = F$, and hence that $\rho|_H$ has only one isotypic component. Let ρ_0 be the simple representation occuring in $\rho|_H$. Then $\mathrm{Hom}_H(\rho_0, \rho)$ is a continuous Γ-representation, and the natural morphism

$$\rho_0 \otimes \mathrm{Hom}_H(\rho_0, \rho) \longrightarrow \rho$$

given by $v \otimes f \mapsto f(v)$ is an isomorphism of ${}^LG$-representations. Let IC_X be the simple perverse sheaf on $\mathrm{Gr}_{G,\bar{F}}$ with $\omega(\mathrm{IC}_X) \simeq \rho_0$. Since ρ has only one isotypic component, the support $X = \mathrm{supp}(\mathrm{IC}_X)$ is Γ-invariant, and hence defined over F. Denote by V the local system on $\mathrm{Spec}(F)$ given by the Γ-representation $\mathrm{Hom}_H(\rho_0, \rho)$. Then $\mathrm{IC}_X \otimes V$ is an object in $P_{L^+G}(\mathrm{Gr}_G)$ such that $\Omega(\mathrm{IC}_X \otimes V) \simeq \rho_0 \otimes \mathrm{Hom}_H(\rho_0, \rho)$. This proves the theorem. $\square$

The proof of Theorem 6.1 also shows the following fact.

Corollary 6.5. *Let $\mathcal{A} \in P_{L^+G}(\mathrm{Gr}_G)$ indecomposable. Let $\{X_i\}_{i\in I}$ be the set of irreducible components of* $\mathrm{supp}(\mathcal{A}_{\bar{F}})$*. Denote by E the minimal finite separable extension of F such that X_i is defined over E for all $i \in I$. Then as perverse sheaves on* Gr_E

$$\mathcal{A}_E \;\simeq\; \bigoplus_{i\in I} \mathrm{IC}_{X_i} \otimes V_i,$$

where V_i are indecomposable local systems on $\mathrm{Spec}(E)$*.*

□

We briefly explain the connection to the full L-group. For more details see [20]. Let $\hat{G}$ be the reductive group over $\bar{\mathbb{Q}}_\ell$ dual to $G_{\bar{F}}$ in the sense of Langlands, i.e. the root datum of $\hat{G}$ is dual to the root datum of $G_{\bar{F}}$. There are two natural actions of Γ_F on $\hat{G}$ as follows. Up to the choice of a pinning $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$ of $\hat{G}$, we have an action $\mathrm{act}^{\mathrm{alg}}$ via

$$\mathrm{act}^{\mathrm{alg}} : \Gamma_F \to \mathrm{Out}(G_{\bar{F}}) \simeq \mathrm{Out}(\hat{G}) \simeq \mathrm{Aut}(\hat{G}, \hat{B}, \hat{T}, \hat{X}) \subset \mathrm{Aut}(\hat{G}), \tag{6.1}$$

where Out(-) denotes the outer automorphisms. On the other hand, we have an action $\mathrm{act}^{\mathrm{geo}} : \Gamma_{\bar{F}} \to \mathrm{Aut}(\hat{G})$ via the Tannakian equivalence from Theorem 5.1. The relation between $\mathrm{act}^{\mathrm{geo}}$ and $\mathrm{act}^{\mathrm{alg}}$ is as follows.
Let $\mathrm{cycl} : \Gamma_F \to \mathbb{Z}_\ell^\times$ be the cyclotomic character of Γ_F defined by the action of Γ_F on the ℓ^∞-roots of unity of $\bar{F}$. Let $\hat{G}_{\mathrm{ad}}$ be the adjoint group of $\hat{G}$. Let ρ be the half sum of positive coroots of $\hat{G}$, which gives rise to a one-parameter group $\rho : \mathbb{G}_m \to \hat{G}_{\mathrm{ad}}$. We define a map

$$\chi : \Gamma_F \xrightarrow{\mathrm{cycl}} \mathbb{Z}_\ell^\times \xrightarrow{\rho} \hat{G}_{\mathrm{ad}}(\bar{\mathbb{Q}}_\ell),$$

which gives a map $\mathrm{Ad}_\chi : \Gamma_F \to \mathrm{Aut}(\hat{G})$ to the inner automorphism of $\hat{G}$.

Proposition 6.6 ([20] Proposition A.4). *For all $\gamma \in \Gamma_F$,*

$$\mathrm{act}^{\mathrm{geo}}(\gamma) \;=\; \mathrm{act}^{\mathrm{alg}}(\gamma) \circ \mathrm{Ad}_\chi(\gamma).$$

□

Remark 6.7. Proposition 6.6 shows that $\mathrm{act}^{\mathrm{geo}}$ only depends on the quasi-split form of G, since the same is true for $\mathrm{act}^{\mathrm{alg}}$. In particular, the Satake category $P_{L^+G}(\mathrm{Gr}_G)$ only depends on the quasi-split form of G whereas the ind-scheme Gr_G does depend on G.

Let ${}^LG^{\mathrm{alg}} = \hat{G}(\bar{\mathbb{Q}}_\ell) \rtimes_{\mathrm{act}^{\mathrm{alg}}} \Gamma_F$ be the *full L-group*. Set ${}^LG^{\mathrm{geo}} = \hat{G}(\bar{\mathbb{Q}}_\ell) \rtimes_{\mathrm{act}^{\mathrm{geo}}} \Gamma_F$.

Corollary 6.8 ([20] Corollary A.5). *The map $(g, \gamma) \mapsto (\mathrm{Ad}_{\chi(\gamma^{-1})}(g), \gamma)$ gives an isomorphism ${}^LG^{\mathrm{alg}} \xrightarrow{\simeq} {}^LG^{\mathrm{geo}}$.*

□

Combining Corollary 6.8 with Theorem 6.1, we obtain the following corollary.

Corollary 6.9. *There is an equivalence of abelian tensor categories*

$$P_{L^+G}(\mathrm{Gr}_G) \;\simeq\; \mathrm{Rep}^{\mathrm{c}}_{\bar{\mathbb{Q}}_\ell}({}^L G^{\mathrm{alg}}),$$

where $\mathrm{Rep}^{\mathrm{c}}_{\bar{\mathbb{Q}}_\ell}({}^L G^{\mathrm{alg}})$ *denotes the full subcategory of the category of finite dimensional continuous ℓ-adic representations of ${}^L G^{\mathrm{alg}}$ such that the restriction to $\hat{G}(\bar{\mathbb{Q}}_\ell)$ is algebraic.*

□

A Perverse Sheaves

For the construction of the category of ℓ-adic perverse sheaves, we refer to the work of Y. Laszlo and M. Olsson [13]. In this appendix we explain our conventions on perverse sheaves on ind-schemes.

Let F be an arbitrary field. Fix a prime $\ell \neq \mathrm{char}(F)$, and denote by $\mathbb{Q}_\ell$ the field of ℓ-adic numbers with algebraic closure $\bar{\mathbb{Q}}_\ell$. For any separated scheme T of finite type over F, we consider the bounded derived category $D^b_c(T, \bar{\mathbb{Q}}_\ell)$ of constructible ℓ-adic sheaves on T. Let $P(T)$ be the abelian $\bar{\mathbb{Q}}_\ell$-linear full subcategory of ℓ-adic perverse sheaves, i.e. the heart of the perverse t-structure on the triangulated category $D^b_c(T, \bar{\mathbb{Q}}_\ell)$.

Now let $(T)_{i\in I}$ be an inductive system of separated schemes of finite type over F with closed immersions as transition morphisms. A fpqc-sheaf $\mathcal{T}$ on the category of F-algebras is called a *strict ind-scheme of ind-finite type over* F if there is an isomorphism of fpqc-sheaves $\mathcal{T} \simeq \varinjlim_i T_i$, for some system $(T)_{i\in I}$ as above. The inductive system $(T)_{i\in I}$ is called an *ind-presentation of* $\mathcal{T}$.

For $i \leq j$, push forward gives transition morphisms $D^b_c(T_i, \bar{\mathbb{Q}}_\ell) \to D^b_c(T_j, \bar{\mathbb{Q}}_\ell)$ which restrict to $P(T_i) \to P(T_j)$, because push forward along closed immersions is t-exact.

Definition A.1. Let $\mathcal{T}$ be a strict ind-scheme of ind-finite type over F, and $(T_i)_{i\in I}$ be an ind-presentation.

(i) The *bounded derived category of constructible ℓ-adic complexes* $D^b_c(\mathcal{T}, \bar{\mathbb{Q}}_\ell)$ on $\mathcal{T}$ is the inductive limit

$$D^b_c(\mathcal{T}, \bar{\mathbb{Q}}_\ell) \overset{\mathrm{def}}{=} \varinjlim_i D^b_c(T_i, \bar{\mathbb{Q}}_\ell).$$

(ii) The *category of ℓ-adic perverse sheaves* $P(\mathcal{T})$ on $\mathcal{T}$ is the inductive limit

$$P(\mathcal{T}) \overset{\mathrm{def}}{=} \varinjlim_i P(T_i).$$

The definition is independent of the chosen ind-presentation of $\mathcal{T}$. The category $D^b_c(\mathcal{T}, \bar{\mathbb{Q}}_\ell)$ inherits a triangulation and a perverse t-structure from the $D^b_c(T_i, \bar{\mathbb{Q}}_\ell)$'s. The heart with respect to the perverse t-structure is the abelian $\bar{\mathbb{Q}}_\ell$-linear full subcategory $P(\mathcal{T})$.

If $f : \mathcal{T} \to \mathcal{S}$ is a morphism of strict ind-schemes of ind-finite type over F, we have the Grothendieck operations $f_*, f_!, f^*, f^!$, and the usual constructions carry over after the choice of ind-presentations.
In Section 3.3 we work with equivariant objects in the category of perverse sheaves. The context is as follows. Let $f : T \to S$ be a morphism of separated schemes of finite type, and let H be a smooth affine group scheme over S with geometrically connected fibers acting on $f : T \to S$. Then a perverse sheaf $\mathcal{A}$ on T is called *H-equivariant* if there is an isomorphism in the derived category

$$\theta : a^*\mathcal{A} \;\simeq\; p^*\mathcal{A}, \tag{A.1}$$

where $a : H\times_S T \to T$ (resp. $p : H\times_S T \to T$) is the action (resp. projection on the second factor). A few remarks are in order: if the isomorphism (A.1) exists, then it can be rigidified such that $e_T^*\theta$ is the identity, where $e_T : T \to H \times_S T$ is the identity section. A rigidified isomorphism θ automatically satisfies the cocycle relation due to the fact that H has geometrically connected fibers.
The subcategory $P_H(T)$ of $P(T)$ of H-equivariant objects together with H-equivariant morphisms is called the *category of H-equivariant perverse sheaves* on T.

Lemma A.2 ([13] Remark 5.5). *Consider the stack quotient $H\backslash T$, an Artin stack of finite type over S. Let $p : T \to H\backslash T$ be the quotient map of relative dimension $d = \dim(T/S)$. Then the pull back functor*

$$p^*[d] : P(H\backslash T) \;\longrightarrow\; P_H(T),$$

is an equivalence of categories. In particular, $P_H(T)$ is abelian and $\bar{\mathbb{Q}}_\ell$-linear.

□

Now let $\mathcal{T}$ be a strict ind-scheme of ind-finite type, and $f : \mathcal{T} \to S$ a morphism to a separated scheme of finite type. Fix an ind-presentation $(T_i)_{i\in I}$ of $\mathcal{T}$. Let $(H_i)_{i\in I}$ be an inverse system of smooth affine group scheme with geometrically connected fibers. Let $\mathcal{H} = \varprojlim_i H_i$ be the inverse limit, an affine group scheme over S, because the transition morphism are affine. Assume that $\mathcal{H}$ acts on $f : \mathcal{T} \to S$ such that the action restricts to the inductive system $(f|_{T_i})_{i\in I}$. Assume that the $\mathcal{H}$-action factors through H_i on $f|_{T_i}$ for every $i \in I$.

Definition A.3. Let $f : \mathcal{T} \to S$, $(T_i)_{i\in I}$ and $\mathcal{H}$ as above. The *category $P_{\mathcal{H}}(\mathcal{T})$ of $\mathcal{H}$-equivariant perverse sheaves on $\mathcal{T}$* is the inductive limit

$$P_{\mathcal{H}}(\mathcal{T}) \overset{\text{def}}{=} \varinjlim_i P_{H_i}(T_i).$$

It follows from Lemma A.2 that the category $P_{\mathcal{H}}(\mathcal{T})$ is an abelian $\bar{\mathbb{Q}}_\ell$-linear category. The following lemma is used throughout the text.

Lemma A.4. *Let $T \to S$ be a $\mathcal{H}$-torsor, and let Y be a S-scheme with $\mathcal{H}$-action. Assume that the action of $\mathcal{H}$ on Y factors over H_i for $i >> 0$. Then there is a canonical isomorphism of fpqc-sheaves*

$$T \times^{\mathcal{H}} Y \xrightarrow{\simeq} T^{(i)} \times^{H_i} Y,$$

where $T^{(i)} = T \times^{\mathcal{H}} H_i$.

□

Remark A.5. In particular, if $T^{(i)} \times^{H_i} Y$ is representable of finite type, then is $T \times^{\mathcal{H}} Y$ is representable of finite type.

A.1 Galois Descent of Perverse Sheaves

Fix a separable closure $\bar{F}$ of F. Let $\Gamma = \mathrm{Gal}(\bar{F}/F)$ be the absolute Galois group. For any complex of sheaves $\mathcal{A}$ on T, we denote by $\mathcal{A}_{\bar{F}}$ its base change to $T_{\bar{F}} = T \otimes \bar{F}$. We define the category of *perverse sheaves with continuous* Γ-*descent datum* $P(T_{\bar{F}})^{\Gamma,c}$ as follows. The objects are pairs $(\mathcal{A}, \{c_\gamma\}_{\gamma\in\Gamma})$, where $\mathcal{A} \in P(T_{\bar{F}})$ and $\{c_\gamma\}_{\gamma\in\Gamma}$ is a family of isomorphisms

$$c_\gamma : \gamma_*\mathcal{A} \xrightarrow{\simeq} \mathcal{A},$$

satisfying the cocycle condition $c_{\gamma'\gamma} = c_{\gamma'} \circ \gamma'_*(c_\gamma)$ such that the datum is continuous in the following sense. For every $i \in \mathbb{Z}$ and every locally closed subscheme $S \subset T$ such that the standard cohomology sheaf $\mathcal{H}^i(\mathcal{A})|_S$ is a local system, and for every $U \to S$ étale, with U separated quasi-compact, the induced ℓ-adic Galois representation on the $U_{\bar{F}}$-sections

$$\Gamma \longrightarrow \mathrm{GL}(\mathcal{H}^i(\mathcal{A})(U_{\bar{F}})),$$

is continuous. The morphisms in $P(T_{\bar{F}})^{\Gamma,c}$ are morphisms in $P(T_{\bar{F}})$ compatible with the c_γ's. For every $\mathcal{A} \in P(T)$, its pullback $\mathcal{A}_{\bar{F}}$ admits a canonical continuous descent datum. Hence, we get a functor

$$\begin{aligned} \Phi : P(T) &\longrightarrow P(T_{\bar{F}})^{\Gamma,c} \\ \mathcal{A} &\longmapsto \mathcal{A}_{\bar{F}}. \end{aligned}$$

Lemma A.6 (SGA 7, XIII, 1.1). *The functor Φ is an equivalence of categories.*

□

B Reconstruction of Root Data

Let G a split connected reductive group over an arbitrary field k. Denote by Rep_G the Tannakian category of algebraic representations of G. If k is algebraically closed of characteristic 0, then D. Kazhdan, M. Larsen and Y. Varshavsky [10, Corollary 2.5] show how to reconstruct the root datum of G from the Grothendieck semiring $K_0^+[G] = K_0^+\mathrm{Rep}_G$. In fact, their construction works over arbitrary fields. This relies on the conjecture of Parthasarathy, Ranga-Rao and Varadarajan (PRV-conjecture) proven by S. Kumar [11] ($\mathrm{char}(k) = 0$) and O. Mathieu [15] ($\mathrm{char}(k) > 0$).

Theorem B.1. *The root datum of G can be reconstructed from the Grothendieck semiring $K_0^+[G]$.*

This means, if H is another split connected reductive group over k, and if $\varphi : K_0^+[H] \to K_0^+[G]$ is an isomorphism of Grothendieck semirings, then there exists an isomorphism of group schemes $\phi : H \to G$ determined uniquely up to inner automorphism such that $\phi = K_0^+[\varphi]$.
Let T be a maximal split torus of G and B a Borel subgroup containing T. We denote by $\langle \text{-}, \text{-} \rangle$ the natural pairing between $X = \mathrm{Hom}(T, \mathbb{G}_m)$ and $X^\vee = \mathrm{Hom}(\mathbb{G}_m, T)$. Let $R \subset X$ be the root system associated to (G, T), and R_+ be the set of positive roots corresponding to B. Let $R^\vee \subset X^\vee$ the dual root system with the bijection $R \to R^\vee$, $\alpha \mapsto \alpha^\vee$. Denote by $R_+^\vee$ the set of positive coroots. Let W the Weyl group of (G, T). Consider the half sum of all positive roots

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha.$$

Let Q (resp. Q_+) the subgroup (resp. submonoid) of X generated by R (resp. R_+). We denote by

$$X_+ = \{\mu \in X \mid \langle \mu, \alpha \rangle \geq 0, \ \forall \alpha \in R_+^\vee\}$$

the cone of dominant characters.
We consider partial orders $\leq$ and $\preceq$ on X defined as follows. For $\lambda, \mu \in X$, we define $\lambda \leq \mu$ if and only if $\mu - \lambda \in Q_+$, and we define $\lambda \preceq \mu$ if and only if $\mu - \lambda = \sum_{\alpha \in \Delta} x_\alpha \alpha$ with $x_\alpha \in \mathbb{R}_{\geq 0}$. The latter order is weaker than the former order in the sense that $\lambda \leq \mu$ implies $\lambda \preceq \mu$, but in general not conversely.

Lemma B.2 ([18]). *For every $\lambda, \mu \in X_+$, then $\lambda \leq \mu$ if and only if $\lambda \preceq \mu$ and the images of λ, μ in X/Q agree.*

Let

$$\mathrm{Dom}_{\preceq \mu} = \{\nu \in X_+ \mid \nu \preceq \mu\}.$$

For a finite subset F of the euclidean vector space $E = X \otimes \mathbb{R}$, we denote by $\mathrm{Conv}(F)$ its convex hull.

Proposition B.3. *For $\lambda, \mu \in X_+$, the following conditions are equivalent:*

(i) $\lambda \preceq \mu$

(ii) $\mathrm{Conv}(W\lambda) \subset \mathrm{Conv}(W\mu)$

(iii) *There exists a finite subset $F \subset X_+$ such that for all $k \in \mathbb{N}$:*

$$\mathrm{Dom}_{\preceq k\lambda} \ \subset \ WF + \sum_{i=1}^{k} W\mu$$

(iv) *There exists a representation* U *such that for every* $k \in \mathbb{N}$*, every irreducible subquotient of* $V_\lambda^{\otimes k}$ *is a subquotient of* $V_\mu^{\otimes k} \otimes U$.

Proof. The equivalence of (i) and (ii) is well-known. The implication (ii)$\Rightarrow$(iii) follows from [10, Lemma 2.4]. Assume (iii), we show that (iv) holds: let $U = \oplus_{\nu \in F} V_\nu$, and suppose V_χ is a irreducible subquotient of $V_\lambda^{\otimes k}$, in particular $\chi \leq k\lambda$. By (iii), χ has the form $w\nu + \sum_{i=1}^k w_i\mu$ with $w, w_1, \ldots, w_k \in W$ and $\nu \in F$. Using the PRV-conjecture [4, Theorem 4.3.2], we conclude that V_χ is a subquotient of $V_\mu^{\otimes k} \otimes V_\nu$, hence also of $V_\mu^{\otimes k} \otimes U$. This shows (iv). The implication (iv)$\Rightarrow$(i) is shown in [10, Proposition 2.2]. □

For $\mu \in X_+$, let v_μ be the corresponding element in $K_0^+[G]$. Let $\mathcal{Q}_+ \subset X$ be the semigroup generated by the set

$$\{\alpha \in X \mid \exists\, \mu \in X_+ :\ 2\mu - \alpha \in X_+ \text{ and } v_\mu^2 - v_{2\mu-\alpha} \in K_0^+[G]\}.$$

Lemma B.4. *There is an equality of semigroups* $\mathcal{Q}_+ = Q_+$.

Proof. It is obvious that $\mathcal{Q}_+ \subset Q_+$, and we show that $\mathcal{Q}_+$ contains the simple roots. Let α be a simple root, and choose some $\mu \in X$ such that $\langle \mu, \alpha^\vee \rangle = 2$. Then $2\mu - \alpha$ paired with any simple root is positive, and hence $\mu + s_\alpha(\mu) = 2\mu - \alpha$ is dominant. By the *PRV*-conjecture [4, Theorem 4.3.2], the representation $V_{2\mu-\alpha}$ appears as an irreducible subquotient in $V_\mu^{\otimes 2}$, i.e. $v_\mu^2 - v_{2\mu-\alpha} \in K_0^+[G]$. □

The proof of Theorem B.1 goes along the lines of [10, Corollary 2.5].

Proof of Theorem B.1. By Lemma B.5 below it is enough to construct the partially ordered semigroup $(X_+, \leq)$ of dominant weights.
The underlying set of dominant weights X_+ is the set of irreducible objects in $K_0^+[G]$. Then the partial order $\preceq$ on X_+ is characterized by Proposition B.3 as follows: for $\lambda, \mu \in X_+$, one has $\lambda \preceq \mu$ if and only if there exists a $u \in K_0^+[G]$ such that for all $k \in \mathbb{N}$ and $\nu \in X_+$,

$$v_\lambda^k - v_\nu \in K_0^+[G] \quad \Longrightarrow \quad v_\mu^k \cdot u - v_\nu \in K_0^+[G].$$

The semigroup structure on X_+ is given by: for $\lambda, \mu \in X_+$, one has $\nu = \lambda + \mu$ if and only if ν is the unique dominant weight which is maximal (w.r.t. $\preceq$) with the property that $v_\lambda \cdot v_\mu - v_\nu \in K_0^+[G]$.
Now X is the group completion of X_+, and by Lemma B.4 we can reconstruct $Q_+ \subset X$. Then Q is the group completion of Q_+, and by Lemma B.2 we can reconstruct $\leq$. This shows that the root datum of G can be reconstructed from $K_0^+[G]$.
Now if H is another split connected reductive group over k, and $\varphi : K_0^+[H] \to K_0^+[G]$ an isomorphism of Grothedieck semirings, then the argument above shows that there is an isomorphism of partially ordered semigroups

$$(X_+^H, \leq^H) \longrightarrow (X_+^G, \leq^G) \tag{B.1}$$

inducing φ on Grothendieck semirings. By Lemma B.5 below, the morphism B.1 prolongs to an isomorphism of the associated based root data. Hence, there exists an isomorphism of group schemes $\phi : H \to G$ inducing the isomorphism of based root data. In particular, $\varphi = K_0^+[\phi]$, and such an isomorphism ϕ is uniquely determined up to inner automorphism. This finishes the proof of Theorem B.1. □

Lemma B.5. *Let $\mathcal{B} = (X, R, \Delta, X^\vee, R^\vee, \Delta^\vee)$ any based root datum. Denote by $(X_+, \leq)$ the partially ordered semigroup of dominant weights. Then the root datum $\mathcal{B}$ can be reconstructed from $(X_+, \leq)$, i.e. if $\mathcal{B}' = (X', R', \Delta', X'^\vee, R'^\vee, \Delta'^\vee)$ is another based root datum with associated dominant weights $(X'_+, \leq')$, then any ismorphism $(X, \leq) \to (X', \leq')$ of partially ordered semigroups prolongs to an isomorphism $\mathcal{B} \to \mathcal{B}'$ of based root data.*

Proof. The weight lattice X is the group completion of X_+, a finite free $\mathbb{Z}$-module. The dominance order $\leq$ extends uniquely to X, also denoted $\leq$. Then $X^\vee = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ is the coweight lattice, and the natural pairing $X \times X^\vee \to \mathbb{Z}$ identifies with $\langle \text{-}, \text{-} \rangle$. The reconstruction of the roots and coroots proceeds in several steps:

(1) *The set of simple roots $\Delta \subset X$*:
A weight $\alpha \in X\backslash\{0\}$ is in Δ if and only if $0 \leq \alpha$, and α is minimal with this property.

(2) *The set of simple coroots $\Delta^\vee \subset X^\vee$*:
An element of $X^\vee$ is uniquely determined by its value on X_+. Fix $\alpha \in \Delta$ with corresponding simple coroot $\alpha^\vee$. Then for any $\mu \in X_+$, the value $\langle \mu, \alpha^\vee \rangle$ is the unique number $m \in \mathbb{N}$ such that $2\mu - m\alpha$ is dominant, but $2\mu - (m+1)\alpha$ is not. Indeed, we have

$$\langle 2\mu - m\alpha, \alpha^\vee \rangle \geq 0 \quad \Leftrightarrow \quad \langle \mu, \alpha^\vee \rangle \geq m,$$

and, for every other simple coroot $\beta^\vee \neq \alpha^\vee$ and every $n \in \mathbb{N}$,

$$\langle 2\mu - n\alpha, \beta^\vee \rangle = 2\langle \mu, \beta^\vee \rangle - n\langle \alpha, \beta^\vee \rangle \geq 2\langle \mu, \beta^\vee \rangle \geq 0,$$

since $\langle \alpha, \beta^\vee \rangle \leq 0$. Hence, $\langle 2\mu - (m+1)\alpha, \alpha^\vee \rangle < 0$ and so $m = \langle \mu, \alpha^\vee \rangle$.

(3) *The sets of roots R and coroots $R^\vee$*:
The Weyl group $W \subset \mathrm{Aut}_{\mathbb{Z}}(X)$ is the finite subgroup generated by the reflections $s_{\alpha,\alpha^\vee}$ associated to the pair $(\alpha, \alpha^\vee) \in \Delta \times \Delta^\vee$. Then $R = W \cdot \Delta$, i.e., the roots are given by the translates of the simple roots under W. Since $\mathrm{Aut}_{\mathbb{Z}}(X^\vee) = \mathrm{Aut}_{\mathbb{Z}}(X)^{\mathrm{op}}$, the Weyl group W acts on $X^\vee$ and $R^\vee = W \cdot \Delta^\vee$. This proves the lemma. □

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On Quadratic Diophantine Equations in Four Variables and Orders Associated with Lattices

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Abstract. This paper treats certain lattices in ternary quadratic spaces, which are obtained from the data of a non-zero element and a maximal lattice in a quaternary space. Each class in the genus of such a lattice with respect to the special orthogonal group corresponds to an isomorphism class in the genus of an order associated with the lattice in a quaternion algebra. Using this result together with the principle of Shimura, we show that the number of classes of the primitive solutions of a quadratic Diophantine equation in four variables coincides with the type number of the order under suitable conditions on the given element. We illustrate this result by explicit examples.

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Introduction

Let (V, φ) be a nondegenerate quadratic space of dimension 4 over a number field F, that is, V is a four-dimensional vector space over F and φ is a nondegenerate symmetric F-bilinear form on V. For an element h of V such that $\varphi[h](=\varphi(h, h)) \neq 0$, we put

$$W = (Fh)^{\perp} = \{x \in V \mid \varphi(x, h) = 0\}$$

and consider a quadratic space (W, ψ) of dimension 3 over F with the restriction ψ of φ to W. In this paper the special orthogonal group SO^{ψ} of ψ is regarded as the subgroup $\{\gamma \in SO^{\varphi} \mid h\gamma = h\}$ of the orthogonal group SO^{φ} of φ. The orthogonal group O^{ψ} of ψ is also identified with O^{φ} in a similar manner. For a maximal lattice L in (V, φ) we put

$$L[q,\ \mathfrak{b}] = \{x \in V \mid \varphi[x] = q,\ \varphi(x, L) = \mathfrak{b}\}, \quad D(L) = \{\alpha \in O^{\varphi}_{\mathbf{A}} \mid L\alpha = L\},$$

where $q = \varphi[h]$, $\mathfrak{b} = \varphi(h, L)$, and the subscript $\mathbf{A}$ means adelization. Since $L[q, \mathfrak{b}]$ is stable under $\Gamma^{\cdot}(L) = O^{\varphi} \cap D(L)$, the set $L[q, \mathfrak{b}]/\Gamma^{\cdot}(L)$ is meaningful in an obvious way.

In the sense of Shimura [9, Introduction I], $L[q, \mathfrak{b}]$ is the set of *primitive solutions* of the equation $\varphi[x] = q$. Our interest in this paper is basically the set

$$L[q, \mathfrak{b}]/\Gamma^{\cdot}(L)$$

consisting of the classes of such solutions. By the principle of [9, Theorem 11.6], *each class of solutions of $L_i[q, \mathfrak{b}]$ modulo $\Gamma^{\cdot}(L_i)$ for $i \in J$ corresponds to a class of O^{ψ} relative to an open subgroup $O^{\psi}_{\mathbf{A}} \cap D(L)$ in $O^{\psi}_{\mathbf{A}}$*. Here $\{L_i\}_{i \in J}$ is a set of representatives for the O^{φ}-classes in the O^{φ}-genus of L for which $L_i[q, \mathfrak{b}] \neq \emptyset$ (see also (4.1) below).

Now we consider the lattice $L \cap W$ in (W, ψ) and the even Clifford algebra $A^+(W)$ of ψ, which is a quaternion algebra over F since the dimension of W is 3. Let $A(N)$ be the order generated by $\mathfrak{g}$ and N in the Clifford algebra of ψ, and put $A^+(N) = A^+(W) \cap A(N)$ for an integral lattice N in (W, ψ). Here $\mathfrak{g}$ is the ring of all integers of F and the terms *integral* and *maximal* are given in §1.1. To $L \cap W$ we can associate an order $\mathfrak{O}$ in $A^+(W)$, containing $A^+(L \cap W)$, whose discriminant is given by (3.22) below. The purpose of this paper is to define such an order $\mathfrak{O}$, to give a bijection of the SO^{ψ}-classes in the SO^{ψ}-genus of $L \cap W$ onto the isomorphism classes in the genus of $\mathfrak{O}$, and to prove

$$\sum_{i \in J} \# \{L_i[q, \mathfrak{b}]/\Gamma^{\cdot}(L_i)\} = t(\mathfrak{O}) \tag{0.1}$$

through the above principle under suitable conditions on $h \in L[q, \mathfrak{b}]$, where the genus of $\mathfrak{O}$ is defined by the set $\{\alpha^{-1}\mathfrak{O}\alpha \mid \alpha \in A^+(W)^{\times}_{\mathbf{A}}\}$ and $t(\mathfrak{O})$ is the type number of $\mathfrak{O}$.

To obtain the order $\mathfrak{O}$, we proceed in a similar manner to [10, §4.6] under mild conditions on $h \in L[q, \mathfrak{b}]$ (Proposition 3.3 (3)). As for the bijection, given a lattice N in the genus of $L \cap W$, put $N = (L \cap W)\tau(\alpha)$ with $\alpha \in A^+(W)^{\times}_{\mathbf{A}}$. Here τ is a surjective homomorphism of $G^+(W)_{\mathbf{A}}$ onto $SO^{\psi}_{\mathbf{A}}$ and $G^+(W)$ is the even Clifford group of ψ, which is given by $A^+(W)^{\times}$. Then our bijection is defined by $N \longmapsto \alpha^{-1}\mathfrak{O}\alpha$ (Theorem 3.4 (2)). This result mainly relies on a description of $L \cap W$ in $A^+(W)$ by means of $A^+(L \cap W)$ (cf. [4, Theorem 2.1]). Now in Proposition 4.3, we shall prove that $O^{\psi}\varepsilon D(L \cap W) = O^{\psi}\varepsilon(O^{\psi}_{\mathbf{A}} \cap D(L))$ for every $\varepsilon \in O^{\psi}_{\mathbf{A}}$ under several assumptions on $h \in L[q, \mathfrak{b}]$. Because W is odd-dimensional, the class number of the genus of $L \cap W$ with respect to O^{ψ} equals that with respect to SO^{ψ}. Hence by virtue of the principle mentioned above, the number $\sum_{i \in J} \# \{L_i[q, \mathfrak{b}]/\Gamma^{\cdot}(L_i)\}$ is equal to the number of SO^{ψ}-classes in the SO^{ψ}-genus of $L \cap W$. Our main result (0.1) follows from this fact and the above bijection.

It should be noted that the genus of $\mathfrak{O}$ is determined by the discriminant if $\mathfrak{O}$ has squarefree discriminant, for instance, if $\mathfrak{O}$ is maximal. When the quaternion algebra is totally definite, there are formulas for computing the type number of such an order; see Peters [5, Satz 14] or Pizer [6, Theorem A and Theorem B], etc.. In Section 5 we will take up totally-positive definite forms φ and employ their numerical tables [5, Tabelle 2] and [6, Table 1] for type numbers. It seems that there are few numerical examples for the type number of the genus of an order whose discriminant is not squarefree and for the class number of the genus of a lattice which is neither maximal nor unimodular with respect to a definite form. Here we assume that h satisfies the conditions in Proposition 4.3. Then (0.1) contains the case of non-maximal (and often non-unimodular) $L \cap W$, more clearly, the case that $\mathfrak{O}$ has at most one higher-power prime $\mathfrak{p}^e$ $(e > 1)$ in its discriminant, where $\mathfrak{p}$ is a prime ideal of F (see also (4.10) below).

To see the existence of such an element h, as an application of Proposition 4.3, let $B_{r,\infty}$ be a definite quaternion algebra over $\mathbf{Q}$ ramified exactly at a prime number r and take a prime number d prime to r so that $d \equiv 1 \pmod 4$. In Theorem 5.1 we shall show:
For every odd prime number p prime to dr and $0 \leq n \in \mathbf{Z}$, except when n is odd and p remains prime in $\mathbf{Q}(\sqrt{d})$, there exists a maximal lattice L in (V, φ) over $\mathbf{Q}$ of invariants $\{4, \mathbf{Q}(\sqrt{d}), B_{r,\infty}, 4\}$ (see (1.5) for the definitions) such that $L[dp^n, 2^{-1}d\mathbf{Z}] \neq \emptyset$. And moreover, formula (0.1) is valid for $h \in L[dp^n, 2^{-1}d\mathbf{Z}]$ with an order $\mathfrak{O}$ in $B_{r,\infty}$ of discriminant $rp^n\mathbf{Z}$.
For example, let us take (V, φ) of invariants $\{4, \mathbf{Q}(\sqrt{29}), B_{2,\infty}, 4\}$. By [9, Theorem 12.14 (vi)] the number of O^φ-classes in the O^φ-genus of maximal lattices in (V, φ) equals the number $\#\{\Lambda[29, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)\}$, where Λ is a maximal lattice in a five-dimensional quadratic space over $\mathbf{Q}$ with respect to the quadratic form defined by the sum of five squares. In [9, §12.15], $\#\{\Lambda[29, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)\}$ was determined, and it is 3. Hence the genus of maximal lattices in (V, φ) consists of three O^φ-classes. For details, see the last part of Section 4.1 in the text. We can also see the representatives $\{L_1, L_2, L_3\}$ of such classes by means of a bijection in Lemma 4.1 applied to the set $\Lambda[29, \mathbf{Z}]$. Then we have the following numerical table:

p	$N_1(29p)$	$N_2(29p)$	$N_3(29p)$	$t(2, p)$	$\mathfrak{c}(29p)$
1	2	0	0	1	1
5	0	2	0	1	1
7	0	0	2	1	1
13	0	2	2	2	2
23	0	0	6	1	1
53	0	10	6	3	3
59	24	8	6	3	3

Here we put $N_i(29p) = \#L_i[29p, 2^{-1} \cdot 29\mathbf{Z}]$, $t(2, p) = t(\mathfrak{O})$ with an order $\mathfrak{O}$

in $B_{2,\infty}$ of discriminant $2p\mathbf{Z}$, and $\mathfrak{c}(29p) = \sum_{i=1}^{3} \#\{L_i[29p,\, 2^{-1}\cdot 29\mathbf{Z}]/\Gamma^{\cdot}(L_i)\}$; we quoted the type number $t(2,\, p)$ in [6, Table 1]. Therefore we have

$$\#\left\{L_i[29\cdot 59,\ 2^{-1}\cdot 29\mathbf{Z}]/\Gamma^{\cdot}(L_i)\right\} = 1 \quad \text{for } i = 1,\ 2,\ 3,$$

for instance. It is noted that $\#\Gamma(L_1) = 48$, $\#\Gamma(L_2) = 8$, and $\#\Gamma(L_3) = 6$, where $\Gamma(L_i) = SO^{\varphi} \cap D(L_i)$. In Section 5.3 we shall give a few numerical tables for $r = 2$ and $d = 5,\, 13,\, 17,\, 29$ including the above cases. As a special case of Theorem 5.1 we have Corollary 5.2, which states that for *any* d as in Theorem 5.1 *only one* O^{φ}-class in the genus satisfies $L[d,\, 2^{-1}d\mathbf{Z}] \neq \emptyset$ with a lattice L in the class and then (0.1) precisely gives $\#\{L[d,\, 2^{-1}d\mathbf{Z}]/\Gamma^{\cdot}(L)\} = 1$, provided the type number of $B_{r,\infty}$ is 1.
The existence of a maximal lattice in Theorem 5.1 can be shown in a similar way to [3, Proposition 4.3] by means of two explicit formulas concerning $\#L[dp^n,\, 2^{-1}d\mathbf{Z}]$ and $\#L[dp^n,\, 2^{-1}\mathbf{Z}]$. Those formulas will be given in (5.2) and (5.3) without detailed proofs because of the length of the paper. The author hopes to prove it in a subsequent paper.

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Notation. We denote by $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$ the ring of rational integers, the field of rational numbers, and the field of real numbers, respectively.
If R is an associative ring with identity element and if M is an R-module, then we write $R^{\times}$ for the group of all invertible elements of R and M_n^m the R-module of $m \times n$-matrices with entries in M. We set $R^{\times 2} = \{a^2 \mid a \in R^{\times}\}$. For a finite set X, we denote by $\#X$ the number of elements in X. We set $[a] = \mathrm{Max}\{n \in \mathbf{Z} \mid n \leq a\}$.
Let V be a vector space over a field F of characteristic 0, and $GL(V)$ the group of all F-linear automorphisms of V. We let $GL(V)$ act on V on the *right*.
Let F be an algebraic number field and $\mathfrak{g}$ the ring of all algebraic integers in F. For a fractional ideal of F we often call it a $\mathfrak{g}$-ideal. Let $\mathbf{a}$, $\mathbf{h}$, and $\mathbf{r}$ be the sets of archimedean primes, nonarchimedean primes, and real archimedean primes of F, respectively. We denote by F_v the completion of F at $v \in \mathbf{a} \cup \mathbf{h}$, and by $F_{\mathbf{A}}$ and $F_{\mathbf{A}}^{\times}$ the adele ring and the idele group of F, respectively. We often identify v with the prime ideal of F corresponding to $v \in \mathbf{h}$, and write x_v for the image of x under the embedding of F into $\mathbf{R}$ over $\mathbf{Q}$ at $v \in \mathbf{r}$. For $v \in \mathbf{h}$ we denote by $\mathfrak{g}_v$, $\mathfrak{p}_v$, and π_v the maximal order, the prime ideal, and a prime element of F_v, respectively. If K is a quadratic extension of F, we denote by $D_{K/F}$ the relative discriminant of K over F, and put $K_v = K \otimes_F F_v$ for $v \in \mathbf{h}$.

By a $\mathfrak{g}$-*lattice*, or simply a *lattice* L in a vector space V over a number field or nonarchimedean local field F, we mean a finitely generated $\mathfrak{g}$-submodule in V containing a basis of V. By an *order* in a quaternion algebra B over F we mean a subring of B containing $\mathfrak{g}$ that is a $\mathfrak{g}$-lattice in B. For a symmetric F-bilinear form φ on V and two subspaces X and Y of V, we denote by $X \oplus Y$ the direct sum of X and Y if $\varphi(x, y) = 0$ for every $x \in X$ and $y \in Y$; then we also denote by $\varphi|_X$ the restriction of φ to X. When X is an object defined over a number field F, we often denote by X_v the localization at a prime v if it is meaningful. For given local objects X_v in the text with $v \in \mathbf{a} \cup \mathbf{h}$, we put $X_{\mathbf{a}} = \prod_{v\in\mathbf{a}} X_v$ and $X_{\mathbf{h}} = \prod_{v\in\mathbf{h}} X_v$.

1 Preliminaries for quadratic forms

1.1 Quadratic spaces and Clifford algebras

Let F be an algebraic number field or its completion at a prime. Throughout the paper we often call the former a global field and the latter a local field when it is nonarchimedean. Let (V, φ) be a quadratic space over F, that is, V is a vector space over F and φ is a symmetric F-bilinear form on V. In this paper we consider only a nondegenerate form φ. We put $\varphi[x] = \varphi(x, x)$ for $x \in V$. We define the orthogonal group and the special orthogonal group of φ by

$$O^\varphi(V) = O^\varphi = \{\gamma \in GL(V) \mid \varphi(x\gamma, y\gamma) = \varphi(x, y)\},$$
$$SO^\varphi(V) = SO^\varphi = \{\gamma \in O^\varphi(V) \mid \det(\gamma) = 1\}.$$

We denote by $A(\varphi) = A(V)$ the Clifford algebra of φ and by $A^+(\varphi) = A^+(V)$ the even Clifford algebra of φ. For $x \in A(V)$ we mean x^* the image of x under the canonical involution of $A(V)$. We define the even Clifford group $G^+(V)$ of (V, φ) by

$$G^+(V) = \{\alpha \in A^+(V)^\times \mid \alpha^{-1}V\alpha = V\}. \tag{1.1}$$

We denote by τ a homomorphism defined as follows:

$$\tau : G^+(V) \longrightarrow SO^\varphi(V) \quad \text{via} \quad x\tau(\alpha) = \alpha^{-1}x\alpha \quad \text{for } x \in V. \tag{1.2}$$

This is surjective and the kernel is $F^\times$; see [9, Theorem 3.6], for example.

For a $\mathfrak{g}$-lattice L in V we put

$$\widetilde{L} = L^\sim = \{x \in V \mid 2\varphi(x, L) \subset \mathfrak{g}\}. \tag{1.3}$$

We call L *integral* with respect to φ if $\varphi[L] \subset \mathfrak{g}$. We note that $L \subset \widetilde{L}$ if L is integral. By a $\mathfrak{g}$-*maximal*, or simply a *maximal*, lattice L with respect to φ, we understand a $\mathfrak{g}$-lattice L in V which is maximal among $\mathfrak{g}$-lattices on which the values $\varphi[x]$ are contained in $\mathfrak{g}$. For an integral lattice L in V with respect to φ, we denote by $A(L)$ the subring of $A(V)$ generated by $\mathfrak{g}$ and L. We also put

$$A^+(L) = A^+(V) \cap A(L). \tag{1.4}$$

Then $A(L)$ (resp. $A^+(L)$) is an order in $A(V)$ (resp. $A^+(V)$) (cf. [9, §8.2]).

For a global field F and $v \in \mathbf{a} \cup \mathbf{h}$, we put $V_v = V \otimes_F F_v$ and denote by φ_v the F_v-bilinear extension of φ to V_v; we then put $(V, \varphi)_v = (V_v, \varphi_v)$. For $v \in \mathbf{h}$ and a $\mathfrak{g}_v$-maximal lattice L_v in V_v, $(V, \varphi)_v$ has a *Witt decomposition* as follows (cf. [9, Lemma 6.5]): $V_v = Z_v \oplus \sum_{i=1}^{r_v}(F_v e_i + F_v f_i)$ and $L_v = N_v + \sum_{i=1}^{r_v}(\mathfrak{g}_v e_i + \mathfrak{g}_v f_i)$ with some elements e_i and f_i $(i = 1, \cdots, r_v)$ such that $\varphi_v(e_i, e_j) = \varphi_v(f_i, f_j) = 0$ and $2\varphi_v(e_i, f_j) = 1$ or 0 according as $i = j$ or $i \neq j$. Here $Z_v = \{z \in V_v \mid \varphi_v(z, e_i) = \varphi_v(z, f_i) = 0 \text{ for every } i\}$, on which φ_v is anisotropic; $N_v = \{x \in Z_v \mid \varphi_v[x] \in \mathfrak{g}_v\}$, which is a unique $\mathfrak{g}_v$-maximal lattice in Z_v with respect to φ_v. The dimension t_v of Z_v is uniquely determined by φ_v and $0 \leq t_v \leq 4$ for $v \in \mathbf{h}$ (cf. [9, Theorem 7.6 (ii)]). We call Z_v a *core subspace* of $(V, \varphi)_v$ and t_v the *core dimension* of (V, φ) at v. For convenience, we also call a subspace U_v of V_v anisotropic if φ_v is so on U_v.

For $\mathfrak{g}$-lattices L and M in V over a global or local field F, we denote by $[L/M]$ a $\mathfrak{g}$-ideal of F generated over $\mathfrak{g}$ by $\det(\alpha)$ of all F-linear automorphisms α of V such that $L\alpha \subset M$. If F is a global field, then $[L/M] = \prod_{v\in\mathbf{h}}[L_v/M_v]$ with the localization $[L/M]_v = [L_v/M_v]$ at each v. Following [11, §6.1], in both global and local F, we call $[\widetilde{L}/L]$ the *discriminant ideal* of (V, φ) if L is a $\mathfrak{g}$-maximal lattice in V with respect to φ. This is independent of the choice of L. If F is a local field, the discriminant ideal of φ coincides with that of a core subspace of φ.

By the *invariants* of (V, φ) over a number field F, we understand a set of data

$$\{n,\ F(\sqrt{\delta}),\ Q(\varphi),\ \{s_v(\varphi)\}_{v\in\mathbf{r}}\}, \tag{1.5}$$

where n is the dimension of V, $F(\sqrt{\delta})$ is the *discriminant field* of φ with $\delta = (-1)^{n(n-1)/2}\det(\varphi)$, $Q(\varphi)$ is the *characteristic quaternion algebra* of φ, and $s_v(\varphi)$ is the *index* of φ at $v \in \mathbf{r}$. For these definitions, the reader is referred to [11, §1.1, 3.1, and 4.1] (cf. also [4, (1.6)]). By virtue of [11, Theorem 4.2], the isomorphism class of (V, φ) is determined by $\{n, F(\sqrt{\delta}), Q(\varphi), \{s_v(\varphi)\}_{v\in\mathbf{r}}\}$ and vice versa.

The characteristic algebra $Q(\varphi_v)$ is also defined for φ_v at $v \in \mathbf{a} \cup \mathbf{h}$ (cf. [11, §3.1]). By [11, Lemma 3.3] the isomorphism class of $(V, \varphi)_v$ is determined by $\{n, F_v(\sqrt{\delta}), Q(\varphi_v)\}$ if $v \in \mathbf{h}$. As for $v \in \mathbf{a}$, it is determined by $\{n, s_v(\varphi)\}$ if $v \in \mathbf{r}$, and by the dimension n if $v \notin \mathbf{r}$. If $v \in \mathbf{r}$, then $Q(\varphi_v)$ is given by [11, (4.2a) and (4.2b)], for example. If $v \notin \mathbf{r}$, then $Q(\varphi_v) = M_2(\mathbf{C})$, where $\mathbf{C}$ is the field of complex numbers.

Let $SO^\varphi(V)_\mathbf{A}$ (resp. $O^\varphi(V)_\mathbf{A}$) be the adelization of $SO^\varphi(V)$ (resp. $O^\varphi(V)$) in the usual sense (cf. [9, §9.6]). For $\alpha \in SO^\varphi(V)_\mathbf{A}$ and a $\mathfrak{g}$-lattice L in V, we denote by $L\alpha$ the $\mathfrak{g}$-lattice in V whose localization at each $v \in \mathbf{h}$ is given by

$L_v\alpha_v$. We put

$$C(L) = \{\alpha \in SO^{\varphi}(V)_{\mathbf{A}} \mid L\alpha = L\}, \quad C(L_v) = SO^{\varphi_v}(V_v) \cap C(L), \quad (v \in \mathbf{h})$$
$$\Gamma(L) = SO^{\varphi}(V) \cap C(L).$$

Then the map $\alpha \longmapsto L\alpha^{-1}$ gives a bijection of $SO^{\varphi}\backslash SO^{\varphi}_{\mathbf{A}}/C(L)$ onto $\{L\alpha \mid \alpha \in SO^{\varphi}_{\mathbf{A}}\}/SO^{\varphi}$. We call $\{L\alpha \mid \alpha \in SO^{\varphi}_{\mathbf{A}}\}$ the SO^{φ}-*genus* of L, $\{L\gamma \mid \gamma \in SO^{\varphi}\}$ the SO^{φ}-*class* of L, and $\#\{SO^{\varphi}\backslash SO^{\varphi}_{\mathbf{A}}/C(L)\}$ the *class number* of SO^{φ} relative to $C(L)$ or the class number of the genus of L with respect to SO^{φ}. It is known that all $\mathfrak{g}$-maximal lattices in V with respect to φ form a single SO^{φ}-genus. Let $A^{+}(V)^{\times}_{\mathbf{A}}$ (resp. $G^{+}(V)_{\mathbf{A}}$) be the adelization of $A^{+}(V)^{\times}$ (resp. $G^{+}(V)$). We can extend τ of (1.2) to a homomorphism of $G^{+}(V)_{\mathbf{A}}$ onto $SO^{\varphi}_{\mathbf{A}}$. We denote it by the same symbol τ (cf. [9, §9.10]).

For a $\mathfrak{g}$-lattice L in V, $q \in F$, and a $\mathfrak{g}$-ideal $\mathfrak{b}$ of F, we put

$$L[q] = \{x \in L \mid \varphi[x] = q\}, \quad L[q,\ \mathfrak{b}] = \{x \in V \mid \varphi[x] = q,\ \varphi(x,\ L) = \mathfrak{b}\}.$$

Here $\varphi(x,\ L) = \{\varphi(x,\ y) \mid y \in L\}$, which becomes a $\mathfrak{g}$-ideal of F. Suppose F is a nonarchimedean local field. Let V have dimension $n > 2$ and L be a $\mathfrak{g}$-maximal lattice in V with respect to φ. Then [9, Theorem 10.5] due to Shimura shows that

$$L[q,\ \mathfrak{b}] = hC(L), \tag{1.6}$$

provided $h \in L[q,\ \mathfrak{b}]$ (cf. also [12, Theorem 1.3]).

For a quaternion algebra B over F, we put $2\beta(x,\ y) = xy^{\iota} + yx^{\iota}$ for $x, y \in B$ with the main involution ι of B. For an order $\mathfrak{o}$ in B it is known that $[\widetilde{\mathfrak{o}}/\mathfrak{o}] = d(\mathfrak{o})^2$ with an integral ideal $d(\mathfrak{o})$ of F. Here $\widetilde{\mathfrak{o}}$ is defined by (1.3) with β. The ideal $d(\mathfrak{o})$ is called the *discriminant* of $\mathfrak{o}$. If F is a number field and $\mathfrak{o}$ is a maximal order, then $d(\mathfrak{o})$ is the product of all prime ideals ramified in B, which is called the discriminant of B and denoted by D_B. We set

$$T(\mathfrak{o}) = \{\alpha \in B^{\times}_{\mathbf{A}} \mid \alpha\mathfrak{o} = \mathfrak{o}\alpha\}, \quad T(\mathfrak{o}_v) = B^{\times}_v \cap T(\mathfrak{o}) \quad (v \in \mathbf{a} \cup \mathbf{h}),$$
$$\Gamma^{*}(\mathfrak{o}) = B^{\times} \cap T(\mathfrak{o}),$$

where $B^{\times}_{\mathbf{A}}$ is the adelization of $B^{\times}$, $B_v = B \otimes_F F_v$, and $\mathfrak{o}_v = \mathfrak{o} \otimes_{\mathfrak{g}} \mathfrak{g}_v$. The number $\#\{T(\mathfrak{o})\backslash B^{\times}_{\mathbf{A}}/B^{\times}\}$ is called the *type number* of $\mathfrak{o}$. Let $U = B^{\times}_{\mathbf{a}} \prod_{v \in \mathbf{h}} \mathfrak{o}^{\times}_v$ in $B^{\times}_{\mathbf{A}}$. Then the number of $U \setminus B^{\times}_{\mathbf{A}}/B^{\times}$ is called the *class number* of $\mathfrak{o}$.

Here we introduce two symbols below, which will be used throughout the paper. Let F be a nonarchimedean local field and $\mathfrak{p}$ the prime ideal of F. For $b \in F^{\times}$ we set

$$(F(\sqrt{b})/\mathfrak{p}) = \begin{cases} 1 & \text{if } F(\sqrt{b}) = F, \\ -1 & \text{if } F(\sqrt{b}) \text{ is an unramified quadratic extension of } F, \\ 0 & \text{if } F(\sqrt{b}) \text{ is a ramified quadratic extension of } F. \end{cases}$$

For a quaternion algebra B over F we set

$$\chi(B) = \begin{cases} 1 & \text{if } B \cong M_2(F), \\ -1 & \text{if } B \text{ is a division algebra.} \end{cases}$$

1.2 Ternary quadratic spaces

We recall some basic facts on 3-dimensional quadratic spaces (W, ψ) over a number field or its completion F. The characteristic algebra $Q(\psi)$ is given by $A^+(W)$ by definition. The core dimension s_v of (W, ψ) at $v \in \mathbf{h}$ is determined by

$$s_v = \begin{cases} 1 & \text{if } Q(\psi_v) = M_2(F_v), \\ 3 & \text{if } Q(\psi_v) \text{ is a division algebra.} \end{cases} \tag{1.7}$$

This can be seen from [11, §3.2] and the proof of [11, Lemma 3.3].
There are isomorphisms of (W, ψ) onto $(A^+(W)^\circ, d\nu^\circ)$ with $d \in F^\times$. Here $A^+(W)^\circ = \{x \in A^+(W) \mid x^* = -x\}$, $\nu[x] = xx^*$ for $x \in A^+(W)$, and ν° is the restriction of ν to $A^+(W)^\circ$. Let us explain such isomorphisms, following [9, §7.3].
Take an orthogonal basis $\{k_1, k_2, k_3\}$ of W with respect to ψ, namely, an F-basis $\{k_i\}$ of W such that $\psi(k_i, k_j) = 0$ for $i \neq j$. Under the identification of W with the corresponding subspace in the Clifford algebra $A(W)$, put $\xi = k_1k_2k_3 \in A(W)^\times$; then $F + F\xi$ is the center of $A(W)$. We see that $A^+(W) = F + Fk_1k_2 + Fk_1k_3 + Fk_2k_3$ and $W\xi = Fk_1k_2 + Fk_1k_3 + Fk_2k_3$. By [9, Theorem 2.8 (ii)], $A^+(W)$ is a quaternion algebra over F; the main involution coincides with the canonical involution $*$ restricted to $A^+(W)$. Then the mapping $x \longmapsto x\xi$ gives an F-linear isomorphism of W onto $A^+(W)^\circ$ such that $(x\xi)(x\xi)^* = \xi\xi^*\psi[x]$ for $x \in W$. Putting $\nu[y] = yy^*$ for $y \in A^+(W)$, we have an isomorphism

$$(W, \psi) \cong (A^+(W)^\circ, (\xi\xi^*)^{-1}\nu^\circ) \quad \text{via } x \longmapsto x\xi. \tag{1.8}$$

We note that $\xi\xi^* \in \det(\psi)F^{\times 2}$, since $\xi\xi^* = \psi[k_1]\psi[k_2]\psi[k_3] \in F^\times$.
Let $G^+(W)$ be the even Clifford group of (W, ψ) as in (1.1) and τ the homomorphism defined in (1.2). By the definition of $A^+(W)^\circ$, $\alpha^{-1}A^+(W)^\circ\alpha = A^+(W)^\circ$ for $\alpha \in A^+(W)^\times$. Hence we have $G^+(W) = A^+(W)^\times$. Moreover, under the isomorphism (1.8) we can understand that

$$x\tau(\alpha) = \alpha^{-1}x\xi\alpha\xi^{-1}$$

for $x \in W$ and $\alpha \in A^+(W)^\times$.
Now, the pair $(A^+(W), \nu)$ can be viewed as a quaternary quadratic space over F. We note that $\nu(x, y) = 2^{-1}Tr_{A^+(W)/F}(xy^*)$ for $x, y \in A^+(W)$. For an integral lattice N in W with respect to ψ, we consider the order $A^+(N)$ in $A^+(W)$ defined by (1.4). Its discriminant $d(A^+(N))$ is given by $[A^+(N)\tilde{}/A^+(N)] = d(A^+(N))^2$, where $A^+(N)\tilde{}$ is defined by (1.3) with ν. By

[4, Lemma 1.1], $d(A^+(N)) = 2^{-1}[\widetilde{N}/N]$. It is noted that *if the order $A^+(N)$ is maximal in $A^+(W)$ for an integral lattice N in (W, ψ), then N is $\mathfrak{g}$-maximal with respect to ψ.* The converse is not true; namely, in general, $A^+(N)$ is not maximal even if N is a maximal lattice.

2 Orthogonal complements in quaternary spaces

2.1 Invariants and discriminant ideals

Let (V, φ) be a 4-dimensional quadratic space over a number field F. The characteristic algebra $Q(\varphi)$ is determined by $A(\varphi) \cong M_2(Q(\varphi))$ by definition. Set $B = Q(\varphi)$ and $K = F(\sqrt{\delta})$ with $\delta = \det(\varphi)$. The core dimension t_v of (V, φ) at $v \in \mathbf{h}$ is determined by

$$t_v = \begin{cases} 0 & \text{if } F_v(\sqrt{\delta}) = F_v \text{ and } Q(\varphi_v) = M_2(F_v), \\ 4 & \text{if } F_v(\sqrt{\delta}) = F_v \text{ and } Q(\varphi_v) \text{ is a division algebra}, \\ 2 & \text{if } F_v(\sqrt{\delta}) \neq F_v. \end{cases} \tag{2.1}$$

This can be seen from [11, §3.2] and the proof of [11, Lemma 3.3].
For $h \in V$ such that $\varphi[h] = q \neq 0$ we put

$$W = (Fh)^\perp = \{x \in V \mid \varphi(x, h) = 0\}. \tag{2.2}$$

Then (W, ψ) is a nondegenerate ternary quadratic space over F with the restriction ψ of φ to W and $(V, \varphi) = (W, \psi) \oplus (Fh, \varphi|_{Fh})$. The invariants of (W, ψ) are given by $\{3,\ F(\sqrt{-\delta q}),\ Q(\psi),\ \{s_v(\psi)\}_{v\in\mathbf{r}}\}$, which are independent of the choice of h so that $\varphi[h] = q$. The characteristic algebra $Q(\psi) = A^+(W)$ is determined by the local algebras $Q(\psi_v)$ for all primes v of F. Then by [2, Theorem 1.1 (1)], $Q(\psi_v) = M_2(F_v)$ holds exactly in the following cases:

$$\begin{aligned} &\delta \in F_v^{\times 2} \text{ and } v \nmid D_B, \\ &\delta \notin F_v^{\times 2}, v \nmid D_B, \text{ and } q \in \kappa_v[K_v^\times], \\ &\delta \notin F_v^{\times 2}, v \mid D_B, \text{ and } q \notin \kappa_v[K_v^\times], \\ &v \in \mathbf{r}, q_v > 0, \text{ and } s_v(\varphi) = 0,\ 2, \\ &v \in \mathbf{r}, q_v < 0, \text{ and } s_v(\varphi) = 0,\ -2, \\ &v \in \mathbf{a} \text{ such that } v \notin \mathbf{r}, \end{aligned}$$

where κ_v is the norm form of K_v. It should be noted that

$$M_2(Q(\varphi)) \cong Q(\psi) \otimes_F \{K,\ q\}, \tag{2.3}$$

where F is a number field or its completion and $\{K, q\}$ is the quaternion algebra over F defined in [4, (1.12)] if $K \neq F$; we set $\{K, q\} = M_2(F)$ if $K = F$ (see also [9, §1.10]). This (2.3) can be seen from [11, Theorem 7.4 (i)]. The index at $v \in \mathbf{r}$ is given by $s_v(\psi) = s_v(\varphi) - 1$ if $q_v > 0$ and $s_v(\psi) = s_v(\varphi) + 1$ if $q_v < 0$. The core dimension of (W, ψ) at $v \in \mathbf{h}$ is determined by (1.7).

The Clifford algebra $A(W)$ can be viewed as a subalgebra of $A(V)$ with the restriction ψ. Then $A^+(W) = \{x \in A^+(V) \mid xh = hx\}$ and $G^+(W) = \{\alpha \in G^+(V) \mid \alpha h = h\alpha\}$ by [9, Lemma 3.16]. The canonical involution of $A(W)$ coincides with $*$ of $A(V)$ restricted to $A(W)$. In particular, such an involution $*$ gives the main involution of the quaternion algebra $A^+(W)$.

Let L and M be $\mathfrak{g}$-maximal lattices in V and W with respect to φ and ψ, respectively. The discriminant ideals of φ and ψ are given as follows:

$$[\widetilde{L}/L] = D_{K/F}\mathfrak{e}^2, \tag{2.4}$$

$$[\widetilde{M}/M] = 2\mathfrak{a}^{-1}D_{Q(\psi)}^2 \cap 2\mathfrak{a}, \tag{2.5}$$

where $\mathfrak{e}$ is the product of all the prime ideals which are ramified in B and which do not ramify in K; we understand $D_{K/F} = \mathfrak{g}$ if $K = F$; we put $\delta q\mathfrak{g} = \mathfrak{a}\mathfrak{b}^2$ with a squarefree integral ideal $\mathfrak{a}$ and a $\mathfrak{g}$-ideal $\mathfrak{b}$ of F. These (2.4) and (2.5) can be obtained by applying [11, Theorem 6.2] to (V, φ) and the complement (W, ψ).
The intersection $L \cap W$ is an integral $\mathfrak{g}$-lattice in W with respect to ψ. It can be seen that $[(L \cap W)\widetilde{\ }/L \cap W] = [M/L \cap W]^2[\widetilde{M}/M]$ and $[M/L \cap W]$ is an integral ideal, which is independent of the choice of M; see [2, Lemma 2.2 (6)]. Moreover there is a $\mathfrak{g}$-ideal $\mathfrak{b}(q)$ of F such that

$$[M/L \cap W] = \mathfrak{b}(q)(2\varphi(h, L))^{-1} \tag{2.6}$$

by [2, Theorem 4.2]. We note that $2\varphi(h, L)$ must contain $\mathfrak{b}(q)$ and that $2\varphi(h, L) \subset \mathfrak{g}$ if $h \in L$. The ideal $\mathfrak{b}(q)$ is determined by

$$2q[\widetilde{L}/L] = \mathfrak{b}(q)^2[\widetilde{M}/M] \tag{2.7}$$

(cf. [2, (4.1)]). Combining these, we obtain $[(L \cap W)\widetilde{\ }/L \cap W] = 2q[\widetilde{L}/L](2\varphi(h, L))^{-2}$. Now to $L \cap W$ we associate the order $A^+(L \cap W)$ defined by (1.4). Its discriminant is given by

$$d(A^+(L \cap W)) = 2^{-1}[(L \cap W)\widetilde{\ }/L \cap W] = q[\widetilde{L}/L](2\varphi(h, L))^{-2}. \tag{2.8}$$

It is noted that the discriminant of $A^+(W)$ divides $q[\widetilde{L}/L](2\varphi(h, L))^{-2}$. We also note that if $d(A^+(L \cap W))$ is squarefree, then $2\varphi(h, L)$ must be $\mathfrak{b}(q)$ in (2.6), that is, $L \cap W$ is maximal in W.

For our later use, let us state a *weak* Witt decomposition of the local space $(V, \varphi)_v$ whose core dimension t_v is 0 or 2. We fix a nonarchimedean prime v of F and drop the subscript v. Let L be a $\mathfrak{g}$-maximal lattice in V with respect to φ. We first note that φ is isotropic as t is 0 or 2. Let K be the discriminant algebra of φ defined by $K = F \times F$ if $t = 0$ and by $K = F(\sqrt{\det(\varphi)})$ if $t = 2$; also let κ be the norm form defined by $2\kappa(x, y) = \kappa[x + y] - \kappa[x] - \kappa[y]$ and

$\kappa[(a,\, b)] = ab$ for $x,\, y,\, (a,\, b) \in K$ if $t = 0$, and by $2\kappa(x,\, y) = xy^\rho + x^\rho y$ for $x,\, y \in K$ with a nontrivial automorphism ρ of K over F if $t = 2$. Because K is embeddable in $A^+(V)$, we identify K with the image of it. Then there is a weak Witt decomposition as follows (cf. [4, (1.19) and (1.20)]):

$$\begin{aligned} V = Kg \oplus (Fe + Ff), &\qquad L = \mathfrak{r}g + (\mathfrak{g}e + \mathfrak{g}f), \\ (Kg,\ \varphi) \cong (K,\ c\kappa) &\quad \text{via } xg \longmapsto x \end{aligned} \tag{2.9}$$

with some elements e and f of V such that $\varphi[e] = \varphi[f] = 0$ and $2\varphi(e,\, f) = 1$, and $g \in V$ such that $g^2 = c \in F^\times$. Here $\mathfrak{r} = \mathfrak{g} \times \mathfrak{g}$ if $t = 0$ and $\mathfrak{r}$ is the maximal order of K if $t = 2$. We may assume that $c = 1$ if $t = 0$, $c \in \mathfrak{g}^\times$ if $(K/\mathfrak{p}) = -1$ and $\chi(Q(\varphi)) = +1$, $c \in \pi\mathfrak{g}^\times$ if $(K/\mathfrak{p}) = -1$ and $\chi(Q(\varphi)) = -1$, and $c \in \mathfrak{g}^\times$ if $(K/\mathfrak{p}) = 0$. We also note that $A^+(Kg) = K$ and $xg = gx^*$ for $x \in K$, where $(a,\, b)^* = (b,\, a)$ for $(a,\, b) \in K$ if $t = 0$ and the involution $*$ gives a nontrivial automorphism of K over F if $t = 2$.

2.2 The genus of $L \cap W$

Let $(V,\, \varphi)$ be a quaternary quadratic space over a number field F and $(W,\, \psi)$ as in §2.1 with a fixed element h of V such that $\varphi[h] \neq 0$.

LEMMA 2.1. *Let L be a $\mathfrak{g}$-maximal lattice in V with respect to φ. Then $A^+(L \cap W) = A^+(L) \cap A^+(W)$ for every $h \in V$ such that $\varphi[h] \neq 0$. The discriminant of $A^+(L \cap W)$ is given by* (2.8).

This follows from the similar result [4, Lemma 3.2] on local orders $A^+(L_v \cap W_v)$ by localization. We next restate [4, Corollary 2.2] which is a conclusion from the main result of [4]:

THEOREM 2.2. *Let $(V,\, \varphi)$ be a quaternary quadratic space over a number field F and L a $\mathfrak{g}$-maximal lattice in V with respect to φ. For $h \in V$ such that $\varphi[h] \neq 0$ put $W = (Fh)^\perp$ and let ψ be the restriction of φ to W. Put $\mathfrak{o} = A^+(L \cap W)$. Then $C(L \cap W) = \tau(T(\mathfrak{o}))$ and $\Gamma(L \cap W) = \tau(\Gamma^*(\mathfrak{o}))$ hold. Consequently, the map $N \longmapsto A^+(N)$ gives a bijection of the $SO^\psi(W)$-classes in the $SO^\psi(W)$-genus of $L \cap W$ onto the conjugacy classes in the genus of $\mathfrak{o}$ which is the set $\{\alpha^{-1}\mathfrak{o}\alpha \mid \alpha \in A^+(W)^\times_{\mathbf{A}}\}$.*

3 An order associated with $L \cap W$

3.1 The local case

We first recall some general notation and results, following [9, §8 Part I]. For a quadratic space $(V,\, \varphi)$ over a local field F, take a $\mathfrak{g}$-maximal lattice L in V with respect to φ. We define a subgroup J_V of $G^+(V)$ by

$$J_V = \{\alpha \in G^+(V) \mid \tau(\alpha) \in C(L),\ \alpha\alpha^* \in \mathfrak{g}^\times\}. \tag{3.1}$$

Put $E_V = G^1(V) \cap J_V$, where $G^1(V) = \{\alpha \in G^+(V) \mid \alpha\alpha^* = 1\}$ is the spin group of φ. If the dimension of V is even more than 2, then by virtue of [9, Theorem 8.9] specialized to this case,

$$[C(L) : \tau(J_V)] = \begin{cases} 1 & \text{if } t = 0, \text{ or } t = 2,\ (K/\mathfrak{p}) = -1, \text{ and } Q(\varphi) = M_2(F), \\ 2 & \text{otherwise}, \end{cases} \tag{3.2}$$

where t is the core dimension of (V, φ) and $K = F(\sqrt{\delta})$ is the discriminant field of φ. If the dimension of V is odd more than 1, then by [9, Theorem 8.9] and [12, Theorem 1.8 (ii)],

$$[C(L) : \tau(J_V)] = \begin{cases} 1 & \text{if } t = 1 \text{ and } \delta \in \mathfrak{g}^\times F^{\times 2}, \\ 2 & \text{otherwise}. \end{cases} \tag{3.3}$$

Let (V, φ) be a quaternary quadratic space over F. For $h \in V$ such that $\varphi[h] = q \neq 0$, put $W = (Fh)^\perp$ and let ψ be the restriction of φ to W. Let $K = F(\sqrt{\delta})$ be the discriminant field of φ. Also let L and M be $\mathfrak{g}$-maximal lattices in V and W with respect to φ and ψ, respectively. We define J_V in $G^+(V)$ by (3.1) with L and J_W in $G^+(W)$ with M. Let S_V^+ (resp. S_W^+) be the order in $A^+(V)$ (resp. $A^+(W)$) generated by E_V and $A^+(L)$ (resp. by E_W and $A^+(M)$) except the case where $t = 2$, $(K/\mathfrak{p}) = -1$, and $\chi(Q(\varphi)) = -1$ (resp. where $t = 0$ and $q \in \pi\mathfrak{g}^\times F^{\times 2}$, or $t = 2$, $\delta q \in \pi\mathfrak{g}^\times F^{\times 2}$, and $Q(\varphi) = M_2(F)$); in which cases we put

$$S_V^+ = A^+(V) \cap S_V \quad \text{if } t = 2,\ (K/\mathfrak{p}) = -1, \text{ and } \chi(Q(\varphi)) = -1, \tag{3.4}$$

$$S_W^+ = A^+(W) \cap S_W \quad \text{if } \begin{cases} t = 0 \text{ and } q \in \pi\mathfrak{g}^\times F^{\times 2}, \\ t = 2,\ \delta q \in \pi\mathfrak{g}^\times F^{\times 2}, \text{ and } Q(\psi) = M_2(F), \end{cases} \tag{3.5}$$

where S_V (resp. S_W) is a unique maximal order in $A(V)$ (resp. $A(W)$) containing E_V and $A(L)$ (resp. E_W and $A(M)$) given by [9, Theorem 8.6 (i)]. By [9, Theorem 8.6 (ii)] these S_V^+ and S_W^+ are maximal orders except in cases (3.4) and (3.5). It should be noted that we can prove this fact in a similar way to the proof of [9, Theorem 8.6 (ii)] even for the case which does not satisfy the assumption [9, (8.1)]. For the same reason we also see that $S_W = A(M)$ in case (3.5). In all cases,

$$J_V = G^+(V) \cap (S_V^+)^\times, \tag{3.6}$$

$$J_W = G^+(W) \cap (S_W^+)^\times = (S_W^+)^\times. \tag{3.7}$$

In fact, [9, Proposition 8.8 (ii)] together with $G^+(W) = A^+(W)^\times$ implies (3.7) except in case (3.5). As for (3.5), there is an order in $A(W)$ containing J_W and M by [9, Lemma 8.4 (ii)]. In view of the uniqueness of S_W and $E_W \subset J_W$, S_W

contains J_W and M, and hence [9, Proposition 8.8 (i)] is applicable to the case (3.5). This proves (3.7). Similarly we have (3.6).
Now, $A^+(W) = \{x \in A^+(V) \mid xh = hx\}$ and $G^+(W) = \{\alpha \in G^+(V) \mid \alpha h = h\alpha\}$ as mentioned in §2.1. It can be seen that

$$G^+(W) \cap J_V = (A^+(W) \cap S_V^+)^\times. \tag{3.8}$$

Thus $G^+(W) \cap J_V$ is the unit group of an order $A^+(W) \cap S_V^+$ in $A^+(W)$.

Lemma 3.1. *In the above setting the following assertions hold:*

(1) $[S_W^+/A^+(M)]$ *is given by*

$$\begin{cases} \mathfrak{p} & \textit{if } t = 4 \textit{ and } q \in \mathfrak{g}^\times F^{\times 2}, \\ & \textit{or } t = 2,\ \delta q \in \mathfrak{g}^\times F^{\times 2}, \textit{ and } \chi(Q(\psi)) = -1, \\ \mathfrak{g} & \textit{otherwise.} \end{cases} \tag{3.9}$$

Here S_W^+ may or may not be maximal when $t = 0$ and $q \in \pi\mathfrak{g}^\times F^{\times 2}$ or when $t = 2$, $\delta q \in \pi\mathfrak{g}^\times F^{\times 2}$, and $Q(\psi) = M_2(F)$.

(2) *Assume that $q \in \mathfrak{g}^\times F^{\times 2}$ if $t = 2$, K/F is unramified, and $Q(\varphi)$ is a division algebra. Then $[A^+(W) \cap S_V^+/A^+(L \cap W)]$ is given by*

$$\begin{cases} \mathfrak{p} & \textit{if } t = 4 \textit{ and } q \in \mathfrak{g}^\times F^{\times 2}, \\ & \textit{or } t = 2,\ (K/\mathfrak{p}) = -1, \textit{ and } \chi(Q(\varphi)) = -1, \\ \mathfrak{g} & \textit{otherwise.} \end{cases} \tag{3.10}$$

Proof. Let s be the core dimension of (W, ψ). In view of (1.7), (2.3), and (2.1), we observe that

$$s = 1 \text{ and } \delta q \in \pi\mathfrak{g}^\times F^{\times 2} \iff \begin{cases} t = 0 \text{ and } q \in \pi\mathfrak{g}^\times F^{\times 2}, \\ t = 2,\ \delta q \in \pi\mathfrak{g}^\times F^{\times 2}, \text{ and } Q(\psi) = M_2(F), \end{cases} \tag{3.11}$$

$$s = 3 \text{ and } \delta q \in \mathfrak{g}^\times F^{\times 2} \iff \begin{cases} t = 4 \text{ and } q \in \mathfrak{g}^\times F^{\times 2}, \\ t = 2,\ \delta q \in \mathfrak{g}^\times F^{\times 2}, \text{ and } \chi(Q(\psi)) = -1. \end{cases} \tag{3.12}$$

Then we can verify that

$$S_V^+ = A^+(L) \text{ if } \begin{cases} t = 0, \\ t = 2 \quad \text{except the case } (K/\mathfrak{p}) = -1 \text{ and } \chi(Q(\varphi)) = -1, \end{cases} \tag{3.13}$$

$$S_W^+ = A^+(M) \iff \begin{cases} s = 1, \\ s = 3 \text{ and } \delta q \in \pi\mathfrak{g}^\times F^{\times 2}. \end{cases} \tag{3.14}$$

In fact, if $s = 1$ and $\delta q \in \pi\mathfrak{g}^\times F^{\times 2}$, then the 'if'-part of (3.14) follows from (3.11), (3.5), and $S_W = A(M)$. If $s = 3$ and $\delta q \in \pi\mathfrak{g}^\times F^{\times 2}$, then $S_W^+ = A^+(M)$ because the discriminant of $A^+(M)$ is $\mathfrak{p}$. If $s = 3$ and $\delta q \in \mathfrak{g}^\times F^{\times 2}$, then since S_W^+ is a maximal order in the division algebra $A^+(W)$, it has discriminant $\mathfrak{p}$. Note that the discriminant of $A^+(M)$ is $\mathfrak{p}^2$. Hence $S_W^+ \neq A^+(M)$. Further, observing $A^+(M) \subset S_W^+ \subset (S_W^+)^{\sim} \subset A^+(M)^{\sim}$ and applying [2, Lemma 2.2 (3)] with the norm form ν of $A^+(W)$, we have $[S_W^+/A^+(M)] = \mathfrak{p}$. The remaining parts follow from [9, Theorem 8.6 (vi)].

From (3.14) and (3.12) we see that

$$S_W^+ \neq A^+(M) \Longleftrightarrow \begin{cases} t = 4 \text{ and } q \in \mathfrak{g}^\times F^{\times 2}, \\ t = 2,\ \delta q \in \mathfrak{g}^\times F^{\times 2}, \text{ and } \chi(Q(\psi)) = -1. \end{cases}$$

In this case $[S_W^+/A^+(M)] = \mathfrak{p}$, which proves (1).

To prove (2), it is sufficient to observe the two cases that $t = 4$ or that $t = 2$, $(K/\mathfrak{p}) = -1$, and $\chi(Q(\varphi)) = -1$ by (3.13) and Lemma 2.1.

If $t = 4$ and $q \in \pi\mathfrak{g}^\times F^{\times 2}$, then $A^+(L \cap W) \subset A^+(W) \cap S_V^+ \subset S_W^+ = A^+(M)$. Thus $A^+(W) \cap S_V^+ = A^+(L \cap W)$ because $L \cap W$ is maximal.

Suppose that $t = 4$ and $q \in \mathfrak{g}^\times F^{\times 2}$. Then $A^+(L \cap W)$ has discriminant $\mathfrak{p}^2$ and by Lemma 2.1, $A^+(L \cap W) \subset A^+(W) \cap S_V^+ \subset S_W^+$ in the division algebra $A^+(W)$. We employ the setting and notation in the case where $q_0 \in \mathfrak{g}^\times$ and $(K_1/\mathfrak{p}) = 0$ in [4, §4.4]. In [4, (3.31)] observing $(g_2g_3)(g_2g_3)^* \in \pi^2\mathfrak{g}^\times$, we set

$$\mathfrak{O} = \mathfrak{g} + \mathfrak{g}g_1g_2 + \mathfrak{g}g_1g_3 + \mathfrak{g}\pi^{-1}g_2g_3. \tag{3.15}$$

This is an order in $A^+(W)$ which contains but does not coincide with $A^+(L \cap W)$. Hence $\mathfrak{O}$ is a unique maximal order S_W^+ in $A^+(W)$. Now in the present setting, $(V, \varphi) = (B, \beta)$ and L is a unique maximal order $\mathfrak{o}$ in $B = Q(\varphi)$ with the norm form β. To see the order S_V^+ in $A^+(V)$, we here recall an F-linear mapping p defined in [9, §7.4 (B)]:

$$p : V \longrightarrow M_2(B) \quad \text{via} \quad p(x) = \begin{bmatrix} 0 & x \\ x^\iota & 0 \end{bmatrix},$$

where ι is the main involution of B. Then $A^+(V)$ and S_V^+ are given by

$$A^+(V) = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \mid x,\ y \in B \right\}, \quad S_V^+ = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \mid x,\ y \in \mathfrak{o} \right\}.$$

Under the identification of V with $p(V)$ and of W with $p(W)$, $A^+(L \cap W)$ and S_W^+ are given by [4, (3.31)] and (3.15), respectively. Then we see that

$$\pi^{-1}g_2g_3 = \frac{1}{\pi} \begin{bmatrix} 0 & g_2 \\ g_2^\iota & 0 \end{bmatrix} \begin{bmatrix} 0 & g_3 \\ g_3^\iota & 0 \end{bmatrix} = \begin{bmatrix} \pi^{-1}g_2g_3^\iota & 0 \\ 0 & \pi^{-1}g_2^\iota g_3 \end{bmatrix},$$

$$\beta[\pi^{-1}g_2g_3^\iota] = \pi^{-2} \cdot \pi ac \cdot \pi^{2-2(\kappa-k)}(1-c) \in \mathfrak{g}^\times.$$

Thus both $\pi^{-1}g_2g_3^\iota$ and $\pi^{-1}g_2^\iota g_3$ belong to $\mathfrak{o}$, so that $\pi^{-1}g_2g_3 \in S_V^+$. Therefore S_V^+ contains S_W^+, which implies that $A^+(W) \cap S_V^+$ is the maximal order S_W^+. For the other cases S_W^+ can be observed in a similar manner; we have then

$$S_W^+ = \begin{cases} \mathfrak{g} + \mathfrak{g}1_B\omega + \mathfrak{g}1_B(v\omega) + \mathfrak{g}\pi^{-1}\omega(v\omega) & \text{if } (K_1/\mathfrak{p}) = -1 \text{ and } \mathfrak{p} \nmid 2, \\ \mathfrak{g} + \mathfrak{g}1_B\omega + \mathfrak{g}1_B(u\omega) + \mathfrak{g}\pi^{-1}\omega(u\omega) & \text{if } (K_1/\mathfrak{p}) = -1 \text{ and } \mathfrak{p} \mid 2, \\ \mathfrak{g} + \mathfrak{g}\sqrt{s}\omega + \mathfrak{g}\sqrt{s}(v\omega) + \mathfrak{g}\pi^{-1}\omega(v\omega) & \text{if } (K_1/\mathfrak{p}) = 1. \end{cases}$$

Here the notation is the same as in each case of [4, §3.4]. Consequently $A^+(W) \cap S_V^+ = S_W^+$ in each case. This settles the case where $t = 4$ and $q \in \mathfrak{g}^\times F^{\times 2}$.
Suppose that $t = 2$, $(K/\mathfrak{p}) = -1$, and $\chi(Q(\varphi)) = -1$. In this case S_V^+ is defined by (3.4) with the maximal order S_V in $A(V)$. Let $q \in \pi^{2\ell}\mathfrak{g}^\times$ with $\ell \in \mathbf{Z}$. Then $\mathfrak{b}(q) = \mathfrak{p}^\ell$ as was seen in the case of $q \notin \varphi[Kg]$ in [4, §3.2] with $g^2 \in \pi\mathfrak{g}^\times$ in (2.9). Since $A^+(W)$ is a division algebra, S_W^+ is a unique maximal order in $A^+(W)$ of discriminant $\mathfrak{p}$. We have by (2.8),

$$\begin{aligned} A^+(L \cap W) \subset A^+(W) \cap S_V \subset S_W^+, \\ \mathfrak{p}^{2\ell+2}(2\varphi(h,\, L))^{-2} \subset d(A^+(W) \cap S_V) \subset \mathfrak{p}. \end{aligned} \tag{3.16}$$

Now put $2\varphi(h,\, L) = \mathfrak{p}^m$, which satisfies $m \le \ell$. We observe that $q\pi^{-m}e + \pi^m f \in L[q,\, 2^{-1}\mathfrak{p}^m] = hC(L)$ by (1.6) with the same notation as in the proof of [4, Lemma 3.1]. Then identifying W with that in [4, (3.1)] and employing the isomorphism Ψ of $A(V)$ in the proof of [4, Lemma 3.1], we can find the structure of $A^+(W) \cap S_V$ as follows:

$$\Psi(A^+(W) \cap S_V) = \mathfrak{r} + \pi^{-m}\mathfrak{r}\eta, \tag{3.17}$$

where $\mathfrak{r} = \mathfrak{g}[\xi]$ is the maximal order of K and η is given by [4, (3.3)]. From this together with [4, (3.4)] we have $[A^+(W) \cap S_V / A^+(L \cap W)] = [\mathfrak{r}/\mathfrak{f}] = \mathfrak{p}$, where $\mathfrak{f} = \mathfrak{g} + g^2\mathfrak{g}\xi$. To see (3.17), we recall by [9, Theorem 8.6 (iii)] that $\Psi(S_V) = M_2(Q)$, where $Q = \mathfrak{r} + \mathfrak{r}g$ is a maximal order in the division algebra $Q(\varphi) = A(Kg) = K + Kg$. Then (3.17) can be seen from this and [4, (3.2)]. This completes the proof. $\square$

Lemma 3.2. *Let the notation be the same as in Lemma 3.1 with h and L. Then the following assertions hold:*

(1) *Define an order $\mathfrak{O}$ in $A^+(W)$ by*

$$\mathfrak{O} = \begin{cases} S_W^+ & \text{if } t = 4, \\ A^+(W) \cap S_V & \text{if } t = 2,\ (K/\mathfrak{p}) = -1, \text{ and } \chi(Q(\varphi)) = -1, \\ A^+(L \cap W) & \text{otherwise.} \end{cases} \tag{3.18}$$

Then $G^+(W) \cap J_V = \mathfrak{O}^\times$.

(2) *Assume that* $q = \varphi[h] \in \mathfrak{g}^\times F^{\times 2}$ *and* $2\varphi(h, L) = \mathfrak{b}(q)$ *if* $t = 2$, K/F *is unramified, and* $Q(\varphi)$ *is a division algebra. Let* $\mathfrak{O}$ *be the order defined by* (3.18). *Then* $\mathfrak{O}$ *is a unique order in* $A^+(W)$, *containing* $A^+(L \cap W)$, *of discriminant*

$$\begin{cases} \mathfrak{p} & \text{if } t = 4, \\ & \text{or } t = 2,\ (K/\mathfrak{p}) = -1, \text{ and } \chi(Q(\varphi)) = -1, \\ q[\widetilde{L}/L](2\varphi(h, L))^{-2} & \text{otherwise.} \end{cases} \tag{3.19}$$

In particular, $\mathfrak{O}$ *is a unique maximal order in the division algebra* $A^+(W)$ *when* $t = 2$, $(K/\mathfrak{p}) = -1$, *and* $\chi(Q(\varphi)) = -1$. *Moreover, if* $L \cap W \subset M$, *then* $\mathfrak{O} \subset S_W^+$.

Proof. To prove (1), let $\mathfrak{O}$ be the order given by (3.18). From Lemma 3.1 it can be seen that $\mathfrak{O} = A^+(W) \cap S_V^+$. Thus we have $G^+(W) \cap J_V = \mathfrak{O}^\times$ by (3.8), which proves (1).

To prove (2), by Lemma 3.1 (2) we see that $A^+(W) \cap S_V^+ \neq A^+(L \cap W)$ if and only if $t = 4$ and $q \in \mathfrak{g}^\times F^{\times 2}$ or if $t = 2$, $(K/\mathfrak{p}) = -1$, and $\chi(Q(\varphi)) = -1$. If $t = 4$ and $q \in \mathfrak{g}^\times F^{\times 2}$, then $A^+(W) \cap S_V^+ = S_W^+$ as seen in the proof of Lemma 3.1 (2). If $t = 2$, $(K/\mathfrak{p}) = -1$, and $\chi(Q(\varphi)) = -1$, then, by our assumption, $q \in \pi^{2\ell}\mathfrak{g}^\times$ and $2\varphi(h, L) = \mathfrak{b}(q) = \mathfrak{p}^\ell$ with $\ell \in \mathbf{Z}$. Thus applying (3.16) to $m = \ell$, we have $\mathfrak{p}^2 \subset d(A^+(W) \cap S_V) \subset \mathfrak{p}$. Because $A^+(W) \cap S_V \neq A^+(L \cap W)$, $A^+(W) \cap S_V$ must be maximal in $A^+(W)$. Consequently, if $\mathfrak{O} \neq A^+(L \cap W)$, it is a maximal order which is uniquely determined by discriminant $\mathfrak{p}$. As for the case of $\mathfrak{O} = A^+(L \cap W)$, the discriminant is given by (2.8). Summing up these, we have the uniqueness of $\mathfrak{O}$. To prove the last assertion, suppose that $L \cap W \subset M$. Then $A^+(L \cap W) \subset A^+(M) \subset S_W^+$, which shows $\mathfrak{O} \subset S_W^+$ when $\mathfrak{O} = A^+(L \cap W)$. If $\mathfrak{O} \neq A^+(L \cap W)$, then $\mathfrak{O}$ is maximal in $A^+(W)$. Since S_W^+ is also maximal, we have $\mathfrak{O} = S_W^+$. Hence $\mathfrak{O} \subset S_W^+$ holds if $L \cap W \subset M$. This proves (2). □

3.2 The global case

Let (V, φ) and (W, ψ) be the quadratic spaces over a number field F in the setting of §2.2 with an element h of V such that $\varphi[h] = q \in F^\times$. Let L and M be $\mathfrak{g}$-maximal lattices in V and W with respect to φ and ψ, respectively. Put

$$J_V = G^+(V)_{\mathbf{a}} \prod_{v \in \mathbf{h}} J_{V_v}, \qquad J_W = G^+(W)_{\mathbf{a}} \prod_{v \in \mathbf{h}} J_{W_v}, \tag{3.20}$$

where J_{V_v} and J_{W_v} are given in §3.1. We have an order S_W^+ in $A^+(W)$ determined by $S_{W_v}^+$ for all $v \in \mathbf{h}$, where $S_{W_v}^+$ is the order in $A^+(W_v)$ given in §3.1; notice that $S_{W_v}^+ = A^+(M_v)$ for almost all v.

Let us here insert a remark on the order in $A^+(W)$ given in [12, Lemma 5.3 (ii)]. By applying that lemma to M, we have an order $\mathfrak{O}_0$ containing $A^+(M)$.

Then $[(\mathfrak{O}_0)_v/A^+(M_v)]$ is the same ideal as in (3.9) for each $v \in \mathbf{h}$. This can be seen in the proof of [12, Lemma 5.3 (ii)]. Hence the order $\mathfrak{O}_0$ coincides with S_W^+ in the present situation.

Proposition 3.3. *Let the notation be the same as above with $h \in V$ and L. Also let $K = F(\sqrt{\delta})$ be the discriminant field of φ. Then the following assertions hold:*

(1) *Let $\mathfrak{O}$ be the order in $A^+(W)$ whose localization at $v \in \mathbf{h}$ is the local order defined by* (3.18). *Then*

$$G^+(W)_{\mathbf{A}} \cap J_V = A^+(W)_{\mathbf{a}}^{\times} \mathfrak{O}_{\mathbf{h}}^{\times}. \tag{3.21}$$

(2) *$T(A^+(M)) = T(S_W^+)$ and $J_W = A^+(W)_{\mathbf{a}}^{\times}(S_W^+)_{\mathbf{h}}^{\times}$. Moreover $J_W \subset T(A^+(M))$ and $G^+(W)_{\mathbf{A}} \cap J_V \subset T(A^+(L \cap W))$.*

(3) *Assume that $q = \varphi[h] \in \mathfrak{g}_v^{\times} F_v^{\times 2}$ and $2\varphi(h, L)_v = \mathfrak{b}(q)_v$ for every $v \in \mathbf{h}$ such that $t_v = 2$, K_v/F_v is unramified, and $Q(\varphi)_v$ is a division algebra. Let $\mathfrak{O}$ be the order given in (1). Then $\mathfrak{O}$ is a unique order in $A^+(W)$, containing $A^+(L \cap W)$, of discriminant*

$$q[\widetilde{L}/L](2\varphi(h, L))^{-2}\mathfrak{f}^{-1}. \tag{3.22}$$

Here $\mathfrak{f}$ is the product of all the prime ideals $\mathfrak{p}$ of F such that $t_{\mathfrak{p}} = 4$ and $q \in \mathfrak{g}_{\mathfrak{p}}^{\times} F_{\mathfrak{p}}^{\times 2}$, or that $t_{\mathfrak{p}} = 2$, $K_{\mathfrak{p}}/F_{\mathfrak{p}}$ is unramified, and $Q(\varphi)_{\mathfrak{p}}$ is a division algebra.

(4) *Under the assumptions of (3) suppose $L \cap W \subset M$. Then $\mathfrak{O} \subset S_W^+$ and $G^+(W)_{\mathbf{A}} \cap J_V \subset J_W$.*

Proof. To prove (1), we see that

$$G^+(W)_{\mathbf{A}} \cap J_V = \\ = G^+(W)_{\mathbf{A}} \cap (G^+(V)_{\mathbf{a}} \prod_{v \in \mathbf{h}} (J_V)_v) = G^+(W)_{\mathbf{a}} \prod_{v \in \mathbf{h}} (G^+(W)_v \cap (J_V)_v).$$

Since $G^+(W)_v \cap (J_V)_v = \mathfrak{O}_v^{\times}$ by Lemma 3.2 (1), we have (3.21).
From (3.14), $(S_W^+)_v$ is generated by $G^1(W)_v$ and $A^+(M)_v$ if $\delta \in \mathfrak{g}^{\times} F_v^{\times 2}$ and $\chi(Q(\psi)_v) = -1$, and $(S_W^+)_v = A^+(M)_v$ otherwise $v \in \mathbf{h}$. Since

$$\alpha^{-1} G^1(W)_v \alpha = G^1(W)_v \quad \text{for every } \alpha \in A^+(W)_v^{\times},$$

we have $T(A^+(M)) \subset T(S_W^+)$. Conversely, for $\alpha \in A^+(W)_{\mathbf{A}}^{\times}$

$$\alpha^{-1} S_W^+ \alpha = S_W^+ \Longrightarrow M\tau(\alpha) = M \Longrightarrow \alpha^{-1} A^+(M)\alpha = A^+(M).$$

This is because $C(M) = \tau(T(S_W^+))$ by [12, Lemma 5.4]. Thus $T(A^+(M)) = T(S_W^+)$. Let x be an element of J_W. Since $\tau(x) \in \tau(J_W) \subset C(M)$, together with $C(M) = \tau(T(A^+(M)))$, there is an element y of $T(A^+(M))$ such that $\tau(x) = \tau(y)$. Hence $x = ay$ with some $a \in F_{\mathbf{A}}^\times$. As $F_{\mathbf{A}}^\times \subset T(A^+(M))$, we have $J_W \subset T(A^+(M))$. Similarly let $x \in G^+(W)_{\mathbf{A}} \cap J_V$. Since $\tau(x) \in SO_{\mathbf{A}}^\psi \cap \tau(J_V) \subset C(L \cap W)$, together with $C(L \cap W) = \tau(T(A^+(L \cap W)))$ by Theorem 2.2, there is an element y of $T(A^+(L \cap W))$ such that $\tau(x) = \tau(y)$. From this, noticing $F_{\mathbf{A}}^\times \subset T(A^+(L \cap W))$, we have $G^+(W)_{\mathbf{A}} \cap J_V \subset T(A^+(L \cap W))$. This proves (2).

To prove (3), we take the order $\mathfrak{O}$ of (1). Since Lemma 3.2 (2) is applicable to $\mathfrak{O}_v$ for each $v \in \mathbf{h}$ under the assumption of (3), $\mathfrak{O}_v$ contains $A^+(L \cap W)_v$ and has the discriminant given by (3.19). Also when $\mathfrak{O}_v \neq A^+(L \cap W)_v$, $[\mathfrak{O}_v / A^+(L \cap W)_v] = \mathfrak{p}_v$ by Lemma 3.1 (2). Thus by applying [2, Lemma 2.2 (3)] to $\mathfrak{O}$ and $A^+(L \cap W)$ with the norm form ν of $A^+(W)$, we have

$$\begin{aligned}[][\widetilde{\mathfrak{O}}/\mathfrak{O}] &= [A^+(L \cap W)^{\sim}/A^+(L \cap W)][\mathfrak{O}/A^+(L \cap W)]^{-2} \\ &= (q[\widetilde{L}/L](2\varphi(h,\, L))^{-2})^2 \prod_{\mathfrak{p}|\mathfrak{f}} \mathfrak{p}^{-2},\end{aligned}$$

where $\mathfrak{f}$ is the ideal in the statement of (3). This gives (3.22). Now, let $\mathfrak{O}'$ be an order in $A^+(W)$, containing $A^+(L \cap W)$, whose discriminant is given by (3.22). Then the localization $\mathfrak{O}'_v$ at $v \in \mathbf{h}$ contains $A^+(L \cap W)_v$ and has the discriminant of (3.19). By Lemma 3.2 (2), $\mathfrak{O}'_v = \mathfrak{O}_v$ for every v. Hence we have $\mathfrak{O}' = \mathfrak{O}$, which shows the uniqueness of $\mathfrak{O}$.

Keeping the assumptions of (3), let $L \cap W \subset M$. Then applying Lemma 3.2 (2) with localization, we have $\mathfrak{O} \subset S_W^+$. Thus $G^+(W)_{\mathbf{A}} \cap J_V \subset J_W$ by (3.8) and (3.7). This proves (4). ∎

Theorem 3.4. *Let the notation and assumption be the same as in Proposition 3.3 (3) and $\mathfrak{O}$ the order in $A^+(W)$ given in that proposition. Then the following assertions hold:*

(1) $C(L \cap W) = \tau(T(\mathfrak{O}))$ *and* $\Gamma(L \cap W) = \tau(\Gamma^*(\mathfrak{O}))$.

(2) *The map $(L \cap W)\tau(\alpha) \longmapsto \alpha^{-1}\mathfrak{O}\alpha$ gives a bijection of the $SO^\psi(W)$-classes in the $SO^\psi(W)$-genus of $L \cap W$ onto the conjugacy classes in the genus of $\mathfrak{O}$ which is the set $\{\alpha^{-1}\mathfrak{O}\alpha \mid \alpha \in A^+(W)_{\mathbf{A}}^\times\}$.*

(3) *The type number of $\mathfrak{O}$ equals the type number of $A^+(L \cap W)$ and consequently is equal to the class number of the genus of $L \cap W$ with respect to $SO^\psi(W)$.*

Proof. In view of Lemma 3.1, $\mathfrak{O}_v \neq A^+(L \cap W)_v$ if and only if $t_v = 4$ and $q \in \mathfrak{g}_v^\times F_v^{\times 2}$ or if $t_v = 2$, $(K/v) = -1$, and $\chi(Q(\varphi)_v) = -1$ for $v \in \mathbf{h}$. Since Lemma 3.2 (2) is applicable in our assumption, $\mathfrak{O}_v$ is a unique maximal order in the division algebra $A^+(W)_v$ in both cases. Furthermore, $(L \cap W)_v$ is a unique maximal lattice in the anisotropic space $(W,\, \psi)_v$ becuase $2\varphi(h,\, L)_v = \mathfrak{b}(q)_v$. Thus it can be found that

$$\alpha^{-1}\mathfrak{O}\alpha = \mathfrak{O} \Longleftrightarrow \alpha^{-1}A^+(L \cap W)\alpha = A^+(L \cap W)$$

for $\alpha \in A^+(W)_{\mathbf{A}}^{\times}$. This combined with Theorem 2.2 proves (1).
To prove (2), let N be an arbitrary $\mathfrak{g}$-lattice in the genus of $L \cap W$. Since N is integral we have the order $A^+(N)$ in $A^+(W)$. Taking $\alpha \in A^+(W)_{\mathbf{A}}^{\times}$ so that $N = (L \cap W)\tau(\alpha)$, we can put $\mathfrak{O}(N) = \alpha^{-1}\mathfrak{O}\alpha$ in $A^+(W)$. In fact, if $N = (L \cap W)\tau(\alpha')$ with some $\alpha' \in A^+(W)_{\mathbf{A}}^{\times}$, then $(L \cap W)\tau(\alpha(\alpha')^{-1}) = L \cap W$, whence $\alpha(\alpha')^{-1}$ belongs to $T(A^+(L \cap W)) = T(\mathfrak{O})$ by (1). This shows $(\alpha')^{-1}\mathfrak{O}\alpha' = \alpha^{-1}\mathfrak{O}\alpha$, namely, $\mathfrak{O}(N)$ is independent of the choice of α. Moreover this is a unique order of discriminant $q[\widetilde{L}/L](2\varphi(h,\, L))^{-2}\mathfrak{f}^{-1}$ containing $A^+(N)$, where $\mathfrak{f}$ is the ideal in (3.22). Indeed, since $\mathfrak{O}$ contains $A^+(L \cap W)$ and has the discriminant given by (3.22), the order $\mathfrak{O}(N)$ contains $A^+(N)$ and has the same discriminant. The uniqueness of $\mathfrak{O}(N)$ can be reduced to that of $\mathfrak{O}$. Our assertion (2) can be verified by using this fact and (1). Assertion (3) is a consequence from (2). This completes the proof. $\square$

4 Quadratic Diophantine equations in four variables

4.1 Quadratic Diophantine equations

Let $(V,\, \varphi)$ be a quadratic space of dimension n over a number field F and L a $\mathfrak{g}$-lattice in V. We recall that

$$L[q,\ \mathfrak{b}] = \{x \in V \mid \varphi[x] = q,\ \varphi(x,\, L) = \mathfrak{b}\},$$

and this set is stable under $\Gamma(L)$.
For $h \in V$ such that $\varphi[h] = q \neq 0$ we set $(W,\, \psi)$ as in (2.2). Assume that L is $\mathfrak{g}$-maximal with respect to φ and $n > 2$. Then

$$\sum_{i \in I} \#\left\{L_i[q,\ \mathfrak{b}]/\Gamma(L_i)\right\} = \#\left\{SO^{\psi} \setminus SO^{\psi}_{\mathbf{A}}/(SO^{\psi}_{\mathbf{A}} \cap C(L))\right\}, \tag{4.1}$$

where $\mathfrak{b} = \varphi(h,\, L)$, $\{L_i\}_{i \in I}$ is a set of representatives for the SO^{φ}-classes in the SO^{φ}-genus of L for which $L_i[q,\, \mathfrak{b}] \neq \emptyset$, and SO^{ψ} is regarded as the subgroup $\{\gamma \in SO^{\varphi} \mid h\gamma = h\}$ of SO^{φ}. This is a consequence from the main theorem of quadratic Diophantine equations due to Shimura [9, Theorem 11.6] (cf. also [12, Theorem 2.2 and (2.7)]). For a $\mathfrak{g}$-lattice N in V we put

$$D(N) = \{\alpha \in O^{\varphi}(V)_{\mathbf{A}} \mid N\alpha = N\}, \qquad \Gamma^{\cdot}(N) = O^{\varphi}(V) \cap D(N)$$

as denoted in the Introduction. Then formula (4.1) is valid for $(O^{\varphi},\, O^{\psi},\, D(L),\, \Gamma^{\cdot}(L_i),\, J)$ in place of $(SO^{\varphi},\, SO^{\psi},\, C(L),\, \Gamma(L_i),\, I)$ by [9, Theorem 11.6 (iii) and (v)], where $\{L_i\}_{i \in J}$ is a set of representatives for the O^{φ}-classes in the O^{φ}-genus of L for which $L_i[q,\, \mathfrak{b}] \neq \emptyset$ and O^{ψ} is regarded as the subgroup $\{\gamma \in O^{\varphi} \mid h\gamma = h\}$ of O^{φ}. We note that the O^{φ}-genus of L coincides with the SO^{φ}-genus of L and that the class number of O^{φ} relative to $D(L)$ equals the class number of SO^{φ} relative to $C(L)$ when n is odd; see [9, Lemma 9.23 (i)], for example.

Now we pay attention to the following; if the number of the right-hand side of (4.1) coincides with $\#\{SO^{\psi} \backslash SO^{\psi}_{\mathbf{A}}/C(L \cap W)\}$, then the left-hand side of (4.1) is given by the class number of the genus of $L \cap W$. Concerning this, there is a result [9, Proposition 11.13] for odd-dimensional spaces and also its analogue [2, Proposition 4.4] for even-dimensional spaces whose discriminant fields are the base fields. In Proposition 4.3 below we shall prove another analogue of [9, Proposition 11.13] to quaternary case.
As for the representatives of classes in the genus of $L \cap W$, by virtue of the principle in [9, Theorem 11.6 (i)], we have the following:

LEMMA 4.1. *Let the notation be as above. Fix an element h of $L[q, \mathfrak{b}]$ $(q \neq 0)$ and set (W, ψ) as in (2.2). Then the map*

$$k\Gamma(L_i) \longmapsto (L_i \cap (Fk)^{\perp})\gamma^{-1}SO^{\psi}$$

defines a well-defined surjection of the union of the sets $L_i[q, \mathfrak{b}]/\Gamma(L_i)$ for $i \in I$ onto the SO^{ψ}-classes in the SO^{ψ}-genus of $L \cap W$ with $\gamma \in SO^{\varphi}$ such that $k = h\gamma$ for $k \in L_i[q, \mathfrak{b}]$ and $i \in I$. In particular, if $SO^{\psi}\varepsilon(SO^{\psi}_{\mathbf{A}} \cap C(L)) = SO^{\psi}\varepsilon C(L \cap W)$ for every $\varepsilon \in SO^{\psi}_{\mathbf{A}}$, then the map is bijective. Moreover the assertions are true for $(O^{\psi}, J, \Gamma^{\cdot}(L_i), D(L), D(L \cap W))$ in place of $(SO^{\psi}, I, \Gamma(L_i), C(L), C(L \cap W))$.

Proof. For $k \in L_i[q, \mathfrak{b}]$ with $i \in I$ there is $\gamma \in SO^{\varphi}$ such that $k = h\gamma$ as $\varphi[k] = \varphi[h]$ by [9, Lemma 1.5 (ii)]. We set $L = L_i\alpha_i$ with $\alpha_i \in SO^{\varphi}_{\mathbf{A}}$. We may assume that $(\alpha_i)_v = 1$ for $v \in \mathbf{a}$. Since $h, k(\alpha_i)_v \in L_v[q, \mathfrak{b}_v]$ for $v \in \mathbf{h}$, by (1.6), $h = k(\alpha_i)_v\alpha_v$ with some $\alpha_v \in C(L_v)$ for each v. Putting $\alpha_v = \gamma_v^{-1}$ for $v \in \mathbf{a}$, we have $\alpha \in C(L)$ whose component is α_v for every prime v. Then by [9, Theorem 11.6 (i)] the map $k \longmapsto \gamma\alpha_i\alpha$ induces a well-defined bijection of $\bigcup_{i \in I} L_i[q, \mathfrak{b}]/\Gamma(L_i)$ onto $SO^{\psi} \setminus SO^{\psi}_{\mathbf{A}}/(SO^{\psi}_{\mathbf{A}} \cap C(L))$. Obviously $\gamma\alpha_i\alpha \longmapsto (L \cap W)(\gamma\alpha_i\alpha)^{-1}$ gives a surjection of $SO^{\psi} \setminus SO^{\psi}_{\mathbf{A}}/(SO^{\psi}_{\mathbf{A}} \cap C(L))$ onto the SO^{ψ}-classes in the genus of $L \cap W$. On the other hand, we can consider a $\mathfrak{g}$-lattice $L_i \cap (Fk)^{\perp}$ in the complement $(Fk)^{\perp}$, which is isomorphic to $(L_i \cap (Fk)^{\perp})\gamma^{-1}$ in W under γ^{-1}. Then by localization $(L \cap W)_v(\gamma\alpha_i\alpha)_v^{-1} = \{L_v(\alpha_i)_v^{-1} \cap (F_vh)^{\perp}(\alpha_i\alpha)_v^{-1}\}\gamma^{-1} = (L_i \cap (Fk)^{\perp})_v\gamma^{-1}$ for every $v \in \mathbf{h}$. This determines $(L \cap W)(\gamma\alpha_i\alpha)^{-1} = (L_i \cap (Fk)^{\perp})\gamma^{-1}$. We have thus the desired surjection. Clearly this map is bijective under the assumption in the statement. The assertions for O^{φ} can be handled in a similar way. □

Here we apply Lemma 4.1 to the quadratic form defined by the sum of five squares; the result will be used in Section 5.3.
Let $X = \mathbf{Q}^1_5$ and define Φ by $\Phi[x] = x \cdot {}^t x$ for $x \in X$. The pair (X, Φ) defines a quadratic space over $\mathbf{Q}$ whose invariants are $\{5, \mathbf{Q}, B_{2,\infty}, 5\}$. These invariants can be determined by [11, (Q.5)] because of $(X, \Phi) \cong (B_{2,\infty}, \beta) \oplus (\mathbf{Q}e, \Phi|_{\mathbf{Q}e})$ with some $e \in X$ so that $\Phi[e] = 1$, where β is the norm form of $B_{2,\infty}$. Let Λ be a $\mathbf{Z}$-maximal lattice in (X, Φ). It is known that $\#\{O^{\Phi} \setminus O^{\Phi}_{\mathbf{A}}/D(\Lambda)\} = 1$; see [9, §12.12], for example. By [9, Lemma 12.13 (i)], $\Lambda[d, \mathbf{Z}] \neq \emptyset$ for every

squarefree positive integer d. Fixing $k_0 \in \Lambda[d, \mathbf{Z}]$, we put $V = (\mathbf{Q}k_0)^\perp$ and $L = \Lambda \cap V$. Then by [9, Theorem 12.14 (ii)], L is a $\mathbf{Z}$-maximal lattice in V with respect to the restriction φ of Φ to V. By virtue of (4.1) for O^Φ, $\#\{\Lambda[d, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)\} = \#\{O^\varphi \backslash O^\varphi_{\mathbf{A}}/(O^\varphi_{\mathbf{A}} \cap D(\Lambda))\}$ holds. Suppose that d is an odd prime number. Then [9, Proposition 11.13 (iii)] is applicable to $k_0 \in \Lambda[d, \mathbf{Z}]$. We have thus

$$O^\varphi \varepsilon (O^\varphi_{\mathbf{A}} \cap D(\Lambda)) = O^\varphi \varepsilon D(L) \tag{4.2}$$

for every $\varepsilon \in O^\varphi_{\mathbf{A}}$. Therefore $\#\{\Lambda[d, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)\}$ equals the number of O^φ-classes in the O^φ-genus of $\mathbf{Z}$-maximal lattices in (V, φ). This result can be found in [9, Theorem 12.14 (vi)]; the class number of SO^φ relative to $C(L)$ equals $\#\{\Lambda[d, \mathbf{Z}]/\Gamma(\Lambda)\}$ by the same theorem. Moreover (V, φ) has invariants $\{4, \mathbf{Q}(\sqrt{d}), B_{2,\infty}, 4\}$, which can be seen by applying [2, Theorem 1.1 (2)] to (X, Φ) and d.

In view of (4.2), by Lemma 4.1 we have a bijection

$$k\Gamma^{\cdot}(\Lambda) \longmapsto (\Lambda \cap (\mathbf{Q}k)^\perp)\gamma^{-1}O^\varphi \tag{4.3}$$

of $\Lambda[d, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)$ onto the O^φ-classes in the genus of L with some $\gamma \in O^\Phi$ so that $k = k_0\gamma$ for every odd prime number d. A method of determining the set $\Lambda[d, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)$ is explained in [9, §12.15]. In that explanation the case of $d = 29$ is treated and the result $\#\{\Lambda[29, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)\} = 3$ is obtained with explicit representatives for $\Lambda[29, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)$. Hence the class number of O^φ relative to $D(L)$ is equal to 3, as mentioned in the Introduction. In Section 5.3 we shall list the representatives for $\Lambda[d, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)$ and the corresponding lattices under the map (4.3) for $d = 5,\ 13,\ 17$, and 29.

4.2 Results for quaternary spaces

To apply our results in the previous section to quadratic Diophantine equations, let us assume $n = 4$ in the setting of §4.1 and take an element h of $L[q, \mathfrak{b}]$. Under suitable conditions on q and $\mathfrak{b}$, we have an order $\mathfrak{O}$ defined in Proposition 3.3 (3). The order satisfies inequalities

$$t(\mathfrak{O}) \leq \#\{SO^\psi \setminus SO^\psi_{\mathbf{A}}/(SO^\psi_{\mathbf{A}} \cap C(L))\} \leq c(\mathfrak{O}). \tag{4.4}$$

Here $t(\mathfrak{O})$ (resp. $c(\mathfrak{O})$) is the type number (resp. the class number) of $\mathfrak{O}$. To show (4.4), we observe that

$$SO^\psi_{\mathbf{A}} \cap \tau(J_V) \subset SO^\psi_{\mathbf{A}} \cap C(L) \subset C(L \cap W).$$

Since the kernel of τ is $F^\times_{\mathbf{A}}$, the class number of $\mathfrak{O}$ is more than $\#\{SO^\psi \setminus SO^\psi_{\mathbf{A}}/(SO^\psi_{\mathbf{A}} \cap \tau(J_V))\}$ by (3.21). Further by Theorem 3.4 the type number of $\mathfrak{O}$ equals $\#\{SO^\psi \setminus SO^\psi_{\mathbf{A}}/C(L \cap W)\}$. This proves (4.4).

Corollary 4.2. *Let the notation and assumption be as in Proposition 3.3 (3) and $\mathfrak{O}$ the order in $A^+(W)$ defined in that proposition with an element h of $L[q,\,\mathfrak{b}]$. Also let I (resp. J) be a set of representatives α for $SO^\varphi \setminus SO^\varphi_{\mathbf{A}}/C(L)$ (resp. $O^\varphi \backslash O^\varphi_{\mathbf{A}}/D(L)$) for which $L\alpha^{-1}[q,\,\mathfrak{b}] \neq \emptyset$. Then the following inequalities hold:*

$$t(\mathfrak{O}) \leq \sum_{\alpha\in J} \#\left\{L\alpha^{-1}[q,\ \mathfrak{b}]/\Gamma^{\cdot}(L\alpha^{-1})\right\} \leq$$
$$\leq \sum_{\alpha\in I} \#\left\{L\alpha^{-1}[q,\ \mathfrak{b}]/\Gamma(L\alpha^{-1})\right\} \leq c(\mathfrak{O}).$$

Moreover, assume that $2\varphi(h,\,L)_v$ contains $\mathfrak{p}_v^{[\nu/2]}$ with $q\mathfrak{g}_v = \mathfrak{p}_v^\nu$ for every $v \in \mathbf{h}$ such that $t_v = 2$ and K_v/F_v is ramified. Then the formula in [10, (1.9)] is applicable to h and it can be given as follows:

$$\sum_y \#\left\{L\tau(y)^{-1}[q,\ \mathfrak{b}]/\tau(G^+(V)\cap yJ_Vy^{-1})\right\} = c(\mathfrak{O}). \tag{4.5}$$

Here y runs over a set of all representatives for $G^+(V)\backslash G^+(V)_{\mathbf{A}}/J_V$ such that $G^+(W)_{\mathbf{A}} \cap G^+(V)yJ_V \neq \emptyset$.

Proof. To prove the first assertion, we recall that $\#\{SO^\psi \setminus SO^\psi_{\mathbf{A}}/C(L\cap W)\} = \#\{O^\psi \setminus O^\psi_{\mathbf{A}}/D(L\cap W)\}$, because W is odd-dimensional. Since $O^\psi_{\mathbf{A}} \cap D(L)$ is contained in $D(L\cap W)$, by formula (4.1) for O^φ, we have the first inequality. Here we may assume that $\{L\alpha^{-1}\}_{\alpha\in J} \subset \{L\alpha^{-1}\}_{\alpha\in I}$. Clearly $\#\left\{L\alpha^{-1}[q,\,\mathfrak{b}]/\Gamma^{\cdot}(L\alpha^{-1})\right\} \leq \#\left\{L\alpha^{-1}[q,\,\mathfrak{b}]/\Gamma(L\alpha^{-1})\right\}$ for every $\alpha \in J$. Then the desired inequalities follow from these and (4.4) combined with (4.1).
To prove (4.5), put $q = q_0\pi_v^{2\ell}$ and $2\varphi(h,\,L)_v = \mathfrak{p}_v^m$ with $q_0 \in \mathfrak{g}_v^\times \cup \pi_v\mathfrak{g}_v^\times$ and $\ell,\ m \in \mathbf{Z}$ for $v \in \mathbf{h}$. In order to apply [10, (1.9)], we have to verify that $hC(L) = h\tau(J_V)$ in $V_{\mathbf{A}} = V \otimes_F F_{\mathbf{A}}$. In view of (3.2) it is sufficient to observe the local cases where (i) $t_v = 4$, (ii) $t_v = 2$ and $(K/v) = 0$, (iii) $t_v = 2$, $(K/v) = -1$, and $\chi(Q(\varphi)_v) = -1$. Our argument is basically the same as in [10, §4.3], and so we give only an outline of the proof to avoid a repetition of the same argument. Put $C_v = C(L_v)$ and $J_v = J_{V_v}$.
(i) Through an isomomorphism of $Q(\varphi)_v$ onto $A^+(W)_v$ we have $hC_v = h\tau(J_v)$ in the same way as in §4.3 (i) of [10]. We note that $C_v = SO_v^\varphi$ and $C(L_v\cap W_v) = SO_v^\psi$.
(ii) Assume that $2\varphi(h,\,L)_v \supset \mathfrak{p}_v^\ell$. In a Witt decomposition of φ in (2.9) with $g^2 \in \mathfrak{g}_v^\times$, take the same element $\omega_v \in K_v^\times = G^+(K_vg)$ as in §4.3 (ii) of [10]. We take $k_v = q\pi_v^{-m}e + \pi_v^m f$; then $k_v\tau(\omega_v) = k_v$. In a similar manner to [10, §4.3 (ii)] we have $\tau(\omega_v) \in C_v$ and $\omega_v \notin J_v$, from which it follows that $k_vC_v = k_v\tau(J_v)$. Since, by our assumption, $k_v \in L_v[q,\,2^{-1}\mathfrak{p}_v^m]$, we have $k_v \in hC_v$ by (1.6). Thus the criterion [10, (1.10)] is applicable to k_v; we have $hC_v = h\tau(J_v)$.
(iii) In a Witt decomposition of φ in (2.9) with $g^2 \in \pi_v\mathfrak{g}_v^\times$, we take $\omega_v =$

$g(q\pi_v^{-m}e-\pi_v^m f)\in A^+(V_v)^\times$ and $k_v=q\pi_v^{-m}e+\pi_v^m f$. Then it can be seen that

$$\nu(\omega_v)=\omega_v\omega_v^*=-qg^2\in\pi_v q\mathfrak{g}_v^\times,\qquad k_v\tau(\omega_v)=k_v,$$
$$L_v\tau(\omega_v)=\mathfrak{r}_v g+\mathfrak{g}_v q^{-1}\pi_v^{2m}f+\mathfrak{g}_v q\pi_v^{-2m}e.$$

Because $m=\ell$ by the assumption on (iii), we have $\nu(\omega_v)\in\pi_v^{2\ell+1}\mathfrak{g}_v^\times$ and $L_v\tau(\omega_v)=L_v$. Hence $k_vC_v=k_v\tau(J_v)$ by the same way as in §4.3 (iii) of [10]. Since $k_v\in L_v[q,\ 2^{-1}\mathfrak{p}_v^\ell]=hC_v$, by [10, (1.10)], we have $hC_v=h\tau(J_v)$. Accordingly $hC(L)=h\tau(J_V)$ holds. Therefore [10, (1.9)] is applicable and the formula is given by

$$\begin{aligned}&\sum_y\#\left\{L\tau(y)^{-1}[q,\ \mathfrak{b}]/\tau(G^+(V)\cap yJ_Vy^{-1})\right\}\\ =\ &\#\left\{G^+(W)\setminus G^+(W)_{\mathbf{A}}/(G^+(W)_{\mathbf{A}}\cap J_V)\right\},\end{aligned}\tag{4.6}$$

where y runs over all representatives for $G^+(V)\setminus G^+(V)_{\mathbf{A}}/J_V$ for which $G^+(W)_{\mathbf{A}}\cap G^+(V)yJ_V\neq\emptyset$. Since $G^+(W)_{\mathbf{A}}\cap J_V=A^+(W)_{\mathbf{a}}^\times\mathfrak{O}_{\mathbf{h}}^\times$ by Proposition 3.3 (3), (4.6) equals the class number of $\mathfrak{O}$. Thus we obtain (4.5). □

Let v be a prime of F in case (i), (ii), or (iii) of the proof of Corollary 4.2. As can be seen in the proof, there is an element ω_v of $G^+(V_v)$ such that $h\tau(\omega_v)=h$, $L_v\tau(\omega_v)=L_v$, and $\omega_v\notin J_{V_v}$. This together with (3.2) shows that

$$[SO_v^\psi\cap C(L_v):SO_v^\psi\cap\tau(J_{V_v})]=[C(L_v):\tau(J_{V_v})]\tag{4.7}$$

for every $v\in\mathbf{h}$ under the two assumptions that $2\varphi(h,\ L)_v\supset\mathfrak{p}_v^{[\nu/2]}$ if $(K/v)=0$ and that $\nu\in 2\mathbf{Z}$ and $2\varphi(h,\ L)_v=\mathfrak{b}(q)_v$ if $(K/v)=-1$ and $\chi(Q(\varphi)_v)=-1$. Here K is the discriminant field of φ and $q\mathfrak{g}_v=\mathfrak{p}_v^\nu$. In the same assumptions we also see that

$$SO_v^\psi\cap\tau(J_{V_v})=\tau(\mathfrak{O}_v^\times).\tag{4.8}$$

These facts (4.7) and (4.8) are often useful in the application to quadratic Diophantine equations with four variables.

As for formula (4.1) for O^φ, we can state the following proposition:

Proposition 4.3. *Let $(V,\ \varphi)$ be a quadratic space of dimension 4 over a number field F and $K=F(\sqrt{\delta})$ the discriminant field of φ. For an element h of V such that $\varphi[h]=q\neq 0$ put $W=(Fh)^\perp$ and let ψ be the restriction of φ to W. Identify $O^\psi(W)$ with $\{\gamma\in O^\varphi(V)\mid h\gamma=h\}$. Let L be a $\mathfrak{g}$-maximal lattice in V with respect to φ. Also let $\mathfrak{f}_1$ be the product of all primes $v\in\mathbf{h}$ such that $2\varphi(h,\ L)_v\neq\mathfrak{b}(q)_v$. Suppose that for $v\in\mathbf{h}$,*

(1) *$v\nmid 2$ and $\varphi(h,\ L)_v^2=q\mathfrak{g}_v$ if $(K/v)=0$ and $\chi(Q(\psi)_v)=-1$.*

(2) *$q\mathfrak{g}_v$ is a square ideal of F_v if $(K/v)=-1$ and $\chi(Q(\varphi)_v)=-1$.*

(3) $\mathfrak{f}_1$ *consists of the primes* v *such that* $t_v = 0$ *or that* $v \nmid 2$, $(K/v) = -1$, $\chi(Q(\varphi)_v) = +1$, *and* $q\mathfrak{g}_v$ *is a square ideal of* F_v.

Here t_v *is the core dimension of* φ *at* v. *Let* λ *be the number of prime factors of* $\mathfrak{f}_1\mathfrak{f}_2$, *where* $\mathfrak{f}_2$ *is the product of all primes* $v \in \mathbf{h}$ *such that* $v \nmid \mathfrak{f}_1$, $t_v \neq 4$, $(K/v) \neq 0$, *and* $q\mathfrak{g}_v$ *is not a square ideal of* F_v. *Then* $[D(L \cap W) : O^\psi_{\mathbf{A}} \cap D(L)] = [C(L \cap W) : SO^\psi_{\mathbf{A}} \cap C(L)] = 2^\lambda$. *Moreover, if* $\lambda \leq 1$, *then* $O^\psi \varepsilon D(L \cap W) = O^\psi \varepsilon(O^\psi_{\mathbf{A}} \cap D(L))$ *for every* $\varepsilon \in O^\psi_{\mathbf{A}}$.

Before stating the proof, we note a simple fact. Let $G(V)$ be the Clifford group of φ. Then the homomorphism τ of (1.2) gives a surjection of $G(V)$ onto $O^\varphi(V)$, because V is even-dimensional.

Proof. In view of assumptions (2) and (3), we can take the order $\mathfrak{O}$ in Proposition 3.3 (3). Put $q\mathfrak{g}_v = \mathfrak{p}_v^{\nu_v}$ with $\nu_v \in \mathbf{Z}$ for $v \in \mathbf{h}$. We note that $v \mid \mathfrak{f}_2$ if and only if $t_v = 0$, $v \nmid \mathfrak{f}_1$, and ν_v is odd, or if $(K/v) = -1$, $v \nmid \mathfrak{f}_1$, and ν_v is odd.
Suppose $v \nmid \mathfrak{f}_1\mathfrak{f}_2$. Then $(L \cap W)_v$ is maximal in $(W, \psi)_v$. If $t_v = 0$, then ψ_v is isotropic and $\delta q\mathfrak{g}_v$ is square, which is because ν_v must be even by $v \nmid \mathfrak{f}_2$. Since $C(L_v) = \tau(J_{V_v})$ by (3.2), we have $SO^\psi_v \cap C(L_v) = \tau(\mathfrak{O}^\times_v)$ by (4.8). Clearly $\mathfrak{O}_v = A^+(L \cap W)_v$ by calculating the discriminant. Note that $C(L_v \cap W_v) = \tau(A^+(L_v \cap W_v)^\times)$ by [12, Lemma 5.4]. Hence we have $C(L_v \cap W_v) = SO^\psi_v \cap C(L_v)$. If $t_v = 4$, then $D(L_v) = O^\varphi_v$ and $C(L_v) = SO^\varphi_v$. Also $D(L_v \cap W_v) = O^\psi_v$ and $C(L_v \cap W_v) = SO^\psi_v$ as ψ_v is anisotropic. Hence we have $D(L_v \cap W_v) = O^\psi_v \cap D(L_v)$ and $C(L_v \cap W_v) = SO^\psi_v \cap C(L_v)$. Assume $t_v = 2$ and $(K/v) = -1$. Then $\delta q\mathfrak{g}_v$ must be square. If $Q(\varphi)_v = M_2(F_v)$, then ψ_v is isotropic. In the same way as in the case $t_v = 0$ we see that $C(L_v \cap W_v) = SO^\psi_v \cap C(L_v)$. If $Q(\varphi)_v$ is a division algebra, then ψ_v is anisotropic and $A^+(W)_v$ is a division algebra. Notice that $C(L_v \cap W_v) = \tau(A^+(W)^\times_v)$ as $L_v \cap W_v$ is maximal. Our order $\mathfrak{O}_v$ has discriminant $\mathfrak{p}_v$ by (3.22), whence it is maximal in $A^+(W)_v$. Observe that $A^+(W)^\times_v = F^\times_v(\mathfrak{O}^\times_v \cup \mathfrak{O}^\times_v\omega)$ with some $\omega \in A^+(W)^\times_v$ so that ω^2 is a prime element of F_v. Since $SO^\psi_v \cap \tau(J_{V_v}) = \tau(\mathfrak{O}^\times_v)$ by (4.8), we have $[C(L_v \cap W_v) : SO^\psi_v \cap \tau(J_{V_v})] = 2$. In view of (4.7) together with $[C(L_v) : \tau(J_{V_v})] = 2$ by (3.2), $SO^\psi_v \cap C(L_v)$ must coincide with $C(L_v \cap W_v)$. Assume $(K/v) = 0$. If $Q(\psi)_v = M_2(F_v)$, we take a Witt decomposition of φ_v in (2.9) with $g \in V_v$ so that $g^2 \in \mathfrak{g}^\times_v$. Then $\mathfrak{r}_v g$ is a maximal lattice in the core subspace $(K_v g, \varphi_v)$. Our assumption $Q(\psi)_v = M_2(F_v)$ implies that there is an element k of $K_v g$ such that $\varphi_v[k] = q = \varphi[h]$. Since the lattice $\mathfrak{r}_v g \cap (F_v k)^\perp$ is maximal in the complement $(F_v k)^\perp$ in $K_v g$ as $(K_v g, \varphi_v)$ is anisotropic, we have $2\varphi_v(k, \mathfrak{r}_v g) = \mathfrak{b}(q)_v = 2\varphi(h, L)_v$. Thus [9, Proposition 11.12 (iv) and (v)] are applicable to h. We have $C(L_v \cap W_v) = SO^\psi_v \cap C(L_v)$ and $D(L_v \cap W_v) = O^\psi_v \cap D(L_v)$. Similarly for the case where $Q(\psi)_v$ is a division algebra, under the assumption (1), we have $C(L_v \cap W_v) = SO^\psi_v \cap C(L_v)$ and $D(L_v \cap W_v) = O^\psi_v \cap D(L_v)$.
Suppose $v \mid \mathfrak{f}_1$. By assumption (3) such a prime satisfies either (i) $t_v = 0$ or (ii) $v \nmid 2$, $(K/v) = -1$, $Q(\varphi)_v = M_2(F_v)$, and ν_v is even. In both cases (i) and (ii),

$\mathfrak{O}_v = A^+(L\cap W)_v$ and $SO^\psi_v \cap C(L_v) = \tau(\mathfrak{O}^\times_v)$. Moreover, we can prove that

$$T(\mathfrak{O}_v) = F^\times_v(\mathfrak{O}^\times_v \cup \mathfrak{O}^\times_v\eta) \tag{4.9}$$

with some element η of $A^+(W)^\times_v$ such that $\eta\mathfrak{O}^\times_v = \mathfrak{O}^\times_v\eta$ and $\eta\eta^*\mathfrak{g}_v = \mathfrak{g}_v$ or $\eta\eta^*\mathfrak{g}_v = \mathfrak{p}_v$ according as ν_v is even or odd. This can be handled in a similar way to the proof of [3, Theorem 3.1] for Case (i) and to [3, §3.4] for Case (ii). (We will determine the index $[A^+(M_v)^\times : A^+(L_v\cap W_v)^\times]$ in a subsequent paper, which may be used in the proof of (4.9).) We have therefore $[C(L_v\cap W_v) : \tau(\mathfrak{O}^\times_v)] = 2$.

Suppose $v \mid \mathfrak{f}_2$. Then $(L\cap W)_v$ is maximal and $\mathfrak{O}_v = A^+(L\cap W)_v$. If $t_v = 0$, we have $SO^\psi_v\cap C(L_v) = \tau(\mathfrak{O}^\times_v)$. Since $q\mathfrak{g}_v$ is not square, by (3.5) and (3.7), $J_{W_v} = A^+(L_v\cap W_v)^\times$. Hence $[C(L_v\cap W_v) : \tau(\mathfrak{O}^\times_v)] = 2$ by (3.3). If $(K/v) = -1$, then $Q(\varphi)_v$ must be $M_2(F_v)$ under the assumption (2) as ν_v is odd. Applying (3.2) and (4.8), we have $SO^\psi_v \cap C(L_v) = \tau(\mathfrak{O}^\times_v)$. Hence by the same way as in the case $t_v = 0$, $[C(L_v\cap W_v) : \tau(\mathfrak{O}^\times_v)] = 2$.

To prove $[D(L_v\cap W_v) : O^\psi_v \cap D(L_v)] = [C(L_v\cap W_v) : SO^\psi_v\cap C(L_v)]$, we shall show that $[O^\psi_v \cap D(L_v) : SO^\psi_v \cap C(L_v)] = 2$ because $[D(L_v\cap W_v) : C(L_v\cap W_v)] = 2$ by [9, Lemma 6.8]. It is sufficient to investigate the following cases; (a) $t_v = 0$, (b) $(K/v) = -1$ and $\chi(Q(\varphi)_v) = +1$, (c) $(K/v) = -1$ and $\chi(Q(\varphi)_v) = -1$. In cases (a) and (b) we can verify the desired fact by the same technique as in the proof of [9, Proposition 11.12 (v)]; see the case $\widetilde{L} = L$ and $t \neq 1$ in that proof. As for case (c), we first note $\nu_v \in 2\mathbf{Z}$ by our assumption (2); put $\ell = \nu_v/2$. Since $(L\cap W)_v$ must be maximal under assumption (3), $2\varphi(h, L)_v = \mathfrak{b}(q)_v = \mathfrak{p}^\ell_v$ by (2.6) and (2.7). Now we take our setting and notation to be those in Case (iii) of the proof of Corollary 4.2. By (1.6), $h\alpha = k_v$ with some $\alpha \in C(L_v)$. Under such an α we may identify h, $(W, \psi)_v$, and $(L\cap W)_v$ with k_v, $K_vg \oplus F_v(q\pi^{-\ell}_v e - \pi^\ell_v f)$, and $\mathfrak{r}_vg + \mathfrak{p}^{-\ell}_v(q\pi^{-\ell}_v e - \pi^\ell_v f)$, respectively. Looking at the lattice $\mathfrak{r}_vg$ in the subspace (K_vg, φ_v) of $(V, \varphi)_v$, we can find $\gamma_0 \in O(K_vg)$ such that $\det(\gamma_0) = -1$ and $(\mathfrak{r}_vg)\gamma_0 = \mathfrak{r}_vg$ by [9, Lemma 6.8]. Extend γ_0 to an element γ of $GL(V_v)$ by setting γ to be the identity map on $(K_vg)^\perp$. Then $\gamma \in O^\varphi_v$, $h\gamma = h$, $\det(\gamma) = -1$, and $L_v\gamma = L_v$. This shows $[O^\psi_v \cap D(L_v) : SO^\psi_v \cap C(L_v)] = 2$. Summing up all these results, we obtain the first assertion.

To prove the second assertion, we borrow the idea of the proof of [9, Proposition 11.13 (ii)]. When there is no prime v dividing $\mathfrak{f}_1\mathfrak{f}_2$, we have $D(L\cap W) = O^\psi_{\mathbf{A}} \cap D(L)$, and so our assertion is obvious. Hereafter we assume $\lambda = 1$. For $\varepsilon \in O^\psi_{\mathbf{A}}$ put $\Lambda = L\varepsilon^{-1}$, which is a $\mathfrak{g}$-maximal lattice in (V, φ). We consider $\tau(h)$ of O^ψ. Put $\mathfrak{a} = \varphi[h](2\varphi(h, L))^{-2}$. Let $c \in F^\times_{\mathbf{A}}$ so that $2c\varphi(h, L) = \mathfrak{g}$; then $2\varphi_v(c_vh, \Lambda_v) = \mathfrak{g}_v$ and $\varphi_v[c_vh]\mathfrak{g}_v = \varphi_v[c_vh]\varphi_v(2c_vh, \Lambda_v)^{-2} = \mathfrak{a}_v$ for every $v \in \mathbf{h}$.

Suppose $\mathfrak{a}_v = \mathfrak{g}_v$. Then $\varphi_v[c_vh] \in \mathfrak{g}^\times_v$ and $2\varphi_v(c_vh, \Lambda_v) = \mathfrak{g}_v$. Hence c_vh belongs to Λ_v and also it is invertible in the order $A(\Lambda)_v$. Since this order contains Λ_v by definition, $A(\Lambda)_v \cap V_v = \Lambda_v$ by [9, Lemma 8.4 (iii)]. Thus we have $\Lambda_v\tau(h) = h^{-1}A(\Lambda)_vh \cap V_v = \Lambda_v$.

Let $v \nmid \mathfrak{f}_1\mathfrak{f}_2$. If $t_v = 0$ or $(K/v) = -1$, then $q\mathfrak{g}_v = \mathfrak{b}(q)_v^2$ as ν_v is even. We have $\mathfrak{a}_v = \mathfrak{g}_v$, whence $\Lambda_v\tau(h) = \Lambda_v$. If $t_v = 4$, then Λ_v is a unique maximal lattice in the anisotropic space $(V, \varphi)_v$. Hence $\Lambda_v\tau(h) = \Lambda_v$. If $(K/v) = 0$ and $Q(\psi)_v = M_2(F_v)$, we take a Witt decomposition of φ_v as in the same case of the proof of the first assertion with Λ_v in place of L_v. Because $2\varphi_v(h, \Lambda_v) = 2\varphi(h, L)_v = \mathfrak{b}(q)_v$, by the same manner as in that proof, we can find $\alpha \in C(\Lambda_v)$ so that $h\alpha = k$ with some $k \in K_v g$. Then $\tau(h) = \alpha\tau(k)\alpha^{-1}$ by [9, Lemma 3.8 (ii)]. We see that

$$\Lambda_v\tau(h) = \{(\mathfrak{r}_v g)\tau(k) + \mathfrak{g}_v e\tau(k) + \mathfrak{g}_v f\tau(k)\}\alpha^{-1} = \Lambda_v.$$

If $(K/v) = 0$ and $Q(\psi)_v$ is a division algebra, then $\mathfrak{a}_v = \mathfrak{g}_v$ by assumption (1), which leads $\Lambda_v\tau(h) = \Lambda_v$.

Let $v \mid \mathfrak{f}_1\mathfrak{f}_2$. We take a weak Witt decomposition of φ_v as in (2.9) with Λ_v in place of L_v. Put $q_v = \varphi_v[c_v h]$ and $k = q_v e + f$. We see that $\mathfrak{a}_v = [M/L \cap W]_v^2$ if $v \mid \mathfrak{f}_1$ and $\nu_v \in 2\mathbf{Z}$, $\mathfrak{a}_v = [M/L \cap W]_v^2\mathfrak{p}_v$ if $t_v = 0$ and $\nu_v \notin 2\mathbf{Z}$, and $\mathfrak{a}_v = \mathfrak{p}_v$ if $v \mid \mathfrak{f}_2$ and $(K/v) = -1$, where M is a maximal lattice in (W, ψ). Since $q_v \in \mathfrak{a}_v$, it belongs to $\mathfrak{g}_v$. Hence we have $k \in \Lambda_v[q_v, 2^{-1}\mathfrak{g}_v]$. By (1.6) there is $\alpha \in C(\Lambda_v)$ so that $(c_v h)\alpha = k$. Moreover $\tau(k) = \alpha^{-1}\tau(h)\alpha$ by [9, Lemma 3.8 (ii)]. Then α gives an isomorphism of W_v onto $W' = (F_v k)^\perp$ such that $(\Lambda_v \cap W_v)\alpha = \Lambda_v \cap W'$. Observe that $\Lambda_v \cap W' = \mathfrak{r}_v g + \mathfrak{g}_v(q_v e - f)$. Employing [9, Lemma 3.10], we can find that

$$(\Lambda_v \cap W')\tau(k) = \{-x - a(q_v e - f) \mid x \in \mathfrak{r}_v g,\ a \in \mathfrak{g}_v\} = \Lambda_v \cap W',$$
$$\Lambda_v\tau(k) = \{-x + q_v ae + q_v^{-1}bf \mid x \in \mathfrak{r}_v g,\ a, b \in \mathfrak{g}_v\} \neq \Lambda_v,$$

because $q_v \in \mathfrak{a}_v \subset \mathfrak{p}_v$ as seen above. Thus we have $(\Lambda_v \cap W_v)\tau(h) = \Lambda_v \cap W_v$ but $\Lambda_v\tau(h) \neq \Lambda_v$. To sum up, $\tau(h)$ is an element of O^ψ such that $(\Lambda \cap W)\tau(h) = \Lambda \cap W$ and $\Lambda\tau(h) \neq \Lambda$.

Now, observe $D(\Lambda \cap W) = \varepsilon D(L \cap W)\varepsilon^{-1}$ and $O^\psi_{\mathbf{A}} \cap D(\Lambda) = \varepsilon(O^\psi_{\mathbf{A}} \cap D(L))\varepsilon^{-1}$. Since $[D(L \cap W) : O^\psi_{\mathbf{A}} \cap D(L)] = 2$ by $\lambda = 1$, we have $[D(\Lambda \cap W) : O^\psi_{\mathbf{A}} \cap D(\Lambda)] = 2$. By our result on $\tau(h)$ we obtain

$$D(\Lambda \cap W) = (O^\psi_{\mathbf{A}} \cap D(\Lambda)) \cup \tau(h)(O^\psi_{\mathbf{A}} \cap D(\Lambda)).$$

Then our assertion follows from this and $\tau(h) \in O^\psi$. □

As a consequence, assuming that $h \in L[q, \mathfrak{b}]$ satisfies all the assumptions with $\lambda \leq 1$ in Proposition 4.3, by formula (4.1) for O^φ together with Proposition 4.3 and Theorem 3.4 (3), we obtain

$$\sum_{\alpha \in J} \#\left\{L\alpha^{-1}[q,\ \mathfrak{b}]/\Gamma^{\cdot}(L\alpha^{-1})\right\} = t(\mathfrak{O}), \tag{4.10}$$

where J is a set of representatives α for $O^\varphi\backslash O^\varphi_{\mathbf{A}}/D(L)$ for which $L\alpha^{-1}[q, \mathfrak{b}] \neq \emptyset$ and $\mathfrak{O}$ is the order in $A^+(W)$ defined in Proposition 3.3 (3) with h. It should be remarked that the discriminant of $\mathfrak{O}$ has at most one higher-power prime $\mathfrak{p}^e$ $(e > 1)$ if h satisfies $\lambda \leq 1$. Note that formula (4.5) permits several such primes in the discriminant of $\mathfrak{O}$ if h satisfies the assumptions of Corollary 4.2. For example, the reader is referred to our notes after the proof of [3, Proposition 4.3], in which $\mathfrak{O}$ has discriminant $2 \cdot 5^2 g^2 \mathbf{Z}$ for a squarefree odd positive integer g prime to 5.

5 Applications and numerical examples

5.1 Applications to $\{4,\ \mathbf{Q}(\sqrt{d}),\ B_{r,\infty},\ 4\}$

THEOREM 5.1. *Let $B_{r,\infty}$ be a definite quaternion algebra over $\mathbf{Q}$ ramified only at a prime number r. Take a quadratic space (V, φ) over $\mathbf{Q}$ whose invariants are $\{4,\ \mathbf{Q}(\sqrt{d}),\ B_{r,\infty},\ 4\}$ with a prime number d prime to r such that $d \equiv 1 \pmod 4$. Then for every odd prime number p prime to dr and $0 \leq n \in \mathbf{Z}$ there exist $\mathbf{Z}$-maximal lattices L and L' in (V, φ) such that*

$$L[dp^n,\ 2^{-1}d\mathbf{Z}] \neq \emptyset, \qquad L'[dp^n,\ 2^{-1}\mathbf{Z}] \neq \emptyset, \tag{5.1}$$

except when $n \notin 2\mathbf{Z}$ and $\left(\frac{d}{p}\right) = -1$. Moreover the following formulas are valid:

$$\sum_{\alpha \in I} \frac{\# L\alpha^{-1}[dp^n,\ 2^{-1}d\mathbf{Z}]}{[\Gamma(L\alpha^{-1}) : 1]} = \frac{r-1}{24} \cdot \begin{cases} 1 & \textit{if } n = 0, \\ p^{n-1}\left(p + \left(\frac{d}{p}\right)\right) & \textit{if } n \geq 1, \end{cases} \tag{5.2}$$

$$\sum_{\alpha \in I} \frac{\# L'\alpha^{-1}[dp^n,\ 2^{-1}\mathbf{Z}]}{[\Gamma(L'\alpha^{-1}) : 1]} = \frac{(r-1)(d^2-1)}{48} \cdot \begin{cases} 1 & \textit{if } n = 0, \\ p^{n-1}\left(p + \left(\frac{d}{p}\right)\right) & \textit{if } n \geq 1, \end{cases} \tag{5.3}$$

$$\sum_{\alpha \in J} \# \left\{ L\alpha^{-1}[dp^n,\ 2^{-1}d\mathbf{Z}] / \Gamma^{\cdot}(L\alpha^{-1}) \right\} = t(\mathfrak{O}). \tag{5.4}$$

Here $\left(\frac{d}{p}\right)$ is the quadratic residue symbol, I (resp. J) is a complete set of representatives for $SO^{\varphi} \setminus SO^{\varphi}_{\mathbf{A}}/C(L)$ (resp. $O^{\varphi} \setminus O^{\varphi}_{\mathbf{A}}/D(L)$), $\mathfrak{O}$ is an order in the algebra $A^+(W)$, which is isomorphic to $B_{r,\infty}$, of discriminant $rp^n\mathbf{Z}$ containing $A^+(L \cap W)$, and $W = (\mathbf{Q}h)^{\perp}$ with $h \in L[dp^n,\ 2^{-1}d\mathbf{Z}]$.

It is noted that $L\alpha^{-1}[dp^n,\ 2^{-1}d\mathbf{Z}]$ or $L'\alpha^{-1}[dp^n,\ 2^{-1}\mathbf{Z}]$ may be empty for some $\alpha \in I$ or some $\alpha \in J$.

Proof. First of all, under the assumption that $L[dp^n,\ 2^{-1}d\mathbf{Z}] \neq \emptyset$ and $L'[dp^n,\ 2^{-1}\mathbf{Z}] \neq \emptyset$ with some maximal L and L' in V, we can derive formulas (5.2) and (5.3). We should mention that the proof will be given in a subsequent paper and that these formulas will be used in the present proof to show (5.1).

By [8, Proposition 1.8], for any positive integer q there is a $\mathbf{Z}$-maximal lattice L in (V, φ) such that $L[q] \neq \emptyset$. Let $h \in L[dp^n]$ with $0 \leq n \in \mathbf{Z}$ and take the complement $(W,\,\psi)$ as in (2.2). Since $d \equiv 1 \pmod 4$ and $\left(\frac{d}{p}\right) = 1$ if n is odd, the quaternion algebra $\{\mathbf{Q}(\sqrt{d}),\, dp^n\}$ is $M_2(\mathbf{Q})$. Hence $Q(\psi) = B_{r,\,\infty}$ by (2.3) and so the invariants of ψ are $\{3,\, \mathbf{Q}(\sqrt{-p^n}),\, B_{r,\,\infty},\, 3\}$. We have then $\mathfrak{b}(dp^n) = dp^\ell$ with $\ell = [n/2]$. Noticing $\mathfrak{b}(dp^n) \subset 2\varphi(h,\, L) \subset \mathbf{Z}$ as noted in §2.1, we see that

$$L[dp^n] = \bigcup_{i=0}^{\ell} \{L[dp^n,\ 2^{-1}dp^i\mathbf{Z}] \cup L[dp^n,\ 2^{-1}p^i\mathbf{Z}]\}. \tag{5.5}$$

Applying the explicit formula of [8, Theorem 1.5 (II)] to $L[dp^n]$, we can derive that

$$\sum_{\alpha\in I} \frac{\#L\alpha^{-1}[dp^n]}{[\Gamma(L\alpha^{-1}) : 1]} = \frac{(r-1)\left(d^2 + \left(\frac{d}{p}\right)^n\right)}{48} \sum_{i=0}^{2\ell} \left(\frac{d}{p}\right)^{n+i} p^i. \tag{5.6}$$

We here recall our assumption that $\left(\frac{d}{p}\right) = 1$ if n is odd. We put

$$R[q] = \sum_{\alpha\in I} \frac{\#L\alpha^{-1}[q]}{\#\Gamma(L\alpha^{-1})}, \qquad R[q,\, \mathfrak{b}] = \sum_{\alpha\in I} \frac{\#L\alpha^{-1}[q,\ \mathfrak{b}]}{\#\Gamma(L\alpha^{-1})} \tag{5.7}$$

for $q \in \mathbf{Z}$ and a $\mathbf{Z}$-ideal $\mathfrak{b}$ of $\mathbf{Q}$.

Suppose $n = 2\ell$ with $0 \le \ell \in \mathbf{Z}$. We shall prove (5.1) by induction on ℓ. If $\ell = 0$, then $\mathfrak{b}(d) = d\mathbf{Z}$ and $L[d] = L[d,\, 2^{-1}d\mathbf{Z}] \cup L[d,\, 2^{-1}\mathbf{Z}]$ by (5.5). Because $L[d] \ne \emptyset$, either $L[d,\, 2^{-1}d\mathbf{Z}]$ or $L[d,\, 2^{-1}\mathbf{Z}]$ must be nonempty. If $L[d,\, 2^{-1}d\mathbf{Z}] \ne \emptyset$, then formula (5.2) is valid as mentioned above. Combining this with (5.6), we have $R[d,\, 2^{-1}\mathbf{Z}] = R[d] - R[d,\, 2^{-1}d\mathbf{Z}] = 48^{-1}(r-1)(d^2-1)$. This implies that there is some $\alpha \in SO^\varphi_{\mathbf{A}}$ so that $L\alpha^{-1}[d,\, 2^{-1}\mathbf{Z}] \ne \emptyset$. Conversely, if $L[d,\, 2^{-1}\mathbf{Z}] \ne \emptyset$, we have $R[d,\, 2^{-1}d\mathbf{Z}] = 24^{-1}(r-1)$ in the same way, whence $L\alpha^{-1}[d,\, 2^{-1}d\mathbf{Z}] \ne \emptyset$ with some $\alpha \in SO^\varphi_{\mathbf{A}}$. As a consequence we can find maximal lattices L and L' in $(V,\, \varphi)$ such that $L[d,\, 2^{-1}d\mathbf{Z}] \ne \emptyset$ and $L'[d,\, 2^{-1}\mathbf{Z}] \ne \emptyset$. This settles the case $\ell = 0$. Suppose $\ell > 0$. In view of (5.5) we have

$$R[dp^n] = \sum_{i=0}^{\ell} \left\{R[dp^n,\ 2^{-1}dp^i\mathbf{Z}] + R[dp^n,\ 2^{-1}p^i\mathbf{Z}]\right\}. \tag{5.8}$$

Observe that the mapping $x \longmapsto xp^i$ gives a bijection of $L\alpha^{-1}[dp^{2(\ell-i)},\, 2^{-1}d\mathbf{Z}]$ onto $L\alpha^{-1}[dp^n,\, 2^{-1}dp^i\mathbf{Z}]$ for $i \ne 0$ and $\alpha \in SO^\varphi_{\mathbf{A}}$ for which $L\alpha^{-1}[dp^n,\, 2^{-1}dp^i\mathbf{Z}] \ne \emptyset$. Similarly $L\alpha^{-1}[dp^{2(\ell-i)},\, 2^{-1}\mathbf{Z}]$ is mapped onto $L\alpha^{-1}[dp^n,\, 2^{-1}p^i\mathbf{Z}]$ under the above bijection if $i \ne 0$ and $L\alpha^{-1}[dp^n,\, 2^{-1}p^i\mathbf{Z}] \ne \emptyset$. By our induction, (5.2) and (5.3) for $2(\ell - i)$ in place of n are valid for

$i \neq 0$. Thus we see that

$$\begin{aligned} & R[dp^n,\ 2^{-1}d\mathbf{Z}] + R[dp^n,\ 2^{-1}\mathbf{Z}] \\ = \quad & R[dp^n] - \sum_{i=0}^{\ell} \left\{ R[dp^{2(\ell-i)},\ 2^{-1}d\mathbf{Z}] + R[dp^{2(\ell-i)},\ 2^{-1}\mathbf{Z}] \right\} \\ = \quad & \frac{(r-1)(d^2+1)}{48} \cdot p^{2\ell-1}\left(p + \left(\frac{d}{p}\right)\right). \end{aligned} \tag{5.9}$$

This shows that either $L_1[dp^n, 2^{-1}d\mathbf{Z}]$ or $L_1[dp^n, 2^{-1}\mathbf{Z}]$ is not empty with some maximal lattice L_1 in V. Now, if $L_1[dp^n, 2^{-1}d\mathbf{Z}] \neq \emptyset$, then formula (5.2) is valid. Combining these results with (5.9), we have $R[dp^n, 2^{-1}\mathbf{Z}] \neq 0$, which implies that $L_1\alpha^{-1}[dp^n, 2^{-1}\mathbf{Z}] \neq \emptyset$ with some $\alpha \in SO^{\varphi}_{\mathbf{A}}$. Conversely, if $L_1[dp^n, 2^{-1}\mathbf{Z}] \neq \emptyset$, we have $L_1\alpha^{-1}[dp^n, 2^{-1}d\mathbf{Z}] \neq \emptyset$ with $\alpha \in SO^{\varphi}_{\mathbf{A}}$ by the same way. Consequently we have maximal lattices L_1 and L_1' in (V, φ) such that $L_1[dp^n, 2^{-1}d\mathbf{Z}] \neq \emptyset$ and $L_1'[dp^n, 2^{-1}\mathbf{Z}] \neq \emptyset$. This completes our induction on $\ell = n/2$.

The case of odd n can be proved similarly, which together with the case of even n shows (5.1) for every integer $n \geq 0$. At the same time we obtain formulas (5.2) and (5.3).

As for (5.4), observe first that the conditions of (1) and (2) in Proposition 4.3 are satisfied for $h \in L[dp^n, 2^{-1}d\mathbf{Z}]$ because r, d, and p are distinct prime numbers. Further $(L \cap W)_v$ is not maximal if and only if $v = p$ as $\mathfrak{b}(dp^n) = dp^{\ell}\mathbf{Z}$, except when $\ell = [n/2] = 0$, that is, when $n = 0$ or 1. Then we easily see that condition (3) of that proposition is satisfied; for instance, if p remains prime in $\mathbf{Q}(\sqrt{d})$, then n must be even by our assumption, and so p satisfies (3). The ideal $\mathfrak{f}_2$ of Proposition 4.3 in the present situation is $\mathbf{Z}$, except when $n = 1$. If $n = 0$ or 1, then $L \cap W$ is maximal. Also $\mathfrak{f}_2 = \mathbf{Z}$ or $p\mathbf{Z}$ according as $n = 0$ or 1. To sum up, Proposition 4.3 is applicable to $h \in L[dp^n, 2^{-1}d\mathbf{Z}]$ for every $0 \leq n \in \mathbf{Z}$. Hence (5.4) follows from (4.10). $\square$

We note that when $n \notin 2\mathbf{Z}$ and $\left(\frac{d}{p}\right) = -1$ in Theorem 5.1, $L[dp^n, 2^{-1}d\mathbf{Z}] = \emptyset$ for any maximal lattice L and $L'[dp^n, 2^{-1}\mathbf{Z}] \neq \emptyset$ with some maximal lattice L' in (V, φ).

Formulas (5.2) and (5.3) can be derived by means of the mass formula due to Shimura [9, (13.18)], combined with a result in a subsequent paper as mentioned in the proof of Theorem 5.1.

It should be remarked about (5.4) that the type number of $\mathfrak{O}$ is not determined by discriminant, but by the genus of $\mathfrak{O}$. (In other words, by Theorem 3.4 (2), the ideal $[(L \cap W)\tilde{}/L \cap W]$ does *not* determine the genus of $L \cap W$.) However, if $L \cap W$ is maximal, that is, if $\mathfrak{O}$ has squarefree discriminant, $t(\mathfrak{O})$ is determined by the discriminant. In fact, $\mathfrak{O}$ is maximal or an order of squarefree discriminant $rp\mathbf{Z}$ according as $n = 0$ or 1. By a result due to Eichler [1, Satz 3], any order $\mathfrak{O}'$ of discriminant $rp\mathbf{Z}$ belongs to the genus of $\mathfrak{O}$ in the sense that $\mathfrak{O}' = y^{-1}\mathfrak{O}y$ with some $y \in A^+(W)^{\times}_{\mathbf{A}}$. The similar fact is true for maximal orders, which have discriminant $r\mathbf{Z}$. Accordingly in either case $n = 0$ or 1 the discriminant $rp^n\mathbf{Z}$ certainly determines the genus of $\mathfrak{O}$. Moreover the discriminant does not depend on d. In view of these together with (5.4), we can conclude

COROLLARY 5.2. *Let the notation be as in Theorem 5.1. Then for $n = 0$ or 1 the number of the left-hand side of* (5.4) *is independent of the choice of d. Especially, if the type number of orders in $B_{r,\infty}$ of discriminant $rp^n\mathbf{Z}$ is* 1, *for any prime number d prime to rp^n such that $d \equiv 1 \pmod 4$ and $\left(\frac{d}{p}\right)^n = 1$ there exists only one O^φ-class in the genus of maximal lattices in (V, φ) of $\{4, \mathbf{Q}(\sqrt{d}), B_{r,\infty}, 4\}$ such that $L_1[dp^n, 2^{-1}d\mathbf{Z}] \neq \emptyset$ and*

$$L_1[dp^n,\ 2^{-1}d\mathbf{Z}] = h\Gamma^{\cdot}(L_1)$$

with a lattice L_1 in the class and $h \in L_1[dp^n, 2^{-1}d\mathbf{Z}]$.

In Table 1 of Section 5.3 below we shall see a few numerical examples for $r = 2$ and $n = 0$ supporting this fact.

5.2 EXAMPLES FOR REAL QUADRATIC FIELDS

Let V be a totally definite quaternion algebra over F of discriminant $\mathfrak{g}$ and φ its norm form, where F is a totally real field of even degree. Taking a nonzero element h of V and a $\mathfrak{g}$-maximal lattice L in (V, φ), we have the complement (W, ψ) of Fh and the lattice $L \cap W$. We see that $A^+(W)$ is isomorphic to the present V as quaternion algebras. Our order $\mathfrak{O}$ is then $A^+(L \cap W)$ and has discriminant $q\mathfrak{b}^{-2}$ with $q = \varphi[h]$ and $\mathfrak{b} = 2\varphi(h, L)$. Let $c(\mathfrak{O})$ denote the class number of $\mathfrak{O}$ as before.

PROPOSITION 5.3. *In the above setting with $h \in L[q, 2^{-1}\mathfrak{b}]$ assume that F has class number* 1. *Then there exists an order $\mathfrak{O}$ of discriminant $q\mathfrak{b}^{-2}$ in V such that $\sum_{i\in I} \#\left\{L_i[q, 2^{-1}\mathfrak{b}]/\Gamma(L_i)\right\} = c(\mathfrak{O})$, where $\{L_i\}_{i\in I}$ is a set of representatives for the SO^φ-classes in the SO^φ-genus of L for which $L_i[q, 2^{-1}\mathfrak{b}] \neq \emptyset$.*

We first note by [8, Proposition 1.8] that, for every totally positive integer q of F, there is a $\mathfrak{g}$-maximal lattice L in (V, φ) such that $L[q] = \{x \in L \mid \varphi[x] = q\} \neq \emptyset$. Moreover if $q\mathfrak{g}$ is squarefree, then $L[q] = L[q, 2^{-1}\mathfrak{g}]$ because of $\mathfrak{b}(q) = \mathfrak{g}$.

Proof. Clearly formula (4.5) is applicable to $h \in L[q, 2^{-1}\mathfrak{b}]$. Since $C(L) = \tau(J_V)$ and F has class number 1, τ of (1.2) gives a bijection of $G^+(V) \setminus G^+(V)_\mathbf{A}/J_V$ onto $SO^\varphi \setminus SO^\varphi_\mathbf{A}/C(L)$. Furthermore we have $\tau(G^+(V) \cap yJ_Vy^{-1}) = \Gamma(L\tau(y)^{-1})$ for every $y \in G^+(V)_\mathbf{A}$. The assertion follows from these combined with (4.5). □

For example, take (V, φ) as in Proposition 5.3 over $F = \mathbf{Q}(\sqrt{d})$ with $d = 5$, 13, or 101. It is known that $\#\{SO^\varphi \setminus SO^\varphi_\mathbf{A}/C(L)\} = 1$ when $d = 5$, 13. As noted above, there is a maximal lattice L in (V, φ) such that $L[q] = L[q, 2^{-1}\mathfrak{g}] \neq \emptyset$ for a given totally-positive squarefree integer q of F. Applying Proposition 5.3 to $h \in L[q]$, we have an order $\mathfrak{O}$ in V of discriminant $q\mathfrak{g}$. Now suppose $q \in \mathfrak{g}^\times$. Then $\mathfrak{O}$ is maximal as $d(\mathfrak{O}) = \mathfrak{g}$. Its class number is 1 if $d = 5, 13$ and is 5 if $d = 101$. These results can be found in [5, Tabelle 2] due to Peters. Therefore by the same proposition, $\#\{L[q, 2^{-1}\mathfrak{g}]/\Gamma(L)\} = 1$ if $d = 5, 13$ and $\sum_{i\in I} \#\{L_i[q, 2^{-1}\mathfrak{g}]/\Gamma(L_i)\} = 5$ if $d = 101$, where $\{L_i\}_{i\in I}$ is a set of representatives of the SO^φ-classes in the genus of L for which $L_i[q, 2^{-1}\mathfrak{g}] \neq \emptyset$. We mention that there is a previous result [10, Theorem 1.11] concerning the application of [10, Theorem 1.6] to the norm forms of definite quaternion algebras over $\mathbf{Q}$.

5.3 Numerical tables for $\{4,\ \mathbf{Q}(\sqrt{d}),\ B_{2,\infty},\ 4\}$

Let d be a prime number such that $d \equiv 1 \pmod 4$. We take a quadratic space (V, φ) over $\mathbf{Q}$ of invariants $\{4, \mathbf{Q}(\sqrt{d}), B_{2,\infty}, 4\}$ and a complete set $\{L_i\}_{i\in J}$ of representatives for the O^φ-classes in the O^φ-genus of maximal lattices in (V, φ). By (5.4) the number $\sum_{i\in J} \#\left\{L_i[dp^n, 2^{-1}d\mathbf{Z}]/\Gamma^{\cdot}(L_i)\right\}$ is given by the type number $t(\mathfrak{O})$ of some order $\mathfrak{O}$ in $B_{2,\infty}$ of discriminant $2p^n\mathbf{Z}$ for an odd prime number p prime to d and $0 \leq n \in \mathbf{Z}$, where we assume $\left(\frac{d}{p}\right) = 1$ if n is odd and remark that $L_i[dp^n, 2^{-1}d\mathbf{Z}]$ may be empty for some $i \in J$. We put $\mathfrak{c}(dp^n) = \sum_{i\in J} \#\left\{L_i[dp^n, 2^{-1}d\mathbf{Z}]/\Gamma^{\cdot}(L_i)\right\}$ for convenience. We restrict ourselves to the case $n = 0$ or 1. In this section we shall not only give the numbers $\mathfrak{c}(dp^n)$ by quoting $t(\mathfrak{O})$, but also present $\#L_i[dp^n, 2^{-1}d\mathbf{Z}]$ for $i \in J$ by taking $\{L_i\}_{i\in J}$ in the case of $d = 5,\ 13,\ 17$, or 29.

To obtain $\{L_i\}_{i\in J}$ for these primes d, we proceed according to the viewpoint explained at the last part of §4.1. Let (X, Φ) be as in that section. We set

$$\Lambda = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}g + \mathbf{Z}e_5,$$

where $\{e_i\}$ is the standard basis of $\mathbf{Q}_5^1$ and $g = 2^{-1}(e_1 + e_2 + e_3 + e_4)$. Then Λ is a $\mathbf{Z}$-maximal lattice in (X, Φ). By (4.3) we have a bijection

$$k_i\Gamma^{\cdot}(\Lambda) \longmapsto (\Lambda \cap (\mathbf{Q}k_i)^\perp)\gamma_i^{-1}O^\varphi$$

of $\Lambda[d, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)$ onto the O^φ-classes in the O^φ-genus of maximal lattices in (V, φ) with some $\gamma_i \in O^\Phi$ so that $k_i = k_0\gamma$ for $i \in J$, where $\{k_i\}_{i\in J}$ is a complete set of representatives for $\Lambda[d, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)$ and k_0 is an arbitrarily fixed element of $\Lambda[d, \mathbf{Z}]$; we put $V = (\mathbf{Q}k_0)^\perp$ and $\varphi = \Phi|_V$. Hence the desired representatives $\{L_i\}_{i\in J}$ can be obtained from explicit elements k_i for $i \in J$ by taking $(\Lambda \cap (\mathbf{Q}k_i)^\perp)\gamma_i^{-1}$ as L_i. A method of determining $\Lambda[d, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)$ is explained in [9, §12.15]; in which $\{k_i\}_{i\in J}$ was found for the case of $d = 29$. We employ that method for our purpose. Once such a set $\{k_i\}_{i\in J}$ is obtained, using the lattice $\Lambda \cap (\mathbf{Q}k_i)^\perp$, we can compute the number $\#L_i[dp^n, 2^{-1}d\mathbf{Z}]$ for every $i \in J$.

Here is a list of the representatives k_i for $\Lambda[d,\, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)$ and the corresponding lattices $\Lambda \cap (\mathbf{Q}k_i)^{\perp}$ for $i \in J$ and $d = 5,\ 13,\ 17,\ 29$:

(1) $d = 5$.

$$\begin{aligned} &k_1 = 2e_1 + e_5. \\ &\Lambda \cap (\mathbf{Q}k_1)^{\perp} = \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}e_4 + \mathbf{Z}(g - e_5). \end{aligned}$$

(2) $d = 13$.

$$\begin{aligned} &k_1 = 2e_1 + 3e_5, \quad k_2 = 2(e_2 + e_3 + e_4) + e_5. \\ &\Lambda \cap (\mathbf{Q}k_1)^{\perp} = \mathbf{Z}(e_2 + e_3 + e_4) + \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}(3g - e_5), \\ &\Lambda \cap (\mathbf{Q}k_2)^{\perp} = \mathbf{Z}e_1 + \mathbf{Z}(e_2 - 2e_5) + \mathbf{Z}(e_3 - 2e_5) + \mathbf{Z}(g - 3e_5). \end{aligned}$$

(3) $d = 17$.

$$\begin{aligned} &k_1 = 4e_4 + e_5, \quad k_2 = 2(e_3 + e_4) + 3e_5. \\ &\Lambda \cap (\mathbf{Q}k_1)^{\perp} = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}(g - 2e_5), \\ &\Lambda \cap (\mathbf{Q}k_2)^{\perp} = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}(3e_3 - 2e_5) + \mathbf{Z}(g - e_3). \end{aligned}$$

(4) $d = 29$.

$$\begin{aligned} &k_1 = 2e_4 + 5e_5, \quad k_2 = 2(e_3 + 2e_4) + 3e_5, \\ &\qquad\qquad k_3 = 2(e_1 + e_2 + e_3 + 2e_4) + e_5. \\ &\Lambda \cap (\mathbf{Q}k_1)^{\perp} = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}(5g - e_5), \\ &\Lambda \cap (\mathbf{Q}k_2)^{\perp} = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}(3e_3 - 2e_5) + \mathbf{Z}(g - e_5), \\ &\Lambda \cap (\mathbf{Q}k_3)^{\perp} = \mathbf{Z}(e_1 - 2e_5) + \mathbf{Z}(e_2 - 2e_5) + \\ &\qquad\qquad\qquad + \mathbf{Z}(e_3 - 2e_5) + \mathbf{Z}(g - 5e_5). \end{aligned}$$

Here we note that the case $d = 5$ can be seen from [10, §4.4, (4.12c)]. It can also be verified that these k_i for $i \in J$ form a complete set of representatives for $\Lambda[d,\, \mathbf{Z}]/\Gamma(\Lambda)$ for $d = 5,\ 13,\ 17,\ 29$. Since [9, Proposition 11.13 (ii)] is also applicable to $k_0 \in \Lambda[d,\, \mathbf{Z}]$, by Lemma 4.1, $\{L_i\}_{i \in J}$ gives a complete set of representatives for the SO^{φ}-classes in the SO^{φ}-genus of maximal lattices in $(V,\, \varphi)$.
We can further determine $[\Gamma(L_i) : 1]$ for $i \in J$. In fact, by Theorem 5.1 we have an explicit formula (5.2) for $R[dp^n,\, 2^{-1}d\mathbf{Z}]$ with the notation of (5.7); then $\#\Gamma(L_i)$ is computable in an elementary way by using this formula combined with the numerical data of $\#L_i[dp^n,\, 2^{-1}d\mathbf{Z}]$ in our tables. For example, if $d = 29$, then we have three maximal lattices $\{L_1,\, L_2,\, L_3\}$ given above. Looking at Table 1 for $d = 29$ and at Table 3 for $d = 29$, $p = 5$ and 7, we have $2 \cdot \#\Gamma(L_1)^{-1} = 24^{-1}$, $2 \cdot \#\Gamma(L_2)^{-1} = 4^{-1}$, and $2 \cdot \#\Gamma(L_3)^{-1} = 3^{-1}$ by (5.2). From these we get $\#\Gamma(L_1) = 48$, $\#\Gamma(L_2) = 8$, and $\#\Gamma(L_3) = 6$. Moreover the

mass of the genus with respect to SO^{φ} is $5/16$, which indeed coincides with the mass derived from the exact formula of [7, Theorem 5.8]. Similarly for $d = 5,\ 13,\ 17$, we have $\#\Gamma(L_1) = 48$ if $d = 5$; $\#\Gamma(L_1) = 48$ and $\#\Gamma(L_2) = 12$ if $d = 13$; $\#\Gamma(L_1) = \#\Gamma(L_2) = 48$ if $d = 17$.

In the numerical tables below, we put $N_i(dp^n) = \#L_i[dp^n,\, 2^{-1}d\mathbf{Z}]$ and denote by $t(2,\, p^n)$ (resp. $c(2,\, p^n)$) the type number (resp. the class number) of $\mathfrak{O}$ in $B_{2,\infty}$ of discriminant $2p^n\mathbf{Z}$. We quote $t(2,\, p^n)$ and $c(2,\, p^n)$ from [6, Table 1] due to Pizer. It is noted by Corollary 4.2 that the number $\sum_{i\in J} \# \left\{L_i[dp^n,\, 2^{-1}d\mathbf{Z}]/\Gamma(L_i)\right\}$ coincides with $c(2,\, p^n)$ if $t(2,\, p^n) = c(2,\, p^n)$.

d	$N_1(d)$	$N_2(d)$	$N_3(d)$	$t(2,\,1)$	$c(2,\,1)$	$\mathfrak{c}(d)$
5	2	$*$	$*$	1	1	1
13	2	0	$*$	1	1	1
17	2	0	$*$	1	1	1
29	2	0	0	1	1	1

Table 1: $\mathfrak{c}(d)$ for $d = 5,\ 13,\ 17,\ 29$

Let us verify our numerical results for $\mathfrak{c}(dp)$ in a straightforward way by using the lattices listed above. As an example, we take up the case of $d = 13$ and $p = 23$. We begin with the 5-dimensional space $(X,\, \Phi)$ and Λ as above. Put $k_1 = 2e_1 + 3e_5$ and $k_2 = 2(e_2 + e_3 + e_4) + e_5$. In our list with $d = 13$, k_1 and k_2 form a complete set of representatives for $\Lambda[13,\, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)$ (and it is true for $\Gamma(\Lambda)$ in place of $\Gamma^{\cdot}(\Lambda)$). Set $V = (\mathbf{Q}k_2)^{\perp}$ and let φ be the restriction of Φ to V. Then $(V,\, \varphi)$ has invariants $\{4,\, \mathbf{Q}(\sqrt{13}),\, B_{2,\infty},\, 4\}$ and $L_2 = \Lambda \cap V$ is $\mathbf{Z}$-maximal in $(V,\, \varphi)$. Since $\{e_1,\, e_2 - 2e_5,\, e_3 - 2e_5,\, g - 3e_5\}$ is a $\mathbf{Z}$-basis of L_2, representing φ by this basis, we may put $V = \mathbf{Q}_4^1$,

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 5 & 4 & 13/2 \\ 0 & 4 & 5 & 13/2 \\ 1/2 & 13/2 & 13/2 & 10 \end{bmatrix},$$

and $L_2 = \mathbf{Z}_4^1$. Under this identification, $\Gamma^{\cdot}(L_2)$ is the subgroup $\{\gamma \in GL_4(\mathbf{Z}) \mid$

d	p	$N_1(dp)$	$N_2(dp)$	$t(2,\ p)$	$c(2,\ p)$	$\mathfrak{c}(dp)$
5	11	24	$*$	1	1	1
5	19	40	$*$	2	3	2
5	29	60	$*$	2	3	2
5	31	64	$*$	2	4	2
5	41	84	$*$	3	4	3
5	59	120	$*$	3	5	3
5	61	124	$*$	4	7	4
5	71	144	$*$	2	6	2
5	79	160	$*$	3	8	3
5	89	180	$*$	5	8	5
5	101	204	$*$	5	9	5
13	3	0	2	1	1	1
13	17	12	6	2	2	2
13	23	0	12	1	2	1
13	29	12	12	2	3	2
13	43	16	18	3	5	3
13	53	12	24	3	5	3
13	61	12	28	4	7	4
13	79	48	28	3	8	3
13	101	60	36	5	9	5

Table 2: $\mathfrak{c}(dp)$ for $d = 5,\ 13$

d	p	$N_1(dp)$	$N_2(dp)$	$N_3(dp)$	$t(2, p)$	$c(2, p)$	$\mathfrak{c}(dp)$
17	13	16	12	$*$	2	3	2
17	19	16	24	$*$	2	3	2
17	43	40	48	$*$	3	5	3
17	47	48	48	$*$	2	4	2
17	53	60	48	$*$	3	5	3
17	59	72	48	$*$	3	5	3
17	67	64	72	$*$	4	7	4
17	83	72	96	$*$	4	7	4
17	89	96	84	$*$	5	8	5
17	101	96	108	$*$	5	9	5
29	5	0	2	0	1	1	1
29	7	0	0	2	1	2	1
29	13	0	2	2	2	3	2
29	23	0	0	6	1	2	1
29	53	0	10	6	3	5	3
29	59	24	8	6	3	5	3
29	67	24	8	8	4	7	4
29	71	0	8	12	2	6	2
29	83	24	16	6	4	7	4
29	103	16	16	12	5	10	5

Table 3: $\mathfrak{c}(dp)$ for $d = 17,\ 29$

$\gamma\varphi \cdot {}^t\gamma = \varphi\}$ of $GL_4(\mathbf{Z})$. Then $L_2[13p^n,\, 2^{-1} \cdot 13\mathbf{Z}]$ is given by

$$\begin{aligned}&L_2[13p^n,\ 2^{-1} \cdot 13\mathbf{Z}] = \{[x_1\ x_2\ x_3\ x_4] \in \mathbf{Z}_4^1 \mid\\ &x_1^2 + 5x_2^2 + 5x_3^2 + 10x_4^2 + x_1x_4 + 8x_2x_3 + 13x_2x_4 + 13x_3x_4 = 13p^n,\\ &(2x_1 + x_4)\mathbf{Z} + (10x_2 + 8x_3 + 13x_4)\mathbf{Z} + (8x_2 + 10x_3 + 13x_4)\mathbf{Z}\\ &+(x_1 + 13x_2 + 13x_3 + 20x_4)\mathbf{Z} = 13\mathbf{Z}\}.\end{aligned}$$

Now for $p = 23$ and $n = 1$ we have all solutions in $L_2[13 \cdot 23,\, 2^{-1} \cdot 13\mathbf{Z}]$:

$$[\pm 5\ \ \mp 13\ \ \mp 13\ \ \pm 16],\ [\pm 8\ 0\ \mp 13\ \ \pm 10],\ [\pm 8\ \ \mp 13\ 0\ \ \pm 10],$$
$$[\pm 18\ 0\ \ \pm 13\ \ \mp 10],\ [\pm 18\ \ \pm 13\ 0\ \ \mp 10],\ [\pm 21\ \ \pm 13\ \ \pm 13\ \ \mp 16].$$

We put

$$\gamma_1 = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & -1 \end{bmatrix},\quad \gamma_2 = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 1 & 1 & 1 & -2\\ 1 & 0 & 0 & -1 \end{bmatrix},\quad \gamma_3 = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

These matrices belong to $\Gamma^{\cdot}(L_2)$. Consider the subgroup U of $\Gamma^{\cdot}(L_2)$ generated by γ_1, γ_2, γ_3, and -1_4, where 1_4 is the identity matrix of size 4. Put $x = [5\ \ -13\ \ -13\ 16]$. Then it can be seen that xU contains all elements of $L_2[13 \cdot 23,\, 2^{-1} \cdot 13\mathbf{Z}]$. Thus we have $L_2[13 \cdot 23,\, 2^{-1} \cdot 13\mathbf{Z}] = x\Gamma^{\cdot}(L_2)$.
Similarly for k_1, we can consider a $\mathbf{Z}$-lattice $\Lambda \cap (\mathbf{Q}k_1)^{\perp}$. Denoting by φ_1 the restriction of Φ to $(\mathbf{Q}k_1)^{\perp}$, we may put $(\mathbf{Q}k_1)^{\perp} = \mathbf{Q}_4^1$,

$$\varphi_1 = \begin{bmatrix} 3 & 1 & 1 & 9/2\\ 1 & 1 & 0 & 3/2\\ 1 & 0 & 1 & 3/2\\ 9/2 & 3/2 & 3/2 & 10 \end{bmatrix},$$

$\Lambda \cap (\mathbf{Q}k_1)^{\perp} = \mathbf{Z}_4^1$, and $\Gamma^{\cdot}(\Lambda \cap (\mathbf{Q}k_1)^{\perp}) = \{\gamma \in GL_4(\mathbf{Z}) \mid \gamma\varphi_1 \cdot {}^t\gamma = \varphi_1\}$ under the identification with respect to a $\mathbf{Z}$-basis $\{e_2 + e_3 + e_4,\, e_2,\, e_3,\, 3g - e_5\}$ of $\Lambda \cap (\mathbf{Q}k_1)^{\perp}$. Let L_1 be the lattice in $(V,\, \varphi)$ corresponding to $\Lambda \cap (\mathbf{Q}k_1)^{\perp}$ under some isomorphism of $(V,\, \varphi)$ onto $((\mathbf{Q}k_1)^{\perp},\, \varphi_1)$. Then the number $\#L_1[13p^n,\, 2^{-1} \cdot 13\mathbf{Z}]$ is equal to

$$\begin{aligned}&\#\{[x_1\ x_2\ x_3\ x_4] \in \mathbf{Z}_4^1 \mid\\ &3x_1^2 + x_2^2 + x_3^2 + 10x_4^2 + 2x_1x_2 + 2x_1x_3 + 9x_1x_4 + 3x_2x_4 + 3x_3x_4 = 13p^n,\\ &(6x_1 + 2x_2 + 2x_3 + 9x_4)\mathbf{Z} + (2x_1 + 2x_2 + 3x_4)\mathbf{Z} + (2x_1 + 2x_3 + 3x_4)\mathbf{Z}\\ &+(9x_1 + 3x_2 + 3x_3 + 20x_4)\mathbf{Z} = 13\mathbf{Z}\}.\end{aligned}$$

For $p = 23$ and $n = 1$ there is no elements of $(\Lambda \cap (\mathbf{Q}k_1)^{\perp})[13 \cdot 23,\, 2^{-1} \cdot 13\mathbf{Z}]$. Hence $\#L_1[13 \cdot 23,\, 2^{-1} \cdot 13\mathbf{Z}] = 0$. Because L_1 and L_2 are not in the same O^{φ}-class as $k_1\Gamma^{\cdot}(\Lambda) \neq k_2\Gamma^{\cdot}(\Lambda)$, we have therefore $\mathfrak{c}(13 \cdot 23) = \#\{L_2[13 \cdot 23,\, 2^{-1} \cdot$

$13\mathbf{Z}]/\Gamma^{\cdot}(L_2)\} = 1$. This coincides with our result in the case of $d = 13$ and $p = 23$ in Table 2.
We note that $\#U = 24$, $\Gamma^{\cdot}(L_2) = U$, and $\Gamma(L_2)$ is generated by γ_1, γ_2, -1_4; furthermore we have $x\Gamma(L_2) = L_2[13\cdot 23,\ 2^{-1}\cdot 13\mathbf{Z}]$, that is, $\#\{L_2[13\cdot 23,\ 2^{-1}\cdot 13\mathbf{Z}]/\Gamma(L_2)\} = 1$. As for $\Lambda\cap(\mathbf{Q}k_1)^{\perp}$, four elements $\delta_1, \cdots, \delta_4$ and -1_4 generate $\Gamma^{\cdot}(\Lambda\cap(\mathbf{Q}k_1)^{\perp})$ and then $\Gamma(\Lambda\cap(\mathbf{Q}k_1)^{\perp})$ is generated by $\delta_1\delta_2$, $\delta_2\delta_3$, δ_4, -1_4, where

$$\delta_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & -1 \end{bmatrix}, \qquad \delta_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\delta_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \delta_4 = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}.$$

We shall show one more example for $d = 13$ and $p = 79$ obtained in the same manner:

$$\#\left\{L_1[13\cdot 79,\ 2^{-1}\cdot 13\mathbf{Z}]/\Gamma^{\cdot}(L_1)\right\} = 1,\ \#\left\{L_2[13\cdot 79,\ 2^{-1}\cdot 13\mathbf{Z}]/\Gamma^{\cdot}(L_2)\right\} = 2,$$
$$\#\left\{L_1[13\cdot 79,\ 2^{-1}\cdot 13\mathbf{Z}]/\Gamma(L_1)\right\} = 1,\ \#\left\{L_2[13\cdot 79,\ 2^{-1}\cdot 13\mathbf{Z}]/\Gamma(L_2)\right\} = 3.$$

Here $L_1'[13\cdot 79,\ 2^{-1}\cdot 13\mathbf{Z}]$, with $L_1' = \Lambda\cap(\mathbf{Q}k_1)^{\perp} \cong L_1$, consists of 48 solutions

$$[\pm 10\ \mp 39\ \mp 26\ \pm 2],\ \cdots,\ [\pm 29\ \mp 13\ \mp 13\ \mp 2]$$

and $L_2[13\cdot 79,\ 2^{-1}\cdot 13\mathbf{Z}]$ of 28 solutions

$$[\pm 3\ \pm 13\ \pm 39\ \mp 32],\ \cdots,\ [\pm 36\ \pm 39\ \pm 26\ \mp 46].$$

Accordingly $\sum_{i=1}^{2}\#\{L_i[13\cdot 79,\ 2^{-1}\cdot 13\mathbf{Z}]/\Gamma(L_i)\}$ is a quantity that differs from both the type number and the class number of $\mathfrak{O}$.

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Counting Polynomials for Linear Codes, Hyperplane Arrangements, and Matroids

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Abstract. Thomas-decomposition of a polynomial systems and the resulting counting polynomials are applied to the theory of linear codes, hyperplane arrangements, and vector matroids to reinterpret known polynomials such as characteristic polynomials and weight enumerator, to introduce a new polynomial counting the matrices defining the same matroid, and to introduce the concept of essential flats revealing a structure which allows to rewrite the rank generating polynomial as a sum of products of univariate polynomials. Our concepts make no essential distinction between finite and infinite fields.

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1 Introduction

This paper is concerned with two topics: Recognizing known polynomial invariants in the theory of codes, hyperplane arrangements, and matroids such as characteristic polynomials, weight enumerators etc. as counting polynomials and finding a further example of counting polynomials, cf. [Ple 09a], [Ple 09b], in this area. Secondly, on the background of this, analysing the structure of the lattice of flats of a matroid by means of the Tutte-polynomial or rather the rank generating polynomial by singling out a special class of flats which we call essential. Though we started with linear codes and hyperplane arrangements, we realized that matroids yield a more appropriate language for our investigation.

The basic ideas of counting polynomials, which are based on the Thomas decomposition for systems of polynomial equations and inequations into disjoint simple systems, cf, [Tho 37], [Ple 09a], [BGLR 11], are briefly described in Section 2, in particular the relevant case of this paper, where the splitting behaviour of the polynomials in the resulting simple systems allows an enumeration of the possibly infinitely many solutions. In this case, the resulting counting polynomial yields the number of solutions of the system in the following cases: For a finite ground field K the number of solutions over any finite extension field E of K are obtained by substituting the number $|E|$ for the indeterminate. For a global field K as ground field the number of solutions over the residue class field F of the valuation ring for almost all discrete valuations of K are found upon substituting the order $|F|$ of the residue class field into the counting polynomial, e. g. $K = \mathbb{Q}$ with the finite prime fields $\mathbb{F}_p$ being the most common example. At the beginning of Section 2, the construction of counting polynomials is summarized as a finitely additive measure defined on the set of solution sets of polynomial systems (of equations and inequations) and taking values in the polynomial ring $\mathbb{Z}[u]$, where u is an indeterminate standing in some sense for the order of the field, even if it is infinite. In this way, for instance, the characteristic polynomial of a hyperplane arrangement gets a less formal and more algebraic-combinatorial meaning in the case of infinite fields, since it is simply the measure of the complement of the arrangement. Also, the critical theorem by Crapo and Rota, cf. [CrR 70] gets an interpretation in the case of infinite fields, so does the (comprehensive) weight enumerator of a linear code which Greene constructed from the Tutte-polynomial, cf. [Gre 76]. Whereas these examples deal with linear inequations, the final example, i. e. the counting polynomial of the set of rank r matrices of degree $k \times n$ requires slightly more background preparation.

Section 3 applies these ideas to introduce the matrix counter of a matroid which counts the "number" of matrices yielding the given matroid. If this is possible, the matroid is called polynomially countable. In this case, the matrix counter is shown to factorize into three factors: Firstly $\mathrm{gl}(k, u)$, where k is the rank of the matroid and $\mathrm{gl}(k, u) := (u^k - 1)(u^k - u) \cdots (u^k - u^{k-1})$ is the counting polynomial of the general linear group $\mathrm{GL}(n, \cdot)$. Secondly a factor $(u - 1)^{n-l}$, where n is the number of elements of the underlying set of the matroid and l is the number of connected components of the matroid. Finally, a factor called orbit counter. If the orbit counter is 1, the matroid seems to be particularly interesting from a geometrical combinatorial point of view. We call the matroid rigid in this case, note however that it is the simplest case from the point of view of the matrix counter. Some examples are discussed such as root systems of type A_n and B_n and the extended Golay code over $\mathbb{F}_2$ of length 24. On the other extreme is the matrix counter of the uniform matroid. Indeed, it would be a challenge to find which uniform matroids are polynomially countable.

Section 4 is a suggestion to reconstruct the lattice of flats out of the rank generating polynomial. The converse direction is well understood, cf. Example 2.4. The rank generating polynomial of a matroid M is defined as a sum over all subsets of the underlying set E of the matroid. By putting together all subsets with the same flat as closure, this sum gets a lot more structured. But then one can also put together all those flats whose complements in E have the same closure with respect to the dual M^* of the original matroid. This common closure has again an M-flat as complement in E, which we call an essential flat. As a result of this the generating polynomial of M becomes a sum over the essential flats only. The summand corresponding to an essential flat X is the product of a polynomial in the first variable x depending only on the minor M/X of M and a polynomial in the second variable y depending only on the restiction $M|X$. This can be used to discover all the essential flats from the rank generating polynomial as described in Remark 4.11. The theory and two examples, the first being the Golay-code of length 24 are discussed in Section 4.

The final Section 5 discusses matroids of rank 3. The counting polynomials are computed for all matroids on up to seven points, and some examples on 8 points are given to demonstrate new phenomena. The tables of this section depend on heavy computer calculations with the program [BLH 13] to compute the Thomas decomposition of a polynomial system of equations and inequations. Various interesting issues come up, such as two nonisomorphic matroids with the same rank generating polynomial but different counting polynomials, different behaviours in different characteristics, factorization properties of the orbit counter, non-split examples where the matrix counter is not defined, etc. We are grateful to the referees to point out very helpful, relevant comments and literature.

2 Counting Polynomials

We first collect the facts from [Ple 09a] and [Ple 09b] relevant for this paper. Let K be a field with algebraic closure $\overline{K}$. Consider subsets of $\overline{K}^n$ of the form $N_p := \{a \in \overline{K}^n | p(a) = 0\}$ with $p \in K[x_1, \ldots, x_n]$, i. e. hypersurfaces defined over K. Denote by $\mathcal{L}(K, n)$ the set of subsets of $\overline{K}^n$ obtained by taking finite intersections, unions, and complements of the N_p for various such $p \in K[x_1, \ldots, x_n]$ iteratively. Clearly, if

$$\pi_n : \overline{K}^n \to \overline{K}^{n-1} : (a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_{n-1})$$

denotes the projection (in case $n > 1$), then $\pi_n(S) \in \mathcal{L}(K, n-1)$ for any $S \in \mathcal{L}(K, n)$. Moreover $\lambda_S(b) := \{a \in \overline{K} | (b, a) \in S\} \in \mathcal{L}(K, 1)$ for each $b \in \pi_n(S)$.

Proposition 2.1. *Let K be a field of characteristic zero. For every $n \in \mathbb{N}$ there is a unique map*

$$c = c_n : \mathcal{L}(K, n) \to \mathbb{Z}[u] : S \mapsto c(S) = c(S, u)$$

(where $c(S, u)$ is called the counting polynomial of S) with the following properties:
1.) For finite sets $S \in \mathcal{L}(K, n)$, one has $c(S, u) = |S|$, the number of elements in S.
2.) For any k-dimensional affine subspace N of $\overline{K}^n$ defined over K one has $c(N, u) = u^k$.
3.) For any $S, T \in \mathcal{L}(K^n)$, one has $c(S, u) + c(T, u) = c(S \cap T, u) + c(S \cup T, u)$, in particular, $c(K^n - S, u) = u^n - c(S, u)$.
4.) In case $n > 1$, for any $S \in \mathcal{L}(K, n)$ where $c_1(\lambda_S(b), u) \in \mathbb{Z}[u]$ is independent of $b \in \pi_n(S)$ one has

$$c_n(S, u) = c_{n-1}(\pi_n(S), u) \cdot c_1(\lambda_S(b), u)$$

The proof is based on a finite decomposition of the systems of equations and inequations into certain triangular systems called simple, which were introduced by J. M. Thomas, cf. [Tho 37]. Various algorithmic refinements of this decomposition algorithm and an implementation are discussed in [BGLR 11]. It is work in progress extending [LMW 10] to show that the above result also holds for fields K of positive characteristic. The cases relevant for this paper, the so called split systems, were discussed in [Ple 09b] and require no assumptions on the characteristic of K. In any case, the implementation in [BLH 13] has worked successfully for all examples of this paper.

Though the counting polynomial in general only says something about the set of solutions over the algebraic closure, for the present investigation we want to use the counting polynomials to count the number of solutions over finite fields. This is not always possible. Namely, if one specifies the free variables in the equations of a triangular system to lie in a fixed field, the resulting univariate polynomials in general do not split over this field. However, if we have split simple systems, i. e. if the polynomials of the simple systems factorize into degree-one-polynomials in their leading variable, cf. [Ple 09b], it becomes possible. To cover as many cases as possible we go beyond [Ple 09b] and distinguish three cases:

Definition 2.2. *Let $S \in \mathcal{L}(K, n)$ for some field K.*
1.) S is called uniformly enumerable *if S can be decomposed into disjoint split simple systems, in the sense of [Ple 09b], where the variables of $K[x_1, \ldots, x_n]$ are taken in the same order for all simple systems.*
2.) S is called enumerable *if S can be decomposed into disjoint systems $S_i \in \mathcal{L}(K, n)$, such that for every i there exists a split simple system $T_i \in \mathcal{L}(K, n)$ and a bijection $T_i \to S_i$ defined by some rational function over*

K.
3.) S is called POLYNOMIALLY COUNTABLE *if S is the union of finitely many systems $S_i \in \mathcal{L}(K,n)$, $i \in I$ such that at least one of $\cap_{i\in J} S_i$ or the complement $\overline{K}^n - \cap_{i\in J} S_i$ is enumerable for each subset J of I.*

Clearly, uniformly enumerable sets are enumerable and enumerable ones are polynomially countable. The notion of polynomial countability becomes especially interesting if K is finite. In case K is a global field, $S \in \mathcal{L}(K,n)$ defines a set $S_L \in \mathcal{L}(L,n)$ for all but finitely many finite residue class fields L corresponding to a non Archimedian valuation of K. In case S satisfies one of the three properties above, so does S_L in all but finitely many residue class fields L.

Proposition 2.3. *Let $S \in \mathcal{L}(K,n)$ be a polynomially countable system over a field K.*
1.) In case K is finite there is a unique polynomial $c(S,u) \in \mathbb{Z}[u]$ satisfying

$$|S \cap L^n| = c(S, |L|)$$

for all finite field extension (L/K).
2.) In case K is a global field, there is a unique polynomial $c(S,u) \in \mathbb{Z}[u]$ satisfying

$$|S_L \cap L^n| = c(S, |L|)$$

for all but finitely many residue class fields L defined by valuations of K.
In both cases, we call $c(S,u)$ the FAITHFUL COUNTING POLYNOMIAL *of S.*

Proof. The uniqueness of the faithful counting polynomial is in both cases clear, since infinitely many values of it are specified. We come to the existence. In the uniformly enumerable case one simply takes the counting polynomial, cf. [Ple 09b], and in the enumerable case the sum of the counting polynomials of the split simple systems T_i. The general case of polynomially countable systems is reduced to the enumerable case via the inclusion exclusion principle. $\square$

Note that the faithful counting polynomial is independent of the ordering of the variables or more generally of the choice of the coordinates (over the ground field K). Whether a faithful counting polynomial is uniquely defined for more general fields is interesting but not relevant for the applications in the present paper. To demonstrate the difference between counting polynomial and faithful counting polynomial, look at $S := N_p$ for $p := x^2 - y \in \mathbb{Q}[x,y]$. Taking the variables in the order $y < x$ yields $2(u-1)+1$ as counting polynomial, which is not faithful, whereas the order $x < y$ yields the faithful counting polynomial u.
The simplest case of a polynomially countable system is one given by linear (degree one) equations and inequations. In fact such a system is uniformly enumerable, but usually one obtains the faithful counting polynomial by the inclusion exclusion principle, since the computation of Thomas-decomposition

becomes rather expensive once a certain number of inequations is involved. We remark that, in this case, no assumptions on the field K are necessary and the counting polynomials are independent of the choice of the coordinate system. Here are some examples.

Example 2.4. *1.) Characteristic polynomial of a central hyperplane arrangement.*
Let V be a K-vector space of dimension k and $\varphi_i \in V^* - \{0\}$ for $i = 1, \dots, n$ be linear forms on V. Then the counting polynomial $c(S, u)$ of the system $\varphi_i(x) \neq 0$ for $i = 1, \dots n$ is called the characteristic polynomial of the hyperplane arrangement of the $\ker(\varphi_i)$. In case K is finite it counts the number of elements in $V - \cup_i \ker(\varphi_i)$ in a faithful way as explained above, i. e. it also counts the corresponding number of elements for any finite extension field of K. It clearly is monic of degree k and the coefficient of u^{k-1} is the negative of the number of different hyperplanes $\ker(\varphi_i)$, cf. [CrR 70], [OrT 92],[Ath 96]. For a recent survey on the interplay of linear codes, hyperplane arrangements, and matroids cf. e. g. [Sta 07].
2.) (Comprehensive or) Support weight enumerator of a code.
Let $A \in K^{k \times n}$ a matrix of rank k and let V be the K-vector space spanned by the rows of A. We want to count the vectors of V (and the scalar extensions of V) having exactly j components zero for $j = 0, \dots, n$. To this aim let φ_i be the projection of the row space of A corresponding to the i-th column. For each subset I of $\underline{n}$ let $S_I \in \mathcal{L}(K, k)$ be the system defined by $\varphi_j(v) = 0$ for $j \in I$ and $\varphi_j(v) \neq 0$ for $j \notin I$. Then the (comprehensive) weight enumerator

$$\omega_A(u, x, y) := \sum_{I \subseteq \underline{n}} c(S_I, u) x^{|I|} y^{n-|I|}$$

gives exactly the weight enumerator for any finite extension field L of K, in case K is finite, if one substitutes $|L|$ for u. (Note however, this weight enumerator also makes sense if K is not finite, even beyond Proposition 2.3.) Note also, the $I \subseteq \underline{n}$ with $c(S_I, u) \neq 0$ are just the flats of the matroid induced by the matrix A, cf. 3.1 below. In a splendid piece of work, it was shown in [Gre 76] how this weight enumerator could be obtained from the Tutte *polynomial $T_A(x, y)$ as follows:*

$$\omega_A(u, x, y) = (1-u)^k u^{n-k} T_A \left(\frac{1 + (u-1)x}{1-x}, \frac{1}{x} \right),$$

cf also [Bri 02]. Conversely, the Tutte*-polynomial is determined by the support weight enumerator, cf. [Jur 12] and [JuP 13], where also the most recent account is given on these results, as well as on the connections between matroids, codes, and hyperplane arrangements. (All these results do not depend on the finiteness of K, as assumed in the original papers.)*

More general systems described by polynomials of degree one in each variable still have some chance to be enumerable or at least polynomially countable.

For instance the $n \times n$-determinant *det* yields the faithful system $\det(x_{ij}) \neq 0$ with polynomial

$$\mathrm{gl}(n,u) := (u^n - 1)(u^n - u) \cdots (u^n - u^{n-1})$$

well known from the order of the general linear group over a finite field. The group theoretic counting of orbits can be used to find faithful counting polynomials. Here is an example from determinantal varieties, where we set $\mathrm{gl}(0,u) := 1$:

Proposition 2.5. *The set of $k \times n$-matrices of rank r defined over a field K is uniformly enumerable. Its (faithful) counting polynomial is given by*

$$\frac{\mathrm{gl}(k,u)\mathrm{gl}(n,u)}{\mathrm{gl}(r,u)\mathrm{gl}(k-r,u)\mathrm{gl}(n-r,u)u^{(k-r)r+(n-r)r}}$$

Proof. It follows from Theorem 2.8 of [Ple 09b] that the system is uniformly enumerable over any field. Therefore one has a faithful counting polynomial. We compute it by viewing the set as the orbit of the matrix

$$\begin{pmatrix} I_r & O_{r\times(n-r)} \\ O_{(k-r)\times r} & O_{(k-r)\times(n-r)} \end{pmatrix}$$

under the group $\mathrm{GL}(k,K) \times \mathrm{GL}(n,K)$ acting on $K^{k\times n}$ via

$$(\mathrm{GL}(k,K) \times \mathrm{GL}(n,K)) \times K^{k\times n} \to K^{k\times n} : ((g,h),m) \mapsto gmh^{-1}.$$

By computing the stabilizer, one gets exactly the denominator of the above number with u substituted by $|K|$ for any finite field K. Since we know that the result must be a polynomial, we have found it via these infinitely many values. $\square$

Note, the degree of the polynomial just derived is $r(-r+n+k)$, which is increasing in r for $r = 0, \ldots, k$, so that the dimension of the so called generic determinantal variety of $k \times n$-matrices of rank $\leq r$ is equal to $r(-r+n+k)$, which is well known.

3 Matrix counters

We proceed into a different direction now, by restricting the group action in the last proof to $\mathrm{GL}(k,K) \times \mathrm{Diag}(n,K)$, where $\mathrm{Diag}(n,K) \leq \mathrm{GL}(n,K)$ is the subgroup of all diagonal matrices of $\mathrm{GL}(n,K)$. For this action one has a finer invariant than the rank, namely the vector matroid represented by the matrices. We use the following notation: For $n \in \mathbb{N}$ let $\underline{n} := \{1, 2, \ldots, n\}$ and $\mathrm{Pot}_k(\underline{n})$ the set of all k-element subsets of $\underline{n}$.

Definition 3.1. *Let $k \leq n$. The map*

$$\mu : K^{k\times n} \to \mathrm{Pot}(\underline{n}) : A \mapsto \{X \in \mathrm{Pot}(\underline{n}) \mid |X| = \mathrm{rank}(A) = \mathrm{rank}(A_{|X})\}$$

is called the matroid map*, where $A_{|X}$ denotes the submatrix of A formed by the columns with column indices in X. For $A \in K^{k\times n}$ of rank r, the pair $(\underline{n}, \mu(A))$ with $\mu(A) \subseteq \mathrm{Pot}_r(\underline{n})$ is called the* (vector) matroid *of A.*

We shall usually assume that the matrix A is of rank k. An (abstract) matroid is a pair consiting of a ground set $\underline{n}$ and a subset B of $\mathrm{Pot}_k(\underline{n})$ satisfying certain axioms similar to the Steinitz exchange properties of bases, cf. [Oxl 11] or [Wel 76]. If the ground set is clear, we only refer to B as the matroid. If B is of the form $\mu(A)$ for some matrix over the field K, the matroid B is called K-representable. It should be noted that the weight enumerator, cf. Example 2.4, of the linear code spanned by the rows of a matrix $A \in K^{k\times n}$ only depends on the matroid $\mu(A)$. These issues are concerned with linear equations and inequations and therefore the counting polynomials in this context are faithful. However, the counting polynomial defined next is defined via polynomials which are of degree at most one in each of their variables, where it is not clear whether or not they are faithful.

Definition 3.2. *1.) Let K be a field,*

$$R_{k,n} := K[x_{1,1}, x_{2,1}, \ldots, x_{k,1}, x_{1,2}, \ldots, x_{k,2}, \ldots, x_{k,n}]$$

and $X := (x_{ij})_{i\in\underline{k}, j\in\underline{n}} \in R_{k,n}^{k\times n}$ denotes a $k \times n$-matrix of indeterminates. Finally $X_{|b} := (x_{i,j}) \in R_{k,n}^{k\times k}$ with $i \in \underline{k}, j \in b$ denotes the submatrix of X with column indices in $b \in \mathrm{Pot}_k(\underline{n})$.
2.) For a non empty subset B of $\mathrm{Pot}_k(\underline{n})$ denote by $S(B) \in \mathcal{L}(K, kn)$ the set of solutions over the algebraic closure $\overline{K}$ of K of the polynomial system

$$\det(X_{|b}) \neq 0 \text{ for } b \in B, \det(X_{|b}) = 0 \text{ for } b \in \mathrm{Pot}_k(\underline{n}) - B.$$

3.) In case $S(B) \neq \emptyset$ we call B uniformly enumerable, enumerable, *resp.* polynomially countable *(over K) if $S(B)$ has this property. In either of these cases the faithful counting polynomial $c(S(B), u) \in \mathbb{Z}[u]$ is called the* full matrix counter *of B and denoted by $c(B, u)$ or $c_B(u)$.*

Hence B is a matroid representable over $\overline{K}$ if and only if the counting polynomial of $S(B)$ with respect to some order of the variables is not zero. Clearly in the above definition, one might assume K to be a prime field.

Example 3.3. *1.) $k := 1$. Any non empty subset B of $\mathrm{Pot}_1(\underline{n})$ is a representable matroid. Its matrix counter is $(u-1)^{|B|}$.*
2.) For $k := 2$ the representable matroids are given as follows: Let $\underline{n} = \biguplus_{i=0}^{s} M_i$ with M_0 (representing the zero columns) possibly empty, but the other M_j (called parallel classes) nonempty and $s \geq 2$. Then

$$B := \{\{a, b\} \mid \text{ there are } i, j \text{ with } 0 < i < j \leq s, a \in M_i, b \in M_j\}$$

and the full matrix counter of B is given by

$$c_B(u) = u \cdot (u+1) \cdot (u-1)^{n-|M_0|} \cdot \prod_{i=1}^{s-2} (u-i),$$

which can easily be obtained in the same way as one computes the order $(u^2 - 1)(u^2 - u)$ of the full linear group: In the critical case $|M_i| = 1$ for $i > 0$ one has

$$\prod_{i=1}^{s} (u^2 - 1 - (i-1)(u-1)) = (u-1)^s \prod_{i=1}^{s} (u + 1 - (i-1)).$$

Note, the characteristic of the underlying field has no relevance in this particular case. However, if K is finite, one might have $c_B(|K|) = 0$.

Here is a first property of the full matrix counter.

Proposition 3.4. *Let $B \subseteq \mathrm{Pot}_k(\underline{n})$ be polynomially countable over any prime field. Then*

$$\mathrm{gl}(k,u) \mid c_B(u), \;\; i.\ e.\ c_B(u) = \mathrm{gl}(k,u) \cdot r_B(u)$$

for some $r_B(u) \in \mathbb{Z}[u]$, which we call reduced matrix counter *of B.*

Proof. If K is of characteristic zero, we may assume without loss of generality $K = \mathbb{Q}$, since the equations and inequations come from determinants and hence only involve integers. Since in the process of computing simple systems, only finitely many denominators come up, we may choose any prime p dividing none of these and pass to the finite field $\mathbb{F}_p$ and still retain the same matroid B. Since B is polynomially countable, $c_B(|L|)$ is equal to the number of matrices $A \in L^{k \times n}$ with $\mu(A) = B$ for any finite extension field L of $\mathbb{F}_p$. Since $\mathrm{GL}(k,L)$ acts semiregularly on this set of matrices, i. e. any stabilizer is trivial and all orbits have length $\mathrm{gl}(k,|L|)$, one easily gets $\mathrm{gl}(k,u) \mid c_B(u)$. □

Often the reduced matrix counter of $B \subseteq \mathrm{Pot}_k(\underline{n})$ is the counting polynomial of $S(B)$ intersected with the set of those $k \times n$-matrices for which certain k columns form the unit matrix. Unfortunately, it is in general not true that a split simple system with an equation of the form $x_i - k$ for some $k \in K$ added can be decomposed into split simple systems. Here is a practical sufficient criterion for B to be polynomially countable.

Proposition 3.5. *Let $B \subseteq \mathrm{Pot}_k(\underline{n})$ and choose some $a \in B$. By $S(a,B) \in \mathcal{L}(K,kn)$ we denote the set of solutions of the system*

$$X_a = I_k, \det(X_{|b}) \neq 0 \text{ for } b \in B, \det(X_{|b}) = 0 \text{ for } b \in \mathrm{Pot}_k(\underline{n}) - B,$$

where I_k denotes the $k \times k$ unit matrix. If $S(a,B)$ is polynomially countable with faithful counting polynomial $c(S(a,B),u)$, then B is polynomially countable with reduced matrix counter $r_B(u) = c(S(a,B),u)$.

Proof. Let $S(a) \in \mathcal{L}(K, kn)$ be the set of solutions of $X_a = I_k$, and $S'(a) \in \mathcal{L}(K, kn)$ be the set of solutions of $\det(X_a) \neq 0$. Then

$$\mathrm{GL}(k, \overline{K}) \times S(a) \to S'(a) : (g, A) \mapsto g \cdot A$$

is a bijective birational map defined over K. Note, $\mathrm{GL}(k, \overline{K})$ is uniformly enumerable by [Ple 09b], say

$$\mathrm{GL}(k, \overline{K}) = \biguplus G_i$$

with finitely many split simple systems G_i. Assume first that $S(a, B)$ is enumerable, say $S(a, B) = \biguplus C_j$. Then above bijection restricts to a birational bijection $G_i \times C_j \to G_i \cdot C_j$ for every pair (i, j). Since

$$S(B) = \mathrm{GL}(k, \overline{K}) \cdot S(a, B) = \biguplus_{i,j} G_i \cdot C_j,$$

the claim follows in this case.
If $S(a, B)$ is only polynomially enumerable, the proof is a slight modification. □

Lemma 3.6. *In the situation of $S(a, B)$ above, for any given pair $(i, j) \in \underline{k} \times (\underline{n} - a)$ one either has $x_{ij} = 0$ for all $X \in S(a, B)$ or $x_{ij} \neq 0$ for all $X \in S(a, B)$.*

Proof. Let k be the unique element of a such that the k-th column of X is the i-th column of the identity matrix. Let $c := (a - \{k\}) \cup \{j\}$. Either $c \in B$, in which case $x_{ij} = \pm \mathrm{Det}(X_{|c}) \neq 0$ or $c \notin B$, in which case $x_{ij} = \pm \mathrm{Det}(X_{|c}) = 0$. □

Beyond the action of the general linear group $\mathrm{GL}(k, K)$ one can take the torus action into account, i. e. the action of $(K^*)^n$ which results in further irreducible factors of the matrix counter. Recall that a vector matroid is called decomposable or disconnected if it is of the form $\pi(\mu(\mathrm{Diag}(A_1, A_2)))$ for some matrices $A_1 \in K^{k' \times n'}, A_2 \in K^{k'' \times n''}$ with $k' + k'' = k$ and $n' + n'' = n$ and for some permutation $\pi \in S_n$.

Proposition 3.7. *If in the notation of Proposition 3.5 $S(a, B)$ is polynomially countable, then $(u-1)^{n-l} | r_B(u)$, where l is the number of connected components of B. The polynomial $o_B(u) := (u-1)^{-(n-l)} r_B(u) \in \mathbb{Z}[u]$ is called the* orbit counter *of B.*

Proof. Since r_B is obviously multiplicative in the components of B, it suffices to assume that B is a connected matroid. Also we may assume $a = \underline{k}$. The group $D_n := (\overline{K}^*)^n$ acts on $S(a, B) \in \mathcal{L}(K, kn)$ by

$$D_n \times S(a, B) \to S(a, B) : (d, A) \mapsto (d_i^{-1} A_{i,j} d_j)_{i \in \underline{k}, j \in \underline{n}},$$

where the factors d_i^{-1} make sure that the submatrix of the first k columns remains the unit matrix. Note, by Lemma 3.6 for any $(i, j) \in \underline{k} \times \underline{n}$ either

$A_{ij} = 0$ for all $A \in S(a, B)$ or $A_{ij} \neq 0$ for all $A \in S(a, B)$. Call $T \subseteq \underline{k} \times (\underline{n} - \underline{k})$ a RIGIDITY FRAME, if
1.) $|T| = n - 1$,
2.) $(i, j) \in T$ implies $A_{ij} \neq 0$ for all $A \in S(a, B)$, and
3.) $\pi_1(T) = \underline{k}, \pi_2(T) = \underline{n} - \underline{k}$, where π_i denotes the projection onto the i-th component for $i = 1, 2$.
Since B is connected, such a rigidity frame T exists. It gives rise to the system of equations

$$A_{i,j} d_j = d_i \qquad (i, j) \in T$$

for the d_i, the solutions of which transforms $A \in S(a, B)$ into a matrix of

$$S_T(a, B) := \{A \in S(a, B) | A_{i,j} = 1 \text{ for all } (i, j) \in T\}.$$

Since the stabilizer of any $A \in S_T(a, B)$ in D_n, which is isomorphic to K^*, acts trivially on $S_T(a, B)$, it follows that $S_T(a, B)$ is a set of representatives of the action of D_n on $S(a, B)$. Hence we have a bijective rational function

$$\tilde{D}_n \times S_T(a, B) \to S(a, B) : (d, A) \mapsto (d_i^{-1} A_{i,j} d_j)_{i \in \underline{k}, j \in \underline{n}}$$

defined over the ground field, where $\tilde{D}_n$ is the subgroup of all $d \in D_n$ with $d_1 = 1$. The claim follows. □

Clearly $o_B(u)$ counts the orbits of $\mathrm{GL}(k, K) \times (K^*)^n$ on $S(B)$. If $o_B(u) = 1$, B is called RIGID. In practice, one often proceeds by the above ideas, however in reversed order:

Corollary 3.8. *In the notation of the last proof let $T \subseteq \underline{k} \times (\underline{n} - \underline{k})$ be a rigidity frame and assume that $S_T(a, B)$ is polynomially countable with faithful counting polynomial $o_B(u)$. Then $S(a, B)$ is polynomially countable with reduced matrix counter $(u-1)^{n-l} \cdot o_B(u)$ and B is polynomially countable with matrix counter $c_B(u) = \mathrm{gl}(k, u) \cdot (u-1)^{n-l} \cdot o_B(u)$.*

Here are some examples demonstrating how one may proceed:

Example 3.9. *The root system A_n viewed as its matrix of positive roots in $K^{n \times \binom{n+1}{2}}$ gives rise to the matroid $\mu(A_n)$ which is rigid, i. e. whose reduced matrix counter is*

$$(u-1)^{\binom{n+1}{2} - 1}.$$

Proof. Let $(e_0, \ldots, e_n)$ be a basis on an $n+1$-dimensional vector space over a field K. Consider the set of vectors $X_n := \{e_i - e_j | 0 \leq i < j \leq n\}$. As basis we choose $(e_0 - e_i | i = 1, \ldots n)$. The coordinate columns of the elements of X_n yield a matrix M with $\mu(M) = \mu(A_n)$. For convenience we index the columns of our matrix by the set of 2-element subsets of $\{0, 1, \ldots, n\}$. In particular the basis part of the matrix has indices $\{0, i\}$ for $i \in \underline{n}$. Call this set a. Now let

$A \in S(a, \mu(A_n))$. We look at the submatrix with column indices in $\mathrm{Pot}_2(\underline{n})$ in the spirit of the last proof. We may choose as rigidity frame the set

$$L := \{(i, \{i, j\}) | i \in \underline{n-1}, i < j\} \cup \{(i, \{1, i\}) | i = 2, \ldots, n\}.$$

We may assume $A_{s,t} = 1$ for $(s,t) \in L$. Clearly A also has zeroes in the positions where M has zeroes. In particular the first unknown entry of A is $A_{3,\{2,3\}}$. The central observation is that $\{\{1,2\}, \{1,3\}, \{2,3\}\}$ is a cycle of our matroid, i. e. the three corresponding column vectors are linearly dependent. We know all the entries of theses three column vectors except $A_{3,\{2,3\}}$. Hence we know exactly what the linear dependence looks like:

$$A_{-,\{1,2\}} - A_{-,\{1,3\}} - A_{-,\{2,3\}} = 0.$$

This determines the unkonwn entry. Similarly all the other entries can be determined and the claim follows. □

Example 3.10. *The root system B_n viewed as its matrix of positive roots in $K^{n \times n^2}$ gives rise to the matroid $\mu(B_n)$ whose reduced matrix counter is*

$$(u-1)^3(u-2) \text{ for } n = 2 \text{ and } (u-1)^{n^2-1} \text{ for } n > 2.$$

Proof. The case $n = 2$ is an easy exercise. We look at the case $n = 3$ from which the general proof will be clear.

$$A := \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 & 1 \end{pmatrix}$$

yields $\mu(B_3)$. Note columns 1,2,3,4,6,8 yield A_3. The decisive linear dependence for the rest is $A_{-,5} - A_{-,6} + A_{-,9} = 0$. Otherwise the proof is similar to the one for A_n. □

The case of root systems D_n for $n \geq 3$ can be reduced to the previous cases and results in the appropriate power of $u - 1$ for the reduced matrix counter. Here is another source of examples for polynomially countable vector matroids. As a third exampel we look at the Golay code.

Example 3.11. *Let $M \in \mathbb{F}_2^{12 \times 24}$ be the generator matrix of the extended* Golay *code of length 24 over $\mathbb{F}_2$. The induced matroid is rigid, i. e. the orbit counter is* 1 *and the reduced matrix counter* $(u-1)^{23}$.

Proof. By Corollary 3.8 and Proposition 3.5, we are free to choose any basis of $\mu(M)$. Since the automophism group M_{24} has exactly two orbits on the bases of the induced matroid $\mu(M)$ of M, cf. Example 4.12, there are essentially two different types of matrices possible for M. In either case we may assume (after

permuting the columns of M appropriately) that the submatrix of the first 12 columns of M is the identity matrix, i. e. $M = (I_{12}|N)$. In the first type of bases each row and column of N has exactly 7 ones and 5 zeroes. Since this type is slightly more awkward to treat, we choose the second type. Here one has apart 11 columns with exactly 7 ones and exactly one pair of a row and a column intersecting in a zero but otherwise consisting of ones (referred to as cross of ones below).
Let K be a field containing $\mathbb{F}_2$. For any matrix $A \in K^{12\times 24}$ with $\mu(A) = \mu(M)$ we may assume the same shape $A = (I_{12}|X)$. By Lemma 3.6, the positions with zeroes in X are exactly the same as in N. As rigidity frame T, we choose the set of positions in the cross of ones in N, where the position of the zero in the crossing is replaced by some index pair (i, j) outside the cross with $N_{ij} = 1$. In X, we may also choose these 23 positions to be one and we remain with $11 \cdot (7-1) - 1$ positions where the entry is not zero, but otherwise not known. We start with the i-th row: We know $X_{ij} = 1$. Let k be such that $X_{ik} \neq 0$ and X_{ik} is not yet known. Let l be the row index of the row of ones in the cross. Then $X_{li} = X_{lk} = 1$ and the four numbers form a submatrix whose determinant is zero, because the determinant of the corresponding submatrix of N is zero. (Note, this submatrix can be complemented to a submatrix of 12 complete columns by choosing from among the first 12 columns of M similarly as in the proof of Lemma 3.6.) Coming back to the submatrix of X, it has three entries equal to 1 and determinant 0, which implies $X_{ik} = 1$ for the last entry. In this way we conclude that all the remaining non zero entries of the i-th row of X are equal to 1. Similarly all the non zero entries of the j-th column of X are equal to 1. With each new position proved to contain a one, by the same argument, its complete row and column has all its non-zero entries equal to 1. Since the matroid is obviously connected, this finally shows that all unknown entries are equal to one. $\square$

Proposition 3.12. *Let $B \subseteq \mathrm{Pot}_k(\underline{n})$ be a matroid and $a \in B$ such that $S(a, B)$ is uniformly enumerable, enumerable, resp. polynomially countable, then so is $S(\underline{n} - a, B^*)$ where $B^* := \{\underline{n} - b | b \in B\}$ is the dual matroid of B. In this case, the reduced matrix counters are equal: $r_B(u) = r_{B^*}(u)$.*

Proof. We may assume $a = \underline{k}$. Then $(I_k|A) \in S(a, B)$ if and only if $(-A^{tr}|I_{n-k}) \in S(\underline{n} - a, B^*)$. The claim follows easily. $\square$

It seems that matroids with few bases have a tendency to be polynomially countable. In [Sko 96] a survey of the number of representations of uniform rank 3 matroids on 7, 8, and 9 point is given, which indicates that polynomial countability for 8 and 9 points is only given if certain univariate quadratic polynomials split over the ground field, though the number of solutions can be described in the other cases as well. This phenomenon in called quasisplit case in [Ple 09b], cf. Example 5.9 1) for simpler examples. One might suspect that sooner or later one gets examples which are not polynomially countable and not even quasisplit. A good candidate for this might be the matroid of rank

3 on 11 points in [Stu 93], pg. 101, where among the equations for $S(a, B)$ the absolutely irreducible polynomial given there turns up. In the last chapter, rank 3 vector matroids on up to 7 points are given with their matrix counters. There are also examples given there with the same Tutte polynomial, but different matrix counters; on the other hand there is also an abundance of pairs of non isomorphic matroids with the same matrix counter.

4 The rank generating polynomial

In this section, M denotes a matroid on the set E of n elements with rank function $\rho : \mathrm{Pot}(E) \to \mathbb{Z}_{\geq 0}$. The rank generating polynomial is defined as

$$S(M; x, y) := \sum_{X \subseteq E} x^{\rho(E)-\rho(X)} y^{|X|-\rho(X)} \in \mathbb{Z}[x, y].$$

A good example is the rank generating polynomial

$$p_{n,k}(x, y) := \binom{n}{k} + \sum_{i=1}^{n-k} \binom{n}{k+i} y^i + \sum_{i=1}^{k} \binom{n}{k-i} x^i$$

of the uniform matroid of rank k, where every k-element subset of E forms a basis of M. The aim of this section is to reduce the summation over all subsets of E to something more manageable. One rather simple approach is to define the deviation polynomial

$$\delta(M; x, y) := p_{n,k}(x, y) - S(M; x, y) \in \mathbb{Z}[x, y]$$

where k is the rank of M. For matroids with a big number of bases, $\delta(M; x, y)$ will have few terms and the rank generating polynomial can be easily recovered form $\delta(M; x, y)$. Since in both $p_{n,k}(x, y)$ and in $S(M; x, y)$ all $\binom{n}{k-s}$ subsets of M with $k - s$ elements are taken into account, one has the following.

Remark 4.1. *Let $\delta(M; x, y) = \sum_{i,j} a_{ij} x^i y^j$, then for each $s \in \mathbb{Z}$ one has*

$$\sum_i a_{s+i,i} = 0$$

A more serious attempt to analyse and understand the sum with the idea of simplification is by grouping together the summands belonging to one flat.

Definition 4.2. *1.) The polynomial*

$$S(M; y) := S(M; 0, y) \in \mathbb{Z}[y]$$

is called the generator generating polynomial *of M.*
2.) For $X \subseteq E$ let

$$\sigma(X) := \{e \in E | \rho(X) = \rho(X \cup \{e\})\}$$

the closure operator *with respect to M and for any* flat $X = \sigma(X)$ *call*

$$\sigma^{-1}(\{X\}) := \{Y \subseteq E | \sigma(Y) = X\}$$

flock *of X. Finally*

$$\mathcal{L}(M) := \{X \subseteq E | \sigma(X) = X\}$$

the set of all flats of M.

Clearly $S(M; y) = \sum_{X \in \sigma^{-1}(\{M\})} y^{|X| - \rho(X)}$. If we simply write $S(X; y)$ for $S(M|X; y)$ for any flat X of M, where $M|X$ denotes the restriction of M to X, then the original definition of the rank generating polynomial becomes:

$$S(M; x, y) = \sum_{X \in \mathcal{L}(M)} x^{\rho(E) - \rho(X)} S(X; y).$$

To proceed further, we exhibit flats with the same generator generating polynomial. Recall that a coloop of M is an element e of E occuring in each basis of M, i. e. $\rho(E - \{e\}) = \rho(E) - 1$.

Definition 4.3. *A flat $X \in \mathcal{L}(M)$ is called* essential *if $M|X$ has no coloop.*

One clearly has the following lemma.

Lemma 4.4. *Let $X \in \mathcal{L}(M)$. Then there exists a unique essential flat Y, called the essential flat $\epsilon(X)$ of X, such that*

$$M|X = M|Y \oplus M|\{e_1\} \oplus \cdots \oplus M|\{e_r\}$$

where $e_1, \ldots, e_r$ are the coloops of $M|X$. Moreover

$$\sigma^{-1}(\{Y\}) \to \sigma^{-1}(\{X\}) : Z \mapsto Z \cup \{e_1, \ldots, e_r\}$$

is a bijection so that

$$S(X; y) = S(Y; y).$$

This leads to the following definiton.

Definition 4.5. *For any essential flat $Y \in \epsilon(\mathcal{L}(M))$ call $\epsilon^{-1}(\{Y\})$ the* cloud *of Y and*

$$S(M, Y; x) := \sum_{X \in \epsilon^{-1}(\{Y\})} x^{\rho(E) - \rho(X)}$$

the cloud polynomial *of Y.*

Summarizing, we have the following.

Proposition 4.6. *Let M be a matroid without loops on the set E. Then*

$$\mathrm{Pot}(E) = \biguplus_{Y \in \epsilon(\mathcal{L}(M))} \sigma^{-1}(\epsilon^{-1}(\{Y\}))$$

with a bijection

$$\sigma^{-1}(\epsilon^{-1}(\{Y\})) \to \epsilon^{-1}(\{Y\}) \times \sigma^{-1}(\{Y\}) : Z \mapsto (\sigma(Z), Z \cap Y)$$

for every essential flat Y. In particular

$$S(M; x, y) = \sum_{Y \in \epsilon(\mathcal{L}(M))} S(M, Y; x) S(Y; y).$$

Whereas the generator generating polynomial $S(Y; y)$ depends on Y or, more precisely, $M|Y$ only, the cloud polynomial $S(M, Y; x)$ depends on the embedding of Y in M. In fact, it depends only on the minor M/Y:

Proposition 4.7. *Let M be a matroid without loops on the set E and Y an essential flat of M. Then $\emptyset$ is an essential flat of the minor M/Y and there is a bijection between the clouds:*

$$\epsilon_M^{-1}(\{Y\}) \to \epsilon_{M/Y}^{-1}(\{\emptyset\}) : X \mapsto X - Y.$$

In particular, $S(M, Y; x) = S(M/Y, \emptyset; x)$.

Proof. $\emptyset$ is an essential flat of M/Y if and only if no element $a \in E - Y$ is dependent, i. e. $\rho_M(Y \cup \{a\}) > \rho_M(Y)$ for all $a \in E - Y$. This however is clear, since Y is a flat. Clearly the map $X \mapsto X - Y$ maps flats (contained in E) with respect to M containing Y to flats (contained in $E - Y$) with respect to M/Y, where $\rho_{M/Y}(X - Y) = \rho_M(X) - \rho_M(Y)$. The claim follows. ☐

To get a better understanding, we connect the result to the passage to the dual matroid M^*. Denote the closure operator on $\mathrm{Pot}(E)$ with respect to M^* by σ^* and the essentiality operator on $\mathcal{L}(M^*)$ by ϵ^*.

Lemma 4.8. *Let $X \subseteq E$ and $Y := \complement X = E - X$. For $a \in E$ the following statements are equivalent:*
1.) $a \in \sigma(X) - X$.
2.) a is a coloop of $M^*|Y$.

Proof. Clearly 1.) is equivalent to a being a loop of the minor $M/X = M.Y$. Hence 1.) holds iff a is a coloop of $(M.Y)^*$, which by Theorem 4.3.2 of [Wel 76] is isomorpic to $M^*|Y$. ☐

If X is an essential flat of M, we need to distinguish between $\epsilon^{-1}(\{X\})$ and $\epsilon^{\uparrow}(X) := \{Z \subseteq E | X \subseteq Z, Z - X \text{ consists of coloops of } M|Z\}$.

Theorem 4.9. *Let M be a matroid on E without loops and coloops. Let $\complement$: $\mathrm{Pot}(E) \to \mathrm{Pot}(E) : X \mapsto E - X$.*
*1.) $\complement$ induces a bijection (*Galois*-correspondence) between $\epsilon(\mathcal{L}(M))$, the set of essential flats of E with respect to M, and $\epsilon^*(\mathcal{L}(M^*))$, the set of essential flats of E with respect to M^*.*
2.) For an essential flat X in E with respect to M, the bijection $\complement$ induces a bijection between $\sigma^{-1}(\{X\})$ and $\epsilon^{*\uparrow}(E-X)$ and a bijection between $\epsilon^{\uparrow}(X)$ and $\sigma^{*-1}(\{E-X\})$.
3.) For an essential flat X in E with respect to M one has $\epsilon^{-1}(\{X\}) \subseteq \epsilon^{\uparrow}(X)$, indeed $\epsilon^{-1}(\{X\})$ consists of all the flats in $\epsilon^{\uparrow}(X)$.

Proof. 1.) Let $X \subseteq E$. Then X is a flat with respect to M, if and only if $M^*|(E-X)$ has no coloops by Lemma 4.8. If $X \subseteq E$ is an M-flat, then X is an essemtial M-flat, if and only if $M|X$ has no coloops. So $X \subseteq E$ is an essential flat of M, iff $M^*|(E-X)$ has no coloops and $M|X$ has no coloops. By applying the same argument in reverse, with the roles of $M|X$ and $M^*|(E-X)$ interchanged, this is again equivalent to $E-X$ being an essential M^*-flat.
2.) Immediately from Lemma 4.8. 3.) Clear by definition. ☐

Here are some examples and characterizations of essential flats. The proofs are straightforward.

Remark 4.10. *Let M be a matroid on the set E without loops and coloops.*
1.) $\emptyset$ and E are essential flats.
2.) If a hyperplanes (of codimension 1) is an essential flat, its cloud polynomial is x.
3.) If $S(M,x,y) = \sum_{i=0}^{\rho(E)} x^i g_i(y)$ and $\deg g_i(y) > \deg g_{i+1}(y)$ for one i, then M has α_i essential flats of dimension $\rho(E)-i$ consisting of $\deg g_i(y)$ elements, where α_i is the leading coefficient of $g_i(y)$.
4.) If $X \subseteq E$ is an essential flat and $M|X = M|A \oplus M|B$ with $X = A \uplus B$, then A, B are essential flats.
5.) If S is a set of circuits of M, then $\sigma(\bigcup_{X\in S} X)$ is an essential flat of M. Every essential flat is of this form for a suitable set S of circuits.
6.) The minimal number $|S|$ of circuits such that $E = \sigma(\bigcup_{X\in S} X)$ may be called the covering number *of M. It measures certain aspects of the complexity of M. For instance the uniform matroid of rank k on n elements has the covering number 1.*

The rank generating polynomial can of course be computed from the information about the essential flats $\neq E$ using the above results, however, we can also get information about the lattice of flats from the rank generating polynomial.

Remark 4.11. *For a polynomial $p(x,y) = \sum_{i,j} a_{i,j} x^i y^j \in \mathbb{Z}[x,y]$ with non negative coefficients $a_{i,j}$, call the exponent (i,j) extreme if $a_{i,j} > 0, a_{i+k,j} = 0, a_{i,j+k} = 0$ for all $k \geq 1$. Starting with the rank generating polynomial $p(x,y) = \sum_{i,j} a_{i,j} x^i y^j$ of M and an extreme exponent (i,j) one knows of the existence of $a_{i,j}$ essential flats. Subtract the contribution of these essential flats* *from $p(x,y)$ and proceed in the same way with the difference polynomials to get the next set of essential flats.*

Example 4.12. *The* Golay*-code C of length 24 and dimension 12 over $\mathbb{F}_2$ gives rise to a selfdual matroid with rank generating polynomial $p_{24,12}(x,y) - \delta(C;x,y)$, where $\delta(C;x,y)$ is given by*

$$(1-xy)(r(y)+644(55xy+2039)+r(x))$$

with

$$r(t) := 759t^4 + 12144t^3 + 91080t^2 + 425040t$$

The automorphism group is known to be the Mathieu*-group M_{24} of order $|M_{24}| = 2^{10}3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ and by means of GAP, cf. [GAP] some relevant orbits of M_{24} in* $\mathrm{Pot}(\underline{24})$ *can easily be computed towards the following results. The essential flats fall into 6 orbits: The empty set, the circuits of length 8, certain 10-dimensional flats of covering number 2, circuits of length 12, the complements of the 8-circuits (the dual of which are isomorphic to the matroid of affine 4-space over $\mathbb{F}_2$), and finally the full set. Here is the rank generating polynomial split up into the corresponing sum of 6 summands, each of which is a product of the length of the orbit, the cloud polynomial (in x), and the generator generating polynomial (in y):*

$$\begin{gathered}
1 \cdot (\textstyle\sum_{i=0}^{6} \binom{24}{i} x^{12-i} + |M_{24}| \cdot (\frac{1}{720}x^5 + \frac{1}{384}x^4 + \frac{1}{432}x^3 + \frac{1}{1440}x^2)) \cdot 1 + \\
759 \cdot (x^5 + 16x^4 + 120x^3) \cdot (y+8) + \\
35420 \cdot x^2 \cdot (y^2 + 12y + 48) + \\
2576 \cdot x \cdot (y+12) + \\
759 \cdot x \cdot (y^5 + 16y^4 + 120y^3 + 560y^2 + 1680y + 2688) + \\
1 \cdot 1 \cdot (\textstyle\sum_{i=0}^{7} \binom{24}{i} y^{12-i} + |M_{24}|(\frac{121}{40320}y^4 + \frac{1}{189}y^3 + \frac{11}{1440}y^2 + \frac{67}{7920}y + \frac{1}{176}))
\end{gathered}$$

Because of the presence of the 12-circuits one easily sees that the covering number of the full matroid is 2. We list some additional information about the orbits of M_{24} on the set of 12-subsets of $\{1,2,\ldots,24\}$ as an interpretation of the coefficients of $(xy)^i$ in the rank generating polynomial above: Two orbits of bases of lengths 1020096 and 370944, one orbit of 12-circuits of length 2576 (stabilizers isomorphic to M_{12}), and one further orbit of length $1275120 = 759 \cdot 1680$, of which each element has one linear relation and has the complement of an octave as its closure. Finally, there is one orbit of essential flats with two relations of length 35420. Of course, the above way to write the rank generating polynomial encodes much more information than the polynomial itself.

We finish this chapter with an example of a non selfdual matroid of rank 8 on 15 points. Its combinatorial structure is sufficiently clear so that its rank generating polynomial as a sum of the products of its cloud and generator generating polynomials can in principle be computed by hand using the results of this chapter. One way to describe it is as the dual of the matroid of the columns of $p_{15}(A_{15})$, where $p_{15}(t)$ is the 15-cyclotomic polynomial and A_{15}

is the permutation matrix of a 15-circuit or, if one prefers it, the companion matrix of $t^{15}-1$. So the example can be considered as a cyclic code over $\mathbb{Q}$. In the actual formulation of the example, a different description is given, which results in a permutation of the ground set, but gives a clearer picture of the structure.

EXAMPLE 4.13. *Let $Z_k \in \mathbb{Q}^{(k-1)\times k}$ the matrix whose first $k-1$ columns form the unit matrix and whose last column has all entries equal to -1. Note, the matroid of the columns of Z_k is a k-circuit. Choose $A \in \mathbb{Q}^{8\times 15}$ to be the* KRONECKER *product $A := Z_5 \otimes Z_3$. The associated matroid M is of rank 8 on $E := \underline{15}$. Its structure is governed by the (essential) circuits given by the sets of entries in one of the rows or colums of the matrix*

$$\kappa := \begin{pmatrix} 1 & 4 & 7 & 10 & 13 \\ 2 & 5 & 8 & 11 & 14 \\ 3 & 6 & 9 & 12 & 15 \end{pmatrix}$$

The automorphism group of M is the direct product $S_3 \times S_5$ whose action on E is induced by the action of S_3 on the rows and of S_5 on the columns of κ. The closure operator σ takes a subset $X \subseteq E$ and obtains $\sigma(X)$ as the union of X, of the column sets of κ whose intersection with X has two elements, and of the row sets of κ whose intersection with X has four elements. The essentiality operator ϵ takes a flat $X \subseteq E$ and removes all elements from X for which neither its full row nor its full column is contained in X. In particular, the essential flats are unions of complete rows (0, 1, or 3) and complete columns (0,1,2,3, or 5). The notation for the $S_3 \times S_5$-orbits is forced upon one: $i_r j_c$ meaning $i \in \mathbb{Z}_{\geq 0}$ rows and $j \in \mathbb{Z}_{\geq 0}$ columns. With this information it is not so difficult to compute the generator generating polynomials and the cloud polynomials except maybe the cloud polynomial for 0_r0_c and the generator generating polynomial for 3_r5_c. However, these con be easily obtained from Remark 4.1, once everything else is computed. Here is the result: The first factor in each summand is the length of the $S_3 \times S_5$-orbit, followed by the cloud polynomial, and finally by the generator generating polynomial. On the right the symbol for the orbit of the essential flat is given.

$$\begin{array}{c|l}
1\cdot 1\cdot (y^4+6y^3+24y^2+50y+75)(3+y)^3+ & 3_r5_c \\
30\cdot x\cdot (y^2+5y+12)(3+y)^2+ & 1_r3_c \\
30\cdot x^2\cdot (y^2+6y+13)(3+y)+ & 1_r2_c \\
10\cdot x^2\cdot (3+y)^3+ & 0_r3_c \\
15\cdot x^3\cdot (14+7y+y^2)+ & 1_r1_c \\
10\cdot (x^4+9x^3+18x^2+6x)\cdot (3+y)^2+ & 0_r2_c \\
3\cdot x^4\cdot (5+y)+ & 1_r0_c \\
5\cdot (x^6+12x^5+54x^4+96x^3+54x^2)\cdot (3+y)+ & 0_r1_c \\
1\cdot (x^8+15x^7+90x^6+270x^5+390x^4+210x^3)\cdot 1 & 0_r0_c
\end{array}.$$

For completeness we add the corresponding information for the dual matroid. The notation for the $S_3 \times S_5$-orbits is kept, though one has to take the comple-

ments of the sets above.

$$\begin{array}{rl|l}
1\cdot 1\cdot (y^8+15y^7+105y^6+450y^5+1305y^4+2670y^3+3870y^2 & & \\
+3780y+2025)+ & & 0_r0_c \\
5\cdot x\cdot (y^6+12y^5+66y^4+216y^3+456y^2+612y+432)+ & & 0_r1_c \\
3\cdot x\cdot (y^4+10y^3+40y^2+80y+80)+ & & 1_r0_c \\
10\cdot x^2\cdot (y^4+9y^3+36y^2+78y+81)+ & & 0_r2_c \\
15\cdot (x^2+x)\cdot (y+4)(y^2+4y+8)+ & & 1_r1_c \\
10\cdot x^3\cdot (12+6y+y^2)+ & & 0_r3_c \\
30\cdot (x^3+2x^2+x)\cdot (12+6y+y^2)+ & & 1_r2_c \\
30\cdot (x^4+3x^3+3x^2+x)\cdot (4+y)+ & & 1_r3_c \\
1\cdot (x^7+15x^6+105x^5+335x^4+495x^3+303x^2+15x)\cdot 1 & & 3_r5_c
\end{array}$$

5 VECTOR MATROIDS OF RANK 3

Call a vector matroid simple [Wel 76] if it has no circuits of length 1 or 2. Any matroid can easily be reduced to such a one, and conversely all the other matroids can easily be constructed from the simple matroids once their automorphism groups are known. Moreover, the matrix counters change by an easily computed power of $u-1$. In this section we compute the matrix counters for the simple matroids of rank three on $n \leq 7$ points. The first aim is to introduce a symbol for each simple matroid of rank 3, cf. [Wel 76] Section 1.11. Our formalization of the graphical notation there has the advantage that permutations of the underlying set E can be easily dealt with, i. e. automorphism groups can almost be read off. The rank three matroids are best understood via their nontrivial hyperplanes.

DEFINITION 5.1. *Let M be a matroid of rank 3 with underlying set E.*
1.) A hyperplane of M of more than 2 elements is called NONTRIVIAL.
2.) For $i \in E$ let $\mathrm{Hy}(i)$ *denote the set of all nontrivial hyperplanes containing i.*
3.) $i \in E$ is called CONGLOMERATOR, *if* $\mathrm{Hy}(i)$ *contains more than one element.*
4.) The set of all conglomerators of M is denoted by $\mathrm{Congl}(M)$.
5.) $i \in E$ is called LAZY *if it is not contained in any nontrivial hyperplane.*

EXAMPLE 5.2. *Let M be a simple matroid of rank* 3.
Let $|E| = 4$, then M has no conglomerators. It either consists only of lazy elements, or has exactly one nontrivial hyperplane, which then necessarily contains 3 elements. So either all elements are lazy or exactly one is lazy.
Let $|E| = 5$, then M has at most one conglomerator. In this case there are no lazy elements and one has two nontrivials hyperplanes of length 3 having the conglomerator in common.

DEFINITION 5.3. *Let M be a simple matroid of rank* 3 *on a set E. The symbol $[M]$ of M is an element of the free abelian group* $\mathbb{Z}[\mathrm{Pot}(E) \times \{3, 4, \ldots |E|-1\}]$

with $\mathbb{Z}$-basis $\mathrm{Pot}(E) \times \{3, 4, \dots |E| - 1\}$*, where for brevity the basis elements are written as* X_a *rather than* (X, a) *for* $X \subseteq E, a \in \{3, 4, \dots |E| - 1\}$*. The symbol* $[M]$ *is then defined as follows:*

$$[M] := \sum_H (H \cap \mathrm{Congl}(M))_{|H|},$$

where the sum is taken over all nontrivial hyperplanes H*. If* E *or the number of elements of* E *are not clear from the context, we write* $[M]_E$ *or* $[M]_{|E|}$ *instead of* $[M]$*.*

Note that M can be recovered from $[M]$ up to the names of the elements not contained in $\mathrm{Congl}(M)$.

Example 5.4. *Let* M *be a simple matroid of rank* 3 *and* $E := \underline{n}$*.*
Let $|E| = 4$*, then* $[M]$ *is either* 0 *for the uniform matroid or* $\emptyset_3$ *for any rank 3 matroid on* E *with exactly one (unspecified) nontrivial hyperplane, which then consists of 3 elements.*
Let $|E| = 5$*, then* $[M]$ *is either* 0*,* $\emptyset_3$*,*$\emptyset_4$*, or* $2\{i\}_3$ *for some* $i \in E$*.*
Note, the matroid on 5 elements with symbol $(\emptyset_3)_5$ *is obtained from the one with symbol* $(\emptyset_3)_4$ *on four elements by adding one lazy element.*

It is clear that the symbol determines the matroid up to isomorphism. To list all rank 3 simple matroids on n elements up to isomorphism by their symbols, we may (and will) restrict to the symbols of matroids M with $\mathrm{Congl}(M) = \underline{a}$ for some $a \leq n$. It remains to deal with the problem of isomorphism for these symbols.

Example 5.5. *Let* $n := 6$*. Then one has exactly 9 isomorphism classes of simple matroids. They are represented by the symbols*

$$0,\ \emptyset_3,\ \emptyset_4,\ 2\{1\}_3$$

obtained from matroids on less than 6 elements by adding lazy elements, further the ones with $|\mathrm{Congl}(M)| \leq 3$*:*

$$\emptyset_5,\ 2\emptyset_3,\ \{1\}_3 + \{1\}_4,\ \{1,2\}_3 + \{1,3\}_3 + \{2,3\}_3,$$

and one with $|\mathrm{Congl}(M)| = 6$*:*

$$\{1,3,4\}_3 + \{1,5,6\}_3 + \{2,3,6\}_3 + \{2,4,5\}_3.$$

It is clear that the symbols satisfy certain obvious conditions, which we list.

Remark 5.6. *Let* $\alpha \in \mathbb{Z}[\mathrm{Pot}(\underline{n}) \times \{3, 4, \dots, n-1\}]$ *be a symbol of a rank 3 matroid. Then*
1.) If S_a *occurs in* α*, then* $|S| \leq a$*.*
2.) If $S_a \neq T_b$ *both occur in* α*, then* $|S \cap T| \leq 1$*. Also the coefficient of* S_a *can be at most 1 unless* $|S| = 1$*.*

3.) Let $\mathrm{Congl}(\alpha)$ *denote the union of the sets in the first component of the terms of* α*. Then for each* $i \in \mathrm{Congl}(\alpha)$ *there occur at least two* S_a*,* T_b *in* α *with* $i \in S$ *and* $i \in T$*.*
4.) The smallest cardinality $|E|$ *for the ground set of a matroid with symbol* α *is*

$$|\mathrm{Congl}(\alpha)| + \sum_{S_a} (a - |S|)$$

where the sum is taken over all terms S_a *(with multiplicities) occuring in* α*.*
5.) The rank generating polynomial can be read off from the symbol α*, more precisely from the indices of the summands of the symbol:*

$$p_{n,3}(x,y) - \delta_\alpha(x,y) \text{ with } \delta_\alpha(x,y) = (1-xy)\sum_{l=0}^{n} y^l \sum_{H \in \mathcal{H}} \binom{|H|}{3+l}$$

where $\mathcal{H}$ *is the set of all nontrivial hyperplanes.*

Existence of vector matroids cannot a priori be read off from the symbol, but usually has to be computed explicitly. Our main interest is to find the matrix counters in the cases where it is possible, including the relevant information on the fields. The following tables were computed as follows: For a given rank 3 matroid, a basis and a rigidity frame, cf. proof of Proposition 3.7, is fixed. This gives an ansatz for the matrix with a unit matrix and a matrix of indeterminates as complemetary submatrices. By the choice of the rigidity frame, certain indeterminates are substituted by 1. Each 3×3-minor results in an equation or inequation, depending on whether we have dependence or a basis in the matroid. This system is put into the AlgebraicThomas-program, cf. [BLH 13]. For many cases a suitable order of the variables yields a faithful counting polynomial, i. e. an orbit counter right away, including information on the characteristics. If not all systems split, one might try a different order of variables. If the system is too big, we use the inclusion-exclusion principle to generate systems of equations only, which often can be used to obtain a faithful counting polynomial of a polynomially countable set. The question of enumerability usually remains open in these cases.

Example 5.7. *For* $|E| = n = 6$*, Table 1 lists the orbit counters of the simple matroids of rank 3 up to* S_n*-action sorted according to the degrees of the matrix counters. These matroids are all indecomposable except for* $\emptyset_5$*, which has two compontents and the orbit counter has to be multiplied by* $(u-1)^4$ *instead of* $(u-1)^5$ *to obtain the reduced matrix counter* $r_B(u)$*. Note also that* $2\emptyset_3$ *and* $2\{1\}_3$*, which have the same rank generating polynomial, are distinguished by their matrix counter. In the uniform case* 0*, the system* $S_T(a,B)$*, cf. Corollary 3.8, is polynomially countable, where all characteristics* $\neq 2$ *can be treated simultaneously, however the final faithful counting polynomial is the same for all characteristics including 2. In all the other cases we have uniform enumerability for* $S_T(a,B)$*, with the restriction that for* $\emptyset_3$ *characteristic 2 has to be treated separately, but again yields the same orbit counter. We note that for*

symbol cf. 5.3	$o_B(u)$ (orbit counter cf. 3.7)	$\|S_6$-orbit$\|$
0	$(u-2)(u-3)(u^2-9u+21)$	1
$\emptyset_3$	$(u-2)(u-3)(u-4)$	20
$2\emptyset_3$	$(u-2)^2$	10
$2\{1\}_3$	$(u-2)(u-3)$	90
$\emptyset_4$	$(u-2)(u-3)$	15
$\{1\}_3+\{1\}_4$	$(u-2)$	60
$\{1,2\}_3+\{1,3\}_3+\{2,3\}_3$	$(u-2)$	120
$\emptyset_5$	$(u-2)(u-3)$	6
$\{1,2,4\}_3+\{1,3,5\}_3+$ $\{2,3,6\}_3+\{4,5,6\}_3$	1	30

Table 1: Orbit counters for simple rank 3-matroids on 6 points, cf. Example 5.7

the uniform matroid 0 *the above mentioned inclusion-exclusion count has been applied. However, in this particular case, it can be avoided by also computing the contribution of the non simple matroids towards the counting polynomial for all rank 3 matrices in* $K^{3\times 6}$*, cf. Proposition 2.5. After division by* $\mathrm{gl}(3,u)$*, this is the product of the 2nd, 4-th, 5-th, and 6-th cyclotomic polynomial*

$$\begin{aligned}&(u+1)(u^2+1)(u^4+u^3+u^2+u+1)(u^2-u+1)\\=\;&10(3u-1)(3u^2-3u+1)\\&+(u^2+2u-5)(u^3+3u^2-10)(u-1)^4\\&+5(u+3)(13u^2-14u+4)(u-1)^2\\&+3(5u-3)(u^3+4u^2+u-11)(u-1)^3\end{aligned}$$

where the i*-th summand gives the contribution* $\sum_B r_B(u)$ *of the matrices whose matroid* B *reduces to a simple matroid on* $2+i$ *elements for* $i=1,2,3,4$.
Finally, the factors of the orbit polynomials can usually be given interpretations. For instance, in the case of the uniform matroid $[0]_6$*, the factors* $u-2$ *and* $u-3$ *mean that there are no representations of the matroid over a field of 2 or three elements. For bigger fields they can be interpreted as follows: Once three columns of the matrix are chosen to form the unit matrix and, say one column and one row of the remaining matrix is chosen to be equal to 1 in each position as rigidity frame, cf. proof of Proposition 3.7, choose a fixed column* C *among the two other columns. The first remaining position of* C *can be chosen to be* $a\neq 1,0$ *and the second remaining position of* C *can be chosen to be* $c\neq 0,1,a$*. Independently of these choices, the matrix can be completed in* $(u^2-9u+21)$ *ways.*

The next example treats the simple rank 3 matroids on 7 points. By Propostion 3.12 this can be turned into (almost all of) the corresponding list of orbit counters of rank 4 matroids on 7 points. The orbit counters for 0 in Examples 5.7 and 5.8 have already been known by different methods, cf. [Sko 96] last section and the references there. Also the more complicated cases of rank 3 matroids on 8 and 9 points are described there, cf. also [ISS 95]. For a more geometric approach to these problems, cf. [Sko 92] and [RoS 96].

Example 5.8. *For $|E| = n = 7$ the matroids of rank 3 are all polynomially countable. Table 2 lists the orbit counters of the simple matroids of rank 3 up to S_n-action sorted according to the degrees of the matrix counters. These matroids are all indecomposable except of $\emptyset_6$, which has two compontents and the orbit counter has to be multiplied by $(u-1)^5$ instead of $(u-1)^6$ to obtain the reduced matrix counter $r_B(u)$. In this case of 7 points for E, one often gets different orbit counters for characteristic 2. Remarkably the orbit counters of the same matroid (of rank 3 on 7 points) for characteristic 2 and characteristic $\neq 2$ differ only by a number. Therefore in Table 2 the δ_2 is 1 if the characteristic of the field is 2, otherwise it is zero. Usually, an orbit counter in characteristic 2 factors similarly to the corresponding one for the other characteristics, e. g. for $\emptyset_3$ we have*

$$6\cdot\delta_2+(u-5)(u-3)(u^3-13u^2+54u-66) = (u-4)(u-2)(u^3-15u^2+75u-123)$$

Since the matroids of rank 3 on less than 7 elements are all polynomially countable with polynomials independent of the characteristic of the field, the contribution of the non simple matroids to the counting polynomial of 3×7-matrices is also independent of the characteristic, and therefore also the contribution of all simple matroids (listed in Table 2) together, since the counting polynomial for all 3×7-matrices of rank 3 is independent of the characteristic, cf. Propsition 2.5. This amounts to saying that the differences of the general reduced matrix counters to the characteristic 2 ones multiplied by the orbit lengths in the last column of the table should add up to zero, because the multiplicities of the factor $u-1$ are the same in all relevant cases. But in fact, these product do not only add up to zero, but (for us unexpectedly) cancel in pairs (zeroes omitted):

$$[30, -210, 630, -840, -210, 210, 840, -630, 210, -30]$$

Concerning the individual orbit counters, the ones for $0, \emptyset_3, 2\emptyset_3, 2\{1\}_3$ were obtained via inclusion-exclusion, in all other bases directly so that at least the transversal there is uniformly enumerable. In the case of the uniform matroid 0, even for the inclusion-exclusion approach to work, one had to change the order of the coordinates for some of the simple systems, i. e. the investigated systems were probably not uniformly enumerable, but only enumerable, resulting in polynomially countable systems for the final result. Note, the last matroid corresponds to the projective plane over $\mathbb{F}_2$.

It is known, cf. [Wel 76] pg. 306, that there are 68 simple rank 3 matroids on 8 points, all listed in the supplement of [BCH 73]. Instead of going through

symbol cf. 5.3	$o_B(u)$ (orbit counter cf. 3.7)	$\lvert S_7$-orbit$\rvert$
0	$-30 \cdot \delta_2 + (u-3)(u-5)\cdot$ $(u^4 - 20u^3 + 148u^2 - 468u + 498)$	1
$\emptyset_3$	$6 \cdot \delta_2 + (u-5)(u-3)\cdot$ $(u^3 - 13u^2 + 54u - 66)$	35
$2\emptyset_3$	$(u-5)(u-2)(u-3)(u-4)$	70
$2\underline{1}_3$	$-2 \cdot \delta_2 + (u-3)(u^3 - 12u^2 + 46u - 54)$	315
$\emptyset_4$	$(u-5)(u-2)(u-3)(u-4)$	35
$\underline{1}_3 + \underline{1}_4$	$(u-2)(u-3)(u-4)$	420
$\underline{12}_3 + \underline{13}_3 + \underline{23}_3$	$\delta_2 + (u-3)(u^2 - 7u + 11)$	840
$\emptyset_5$	$(u-2)(u-3)(u-4)$	21
$3\underline{1}_3$	$2 \cdot \delta_2 + (u-3)^2(u-4),$	105
$\emptyset_3 + \emptyset_4$	$(u-3)(u-2)^2$	35
$\underline{1}_3 + \underline{2}_3 + \underline{12}_3$	$(u-4)(u-3)(u-2)$	630
$\underline{124}_3 + \underline{135}_3 +$ $\underline{236}_3 + \underline{456}_3$	$-\delta_2 + (u-3)^2$	210
$\underline{1}_3 + \underline{1}_5$	$(u-2)(u-3)$	105
$\underline{12}_3 + \underline{13}_3 + \underline{23}_4$	$(u-3)(u-2)$	1260
$\underline{1}_4 + \underline{1}_4$	$(u-2)^2$	70
$\emptyset_6$	$(u-2)(u-3)(u-4)$	7
$\underline{12}_3 + \underline{13}_3 +$ $\underline{14}_3 + \underline{234}_3$	$-\delta_2 + (u-3)^2$	840
$\underline{23}_3 + \underline{45}_3 +$ $\underline{124}_3 + \underline{135}_3$	$(u-3)(u-2)$	1260
$\underline{124}_4 + \underline{136}_3 +$ $\underline{256}_3 + \underline{345}_3$	$(u-2)(u-3)$	840
$\underline{12}_3 + \underline{135}_3 + \underline{146}_3 +$ $\underline{234}_3 + \underline{256}_3$	$\delta_2 + (u-3)$	630
$\underline{124}_3 + \underline{135}_3 + \underline{167}_3 +$ $\underline{236}_3 + \underline{457}_3$	$(u-2)$	420
$\underline{126}_3 + \underline{135}_3 + \underline{147}_3 +$ $\underline{237}_3 + \underline{245}_3 + \underline{346}_3$	$-\delta_2 + 1$	210
$\underline{126}_3 + \underline{135}_3 + \underline{147}_3 +$ $\underline{237}_3 + \underline{245}_3 + \underline{346}_3 +$ $\underline{567}_3$	δ_2	30

Table 2: Orbit counters for simple rank 3-matroids on 7 points, cf. Example 5.8, where $\{i, j, k\}$ in the symbol is abbreviated as $\underline{ijk}$.

all possibilities, we only give two examples demonstrating phenomena not yet occurring in the case of $|E| = 7$ points.

Example 5.9. *Let $|E| = 8$, i. e. we consider some examples of rank 3 matroids on 8 points.*
1.) The matroid $B := \{1,2\}_3 + \{1,4\}_3 + \{2,3\}_3 + \{3,4\}_3$ gives rise to a system $S_T(\{1,2,3\}, B)$ for a suitable rigidity frame T, cf. proof of Proposition 3.7, saying that all the 3×3-minors of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & x_{2,2} & x_{2,3} & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & x_{2,3} & x_{3,4} & x_{3,5} \end{bmatrix}$$

which do not vanish identically in the four variables are not equal to zero. The system is too big to be treated directly so that one has to use the inclusion-exclusion count. In doing that it turns out that at characteristic 2 one has a different behaviour, but otherwise all simple system coming up are split, except for one, which is only quasisplit in the sense of [Ple 09b]:

$$x_{3,5} - x_{2,2} = 0, x_{2,3} - x_{2,2} = 0, x_{3,4} - x_{2,2} = 0, x_{2,2}^2 - x_{2,2} + 1 = 0,$$

i. e. becomes split after a suitable finite field extension. The counting polynomial for the whole system is

$$c(u) := u^4 - 16u^3 + 93u^2 - 231u + 208.$$

Now the interpretation is slightly more complicated: If the field in question does not contain a primitive sixth root of unity, the number of solutions (or rather the counting polynomial for these solutions) is $c(u) - 2$. If it contains a primitive sixth root of unity and the characteristic is not (2 or) 3, then it is $c(u)$ as it stands. If the characteristic is 3, then it is clearly $c(u) - 1 = (u-3)(u^3 - 13u^2 + 54u - 69)$, since $x_{2,2}^2 - x_{2,2} + 1$ has a double root then. Finally the case of characteristic 2 has to be treated separately in the same manner. One obtains the counting polynomial $c(u) - 4$, which is correct if the ground field contains a primitive third root of unity and $c(u) - 6$ if not. Of course more complicated systems which cannot be decomposed into quasisplit systems sooner or later come in abundance.

2.) The matroid $2\{1\}_3 + \{1\}_4$ has the orbit counter $o_B(u) = (u-5)(u-3)(u-4)^2$ in every characteristic $\neq 2$ and in characteristi 2 it has $(u-2)(u-4)(u^2 - 10u + 27)$ as orbit counter, both obtainable via the inclusion exclusion count. The difference of the two is $-6u + 24$, which is no longer a constant like in the matroids on 7 points.

3.) Here is a list of isomorphism classes of the rigid rank 3 matroids on 8 points:

a) $\{1,2,4\}_3+\{1,3,7\}_4+\{1,5,6\}_3+\{2,3,5\}_3+\{2,6,7\}_3+\{3,4,6\}_3+\{4,5,7\}_3$ *in characteristic 2 (length of orbit under* S_8 *is 1680).*
b) $\{1,2,4\}_3+\{1,5,6\}_3+\{2,3,5\}_3+\{2,6,7\}_3+\{3,4,6\}_3+\{4,5,8\}_3+\{1,3,7,8\}_4$ *in any characteristic (length of orbit under* S_8 *is 5040).*
c) $\{1,2,4\}_3+\{1,3,7\}_3+\{1,6,8\}_3+\{2,3,8\}_3+\{2,5,6\}_3+\{3,4,5\}_3+\{4,6,7\}_3+\{5,7,8\}_3$ *in characteristic 3.*
The last matroid is not rigid in in characteristics $\neq 3$*: In characteristic 2 one has no solutions and in characteristics* $\neq 2,3$ *one has* 2 *or* 0 *solutions, depending on whether* x^2-x+1 *does or does not split over the ground field, similarly to part 1) of this example. One is tempted to call this situation* Galois*-rigid, since the* Galois *group acts transitively on the solutions. (Length of orbit under* S_8 *is 840.)*

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The First L^2-Betti Number and Approximation in Arbitrary Characteristic[1]

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Abstract. Let G be a finitely generated group and $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$ a descending chain of finite index normal subgroups of G. Given a field K, we consider the sequence $\frac{b_1(G_i;K)}{[G:G_i]}$ of normalized first Betti numbers of G_i with coefficients in K, which we call a K-approximation for $b_1^{(2)}(G)$, the first L^2-Betti number of G. In this paper we address the questions of when $\mathbb{Q}$-approximation and $\mathbb{F}_p$-approximation have a limit, when these limits coincide, when they are independent of the sequence (G_i) and how they are related to $b_1^{(2)}(G)$. In particular, we prove the inequality $\lim_{i\to\infty} \frac{b_1(G_i;\mathbb{F}_p)}{[G:G_i]} \geq b_1^{(2)}(G)$ under the assumptions that $\cap G_i = \{1\}$ and each G/G_i is a finite p-group.

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1. Introduction

1.1. $\mathbb{Q}$-approximation for the first L^2-Betti number. Let G be a finitely generated group. Given a field K, we let $b_1(G;K) = \dim_K(H_1(G;K))$ be the first Betti number of G with coefficients in K and $b_1(G) = b_1(G;\mathbb{Q})$ where $\mathbb{Q}$ denotes the field of rational numbers. Denote by $b_1^{(2)}(G)$ the *first L^2-Betti number of G*. Assuming that G is finitely presented and residually finite, by Lück Approximation Theorem (see [13]), $b_1^{(2)}(G)$ can be approximated by normalized rational first Betti numbers of finite index subgroups of G:

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Theorem 1.1 (Lück approximation theorem). *Let G be a finitely presented residually finite group and $G = G_0 \supseteq G_1 \supseteq \dots$ a descending chain of finite index normal subgroups of G, with $\cap_{i\in\mathbb{N}} G_i = \{1\}$. Then*

$$b_1^{(2)}(G) = \lim_{i\to\infty} \frac{b_1(G_i)}{[G:G_i]}. \tag{1.2}$$

In the sequel we will occasionally refer to a descending chain (G_i) of finite index normal subgroups of G as a *finite index normal chain* in G and to the associated sequence $\left(\frac{b_1(G_i)}{[G:G_i]}\right)_i$ as $\mathbb{Q}$-*approximation.*

If we drop the assumption that G is finitely presented, but still require that $\cap_{i\in\mathbb{N}} G_i = \{1\}$, one still has inequality $b_1^{(2)}(G) \geq \limsup_{i\to\infty} \frac{b_1(G_i)}{[G:G_i]}$ by [16, Theorem 1.1], but equality need not hold [16, Theorem 1.2]. The latter is proved in [16] by constructing an example where $b_1^{(2)}(G) > 0$, but $\limsup_{i\to\infty} \frac{b_1(G_i)}{[G:G_i]} = 0$ for any chain (G_i) as above. In Section 5 we will describe a variation of this construction showing that the $\mathbb{Q}$-approximation $\left(\frac{b_1(G_i)}{[G:G_i]}\right)_i$ may not even have a limit:

Theorem 1.3. *There exists a finitely generated residually finite group G and a descending chain $(G_i)_{i\in\mathbb{N}}$ of finite index normal subgroups of G, with $\cap_{i\in\mathbb{N}} G_i = \{1\}$, such that $\lim_{i\to\infty} \frac{b_1(G_i)}{[G:G_i]}$ does not exist.*

Another sequence we shall be interested in is $\mathbb{F}_p$-*approximation*, that is, $\left(\frac{b_1(G_i;\mathbb{F}_p)}{[G:G_i]}\right)_i$, where $\mathbb{F}_p$ is the finite field of prime order p. This sequence is particularly important under the additional assumption that (G_i) is a p-*chain*, that is, each G_i has p-power index (equivalently, G/G_i is a finite p-group). In this case, $\left(\frac{b_1(G_i;\mathbb{F}_p)}{[G:G_i]}\right)_i$ is monotone decreasing and therefore has a limit, often called *p-gradient* or *mod p homology gradient* (see, e.g., [11]).

Since obviously $b_1(H) \leq b_1(H;\mathbb{F}_p)$ for any group H, one always has inequality

$$\limsup_{i\to\infty} \frac{b_1(G_i)}{[G:G_i]} \leq \limsup_{i\to\infty} \frac{b_1(G_i;\mathbb{F}_p)}{[G:G_i]}, \tag{1.4}$$

and it is natural to ask for sufficient conditions under which equality holds. Of particular interest is the case when G is finitely presented and $\cap_{i\in\mathbb{N}} G_i = \{1\}$ when $\mathbb{Q}$-approximation does have a limit by Theorem 1.1.

Question 1.5 ($\mathbb{Q}$-approximation and $\mathbb{F}_p$-approximation). For which finitely presented groups G and finite index normal chains (G_i) with $\cap_{i\in\mathbb{N}} G_i = \{1\}$ do we have equality

$$\lim_{i\to\infty} \frac{b_1(G_i)}{[G:G_i]} = \lim_{i\to\infty} \frac{b_1(G_i;\mathbb{F}_p)}{[G:G_i]}?$$

If G is not finitely presented, the above equality need not hold even if we require that (G_i) is a p-chain. Indeed, as proved in [18] and independently in [20], there exists a p-torsion residually-p group G with $\lim_{i\to\infty} \frac{b_1(G_i;\mathbb{F}_p)}{[G:G_i]} > 0$

for any p-chain (G_i) in G (and since G is residually-p, we can choose a p-chain with $\cap G_i = \{1\}$). Since $b_1(H) = 0$ for any torsion group H, we have $\lim_{i\to\infty} \frac{b_1(G_i)}{[G:G_i]} = 0$ for such group G.
In Section 4 we give an example showing that the answer to Question 1.5 would also become negative if we drop the assumption $\cap_{i\in\mathbb{N}} G_i = \{1\}$, even if G is finitely presented and (G_i) is a p-chain which has infinitely many distinct terms.

1.2. Comparing $\mathbb{F}_p$-approximation and first L^2-Betti number. Since both $\mathbb{F}_p$-approximation and the first L^2-Betti number provide upper bounds for $\mathbb{Q}$-approximation, it is natural to ask how the former two quantities are related to each other. We address this question in the case of p-chains.

Theorem 1.6. *Let p be a prime number. Let G be a finitely generated group and $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$ a descending chain of normal subgroups of G of p-power index. Then*

(1) The sequence $\left(\frac{b_1(G_i;\mathbb{F}_p)}{[G:G_i]}\right)_i$ is monotone decreasing and therefore converges;
(2) Assume that $\bigcap_{i\in\mathbb{N}} G_i = \{1\}$. Then

$$b_1^{(2)}(G) \leq \lim_{i\to\infty} \frac{b_1(G_i;\mathbb{F}_p)}{[G:G_i]}.$$

We note that for finitely presented groups Theorem 1.6(2) is a straightforward consequence of Theorem 1.1.
We provide two different proofs of Theorem 1.6. First, Theorem 1.6 is a special case of Theorem 2.2, which will be proved in Section 2. An alternative proof of Theorem 1.6 given in Section 3 will be based on Theorem 3.1. The latter may be of independent interest and has another important corollary, which can be considered as an extension of Theorem 1.1 to groups which are finitely presented, but not necessarily residually finite. Here is a slightly simplified version of Theorem 3.1.

Theorem 1.7. *Let G be a finitely presented group, and let K be the kernel of the canonical map from G to its profinite completion or pro-p completion for some prime p. Let (G_i) be a descending chain of finite index normal subgroups of G such that $\cap_{i\in\mathbb{N}} G_i = K$ (note that such a chain always exists). Then*

$$b_1^{(2)}(G/K) = \lim_{i\to\infty} \frac{b_1(G_i)}{[G:G_i]}.$$

1.3. Connection with rank gradient. Let G be a finitely generated group. In the sequel we denote by $d(G)$ the minimal number of generators, sometimes also called *the rank of* G. Let $(G_i)_{i\in\mathbb{N}}$ be a descending chain of finite index normal subgroups of G. The *rank gradient of* G (with respect to (G_i)), denoted by $\mathrm{RG}(G;(G_i))$, is defined by

$$\mathrm{RG}(G;(G_i)) = \lim_{i\to\infty} \frac{d(G_i)-1}{[G:G_i]}. \tag{1.8}$$

The above limit always exists since for any finite index subgroup H of G one has $\frac{d(H)-1}{[G:H]} \leq d(G) - 1$ by the Schreier index formula.
Rank gradient was originally introduced by Lackenby [10] as a tool for studying 3-manifold groups, but is also interesting from a purely group-theoretic point of view (see, e.g., [1, 2, 18, 20]).
Provided that G is infinite and $\bigcap_{i\in\mathbb{N}} G_i = \{1\}$, the following inequalities are known to hold:

$$\mathrm{RG}(G;(G_i)) \geq \mathrm{cost}(G) - 1 \geq b_1^{(2)}(G). \tag{1.9}$$

The first inequality was proved by Abért and Nikolov [2, Theorem 1], and the second one is due to Gaboriau [8, Corollaire 3.16, 3.23] (see [7, 8, 9] for the definition and some key results about cost).
It is not known if either inequality in (1.9) can be strict. In particular, the following question is open.

QUESTION 1.10. Let G be an infinite finitely generated residually finite group and (G_i) a descending chain of finite index normal subgroups of G with $\cap_{i\in\mathbb{N}} G_i = \{1\}$. Is it always true that

$$\mathrm{RG}(G;(G_i)) = b_1^{(2)}(G)?$$

Theorem 1.6 provides a potentially new approach for answering Question 1.10 in the negative, as explained below.
In view of the obvious inequality $d(H) \geq b_1(H;K)$ for any group H and any field K, one always has $\mathrm{RG}(G;(G_i)) \geq \limsup_{i\to\infty} \frac{b_1(G_i;K)}{[G:G_i]}$.

QUESTION 1.11. For which infinite finitely generated groups G, finite index normal chains $(G_i)_{i\in\mathbb{N}}$ with $\bigcap_{i\in\mathbb{N}} G_i = \{1\}$ and fields K, do we have

$$\mathrm{RG}(G;(G_i)) = \limsup_{i\to\infty} \frac{b_1(G_i;K)}{[G:G_i]}? \tag{1.12}$$

REMARK 1.13. Since for a group H, the first Betti number $b_1(H;K)$ depends only on the characteristic of K, one can assume that $K = \mathbb{Q}$ or $K = \mathbb{F}_p$ for some p. The same remark applies to Question 1.14 below.

Note that if $K = \mathbb{Q}$, equality (1.12) does not hold in general – if it did, Theorem 1.3 would have implied the existence of a group G and a finite index normal chain (G_i) in G for which the sequence $\left(\frac{d(G_i)-1}{[G:G_i]}\right)_i$ has no limit, which is impossible since this sequence is monotone decreasing. If one can find a group G for which (1.12) fails with $K = \mathbb{F}_p$ and (G_i) a p-chain, then in view of Theorem 1.6 such group G would answer Question 1.10 in the negative.
The answer to Question 1.11 would become negative if we drop the assumption $\cap G_i = \{1\}$ even if G is finitely presented and (G_i) is a p-chain (with infinitely many distinct terms), as we will see in Section 4.

1.4. Independence of the chain. So far we discussed the dependence of the quantity $\limsup_{i\to\infty} \frac{b_1(G_i;K)}{[G:G_i]}$ on the field K, but perhaps an even more important question is when it is independent of the chain. Again it is reasonable to require that $\bigcap_{i\in\mathbb{N}} G_i = \{1\}$ since without this restriction the answer would be negative already for very nice groups like $F \times \mathbb{Z}$, where F is a non-abelian free group. Note that independence of $\limsup_{i\to\infty} \frac{b_1(G_i;K)}{[G:G_i]}$ of the chain (G_i) as above automatically implies that $\lim_{i\to\infty} \frac{b_1(G_i;K)}{[G:G_i]}$ must exist.

Question 1.14. For which finitely generated residually finite groups G and fields K does the limit $\lim_{i\to\infty} \frac{b_1(G_i;K)}{[G:G_i]}$ exist for all finite index normal chains $(G_i)_{i\in\mathbb{N}}$ with $\bigcap_{i\in\mathbb{N}} G_i = \{1\}$ and is independent of the choice of the chain (G_i)?

The answer to Question 1.14 is known to be positive if $K = \mathbb{Q}$ and either G is finitely presented (by Theorem 1.1) or G is a limit of left orderable amenable groups in the space of marked group presentations, in which case equality (1.2) holds by [19, Corollary 1.5]. Question 1.14 remains open if G is finitely presented and $K = \mathbb{F}_p$. If G is arbitrary, the answer may be negative for any K – this follows directly from Theorem 1.3 if $K = \mathbb{Q}$ and from its stronger version Theorem 5.1 if $K = \mathbb{F}_p$. In the latter case, however, it is natural to impose the additional assumption that (G_i) is a p-chain, which does not hold in our examples.

Essentially the only case when answer to Question 1.14 is known to be positive for all fields is when G contains a normal infinite amenable subgroup (e.g., if G itself is infinite amenable). In this case, $\mathrm{RG}(G;(G_i)) = 0$ for all finite index normal chains (G_i) with trivial intersection, as proved by Lackenby [10, Theorem 1.2] when G is finitely presented and by Abért and Nikolov [2, Theorem 3] in general. This, of course, implies that in such groups $\lim_{i\to\infty} \frac{b_1(G_i;K)}{[G:G_i]} = 0$ for any such chain (G_i) and hence the answer to Questions 1.11 and 1.14 is positive.

Finally, we comment on the status of a more general version of Question 1.14:

Question 1.15. For which residually finite groups G, fields K, finite index normal chains (G_i) with $\bigcap_{i\in\mathbb{N}} G_i = \{1\}$, free G-CW-complexes X of finite type and natural numbers n, does the limit $\lim_{i\to\infty} \frac{b_n(G_i\backslash X;K))}{[G:G_i]}$ exist and is independent of the chain?

Again, if K has characteristic zero, the answer is always yes and the limit can be identified with the n-th L^2-Betti number $b_n^{(2)}(X;\mathcal{N}(G))$ (see [13] or [14, Theorem 13.3 (2) on page 454], which is a generalization of Theorem 1.1). If K has positive characteristic, the answer is yes if G is virtually torsion-free elementary amenable, in which case the limit can be identified with the Ore dimension of $H_n(X;K)$ (see [12, Theorem 5.3]); the answer is also yes for any finitely generated amenable group G – this follows from [1, Theorem 17] or [12, Theorem 2.1] – and the limit can be described using Elek dimension function (see [5]). There are examples for $G = \mathbb{Z}$ of finite G-CW-complexes X where the limits $\lim_{i\to\infty} \frac{b_n(G_i\backslash X;K))}{[G:G_i]}$ are different for $K = \mathbb{Q}$ and $K = \mathbb{F}_p$ (but X is not EG), see [12, Example 6.2].

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2. The first L^2-Betti number and approximation in prime characteristic

If G is a group and X a G-CW-complex, we denote by

$$b_n^{(2)}(X;\mathcal{N}(G)) \quad = \quad \dim_{\mathcal{N}(G)}\big(H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*(X))\big) \tag{2.1}$$

its *n-th L^2-Betti number*. Here $C_*(X)$ is the cellular $\mathbb{Z}G$-chain complex of X, $\mathcal{N}(G)$ is the group von Neumann algebra and $\dim_{\mathcal{N}(G)}$ is the dimension function for (algebraic) $\mathcal{N}(G)$-modules in the sense of [14, Theorem 6.7 on page 239]. Notice that $b_1^{(2)}(G) = b_1^{(2)}(EG;\mathcal{N}(G))$.

The goal of this section is to prove the following theorem which generalizes Theorem 1.6:

Theorem 2.2 (The first L^2-Betti number and $\mathbb{F}_p$-approximation). *Let p be a prime number. Let G be a finitely generated group and (G_i) a descending chain of normal subgroups of p-power index in G. Let $K = \bigcap_{i\in\mathbb{N}} G_i$. Then the sequence $\left(\frac{b_1(G_i;\mathbb{F}_p)}{[G:G_i]}\right)_i$ is monotone decreasing, the limit $\lim_{i\to\infty} \frac{b_1(G_i;\mathbb{F}_p)}{[G:G_i]}$ exists and satisfies*

$$b_1^{(2)}\big(K\backslash EG;\mathcal{N}(G/K)\big) \le \lim_{i\to\infty} \frac{b_1(G_i;\mathbb{F}_p)}{[G:G_i]}.$$

For its proof we will need the following lemma, which is proved in [3, Lemma 4.1], although it was probably well known before.

Lemma 2.3. *Let p be a prime and m,n positive integers. Let H be a finite p-group. Consider an $\mathbb{F}_pH$-map $\alpha\colon \mathbb{F}_pH^m \to \mathbb{F}_pH^n$. Define the $\mathbb{F}_p$-map*

$$\overline{\alpha} = \mathrm{id}_{\mathbb{F}_p} \otimes_{\mathbb{F}_pH} \alpha\colon \mathbb{F}_p^m = \mathbb{F}_p \otimes_{\mathbb{F}_pH} \mathbb{F}_pH^m \to \mathbb{F}_p^n = \mathbb{F}_p \otimes_{\mathbb{F}_pH} \mathbb{F}_pH^n,$$

where we consider $\mathbb{F}_p$ as $\mathbb{F}_pH$-module by the trivial H-action. Then

$$\dim_{\mathbb{F}_p}(\mathrm{im}(\alpha)) \ge |H| \cdot \dim_{\mathbb{F}_p}(\mathrm{im}(\overline{\alpha})).$$

Notice that the assertion of Lemma 2.3 is not true if we do not require that H is a p-group or if we replace $\mathbb{F}_p$ by a field of characteristic not equal to p.

Proof of Theorem 2.2. Since G is finitely generated, there is a CW-model for BG with one 0-cell and a finite number, let us say s, of 1-cells. Let $EG \to BG$ be the universal covering. Put $X = K\backslash EG$ and $Q = G/K$. Then X is a free Q-CW-complex with finite 1-skeleton. Its cellular $\mathbb{Z}Q$-chain complex $C_*(X)$ looks like

$$\cdots \to C_2(X) = \bigoplus_{j=1}^{r} \mathbb{Z}Q \xrightarrow{c_2} C_1(X) = \bigoplus_{j=1}^{s} \mathbb{Z}Q \xrightarrow{c_1} C_0(X) = \mathbb{Z}Q$$

where r is a finite number or infinity.
For $m = 0,1,2,\dots$ we define a $\mathbb{Z}Q$-submodule of $C_2(X)$ by $C_2(X)|_m = \bigoplus_{j=1}^{\max\{m,r\}} \mathbb{Z}Q$. Denote by $c_2|_m\colon C_2(X)|_m \to C_1(X)$ the restriction of c_2 to $C_2(X)|_m$.
Consider a $\mathbb{Z}Q$-map $f\colon M \to N$. Denote by $f^{(2)}\colon M^{(2)} \to N^{(2)}$ the $\mathcal{N}(Q)$-homomorphism $\mathrm{id}_{\mathcal{N}(G)} \otimes_{\mathbb{Z}Q} f\colon \mathcal{N}(Q) \otimes_{\mathbb{Z}Q} M \to \mathcal{N}(Q) \otimes_{\mathbb{Z}Q} N$. Put $Q_i = G_i/K$. Let $f[i]\colon M[i] \to N[i]$ be the $\mathbb{Q}$-homomorphism $\mathrm{id}_{\mathbb{Q}} \otimes f\colon \mathbb{Q} \otimes_{\mathbb{Z}[Q_i]} M \to \mathbb{Q} \otimes_{\mathbb{Z}[Q_i]} N$. Denote by $f[i,p]\colon M[i,p] \to N[i,p]$ the $\mathbb{F}_p$-homomorphism $\mathrm{id}_{\mathbb{F}_p} \otimes_{\mathbb{Z}[Q_i]} f\colon \mathbb{F}_p \otimes_{\mathbb{Z}[Q_i]} M \to \mathbb{F}_p \otimes_{\mathbb{Z}[Q_i]} N$. If $M = \bigoplus_{j=1}^{t} \mathbb{Z}Q$, then $M^{(2)} = \bigoplus_{j=1}^{t} \mathcal{N}(Q)$, $M[i] = \bigoplus_{j=1}^{t} \mathbb{Z}[Q/Q_i]$ and $M[i,p] = \bigoplus_{j=1}^{t} \mathbb{F}_p[Q/Q_i]$.
Note that

$$b_1(Q_i\backslash X;\mathbb{F}_p) = b_1(G_i\backslash EG;\mathbb{F}_p) = b_1(BG_i;\mathbb{F}_p) = b_1(G_i;\mathbb{F}_p).$$

Since all dimension functions are additive (see [14, Theorem 6.7 on page 239]), we conclude

$$b_1^{(2)}\big(X;\mathcal{N}(Q)\big) = s - 1 - \dim_{\mathcal{N}(Q)}\big(\mathrm{im}(c_2^{(2)})\big); \tag{2.4}$$

$$\frac{b_1\big(G_i;\mathbb{F}_p\big)}{[Q:Q_i]} = s - 1 - \frac{\dim_{\mathbb{F}_p}\big(\mathrm{im}(c_2[i,p])\big)}{[Q:Q_i]}; \tag{2.5}$$

$$\dim_{\mathcal{N}(Q)}\big(\mathrm{im}(c_2|_m^{(2)})\big) = m - \dim_{\mathcal{N}(Q)}\big(\ker(c_2|_m^{(2)})\big); \tag{2.6}$$

$$\frac{\dim_{\mathbb{Q}}\big(\mathrm{im}(c_2|_m[i])\big)}{[Q:Q_i]} = m - \frac{\dim_{\mathbb{Q}}\big(\ker(c_2|_m[i])\big)}{[Q:Q_i]}; \tag{2.7}$$

$$\frac{\dim_{\mathbb{F}_p}\big(\mathrm{im}(c_2|_m[i,p])\big)}{[Q:Q_i]} = m - \frac{\dim_{\mathbb{F}_p}\big(\ker(c_2|_m[i,p])\big)}{[Q:Q_i]}. \tag{2.8}$$

There is an isomorphism of $\mathbb{F}_p$-chain complexes $\mathbb{F}_p \otimes_{\mathbb{F}_p[Q_{i+1}\backslash Q_i]} C_*(X)[(i+1),p] \xrightarrow{\cong} C_*(X)[i,p]$, where the $Q_{i+1}\backslash Q_i$-operation on $C_*(X)[i+1]$ comes from the identification $C_*(X)[i+1] = \mathbb{F}_p \otimes_{\mathbb{F}_p[Q_{i+1}]} C_*(X) = \mathbb{F}_p[Q_{i+1}\backslash Q] \otimes_{\mathbb{F}_pQ} C_*(X)$. This is compatible with the passage from $C_2(X)$ to $C_2(X)|_m$. Hence $c_2|_m[i,p]$ can be identified with $\mathrm{id}_{\mathbb{F}_p} \otimes_{\mathbb{F}_p[Q_{i+1}\backslash Q_i]} c_2|_m[(i+1),p]$. Since $Q_{i+1}\backslash Q_i$ is a finite p-group, Lemma 2.3 implies

$$\dim_{\mathbb{F}_p}\big(\mathrm{im}(c_2|_m[(i+1),p])\big) \geq [Q_i : Q_{i+1}] \cdot \dim_{\mathbb{F}_p}\big(\mathrm{im}(c_2|_m[i,p])\big).$$

We conclude

$$\frac{\dim_{\mathbb{F}_p}\big(\mathrm{im}(c_2|_m[(i+1),p])\big)}{[Q:Q_{i+1}]} \geq \frac{\dim_{\mathbb{F}_p}\big(\mathrm{im}(c_2|_m[i,p])\big)}{[Q:Q_i]}. \tag{2.9}$$

Since $\mathrm{im}(c_2^{(2)}) = \bigcup_m \mathrm{im}(c_2|_m^{(2)})$ and $\mathrm{im}(c_2[i,p]) = \bigcup_m \mathrm{im}(c_2|_m[i,p])$ and the dimension functions are compatible with directed unions (see [14, Theorem 6.7 on page 239]), we get

$$\dim_{\mathcal{N}(Q)}\bigl(\mathrm{im}(c_2^{(2)})\bigr) = \lim_{m\to\infty} \dim_{\mathcal{N}(Q)}\bigl(\mathrm{im}(c_2|_m^{(2)})\bigr); \tag{2.10}$$

$$\dim_{\mathbb{F}_p}\bigl(\mathrm{im}(c_2[i,p])\bigr) = \lim_{m\to\infty} \dim_{\mathbb{F}_p}\bigl(\mathrm{im}(c_2|_m[i,p])\bigr). \tag{2.11}$$

We conclude from [14, Theorem 13.3 (2) on page 454 and Lemma 13.4 on page 455]

$$\lim_{i\to\infty} \frac{\dim_{\mathbb{Q}}\bigl(\ker(c_2|_m[i])\bigr)}{[Q:Q_i]} = \dim_{\mathcal{N}(Q)}\bigl(\ker(c_2|_m^{(2)})\bigr).$$

This implies together with (2.6) and (2.7)

$$\lim_{i\to\infty} \frac{\dim_{\mathbb{Q}}\bigl(\mathrm{im}(c_2|_m[i])\bigr)}{[Q:Q_i]} = \dim_{\mathcal{N}(Q)}\bigl(\mathrm{im}(c_2|_m^{(2)})\bigr). \tag{2.12}$$

Finally, it is easy to see that

$$\dim_{\mathbb{Q}}\bigl(\mathrm{im}(c_2|_m[i])\bigr) \geq \dim_{\mathbb{F}_p}\bigl(\mathrm{im}(c_2|_m[i,p])\bigr). \tag{2.13}$$

Putting everything together, we can now prove both assertions of Theorem 2.2.

First, for a fixed m, the sequence $\left(\frac{\dim_{\mathbb{F}_p}\left(\mathrm{im}(c_2|_m[i,p])\right)}{[Q:Q_i]}\right)_i$ is monotone increasing by (2.9), whence the sequence $\left(\frac{\dim_{\mathbb{F}_p}\left(\mathrm{im}(c_2[i,p])\right)}{[Q:Q_i]}\right)_i$ is also monotone increasing by (2.11) and therefore the sequence $\left(\frac{b_1(G_i;\mathbb{F}_p)}{[Q:Q_i]}\right)_i$ is monotone decreasing by (2.5). This proves the first assertion of Theorem 2.2 since clearly $[Q:Q_i] = [G:G_i]$.

Inequality (2.9) also implies that $\lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p}\left(\mathrm{im}(c_2|_m[i,p])\right)}{[Q:Q_i]} \geq \frac{\dim_{\mathbb{F}_p}\left(\mathrm{im}(c_2|_m[j,p])\right)}{[Q:Q_j]}$ for any fixed j and m, and so

$$\lim_{m\to\infty}\lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p}\bigl(\mathrm{im}(c_2|_m[i,p])\bigr)}{[Q:Q_i]} \geq \sup_{i\geq 0}\left\{\lim_{m\to\infty} \frac{\dim_{\mathbb{F}_p}\bigl(\mathrm{im}(c_2|_m[i,p])\bigr)}{[Q:Q_i]}\right\}. \tag{2.14}$$

Therefore,

$$\begin{aligned}
b_1^{(2)}(X;\mathcal{N}(Q)) &\overset{(2.4)}{=} s-1-\dim_{\mathcal{N}(Q)}\bigl(\mathrm{im}(c_2^{(2)})\bigr)\\
&\overset{(2.10)}{=} s-1-\lim_{m\to\infty}\dim_{\mathcal{N}(Q)}\bigl(\mathrm{im}(c_2|_m^{(2)})\bigr)\\
&\overset{(2.12)}{=} s-1-\lim_{m\to\infty}\lim_{i\to\infty}\frac{\dim_{\mathbb{Q}}\bigl(\mathrm{im}(c_2|_m[i])\bigr)}{[Q:Q_i]}\\
&\overset{(2.13)}{\leq} s-1-\lim_{m\to\infty}\lim_{i\to\infty}\frac{\dim_{\mathbb{F}_p}\bigl(\mathrm{im}(c_2|_m[i,p])\bigr)}{[Q:Q_i]}\\
&\overset{(2.14)}{\leq} s-1-\sup_{i\geq 0}\left\{\lim_{m\to\infty}\frac{\dim_{\mathbb{F}_p}\bigl(\mathrm{im}(c_2|_m[i,p])\bigr)}{[Q:Q_i]}\right\}\\
&\overset{(2.11)}{=} s-1-\sup_{i\geq 0}\left\{\frac{\dim_{\mathbb{F}_p}\bigl(\mathrm{im}(c_2[i,p])\bigr)}{[Q:Q_i]}\right\}\\
&= \inf_{i\geq 0}\left\{s-1-\frac{\dim_{\mathbb{F}_p}\bigl(\mathrm{im}(c_2[i,p])\bigr)}{[Q:Q_i]}\right\}\\
&\overset{(2.5)}{=} \inf_{i\geq 0}\left\{\frac{b_1(G_i;\mathbb{F}_p)}{[Q:Q_i]}\right\}.
\end{aligned}$$

This finishes the proof of Theorem 2.2. □

3. Alternative proof of Theorem 1.6

In this section we give an alternative proof of Theorem 1.6. Namely, Theorem 1.6 is an easy consequence of the following result, which may be useful in its own right.

Theorem 3.1. *Let G be a finitely presented group, let (G_i) be a descending chain of finite index normal subgroups of G, and let $K=\bigcap_{i=1}^{\infty}G_i$.*

(1) The following inequalities hold:

$$\lim_{i\to\infty}\frac{b_1(G_i/K)}{[G:G_i]}\leq b_1^{(2)}(G/K)\leq b_1^{(2)}\bigl(K\backslash EG;\mathcal{N}(G/K)\bigr)=\lim_{n\to\infty}\frac{b_1(G_i)}{[G:G_i]}.$$

(2) Let $\mathcal{C}$ be any class of finite groups which is closed under subgroups, extensions (and isomorphisms) and contains at least one non-trivial group (for instance, $\mathcal{C}$ could be the class of all finite groups or all finite p-groups for a fixed prime p). Assume that K is the kernel of the canonical map from G to its pro-$\mathcal{C}$ completion. Then

$$b_1^{(2)}(G/K)=\lim_{i\to\infty}\frac{b_1(G_i)}{[G:G_i]}.$$

If in addition all groups G/G_i are in $\mathcal{C}$, then

(3.2)
$$\lim_{i\to\infty} \frac{b_1(G_i/K)}{[G:G_i]} = b_1^{(2)}(G/K) = b_1^{(2)}\big(K\backslash EG;\mathcal{N}(G/K)\big) = \lim_{i\to\infty} \frac{b_1(G_i)}{[G:G_i]}.$$

Proof. (1) Since G is finitely presented, there is a G-CW-model for the classifying space BG whose 2-skeleton is finite. Let $EG \to BG$ be the universal covering. Then EG is a free G-CW-complex with finite 2-skeleton. Put

$$Q = G/K;$$
$$Q_i = G_i/K.$$

Then $Q = Q_0 \supseteq Q_1 \supseteq \cdots$ is a descending chain of finite index normal subgroups of Q with $\bigcap_{i=0}^{\infty} Q_i = \{1\}$ and we have for $i = 0, 1, 2, \ldots$

$$[G:G_i] \quad = \quad [Q:Q_i]. \tag{3.3}$$

The quotient $X = K\backslash EG$ is a free Q-CW-complex whose 2-skeleton is finite. Let X_2 be the 2-skeleton of X. Since the first L^2-Betti number and the first Betti number depend only on the 2-skeleton, from [13, Theorem 0.1] applied to the G-covering $X_2 \to X_2/G$ (we do not need X_2 to be simply connected) or directly from [14, Theorem 13.3 on page 454], we obtain

$$b_1^{(2)}(X;\mathcal{N}(Q)) \quad = \quad \lim_{i\to\infty} \frac{b_1(Q_i\backslash X)}{[Q:Q_i]}. \tag{3.4}$$

Let $f\colon X \to EQ$ be the classifying map. Since EQ is simply connected, this map is 1-connected. This implies by [14, Theorem 6.54 (1a) on page 265]

$$b_1^{(2)}(X;\mathcal{N}(Q)) \quad \geq \quad b_1^{(2)}(EQ;\mathcal{N}(Q)). \tag{3.5}$$

The group Q is finitely generated (but not necessarily finitely presented), so by [16, Theorem 1.1] we have

$$\lim_{i\to\infty} \frac{b_1(Q_i)}{[Q:Q_i]} \quad \leq \quad b_1^{(2)}(Q). \tag{3.6}$$

Notice that $b_1^{(2)}(Q) = b_1^{(2)}(EQ;\mathcal{N}(Q))$ by definition and we obviously have $Q_i\backslash X = G_i\backslash EG = BG_i$ and hence $b_1(Q_i\backslash X) = b_1(G_i)$. Combining (3.3), (3.4), (3.5), and (3.6), we get

$$\lim_{i\to\infty} \frac{b_1(Q_i)}{[Q:Q_i]} \leq b_1^{(2)}(Q) \leq b_1^{(2)}(X;\mathcal{N}(Q)) = \lim_{i\to\infty} \frac{b_1(Q_i\backslash X)}{[Q:Q_i]} = \lim_{i\to\infty} \frac{b_1(G_i)}{[G:G_i]}.$$

This finishes the proof of assertion (1).

(2) First observe that since $b_1^{(2)}\big(K\backslash EG;\mathcal{N}(G/K)\big) = \lim\limits_{i\to\infty} \frac{b_1(G_i)}{[G:G_i]}$ by (1), the limit $\lim\limits_{i\to\infty} \frac{b_1(G_i)}{[G:G_i]}$ is the same for all finite index normal chains (G_i) with $\cap_{i\in\mathbb{N}} G_i = K$.By definition of K, there exists at least one such chain with $G/G_i \in \mathcal{C}$ for all i (e.g., we can let (G_i) be a base of neighborhoods of 1 for the pro-$\mathcal{C}$ topology on G), so it suffices to prove (3.2). Thus, from now on we will assume that $G/G_i \in \mathcal{C}$ for $i \in \mathbb{N}$.

For a finitely generated group H we denote by H' the kernel of the composite of canonical projections $H \to H_1(H) \to H_1(H)/\operatorname{tors}(H_1(H))$, so that H/H' is a free abelian group of rank $b_1(H)$.
As in the proof of (1), we put $Q_i = G_i/K$ for $i \in \mathbb{N}$. It is sufficient to prove that that $K \subseteq G_i'$ for $i \in \mathbb{N}$. Indeed, this would imply that $Q_i/Q_i' \cong G_i/G_i'$, whence $b_1(Q_i) = b_1(G_i)$ and therefore $\lim_{i\to\infty} \frac{b_1(Q_i)}{[G:G_i]} = \lim_{i\to\infty} \frac{b_1(G_i)}{[G:G_i]}$, which proves (2) in view of (1).

Fix $i \in \mathbb{N}$ and let $H = G_i$. Since $\mathcal{C}$ contains at least one non-trivial finite group and is closed under subgroups, it contains a finite cyclic group, say of order k. Since $\mathcal{C}$ is closed under extensions, it contains $(\mathbb{Z}/k^m\mathbb{Z})^b$ for all $m, b \in \mathbb{N}$. Setting $b = b_1(H)$, we get that $H/H'H^{k^m} \in \mathcal{C}$ for all $m \in \mathbb{N}$, and since $\mathcal{C}$ is closed under extensions, we obtain $G/H'H^{k^m} \in \mathcal{C}$. By definition, K is the intersection of all normal subgroups L of G with $G/L \in \mathcal{C}$. Therefore, $K \subseteq \bigcap\limits_{m\in\mathbb{N}} H'H^{k^m} = H'$. □

Second proof of Theorem 1.6.
(1) This is a direct consequence of the following well-known fact: if H is a normal subgroup of p-power index in G, then $b_1(H;\mathbb{F}_p)-1 \leq [G:H](b_1(G;\mathbb{F}_p)-1)$ (see, e.g., [11, Proposition 3.7]).

(2) Choose an epimorphism $\pi\colon F \to G$, where F is a finitely generated free group. Fix $n \in \mathbb{N}$, let $F_n = \pi^{-1}(G_n)$ and $H = [F_n, F_n]F_n^p$. Then H is a finite index subgroup of F, so we can choose a presentation (X, R) of G associated with π such that $R = R_1 \sqcup R_2$, where R_1 is finite and $R_2 \subseteq H$.
Consider the finitely presented group $\widetilde{G} = \langle X \mid R_1\rangle$. We have natural epimorphisms $\phi\colon \widetilde{G} \to G$ and $\psi\colon F \to \widetilde{G}$, with $\phi\psi = \pi$. If we let $\widetilde{G}_i = \phi^{-1}(G_i)$ and $\widetilde{K} = \bigcap_{i=1}^{\infty} \widetilde{G}_i$, then $\widetilde{G}/\widetilde{K} \cong G$. Thus, applying Theorem 3.1 (1) to the group $\widetilde{G}$ and its subgroups $(\widetilde{G}_i)$, we get $b_1^{(2)}(G) \leq \lim_{i\to\infty} \frac{b_1(\widetilde{G}_i)}{[\widetilde{G}:\widetilde{G}_i]}$. Clearly, $\lim_{i\to\infty} \frac{b_1(\widetilde{G}_i)}{[\widetilde{G}:\widetilde{G}_i]} \leq \lim_{i\to\infty} \frac{b_1(\widetilde{G}_i;\mathbb{F}_p)}{[\widetilde{G}:\widetilde{G}_i]}$, and by assertion (1),

$$\lim_{i\to\infty} \frac{b_1(\widetilde{G}_i;\mathbb{F}_p)}{[\widetilde{G}:\widetilde{G}_i]} \leq \frac{b_1(\widetilde{G}_n;\mathbb{F}_p)}{[\widetilde{G}:\widetilde{G}_n]} = \frac{b_1(\widetilde{G}_n;\mathbb{F}_p)}{[G:G_n]}.$$

Since $G \cong \widetilde{G}/\langle\langle\psi(R_2)\rangle\rangle$ and by construction $\psi(R_2) \subseteq \psi(H) = [\widetilde{G}_n, \widetilde{G}_n]\widetilde{G}_n^p$, we have $\ker\phi \subseteq [\tilde{G}_n, \tilde{G}_n]\tilde{G}_n^p$, and therefore $b_1(\widetilde{G}_n;\mathbb{F}_p) = b_1(\phi(\widetilde{G}_n);\mathbb{F}_p) = b_1(G_n;\mathbb{F}_p)$.
Combining these inequalities, we get $b_1^{(2)}(G) \leq \frac{b_1(G_n;\mathbb{F}_p)}{[G:G_n]}$. Since n is arbitrary, the proof is complete. □

4. A counterexample with non-trivial intersection

In this section we show that the answer to Questions 1.5 and 1.11 could be negative for a finitely presented group G and a strictly descending chain $(G_i)_{i\in\mathbb{N}}$

of normal subgroups of p-power index if the intersection $\cap_{i\in\mathbb{N}}G_i$ is non-trivial (see inequalities (4.2) below).

We start with a finitely generated group H (which will be specified later) and let $G = H * \mathbb{Z}$. Choose a strictly increasing sequence of positive integers $n_1, n_2, \ldots$ with $n_i \mid n_{i+1}$ for each i, and let $G_i \subseteq G$ be the preimage of $n_i \cdot \mathbb{Z}$ under the natural projection $\mathrm{pr}\colon G = \mathbb{Z} * H \to \mathbb{Z}$. Then $(G_i)_{i\in\mathbb{N}}$ is a descending chain of normal subgroups of G with $\bigcap_{i\geq 1} G_i = \ker(\mathrm{pr})$. Let $BG_i \to BG$ be the covering of BG associated to $G_i \subseteq G$. Then BG_i is homeomorphic to $S^1 \vee \left(\bigvee_{j=1}^{n_i} BH\right)$. We have

$$G_i \cong \pi_1(BG_i) \cong \pi_1\left(S^1 \vee \left(\bigvee_{j=1}^{n_i} BH\right)\right) \cong \mathbb{Z} * (*_{j=1}^{n_i} H).$$

Since for any groups A and B we have $A*B/[A*B, A*B] \cong A/[A,A] \oplus B/[B,B]$ and $d(A*B) = d(A) + d(B)$ by Grushko-Neumann theorem (see [4, Corollary 2 in Section 8.5 on page 227], we conclude

$$\begin{aligned}
H_1(G_i;K) &= K \oplus \bigoplus_{j=1}^{n_i} H_1(H;K);\\
H_1(G_i) &= \mathbb{Z} \oplus \bigoplus_{j=1}^{n_i} H_1(H);\\
d(G_i) &= 1 + n_i \cdot d(H);\\
\lim_{i\to\infty} \frac{b_1(G_i;K)}{n_i} &= b_1(H;K);\\
\lim_{i\to\infty} \frac{d(H_1(G_i))}{n_i} &= d(H_1(H));\\
\mathrm{RG}(G;(G_i)_{i\geq 1}) &= d(H).
\end{aligned}$$

Now let $p \neq q$ be distinct primes and $H = \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$. Clearly we have

$$b_1(H) = 0, \quad b_1(H;\mathbb{F}_p) = 1, \quad d(H_1(H)) = 2, \quad d(H) = 3. \tag{4.1}$$

Hence we obtain

$$\lim_{i\to\infty} \frac{b_1(G_i)}{[G:G_i]} < \lim_{i\to\infty} \frac{b_1(G_i;\mathbb{F}_p)}{[G:G_i]} < \lim_{i\to\infty} \frac{d(H_1(G_i))}{[G:G_i]} < \mathrm{RG}(G;(G_i)_{i\geq 1}). \tag{4.2}$$

Using a different H we can produce an example of this type where G has a very strong finiteness property, namely, G has finite 2-dimensional BG. The construction below is due to Denis Osin and is simpler and more explicit than the original version of our example.

Again, let $p \neq q$ be two primes. Consider the group

$$H = \langle x, y, z \mid x^p = u, y^q = v, z^q = w\rangle,$$

where u, v, w are words from the commutator subgroup of the free group F with basis x, y, z such that the presentation of H satisfies the $C'(1/6)$ small

cancellation condition. Such words are easy to find explicitly. Note that $G = H * \mathbb{Z}$ is a torsion-free $C'(1/6)$ group, hence it has a finite 2-dimensional BG. Since $u, v, w \in [F, F]$, we have $b_1(H) = 0$, $b_1(H; \mathbb{F}_p) = 1$, $d(H_1(H)) = 2$. Further it follows from [6, Corollary 2] that the exponential growth rate of H can be made arbitrarily close to $2 \cdot 3 - 1 = 5$, the exponential growth rate of the free group of rank 3, by taking sufficiently long words u, v, w. As the exponential growth rate of an m-generated group is bounded from above by $2m - 1$, we obtain $d(H) = 3$ whenever u, v, w are sufficiently long. (For details about the exponential growth rate we refer to [6].)
By using a more elaborated construction from [21], one can make such a group G the fundamental group of a compact 2-dimensional $CAT(-1)$ CW-complex. Other examples of this type can be found in [3] and [15].

5. $\mathbb{Q}$-approximation without limit

In this section we prove the following theorem, which trivially implies Theorem 1.3.

Theorem 5.1. *Let $d \geq 2$ be a positive integer, let p be a prime and let ε be a real number satisfying $0 < \varepsilon < 1$. Then there exist a group G with d generators and a descending chain $G = G_0 \supseteq G_1 \supseteq G_2 \ldots$ of normal subgroups of G of p-power index with $\bigcap_{i=1}^{\infty} G_i = \{1\}$ with the following properties:*

(i) $\liminf_{i\to\infty} \frac{b_1(G_{2i})}{[G:G_{2i}]} \geq d - 1 - \varepsilon$;

(ii) $\lim_{i\to\infty} \frac{b_1(G_{2i-1})}{[G:G_{2i-1}]} = 0$.

Moreover, if q is a prime different from p, we can replace (ii) by a stronger condition (ii)':

(ii') $\lim_{i\to\infty} \frac{b_1(G_{2i-1};\mathbb{F}_q)}{[G:G_{2i-1}]} = 0$.

Note that the last assertion of Theorem 5.1 shows that the answer to Question 1.14 can be negative when $\mathrm{char}(K) = q > 0$ if we do not require that (G_i) is a q-chain.

5.1. Preliminaries. Throughout this section p will be a fixed prime number. Given a finitely generated group G, we will denote by $G_{\hat{p}}$ the pro-p completion of G and by $G_{(p)}$ the image of G in $G_{\hat{p}}$ (which is isomorphic to the quotient of G by the intersection of normal subgroups of p-power index). Given a set X, by $F(X)$ we denote the free group on X.
Let F be a free group and $w \in F$ a non-identity element. Given $n \in \mathbb{N}$, denote by $\sqrt[n]{w}$ the unique element of F whose n^{th} power is equal to w (if such element exists). Define $e_p(w, F)$ to be the largest natural number e with the property that $\sqrt[p^e]{w}$ exists in F.

Lemma 5.2. *Let (X, R) be a presentation of a group G with X finite, $F = F(X)$ and $\pi\colon F \to G$ the natural projection. Let H be a normal subgroup of p-power index in G, and let $F_H = \pi^{-1}(H)$. Then $H = F_H/\langle\langle R_H\rangle\rangle$ where R_H contains $\frac{[G:H]}{p^{e_p(r,F)-e_p(r,F_H)}}$ F-conjugates of r for each $r \in R$ and no other elements.*

Proof. Very similar results are proved in both [18] and [20], but for completeness we give a proof. For each $r \in R$, write $r = w(r)^{p^{e_p(r,F)}}$, and choose a right transversal $T = T(r)$ for $\langle w(r)\rangle F_H$ in F. Then, since $w(r)$ commutes with r, by [17, Lemma 2.3] we have $\langle r\rangle^F = \langle\{t^{-1}rt : t \in T\}\rangle^{F_H}$. Hence $\langle\{t^{-1}rt : r \in R, t \in T(R)\}\rangle^{F_H} = \langle R\rangle^F = \ker\pi = \ker(F_H \to H)$, and so it suffices to prove that $|T(r)| = \frac{[G:H]}{p^{e_p(r,F)-e_p(r,F_H)}}$.
We have

$$|T(r)| = [F : \langle w(r)\rangle F_H] = \frac{[F : F_H]}{[\langle w(r)\rangle F_H : F_H]} = \frac{[G : H]}{[\langle w(r)\rangle : \langle w(r)\rangle \cap F_H]}$$

Finally note that $[\langle w(r)\rangle : \langle w(r)\rangle \cap F_H]$ is equal to p^k for some k (as it divides $[F : F_H] = p^n$), so $\langle w(r)\rangle \cap F_H = \langle w(r)^{p^k}\rangle$. But then from definition of $e_p(r, F_H)$ we easily conclude that $((w(r)^{p^k})^{p^{e_p(r,F_H)}} = r = w(r)^{p^{e_p(r,F)}}$. Hence $k = e_p(r, F) - e_p(r, F_H)$ and $|T(r)| = \frac{[G:H]}{p^{e_p(r,F)-e_p(r,F_H)}}$, as desired. □

The following definition was introduced by Schlage-Puchta in [20].

Definition 5.3. Given a group presentation by generators and relators (X, R), where X is finite, its *p-deficiency* $\mathrm{def}_p(X, R) \in \mathbb{R} \cup \{-\infty\}$ is defined by

$$\mathrm{def}_p(X, R) = |X| - 1 - \sum_{r\in R} \frac{1}{p^{e_p(r,F(X))}}.$$

The p-deficiency of a finitely generated group G is the supremum of the set $\{\mathrm{def}_p(X, R)\}$ where (X, R) ranges over all presentations of G.

The main motivation for introducing p-deficiency in [20] was to construct a finitely generated p-torsion group with positive rank gradient. Indeed, it is clear that there exist p-torsion groups with positive p-deficiency, and in [20] it is proved that a group with positive p-deficiency has positive rank gradient (in fact, positive p-gradient). This is one of the results indicating that groups of positive p-deficiency behave similarly to groups of deficiency greater than 1 (all of which trivially have positive p-deficiency for any p).
Lemma 5.5 below shows that a finitely presented group G of positive p-deficiency actually contains a normal subgroup of p-power index with deficiency greater than 1, provided that the presentation of G yielding positive p-deficiency is finite and satisfies certain technical condition.

Definition 5.4. A presentation (X, R) of a group G will be called *p-regular* if for any $r \in R$ such that $\sqrt[p]{r}$ exists in $F(X)$, the image of $\sqrt[p]{r}$ in $G_{(p)}$ is non-trivial. This is equivalent to saying that if we write each $r \in R$ as $r = v^{p^e}$, where v is not a p^{th} power in $F(X)$, then the image of v in $G_{(p)}$ has order p^e.

Lemma 5.5. *Let (X, R) be a finite p-regular presentation of a group G. Then there exists a normal subgroup of p-power index H of G with $\frac{\mathrm{def}(H)-1}{[G:H]} \geq \mathrm{def}_p(X, R)$.*

Proof. Let $F = F(X)$. Let $r_1, \ldots, r_m$ be the elements of R and let $s_i = \sqrt[p]{r_i}$, whenever it is defined in $F(X)$.
Let $\pi\colon F \to G_{(p)}$ be the natural projection. Since the presentation (X, R) is p-regular, $\pi(s_i)$ is non-trivial whenever s_i is defined, and since the group $G_{(p)}$ is residually-p, there exists a normal subgroup H' of $G_{(p)}$ of p-power index which contains none of the elements $\pi(s_i)$.
Let $F_H = \pi^{-1}(H')$. By construction, $s_i \notin F_H$, but $r_i \in F_H$, and therefore $e_p(r_i, F_H) = 0$ for each i. Let H be the image of F_H in G. Then by Lemma 5.2, H has a presentation with $d(F_H)$ generators and $\sum_{i=1}^m \frac{[G:H]}{p^{e_p(r_i,F)}}$ relators. Since $d(F_H) - 1 = (|X| - 1)[F : F_H] = (|X| - 1)[G : H]$ by the Schreier formula, we get

$$\operatorname{def}(H) - 1 \geq [G : H] \cdot \left(|X| - 1 - \sum_{i=1}^{m} p^{-e_p(r_i,F)} \right) = [G : H] \cdot \operatorname{def}_p(X, R).$$

□

Lemma 5.6. *Let (X, R) be a finite p-regular presentation, and let $G = \langle X|R \rangle$. Let $f \in F(X)$ be such that the image of f in the pro-p completion of G has infinite order. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ the presentation $(X, R \cup \{f^{p^n}\})$ is p-regular.*

Proof. Let $r_1, \ldots, r_m$ be the elements of R. By assumption there is a normal subgroup of p-power index H of G such that $\sqrt[p]{r_i}$ does not vanish in G/H (whenever $\sqrt[p]{r_i}$ exists in $F(X)$). Let $\pi\colon F(X) \to G$ be the natural projection, and choose $N \in \mathbb{N}$ satisfying $\pi(f^{p^N}) \in H$.
Let $n \geq N$, let $g = \pi(f)$, and let $G' = G/\langle\langle g^{p^n} \rangle\rangle = \langle X | R \cup \{f^{p^n}\} \rangle$. We claim that the presentation $(X, R \cup \{f^{p^n}\})$ is p-regular. We need to check that

(i) each $\sqrt[p]{r_i}$ does not vanish in $G'_{\hat{p}}$
(ii) $f^{p^{n-1}}$ does not vanish in $G'_{\hat{p}}$

The kernel of the natural map $G \to G'_{\hat{p}}$ is contained in H since $g^{p^n} \in H$ and G/H is a finite p-group. Since $\pi(\sqrt[p]{r_i}) \notin H$, this implies (i). Further, an element $x \neq 1$ of a pro-p group cannot lie in the closed normal subgroup generated by x^p. Hence if $\hat{g}$ is the image of g (also the image of f) in $G_{\hat{p}}$, then $\hat{g}^{p^{n-1}}$ does not lie in the closed normal subgroup of $G_{\hat{p}}$ generated by $\hat{g}^{p^n}$, call this subgroup C. Finally, by definition of G', there is a canonical isomorphism from $G_{\hat{p}}/C$ to $G'_{\hat{p}}$, which maps the image of f in $G_{\hat{p}}/C$ to the image of f in $G'_{\hat{p}}$. Thus, we verified (ii). □

Corollary 5.7. *Let (X, R) be a finite p-regular presentation, and let $G = \langle X \mid R \rangle$. Let $H \subseteq K$ be normal subgroups of $F(X)$ of p-power index, and let $\delta > 0$ be a real number. Then there exists a finite set $R' \subset [K, K]$ with $\sum\limits_{r \in R'} p^{-e_p(r,F(X))} < \delta$ such that*

(1) the presentation $(X, R \cup R')$ is p-regular;

(2) if $G' = \langle X \mid R \cup R' \rangle$ and H' is the image of H in G', then $b_1(H') \leq d(K)$.

Moreover, if q is a prime different from p, we can require that $b_1(H'; \mathbb{F}_q) \leq d(K)$.

Proof. If $b_1(H; \mathbb{F}_q) \leq d(K)$, we can choose $R' = \emptyset$. Hence we can assume without loss of generality that $b_1(H; \mathbb{F}_q) > d(K)$. Clearly, it suffices to prove a weaker statement, where inequality $b_1(H'; \mathbb{F}_q) \leq d(K)$ is replaced by $b_1(H'; \mathbb{F}_q) < b_1(H; \mathbb{F}_q)$. The assertion of Corollary 5.7 then follows by repeated applications with δ replaced by $\delta/(b_1(H, \mathbb{F}_q) - d(K))$.
Let Y be any free generating set for H. Obviously $K/[K, K]$ is a free abelian group of rank $d(K)$. Any (finite) matrix over the integers can be transformed by elementary row and column operations to a diagonal matrix. Hence by applying elementary transformations to Y, we can arrange that Y is a disjoint union $Y_1 \sqcup Y_2$ where $|Y_1| \leq d(K)$ and $Y_2 \subseteq [K, K]$.
Let $L = \langle Y_2 \rangle$, the subgroup generated by Y_2. Since $b_1(H; \mathbb{F}_q) > d(K)$, there exists $f \in Y_2$ whose image in $H/[H, H]H^q \cong H_1(H, \mathbb{F}_q)$ is non-trivial. Now apply Lemma 5.6 to this f, choose n such that $\frac{1}{p^n} < \delta$ and let $R' = \{f^{p^n}\}$. The choice of f ensures that $b_1(H'; \mathbb{F}_q) < b_1(H; \mathbb{F}_q)$, so R' has the required properties. □

5.2. Proof of Theorem 5.1. To simplify the notations, we will give a proof of the main part of Theorem 5.1. The last part of Theorem 5.1 is proved in the same way by using the last assertion of Corollary 5.7.
We start by giving an outline of the construction. Let $F = F(X)$ be a free group of rank $d = |X|$. Below we shall define a descending chain $F = F_0 \supseteq F_1 \supseteq \dots$ of normal subgroups of F of p-power index and a sequence of finite subsets $R_1, R_2, \dots$ of F. Let $R = \bigcup_{i=1}^{\infty} R_n$. For each $n \in \mathbb{Z}_{\geq 0}$ we let $G(n) = F/\langle\langle \bigcup_{i=1}^{n} R_i \rangle\rangle$, $G(\infty) = \varinjlim G(i) = F/\langle\langle R \rangle\rangle$ and let G be the image of $G(\infty)$ in its pro-p completion. Denote by $G(n)_i$, $G(\infty)_i$ and G_i the canonical image of F_i in $G(n), G(\infty)$ and G, respectively. We will show that the group G and its subgroups (G_i) satisfy the conclusion of Theorem 5.1.
Fix a sequence of positive real numbers (δ_n) which converges to zero and a descending chain (Φ_n) of normal subgroups of p-power index in F which form a base of neighborhoods of 1 for the pro-p topology. The subgroups F_n and relator sets R_n will be constructed inductively so that the following properties hold:

(i) For $n \geq 0$ we have
$$\frac{b_1(G(n)_{2n})}{[G(n) : G(n)_{2n}]} > d - 1 - \varepsilon;$$

(ii) For $n \geq 1$ we have
$$\frac{b_1(G(n)_{2n-1})}{[G(n) : G(n)_{2n-1}]} < \delta_n;$$

(iii) R_n is contained in $[F_{2n-2}, F_{2n-2}]$ for $n \geq 1$;

(iv) $F_{2n} \subseteq \Phi_n$ for $n \geq 1$;
(v) $\operatorname{def}_p(X, \cup_{i=1}^n R_i) > d-1-\varepsilon$ for $n \geq 1$;
(vi) The presentation $(X, \cup_{i=1}^n R_i)$ is p-regular for $n \geq 1$.

We first explain why properties (i)-(vi) will imply that the group G and its subgroups (G_n) have the desired properties. Each G_n is normal of p-power index in G since F_n is normal of p-power index in F. Condition (iv) implies that (G_n) is a base of neighborhoods of 1 for the pro-p topology on G, and since G is residually-p by construction, we have $\bigcap_{n=1}^\infty G_n = \{1\}$.
Condition (iii) implies that $[G(n) : G(n)_i] = [G(\infty) : G(\infty)_i]$ and $b_1(G(n)_i) = b_1(G(\infty)_i)$ for $i \leq 2n$. Since $G(\infty)_i$ is normal of p-power index in $G(\infty)$, the group $G(\infty)/[G(\infty)_i, G(\infty)_i]$ is residually-p, so both the index and the first Betti number of $G(\infty)_i$ do not change under passage to the image in the pro-p completion of $G(\infty)$: $[G : G_i] = [G(\infty) : G(\infty)_i]$ and $b_1(G_i) = b_1(G(\infty)_i)$. In view of these equalities, conditions (i) and (ii) yield the corresponding conditions in Theorem 5.1.
We now describe the construction of the sets R_n and subgroups F_n. The base case $n = 0$ is obvious: we set $F_0 = F$ and $G(0) = F$, and the only condition we require for $n = 0$ (condition (i)) clearly holds.
Suppose now that $N \in \mathbb{N}$ and we constructed subsets $(R_i)_{i=1}^N$ and subgroups $(F_i)_{i=1}^{2N}$ such that (i)-(vi) hold for all $n \leq N$.
Let $F_{2N+1} = [F_{2N}, F_{2N}]F_{2N}^{p^e}$ where e is specified below. Then F_{2N+1} is a normal subgroup of p-power index in F and $F_{2N} \supseteq F_{2N+1} \supset [F_{2N}, F_{2N}]$. Since $b_1(G(N)_{2N}) > 0$ by (i) for $n = N$ and hence

$$
\begin{aligned}
p^e &\leq \left|H_1(G(N)_{2N})/p^e \cdot H_1(G(N)_{2N})\right| \\
&= \left|G(N)_{2N}/[G(N)_{2N}, G(N)_{2N}]G(N)_{2N}^{p^e}\right| \\
&= |G(N)_{2N}/G(N)_{2N+1}| \\
&= [G(N)_{2N} : G(N)_{2N+1}] \\
&\leq [G(N) : G(N)_{2N+1}],
\end{aligned}
$$

so we can arrange

$$\frac{d(F_{2N})}{[G(N) : G(N)_{2N+1}]} < \delta_{N+1}$$

by choosing e large enough.
Now applying Corollary 5.7 with $H = F_{2N+1}$, $K = F_{2N}$ and $\delta = \operatorname{def}_p(X, \cup_{i=1}^N R_i) - (d-1-\varepsilon)$, we get that there is a finite subset $R_{N+1} \subseteq [F_{2N}, F_{2N}]$ such that the presentation $(X, \cup_{i=1}^{N+1} R_i)$ is p-regular and $\operatorname{def}_p(X, \cup_{i=1}^{N+1} R_i) > d-1-\varepsilon$. Hence conditions (iii),(v),(vi) hold for $n = N+1$. The subgroup H' in the notations of Corollary 5.7 is equal to $G(N+1)_{2N+1}$, so $b_1(G(N+1)_{2N+1}) \leq d(F_{2N})$. Since condition (iii) implies $[G(N+1) : G(N+1)_{2N+1}] = [G(N) : G(N)_{2N+1}]$, we conclude

$$\frac{b_1(G(N+1)_{2N+1})}{[G(N+1) : G(N+1)_{2N+1}]} \leq \frac{d(F_{2N})}{[G(N) : G(N)_{2N+1}]} < \delta_{N+1}.$$

Thus we have shown that conditions (ii),(iii),(v),(vi) hold for $n = N+1$.

It remains to construct F_{2N+2} and to verify (i) and (iv) for $n = N+1$. We apply Lemma 5.5 to $G(N+1) = \langle X \mid \cup_{i=1}^{N+1} R_i \rangle$ and obtain using (v) a normal subgroup H of $G(N+1)$ of p-power index satisfying

$$\frac{\operatorname{def}(H) - 1}{[G(N+1) : H]} > d - 1 - \varepsilon.$$

Let $F_{2N+2} \subseteq F_{2N+1} \cap \Phi_{N+1}$ be the intersection of the preimage of H under the projection $p_{N+1} \colon F_{N+1} \to G(N+1)$ with $F_{2N+1} \cap \Phi_{N+1}$. Obviously (iv) for holds $n = N+1$. Then $G(N+1)_{2N+2}$ is a subgroup of H of finite index. The quantity $\operatorname{def}(\cdot) - 1$ is supermultiplicative, i.e., if L is a finite index subgroup of H, then $\operatorname{def}(L) - 1 \geq [H : L] \cdot (\operatorname{def}(H) - 1)$, see for instance [18, Lemma 2.2]. Hence we conclude

$$\frac{\operatorname{def}(G(N+1)_{2N+2}) - 1}{[G(N+1) : G(N+1)_{2N+2})]} \geq \frac{\operatorname{def}(H) - 1}{[G(N+1) : H]} > d - 1 - \varepsilon.$$

Since $b_1(G(N+1)_{2N+2}) \geq \operatorname{def}(G(N+1)_{2N+2})$, condition (i) holds for $n = N+1$. This finishes the proof of Theorem 5.1.

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A Global Quantum Duality Principle for Subgroups and Homogeneous Spaces

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Abstract. For a complex or real algebraic group G, with $\mathfrak{g} := \mathrm{Lie}(G)$, quantizations of *global* type are suitable Hopf algebras $F_q[G]$ or $U_q(\mathfrak{g})$ over $\mathbb{C}\big[q,q^{-1}\big]$. Any such quantization yields a structure of Poisson group on G, and one of Lie bialgebra on $\mathfrak{g}$: correspondingly, one has dual Poisson groups G^* and a dual Lie bialgebra $\mathfrak{g}^*$. In this context, we introduce suitable notions of *quantum subgroup* and, correspondingly, of *quantum homogeneous space*, in three versions: *weak*, *proper* and *strict* (also called *flat* in the literature). The last two notions only apply to those subgroups which are coisotropic, and those homogeneous spaces which are Poisson quotients; the first one instead has no restrictions whatsoever.

The global quantum duality principle (GQDP), as developed in [F. Gavarini, *The global quantum duality principle*, Journ. für die Reine Angew. Math. 612 (2007), 17–33.], associates with any global quantization of G, or of $\mathfrak{g}$, a global quantization of $\mathfrak{g}^*$, or of G^*. In this paper we present a similar GQDP for quantum subgroups or quantum homogeneous spaces. Roughly speaking, this associates with every quantum subgroup, resp. quantum homogeneous space, of G, a quantum homogeneous space, resp. a quantum subgroup, of G^*. The construction is tailored after four parallel paths — according to the different ways one has to algebraically describe a subgroup or a homogeneous space — and is "functorial", in a natural sense.

Remarkably enough, the output of the constructions are always quantizations of *proper* type. More precisely, the output is related to the input as follows: the former is the *coisotropic dual* of the coisotropic interior of the latter — a fact that extends the occurrence of Poisson duality in the original GQDP for quantum groups. Finally, when the input is a strict quantization then the output is strict as well — so the special rôle of strict quantizations is respected.

We end the paper with some explicit examples of application of our recipes.

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1 Introduction

In this paper we work with quantizations of (algebraic) complex and real groups, their subgroups and homogeneous spaces, and a special symmetry among such quantum objects which we refer to as the "Global Quantum Duality Principle". This is just a last step in a process, which is worth recalling in short.

In any possible sense, quantum groups are suitable deformations of some algebraic objects attached with algebraic groups, or Lie groups. Once and for all, we adopt the point of view of algebraic groups: nevertheless, all our analysis and results can be easily converted in the language of Lie groups.

The first step to deal with is describing an algebraic group G via suitable algebraic object(s). This can be done following two main approaches, a *global* one or a *local* one.

In the *global geometry* approach, one considers $U(\mathfrak{g})$ — the universal enveloping algebra of the tangent Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ — and $F[G]$ — the algebra of regular functions on G. Both these are Hopf algebras, and there exists a non-degenerate pairing among them so that they are dual to each other. Clearly, $U(\mathfrak{g})$ only accounts for the local data of G encoded in $\mathfrak{g}$, whereas $F[G]$ instead totally describes G: thus $F[G]$ yields a global description of G, which is why we speak of "global geometry" approach.

In this context, one describes (globally) a subgroup K of G — always assumed to be Zariski closed — via the ideal in $F[G]$ of functions vanishing on it; alternatively, an infinitesimal description is given taking in $U(\mathfrak{g})$ the subalgebra $U(\mathfrak{k})$, where $\mathfrak{k} := \mathrm{Lie}(K)$.

For a homogeneous G–space, say M, one describes it in the form $M \cong G/K$ — which amounts to fixing some point in M and its stabilizer subgroup K in G. After this, a local description of $M \cong G/K$ is given by representing its left-invariant differential operators as $U(\mathfrak{g})/U(\mathfrak{g})\,\mathfrak{k}$: therefore, we can select $U(\mathfrak{g})\,\mathfrak{k}$ — a left ideal, left coideal in $U(\mathfrak{g})$ — as algebraic object to encode $M \cong G/K$, at least infinitesimally. For a global description instead, obstructions might occur. Indeed, we would like to describe $M \cong G/K$ via some algebra $F[M] \cong F\big[G/K\big]$ strictly related with $F[G]$. This varies after the nature of $M \cong G/K$

— hence of K — and in general might be problematic. Indeed, there exists a most natural candidate for this job, namely the set $F[G]^K$ of K–invariants of $F[G]$, which is a subalgebra and left coideal. The problem is that $F[G]^K$ permits to recover exactly G/K if and only if $M \cong G/K$ is a quasi-affine variety (which is not always the case). This yields a genuine obstruction, in the sense that this way of (globally) encoding the space $M \cong G/K$ only works with quasi-affine G–spaces; for the other cases, we just drop this approach — however, for a complete treatment of the case of *projective* G–spaces see [6].

In contrast, the approach of *formal geometry* is a looser one: one replaces $F[G]$ with a topological algebra $F[[G]] = F\big[\big[G_f\big]\big]$ — the algebra of "regular functions on the formal group G_f" associated with G — which can be realized either as the suitable completion of the local ring of G at its identity or as the (full) linear dual of $U(\mathfrak{g})$. In any case, both algebraic objects taken into account now only encode the local information of G.

In this formal geometry context, the description of (formal) subgroups and (formal) homogeneous spaces goes essentially the same. However, in this case no problem occurs with (formal) homogeneous space, as any one of them can be described via a suitably defined subalgebra of invariants $F\big[\big[G_f\big]\big]^{K_f}$: in a sense, "all formal homogeneous spaces are quasi-affine". As a consequence, the overall description one eventually achieves is entirely symmetric.

When dealing with quantizations, Poisson structures arise (as semiclassical limits) on groups and Lie algebras, so that we have to do with Poisson groups and Lie bialgebras. In turn, there exist distinguished subgroups and homogeneous spaces — and their infinitesimal counterparts — which are "well-behaving" with respect to these extra structures: these are *coisotropic subgroups* and *Poisson quotients*. Moreover, the well-known Poisson duality — among Poisson groups G and G^* and among Lie bialgebras $\mathfrak{g}$ and $\mathfrak{g}^*$ — extends to similar dualities among coisotropic subgroups (of G and G^*) and among Poisson quotients (of G and G^* again). It is also useful to notice that each subgroup contains a maximal coisotropic subgroup (its "coisotropic interior"), and accordingly each homogeneous space has a naturally associated Poisson quotient.

As to the algebraic description, all properties concerning Poisson (or Lie bialgebra) structures on groups, Lie algebras, subgroups and homogeneous spaces have unique characterizations in terms of the algebraic codification one adopts for these geometrical objects. Details change a bit according to whether one deals with global or formal geometry, but everything goes in parallel in either context.

By (complex) "quantum group" of *formal type* we mean any topological Hopf algebra $H_\hbar$ over the ring $\mathbb{C}[[\hbar]]$ whose semiclassical limit at $\hbar = 0$ — i.e., $H_\hbar/\hbar H_\hbar$ — is of the form $F\big[\big[G_f\big]\big]$ or $U(\mathfrak{g})$ for some formal group G_f or Lie algebra $\mathfrak{g}$. Accordingly, one writes $H_\hbar := F_\hbar\big[\big[G_f\big]\big]$ or $H_\hbar := U_\hbar(\mathfrak{g})$, calling the former a QFSHA and the latter a QUEA. If such a quantization (of either type) exists, the formal group G_f is Poisson and $\mathfrak{g}$ is a Lie bialgebra; accordingly, a dual formal Poisson group G_f^* and a dual Lie bialgebra $\mathfrak{g}^*$ exist too.

In this context, as formal quantizations of subgroups or homogeneous spaces one typically considers suitable subobjects of either $F_\hbar\big[\big[G_f\big]\big]$ or $U_\hbar(\mathfrak{g})$ such that: *(1)* with respect to the containing formal Hopf algebra, they have the same relation as a in the "classical" setting — such as being a one-sided ideal, a subcoalgebra, etc.; *(2)* taking their specialization at $\hbar = 0$ is the same as restricting to them the specialization of the containing algebra (this is typically mentioned as a "flatness" property). This second requirement has a key consequence, i.e. the semiclassical limit object is necessarily "good" w.r. to the Poisson structure: namely, if we are quantizing a subgroup, then the latter is necessarily coisotropic, while if we are quantizing a homogeneous space then it is indeed a Poisson quotient.

In the spirit of global geometry, by (complex) "quantum group" of *global type* we mean any Hopf algebra H_q over the ring $\mathbb{C}\big[q,q^{-1}\big]$ whose semiclassical limit at $q=1$ — i.e., $H_q\big/(q-1)\,H_q$ — is of the form $F[G]$ or $U(\mathfrak{g})$ for some algebraic group G or Lie algebra $\mathfrak{g}$. Then one writes $H_q := F_q[G]$ or $H_q := U_\hbar(\mathfrak{g})$, calling the former a QFA and the latter a QUEA. Again, if such a quantization (of either type) exists the group G is Poisson and $\mathfrak{g}$ is a Lie bialgebra, so that dual formal Poisson groups G^* and a dual Lie bialgebra $\mathfrak{g}^*$ exist too.

As to subgroups and homogeneous spaces, global quantizations can be defined via a sheer reformulation of the same notions in the formal context: we refer to such quantizations as *strict*. In this paper, we introduce two more versions of quantizations, namely *proper* and *weak* ones, ordered by increasing generality, namely $\{\textit{strict}\} \subsetneq \{\textit{proper}\} \subsetneq \{\textit{weak}\}$. This is achieved by suitably weakening the condition *(2)* above which characterizes a quantum subgroup or quantum homogeneous space. Remarkably enough, one finds that now the existence of a *proper* quantization is already enough to force a subgroup to be coisotropic, or a homogeneous space to be a Poisson quotient.

The *Quantum Duality Principle* (=QDP) was first developed by Drinfeld (cf. [7], §7) for formal quantum groups (see [10] for details). It provides two functorial recipes, inverse to each other, acting as follows: one takes as input a QFSHA for G_f and yields as output a QUEA for $\mathfrak{g}^*$; the other one as input a QUEA for $\mathfrak{g}$ and yields as output a QFSHA for G_f^*.

The *Global Quantum Duality Principle* (=GQDP) is a version of the QDP tailored for global quantum groups (see [11, 12]): now one functorial recipe takes as input a QFA for G and yields a QUEA for $\mathfrak{g}^*$, while the other takes a QUEA for $\mathfrak{g}$ and provides a QFA for G^*.

An appropriate version of the QDP for formal subgroups and formal homogeneous spaces was devised in [5]. Quite in short, the outcome there was an explicit recipe which taking as input a formal quantum subgroup, or a formal quantum homogeneous space, respectively, of G_f provides as output a quantum formal homogeneous space, or a formal quantum subgroup, respectively, of G_f^*. In short, these recipes come out as direct "restriction" (to formal quantum subgroups or formal quantum homogeneous spaces) of those in the QDP for formal quantum groups. This four-fold construction is fully symmetric, in particular all duality or orthogonality relations possibly holding among different quantum objects are preserved. Finally, Poisson duality is still involved, in that the semiclassical limit of the output quantum object is always the coisotropic dual of the semiclassical limit of the input quantum object.

The main purpose of the present work is to provide a suitable version of the GQDP for global quantum subgroups and global quantum homogeneous spaces — extending the GQDP for global quantum groups — as much general as possible. The inspiring idea, again, is to "adapt" (by restriction, in a sense) to these more general quantum objects the functorial recipes available from the GQDP for global quantum groups. Remarkably enough, this approach is fully successful: indeed, it does work properly not only with *strict* quantizations (which should sound natural) but also for *proper* and for *weak* ones. Even more, the output objects always are global quantizations (of subgroups or homogeneous spaces) of *proper* type — which gives an independent motivation to introduce the notion of proper quantization.

Also in this setup, Poisson duality, in a generalized sense, shows up again as the link between the input and the output of the GQDP recipes: namely, the semiclassical limit of the output quantum object is always the coisotropic dual of the coisotropic interior of the semiclassical limit of the input quantum object.

Besides the wider generality this GQDP applies to (in particular, involving also non-coisotropic subgroups, or homogeneous spaces which are not Poisson quotients), we pay a drawback in some lack of symmetry for the final result — compared to what one has in the formal quantization context. Nevertheless, such a symmetry is almost entirely recovered if one restricts to dealing with *strict* quantizations, or to dealing with "double quantizations" — involving simultaneously a QFA and a QUEA in perfect (i.e. non-degenerate) pairing.

At the end of the paper (Section 6) we present some applications of our GQDP: this is to show how it effectively works, and in particular that it does provide explicit examples of global quantum subgroups and global quantum homogeneous spaces. Among these, we also provide an example of a quantization which is *proper* but is *not strict* — which shows that the former notion is a non-trivial generalization of the latter.

2 General Theory

The main purpose of the present section is to collect some classical material about Poisson geometry for groups and homogeneous spaces. Everything is standard, we just need to fix the main notions and notations we shall deal with.

2.1 Subgroups and homogeneous spaces

Let G be a complex affine algebraic group and let $\mathfrak{g}$ be its tangent Lie algebra. Let us denote by $F[G]$ its algebra of regular functions and by $U(\mathfrak{g})$ its universal enveloping algebra. Both such algebras are Hopf algebras, and there

exists a natural pairing of Hopf algebras between them, given by evaluation of differential operators onto functions. This pairing is perfect if and only if G is connected, which we will always assume in what follows.
A *real form* of either G or $\mathfrak{g}$ is given once a Hopf $*$–algebra structure is fixed on either $F[G]$ or $U(\mathfrak{g})$ — and in case one take such a structure on both sides, the two of them must be dual to each other. Thus by *real algebraic group* we will always mean a complex algebraic group endowed with a suitable $*$–structure.
A subgroup K of G will always be considered as Zariski–closed and algebraic. For any such subgroup, the quotient G/K is an algebraic left homogeneous G–space, which is quasi-projective as an algebraic variety. Given an algebraic left homogeneous G–space M and choosing $m \in M$, the stabilizer subgroup K_m will be a closed algebraic subgroup of G such that $G/K_m \simeq M$; changing point will change the stabilizer within a single conjugacy class.
We shall describe the subgroup K , or the homogeneous space G/K, through either an algebraic subset of $F[G]$ — to which we will refer as a *global* coding — or an algebraic subset of $U(\mathfrak{g})$ — to which we will refer as a *local* coding. The complete picture is the following:

— SUBGROUP K :

(local) letting $\mathfrak{k} = \mathrm{Lie}(K)$ we can consider its enveloping algebra $U(\mathfrak{k})$ which is a Hopf subalgebra of $U(\mathfrak{g})$; we then set $\mathfrak{C} \equiv \mathfrak{C}(K) := U(\mathfrak{k})$;

(global) functions which are 0 on K form a Hopf ideal $\mathcal{I} \equiv \mathcal{I}(K)$ inside $F[G]$, such that $F[K] \simeq F[G]/\mathcal{I}$.

— HOMOGENEOUS SPACE G/K :

(local) let $\mathfrak{I} \equiv \mathfrak{I}(K) = U(\mathfrak{g}) \cdot \mathfrak{k}$: this is a left ideal and two-sided coideal in $U(\mathfrak{g})$, and $U(\mathfrak{g})/\mathfrak{I}$ is the set of left–invariant differential operators on G/K .

(global) regular functions on the homogeneous space G/K may be identified with K–invariant regular functions on G . We will let $\mathcal{C} = \mathcal{C}(K) = F[G]^K$; this is a subalgebra and left coideal in $F[G]$.

Warning: this needs clarification! The point is: can one recover the homogeneous space G/K from $\mathcal{C}(K) = F[G]^K$? The answer depends on geometric properties of G/K itself — or (equivalently) of K — which we explain later on.

For any Hopf algebra $\mathcal{H}$ we introduce the following notations: $\leq^1$ will stand for "unital subalgebra", $\trianglelefteq$ for "two-sided ideal", $\trianglelefteq_l$ for "left ideal" and similarly $\dot{\leq}$ will stand for "subcoalgebra", $\dot{\trianglelefteq}$ for "two-sided coideal" and $\dot{\trianglelefteq}_\ell$ for "left coideal". When the same symbols will be decorated by a subindex referring to a specific algebraic structure their meaning should be modified accordingly, e.g. $\trianglelefteq_{\mathcal{H}}$ will stand for "Hopf ideal" and $\leq_{\mathcal{H}}$ for "Hopf subalgebra".

With such notations, with any subgroup K of G there is associated one of the following algebraic objects:

$$(a)\ \mathcal{I} \trianglelefteq_{\mathcal{H}} F[G]\ ,\ \ (b)\ \mathcal{C} \leq^1 \dot{\trianglelefteq}_\ell\, F[G]\ ,\ \ (c)\ \mathfrak{I} \trianglelefteq_l\ \dot{\trianglelefteq}\, U(\mathfrak{g})\ ,\ \ (d)\ \mathfrak{C} \leq_{\mathcal{H}} U(\mathfrak{g}) \qquad (2.1)$$

In the real case, one has to consider, together with (2.1), additional requirements involving the $*$ structure and the antipode S, namely

$$(a)\quad \mathcal{I}^* = \mathcal{I}\ \ ,\quad (b)\quad S(\mathcal{C})^* = \mathcal{C}\ \ ,\quad (c)\quad S(\mathfrak{I})^* = \mathfrak{I}\ \ ,\quad (d)\quad \mathfrak{C}^* = \mathfrak{C} \qquad (2.2)$$

In the connected case algebraic objects of type $\mathcal{I}$, $\mathfrak{I}$ and $\mathfrak{C}$ in (2.1) are enough to reconstruct either K or G/K:

$$K \,=\, \mathrm{Spec}\big(F[G]/\mathcal{I}\big) \,=\, \exp\big(\mathrm{Prim}(\mathfrak{C})\big) \,=\, \exp\big(\mathrm{Prim}(\mathfrak{I})\big)$$

where $\mathrm{Prim}(X)$ denotes the set of primitive elements of a bialgebra X.
In contrast, $\mathcal{C}(K) = F[G]^K$ might be not enough to reconstruct K, due to lack of enough global algebraic functions; this happens, for example, when G/K is projective and therefore $\mathcal{C}(K) = \mathbb{C}$. Any group K which can be reconstructed from its associated $\mathcal{C}$ is called *observable*: we shall now make this notion more precise. Let us call τ the map that to any subgroup K associates the algebra of invariant functions $F[G]^K$ and let us call σ the map that to any subalgebra A of $F[G]$ associates its stabilizer $\sigma(A) = \big\{\, g \in G \,\big|\, g \cdot f = f \ \ \forall\, f \in A \,\big\}$. These two maps are obviously inclusion–reversing. Furthermore they establish what is also known as a *simple Galois correspondence*: namely, for any subgroup K and any subalgebra A one has

$$(\sigma \circ \tau)(K) \supseteq K\ , \qquad (\tau \circ \sigma)(A) \supseteq A$$

so that $(\tau \circ \sigma \circ \tau)\,(K) = \tau(K)$, $(\sigma \circ \tau \circ \sigma)\,(A) = \sigma(A)$. A subgroup K of G such that $(\sigma \circ \tau)\,(K) = K$ is said to be *observable*: this means exactly that such a subgroup can be fully recovered from its algebra of invariant functions $\tau(K)$. If K is any subgroup, then $\widehat{K} := (\sigma \circ \tau)\,(K)$ is the smallest observable subgroup containing K; we will call it the *observable hull* of K. Remark then that $\mathcal{C}(K) = \mathcal{C}\big(\widehat{K}\big)$.
The following fact (together with many properties of observable subgroups), which gives a characterization of observable subgroups in purely geometrical terms, may be found in [13]:
Fact: *a subgroup K of G is observable if and only if G/K is quasi–affine.*
Let us now clarify how to pass from algebraic objects directly associated with subgroups to those corresponding to homogeneous spaces. Let H be a Hopf algebra, with counit ε and coproduct Δ. For any submodule $M \subseteq H$ define

$$M^+ := M \cap Ker(\varepsilon)\ \ ,\quad H^{coM} := \big\{\, y \in H \,\big|\, (\Delta(y) - y \otimes 1) \in H \otimes M \,\big\} \qquad (2.3)$$

Let C be a (unital) subalgebra and left coideal of H and define $\Psi(C) = H \cdot C^+$. Then $\Psi(C)$ is a left ideal and two-sided coideal in H. Conversely, let I be a left ideal and two-sided coideal in H and define $\Phi(I) := H^{coI}$. Then $\Phi(I)$ is a unital subalgebra and left coideal in H. Also, this pair of maps (Φ, Ψ) defines a simple Galois correspondence, that is to say *(a)* Ψ and Φ are inclusion-preserving;

(b) $(\Phi \circ \Psi)(C) \supseteq C \; , \quad (\Psi \circ \Phi)(I) \subseteq I \; ;$

(c) $\Phi \circ \Psi \circ \Phi = \Phi \; , \quad \Psi \circ \Phi \circ \Psi = \Psi \; .$

(where the third property follows from the previous ones; see [19, 21, 22] for further details).
Let now K be a subgroup of G and let $\mathcal{I}$, $\mathcal{C}$, $\mathfrak{I}$, $\mathfrak{C}$ the corresponding algebraic objects as described in (2.1). We can thus establish the following relations among them:

SUBGROUP VS. HOMOGENEOUS SPACE: objects directly related to the subgroup (namely, $\mathcal{I}$ and $\mathfrak{C}$) and objects directly related to the homogeneous space (namely, $\mathcal{C}$ and $\mathfrak{I}$) are linked by Ψ and Φ as follows:

$$\mathfrak{I} = \Psi(\mathfrak{C}) \; , \qquad \mathfrak{C} = \Phi(\mathfrak{I}) \; , \qquad \mathcal{I} \supseteq \Psi(\mathcal{C}) \; , \qquad \mathcal{C} = \Phi(\mathcal{I}) \tag{2.4}$$

In particular, K is observable if and only if $\mathcal{I} = \Psi(\mathcal{C})$; on the other hand, we have in general $\Psi(\mathcal{C}(K)) = \mathcal{I}(\widehat{K})$.

ORTHOGONALITY with respect to the natural pairing between $F[G]$ and $U(\mathfrak{g})$: this is expressed by the relations

$$\mathcal{I} = \mathfrak{C}^{\perp} \; , \qquad \mathfrak{C} = \mathcal{I}^{\perp} \; , \qquad \mathcal{C} = \mathfrak{I}^{\perp} \; , \qquad \mathfrak{I} \subseteq \mathcal{C}^{\perp} \tag{2.5}$$

In particular, K is observable if and only if $\mathfrak{I} = \mathcal{C}^{\perp}$; on the other hand, we have in general $\mathcal{C}(K)^{\perp} = \mathfrak{I}\big(\widehat{K}\big)$.

Let us also remark that orthogonality intertwines the local and global description.

THE "FORMAL" VS. "GLOBAL" GEOMETRY APPROACH. In the present approach we are dealing with geometrical objects — groups, subgroups and homogeneous spaces — which we describe via suitably chosen algebraic objects. When doing that, universal enveloping algebras or subsets of them only provide a *local* description — around a distinguished point: the unit element in a (sub)group, or its image in a coset (homogeneous) space. Instead, function algebras yield a *global* description, i.e. they do carry information on the whole geometrical object; for this reason, we refer to the present approach as the "global" one.
The "formal geometry" approach instead only aims to describe a group by a topological Hopf algebra, which can be realized as an algebra of formal power series; in short, this is summarized by saying that we are dealing with a "formal group". Subgroups and homogeneous spaces then are described by suitable subsets in such a formal series algebra (or in the universal enveloping algebra, as above): this again yields only a local description — in a formal neighborhood

of a distinguished point — rather than a global one. Now, the analysis above shows that an asymmetry occurs when we adopt the global approach. Indeed, we might have problems when describing a homogeneous space by means of (a suitably chosen subalgebra of invariant) functions: technically speaking, this shows up as the occurrence of *inclusions* — rather than identities! — in formulas 2.4 and 2.5. This is a specific, unavoidable feature of the problem, due to the fact that homogeneous spaces (for a given group) do not necessarily share the same geometrical nature — beyond being all quasi-projective — in particular they are not necessarily quasi-affine.
The case of those homogeneous spaces which are *projective* is treated in [6], where their quantizations are studied; in particular, there a suitable method to solve the problematic "$\mathcal{C}$–side" of the QDP in that case is worked out, still in terms of "global geometry" but with a different tool (semi-invariant functions, rather than invariant ones). In contrast, in the formal geometry approach such a lack of symmetry does not occur: in other words, it happens that *every formal (closed) subgroup is observable*, or *every formal homogeneous space is quasi-affine*. This means that there is no need of worrying about observability, and the full picture — for describing a subgroup or homogeneous space, in four different ways — is entirely symmetric. This was the point of view adopted in [5], where this complete symmetry of the formal approach is exploited to its full extent.

2.2 Poisson subgroups and Poisson quotients

Let us now assume that G is endowed with a complex Poisson group structure corresponding to a Lie bialgebra structure on $\mathfrak{g}$, whose Lie cobracket is denoted $\delta : \mathfrak{g} \longrightarrow \mathfrak{g} \wedge \mathfrak{g}$. At the Hopf algebra level this means that $F[G]$ is a Poisson–Hopf algebra and $U(\mathfrak{g})$ a co-Poisson Hopf algebra, in such a way that the duality pairing is compatible with these additional structures (see [4] for basic definitions). Let us recall that the linear dual $\mathfrak{g}^*$ inherits a Lie algebra structure; on the other hand, it has a natural Lie coalgebra structure, whose cobracket $\delta : \mathfrak{g}^* \longrightarrow \mathfrak{g}^* \wedge \mathfrak{g}^*$ is the dual map to the Lie bracket of $\mathfrak{g}$. Altogether, this makes $\mathfrak{g}^*$ into a Lie bialgebra, which said to be *dual* to $\mathfrak{g}$. Therefore, there exist Poisson groups whose tangent Lie bialgebra is $\mathfrak{g}^*$; we will assume one such connected group is fixed, we will denote it with G^* and call it the dual Poisson group of G. In the real case the involution in $F[G]$ is a Poisson algebra antimorphism and the one in $U(\mathfrak{g})$ is a co-Poisson algebra antimorphism.
A closed subgroup K of G is called *coisotropic* if its defining ideal $\mathcal{I}(K)$ is a Poisson subalgebra, while it is called a *Poisson subgroup* if $\mathcal{I}(K)$ is a Poisson ideal, the latter condition being equivalent to $K \hookrightarrow G$ being a Poisson map. Connected coisotropic subgroups can be characterized, at an infinitesimal level, by one of the following conditions on $\mathfrak{k} \subseteq \mathfrak{g}$:

(C-i) $\delta(\mathfrak{k}) \subseteq \mathfrak{k} \wedge \mathfrak{g}$, that is $\mathfrak{k}$ is a Lie coideal in $\mathfrak{g}$,
(C-ii) $\mathfrak{k}^\perp$ is a Lie subalgebra of $\mathfrak{g}^*$,

while analogous characterizations of Poisson subgroups correspond to $\mathfrak{k}$ being a Lie subcoalgebra or $\mathfrak{k}^\perp$ being a Lie ideal.
The most important features of coisotropic subgroups, in this setting, is the fact that G/K naturally inherits a Poisson structure from that of G. Actually, a Poisson manifold $(M\,,\omega_M)$ is called a *Poisson homogeneous G–space* if there exists a smooth, homogeneous G–action $\phi : G \times M \longrightarrow M$ which is a Poisson map (w.r. to the product Poisson structure on the domain). In particular, we will say that (M, ω_M) is a *Poisson quotient* if it verifies one of the following equivalent conditions (cf. [26]):

(P-i) there exists $x_0 \in M$ whose stabilizer G_{x_0} is coisotropic in G ;

(P-ii) there exists $x_0 \in M$ such that $\phi_{x_0} : G \to M\,,\ \phi(x_0, g) = \phi(x, g)\,,$ is a Poisson map ;

(P-iii) there exists $x_0 \in M$ such that $\omega_M(x_0) = 0$.

It is important to remark here that inside the same conjugacy class of subgroups of G there may be subgroups which are Poisson, coisotropic, or non coisotropic. Therefore, on the same homogeneous space there may exist many Poisson homogeneous structures, some of which make it into a Poisson quotient while some others do not.
For a fixed connected subgroup K of a Poisson group G, with Lie algebra $\mathfrak{k}$, one can consider the following descriptions in terms of the Poisson Hopf algebra $F[G]$ or of the co-Poisson Hopf algebra $U(\mathfrak{g})$:

$$\mathcal{I} \leq_{\mathcal{P}} F[G]\,, \qquad \mathcal{C} \leq_{\mathcal{P}} F[G] \tag{2.6}$$

$$\mathfrak{I} \,\dot{\trianglelefteq}_{\mathcal{P}}\, U(\mathfrak{g})\,, \qquad \mathfrak{C} \,\dot{\trianglelefteq}_{\mathcal{P}}\, U(\mathfrak{g}) \tag{2.7}$$

where on first line we have global conditions and on second line local ones. Conversely each one of these conditions imply coisotropy of G with the exception of the condition on $\mathcal{C}$, which implies only that the observable hull $\widehat{K}$ is coisotropic. Therefore a connected, observable, coisotropic subgroup of G is identified by one of the following algebraic objects:

$$\mathcal{I} \trianglelefteq_{\mathcal{H}} \leq_{\mathcal{P}} F[G]\,, \qquad \mathcal{C} \leq^1 \dot{\trianglelefteq}_\ell \leq_{\mathcal{P}} F[G] \tag{2.8}$$

$$\mathfrak{I} \trianglelefteq_l \dot{\trianglelefteq} \,\dot{\trianglelefteq}_{\mathcal{P}}\, U(\mathfrak{g})\,, \qquad \mathfrak{C} \leq_{\mathcal{H}} \dot{\trianglelefteq}_{\mathcal{P}}\, U(\mathfrak{g}) \tag{2.9}$$

(still with the usual, overall restriction on the use of $\mathcal{C}$, which in general only describes the observable hull $\widehat{K}$).
Thanks to self-duality in the notion of Lie bialgebra, with any Poisson group there is associated a natural *Poisson dual*, which is fundamental in the QDP; note that a priori many such dual groups are available, but when dealing with the QDP such an (apparent) ambiguity will be solved. As we aim to extend the QDP to coisotropic subgroups, we need to introduce a suitable notion of (Poisson) duality for coisotropic subgroups as well.

Definition 2.1. *Let G be a Poisson group and G^* a fixed Poisson dual.*

1. *If K is coisotropic in G we call* complementary dual *of K the unique connected subgroup $K^\perp$ in G^* such that $\mathrm{Lie}(K^\perp) = \mathfrak{k}^\perp$.*

2. *If M is a Poisson quotient and $M \simeq G/K_M$ we call* complementary dual *of M the Poisson G^*–quotient $M^\perp := G^*/K_M^\perp$.*

3. *For any subgroup H of G we call* coisotropic interior *of H the unique maximal, closed, connected, coisotropic subgroup $\overset{\circ}{H}$ of G contained in H .*

Remarks:

1. The complementary dual of a coisotropic subgroup is, trivially, a coisotropic subgroup whose complementary dual is the connected component of the one we started with:. Similarly, the complementary dual of a Poisson quotient is a Poisson quotient, and if we start with a Poisson quotient whose coisotropy subgroup (w.r. to any point) is connected then taking twice the complementary dual brings back to the original Poisson quotient.

2. The coisotropic interior may be characterized, at an algebraic level, as the unique closed subgroup whose Lie algebra is maximal between Lie subalgebras of $\mathfrak{h}$ which are Lie coideals in $\mathfrak{g}$.

Proposition 2.2. *Let K be any subgroup of G and let $K^{\langle\perp\rangle} := \big\langle \exp(\mathfrak{k}^\perp) \big\rangle$ be the closed, connected, subgroup of G^* generated by $\exp(\mathfrak{k}^\perp)$. Then:*

(a) the Lie algebra $\mathfrak{k}^{\langle\perp\rangle}$ of $K^{\langle\perp\rangle}$ is the Lie subalgebra of $\mathfrak{g}^*$ generated by $\mathfrak{k}^\perp$;

(b) $\mathfrak{k}^{\langle\perp\rangle}$ is a Lie coideal of $\mathfrak{g}^*$, hence $K^{\langle\perp\rangle}$ is a coisotropic subgroup of G^*;

(c) $K^{\langle\perp\rangle} = (\overset{\circ}{K})^\perp$; in particular if K is coisotropic then $K^{\langle\perp\rangle} = K^\perp$;

(d) $(K^{\langle\perp\rangle})^{\langle\perp\rangle} = \overset{\circ}{K}$ and K is coisotropic if and only if $(K^{\langle\perp\rangle})^{\langle\perp\rangle} = K$.

Proof. Part *(a)* is trivial. As for *(b)*, since $\mathfrak{k} = (\mathfrak{k}^\perp)^\perp$ is a Lie subalgebra of $\mathfrak{g}$, we have that $\mathfrak{k}^\perp$ is a Lie coideal in $\mathfrak{g}^*$: therefore, due to the identity

$$\delta\big([x,y]\big) \;=\; \sum_{[y]}\big([x,y_{[1]}]\otimes y_{[2]} + y_{[1]}\otimes [x,y_{[2]}]\big) + \sum_{[x]}\big([x_{[1]},y]\otimes x_{[2]} + x_{[1]}\otimes [x_{[2]},y]\big)$$

(where $\delta(z) = \sum_{[z]} z_{[1]} \otimes z_{[2]}$ for $z \in \mathfrak{g}^*$), the Lie subalgebra $\langle\mathfrak{k}^\perp\rangle$ of $\mathfrak{g}^*$ generated by $\mathfrak{k}^\perp$ is a Lie coideal too. It follows then by claim *(a)* that $K^{\langle\perp\rangle}$ is coisotropic. Thus *(b)* is proved.

As for part *(c)* we have

$$\left(\mathfrak{k}^{\langle\perp\rangle}\right)^{\perp} \;=\; \langle\mathfrak{k}^{\perp}\rangle^{\perp} \;=\; \Bigg(\bigcap_{\substack{\mathfrak{h}\leq_{\mathcal{L}}\,\mathfrak{g}^* \\ \mathfrak{h}\supseteq\mathfrak{k}^{\perp}}} \mathfrak{h}\Bigg)^{\perp} \;=\; \sum_{\substack{\mathfrak{h}\leq_{\mathcal{L}}\,\mathfrak{g}^* \\ \mathfrak{h}\supseteq\mathfrak{k}^{\perp}}} \mathfrak{h} \;=\; \sum_{\substack{\mathfrak{f}\,\dot{\trianglelefteq}_{\mathcal{L}}\,\mathfrak{g} \\ \mathfrak{f}\supseteq\mathfrak{k}}} \mathfrak{f} \;=\; \mathring{\mathfrak{k}}$$

(with $\leq_{\mathcal{L}}$ meaning "Lie subalgebra" and $\dot{\trianglelefteq}_{\mathcal{L}}$ meaning "Lie coideal") where $\mathring{\mathfrak{k}}$ is exactly the maximal Lie subalgebra and Lie coideal of $\mathfrak{g}$ contained in $\mathfrak{k}$. To be precise, this last statement follows from the above formula for $\delta\big([x,y]\big)$, since that formula implies that the Lie subalgebra generated by a family of Lie coideals is still a Lie coideal.

Now $\mathring{\mathfrak{k}} = \mathrm{Lie}(\mathring{\mathrm{K}})$, so $\mathrm{Lie}(K^{\langle\perp\rangle}) = \mathfrak{k}^{\langle\perp\rangle} = \Big(\big(\mathfrak{k}^{\langle\perp\rangle}\big)^{\perp}\Big)^{\perp} = \big(\,\mathring{\mathfrak{k}}\,\big)^{\perp} = \mathrm{Lie}(\mathring{\mathrm{K}})^{\perp}$ implies $K^{\langle\perp\rangle} = (\mathring{\mathfrak{k}})^{\perp}$ as we wished to prove. If, in addition, K is coisotropic then, obviously, $K^{\langle\perp\rangle} = K$. All other statements follow easily. $\square$

3 Strict, proper, weak quantizations

The purpose of this section is to fix some terminology concerning the meaning of the word "quantization" and to describe some possible ways of quantizing a (closed) subgroup, or a homogeneous space. We set the algebraic machinery needed for talking of "quantization" and "specialization": these notions must be carefully specified before approaching the construction of Drinfeld's functors.

Let q be an indeterminate, $\mathbb{C}\big[q,q^{-1}\big]$ the ring of complex-valued Laurent polynomials in q, and $\mathbb{C}(q)$ the field of complex-valued rational functions in q. Denote by $\mathcal{HA}$ the category of all Hopf algebras over $\mathbb{C}\big[q,q^{-1}\big]$ which are torsion-free as $\mathbb{C}\big[q,q^{-1}\big]$–modules.

Given a Hopf algebra H over the field $\mathbb{C}(q)$, a subset $\overline{H} \subseteq H$ is called *a* $\mathbb{C}\big[q,q^{-1}\big]$*–integral form* (or simply *a* $\mathbb{C}\big[q,q^{-1}\big]$*–form*) if it is a $\mathbb{C}\big[q,q^{-1}\big]$–Hopf subalgebra of H and $H_F := \mathbb{C}(q) \otimes_{\mathbb{C}[q,q^{-1}]} \overline{H} = H$. Then $\overline{H}$ is torsion-free as a $\mathbb{C}\big[q,q^{-1}\big]$–module, hence $\overline{H} \in \mathcal{HA}$.

For any $\mathbb{C}\big[q,q^{-1}\big]$–module M, we set $M_1 := M\big/(q-1)M = \mathbb{C} \otimes_{\mathbb{C}[q,q^{-1}]} M$: this is a $\mathbb{C}$–module (via $\mathbb{C}\big[q,q^{-1}\big] \to \mathbb{C}\big[q,q^{-1}\big]\Big/(q-1) = \mathbb{C}$), called *specialization of* M *at* $q=1$.

Given two $\mathbb{C}(q)$–modules A and B and a $\mathbb{C}(q)$–bilinear pairing $A \times B \longrightarrow F$, for any $\mathbb{C}\big[q,q^{-1}\big]$–submodule $A_\times \subseteq A$ we set:

$$A_\times^{\bullet} \;:=\; \Big\{\, b \in B \,\Big|\, \big\langle A_\times, b \big\rangle \subseteq \mathbb{C}\big[q,q^{-1}\big] \Big\} \tag{3.1}$$

In such a setting, we call $A_\times^{\bullet}$ *the* $\mathbb{C}\big[q,q^{-1}\big]$*–dual of* $A_\times$.

We will call *quantized universal enveloping algebra* (or, in short, QUEA) any $U_q \in \mathcal{HA}$ such that $U_1 := (U_q)_1$ is isomorphic to $U(\mathfrak{g})$ for some Lie algebra $\mathfrak{g}$, and we will call *quantized function algebra* (or, in short, QFA) any $F_q \in \mathcal{HA}$ such that $F_1 := (F_q)_1$ is isomorphic to $F[G]$ for some connected algebraic group G and, in addition, the following technical condition holds:

$$\bigcap_{n \geq 0} (q-1)^n F_q \;=\; \bigcap_{n \geq 0} \big((q-1) F_q \,+\, Ker(\epsilon_{F_q})\big)^n$$

We will add the specification that such quantum algebras are *real* whenever the starting object is a $*$–Hopf algebra. As a matter of notation, we write

$$\mathbb{U}_q \;:=\; \mathbb{C}(q) \otimes_{\mathbb{C}[q,q^{-1}]} U_q \quad , \qquad \mathbb{F}_q \;:=\; \mathbb{C}(q) \otimes_{\mathbb{C}[q,q^{-1}]} F_q \quad .$$

When U_q is a (real) QUEA, its specialization U_1 is a (real) co–Poisson Hopf algebra so that $\mathfrak{g}$ is in fact a (real) Lie bialgebra. Similarly, for any (real) QFA F_q the specialization F_1 is a (real) Poisson-Hopf algebra and therefore G is a (real) Poisson group (see [4] for details).
On occasions it is useful to consider simultaneous quantizations of both the universal enveloping algebra and the function algebra, or, in a larger generality, of a pair of dual Hopf algebra. Let $H, K \in \mathcal{HA}$ and assume that there exists a pairing of Hopf algebras $\langle\ ,\ \rangle : H \times K \longrightarrow \mathbb{C}\big[q, q^{-1}\big]$. If the pairing is such that

(a) $H = K^\bullet$, $K = H^\bullet$ (notation of (3.1)) w.r.t. the pairing $\mathbb{H} \times \mathbb{K} \to \mathbb{C}(q)$, for $\mathbb{H} := \mathbb{C}(q) \otimes_{\mathbb{C}[q,q^{-1}]} H$, $\mathbb{K} := \mathbb{C}(q) \otimes_{\mathbb{C}[q,q^{-1}]} K$, induced from $H \times K \to \mathbb{C}(q)$

(b) the Hopf pairing $H_1 \times K_1 \to \mathbb{C}$ given by specialization at $q = 1$ is perfect (i.e. non-degenerate)

then we will say that H *and* K *are dual to each other.* Note that all these assumptions imply that the initial pairing between H and K is perfect. When $H = U_q(\mathfrak{g})$ is a QUEA and $K = F_q[G]$ is a QFA, if the specialized pairing at 1 is the natural pairing between $U(\mathfrak{g})$ and $F[G]$ we will say that *the pair* $(U_q(\mathfrak{g}), F_q[G])$ *is a double quantization of* $(G, \mathfrak{g})$.

Let us now move to the case in which G is a Poisson group and K a subgroup. We want to define a reasonable notion of "quantization" of K and of the corresponding homogeneous space $G\big/K$. There is a standard way to implement this, which actually implies — cf. Lemma 3.3 and Proposition 3.5 later on — the additional constraint that K be *coisotropic*.

Definition 3.1. *Let $F_q[G]$ and $U_q(\mathfrak{g})$ be a QFA and a QUEA for G and $\mathfrak{g}$ and let*

$$\pi_{F_q} :\; F_q[G] \longrightarrow\!\!\!\!\!\rightarrow F_q[G]\Big/(q-1) F_q[G] \;\cong\; F[G]$$
$$\pi_{U_q} : U_q(\mathfrak{g}) \longrightarrow\!\!\!\!\!\rightarrow U_q(\mathfrak{g})\Big/(q-1)\, U_q(\mathfrak{g}) \cong U(\mathfrak{g})$$

*be the specialization maps. Let $\mathcal{I}$, $\mathcal{C}$, $\mathfrak{I}$ and $\mathfrak{C}$ be the algebraic objects associated with the subgroup K of G (see 2.1). We call "*strict quantization*" (and sometimes we shall drop the adjective "*strict*") of each of them any object $\mathcal{I}_q$, $\mathcal{C}_q$, $\mathfrak{I}_q$ or $\mathfrak{C}_q$ respectively, such that*

$$\begin{array}{llll} (a) & \mathcal{I}_q \trianglelefteq_\ell \dot{\trianglelefteq} F_q[G] \;, & \pi_{F_q}(\mathcal{I}_q) = \mathcal{I} \;, & \pi_{F_q}(\mathcal{I}_q) \cong \mathcal{I}_q \big/ (q-1)\,\mathcal{I}_q \\ (b) & \mathcal{C}_q \leq^1 \dot{\trianglelefteq}_\ell F_q[G] \;, & \pi_{F_q}(\mathcal{C}_q) = \mathcal{C} \;, & \pi_{F_q}(\mathcal{C}_q) \cong \mathcal{C}_q \big/ (q-1)\,\mathcal{C}_q \\ (c) & \mathfrak{I}_q \trianglelefteq_\ell \dot{\trianglelefteq} U_q(\mathfrak{g}) \;, & \pi_{U_q}(\mathfrak{I}_q) = \mathfrak{I} \;, & \pi_{U_q}(\mathfrak{I}_q) \cong \mathfrak{I}_q \big/ (q-1)\,\mathfrak{I}_q \\ (d) & \mathfrak{C}_q \leq^1 \dot{\trianglelefteq}_\ell U_q(\mathfrak{g}) \;, & \pi_{U_q}(\mathfrak{C}_q) = \mathfrak{C} \;, & \pi_{U_q}(\mathfrak{C}_q) \cong \mathfrak{C}_q \big/ (q-1)\,\mathfrak{C}_q \end{array} \tag{3.2}$$

In order to explain this definition let us start by considering the first two conditions in each line of (3.2).

a) A left ideal and two-sided coideal in a QFA quantizes the Hopf ideal of functions which are zero on a (closed) *subgroup*;

b) a left coideal subalgebra in a QFA quantizes the algebra of invariant functions on a *homogeneous space*;

c) a left ideal and two-sided coideal in a QUEA quantizes the infinitesimal algebra on a *homogeneous space*;

d) a left coideal subalgebra in a QUEA quantizes the universal enveloping subalgebra of a *subgroup*.

Once again, we must stress the fact that $\mathcal{C}_q$, as was explained in Proposition 2.4, has to be seen as a quantization of the observable hull $\widehat{K}$ rather than of K itself.

Let us now be more precise about the last condition in the previous definition. By asking $\mathcal{I}_q \big/ (q-1)\mathcal{I}_q \cong \pi_{F_q}(\mathcal{I}_q) = \mathcal{I}$ we mean the following: the specialization map sends $\mathcal{I}_q$ inside $F[G]$. This map factors through $\mathcal{I}_q \big/ (q-1)\mathcal{I}_q$; in addition, we require that the induced map $\mathcal{I}_q \big/ (q-1)\mathcal{I}_q \longrightarrow F[G]$ be a bijection on $\mathcal{I}$. Of course this bijection will respect the whole Hopf structure, since π_{F_q} does. Now, since

$$\pi_{F_q}(\mathcal{I}_q) = \mathcal{I}_q \Big/ \big(\mathcal{I}_q \cap (q-1) F_q[G]\big)$$

this property may be equivalently rephrased by saying that $\mathcal{I}_q \cap (q-1)\, F_q[G] = (q-1)\,\mathcal{I}_q$ as well. The previous discussions may be repeated unaltered for all four algebraic objects under consideration. *An equivalent definition of strict quantizations is therefore the following:*

$$\begin{array}{llll} (a) & \mathcal{I}_q \trianglelefteq_\ell \dot{\trianglelefteq} F_q[G] \;, & \pi_{F_q}(\mathcal{I}_q) = \mathcal{I} \;, & \mathcal{I}_q \cap (q-1) F_q[G] = (q-1)\,\mathcal{I}_q \\ (b) & \mathcal{C}_q \leq^1 \dot{\trianglelefteq}_\ell F_q[G] \;, & \pi_{F_q}(\mathcal{C}_q) = \mathcal{C} \;, & \mathcal{C}_q \cap (q-1) F_q[G] = (q-1)\,\mathcal{C}_q \\ (c) & \mathfrak{I}_q \trianglelefteq_\ell \dot{\trianglelefteq} U_q(\mathfrak{g}) \;, & \pi_{U_q}(\mathfrak{I}_q) = \mathfrak{I} \;, & \mathfrak{I}_q \cap (q-1) U_q(\mathfrak{g}) = (q-1)\,\mathfrak{I}_q \\ (d) & \mathfrak{C}_q \leq^1 \dot{\trianglelefteq}_\ell U_q(\mathfrak{g}) \;, & \pi_{U_q}(\mathfrak{C}_q) = \mathfrak{C} \;, & \mathfrak{C}_q \cap (q-1) U_q(\mathfrak{g}) = (q-1)\,\mathfrak{C}_q \end{array} \tag{3.3}$$

The purpose of the last condition — which is often mentioned by saying that $\mathfrak{C}_q$ is a *flat* quantization (typically, in the literature on deformation quantization) — should be clear: indeed, removing it means losing any control on what is contained, in quantization, inside the kernel of the specialization map.

Although the just mentioned notion of quantization appears to be, in many respect, the "correct" one — and indeed is typically the one considered in literature — another notion of quantization naturally appears when one has to deal with quantum duality principle.

Definition 3.2. *Let $F_q[G]$ and $U_q(\mathfrak{g})$ be a QFA and a QUEA for G and $\mathfrak{g}$ and let*

$$\pi_{F_q} \colon\, F_q[G] \twoheadrightarrow F_q[G] \Big/ (q-1) F_q[G] \;\cong\; F[G]$$

$$\pi_{U_q} \colon U_q(\mathfrak{g}) \twoheadrightarrow U_q(\mathfrak{g}) \Big/ (q-1)\, U_q(\mathfrak{g}) \;\cong\; U(\mathfrak{g})$$

*be the specialization maps. Let $\nabla := \Delta - \Delta^{\mathrm{op}}$. Let $\mathcal{I}$, $\mathcal{C}$, $\mathfrak{I}$ and $\mathfrak{C}$ be the algebraic objects associated with the subgroup K of G (see 2.1). We call "*proper quantization*" of each of them any object $\mathcal{I}_q$, $\mathcal{C}_q$, $\mathfrak{I}_q$ or $\mathfrak{C}_q$ respectively, such that*

$$\begin{array}{lllll} (a) & \mathcal{I}_q \trianglelefteq_\ell \dot{\trianglelefteq}\, F_q[G]\;, & \pi_{F_q}(\mathcal{I}_q) = \mathcal{I}\;, & \big[\mathcal{I}_q\,, \mathcal{I}_q\big] \subseteq (q-1)\,\mathcal{I}_q \\ (b) & \mathcal{C}_q \leq^1 \dot{\trianglelefteq}_\ell\, F_q[G]\;, & \pi_{F_q}(\mathcal{C}_q) = \mathcal{C}\;, & \big[\mathcal{C}_q\,, \mathcal{C}_q\big] \subseteq (q-1)\,\mathcal{C}_q \\ (c) & \mathfrak{I}_q \trianglelefteq_\ell \dot{\trianglelefteq}\; U_q(\mathfrak{g})\;, & \pi_{U_q}(\mathfrak{I}_q) = \mathfrak{I}\;, & \nabla(\mathfrak{I}_q) \subseteq (q-1)\, U_q(\mathfrak{g}) \wedge \mathfrak{I}_q \\ (d) & \mathfrak{C}_q \leq^1 \dot{\trianglelefteq}_\ell\; U_q(\mathfrak{g})\;, & \pi_{U_q}(\mathfrak{C}_q) = \mathfrak{C}\;, & \nabla(\mathfrak{C}_q) \subseteq (q-1)\, U_q(\mathfrak{g}) \wedge \mathfrak{C}_q \end{array} \tag{3.4}$$

The link between these two notions of quantization is the following:

Lemma 3.3. *Any strict quantization is a proper quantization.*

Proof. This is an easy consequence of definitions. Indeed, let K be a subgroup of G. If $\mathcal{I}_q := \mathcal{I}\big(\widehat{K}\big)$ is any strict quantization of $\mathcal{I}(K)$, we have

$$\mathcal{I}_q \cap (q-1)\, F_q \;=\; (q-1)\,\mathcal{I}_q$$

by assumption, and moreover $\big[F_q\,, F_q\big] \subseteq (q-1)\, F_q$. Then

$$\big[\mathcal{I}_q\,, \mathcal{I}_q\big] \;\subseteq\; \mathcal{I}_q \cap \big[F_q\,, F_q\big] \;\subseteq\; \mathcal{I}_q \cap (q-1)\, F_q \;=\; (q-1)\,\mathcal{I}_q$$

thus $\big[\mathcal{I}_q\,, \mathcal{I}_q\big] \subseteq (q-1)\,\mathcal{I}_q$, i.e. $\mathcal{I}_q$ is proper. A similar argument works for quantizations of type $\mathcal{C}_q(K)$. Also, if $\mathfrak{I}_q(K)$ is any strict quantization of $\mathfrak{I}(K)$, then we have $\mathfrak{I}_q \cap (q-1)\, U_q \;=\; (q-1)\,\mathfrak{I}_q$ by assumption, and moreover $\nabla(U_q) \subseteq (q-1)\, U_q^{\wedge 2}$. Then

$$\nabla(\mathfrak{I}_q) \;\subseteq\; \big(U_q \wedge \mathfrak{I}_q\big) \cap \nabla(U_q) \;\subseteq\; \big(U_q \wedge \mathfrak{I}_q\big) \cap (q-1)\, U_q^{\wedge 2} \;\subseteq\; (q-1)\, U_q \wedge \mathfrak{I}_q$$

so that $\mathfrak{I}_q$ is proper. A similar argument works for quantizations of type $\mathfrak{C}_q(K)$ as well. $\square$

Remark 3.4. *The converse to Lemma 3.3 here above is false.* Indeed, there exist quantizations (of subgroups / homogeneous spaces) which are proper but *not* strict: we present an explicit example — of type $\mathcal{C}_q$ — in Subsection 6.3 later on.
This means that giving two different versions of "quantization" does make sense, in that they actually capture two *inequivalent* notions — hierarchically related via Lemma 3.3.

The following statement clarifies why such definitions actually apply only to the (restricted) case of coisotropic subgroups (this result can be traced back to [18], where it is mentioned as *coisotropic creed*).

Proposition 3.5. *Let K be a subgroup of G and assume a proper quantization of it exists. Then K is coisotropic or, in case the quantization is $\mathcal{C}_q$, its observable hull $\widehat{K}$ is coisotropic.*

Proof. Assume $\mathcal{I}_q$ exists. Let $f, g \in \mathcal{I}$, and let $\varphi, \gamma \in \mathcal{I}_q$ with $\pi_{F_q}(\varphi) = f$, $\pi_{F_q}(\gamma) = g$. Then by definition $\{f, g\} = \pi_{F_q}\big((q-1)^{-1}[\varphi, \gamma]\big)$. But

$$[\varphi, \gamma] \in \big[\mathcal{I}_q, \mathcal{I}_q\big] \subseteq (q-1)\,\mathcal{I}_q$$

by assumption, hence $(q-1)^{-1}[\varphi, \gamma] \in \mathcal{I}_q$, thus $\{f, g\} = \pi_{F_q}\big((q-1)^{-1}[\varphi, \gamma]\big) \in \pi_{F_q}(\mathcal{I}_q) = \mathcal{I}$, which means that $\mathcal{I}$ is closed for the Poisson bracket. Thus (see (2.6)) K is coisotropic.
Similar arguments work when dealing with $\mathcal{C}_q$, $\mathfrak{I}_q$ or $\mathfrak{C}_q$. We shall only remark that working with $\mathcal{C}_q$ we end up with $\mathcal{C}\big(\widehat{K}\big) = \mathcal{C}(K) \leq_{\mathcal{P}} F[G]$, whence $\widehat{K}$ is coisotropic. $\square$

Since we would like to show also what happens in the non coisotropic case, we will consider, also, the weakest possible — naïve — version of quantization.

Definition 3.6. *Let $F_q[G]$ and $U_q(\mathfrak{g})$ be a QFA and a QUEA for G and $\mathfrak{g}$ and let*

$$\pi_{F_q} \colon\, F_q[G] \longrightarrow\!\!\!\!\!\rightarrow F_q[G]\Big/(q-1)F_q[G] \;\cong\; F[G]$$
$$\pi_{U_q} \colon U_q(\mathfrak{g}) \longrightarrow\!\!\!\!\!\rightarrow U_q(\mathfrak{g})\Big/(q-1)\,U_q(\mathfrak{g}) \;\cong\; U(\mathfrak{g})$$

be the specialization maps. Let $\mathcal{I}$, $\mathcal{C}$, $\mathfrak{I}$ and $\mathfrak{C}$ be the algebraic objects associated with the subgroup K of G (see 2.1). We call "weak quantization" *of each of them any object $\mathcal{I}_q$, $\mathcal{C}_q$, $\mathfrak{I}_q$ or $\mathfrak{C}_q$ respectively, such that*

$$\begin{array}{lll} (a) & \mathcal{I}_q \trianglelefteq_\ell \dot{\trianglelefteq}\, F_q[G] \;, & \pi_{F_q}(\mathcal{I}_q) \,=\, \mathcal{I} \\ (b) & \mathcal{C}_q \leq^1 \dot{\trianglelefteq}_\ell\, F_q[G] \;, & \pi_{F_q}(\mathcal{C}_q) \,=\, \mathcal{C} \\ (c) & \mathfrak{I}_q \trianglelefteq_\ell \dot{\trianglelefteq}\, U_q(\mathfrak{g}) \;, & \pi_{U_q}(\mathfrak{I}_q) \,=\, \mathfrak{I} \\ (d) & \mathfrak{C}_q \leq^1 \dot{\trianglelefteq}_\ell\, U_q(\mathfrak{g}) \;, & \pi_{U_q}(\mathfrak{C}_q) \,=\, \mathfrak{C} \end{array} \tag{3.5}$$

It is obvious that strict or proper quantizations are weak. Let us remark that every subgroup of G is quantizable in the weak sense, since we may just consider e.g. $\mathcal{I}_q := \pi_{F_q}^{-1}(\mathcal{I})$ to be a quantization of $\mathcal{I}$. As naïf as it may seem, this remark will play a rôle in what follows.

Let us lastly remark how the real case should be treated.

Definition 3.7. *Let $(F_q[G], *)$ and $(U_q(\mathfrak{g}), *)$ be a real QFA and a real QUEA for G and $\mathfrak{g}$. Let $\mathcal{I}_q$, $\mathcal{C}_q$, $\mathfrak{I}_q$ and $\mathfrak{C}_q$ be subgroup quantizations (either strict, proper or weak). Then such quantizations are called real if*

$$\big(S(\mathcal{I}_q)\big)^\star = \mathcal{I}_q \,, \qquad \mathcal{C}_q^\star = \mathcal{C}_q \,, \qquad \big(S(\mathfrak{I}_q)\big)^\star = \mathfrak{I}_q \,, \qquad \mathfrak{C}_q^\star = \mathfrak{C}_q \tag{3.6}$$

3.8. The formal quantization approach. In the present work we are dealing with global quantizations. In [5] instead we treated *formal quantizations*: these are topological Hopf $\mathbb{C}[[h]]$–algebras which for $h = 0$ yield back the (formal) Hopf algebras associated with a (formal) group. In this case, such objects as $\mathcal{I}_q$, $\mathcal{C}_q$, $\mathfrak{I}_q$ and $\mathfrak{C}_q$ are defined in the parallel way. However, in [5] we did *not* consider the notions of *proper* nor *weak* quantizations but only dealt with strict quantizations. Actually, one can consider the notions of proper or weak quantizations in the formal quantization setup as well; then the relation between these and strict quantizations will be again the same as we showed here above.
We point out also that the semiclassical limits of formal quantizations are just formal Poisson groups, or their universal enveloping algebras, or subgroups, homogeneous spaces, etc. In any case, this means — see the end of Subsection 2.1 — that no restrictions on subgroups apply (all are "observable") nor on homogeneous spaces (all are "quasi-affine").

4 Quantum duality principle

Drinfeld's quantum duality principle (cf. [7], §7; see also [10] for a proof) has a stronger version (see [12]) best suited for *our* quantum groups — in the sense of Section 3.

Let H be any Hopf algebra in $\mathcal{HA}$ and let

$$I := Ker\Big(H \overset{\epsilon}{\twoheadrightarrow} \mathbb{C}\big[q,q^{-1}\big] \xrightarrow{ev_1} \mathbb{C}\Big) = Ker\Big(H \xrightarrow{ev_1} H/(q-1)\,H \overset{\bar{\epsilon}}{\twoheadrightarrow} \mathbb{C}\Big) \tag{4.1}$$

Then I is a Hopf ideal of H. We define

$$H^\vee := \sum_{n \geq 0} (q-1)^{-n} I^n = \bigcup_{n \geq 0} \big((q-1)^{-1} I\big)^n \quad \Big(\subseteq \mathbb{C}(q) \otimes_{\mathbb{C}[q,q^{-1}]} H\Big) \tag{4.2}$$

Notice that, setting $J := Ker\Big(H \overset{\epsilon}{\twoheadrightarrow} \mathbb{C}\big[q,q^{-1}\big]\Big)$, one has $I = (q-1)\cdot 1_{H} + J$, so that

$$H^\vee \;=\; \sum\nolimits_{n\geq 0} (q-1)^{-n} J^n \;=\; \sum\nolimits_{n\geq 0} \big((q-1)^{-1} J\big)^n \tag{4.3}$$

Consider, now, for every $n \in \mathbb{N}$ the iterated coproduct $\Delta^n \colon H \to H^{\otimes n}$ where

$$\Delta^0 := \epsilon \qquad \Delta^1 := \mathrm{id}_H \qquad \Delta^n := \big(\Delta \otimes \mathrm{id}_H^{\otimes (n-2)}\big) \circ \Delta^{n-1} \quad \text{if } \; n \geq 2 \;\; .$$

For any ordered subset $\Sigma = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ with $i_1 < \cdots < i_k$, define the morphism $j_\Sigma : H^{\otimes k} \longrightarrow H^{\otimes n}$ by

$$j_\Sigma(a_1 \otimes \cdots \otimes a_k) := b_1 \otimes \cdots \otimes b_n \text{ where } \begin{cases} b_i := 1 & \text{if} \quad i \notin \Sigma \\ b_{i_m} := a_m & \text{if} \quad 1 \leq m \leq k \end{cases}$$

then set $\Delta_\Sigma := j_\Sigma \circ \Delta^k$, $\Delta_\emptyset := \Delta^0$, and $\delta_\Sigma := \sum_{\Sigma' \subseteq \Sigma} (-1)^{n-|\Sigma'|} \Delta_{\Sigma'}$, $\delta_\emptyset := \epsilon$. By the inclusion-exclusion principle, the inverse formula $\Delta_\Sigma = \sum_{\Psi \subseteq \Sigma} \delta_\Psi$ holds. We shall use notation $\delta_0 := \delta_\emptyset$, $\delta_n := \delta_{\{1,2,\dots,n\}}$, and the key identity $\delta_n = (\mathrm{id_H} - \epsilon)^{\otimes n} \circ \Delta^n$, for all $n \in \mathbb{N}_+$. Given $H \in \mathcal{H}$, we define

$$H' := \big\{\, a \in H \,\big|\, \delta_n(a) \in (q-1)^n H^{\otimes n}, \; \forall \; n \in \mathbb{N} \,\big\} \quad \big(\subseteq H\,\big) \;\; . \tag{4.4}$$

Theorem 4.1 (Global Quantum Duality Principle). *(cf. [12]) For any $H \in \mathcal{HA}$ one has:*

(a) $H^\vee$ is a QUEA and H' is a QFA. Moreover the following inclusions hold:

$$H \subseteq \big(H^\vee\big)' \;, \quad H \supseteq \big(H'\big)^\vee \;, \quad H^\vee = \big(\big(H^\vee\big)'\,\big)^\vee \;, \quad H' = \big(\big(H'\big)^\vee\big)' \tag{4.5}$$

(b) $H = \big(H^\vee\big)' \iff H$ is a QFA, and $H = \big(H'\big)^\vee \iff H$ is a QUEA;

(c) If G is a Poisson group with Lie bialgebra $\mathfrak{g}$, then

$$F_q[G]^\vee \Big/ (q-1) F_q[G]^\vee = U(\mathfrak{g}^*) \qquad U_q(\mathfrak{g})' \big/ (q-1) U_q(\mathfrak{g})' = F[G^*]$$

where G^* is some connected Poisson group dual to G;

(d) Let $F_q[G]$ and $U_q(\mathfrak{g})$ be dual to each other w.r. to some perfect Hopf pairing. Then $F_q[G]^\vee$ and $U_q(\mathfrak{g})'$ are dual to each other w.r. to the same pairing.

A number of remarks are due, at this point:

1. The Poisson group G^* dual to G appearing in *(c)* of Theorem 4.1 does depend on $U_q(\mathfrak{g})$ which is given as a data. Different choices of $U_q(\mathfrak{g})$, though associated with the same Lie bialgebra $\mathfrak{g}$ may give rise to a different connected Poisson dual group G^*.

2. For all Hopf $\mathbb{C}(q)$–algebra $\mathbb{H}$ the existence of a $\mathbb{C}\big[q,q^{-1}\big]$-integral form H_f which is a QUEA at $q=1$ is equivalent to the existence of a $\mathbb{C}\big[q,q^{-1}\big]$–integer form H_u which is a QFA at $q=1$.

3. All claims above have obvious analogues in the real case.

4. If H is a Hopf algebra and $\Phi \subseteq \mathbb{N}$ is a finite subset, then ([16], Lemma 3.2)
$$\delta_\Phi(ab) \;=\; \sum_{\Lambda\cup Y=\Phi} \delta_\Lambda(a)\,\delta_Y(b) \qquad \forall\ a,b\in H \tag{4.6}$$
furthermore, if $\Phi\neq\emptyset$ we have
$$\delta_\Phi(ab-ba) \;=\; \sum_{\substack{\Lambda\cup Y=\Phi \\ \Lambda\cap Y\neq\emptyset}} \big(\delta_\Lambda(a)\,\delta_Y(b)-\delta_Y(b)\,\delta_\Lambda(a)\big) \qquad \forall\ a,b\in H \tag{4.7}$$
The above formulas will be used frequently in what follows

Having clarified the exact statement of quantum duality principle that we have in mind, let us extend it to objects of subgroup type as in Definition 3.6, i.e. to left coideal subalgebras and to left ideals and two-sided coideals — either in $F_q[G]$ or in $U_q(\mathfrak{g})$. This was already done in [5] where we only considered *local* (i.e. over $\mathbb{C}[[h]]$) quantizations. Let us remark that the quantum duality principle we have in mind not only exchanges the rôle of algebras of functions with that of universal enveloping algebras, but also exchanges the rôle of subgroups with that of homogeneous spaces. At the semiclassical level, the pair of dual objects is given by a coisotropic subgroup H and a Poisson quotient $G^*\big/H^\perp$. When H is a Poisson subgroup, its orthogonal $H^\perp$ turns out to be normal in G^* and $G^*\big/H^\perp \cong H^*$ as a Poisson group, thus recovering the usual quantum duality principle. In particular, we will consider a process moving along the following draft:

$$\begin{array}{ll}
(a) & \mathcal{I}\,\big(\subseteq F[G]\,\big) \xrightarrow{(1)} \mathcal{I}_q\,\big(\subseteq F_q[G]\,\big) \xrightarrow{(2)} {\mathcal{I}_q}^{\curlyvee}\,\big(\subseteq F_q[G]^\vee\,\big) \xrightarrow{(3)} {\mathcal{I}_1}^{\curlyvee}\,\big(\subseteq U(\mathfrak{g}^*)\big) \\
(b) & \mathcal{C}\,\big(\subseteq F[G]\,\big) \xrightarrow{(1)} \mathcal{C}_q\,\big(\subseteq F_q[G]\,\big) \xrightarrow{(2)} \mathcal{C}_q^{\,\triangledown}\,\big(\subseteq F_q[G]^\vee\,\big) \xrightarrow{(3)} \mathcal{C}_1^{\,\triangledown}\,\big(\subseteq U(\mathfrak{g}^*)\big) \\
(c) & \mathfrak{I}\,\big(\subseteq U(\mathfrak{g})\,\big) \xrightarrow{(1)} \mathfrak{I}_q\,\big(\subseteq U_q(\mathfrak{g})\,\big) \xrightarrow{(2)} \mathfrak{I}_q^{\,!}\,\big(\subseteq U_q(\mathfrak{g})'\,\big) \xrightarrow{(3)} \mathfrak{I}_1^{\,!}\,\big(\subseteq F[G^*]\big) \\
(d) & \mathfrak{C}\,\big(\subseteq U(\mathfrak{g})\,\big) \xrightarrow{(1)} \mathfrak{C}_q\,\big(\subseteq U_q(\mathfrak{g})\,\big) \xrightarrow{(2)} \mathfrak{C}_q^{\,\Lsh}\,\big(\subseteq U_q(\mathfrak{g})'\,\big) \xrightarrow{(3)} \mathfrak{C}_1^{\,\Lsh}\,\big(\subseteq F[G^*]\big)
\end{array}$$

where arrows (1) are quantizations, arrows (3) are specializations at $q=1$ and the definition of arrows (2) will be the core of what follows. It will turn out that:

1. each one of the right-hand-side objects above is one of the four algebraic objects which describe a closed connected subgroup of G^*: namely, the correspondence is
$$(a) \Longrightarrow (c)\,, \qquad (b) \Longrightarrow (d)\,, \qquad (c) \Longrightarrow (a)\,, \qquad (d) \Longrightarrow (b)\,.$$

2. the four quantizations of subgroups of G^* so obtained are always *proper* — hence the subgroups of G^* associated with them are *coisotropic*.

3. if we begin with *strict* quantizations, and we start from a subgroup K, then the quantization of the unique coisotropic closed connected subgroup of G^* mentioned above is *strict* as well, and the subgroup itself is $K^{\perp}$ (cf. Definition 2.1), with some care in case *(b)*, i.e. if we start from $\mathcal{C}(K)$. This will partially generalize to *weak* quantizations, for which, starting from a subgroup K of G, the unique coisotropic closed connected subgroup of G^* obtained above is $K^{\langle\perp\rangle}$ (cf. Proposition 2.2).

Let us fix, in what follows, quantizations $U_q(\mathfrak{g})$ and $F_q[G]$ as in Section 3. Unless explicitly mentioned we will not assume that this is a double quantization. To simplify notations, let us set

$$\begin{array}{lll} \mathbb{U}_q := \mathbb{U}_q(\mathfrak{g}) \quad , & U_q := U_q(\mathfrak{g}) \quad , & {U_q}' := U_q(\mathfrak{g})' \\ \mathbb{F}_q := \mathbb{F}_q[G] \quad , & F_q := F_q[G] \quad , & {F_q}^\vee := F_q[G]^\vee \end{array}$$

As mentioned in the first remark after Theorem 4.1, this implies that a specific connected Poisson dual G^* of G is selected (it depends on the choice of $U_q := U_Q(\mathfrak{g})$, not only on $\mathfrak{g}$ itself). Let us consider quantum subgroups $\mathcal{I}_q$, $\mathcal{C}_q$, $\mathfrak{I}_q$ and $\mathfrak{C}_q$ as defined in 3.6.

Definition 4.2. *Using notations as in (4.1) we define:*

(a) $${\mathcal{I}_q}^{\curlyvee} := \sum_{n=1}^{\infty} (q-1)^{-n} \cdot I^{\,n-1} \cdot \mathcal{I}_q = \sum_{n=1}^{\infty} (q-1)^{-n} \cdot J^{\,n-1} \cdot \mathcal{I}_q$$

(b) $${\mathcal{C}_q}^{\triangledown} := \sum_{n=0}^{\infty} (q-1)^{-n} \cdot \big(\mathcal{C}_q \cap I\,\big)^n = \sum_{n=0}^{\infty} (q-1)^{-n} \cdot \big(\mathcal{C}_q \cap J\big)^n$$

(c) $${\mathfrak{I}_q}^{!} := \Big\{ x \in \mathfrak{I}_q \,\Big|\, \delta_n(x) \in (q-1)^n \sum_{s=1}^{n} {U_q}^{\otimes (s-1)} \otimes \mathfrak{I}_q \otimes {U_q}^{\otimes (n-s)}, \ \forall\, n \in \mathbb{N}_+ \Big\}$$

(d) $${\mathfrak{C}_q}^{\Lsh} := \Big\{ x \in \mathfrak{C}_q \,\Big|\, \delta_n(x) \in (q-1)^n \, {U_q}^{\otimes (n-1)} \otimes \mathfrak{C}_q \,, \ \forall\, n \in \mathbb{N}_+ \Big\}$$

Let us remark that the following inclusions hold directly by definitions:

$$(i)\ \ {\mathcal{I}_q}^{\curlyvee} \supseteq \mathcal{I}_q \,, \qquad (ii)\ \ {\mathcal{C}_q}^{\triangledown} \supseteq \mathcal{C}_q \,, \qquad (iii)\ \ {\mathfrak{I}_q}^{!} \subseteq \mathfrak{I}_q \,, \qquad (iv)\ \ {\mathfrak{C}_q}^{\Lsh} \subseteq \mathfrak{C}_q \,. \qquad (4.8)$$

5 Duality maps

In the present section we will prove properties of the four Drinfeld–type maps defined in the previous section, namely the maps $\mathcal{I}_q \mapsto {\mathcal{I}_q}^{\curlyvee}$, $\mathcal{C}_q \mapsto {\mathcal{C}_q}^{\triangledown}$, $\mathfrak{I}_q \mapsto {\mathfrak{I}_q}^{!}$ and $\mathfrak{C}_q \mapsto {\mathfrak{C}_q}^{\Lsh}$. Let us recall that such maps do not change, as

we will see, the algebraic properties of subobjects, but interchanges quantized function algebra with quantum enveloping algebra and therefore quantizations of coisotropic subgroups will be sent to quantizations of (embeddable) homogeneous spaces — of the dual quantum group — and viceversa.
Let us start by considering the map $\mathcal{I}_q \mapsto {\mathcal{I}_q}^\curlyvee$.

Proposition 5.1. *Let $\mathcal{I}_q = \mathcal{I}_q(K)$ be a left ideal and two-sided coideal in $F_q[G]$, that is a weak quantization (of type $\mathcal{I}$) of some subgroup K of G. Then*

1. *${\mathcal{I}_q}^\curlyvee$ is a left ideal and two-sided coideal in ${F_q[G]}^\vee$;*

2. *if $\mathcal{I}_q$ is strict, then ${\mathcal{I}_q}^\curlyvee$ is strict too, i.e. ${\mathcal{I}_q}^\curlyvee \bigcap (q-1)\, {F_q[G]}^\vee = (q-1)\, {\mathcal{I}_q}^\curlyvee$;*

3. *there exists a coisotropic subgroup L of G^* such that ${\mathcal{I}_q(K)}^\curlyvee = \mathfrak{I}_q(L)$: namely, ${\mathcal{I}_q(K)}^\curlyvee$ is a proper quantization, of type $\mathfrak{I}$, of some coisotropic subgroup L of G^*;*

4. *in the real case, i.e. if the quantization $\mathcal{I}_q$ is a real one, ${\mathcal{I}_q}^\curlyvee$ is real too, i.e. $\big(S({\mathcal{I}_q}^\curlyvee)\big)^* = {\mathcal{I}_q}^\curlyvee$. Therefore claims (1–3) still hold in the framework of real quantum subgroups.*

Proof. (1) Consider that ${\mathcal{I}_q}^\curlyvee$ is the left ideal of ${F_q}^\vee$ generated by $(q-1)^{-1}\mathcal{I}_q$; therefore, in order to prove ${\mathcal{I}_q}^\curlyvee \,\dot{\trianglelefteq}\, {F_q}^\vee$ it is enough to show that $\Delta\big((q-1)^{-1}\mathcal{I}_q\big) \subseteq {F_q}^\vee \otimes {\mathcal{I}_q}^\curlyvee + {\mathcal{I}_q}^\curlyvee \otimes {F_q}^\vee$. Since $\mathcal{I}_q$ is a coideal of F_q, we have

$$\begin{aligned} &\Delta\big((q-1)^{-1}\mathcal{I}_q\big) \subseteq \\ &\quad \subseteq F_q \otimes (q-1)^{-1}\mathcal{I}_q + (q-1)^{-1}\mathcal{I}_q \otimes F_q \subseteq {F_q}^\vee \otimes {\mathcal{I}_q}^\curlyvee + {\mathcal{I}_q}^\curlyvee \otimes {F_q}^\vee \end{aligned} \tag{5.1}$$

whence ${\mathcal{I}_q}^\curlyvee \,\dot{\trianglelefteq}\, {F_q}^\vee$ follows, and the first claim is proved. *(2)* Assume $\mathcal{I}_q$ to be a *strict* quantization, so that $\mathcal{I}_q \bigcap (q-1)\, F_q = (q-1)\, \mathcal{I}_q$.
Let $J := Ker\big(\epsilon :\, F_q \longrightarrow \mathbb{C}\big[q, q^{-1}\big]\big)$. Then

$$J \mod (q-1)F_q = Ker(\epsilon)\big|_{F[G]} = \mathfrak{m}_e$$

and $\mathfrak{m}_e \Big/ {\mathfrak{m}_e}^2 = \mathfrak{g}^*$, the cotangent Lie bialgebra of G. Let $\{y_1, \dots, y_n\}$ be a subset of $\mathfrak{m}_e$ whose image in the local ring of G at the identity e is a local system of parameters, and pull it back to a subset $\{j_1, \dots, j_n\}$ of J. Let $\widehat{F}_q$ be the J–adic completion of F_q. From [12], Lemma 4.1, we know that the set of ordered monomials $\big\{ j^{\underline{e}} \,\big|\, \underline{e} \in \mathbb{N}^n \big\}$ (where hereafter $j^{\underline{e}} := \prod_{s=1}^n j_s^{\underline{e}(i)}$, for all $\underline{e} \in \mathbb{N}^n$) is a $\mathbb{C}\big[q, q^{-1}\big]$–pseudobasis of $\widehat{F}_q$, which means that each element of $\widehat{F}_q$ has a unique expansion as a formal infinite linear combination of the $j^{\underline{e}}$'s. In a similar way, the $(q-1)$–adic completion of ${F_q}^\vee$ admits $\big\{ (q-1)^{-|\underline{e}|} j^{\underline{e}} \,\big|\, \underline{e} \in \mathbb{N}^n \big\}$ as a $\mathbb{C}\big[q, q^{-1}\big]$–pseudobasis, where $|\underline{e}| := \sum_{i=1}^n \underline{e}(i)$.

For our purposes we need a special choice of the set $\{j_1,\dots,j_n\}$ adapted to the smooth subvariety K of G. By general theory we can choose $\{y_1,\dots,y_n\}$ so that $y_1, \dots, y_k \in \mathfrak{m}_e$ and $y_{k+1},\dots,y_n \in \mathcal{I}(K)$, where $k = dim(K)$. We can also choose the lift $\{j_1,\dots,j_n\}$ of $\{y_1,\dots,y_n\}$ inside J so that j_s is a lift of y_s, for all $s = 1,\dots,k$, and $j_{k+1},\dots,j_n \in \mathcal{I}_q$. With these assumptions, it's easy to see that

$$\varphi \in {\mathcal{I}_q}^{\curlyvee} \bigcap (q-1)\, {F_q}^{\vee} \implies (q-1)^n \varphi \in \big(J^{n-1} \cdot \mathcal{I}_q\big) \bigcap (q-1)\, J^n$$

for some $n \in \mathbb{N}$, which in turn yields $(q-1)^n \varphi \in J^{n-1} \cdot \big(\mathcal{I}_q \bigcap (q-1)\, J\big)$. Since

$$\mathcal{I}_q \bigcap (q-1)\, J \subseteq \mathcal{I}_q \bigcap (q-1)\, F_q = (q-1)\, \mathcal{I}_q$$

we conclude that $(q-1)^n \varphi \in (q-1)\, J^{n-1} \cdot \mathcal{I}_q$, whence $\varphi \in (q-1)\, {\mathcal{I}_q}^{\curlyvee}$. The converse inclusion ${\mathcal{I}_q}^{\curlyvee} \bigcap (q-1)\, {F_q}^{\vee} \supseteq (q-1)\, {\mathcal{I}_q}^{\curlyvee}$ is obvious, hence claim *(2)* is proved. *(3)* It is an obvious statement that ${\mathcal{I}_q}^{\curlyvee}$ is a weak quantization of its image $\pi_{{F_q}^\vee}\big({\mathcal{I}_q}^{\curlyvee}\big)$: in particular, $\pi_{{F_q}^\vee}\big({\mathcal{I}_q}^{\curlyvee}\big) \trianglelefteq_\ell \dot{\trianglelefteq}\, \pi_{{F_q}^\vee}\big({F_q}^{\vee}\big) = U\big(\mathfrak{g}^*\big)$ implies that $\pi_{{F_q}^\vee}\big({\mathcal{I}_q}^{\curlyvee}\big) = \mathfrak{I}(L)$ for some subgroup L of G^*. Thus ${\mathcal{I}_q}^{\curlyvee}$ is a weak quantization, to be called $\mathfrak{I}_q(L)$, of $\mathfrak{I}(L)$, and it is even strict if $\mathcal{I}_q$ itself is strict, as we've just seen. Now we show that such quantization $\mathfrak{I}_q(L)$ turns out to be always *proper*.
In fact, (5.1) implies $\nabla\big((q-1)^{-1}\, \mathcal{I}_q\big) \subseteq (q-1)^{-1} \big(F_q \wedge \mathcal{I}_q\big)$. On the other hand $F_q \wedge \mathcal{I}_q \subseteq J \wedge \mathcal{I}_q \subseteq (q-1)^2\, {F_q}^{\vee} \wedge {\mathcal{I}_q}^{\curlyvee}$, thus, finally, $\nabla\big({\mathcal{I}_q}^{\curlyvee}\big) \in (q-1)\, {F_q}^{\vee} \wedge {\mathcal{I}_q}^{\curlyvee}$, which means that ${\mathcal{I}_q}^{\curlyvee}$ is proper and *(3)* holds. *(4)* This is an obvious consequence of definitions. □

Remark 5.2. In functorial language we may say that the map $\mathcal{I}_q \mapsto {\mathcal{I}_q}^{\curlyvee}$ establishes a functor between quantizations of coisotropic subgroups of G and quantizations of (embeddable) homogeneous spaces of G^*, moving from a global to a local description, sending each type of quantization in a proper one and preserving strictness. Indeed, we should make precise what are the "arrows" in our categories of "quantum subgroups" or "quantum homogeneous spaces", and how the functor acts on these: we leave these details to the interested reader.

Let us move on to properties of the map $\mathcal{C}_q \mapsto {\mathcal{C}_q}^{\triangledown}$.

Proposition 5.3. *Let $\mathcal{C}_q = \mathcal{C}_q(K)$ be a left coideal subalgebra in $F_q[G]$. Then*

1. *${\mathcal{C}_q}^{\triangledown}$ is a left coideal subalgebra in $F_q[G]^{\vee}$;*

2. *if $\mathcal{C}_q$ is strict, then ${\mathcal{C}_q}^{\triangledown}$ is strict too, i.e. ${\mathcal{C}_q}^{\triangledown} \bigcap (q-1)\, F_q[G]^{\vee} = (q-1)\, {\mathcal{C}_q}^{\triangledown}$.*

3. *there exists a coisotropic subgroup L of G^* such that ${\mathcal{C}_q(K)}^{\triangledown} = \mathfrak{C}_q(L)$: namely, ${\mathcal{C}_q(K)}^{\triangledown}$ is a proper quantization, of type $\mathfrak{C}$, of some coisotropic subgroup L of G^*;*

4. in the real case, i.e. if the quantization $\mathcal{C}_q$ is a real one, $\mathcal{C}_q(K)^{\triangledown}$ is real too, i.e. $\big(\mathcal{C}_q^{\triangledown}\big)^* = \mathcal{C}_q^{\triangledown}$. Therefore claims (1–3) still hold in the framework of real quantum subgroups.

Proof. The proof uses essentially the same arguments as the previous one. *(1)* By the very definitions $\mathcal{C}_q^{\triangledown} \leq^1 F_q^{\vee} := F_q[G]^{\vee}$. More precisely, $\mathcal{C}_q^{\triangledown}$ is (by construction) the unital $\mathbb{C}\big[q,q^{-1}\big]$–subalgebra of $F_q^{\vee}$ generated by $(q-1)^{-1}\,(\mathcal{C}_q)^+$, where $(\mathcal{C}_q)^+ := \mathcal{C}_q \bigcap J$. So to get $\mathcal{C}_q^{\triangledown} \dot{\trianglelefteq}_\ell F_q^{\vee}$ we must only prove $\Delta\big((q-1)^{-1}(\mathcal{C}_q)^+\big) \subseteq F_q^{\vee} \otimes \mathcal{C}_q^{\triangledown}$. But $\mathcal{C}_q \dot{\trianglelefteq}_\ell F_q$, so:

$$\Delta\big((q-1)^{-1}(\mathcal{C}_q)^+\big) \subseteq F_q \otimes (q-1)^{-1}(\mathcal{C}_q)^+ \subseteq F_q^{\vee} \otimes \mathcal{C}_q^{\triangledown} \tag{5.2}$$

therefore $\mathcal{C}_q^{\triangledown} \dot{\trianglelefteq}_\ell F_q^{\vee}$, and claim *(1)* is proved. *(2)* Now suppose $\mathcal{C}_q$ to be a *strict* quantization, i.e. $\mathcal{C}_q \bigcap (q-1)\,F_q = (q-1)\,\mathcal{C}_q$. We need an explicit description of $F_q^{\vee}$ and of $\mathcal{C}_q^{\triangledown}$. This goes along the same lines followed to describe $\mathcal{I}_q^{\curlyvee}$ in the proof of Proposition 5.1: but now the choice of the subset $\{j_1,\dots,j_n\}$ of J is different.
First, since $\mathcal{C}(K) = \mathcal{C}\big(\widehat{K}\big)$ we can assume that $K = \widehat{K}$, i.e. K is observable. Then we can choose $\big\{j_1,\dots,j_n\big\}$ so that $j_{k+1},\dots,j_n \in J \bigcap \mathcal{C}_q = \mathcal{C}_q^+$ (where again $k = \mathit{dim}(K)$) and, letting $y_s := j_s \mod (q-1)\,F_q$, the set $\big\{y_1,\dots,y_n\big\}$ yields a local system of parameters at $e \in G$ (in the localized ring), as before; now in addition we have $y_{k+1},\dots,y_n \in \mathfrak{m}_e \bigcap \mathcal{C}(K) =: \mathcal{C}(K)^+$. With these assumptions, the $(q-1)$–adic completion of $F_q^{\vee}$ admits $\big\{(q-1)^{-|\underline{e}|} j^{\,\underline{e}} \;\big|\; \underline{e} \in \mathbb{N}^n\big\}$ as a $\mathbb{C}\big[q,q^{-1}\big]$–pseudobasis, like before, but in addition the same analysis can be done for the $(q-1)$–adic completion of $\mathcal{C}_q^{\triangledown}$ (just because $\mathcal{C}_q$ is *strict*), which then has $\mathbb{C}\big[q,q^{-1}\big]$–pseudobasis $\big\{\prod_{s=k+1}^n j_s^{e_s} \,\big|\, (e_{k+1},\dots,e_n) \in \mathbb{N}^{n-k}\big\}$. From these description of the completions, and comparing the former with $F_q^{\vee}$ and $\mathcal{C}_q$, we easily see that $\mathcal{C}_q^{\triangledown} \bigcap (q-1)\,F_q^{\vee} \subseteq (q-1)\,\mathcal{C}_q^{\triangledown}$. The converse is trivial, hence claim *(1)* is proved. *(3)* It follows directly from *(1)* that $\mathcal{C}_q^{\triangledown}$ is a weak quantization of its image $\pi_{F_q^{\vee}}\big(\mathcal{C}_q^{\triangledown}\big)$: in particular, $\pi_{F_q^{\vee}}\big(\mathcal{C}_q^{\triangledown}\big) \leq^1 \dot{\trianglelefteq}_\ell \, \pi_{F_q^{\vee}}\big(F_q^{\vee}\big) = U\big(\mathfrak{g}^*\big)$ means that $\pi_{F_q^{\vee}}\big(\mathcal{C}_q^{\triangledown}\big) = \mathfrak{C}(L)$ for some subgroup L of G^*. Thus $\mathcal{C}_q^{\triangledown}$ is a weak quantization — to be called $\mathfrak{C}_q(L)$ — of $\mathfrak{C}(L)$, and it is even strict if $\mathcal{C}_q$ itself is strict, by claim *(1)*. Now in addition we show that, in any case, such a quantization $\mathfrak{C}_q(L)$ is always *proper*.
From (5.2) we have

$$\begin{aligned}\nabla\big((q-1)^{-1}\,(\mathcal{C}_q)^+\big) &\subseteq (q-1)^{-1} J \wedge (\mathcal{C}_q)^+ \subseteq \\ &\subseteq (q-1)^{-1+2}\, F_q^{\vee} \wedge \mathcal{C}_q^{\triangledown} = (q-1)\, F_q^{\vee} \wedge \mathcal{C}_q^{\triangledown}\end{aligned}$$

which implies exactly that $\mathcal{C}_q^{\triangledown}$ — which by definition is the unital subalgebra generated by $(q-1)^{-1}\,(\mathcal{C}_q)^+$ — is proper. *(4)* This follows directly from definitions and from $\mathcal{C}_q^* = \mathcal{C}_q$, which holds by assumption. ∎

Remark 5.4. In functorial language we may say that the map $\mathcal{C}_q \mapsto \mathcal{C}_q^{\triangledown}$ establishes a functor between quantized homogeneous spaces of G and quantizations of coisotropic subgroups of G^*, moving from a global to a local description, sending each type of quantization in a proper one and preserving strictness. Again, to be precise, several details need to be fixed, and are left to the reader.

The third step copes with the map $\mathfrak{I}_q \mapsto \mathfrak{I}_q^{\,!}$.

Proposition 5.5. *Let $\mathfrak{I}_q = \mathfrak{I}_q(K)$ be a left ideal and two-sided coideal in $U_q(\mathfrak{g})$, weak quantization (of type $\mathfrak{I}$) of some coisotropic subgroup K of G. Then:*

1. *$\mathfrak{I}_q^{\,!}$ is a left ideal and two-sided coideal in $U_q(\mathfrak{g})'$;*
2. *if $\mathfrak{I}_q$ is strict, then $\mathfrak{I}_q^{\,!}$ is strict too, i.e. $\mathfrak{I}_q^{\,!} \bigcap (q-1)\, U_q(\mathfrak{g})' \,=\, (q-1)\, \mathfrak{I}_q^{\,!}$;*
3. *there exists a coisotropic subgroup L in G^* such that $\mathfrak{I}_q(K)^! = \mathcal{I}_q(L)$: namely, $\mathfrak{I}_q(K)^!$ is a proper quantization, of type $\mathcal{I}$, of some coisotropic subgroup L of G^* ;*
4. *in the real case, i.e. if the quantization $\mathfrak{I}_q$ is a real one, $\mathfrak{I}_q^{\,!}$ is real too, i.e. $\big(S(\mathfrak{I}_q^{\,!})\big)^* = \mathfrak{I}_q^{\,!}$. Therefore claims (1–3) still hold in the framework of real quantum subgroups.*

Proof. *(1)* Let $a \in {U_q}'$ and $b \in \mathfrak{I}_q^{\,!}$: by definition of $\mathfrak{I}_q^{\,!}$, from $\mathfrak{I}_q \trianglelefteq_\ell U_q$ and from (4.6) we get

$$\delta_n(ab) \,\in\, (q-1)^n \sum_{s=1}^{n} {U_q}^{\otimes (s-1)} \otimes \mathfrak{I}_q \otimes {U_q}^{\otimes (n-s)}$$

so $a\, b \in \mathfrak{I}_q^{\,!}$, thus $\mathfrak{I}_q^{\,!} \trianglelefteq_\ell {U_q}'$.
As to the coideal property, it is proven resorting to $(q-1)$–adic completions, arguing as in the proof of Proposition 3.5 in [12], and basing on the fact that $\mathfrak{I}_q \dot{\trianglelefteq}\, U_q$. Details are left to the reader. *(2)* Assume now $\mathfrak{I}_q$ to be *strict*. The inclusion

$$\mathfrak{I}_q^{\,!} \bigcap (q-1)\, U_q(\mathfrak{g})' \,\supseteq\, (q-1)\, \mathfrak{I}_q^{\,!}$$

is trivially true, and we must prove the converse. Let $\eta \in \mathfrak{I}_q^{\,!} \bigcap (q-1)\, U_q(\mathfrak{g})'$. We have

$$\delta_n(\eta) \,\in\, (q-1)^n \left(\left(\textstyle\sum_{s=1}^{n} {U_q}^{\otimes (s-1)} \otimes \mathfrak{I}_q \otimes {U_q}^{\otimes (n-s)} \right) \bigcap (q-1)\, {U_q}^{\otimes n} \right)$$

for all $n \in \mathbb{N}_+$. But then our assumption gives

$$\left(\sum_{s=1}^{n} {U_q}^{\otimes (s-1)} \otimes \mathfrak{I}_q \otimes {U_q}^{\otimes (n-s)} \right) \bigcap\, (q-1)\, {U_q}^{\otimes n} =$$

$$= \sum_{s=1}^{n} {U_q}^{\otimes (s-1)} \otimes \Big(\mathfrak{I}_q \bigcap (q-1)\, U_q \Big) \otimes {U_q}^{\otimes (n-s)} =$$

$$= (q-1)^{n+1} \sum_{s=1}^{n} {U_q}^{\otimes (s-1)} \otimes \mathfrak{I}_q \otimes {U_q}^{\otimes (n-s)}$$

which, in turn, means $\eta \in (q-1)\,\mathfrak{I}_q^{\,!}$. Thus $\mathfrak{I}_q^{\,!} \bigcap (q-1)\,U_q(\mathfrak{g})' \subseteq (q-1)\,\mathfrak{I}_q^{\,!}$, as expected. *(3)* Claim *(1)* implies that $\mathfrak{I}_q^{\,!}$ is a weak quantization of its image, therefore there exists a subgroup L of G^* such that $\pi_{U_q'}\big(\mathfrak{I}_q^{\,!}\big) = \mathcal{I}(L)$. This quantization is even strict if $\mathfrak{I}_q$ itself is strict, by the previous. Now we show that this quantization $\mathcal{I}_q(L)$ is always *proper* — hence the subgroup L is coisotropic, by Lemma 3.5. Recall that, by definition, $\mathcal{I}_q(L)$ is proper if and only if $[x,y] \in (q-1)\,\mathfrak{I}_q^{\,!}$ for all $x, y \in \mathfrak{I}_q^{\,!}$. From definitions we have

$$[x,y] \in (q-1)\,\mathfrak{I}_q^{\,!} \iff \delta_n\big([x,y]\big) \in (q-1)^{n+1} \textstyle\sum_{s=1}^n {U_q}^{\otimes (s-1)} \otimes \mathfrak{I}_q \otimes {U_q}^{\otimes (n-s)}$$

for all $n \in \mathbb{N}$. Then by formula (4.7) we have (for all $n \in \mathbb{N}$)

$$\delta_n\big([x,y]\big) \;=\; \textstyle\sum_{\substack{\Lambda \cup Y = \{1,\dots,n\} \\ \Lambda \cap Y \neq \emptyset}} \big(\, \delta_\Lambda(x)\, \delta_Y(y) \;-\; \delta_Y(y)\, \delta_\Lambda(x) \,\big) \tag{5.3}$$

while (with notation of §4)

$$\delta_\Lambda(x) \in (q-1)^{|\Lambda|} \cdot j_\Lambda \left(\textstyle\sum_{s=1}^{|\Lambda|} {U_q}^{\otimes (s-1)} \otimes \mathfrak{I}_q \otimes {U_q}^{\otimes (|\Lambda|-s)} \right) ,$$
$$\delta_Y(y) \in (q-1)^{|Y|} \cdot j_Y \left(\textstyle\sum_{s=1}^{|Y|} {U_q}^{\otimes (s-1)} \otimes \mathfrak{I}_q \otimes {U_q}^{\otimes (|Y|-s)} \right) ;$$

since $\Lambda \cup Y = \{1,\dots,n\}$ and $\Lambda \cap Y \neq \emptyset$ we have $|\Lambda| + |Y| \geq n+1$; moreover, for each index $i \in \{1,\dots,n\}$ we have $i \in \Lambda$ (and otherwise $Im\,(j_\Lambda)$ has 1 in the i–th spot) or $i \in Y$ (with the like remark on $Im\,(j_Y)$ if not). As $\mathfrak{I}_q$ is a left ideal of U_q, we conclude

$$\begin{aligned} \delta_\Lambda(x) \cdot \delta_Y(y)\,,\; \delta_Y(y) \cdot \delta_\Lambda(x) \;\in\; & (q-1)^{|\Lambda|+|Y|} \textstyle\sum_{s=1}^n {U_q}^{\otimes (s-1)} \otimes \mathfrak{I}_q \otimes {U_q}^{\otimes (n-s)} \\ \subseteq\; & (q-1)^{n+1} \textstyle\sum_{s=1}^n {U_q}^{\otimes (s-1)} \otimes \mathfrak{I}_q \otimes {U_q}^{\otimes (n-s)} \end{aligned}$$

so that (5.3) gives $\delta_n\big([x,y]\big) \in (q-1)^{n+1} \sum_{s=1}^n {U_q}^{\otimes (s-1)} \otimes \mathfrak{I}_q \otimes {U_q}^{\otimes (n-s)}$, as expected. *(4)* In the real case, $\big(S\big(\mathfrak{I}_q^{\,!}\big)\big)^* = \mathfrak{I}_q^{\,!}$ follows at once from definitions and from the identity $\big(S(\mathfrak{I}_q)\big)^* = \mathfrak{I}_q$. ∎

Remark 5.6. In functorial language we may say that the map $\mathfrak{I}_q \mapsto \mathfrak{I}_q^{\,!}$ establishes a functor between quantized homogeneous spaces of G and quantizations of coisotropic subgroups of G^*, moving from a local to a global description, sending each type of quantization in a proper one and preserving strictness. Once more, details are left to the interested reader.

The fourth and last step is devoted to the map $\mathfrak{C}_q \mapsto \mathfrak{C}_q^{\,\curlyvee}$.

Proposition 5.7. *Let $\mathfrak{C}_q = \mathfrak{C}_q(K)$ be a subalgebra and left coideal in $U_q(\mathfrak{g})$, weak quantization (of type $\mathfrak{C}$) of some subgroup K of G. Then:*

1. *$\mathfrak{C}_q^{\,\curlyvee}$ is a subalgebra and left coideal in $U_q(\mathfrak{g})'$;*

2. *if* $\mathfrak{C}_q$ *is strict, then* $\mathfrak{I}_q^!$ *is strict too, i.e.* $\mathfrak{C}_q^{\,\Lsh} \bigcap (q-1)\, U_q(\mathfrak{g})' \;=\; (q-1)\, \mathfrak{C}_q^{\,\Lsh}$ *;*

3. *there exists a coisotropic subgroup* L *in* G^* *such that* $\mathfrak{C}_q(K)^{\Lsh} = \mathcal{C}_q(L)$ *: namely,* $\mathfrak{C}_q(K)^{\Lsh}$ *is a proper quantization, of type* $\mathcal{C}$ *, of some coisotropic subgroup* L *of* G^* *;*

4. *in the real case, i.e. if the quantization* $\mathfrak{C}_q$ *is a real one,* $\mathfrak{C}_q(K)^{\Lsh}$ *is real too, i.e.* $\big(\mathfrak{C}_q^{\,\Lsh}\big)^* = \mathfrak{C}_q^{\,\Lsh}$ *. Therefore claims (1–3) still hold in the framework of real quantum subgroups.*

Proof. The whole proof is very similar to that of Proposition 5.5. *(1)* By definitions, $1 \in \mathfrak{C}_q$ and $\delta_n(1) = 0$ for all $n \in \mathbb{N}$, so $1 \in \mathfrak{C}_q^{\,\Lsh}$. Let $x, y \in \mathfrak{C}_q^{\,\Lsh}$ and $n \in \mathbb{N}$; by (4.6) we have $\delta_n(xy) = \sum_{\Lambda \cup Y = \{1,\dots,n\}} \delta_\Lambda(x)\, \delta_Y(y)$. Each of the factors $\delta_\Lambda(x)$ belongs to a module $(q-1)^{|\Lambda|}\, {U_q}^{\otimes(|\Lambda|-1)} \otimes X$ where the last tensor factor is either $X = \mathfrak{C}_q$ (if $n \in \Lambda$) or $X = \{1\} \subset \mathfrak{C}_q$ (if $n \notin \Lambda$), and similarly for $\delta_Y(y)$; in addition $\Lambda \cup Y = \{1,\dots,n\}$ implies $|\Lambda| + |Y| \geq n$, and summing up $\delta_n(xy) \in (q-1)^n {U_q}^{\otimes(n-1)} \otimes \mathfrak{C}_q$, whence $x\,y \in \mathfrak{C}_q^{\,\Lsh}$. Thus $\mathfrak{C}_q^{\,\Lsh}$ is a subalgebra of ${U_q}'$.
In order to prove that $\mathfrak{C}_q^{\,\Lsh}$ is a left coideal in U_q', one can again resort to $(q-1)$–adic completions, with exactly the same arguments as in the proof of Proposition 3.5 in [5], starting from the fact that $\mathfrak{C}_q \,\dot{\trianglelefteq}_\ell\, U_q$. Details are left to the reader. *(2)* Assume, now, that $\mathfrak{C}_q$ is a *strict* quantization, i.e. $\mathfrak{C}_q \bigcap (q-1)\, F_q \;=\; (q-1)\, \mathfrak{C}_q$. Then clearly $\mathfrak{C}_q^{\,\Lsh} \bigcap (q-1)\, U_q(\mathfrak{g})' \supseteq (q-1)\, \mathfrak{C}_q^{\,\Lsh}$, and we must prove the converse inclusion. Let $\kappa \in \mathfrak{C}_q^{\,\Lsh} \bigcap (q-1)\, U_q(\mathfrak{g})'$. Then:

$$\begin{aligned}\delta_n(\kappa) \;\in\; (q-1)^n \Big(\Big({U_q}^{\otimes(n-1)} \otimes \mathfrak{C}_q \Big) \bigcap (q-1)\, {U_q}^{\otimes n} \Big) \;=\; \\ =\; (q-1)^n \Big({U_q}^{\otimes(n-1)} \otimes \big(\mathfrak{C}_q \bigcap (q-1)\, U_q \big) \Big) \;=\; (q-1)^{n+1} \cdot {U_q}^{\otimes(n-1)} \otimes \mathfrak{C}_q\end{aligned}$$

which means $\kappa \in (q-1)\, \mathfrak{C}_q^{\,\Lsh}$. Therefore $\mathfrak{C}_q^{\,\Lsh} \bigcap (q-1)\, U_q(\mathfrak{g})' \subseteq (q-1)\, \mathfrak{C}_q^{\,\Lsh}$, as claimed. *(3)* The above algebraic properties show that $\mathfrak{C}_q^{\,\Lsh}$ is a weak quantization of its image $\pi_{U_q'}\big(\mathfrak{C}_q^{\,\Lsh}\big)$; thus there exists a coisotropic subgroup L of G^* such that: $\pi_{U_q'}\big(\mathfrak{C}_q^{\,\Lsh}\big) = \mathcal{C}(L)$. Thus $\mathfrak{I}_q^!$ is a weak quantization — to be called $\mathcal{I}_q(L)$ — of $\mathcal{I}(L)$, and it is even strict if $\mathfrak{I}_q$ itself is strict, by the previous. Now we show first that this quantization $\mathcal{I}_q(L)$ is always *proper* — hence the subgroup L is coisotropic, by Lemma 3.5. Proving that $\mathcal{I}_q(L)$ is proper amounts to show that $[x, y] \in (q-1)\, \mathfrak{C}_q^{\,\Lsh}$ for all $x, y \in \mathfrak{C}_q^{\,\Lsh}$. By definition we have

$$[x,y] \;\in\; (q-1)\, \mathfrak{C}_q^{\,\Lsh} \quad\Longleftrightarrow\quad \delta_n\big([x,y]\big) \;\in\; (q-1)^{n+1} {U_q}^{\otimes(n-1)} \otimes \mathfrak{C}_q \quad \forall\, n \in \mathbb{N}$$

and formula (4.7) gives, for all $n \in \mathbb{N}$,

$$\delta_n\big([x,y]\big) \;=\; \sum\nolimits_{\substack{\Lambda \cup Y = \{1,\dots,n\} \\ \Lambda \cap Y \neq \emptyset}} \big(\delta_\Lambda(x)\, \delta_Y(y) \;-\; \delta_Y(y)\, \delta_\Lambda(x) \big) \tag{5.4}$$

while

$$\delta_\Lambda(x) \in (q-1)^{|\Lambda|} j_\Lambda\Big(U_q^{\otimes(|\Lambda|-1)} \otimes \mathfrak{C}_q\Big) \ , \ \ \delta_Y(y) \in (q-1)^{|Y|} j_Y\Big(U_q^{\otimes(|Y|-1)} \otimes \mathfrak{C}_q\Big)$$

Now, $\Lambda \cup Y = \{1,\dots,n\}$ and $\Lambda \cap Y \neq \emptyset$ give $|\Lambda|+|Y| \geq n+1$, and since $\mathfrak{C}_q$ is a subalgebra of U_q we get

$$\begin{aligned}\delta_\Lambda(x)\,\delta_Y(y)\,,\ \delta_Y(y)\,\delta_\Lambda(x) \in (q-1)^{|\Lambda|+|Y|}\, U_q^{\otimes(n-1)} \otimes \mathfrak{C}_q \subseteq \\ \subseteq (q-1)^{n+1} U_q^{\otimes(n-1)} \otimes \mathfrak{C}_q\end{aligned}$$

so that (5.4) yields

$$\delta_n\big([x,y]\big) \in (q-1)^{n+1} U_q^{\otimes(n-1)} \otimes \mathfrak{C}_q$$

thus $[x,y] \in (q-1)\,\mathfrak{C}_q^{\,\curvearrowleft}$. *(4)* In the real case $(\mathfrak{C}_q)^* = \mathfrak{C}_q$: this and the very definitions imply the claim. $\square$

Remark 5.8. In functorial language we may say that the map $\mathfrak{C}_q \mapsto \mathfrak{C}_q^{\,\curvearrowleft}$ establishes a functor between quantization of coisotropic subgroups of G and quantizations of Poisson homogeneous spaces of G^*, moving from a local to a global description, sending each type of quantization in a proper one and preserving strictness. We leave to the interested reader all details which still need to be fixed.

We now move to connectedness properties of the coisotropic subgroup L identified in Propositions 5.5 and 5.7.

Proposition 5.9.

1. *Let $\mathfrak{I}_q(K)$ be a strict quantization (of type $\mathfrak{I}$) of a (coisotropic) subgroup K in G. Then the subgroup L of G^* such that $\mathfrak{I}_q(K)^! = \mathcal{I}_q(L)$ is connected.*

2. *Let $\mathfrak{C}_q(K)$ be a strict quantization of type $\mathfrak{C}$ of a (coisotropic) subgroup K of G. Then the subgroup L of G^* such that $\mathfrak{C}_q(K)^! = \mathcal{C}_q(L)$ is connected.*

Proof. (1) Saying that the (closed) subgroup L is connected is equivalent to saying that its function algebra $F[L] = F\big[G^*\big]\Big/\mathcal{I}(L)$ has no non-trivial idempotents. Note that, since $F\big[G^*\big]$ is the specialization of U_q' at $q=1$ and $\mathcal{I}(L)$ is the similar specialization of $\mathfrak{I}_q^{\,!}$, the quotient $F[L] = F\big[G^*\big]\Big/\mathcal{I}(L)$ is canonically isomorphic to the specialization at $q=1$ of $U_q'\Big/\mathfrak{I}_q^{\,!}$. Let $\overline{a}$ be an idempotent in $F[L]$: if we take any lift of it in $U_q'\Big/\mathfrak{I}_q^{\,!}$, i.e. any $a \in U_q'\Big/\mathfrak{I}_q^{\,!}$ such that $\overline{a} = a \mod (q-1)\,U_q'\Big/\mathfrak{I}_q^{\,!}$. We must prove:

$$a^2 \equiv a \mod (q-1)\,U_q'\Big/\mathfrak{I}_q^{\,!} \implies a \mod (q-1)\,U_q'\Big/\mathfrak{I}_q^{\,!} \in \big\{0,1\big\} \quad (5.5)$$

We can clearly reduce to the case when $\epsilon(\overline{a}) = 0$: in fact, if $\overline{a}^{\,2} = \overline{a}$ then $\epsilon(\overline{a})$ is necessarily 0 or 1 (for it is unipotent too), and in the latter case we then find that $\overline{a}_0 := 1 - \overline{a}$ is idempotent and $\epsilon(\overline{a}_0) = 0$. Also the lift $a \in U_q' \big/ \mathfrak{I}_q^{\,!}$ can be chosen, in this case, such that: $\epsilon(a) = 0$. To simplify notation, we set $H := U_q \big/ \mathfrak{I}_q$ and $H' := U_q' \big/ \mathfrak{I}_q^{\,!}$. We shall prove that, if $a \in H'$, $\epsilon(a) = 0$ and $a^2 \equiv a \mod (q-1)\, H'$, then $a \equiv 0 \mod (q-1)\, H'$, i.e. $a \in (q-1)\, H'$; in fact, this will give (5.5).

Having assumed that $\mathfrak{I}_q$ to be strict, H' identifies with a $\mathbb{C}\big[q, q^{-1}\big]$–submodule of H given in terms of the coalgebra structure of the latter: the embedding is the one canonically induced by the maps $U_q' \longhookrightarrow U_q \twoheadrightarrow U_q \big/ \mathfrak{I}_q$. In fact, the kernel of the latter map is $U_q' \bigcap \mathfrak{I}_q$ (by strictness assumption). It is easy to see from definitions that $U_q' \bigcap \mathfrak{I}_q = \mathfrak{I}_q^{\,!}$. Thus H' does embed into H:

$$H' = \left\{ \eta \in H \;\middle|\; \delta_n(\eta) \in (q-1)^n \, H^{\otimes\, n} \,,\ \forall\, n \in \mathbb{N} \right\}. \tag{5.6}$$

Now, $a^2 \equiv a \mod (q-1)H'$ means $a = a^2 + (q-1)\, c$ for some $c \in H'$; since $\epsilon(a) = 0$, we have $\epsilon(c) = 0$ as well. Applying δ_n to the identity $a = a^2 + (q-1)\, c$ and using formula (4.6) we get

$$\delta_n(a) \;=\; \delta_n\big(a^2\big) \;+\; (q-1)\, \delta_n(c) \;=\; \sum_{\Lambda \cup Y = \{1,\dots,n\}} \delta_\Lambda(a)\, \delta_Y(a) \;+\; (q-1)\, \delta_n(c)$$

for all $n \in \mathbb{N}$, which — noting that $\delta_0(a) := \epsilon(a) = 0$ yields:

$$\delta_n(a) \;=\; \sum_{\substack{\Lambda \cup Y = \{1,\dots,n\} \\ \Lambda, Y \neq \emptyset}} \delta_\Lambda(a)\, \delta_Y(a) \;+\; (q-1)\, \delta_n(c) \tag{5.7}$$

Since $c \in H'$, the last summand $(q-1)\, \delta_n(c)$ in right-hand side of (5.7) belongs to $(q-1)^{n+1} H^{\otimes n}$, thanks to (5.6). Similarly, since $a \in H'$ we have $\delta_k(a) \in (q-1)^k H^{\otimes k}$ for all $k \in \mathbb{N}$, by (5.6) again: therefore each summand $\delta_\Lambda(a)\, \delta_Y(a)$ in right-hand side of (5.7) belongs to $(q-1)^{n+1} H^{\otimes n}$ as well. But then (5.7) yields $\delta_n(a) \in (q-1)^{n+1} H^{\otimes n}$ for all $n \in \mathbb{N}$, which, again by (5.6), means exactly that $a \in (q-1)\, H'$. This ends the proof of the first claim. *(2)* We will use similar arguments to show this claim: $F[L] = F\big[G^*\big] \big/ \mathcal{I}(L)$ has no non-trivial idempotents. Since $\mathfrak{C}_q^{\,\urcorner} = \mathcal{C}_q(L)$ and $\mathcal{C}(L) = \mathcal{C}\big(\widehat{L}\,\big)$, we can assume $L = \widehat{L}$, i.e. L is observable. This implies $\mathcal{I}(L) = \Psi\big(\mathcal{C}(L)\big)$, which is clearly the specialization at $q = 1$ of $\Psi\big(\mathcal{C}(L)\big) = U_q'\, \mathfrak{C}_q^{\,\urcorner}$; therefore, $F[L] = F\big[G^*\big] \big/ \mathcal{I}(L)$ is canonically isomorphic to the specialization at $q = 1$ of $U_q' \big/ U_q'\, \mathfrak{C}_q^{\,\urcorner}$.

From now on, one can mimic step by step the proof of part *(1)*. The only detail to modify is that one must take $U_q\, {\mathfrak{C}_q}^{+} =: \Psi(\mathfrak{C}_q)$ in place of $\mathfrak{I}_q$, and $U_q' \, \big(\mathfrak{C}_q^{\,\urcorner}\big)^{+} =: \Psi\big(\mathfrak{C}_q^{\,\urcorner}\big)$ in place of $\mathfrak{I}_q^{\,!}$. Letting $H := U_q \big/ \Psi(\mathfrak{C}_q)$, and $H' :=$

$U_q' \big/ \Psi\big(\mathfrak{C}_q^{\,\daleth}\big)$, the thesis amounts to prove that

$$a \in H\text{'}\,, \quad a^2 \equiv a \mod (q-1)\, H\text{'} \Rightarrow a \equiv 0 \mod (q-1)\, H\text{'}$$

(In fact also $a \equiv 1 \mod (q-1)\, H$' would be ok, but, arguing as before, we'll restrict to the case $\epsilon(a) = 0$).
As $\mathfrak{C}_q$ is strict, it is easy to see from definitions that $\mathfrak{C}_q^{\,\daleth} = U_q' \bigcap \mathfrak{C}_q$, hence $\Psi\big(\mathfrak{C}_q^{\,\daleth}\big) := U_q' \big(\mathfrak{C}_q^{\,\daleth}\big)^+ = U_q'\big(U_q' \cap \mathfrak{C}_q\big)^+$: the latter is the kernel of the map $U_q' \hookrightarrow U_q \twoheadrightarrow U_q \big/ U_q\, \mathfrak{C}_q^{\,+}$, so H' embeds as a $\mathbb{C}\big[q,q^{-1}\big]$–submodule of H, namely

$$H\text{'} = \left\{ \eta \in H \,\middle|\, \delta_n(\eta) \in (q-1)^n\, H^{\otimes\, n}\,,\ \forall\, n \in \mathbb{N} \right\}.$$

With this description at hand, computations are as in the proof of claim *(1)*. $\square$

Our next results are about the behavior of quantum subgroups under composition of Drinfeld-like maps.

Proposition 5.10. *Let $\mathcal{I}_q$, $\mathcal{C}_q$, $\mathfrak{I}_q$, $\mathfrak{C}_q$ be weak quantizations of a subgroup K of G . Then:*

1. $\mathcal{I}_q \subseteq \big(\mathcal{I}_q^{\,\curlyvee}\big)^!$, $\mathcal{C}_q \subseteq \big(\mathcal{C}_q^{\,\triangledown}\big)^{\daleth}$;
2. $\mathfrak{C}_q \supseteq \big(\mathfrak{C}_q^{\,\daleth}\big)^{\triangledown}$, $\mathfrak{I}_q \supseteq \big(\mathfrak{I}_q^{\,!}\big)^{\curlyvee}$.

Proof. (1) By the very definitions, for any $n \in \mathbb{N}$ we have

$$\delta_n\big(\mathcal{I}_q\big) \subseteq J_{F_q}^{\;\otimes n} \bigcap \left(\sum_{s=0}^{n} F_q^{\otimes s} \otimes \mathcal{I}_q \otimes F_q^{\otimes (n-s-1)} \right) =$$
$$= \sum_{s=0}^{n} J_{F_q}^{\;\otimes s} \otimes \mathcal{I}_q \otimes J_{F_q}^{\;\otimes (n-s-1)} \subseteq (q-1)^n \cdot \sum_{s=0}^{n} \big(F_q^{\,\vee}\big)^{\otimes s} \otimes \mathcal{I}_q^{\,\curlyvee} \otimes \big(F_q^{\,\vee}\big)^{\otimes (n-s-1)}$$

which means exactly $\mathcal{I}_q \subseteq \big(\mathcal{I}_q^{\,\curlyvee}\big)^!$. Similarly we can remark that:

$$\delta_n\big(\mathcal{C}_q\big) \subseteq J_{F_q}^{\;\otimes n} \bigcap \big(F_q^{\otimes (n-1)} \otimes \mathcal{C}_q\big) =$$
$$= J_{F_q}^{\;\otimes (n-1)} \otimes \big(\mathcal{C}_q \bigcap J_{F_q}\big) \subseteq (q-1)^n \big(F_q^{\,\vee}\big)^{\otimes (n-1)} \otimes \mathcal{C}_q^{\,\triangledown}$$

which means $\mathcal{C}_q \subseteq \big(\mathcal{C}_q^{\,\triangledown}\big)^{\daleth}$. Therefore claim *(1)* is proved. *(2)* As $\big(\mathfrak{C}_q^{\,\daleth}\big)^{\triangledown}$ is generated — as an algebra — by $(q-1)^{-1}\mathfrak{C}_q^{\,\daleth} \bigcap J_{U_q'}$, it is enough to show that the latter space is contained in $\mathfrak{C}_q$. Let, then, $x' \in \mathfrak{C}_q^{\,\daleth} \bigcap J_{U_q'}$. Surely $\delta_1\big(x'\big) \in (q-1)\,\mathfrak{C}_q$, hence $x' = \delta_1\big(x'\big) + \epsilon\big(x'\big) \in (q-1)\,\mathfrak{C}_q$. Therefore $(q-1)^{-1}x' \in \mathfrak{C}_q$, q.e.d. Similarly, $\big(\mathfrak{I}_q^{\,!}\big)^{\curlyvee}$ is the left ideal of U_q' generated by $(q-1)^{-1}\,\mathfrak{I}_q^{\,!} \bigcap J_{U_q'}$, thus — since $U_q' \subseteq U_q$ — we must only prove that $(q-1)^{-1}\,\mathfrak{I}_q^{\,!} \bigcap J_{U_q'}$ is contained in U_q. Again, if $y' \in \mathfrak{I}_q^{\,!} \bigcap J_{U_q'}$ then $y' = \delta_1\big(y'\big) + \epsilon\big(y'\big) \in (q-1)\,\mathfrak{I}_q$. Thus we get $(q-1)^{-1}y' \in \mathfrak{I}_q$, and *(2)* is proved. $\square$

Remarks:

(a) By repeated applications of the previous proposition it is easily proved that:

$$\mathcal{I}_q^{\curlyvee} = \Big(\big(\mathcal{I}_q^{\curlyvee}\big)^{!} \Big)^{\curlyvee} \; , \quad \mathcal{C}_q^{\triangledown} = \Big(\big(\mathcal{C}_q^{\triangledown}\big)^{\Lsh} \Big)^{\triangledown} \; , \quad \mathfrak{C}_q^{\Lsh} = \Big(\big(\mathfrak{C}_q^{\Lsh}\big)^{\triangledown} \Big)^{\Lsh} \; , \quad \mathfrak{I}_q^{!} = \Big(\big(\mathfrak{I}_q^{!}\big)^{\curlyvee} \Big)^{!}$$

(b) Since we proved that Drinfeld-like maps always produce *proper* quantizations, and that proper quantizations specialize to *coisotropic* subgroups (cf. Proposition 3.5), the following holds:

1. if $\mathcal{I}_q = \big(\mathcal{I}_q^{\curlyvee}\big)^{!}$ then $\mathcal{I}_q$ is a proper quantization (of type $\mathcal{I}$) of a coisotropic subgroup of G ;
2. if $\mathcal{C}_q = \big(\mathcal{C}_q^{\triangledown}\big)^{\Lsh}$ then $\mathcal{C}_q$ is a proper quantization (of type $\mathcal{C}$) of a coisotropic subgroup of G ;
3. if $\mathfrak{I}_q = \big(\mathfrak{I}_q^{!}\big)^{\curlyvee}$ then $\mathfrak{I}_q$ is a proper quantization (of type $\mathfrak{I}$) of a coisotropic subgroup of G ;
4. if $\mathfrak{C}_q = \big(\mathfrak{C}_q^{\Lsh}\big)^{\triangledown}$ then $\mathfrak{C}_q$ is a proper quantization (of type $\mathfrak{C}$) of a coisotropic subgroup of G .

(c) Since the whole construction is independent of the existence of real structures all the above claims hold true in the *real* framework as well.

Next result reads as a converse of the previous one, holding for Drinfeld maps applied to *strict* quantizations:

Theorem 5.11.

(a) if $\mathcal{I}_q$ is a strict quantization of a coisotropic subgroup of G then one has $\mathcal{I}_q = \big(\mathcal{I}_q^{\curlyvee}\big)^{!}$;

(b) if $\mathcal{C}_q$ is a strict quantization of a coisotropic subgroup of G then one has $\mathcal{C}_q = \big(\mathcal{C}_q^{\triangledown}\big)^{\Lsh}$;

(c) if $\mathfrak{I}_q$ is a strict quantization of a coisotropic subgroup of G then one has $\mathfrak{I}_q = \big(\mathfrak{I}_q^{!}\big)^{\curlyvee}$;

(d) if $\mathfrak{C}_q$ is a strict quantization of a coisotropic subgroup of G then one has $\mathfrak{C}_q = \big(\mathfrak{C}_q^{\Lsh}\big)^{\triangledown}$;

(e) The above claims hold true in the real framework as well.

Proof. (a) Let $\mathcal{I}_q$ be a strict quantization; by Proposition 5.10*(1)*, it is enough to prove $\mathcal{I}_q \supseteq \big(\mathcal{I}_q^{\curlyvee}\big)^!$. For this we apply the argument used in [12], Proposition 4.3, to prove that $F_q \supseteq \big(F_q^{\vee}\big)'$.

We denote by L the closed, coisotropic, connected subgroup of G^* such that $\mathcal{I}_q^{\curlyvee} = \mathfrak{I}_q(L)$, as in Proposition 5.1, and with $\mathfrak{l}$ its Lie algebra.

Let $y' \in \big(\mathcal{I}_q^{\curlyvee}\big)^!$. Then there is $n \in \mathbb{N}$ and $y^\vee \in \mathcal{I}_q^{\curlyvee} \setminus (q-1)\,\mathcal{I}_q^{\curlyvee}$ such that $y' = (q-1)^n y^\vee$. As we have seen strictness of $\mathcal{I}_q$ implies strictness of $\mathcal{I}_q^{\curlyvee}$ and therefore $y^\vee \notin (q-1)\,F_q^\vee$, and so for $\overline{y^\vee} := y^\vee \mod (q-1)\,F_q^\vee$ we have $\overline{y^\vee} \neq 0 \in F_q^\vee\Big|_{q=1} = U\big(\mathfrak{g}^*\big)$.

As $F_q^\vee$ is a quantization of $U\big(\mathfrak{g}^*\big)$, we can pick an ordered basis $\{b_\lambda\}_{\lambda\in\Lambda}$ of $\mathfrak{g}^*$, and a subset $\big\{x_\lambda^\vee\big\}_{\lambda\in\Lambda}$ of $(q-1)^{-1}J_{F_q}$ so that $x_\lambda^\vee \mod (q-1)\,F_q^\vee = b_\lambda$ for all $\lambda \in \Lambda$; therefore $x_\lambda^\vee = (q-1)^{-1}x_\lambda$ for some $x_\lambda \in J_{F_q}$, for all λ (like in the proof of [12] Proposition 4.3). In addition, we choose now the basis and its lift so that a subset $\{b_\theta\}_{\theta\in\Theta}$ (for some suitable $\Theta \subseteq \Lambda$) is a basis of $\mathfrak{l}$, and, correspondingly, $\big\{x_\theta^\vee\big\}_{\theta\in\Theta} \subseteq \mathcal{I}_q^{\curlyvee}$. Since $\overline{y^\vee} \neq 0 \in F_q^\vee\big|_{q=1} = U\big(\mathfrak{g}^*\big)$, by the Poincaré-Birkhoff-Witt theorem there is a non-zero polynomial $P\big(\{b_\theta\}_{\theta\in\Theta}\big)$ in the b_θ's such that $\overline{y^\vee} = P\big(\{b_\theta\}_{\theta\in\Theta}\big)$, hence

$$y^\vee - P\big(\big\{x_\theta^\vee\big\}_{\theta\in\Theta}\big) \;\in\; \mathcal{I}_q^{\curlyvee} \bigcap\, (q-1)\,F_q^\vee \;=\; (q-1)\,\mathcal{I}_q^{\curlyvee} \quad .$$

This implies $y^\vee = P\big(\big\{x_\theta^\vee\big\}_{\theta\in\Theta}\big) + (q-1)^\nu\, y_1^\vee$ for some $\nu \in \mathbb{N}_+$ where $y_1^\vee \in \mathcal{I}_q^{\curlyvee} \setminus (q-1)\,\mathcal{I}_q^{\curlyvee}$.

One can see, like in [9], Lemma 4.12, that the polynomial P has degree not greater than n. Thus $y' = (q-1)^n y^\vee = (q-1)^n P\big(\big\{x_\theta^\vee\big\}_{\theta\in\Theta}\big) + (q-1)^{n+\nu} y_1^\vee$, and

$$(q-1)^n P\big(\big\{x_\theta^\vee\big\}_{\theta\in\Theta}\big) = (q-1)^n P\big(\big\{(q-1)^{-1}\,x_\theta\big\}_{\theta\in\Theta}\big) \in \mathcal{I}_q$$

by a degree argument. But now, Proposition 5.10 gives $\mathcal{I}_q \subseteq \big(\mathcal{I}_q^{\curlyvee}\big)^!$. Then

$$y_1' \,:=\, y' - (q-1)^n P\big(\big\{x_\mu^\vee\big\}_{\theta\in\Theta}\big) \in \big(\mathcal{I}_q^{\curlyvee}\big)^! \;\text{ and }\; y_1' \,=\, (q-1)^{n+\nu} y_1^\vee = (q-1)^{n_1} y_1^\vee$$

where $n_1 := n + \nu > n$, and $y_1^\vee \in \mathcal{I}_q^{\curlyvee} \setminus (q-1)\,\mathcal{I}_q^{\curlyvee}$. We can then repeat the construction, with y_1' instead of y', n_1 instead of n, etc.: iterating, we find an increasing sequence of numbers $\big\{n_s\big\}_{s\in\mathbb{N}}$ (with $n_0 := n$) and a sequence of polynomials $\big\{P_s\big(\{X_\theta\}_{\theta\in\Theta}\big)\big\}_{s\in\mathbb{N}}$ (again $P_0 := P$) such that the degree of $P_s\big(\{X_\theta\}_{\theta\in\Theta}\big)$ is at most n_s, and the formal identity $y' = \sum_{s\in\mathbb{N}} (q-1)^{n_s} P_s\big(\big\{x_\theta^\vee\big\}_{\theta\in\Theta}\big)$ holds. Now set $I_n := \sum_{k=1}^n (q-1)^{n-k}\,\mathcal{I}_q^k$ (for all $n \in \mathbb{N}$), and let $\widehat{\mathcal{I}}_q$ be the topological completion of $\mathcal{I}_q$ with respect to the filtration provided by the I_n's. Then, by construction, $(q-1)^{n_s} P_s\big(\big\{x_\theta^\vee\big\}_{\theta\in\Theta}\big) \in I_n$ for all $s \in \mathbb{N}$. This yields

$$\sum_{s\in\mathbb{N}} (q-1)^{n_s} P_s\big(\big\{x_\theta^\vee\big\}_{\theta\in\Theta}\big) \,\in\, \widehat{\mathcal{I}}_q \qquad \text{and} \qquad y' = \sum_{s\in\mathbb{N}} (q-1)^{n_s} P_s\big(\big\{x_\theta^\vee\big\}_{\theta\in\Theta}\big)$$

where the last is an identity in $\widehat{\mathcal{I}}_q$. Thus $y' \in \big(\mathcal{I}_q^{\curlyvee}\big)^{!} \bigcap \widehat{\mathcal{I}}_q$. Again with the same arguments as in [12], we see that $\mathcal{I}_q \bigcap (q-1)^{\ell}\, \widehat{\mathcal{I}}_q = (q-1)^{\ell}\, \mathcal{I}_q$ for any $\ell \in \mathbb{N}$. This together with $y' \in \big(\mathcal{I}_q^{\curlyvee}\big)^{!} \bigcap \widehat{\mathcal{I}}_q$ give $y' = (q-1)^{-m}\eta$ for some $m \in \mathbb{N}$ and $\eta \in \mathcal{I}_q$; thus

$$\eta = (q-1)^m\, y' \in \mathcal{I}_q \bigcap (q-1)^m\, \widehat{\mathcal{I}}_q = (q-1)^m\, \mathcal{I}_q\,,$$

whence $y' \in \mathcal{I}_q$, q.e.d.

(b) Assume that $\mathcal{C}_q$ is a strict quantization; by Proposition 5.10*(2)*, it is enough to prove $\mathcal{C}_q \supseteq \big(\mathcal{C}_q^{\triangledown}\big)^{\Lsh}$. To do that, we resume the argument used in [12], Proposition 4.3, to show that $F_q \supseteq \big(F_q^{\vee}\big)'$.

We denote by L the closed, coisotropic, connected subgroup of G^* such that $\mathcal{C}_q^{\triangledown} = \mathfrak{C}_q(L)$ and with $\mathfrak{l}$ its Lie algebra.

Let $c' \in \big(\mathcal{C}_q^{\triangledown}\big)^{\Lsh}$. Then there exist $n \in \mathbb{N}$ and $c^\vee \in \mathcal{C}_q^{\triangledown} \setminus (q-1)\, \mathcal{C}_q^{\triangledown}$ such that $c' = (q-1)^n c^\vee$. Note that strictness of $\mathcal{C}_q$ implies strictness of $\mathcal{C}_q^{\triangledown}$; hence $c^\vee \not\in (q-1)\, F_q^{\vee}$, so that for $\overline{c^\vee} := c^\vee \mod (q-1)\, F_q^{\vee}$ we have $\overline{c^\vee} \neq 0 \in F_q^{\vee}\big|_{q=1} = U\big(\mathfrak{g}^*\big)$. Moreover, $\overline{c^\vee} \in \mathcal{C}_q^{\triangledown}\big|_{q=1} = \mathfrak{C}(L) = U(\mathfrak{l}) \subseteq U\big(\mathfrak{g}^*\big)$.

Since $F_q^{\vee}$ is a quantization of $U\big(\mathfrak{g}^*\big)$, we can fix an ordered basis $\{b_\lambda\}_{\lambda\in\Lambda}$ of $\mathfrak{g}^*$, and a subset $\big\{x_\lambda^\vee\big\}_{\lambda\in\Lambda}$ of $(q-1)^{-1} J_{F_q}$ such that $x_\lambda^\vee \mod (q-1)\, F_q^{\vee} = b_\lambda$ for all $\lambda \in \Lambda$; so $x_\lambda^\vee = (q-1)^{-1} x_\lambda$ for some $x_\lambda \in J_{F_q}$, for all λ (as in the proof of [12] Proposition 4.3). We can choose both the basis and its lift so that a subset $\{b_\mu\}_{\mu\in M}$ is a basis of $\mathfrak{l}$ (here $M \subseteq \Lambda$), and, correspondingly, $\big\{x_\mu^\vee\big\}_{\mu\in M} \subseteq (q-1)^{-1} J_{F_q} \bigcap \mathcal{C}_q^{\triangledown}$. Since $\overline{c^\vee} \neq 0 \in F_q^{\vee}\Big|_{q=1} = U\big(\mathfrak{g}^*\big)$, by the Poincaré-Birkhoff-Witt theorem there exists a non-zero polynomial $P\big(\{b_\mu\}_{\mu\in M}\big)$ in variables b_μ's such that $\overline{c^\vee} = P\big(\{b_\mu\}_{\mu\in M}\big)$, hence:

$$c^\vee - P\big(\big\{x_\mu^\vee\big\}_{\mu\in M}\big) \;\in\; \mathcal{C}_q^{\triangledown} \bigcap (q-1)\, F_q^{\vee} \;=\; (q-1)\, \mathcal{C}_q^{\triangledown}\,.$$

Therefore, $c^\vee = P\big(\big\{x_\mu^\vee\big\}_{\mu\in M}\big) + (q-1)^\nu\, c_1^\vee$ for some $\nu \in \mathbb{N}_+$ where $c_1^\vee \in \mathcal{C}_q^{\triangledown} \setminus (q-1)\, \mathcal{C}_q^{\triangledown}$.

Now, we can see — like in [9], Lemma 4.12 — that the degree of P is not greater than n. Then

$$c' = (q-1)^n c^\vee = (q-1)^n P\big(\big\{x_\mu^\vee\big\}_{\mu\in M}\big) + (q-1)^{n+\nu} c_1^\vee$$

with $(q-1)^n P\big(\big\{x_\mu^\vee\big\}_{\mu\in M}\big) = (q-1)^n P\big(\big\{(q-1)^{-1}\, x_\mu\big\}_{\mu\in M}\big) \in \mathcal{C}_q$ because P has degree bounded (from above) by n. As $\mathcal{C}_q \subseteq \big(\mathcal{C}_q^{\triangledown}\big)^{\Lsh}$, by Proposition 5.10, we get

$$c_1' := c' - (q-1)^n P\big(\big\{x_\mu^\vee\big\}_{\mu\in M}\big) \in \big(\mathcal{C}_q^{\triangledown}\big)^{\Lsh} \quad \text{and} \quad c_1' = (q-1)^{n+\nu} c_1^\vee = (q-1)^{n_1} c_1^\vee$$

with $n_1 := n + \nu > n$, and $c_1^\vee \in \mathcal{C}_q^{\triangledown} \setminus (q-1)\, \mathcal{C}_q^{\triangledown}$. We can repeat this construction with c_1' in place of c', n_1 in place of n, etc.. Iterating, we get an

increasing sequence of numbers $\{n_s\}_{s\in\mathbb{N}}$ $(n_0 := n)$ and a sequence of polynomials $\{P_s(\{X_\mu\}_{\mu\in M})\}_{s\in\mathbb{N}}$ $(P_0 := P)$ such that the degree of $P_s(\{X_\mu\}_{\mu\in M})$ is at most n_s, and $c' = \sum_{s\in\mathbb{N}} (q-1)^{n_s} P_s(\{x_\mu^\vee\}_{\mu\in M})$.
Consider

$$I_{\mathcal{C}_q} := Ker\Big(\mathcal{C}_q \xrightarrow{\ \epsilon\ }\!\!\!\!\rightarrow \mathbb{C}\big[q,q^{-1}\big] \xrightarrow{\ ev_1\ }\!\!\!\!\rightarrow \mathbb{C}\Big) = Ker\Big(\mathcal{C}_q \xrightarrow{\ ev_1\ }\!\!\!\!\rightarrow \mathcal{C}_q/(q-1)\,\mathcal{C}_q \xrightarrow{\ \bar{\epsilon}\ }\!\!\!\!\rightarrow \mathbb{C}\Big)$$

By construction, we have $(q-1)^{n_s} P_s(\{x_\mu^\vee\}_{\mu\in M}) \in I_{\mathcal{C}_q}^{\,n_s}$ for all $s \in \mathbb{N}$; in turn, this means that $\sum_{s\in\mathbb{N}} (q-1)^{n_s} P_s(\{x_\mu^\vee\}_{\mu\in M}) \in \widehat{\mathcal{C}}_q$, the latter being the $I_{\mathcal{C}_q}$–adic completion of $\mathcal{C}_q$, and the formal expression $c' = \sum_{s\in\mathbb{N}} (q-1)^{n_s} P_s(\{x_\mu^\vee\}_{\mu\in M})$ is an identity in $\widehat{\mathcal{C}}_q$: therefore $c' \in \big(\mathcal{C}_q^{\,\triangledown}\big)^{\urcorner} \bigcap \widehat{\mathcal{C}}_q$. Acting as in [12], again, we see that $\mathcal{C}_q \bigcap (q-1)^\ell\, \widehat{\mathcal{C}}_q = (q-1)^\ell\, \mathcal{C}_q$ for all $\ell \in \mathbb{N}$. Getting back to $c' \in \big(\mathcal{C}_q^{\,\triangledown}\big)^{\urcorner} \bigcap \widehat{\mathcal{C}}_q$, we have $c' = (q-1)^{-m}\kappa$ for some $m \in \mathbb{N}$ and $\kappa \in \mathcal{C}_q$; thus $\kappa = (q-1)^m\, c' \in \mathcal{C}_q \bigcap (q-1)^m\, \widehat{\mathcal{C}}_q = (q-1)^m\, \mathcal{C}_q$, whence $c' \in \mathcal{C}_q$, q.e.d.
(c) Let $\mathfrak{I}_q$ be a strict quantization: by Proposition 5.10*(2)* it is enough to prove $\mathfrak{I}_q \subseteq \big(\mathfrak{I}_q^{\,!}\big)^{\curlyvee}$; so given $y \in \mathfrak{I}_q$, we must prove that $y \in \big(\mathfrak{I}_q^{\,!}\big)^{\curlyvee}$. Recall that $\mathfrak{I}_q \subseteq U_q = \big(U_q'\big)^\vee$, the last identity following from Theorem 4.1. By construction,

$$\big(U_q'\big)^\vee \; = \; \textstyle\sum_{n\geq 0} (q-1)^{-n}\, I_{U_q'}^{\,n} \quad , \qquad I_{U_q'} \; := \; \big(U_q'\big)^+ + (q-1)\, U_q'$$

so for $y \in \mathfrak{I}_q \subseteq U_q = \big(U_q'\big)^\vee$ there exists $N \in \mathbb{N}$ such that

$$y_+ \; := \; (q-1)^N\, y \; \in \; I_{U_q'}^{\,N} \; \subseteq \; U_q' \tag{5.8}$$

Strictness of $\mathfrak{I}_q$, i.e. $\mathfrak{I}_q \bigcap (q-1)\, U_q = (q-1)\, \mathfrak{I}_q$, implies

$$\Big(\sum_{s=1}^{n} U_q^{\otimes(s-1)} \otimes \mathfrak{I}_q \otimes U_q^{\otimes(n-s)}\Big) \bigcap \big((q-1)^n\, U_q^{\otimes n}\big) \; = \; (q-1)^n \Big(\sum_{s=1}^{n} U_q^{\otimes(s-1)} \otimes \mathfrak{I}_q \otimes U_q^{\otimes(n-s)}\Big)$$

for all $n \in \mathbb{N}_+$; then, by the very definitions, the latter yields $\mathfrak{I}_q^{\,!} = \mathfrak{I}_q \bigcap U_q'$. If in (5.8) $N = 1$, then $y_+ = y \in U_q'$, thus $y \in \mathfrak{I}_q \bigcap U_q' = \mathfrak{I}_q^{\,!}$, q.e.d. If $N > 1$ instead, then formula (5.8), along with $\mathfrak{I}_q \mathrel{\dot{\trianglelefteq}} U_q$, yields

$$\delta_n(y_+) \in \Big((q-1)^N \cdot \sum_{s=1}^{n} U_q^{\otimes(s-1)} \otimes \mathfrak{I}_q \otimes U_q^{\otimes(n-s)}\Big) \bigcap \big((q-1)^n\, U_q^{\otimes n}\big) \tag{5.9}$$

for all $n \in \mathbb{N}_+$, and since $\mathfrak{I}_q$ is strict, from (5.9) one gets

$$\delta_n(y_+) \in (q-1)^n \sum_{s=1}^{n} U_q^{\otimes(s-1)} \otimes \mathfrak{I}_q \otimes U_q^{\otimes(n-s)} \qquad \forall\, n \in \mathbb{N}$$

which means $y_+ \in \mathfrak{I}_q^{\,!}$. Eventually, we have found $y_+ \in \mathfrak{I}_q^{\,!} \bigcap I_{U_q'}^N$.
Now look at $I_{\mathfrak{I}_q^!} := I_{U_q'} \bigcap \mathfrak{I}_q^{\,!}$. Using the fact that $U_q' = U_q(\mathfrak{g})' = F\big[G^*\big]$ — from Theorem 4.1 — and $\mathfrak{I}_q^{\,!} = \mathfrak{I}_q(K)^! = \mathcal{I}_q(L)$ for some coisotropic subgroup L in G^* — as granted by Proposition 5.5 — and still taking into account strictness, by an easy geometrical argument (via specialization at $q=1$) we see that

$$I_{U_q'}^n \bigcap \mathfrak{I}_q^{\,!} \equiv I_{\mathfrak{I}_q^!}^n \mod (q-1)\, U_q' \qquad \forall\, n \in \mathbb{N}_+ \,.$$

This, together with $\mathfrak{I}_q \bigcap (q-1)\, U_q = (q-1)\, \mathfrak{I}_q$, yields also

$$I_{U_q'}^n \bigcap \mathfrak{I}_q^{\,!} \equiv I_{\mathfrak{I}_q^!}^n \mod (q-1)\, \mathfrak{I}_q^{\,!} \qquad \forall\, n \in \mathbb{N}_+$$

Finally, by suitable, iterated cancelation of factors $(q-1)$, which is possible because of the condition $\mathfrak{I}_q \bigcap (q-1) U_q = (q-1)\, \mathfrak{I}_q$, we eventually obtain

$$I_{U_q'}^n \bigcap \mathfrak{I}_q^{\,!} \equiv I_{\mathfrak{I}_q^!}^n \mod (q-1)^n\, \mathfrak{I}_q^{\,!} \qquad \forall\, n \in \mathbb{N}_+ \,.$$

To sum up, we have $y_+ \in I_{U_q'}^N \bigcap \mathfrak{I}_q^{\,!} = I_{\mathfrak{I}_q^!}^N$; therefore, by definitions,

$$y = (q-1)^{-N}\, y_+ \;\in\; (q-1)^{-N}\, I_{\mathfrak{I}_q^!}^N \;\subseteq\; \big(\mathfrak{I}_q^{\,!}\big)^{\curlyvee} \,.$$

(d) Let $\mathfrak{C}_q$ be a strict quantization: by Proposition 5.10*(2)* it is enough to prove $\mathfrak{C}_q \subseteq \big(\mathfrak{C}_q^{\,\Lsh}\big)^{\triangledown}$. We follow the same arguments used for claim *(c)*. Let $c \in \mathfrak{C}_q$, since $\mathfrak{C}_q \subseteq U_q = \big(U_q'\big)^\vee$ — from Theorem 4.1 — and $\big(U_q'\big)^\vee = \sum_{n \geq 0} (q-1)^{-n}\, I_{U_q'}^n$, (notation as above) for $c \in \mathfrak{C}_q \subseteq U_q = \big(U_q'\big)^\vee$ there exists $N \in \mathbb{N}$ such that $c_+ := (q-1)^N c \in I_{U_q'}^N \subseteq U_q'$.
Now, strictness of $\mathfrak{C}_q$ implies

$$\big(U_q^{\otimes(n-1)} \otimes \mathfrak{C}_q\big) \bigcap (q-1)^n\, U_q^{\otimes n} = (q-1)^n \,\big(U_q^{\otimes(n-1)} \otimes \mathfrak{C}_q\big) \qquad \forall\, n \in \mathbb{N}_+$$

hence $\mathfrak{C}_q^{\,\Lsh} = \mathfrak{C}_q \bigcap U_q'$. If the above N is 1, then $c_+ = c \in U_q'$, thus $c \in \mathfrak{C}_q \bigcap U_q' = \mathfrak{C}_q^{\,\Lsh}$, q.e.d. If instead $N > 1$, then

$$\delta_n(c_+) \;\in\; \big((q-1)^N \cdot U_q^{\otimes n-1} \otimes \mathfrak{C}_q\big) \bigcap \big((q-1)^n\, U_q^{\otimes n}\big) \qquad \forall\, n \in \mathbb{N}_+$$

and, since $\mathfrak{C}_q$ is strict, $\delta_n(c_+) \in (q-1)^n \cdot U_q^{\otimes n-1} \otimes \mathfrak{C}_q$ for all $n \in \mathbb{N}_+$, which means $c_+ \in \mathfrak{C}_q^{\,\Lsh}$. Thus, eventually, we have $c_+ \in \mathfrak{C}_q^{\,\Lsh} \bigcap I_{U_q'}^N$.
Let us look, now, at $I_{\mathfrak{C}_q^{\Lsh}} := I_{U_q'} \bigcap \mathfrak{C}_q^{\,\Lsh}$. Again in force of strictness of $\mathfrak{C}_q$, a geometrical argument (at $q=1$) as before leads us to

$$I_{U_q'}^n \bigcap \mathfrak{C}_q^{\,\Lsh} \equiv I_{\mathfrak{C}_q^{\Lsh}}^n \mod (q-1)^n\, \mathfrak{C}_q^{\,\Lsh} \;, \qquad \forall\, n \in \mathbb{N}_+$$

from which we conclude that $c_+ \in I_{U_q'}^N \bigcap \mathfrak{C}_q^{\,\Lsh} = I_{\mathfrak{C}_q^{\Lsh}}^N$. Therefore, by the very definitions,

$$c \;=\; (q-1)^{-N}\, c_+ \;\in\; (q-1)^{-N}\, I_{\mathfrak{C}_q^{\Lsh}}^N \;\subseteq\; \big(\mathfrak{C}_q^{\,\Lsh}\big)^{\triangledown} \;, \qquad \text{q.e.d.}$$

(e) This is a direct consequence of claims from *(a)* through *(d)*. *(f)* Once again, this is true because the whole construction is independent of the existence of real structures. □

It is now time to clarify how the coisotropic subgroup L of G^* is linked to the coisotropic subgroup K of G. We will give this relation in the weak quantization case first, and show how it improves under stronger hypothesis.

Theorem 5.12. *Let K be a subgroup of G, and let $\mathcal{I}_q(K)$, $\mathcal{C}_q(K)$, $\mathfrak{I}_q(K)$ and $\mathfrak{C}_q(K)$ be weak quantizations as in Definition 3.6. Then (with notation of Proposition 2.2)*

(a) $\mathcal{I}_q(K)^{\curlyvee} = \mathfrak{I}_q\big(K^{\langle\perp\rangle}\big)$;

(b) $\mathcal{C}_q(K)^{\triangledown} = \mathfrak{C}_q\big(K^{\langle\perp\rangle}\big)$;

(c) *if* $\mathfrak{I}_q(K) = \big(\mathfrak{I}_q(K)^{!}\big)^{\curlyvee}$, *then* $\mathfrak{I}_q(K)^{\,!} = \mathcal{I}_q\big(K^{\langle\perp\rangle}\big)$*; in particular, this holds if the quantization* $\mathfrak{I}_q(K)$ *is strict;*

(d) *if* $\mathfrak{C}_q = \big(\mathfrak{C}_q(K)^{\Lsh}\big)^{\triangledown}$, *then* $\mathfrak{C}_q(K)^{\Lsh} = \mathcal{C}_q\big(K^{\langle\perp\rangle}\big)$*; in particular, this holds if the quantization* $\mathfrak{C}_q(K)$ *is strict;*

(e) *claims (a–d) hold as well in the framework of real quantum subgroups.*

Proof. *(a)* By Proposition 5.1 we already have $\mathcal{I}_q(K)^{\curlyvee} = \mathfrak{I}_q(L)$ for some subgroup $L \subseteq G^*$. In order to show that $L = K^{\langle\perp\rangle}$, we will proceed much like in the proof of $F_q^{\vee} \Big/ (q-1)\, F_q^{\vee} \cong U(\mathfrak{g}^*)$, as given in [12], Theorem 4.7.

Let us fix a subset $\{j_1, \dots, j_n\}$ of J adapted to K as in the proof of Proposition 5.1. Let $J^{\vee} := (q-1)^{-1} J \subset F_q^{\vee}$ and $j^{\,\vee} := (q-1)^{-1} j$ for all $j \in J$. From the discussion in that proof, we argue also that $\big\{ (q-1)^{-|\underline{e}|} j^{\,\underline{e}} \mod (q-1)\, F_q^{\vee} \;\big|\; \underline{e} \in \mathbb{N}^{\,n} \big\}$, where $j^{\,\underline{e}} = \prod_{s=1}^{n} j_s^{\underline{e}(i)}$, is a $\mathbb{C}$–basis of $F_1^{\vee}$, and $\big\{ j_1^{\vee}, \dots, j_n^{\vee} \big\}$ is a $\mathbb{C}$–basis of $\mathfrak{t} = J^{\vee} \mod (q-1)\, F_q^{\vee}$.

Now, $j_\mu\, j_\nu - j_\nu\, j_\mu \in (q-1)\, J$ (for $\mu, \nu \in \{1, \dots, n\}$) implies that:

$$j_\mu\, j_\nu - j_\nu\, j_\mu \;=\; (q-1) \textstyle\sum_{s=1}^{n} c_s\, j_s \,+\, (q-1)^2\, \gamma_1 \,+\, (q-1)\, \gamma_2$$

for some $c_s \in \mathbb{C}\big[q, q^{-1}\big]$, $\gamma_1 \in J$ and $\gamma_2 \in J^2$. Therefore

$$\begin{aligned} \big[j_\mu^{\vee}, j_\nu^{\vee}\,\big] \;:=\; j_\mu^{\vee}\, j_\nu^{\vee} - j_\nu^{\vee}\, j_\mu^{\vee} \;=\; \textstyle\sum_{s=1}^{n} c_s\, j_s^{\vee} + \gamma_1 + (q-1)\, \gamma_2^{\vee} \;\equiv \\ \equiv\; \textstyle\sum_{s=1}^{n} c_s\, j_s^{\vee} \mod (q-1)\, F_q^{\vee} \end{aligned}$$

(where we set $\gamma_2^{\vee} := (q-1)^{-2} \gamma_2 \in (q-1)^{-2} \big(J^{\vee}\big)^2 \subseteq F_q^{\vee}$) thus the subspace $\mathfrak{t} := J^{\vee} \mod (q-1)\, F_q^{\vee}$ is a Lie subalgebra of $F_1^{\vee}$. But then it should be $F_1^{\vee} \cong U(\mathfrak{t})$ as Hopf algebras, by the above description of $F_1^{\vee}$ and PBW theorem.

Now for the second step. The specialization map $\pi^{\vee} \colon F_q^{\vee} \longrightarrow\!\!\!\!\!\rightarrow F_1^{\vee} = U(\mathfrak{t})$ actually restricts to $\eta :\; J^{\vee} \twoheadrightarrow \mathfrak{t} = J^{\vee} \Big/ J^{\vee} \bigcap \big((q-1)\, F_q^{\vee}\big) = J^{\vee} \Big/ \big(J + J^{\vee} J\big)$, because $J^\vee \bigcap \big((q-1)\,F_q^{\,\vee}\big) = J^\vee \bigcap (q-1)^{-1} {I_{F_q}}^2 = J + J^\vee J$. Also, multiplication by $(q-1)^{-1}$ yields a $\mathbb{C}\big[q,q^{-1}\big]$–module isomorphism $\mu : J \overset{\cong}{\hookrightarrow\!\!\!\rightarrow} J^\vee$. Let $\rho : \mathfrak{m}_e \twoheadrightarrow \mathfrak{m}_e\big/{\mathfrak{m}_e}^2 = \mathfrak{g}^*$ be the natural projection map, and $\nu : \mathfrak{g}^* \hookrightarrow \mathfrak{m}_e$ a section of ρ. The specialization map $\pi : F_q \twoheadrightarrow F_1$ restricts to a map $\pi' : J \twoheadrightarrow J\big/\big(J \bigcap (q-1)\,F_q\big) = \mathfrak{m}_e$. Let's fix a section $\gamma : \mathfrak{m}_e \hookrightarrow J$ of π' and consider the composition $\sigma := \eta \circ \mu \circ \gamma \circ \nu : \mathfrak{g}^* \longrightarrow \mathfrak{t}$: this is a well-defined Lie bialgebra morphism, independent of the choice of ν and γ.

In the proof of Proposition 5.1 we made a particular choice for the subset $\{j_1, \dots, j_n\}$. As a consequence, the above analysis to prove that $\sigma : \mathfrak{g}^* \cong \mathfrak{t}$ shows also that the left ideal ${\mathcal{I}_1}^\curlyvee := {\mathcal{I}_q}^\curlyvee \mod (q-1)\,{F_q}^\vee$ of $U(\mathfrak{t})$ is generated by

$$\eta\big({\mathcal{I}_q}^\curlyvee\big) = (\eta \circ \mu)\big(\mathcal{I}_q\big) = (\sigma \circ \rho \circ \pi)\big(\mathcal{I}_q\big) = \sigma\big(\rho(\mathcal{I})\big) = \sigma\big(\mathfrak{k}^\perp\big)\,.$$

So ${\mathcal{I}_1}^\curlyvee = U(\mathfrak{g}^*) \cdot \mathfrak{k}^\perp = U(\mathfrak{g}^*) \cdot \langle \mathfrak{k}^\perp \rangle = \mathfrak{I}\big(K^{\langle \perp \rangle}\big)$ — where we are identifying $\mathfrak{g}^*$ with its image via σ — which eventually means $\mathfrak{l} = \langle \mathfrak{k}^\perp \rangle$. *(b)* By Proposition 5.3 we have $\mathcal{C}_q(K)^\triangledown = \mathfrak{C}_q(L)$ for some coisotropic subgroup L in G^*. We must prove that $L = K^{\langle \perp \rangle}$. Once again, we mimic the procedure of the proof of Proposition 5.3, and we fix a subset $\big\{j_1, \dots, j_n\big\}$ of J as in the proof of such Proposition. Then, tracking the analysis we did there to prove that $\sigma : \mathfrak{g}^* \cong \mathfrak{t}$, we see also that the unital subalgebra $\mathcal{C}_1^\triangledown := \mathcal{C}_q^\triangledown \mod (q-1)\,{F_q}^\vee$ of $U\big(\mathfrak{g}^*\big)$ is generated by $\eta\big(\mathcal{C}_q^\triangledown\big) = (\mu \circ \eta)\big(\mathcal{C}_q\big) = (\sigma \circ \rho \circ \pi)\big(\mathcal{C}_q\big) = \sigma\big(\rho(\mathcal{C})\big) = \sigma\big(\mathfrak{k}^\perp\big)$. Thus $\mathcal{C}_1^\triangledown$ is the subalgebra of $U\big(\mathfrak{g}^*\big)$ generated by $\mathfrak{k}^\perp$, hence $\mathcal{C}_1^\triangledown = \big\langle \mathfrak{k}^\perp \big\rangle_{Alg} = U\big(\big\langle \mathfrak{k}^\perp \big\rangle_{Lie}\big) = U\big(\mathfrak{k}^{\langle \perp \rangle}\big) = \mathfrak{C}\big(K^{\langle \perp \rangle}\big)$, which means $\mathfrak{l} = \langle \mathfrak{k}^\perp \rangle$, q.e.d. *(c)* Thanks to Proposition 5.5 we already know that $\mathfrak{I}_q(K)^! = \mathcal{I}_q(L)$ for some coisotropic subgroup L in G^*. Again, we must prove that $L = K^{\langle \perp \rangle}$. Note that we can assume K to be connected, as its relationship with $\mathfrak{I}_q(K)$ passes through $\mathfrak{k}$ alone; thus in the end we simply have to prove that $\mathfrak{l} := Lie\,(L) = \mathfrak{k}^{\langle \perp \rangle} = \mathfrak{k}^\perp$, taking into account that $\mathfrak{k}^{\langle \perp \rangle} = \mathfrak{k}^\perp$ because $\mathfrak{k}$ is coisotropic, by a remark following Proposition 5.10. By assumption $\mathfrak{I}_q(K) = \big(\mathfrak{I}_q(K)^!\big)^\curlyvee$; this and *(a)* together give

$$\mathfrak{I}_q(K) \;=\; \big(\mathfrak{I}_q(K)^!\big)^\curlyvee \;=\; \mathcal{I}_q(L)^\curlyvee \;=\; \mathfrak{I}_q\big(L^{\langle \perp \rangle}\big) \;=\; \mathfrak{I}_q\big(L^\perp\big)$$

where $L^{\langle \perp \rangle} = L^\perp$ because L is coisotropic as well: at $q = 1$ this implies $\mathfrak{k} = \mathfrak{l}^\perp$, q.e.d. *(d)* We must prove that $L = K^{\langle \perp \rangle}$: as above we can assume K to be connected, so we only have to prove that $\mathfrak{l} := Lie\,(L) = \mathfrak{k}^{\langle \perp \rangle} = \mathfrak{k}^\perp$ (as $\mathfrak{k}$ is coisotropic, by Proposition 5.11.

By assumption $\mathfrak{C}_q = \big(\mathfrak{C}_q(K)^\Lsh\big)^\triangledown$; this along with *(c)* gives

$$\mathfrak{C}_q(K) \;=\; \big(\mathfrak{C}_q(K)^\Lsh\big)^\triangledown \;=\; \mathcal{C}_q(L)^\triangledown \;=\; \mathfrak{C}_q\big(L^{\langle \perp \rangle}\big) \;=\; \mathfrak{C}_q\big(L^\perp\big)$$

with $L^{\langle \perp \rangle} = L^\perp$ since L is coisotropic too: specializing at $q = 1$, this eventually yields $\mathfrak{k} = \mathfrak{l}^\perp$. *(e)* This is clear again since all arguments pass through unchanged in the real setup. $\square$

Corollary 5.13. *Let $\mathcal{I}_q(K)$ and $\mathcal{C}_q(K)$ be weak quantizations of a (not necessarily) coisotropic subgroup K of G, of type $\mathcal{I}$ and $\mathcal{C}$ respectively. Then, with notation of Definition 2.1, we have*

$$\big(\mathcal{I}_q(K)^{\curlyvee}\big)^{!} = \mathcal{I}_q\big(\,\overset{\circ}{K}\,\big) \;\; , \qquad \big(\mathcal{C}_q(K)^{\triangledown}\big)^{\Lsh} = \mathcal{C}_q\big(\,\overset{\circ}{K}\,\big) \;\; .$$

Proof. Theorem 5.12*(a)* gives $\mathcal{I}_q(K)^{\curlyvee} = \mathfrak{I}_q\big(K^{\langle\perp\rangle}\big)$, and Proposition 5.10 yields

$$\Big(\mathfrak{I}_q\big(K^{\langle\perp\rangle}\big)^{!}\Big)^{\curlyvee} = \Big(\big(\mathcal{I}_q(K)^{\curlyvee}\big)^{!}\Big)^{\curlyvee} = \mathcal{I}_q(K)^{\curlyvee} = \mathfrak{I}_q\big(K^{\langle\perp\rangle}\big)$$

so that $\Big(\mathfrak{I}_q\big(K^{\langle\perp\rangle}\big)^{!}\Big)^{\curlyvee} = \mathfrak{I}_q\big(K^{\langle\perp\rangle}\big)$. Then Theorem 5.12 gives

$$\mathfrak{I}_q\big(K^{\langle\perp\rangle}\big)^{!} = \mathcal{I}_q\big((K^{\langle\perp\rangle})^{\langle\perp\rangle}\big) = \mathcal{I}_q\big(\overset{\circ}{\mathrm{K}}\,\big)$$

by Proposition 2.2. Therefore $\big(\mathcal{I}_q(K)^{\curlyvee}\big)^{!} = \mathfrak{I}_q\big(K^{\langle\perp\rangle}\big)^{!} = \mathcal{I}_q\big(\overset{\circ}{\mathrm{K}}\,\big)$ as claimed. Similarly, Theorem 5.12*(b)* gives $\mathcal{C}_q(K)^{\triangledown} = \mathfrak{C}_q\big(K^{\langle\perp\rangle}\big)$, and the first remark after Proposition 5.10 yields

$$\Big(\mathfrak{C}_q\big(K^{\langle\perp\rangle}\big)^{\Lsh}\Big)^{\triangledown} = \Big(\big(\mathcal{C}_q(K)^{\triangledown}\big)^{\Lsh}\Big)^{\triangledown} = \mathcal{C}_q(K)^{\triangledown} = \mathfrak{C}_q\big(K^{\langle\perp\rangle}\big)$$

so that $\Big(\mathfrak{C}_q\big(K^{\langle\perp\rangle}\big)^{\Lsh}\Big)^{\triangledown} = \mathfrak{C}_q\big(K^{\langle\perp\rangle}\big)$. Then again by Theorem 5.12*(d)* we get

$$\mathfrak{C}_q\big(K^{\langle\perp\rangle}\big)^{\Lsh} = \mathcal{C}_q\big((K^{\langle\perp\rangle})^{\langle\perp\rangle}\big) = \mathcal{C}_q\big(\overset{\circ}{\mathrm{K}}\,\big)$$

still by Proposition 2.2. Thus $\big(\mathcal{C}_q(K)^{\triangledown}\big)^{\Lsh} = \mathfrak{C}_q\big(K^{\langle\perp\rangle}\big)^{\Lsh} = \mathcal{C}_q\big(\overset{\circ}{\mathrm{K}}\,\big)$ as claimed. $\square$

Remark 5.14. One might guess that the analogue to this Corollary holds true for weak quantizations of type $\mathfrak{I}$ and $\mathfrak{C}$ as well: actually, we have no clue about that, in either sense.

We now consider the "compatibility" among different Drinfeld-like maps acting on quantizations of different types over a single pair *(subgroup, space)*. Indeed, we show that Drinfeld's functors preserve the subgroup-space correspondence — Proposition 5.15 — and the orthogonality correspondence — Proposition 5.17 — (if either occurs at the beginning) between different quantizations as mentioned.

Proposition 5.15. *Let K be a closed subgroup of G, and let Ψ and Φ be the map mentioned in §2.1. Then the following holds:*

(a) Let $\mathcal{C}_q$ and $\mathcal{I}_q$ be as in Section 3. If $\Psi(\mathcal{C}_q) = \mathcal{I}_q$ *, then* $\Psi\big(\mathcal{C}_q^{\triangledown}\big) = \mathcal{I}_q^{\curlyvee}$.

(b) Let $\mathcal{I}_q$ and $\mathcal{C}_q$ be as in Section 3. If $\Phi(\mathcal{I}_q) = \mathcal{C}_q$ *, then* $\Phi\big(\mathcal{I}_q^{\curlyvee}\big) = \mathcal{C}_q^{\triangledown}$.

(c) Let $\mathfrak{C}_q$ and $\mathfrak{I}_q$ be as in Section 3. If $\Psi(\mathfrak{C}_q) = \mathfrak{I}_q$ *, then* $\Psi\big(\mathfrak{C}_q^{\Lsh}\big) \subseteq \mathfrak{I}_q^{!}$.

(d) Let $\mathfrak{I}_q$ and $\mathfrak{C}_q$ be as in Section 3. If $\Phi(\mathfrak{I}_q) = \mathfrak{C}_q$ *, then* $\Phi\big(\mathfrak{I}_q^{!}\big) = \mathfrak{C}_q^{\Lsh}$.

Proof. Claims *(a)* and *(c)* both follow trivially from definitions.
As to claim *(b)*, let $\eta \in \mathcal{C}_q^+ = \Phi(\mathcal{I}_q)^+$, so that $\Delta(\eta) \in \eta \otimes 1 + F_q \otimes \mathcal{I}_q$. Then $\eta^\vee := (q-1)^{-1}\eta$ enjoys

$$\Delta\big(\eta^\vee\big) \in \eta^\vee \otimes 1 + F_q \otimes (q-1)^{-1}\,\mathcal{I}_q \subseteq \eta^\vee \otimes 1 + {F_q}^\vee \otimes {\mathcal{I}_q}^{\curlyvee}$$

whence $\eta^\vee \in \big({F_q}^\vee\big)^{co\mathcal{I}_q^{\curlyvee}} =: \Phi\big({\mathcal{I}_q}^{\curlyvee}\big)$. Since $\mathcal{C}_q^{\triangledown}$ is generated (as a subalgebra) by $(q-1)^{-1}\,\mathcal{C}_q^+$, we conclude that $\mathcal{C}_q^{\triangledown} \subseteq \Phi\big({\mathcal{I}_q}^{\curlyvee}\big)$.
Conversely, let $\varphi \in \Phi\big({\mathcal{I}_q}^{\curlyvee}\big)$. Then $\Delta(\varphi) \in \varphi \otimes 1 + {F_q}^\vee \otimes {\mathcal{I}_q}^{\curlyvee}$, and there exists $n \in \mathbb{N}$ such that $\varphi_+ := (q-1)^n \varphi \in \mathcal{I}_q$, so that $\Delta(\varphi_+) \in F_q \otimes \mathcal{I}_q + \mathcal{I}_q \otimes F_q$ (since $\mathcal{I}_q \dot{\trianglelefteq} F_q$). Then

$$\Delta(\varphi_+) \in \big(\varphi_+ \otimes 1 + (q-1)^n {F_q}^\vee \otimes {\mathcal{I}_q}^{\curlyvee}\big) \bigcap \big(F_q \otimes \mathcal{I}_q + \mathcal{I}_q \otimes F_q\big)$$

or equivalently

$$\Delta(\varphi_+) \; - \; \varphi_+ \otimes 1 \;\in \big((q-1)^n {F_q}^\vee \otimes {\mathcal{I}_q}^{\curlyvee}\big) \bigcap \big(F_q \otimes \mathcal{I}_q + \mathcal{I}_q \otimes F_q\big) \tag{5.10}$$

Now, the description of ${\mathcal{I}_q}^{\curlyvee}$ given in the proof of Proposition 5.1 implies that

$$\big((q-1)^n {F_q}^\vee \otimes {\mathcal{I}_q}^{\curlyvee}\big) \bigcap \big(F_q \otimes \mathcal{I}_q + \mathcal{I}_q \otimes F_q\big) \;=\; F_q \otimes \mathcal{I}_q$$

this together with (5.10) yields $\Delta(\varphi_+) \in \varphi_+ \otimes 1 + F_q \otimes \mathcal{I}_q$, hence $\varphi_+ \in {F_q}^{co\mathcal{I}_q} =: \Phi(\mathcal{I}_q) = \mathcal{C}_q$ and so $\varphi \in (q-1)^n\,\mathcal{C}_q \bigcap {F_q}^\vee$. On the other hand, the description of $\mathcal{C}_q^{\triangledown}$ in the proof of Proposition 5.3 implies that $(q-1)^{-n}\,\mathcal{C}_q \bigcap {F_q}^\vee \subseteq \mathcal{C}_q^{\triangledown}$, hence we get $\varphi \in \mathcal{C}_q^{\triangledown}$, q.e.d.
We finish with claim *(d)*. For the inclusion $\Phi\big(\mathfrak{I}_q^{!}\big) \supseteq \mathfrak{C}_q^{\Lsh}$, let $\kappa \in \mathfrak{C}_q^{\Lsh}$. Since $\Phi\big(\mathfrak{I}_q^{!}\big)$ contains the scalars, we may assume that $\kappa \in Ker(\epsilon)$, thus $\Delta(\kappa) = \kappa \otimes 1 + 1 \otimes \kappa + \delta_2(\kappa)$. By Proposition 5.7, we have $\mathfrak{C}_q^{\Lsh} \dot{\trianglelefteq}_\ell U_q'$; thus $\Delta(\kappa) - \kappa \otimes 1 = 1 \otimes \kappa + \delta_2(\kappa) \in U_q' \otimes \mathfrak{C}_q^{\Lsh}$, and more precisely

$$\Delta(\kappa) - \kappa \otimes 1 = 1 \otimes \kappa + \delta_2(\kappa) \in U_q' \otimes \big(\mathfrak{C}_q^{\Lsh}\big)^+ \quad .$$

Since $\mathfrak{C}_q^{\Lsh} \subseteq \Psi\big(\mathfrak{C}_q^{\Lsh}\big) \subseteq \mathfrak{I}_q^{!}$, by claim *(c)*, we get $\Delta(\kappa) - \kappa \otimes 1 \in U_q' \otimes \mathfrak{I}_q^{!}$, so $\kappa \in \big(U_q'\big)^{co\mathfrak{I}_q^{!}} =: \Phi\big(\mathfrak{I}_q^{!}\big)$. Thus $\mathfrak{C}_q^{\Lsh} \subseteq \Phi\big(\mathfrak{I}_q^{!}\big)$. For the converse inclusion, let $\eta \in \Phi\big(\mathfrak{I}_q^{\,!}\big)$; again, we can assume $\eta \in Ker(\epsilon)$ too. As $\mathfrak{I}_q^{\,!} \subseteq \mathfrak{I}_q$, we get $\eta \in \Phi\big(\mathfrak{I}_q^{\,!}\big) \subseteq \Phi\big(\mathfrak{I}_q\big) = \mathfrak{C}_q$. Then $\delta_n(\eta) \in U_q^{\otimes n} \otimes \mathfrak{C}_q$ for all $n \in \mathbb{N}_+$, so

$$\delta_n(\eta) \;\in\; (q-1)^n \Bigg(\sum_{s=1}^{n-1} U_q^{\otimes (s-1)} \otimes \mathfrak{I}_q \otimes U_q^{\otimes (n-s)}\Bigg) \bigcap \, \big(U_q^{\otimes (n-1)} \otimes \mathfrak{C}_q\big) \;\subseteq$$
$$\subseteq\; (q-1)^n \, U_q^{\otimes (n-1)} \otimes \mathfrak{C}_q$$

hence $\delta_n(\eta) \in (q-1)^n \, U_q^{\otimes (n-1)} \otimes \mathfrak{C}_q$ ($n \in \mathbb{N}_+$) and $\eta \in \mathfrak{C}_q$, which means that $\eta \in \mathfrak{C}_q^{\,\Lsh}$. ∎

Remark 5.16. The inclusion $\Psi\big(\mathfrak{C}_q^{\,\Lsh}\big) \subseteq \mathfrak{I}_q^{\,!}$ of Proposition 5.15*(c)* is *not* an identity in general — indeed, *counterexamples do exist.*

Finally, we look at what happens when our Drinfeld-like recipes are applied to a pair of quantizations associated with a same subgroup / homogeneous spaces with respect to some fixed double quantization (in the sense of Section 3). The result reads as follows:

Proposition 5.17. *Let $\big(F_q[G]\,, U_q(\mathfrak{g})\big)$ be a double quantization of $(G,\mathfrak{g})$. Then:*

(a) Let $\mathcal{C}_q$ and $\mathfrak{I}_q$ be weak quantizations and assume that $\mathcal{C}_q = \mathfrak{I}_q^{\,\perp}$ and $\mathfrak{I}_q = \mathcal{C}_q^{\,\perp}$. Then $\mathfrak{I}_q^{\,!} = \big(\mathcal{C}_q^{\,\triangledown}\big)^{\perp}$ and $\mathcal{C}_q^{\,\triangledown} \subseteq \big(\mathfrak{I}_q^{\,!}\big)^{\perp}$. If, in addition, either one of $\mathcal{C}_q$ or $\mathfrak{I}_q$ is strict, then also $\mathcal{C}_q^{\,\triangledown} = \big(\mathfrak{I}_q^{\,!}\big)^{\perp}$.

(b) Let $\mathfrak{C}_q$ and $\mathcal{I}_q$ be weak quantizations and assume that $\mathcal{I}_q = \mathfrak{C}_q^{\,\perp}$ and $\mathfrak{C}_q = \mathcal{I}_q^{\,\perp}$. Then $\mathfrak{C}_q^{\,\Lsh} = \big(\mathcal{I}_q^{\,\curlyvee}\big)^{\perp}$ and $\mathcal{I}_q^{\,\curlyvee} \subseteq \big(\mathfrak{C}_q^{\,\Lsh}\big)^{\perp}$. If, in addition, either one of $\mathfrak{C}_q$ or $\mathcal{I}_q$ is strict, then also $\mathcal{I}_q^{\,\curlyvee} = \big(\mathfrak{C}_q^{\,\Lsh}\big)^{\perp}$.

Proof. Both in claim *(a)* and in claim *(b)* the orthogonality relations between $\mathfrak{C}_q$ and $\mathcal{I}_q$ and between $\mathcal{C}_q$ and $\mathfrak{I}_q$ are considered w.r.t. the pairing between $F_q[G]$ and $U_q(\mathfrak{g})$, and the subsequent orthogonality relations are meant w.r.t. the pairing between $F_q[G]^\vee$ and $U_q(\mathfrak{g})'$. Indeed, by Theorem 4.1, $\Big(U_q(\mathfrak{g})'\,, F_q[G]^\vee\Big)$ is a double quantization of $\big(G^*,\, \mathfrak{g}^*\big)$. *(a)* First, $\epsilon(\mathfrak{I}_q) = 0$ because $\mathfrak{I}_q$ is a coideal. Then $x = \delta_1(x) \in (q-1)\, U_q$ for all $x \in \mathfrak{I}_q^{\,!}$, hence $\mathfrak{I}_q^{\,!} \subseteq (q-1)\, U_q$. Thus we have

$$\big\langle \mathcal{C}_q, \mathfrak{I}_q^{\,!} \big\rangle \subseteq (q-1)\, \mathbb{C}\big[q, q^{-1}\big] \;\; .$$

Now let $J = J_{F_q}$ be the ideal of F_q, and take $c_i \in \mathcal{C}_q \cap J$ $(i = 1, \dots, n)$; then $\langle c_i, 1\rangle = \epsilon(c_i) = 0$ $(i = 1, \dots, n)$. Given $y \in \mathfrak{I}_q^{\,!}$, look at

$$\Big\langle \prod_{i=1}^n c_i \,,\, y \Big\rangle \;=\; \Big\langle \mathop{\otimes}_{i=1}^n c_i \,, \Delta^n(y) \Big\rangle \;=\; \Big\langle \mathop{\otimes}_{i=1}^n c_i \,, \sum_{\Psi \subseteq \{1,\dots,n\}} \delta_\Psi(y) \Big\rangle \;=$$
$$= \sum_{\Psi \subseteq \{1,\dots,n\}} \Big\langle \mathop{\otimes}_{i=1}^n c_i \,, \delta_\Psi(y) \Big\rangle$$

Consider the summands in the last term of the above formula. Let $|\Psi| = t$ $(t \leq n)$, then

$$\Big\langle \mathop{\otimes}_{i=1}^{n} c_i \, , \, \delta_\Psi(y) \Big\rangle \; = \; \Big\langle \mathop{\otimes}_{i \in \Psi} c_i \, , \, \delta_t(y) \Big\rangle \cdot \prod_{j \not\in \Psi} \big\langle \, c_j \, , 1 \, \big\rangle$$

by definition of δ_Ψ. Thanks to the previous analysis, we have $\prod_{j \not\in \Psi} \langle c_j \, , 1 \rangle = 0$ unless $\Psi = \{1, \dots, n\}$, and in the latter case

$$\delta_\Psi(y) \; = \; \delta_n(y) \; \in \; (q-1)^n \sum_{s=1}^{n} {U_q}^{\otimes (s-1)} \otimes \mathfrak{I}_q \otimes {U_q}^{\otimes (n-s)} \quad .$$

The outcome is

$$\Big\langle \mathop{\otimes}_{i=1}^{n} c_i \, , \, y \Big\rangle \; = \; \Big\langle \mathop{\otimes}_{i=1}^{n} c_i \, , \, \delta_n(y) \Big\rangle \; \in$$
$$\in \; \Big\langle \mathop{\otimes}_{i=1}^{n} c_i \, , \, (q-1)^n \sum_{s=1}^{n} {U_q}^{\otimes (s-1)} \otimes \mathfrak{I}_q \otimes {U_q}^{\otimes (n-s)} \Big\rangle \; = \; 0$$

because $y \in \mathfrak{I}_q^{\,!}$ and $\mathfrak{I}_q = {\mathcal{C}_q}^\perp$ by assumption. Therefore one has $\big\langle (q-1)^{-n} \big(\mathcal{C}_q \bigcap J \big)^n , \, \mathfrak{I}_q^{\,!} \big\rangle = 0$, for all $n \in \mathbb{N}_+$. In addition, $\big\langle 1 \, , \, \mathfrak{I}_q^{\,!} \big\rangle = \epsilon\big(\mathfrak{I}_q^{\,!}\big) = 0$. The outcome is $\big\langle \mathcal{C}_q^{\,\triangledown} , \, \mathfrak{I}_q^{\,!} \big\rangle = 0$, whence $\mathfrak{I}_q^{\,!} \subseteq \big(\mathcal{C}_q^{\,\triangledown}\big)^\perp$ and $\mathcal{C}_q^{\,\triangledown} \subseteq \big(\mathfrak{I}_q^{\,!}\big)^\perp$.

Now we prove also $\big(\mathcal{C}_q^{\,\triangledown}\big)^\perp \subseteq \mathfrak{I}_q^{\,!}$. Notice that $\mathcal{C}_q^{\,\triangledown} \supseteq \mathcal{C}_q$, whence $\big(\mathcal{C}_q^{\,\triangledown}\big)^\perp \subseteq {\mathcal{C}_q}^\perp = \mathfrak{I}_q$; therefore $\big(\mathcal{C}_q^{\,\triangledown}\big)^\perp \subseteq \mathfrak{I}_q$. Pick now $\eta \in \big(\mathcal{C}_q^{\,\triangledown}\big)^\perp$ (inside ${U_q}'$). Since $\eta \in {U_q}'$, for all $n \in \mathbb{N}_+$ we have $\delta_n(\eta) \in (q-1)^n {U_q}^{\otimes n}$, and from $\eta \in \big(\mathcal{C}_q^{\,\triangledown}\big)^\perp$ we get also that $\eta_+ := (q-1)^{-n} \delta_n(\eta)$ enjoys $\Big\langle \big(\mathcal{C}_q \bigcap J_{F_q} \big)^{\otimes n} , \, \eta_+ \Big\rangle = 0$ — acting as before — so that

$$\eta_+ \in \Big(\big(\mathcal{C}_q \bigcap J_{F_q} \big)^{\otimes n} \Big)^\perp \; = \sum_{r+s=n-1} {U_q}^{\otimes r} \otimes \big(\mathcal{C}_q \bigcap J_{F_q} \big)^\perp \otimes {U_q}^{\otimes s} \quad .$$

Moreover $\delta_n(\eta) \in {J_{U_q}}^{\otimes n}$, hence $\delta_n(\eta) \in \big((q-1)^n {U_q}^{\otimes n} \big) \bigcap {J_{U_q}}^{\otimes n} = (q-1)^n {J_{U_q}}^{\otimes n}$, so

$$\eta_+ \; \in \; \Big(\big(\mathcal{C}_q \bigcap J_{F_q} \big)^{\otimes n} \Big)^\perp \bigcap {J_{U_q}}^{\otimes n} \; =$$
$$= \; \Big(\sum_{r+s=n-1} {U_q}^{\otimes r} \otimes \big(\mathcal{C}_q \bigcap J_{F_q} \big)^\perp \otimes {U_q}^{\otimes s} \Big) \bigcap {J_{U_q}}^{\otimes n} \; =$$
$$= \sum_{r+s=n-1} {J_{U_q}}^{\otimes r} \otimes \Big(\big(\mathcal{C}_q \bigcap J_{F_q} \big)^\perp \bigcap J_{U_q} \Big) \otimes {J_{U_q}}^{\otimes s} \quad .$$

Since $\big(\mathcal{C}_q \bigcap J_{F_q} \big)^\perp \bigcap J_{U_q} = {\mathcal{C}_q}^\perp \bigcap J_{U_q} = \mathfrak{I}_q \bigcap J_{U_q} = \mathfrak{I}_q$, we have

$$\eta_+ \; \in \sum_{r+s=n-1} {J_{U_q}}^{\otimes r} \otimes \mathfrak{I}_q \otimes {J_{U_q}}^{\otimes s}$$

whence

$$\delta_n(\eta) \in (q-1)^n \sum_{r+s=n-1} {U_q}^{\otimes r} \otimes \mathfrak{I}_q \otimes {U_q}^{\otimes s} \qquad \forall\, n \in \mathbb{N}_+ \ .$$

Being, in addition, $\eta \in \mathfrak{I}_q$, for we proved that $\big(\mathcal{C}_q^\triangledown\big)^\perp \subseteq \mathfrak{I}_q$, we get $\eta \in \mathfrak{I}_q^{\,!}$. Therefore $\big(\mathcal{C}_q^\triangledown\big)^\perp \subseteq \mathfrak{I}_q^{\,!}$, q.e.d.

Finally, assume that $\mathcal{C}_q$ or $\mathfrak{I}_q$ are strict quantizations. Then we must still prove that $\mathcal{C}_q^\triangledown = \big({\mathfrak{I}_q}^!\big)^\perp$. Since $\mathcal{C}_q = {\mathfrak{I}_q}^\perp$ and $\mathfrak{I}_q = {\mathcal{C}_q}^\perp$, it is easy to check that $\mathcal{C}_q$ is strict if and only if $\mathfrak{I}_q$ is; therefore, we can assume that $\mathfrak{I}_q$ is strict.

The assumptions and Theorem 5.11 *(b)* give $\mathfrak{I}_q = \big({\mathfrak{I}_q}^!\big)^\curlyvee$; moreover, $\mathcal{I}_q := \mathfrak{I}_q^{\,!}$ is strict. Then we can apply the first part of claim *(b)* — which is proved, later on, in a way independent of the present proof of claim *(a)* itself — and get $\big({\mathcal{I}_q}^\curlyvee\big)^\perp = \big({\mathcal{I}_q}^\perp\big)^\Lsh$. Therefore

$$\mathcal{C}_q^\triangledown \;=\; \big({\mathfrak{I}_q}^\perp\big)^\triangledown \;=\; \Big(\Big(\big({\mathfrak{I}_q}^!\big)^\curlyvee\Big)^\perp\Big)^\triangledown \;=\; \Big(\big({\mathcal{I}_q}^\curlyvee\big)^\perp\Big)^\triangledown \;=\; \Big(\big({\mathcal{I}_q}^\perp\big)^\Lsh\Big)^\triangledown \quad . \tag{5.11}$$

Now, it is straightforward to prove that $\mathcal{I}_q$ strict implies that ${\mathcal{I}_q}^\perp$ is strict as well. Then Proposition 5.11*(d)* ensures $\Big(\big({\mathcal{I}_q}^\perp\big)^\Lsh\Big)^\triangledown = {\mathcal{I}_q}^\perp$. This along with (5.11) yields $\mathcal{C}_q^\triangledown = \Big(\big({\mathcal{I}_q}^\perp\big)^\Lsh\Big)^\triangledown = {\mathcal{I}_q}^\perp = \big({\mathfrak{I}_q}^!\big)^\perp$, ending the proof of *(a)*. *(b)* With much the same arguments as for *(a)*, we find as well that

$$\big\langle {\mathcal{I}_q}^\curlyvee, \mathfrak{C}_q^\Lsh \big\rangle \;\in\; \big\langle J^{\otimes(n-1)} \otimes \mathcal{I}_q \,,\, {U_q}^{\otimes(n-1)} \otimes \mathfrak{C}_q \big\rangle \;\subseteq\; \big\langle \mathcal{I}_q \,, \mathfrak{C}_q \big\rangle \;=\; 0$$

because $\mathcal{I}_q = {\mathfrak{C}_q}^\perp$; this means that

$${\mathcal{I}_q}^\curlyvee \subseteq \big(\mathfrak{C}_q^\Lsh\big)^\perp \quad , \qquad\qquad \mathfrak{C}_q^\Lsh \subseteq \big({\mathcal{I}_q}^\curlyvee\big)^\perp \quad . \tag{5.12}$$

Let now $\kappa \in \big({\mathcal{I}_q}^\curlyvee\big)^\perp_q \ \big(\subseteq {U_q}'\big)$. Since $\kappa \in {U_q}'$, we have $\delta_n(\kappa) \in (q-1)^n {U_q}^{\otimes n}$ for all $n \in \mathbb{N}$; moreover, from $\kappa \in \big({\mathcal{I}_q}^\curlyvee\big)^\perp$ it follows that $\kappa_+ := (q-1)^{-n}\delta_n(\kappa) \in {U_q}^{\otimes n}$ enjoys $\Big\langle J^{\otimes(n-1)} \otimes \mathcal{I}_q \,,\, \kappa_+ \Big\rangle = 0$, so that

$$\kappa_+ \in \Big(J^{\otimes(n-1)} \otimes \mathcal{I}_q\Big)^\perp \;=\; \textstyle\sum_{r+s=n-2} {U_q}^{\otimes r} \otimes J^\perp \otimes {U_q}^{\otimes s} \otimes U_q + {U_q}^{\otimes(n-1)} \otimes {\mathcal{I}_q}^\perp \ .$$

In addition, $\delta_n(\kappa) \in {J_{U_q}}^{\otimes n}$, where $J_{U_q} := Ker\big(\epsilon\colon U_q \longrightarrow \mathbb{C}\big[q,q^{-1}\big]\big)$; therefore $\delta_n(\kappa) \in \big((q-1)^n {U_q}^{\otimes n}\big) \bigcap {J_{U_q}}^{\otimes n} = (q-1)^n {J_{U_q}}^{\otimes n}$, which together with the above formula yields

$$\begin{aligned}
&\kappa_+ \in \Big(J^{\otimes(n-1)} \otimes \mathcal{I}_q\Big)^{\perp} \bigcap {J_{U_q}}^{\otimes n} = \\
&= \Big(\sum_{r+s=n-2} {U_q}^{\otimes r} \otimes J^{\perp} \otimes {U_q}^{\otimes s} \otimes U_q\Big) \bigcap {J_{U_q}}^{\otimes n} + \Big({U_q}^{\otimes(n-1)} \otimes {\mathcal{I}_q}^{\perp}\Big) \bigcap {J_{U_q}}^{\otimes n} = \\
&\quad = \sum_{r+s=n-2} {J_{U_q}}^{\otimes r} \otimes \Big(J^{\perp} \bigcap J_{U_q}\Big) \otimes {J_{U_q}}^{\otimes s} \otimes J_{U_q} + {J_{U_q}}^{\otimes(n-1)} \otimes \Big({\mathcal{I}_q}^{\perp} \bigcap J_{U_q}\Big) = \\
&\quad = {J_{U_q}}^{\otimes(n-1)} \otimes \Big({\mathcal{I}_q}^{\perp} \bigcap J_{U_q}\Big) = {J_{U_q}}^{\otimes(n-1)} \otimes \Big(\mathfrak{C}_q \bigcap J_{U_q}\Big) \subseteq {U_q}^{\otimes(n-1)} \otimes \mathfrak{C}_q
\end{aligned}$$

where in the third equality we used the fact that $J^{\perp} \bigcap J_{U_q} = \{0\}$. So $\kappa_+ \in {U_q}^{\otimes(n-1)} \otimes \mathfrak{C}_q$, hence $\delta_n(\kappa) \in (q-1)^n {U_q}^{\otimes(n-1)} \otimes \mathfrak{C}_q$ for all $n \in \mathbb{N}_+$: thus $\kappa \in \mathfrak{C}_q^{\,\dagger}$. Therefore $\big({\mathcal{I}_q}^{\curlyvee}\big)^{\perp} \subseteq \mathfrak{C}_q^{\,\dagger}$, which together with the right-hand side inequality in (5.12) gives $\mathfrak{C}_q^{\,\dagger} = \big({\mathcal{I}_q}^{\curlyvee}\big)^{\perp}$.

In the end, suppose also that one between $\mathfrak{C}_q$ and $\mathcal{I}_q$ is strict. As $\mathcal{I}_q = {\mathfrak{C}_q}^{\perp}$ and $\mathfrak{C}_q = {\mathcal{I}_q}^{\perp}$, one sees easily that $\mathcal{I}_q$ is strict if and only if $\mathfrak{C}_q$ is; then we can assume that $\mathfrak{C}_q$ is strict. We want to show that ${\mathcal{I}_q}^{\curlyvee} = \big(\mathfrak{C}_q^{\,\dagger}\big)^{\perp}$.

The assumptions and Theorem 5.11*(d)* give $\mathfrak{C}_q = \big(\mathfrak{C}_q^{\,\dagger}\big)^{\triangledown}$. Moreover, we have that $\mathcal{C}_q$ is strict by Proposition 5.3*(3)* and Proposition 5.7 *(3)*. Then we can apply the first part of claim *(a)*, thus getting $\big(\mathcal{C}_q^{\triangledown}\big)^{\perp} = \big({\mathcal{C}_q}^{\perp}\big)^{!}$. Therefore

$$ {\mathcal{I}_q}^{\curlyvee} = \big({\mathfrak{C}_q}^{\perp}\big)^{\curlyvee} = \Big(\Big(\big(\mathfrak{C}_q^{\,\dagger}\big)^{\triangledown}\Big)^{\perp}\Big)^{\curlyvee} = \Big(\big(\mathcal{C}_q^{\triangledown}\big)^{\perp}\Big)^{\curlyvee} = \Big(\big({\mathcal{C}_q}^{\perp}\big)^{!}\Big)^{\curlyvee} \tag{5.13} $$

Now, one proves easily that $\mathcal{C}_q$ strict implies ${\mathcal{C}_q}^{\perp}$ strict. Then Theorem 5.11*(c)* yields $\Big(\big({\mathcal{C}_q}^{\perp}\big)^{!}\Big)^{\curlyvee} = {\mathcal{C}_q}^{\perp}$. This and (5.13) give ${\mathcal{I}_q}^{\curlyvee} = \Big(\big({\mathcal{C}_q}^{\perp}\big)^{!}\Big)^{\curlyvee} = {\mathcal{C}_q}^{\perp} = \big(\mathfrak{C}_q^{\,\dagger}\big)^{\perp}$, which eventually ends the proof of *(b)*. □

6 Examples

In this last section we will give some examples showing how our general constructions may be explicitly implemented. Some of the examples may look rather singular, but our aim here is mainly to draw the reader's attention on how even badly behaved cases can produce reasonable results. It has to be remarked that a wealth of new examples of coisotropic subgroups of Poisson groups have been recently produced ([25]), to which our recipes could be interestedly applied.

N.B.: for the last two examples — Subsections 6.2 and 6.3 — one can perform the explicit computations (that we just sketch) using definitions, formulas and notations as in [5], §6, and in [11], §7.

6.1 Quantization of Stokes matrices as a GL_n^*–space

As a first example, we mention the following. A well-known structure of Poisson group, typically known as the *standard* one, is defined on SL_n ; then one can consider its (connected) dual Poisson group SL_n^* , which in turn is a Poisson group as well. The set of *Stokes matrices* — i.e. upper triangular, unipotent matrices — of size n bears a natural structure of Poisson homogeneous space, and even Poisson quotient, for SL_n^* . In [5], Section 6, it was shown that one can find an explicit quantization, of formal type, of this Poisson quotient by a suitable application of the QDP procedure for formal quantizations developed in that paper.
Now, let us look at the explicit presentation of the formal quantization $U_\hbar(\mathfrak{sl}_n)$ considered in [*loc. cit.*]. One sees easily that this can be turned into a presentation of a *global* quantization (of $\mathfrak{sl}_n$ again), i.e. a QUEA $U_q(\mathfrak{sl}_n)$ in the sense of Section 3. Similarly, Drinfeld's QDP (for quantum groups) applied to $U_\hbar(\mathfrak{sl}_n)$ provides a formal quantization $F_\hbar[[SL_n^*]] := U_\hbar(\mathfrak{sl}_n)'$ of the function algebra over the formal group SL_n^* ; but then the analogous functor for the global version of QDP yields (cf. Theorem 4.1) a global quantization $F_q[SL_n^*] := U_q(\mathfrak{sl}_n)'$ of the function algebra over SL_n^* . In a nutshell, $F_q[SL_n^*]$ is nothing but (a suitable renormalization of) an obvious $\mathbb{C}\big[q,q^{-1}\big]$–integral form of $F_\hbar[[SL_n^*]]$.
Carrying further on this comparison, one can easily see that the whole analysis performed in [5] can be converted into a similar analysis for the global context, yielding parallel results; in particular, *one ends up with a global quantization — of type $\mathcal{C}$, in the sense of Section 3 — of the space of Stokes matrices.* More in detail, this quantization is a strict one, as such is the quantum subobject one starts with.
Since all this does not require more than a word by word translation, we refrain from filling in details.

6.2 A parametrized family of real coisotropic subgroups

Coisotropic subgroups may come in families, in some cases inside the same conjugacy class (which is responsible for different Poisson homogeneous bivectors on the same underlying manifold). An example in the real case was described in detail in [2]. The setting is the one of standard Poisson $SL_2(\mathbb{R})$, which contains a two parameter family of 1–dimensional coisotropic subgroups described, globally, by the right ideal and two-sided ideal

$$\mathcal{I}_{\mu,\nu} := \Big\{ a - d + 2\, q^{\frac{1}{2}} \mu b \;,\; q\,\nu b + c \Big\} \cdot F_q\big[SL_2(\mathbb{R})\big] \tag{6.1}$$

where a,b,c,d are the usual matrix elements generating $F_q\big[SL_2(\mathbb{R})\big]$, with $*$–structure in which they are all real (thus $q^* = q^{-1}$) and $\mu,\nu \in \mathbb{R}$. The corresponding family of coisotropic subgroups of classical $SL_2(\mathbb{R})$ may be described as

$$K_{\mu,\nu} := \left\{ \begin{pmatrix} d - 2\mu b & b \\ -\nu b & d \end{pmatrix} \,\middle|\, b,d \in \mathbb{R}\,,\; d^2 + \nu b^2 = 1 \right\}$$

(adapting our main text arguments to the case of *right* quantum coisotropic subgroups, this is quite trivial and we will do it without further comments). The corresponding $SL_2(\mathbb{R})$–quantum homogeneous spaces have local description given as follows: $\mathcal{C}_{\mu,\nu}$ is the subalgebra generated by

$$\begin{aligned} z_1 \,&=\, q^{-\frac{1}{2}}(ac+\nu bd)+2\mu bc\,, \quad z_2 \,=\, c^2+\nu d^2+2\mu q^{-\frac{1}{2}}cd\,, \\ z_3 \,&=\, a^2+\nu b^2+2\mu q^{-\frac{1}{2}}ab\,. \end{aligned} \tag{6.2}$$

Using commutation relations — see (12) in [3] — it is easily seen that $\mathcal{C}_{\mu,\nu}$ has a linear basis given by $\big\{\, z_1^p z_2^q\,,\, z_1^p z_3^r \,\big|\, p,q,r\in\mathbb{N}\,\big\}$.

Proposition 6.1. *The subalgebra $\mathcal{C}_{\mu,\nu}$ is a right coideal of $F_q\big[SL_2(\mathbb{R})\big]$ and is a strict quantization — of type $\mathcal{C}$ — of $K_{\mu,\nu}$.*

Proof. The first statement is proven in [3]. As for the second we will first show that $z_1^p z_2^q\,, z_1^p z_3^r \not\in (q-1)F_q\big[SL_2(\mathbb{R})\big]$ for any $p,q,r\in\mathbb{N}$. This may done by considering their expression in terms of the usual basis $\big\{\, a^p b^r c^s\,,\, b^h c^k d^i \,\big\}$ of $F_q\big[SL_2(\mathbb{R})\big]$. In fact we do not need a full expression of monomials $z_1^p z_2^r$ or $z_1^p z_3^r$ in terms of this basis, which would lead to quite heavy computations. It is enough to remark that, for example, since

$$z_1^p z_2^r \;=\; \Big(q^{-\frac{1}{2}}ac+b(\nu d+2\mu c)\Big)^p\,\Big(c^2+(\nu d+2\mu q^{-\frac{1}{2}}c)d\Big)^r$$

we can get an element multiple of $a^p c^{p+2r}$ only from $(ac)\cdot\dots\cdot(ac)\cdot c\cdot\dots\cdot c$, which is of the form $q^h a^p c^{p+2r}\not\in F_q\big[SL_2(\mathbb{R})\big]$. Since no other elements may add up with this one, we have $z_1^p z_2^r \not\in (q-1)F_q\big[SL_2(\mathbb{R})\big]$. A similar argument works for $z_1^p z_3^r$.
In a similar way we prove that any $\mathbb{C}\big[q,q^{-1}\big]$–linear combination of the $z_1^p z_2^q$'s and the $z_1^s z_3^r$'s is in $(q-1)F_q\big[SL_2(\mathbb{R})\big]$ if and only if all coefficients are in $(q-1)\mathbb{C}\big[q,q^{-1}\big]$. Therefore $\mathcal{C}_q$ is strict, q.e.d. ∎

It makes therefore sense to compute $\mathcal{C}_{\mu,\nu}^{\,\triangledown}$; to this end, we can resume a detailed description of $U_q(\mathfrak{sl}_2^*) := F_q\big[SL_2(\mathbb{R})\big]^\vee$ — apart for the real structure, which is not really relevant here — from [11], §7.7. From our PBW-type basis we have that $\mathcal{C}_{\mu,\nu}^{\,\triangledown}$ is the subalgebra of $F_q\big[SL_2(\mathbb{R})\big]^\vee$ generated by the elements $\zeta_i := \frac{1}{q-1}\big(z_i-\varepsilon(z_i)\big)\in F_q\big[SL_2(\mathbb{R})\big]^\vee\;(i=1,2,3)$. Since we know that

$$H_+ \,:=\, \frac{a-1}{q-1}\;, \qquad E \,:=\, \frac{b}{q-1}\;, \qquad F \,:=\, \frac{c}{q-1}\;, \qquad H_- \,:=\, \frac{d-1}{q-1}$$

are algebra generators of $U_q(\mathfrak{sl}_2^*) := F_q\big[SL_2(\mathbb{R})\big]^\vee$, we deduce that

$$\begin{aligned}
\frac{\zeta_1}{q-1} &= q^{-\frac{1}{2}}(F+\nu E) + (q-1)\Big(q^{-\frac{1}{2}}H_+F + q^{-\frac{1}{2}}\nu E H_- + 2\mu EF\Big) \\
\frac{\zeta_2-\nu}{q-1} &= 2\,(\nu H_- + \mu q^{-\frac{1}{2}}F) + (q-1)\Big(F^2 + \nu H_-^2 + 2\mu q^{-\frac{1}{2}}FH_-\Big) \\
\frac{\zeta_3-1}{q-1} &= 2\,(H_+ + \mu q^{-\frac{1}{2}}E) + (q-1)\Big(H_+^2 + \nu E^2 + 2\mu q^{-\frac{1}{2}}H_+E\Big)
\end{aligned} \tag{6.3}$$

In the semiclassical specialization $U_q(\mathfrak{sl}_2^*) \xrightarrow{q\longrightarrow 1} U_q(\mathfrak{sl}_2^*)\Big/(q-1)U_q(\mathfrak{sl}_2^*)$ one has that $E \mapsto \mathrm{e}$, $F \mapsto \mathrm{f}$, $H_\pm \mapsto \pm\mathrm{h}$, where $\mathrm{h}, \mathrm{e}, \mathrm{f}$ are Lie algebra generators of $\mathfrak{sl}_2^*$; therefore the semiclassical limit of the right hand side of (6.3) is the Lie subalgebra generated by $\mathrm{f} + \nu\,\mathrm{e}$, $-\nu\,\mathrm{h} + \mu\,\mathrm{e}$, $\mathrm{h} + \mu\,\mathrm{e}$, or, equivalently, the 2–dimensional Lie subalgebra generated by $\mathrm{f}+\nu\,\mathrm{e}$ and $\mathrm{h}+\mu\,\mathrm{e}$ (the three elements above being linearly dependent) with relation $[\,\mathrm{h}+\mu\,\mathrm{e}\,,\mathrm{f}+\nu\,\mathrm{e}\,] = \mathrm{f}+\nu\,\mathrm{e}$. The quantization of this coisotropic subalgebra of $\mathfrak{sl}_2^*$ is therefore the subalgebra generated inside $U_q(\mathfrak{sl}_2^*)$ by the quadratic elements (6.3).
Similar computations can be performed starting from $\mathcal{I}_{\mu,\nu}$. The transformed $\mathcal{I}_{\mu,\nu}^{\curlyvee}$ is the right ideal generated by the image of $a - d + 2\,q^{\frac{1}{2}}\mu b$ and $q\nu b + c$, i.e. the right ideal generated by $H_+ - H_- + 2\,q^{\frac{1}{2}}\mu E$ and $q\,\nu E + F$; also, from its semiclassical limit it is easily seen that this again corresponds to the same coisotropic subgroup of the dual Poisson group $SL_2(\mathbb{R})^*$.
All this gives a local — i.e., infinitesimal — description of the (2–dimensional) coisotropic subgroups $K_{\mu,\nu}^{\perp}$ in $SL_2(\mathbb{R})^*$.

6.3 The non coisotropic case

Let us finally consider the case of a non coisotropic subgroup. We will consider the embedding of $SL_2(\mathbb{C})$ into $SL_3(\mathbb{C})$ corresponding to a non simple root, which easily generalizes to higher dimensions. Computations will only be sketched.
Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{sl}_3(\mathbb{C})$ spanned by $E_{1,3}$, $F_{1,3}$, $H_{1,3} = H_1 + H_2$. Easy computations show that the standard cobracket values are

$$\begin{aligned}
\delta(E_{13}) &= E_{13}\wedge(H_1+H_2) + 2E_{23}\wedge E_{12} \\
\delta(F_{13}) &= F_{13}\wedge(H_1+H_2) - 2F_{23}\wedge F_{12} \\
\delta(H_1+H_2) &= 0
\end{aligned} \tag{6.4}$$

and, therefore, the corresponding embedding $SL_2(\mathbb{C}) \longrightarrow SL_3(\mathbb{C})$ is *not* coisotropic. To compute the coisotropic interior $\overset{\circ}{\mathfrak{h}}$ of $\mathfrak{h}$, consider that $\langle H_1+H_2\rangle$ is, trivially, a subbialgebra of $\mathfrak{h}$, thus contained in $\overset{\circ}{\mathfrak{h}}$. Let $X := (H_1+H_2) + \alpha E_{13} + \beta F_{13}$: then

$$\delta(X) \;=\; X\wedge(H_1+H_2) + 2\,(\alpha E_{23}\wedge E_{12} - \beta F_{23}\wedge F_{12})$$

shows that no such X is in $\overset{\circ}{\mathfrak{h}}$, unless $\alpha = 0 = \beta$. The outcome is that we have

$$\overset{\circ}{H} = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma^{-1} \end{pmatrix} \subseteq SL_3(\mathbb{C})$$

with $\gamma \in \mathbb{C}^*$. Correspondingly

$$\mathfrak{h}^{\langle\perp\rangle} = \Big(\overset{\circ}{\mathfrak{h}}\Big)^{\perp} = \big\langle \mathrm{e}_{1,2}\,, \mathrm{e}_{1,3}\,, \mathrm{e}_{2,3}\,, \mathrm{f}_{1,2}\,, \mathrm{f}_{1,3}\,, \mathrm{f}_{2,3}\,, \mathrm{h}_{2.2} \big\rangle \qquad \Big(\subseteq \mathfrak{sl}_3(\mathbb{C})^* \Big)$$

and, thus $SL_3(\mathbb{C})^* \Big/ H^{\langle\perp\rangle}$ is a 1–dimensional Poisson homogeneous space — with, of course, zero Poisson bracket.
Let us consider now any weak quantization $\mathfrak{C}_q(H)$ of H. It should certainly contain the subalgebra of $U_q(\mathfrak{sl}_3)$ generated by the root vectors $E_{1,3}$, $F_{1,3}$, together with $K_1 K_3^{-1}$ and $\widehat{H}_{1,3} := \big(K_1 K_3^{-1} - 1\big)\Big/(q-1)$. The equality

$$\Delta(E_{1,3}) = E_{1,3} \otimes K_1 K_3^{-1} + 1 \otimes E_{1,3} + (q-1) E_{1,2} \otimes E_{2,3}$$

tells us that, in order to be a left coideal, such a quantization should also contain either $(q-1)E_{1,2}$ or $(q-1)E_{2,3}$ (and thus, as expected, it cannot be strict). Let us try to compute some elements in $\mathfrak{C}_q(H)^{\curlyvee}$. Certainly, since

$$\delta_2\big(\widehat{H}_{1,3}\big) = \widehat{H}_{1,3} \otimes \big(K_1 K_3^{-1} - 1\big) = (q-1)\,\widehat{H}_{1,3} \otimes \widehat{H}_{1,3}$$

we can conclude that $(q-1)\widehat{H}_{1,3} \in \mathfrak{C}_q(H)^{\curlyvee}$. On the other hand,

$$\delta_2(E_{1,3}) = (q-1) E_{1,3} \otimes \widehat{H}_{1,3} + (q-1) E_{1,2} \otimes E_{2,3}$$

implies that $(q-1)E_{1,3} \notin \mathfrak{C}_q(H)^{\curlyvee}$, while $(q-1)^2 E_{1,3} \in \mathfrak{C}_q(H)^{\curlyvee}$.
All this means the following.
Within $\mathfrak{C}_q(H)^{\curlyvee}$ we find a non-diagonal matrix element of the form $(q-1)\,t_{1,3}$: it belong to $(q-1)U_q(\mathfrak{sl}_3)'$ but not to $(q-1)\mathfrak{C}_q(H)^{\curlyvee}$, so that

$$\mathfrak{C}_q(H)^{\curlyvee} \bigcap (q-1)\, U_q(\mathfrak{sl}_3)' \supsetneqq (q-1)\, \mathfrak{C}_q(H)^{\curlyvee}$$

which means that the quantization $\mathfrak{C}_q(H)^{\curlyvee}$ *is not strict.* On the other hand, we know by Proposition 5.7*(3)* that $\mathfrak{C}_q(H)^{\curlyvee}$ is *proper*. Therefore, *we have an example of a quantization (of type $\mathcal{C}_q$, still by Proposition 5.7(3)) which is proper, yet it is not strict.* In addition, in the specialization map $\pi : U_q(\mathfrak{sl}_3)' \twoheadrightarrow U_q(\mathfrak{sl}_3)' \Big/ (q-1) U_q(\mathfrak{sl}_3)'$ the element $(q-1)\,t_{1,3}$ is mapped to zero, i.e. it yields a trivial contribution to the semiclassical limit of $\mathfrak{C}_q(H)^{\curlyvee}$ — which here is meant as being $\pi\big(\mathfrak{C}_q(H)^{\curlyvee}\big) = \mathfrak{C}_q(H)^{\curlyvee} \Big/ \mathfrak{C}_q(H)^{\curlyvee} \bigcap (q-1)\, U_q(\mathfrak{sl}_3)'$. With similar computations it is possible to prove, in fact, that the only generating element in

$\mathfrak{C}(H)^{\urcorner}$ having a non-trivial semiclassical limit is $(q-1)\widehat{H}_{1,3}$. Therefore, through specialization at $q = 1$, from $\mathfrak{C}(H)^{\urcorner}$ one gets only $\pi\big(\mathfrak{C}_q(H)^{\urcorner}\big) = \mathbb{C}\big[t_{2,2}\big]$: indeed, this in turn tells us exactly that $\mathfrak{C}_q(H)^{\urcorner}$ is a quantization, of *proper* type, of the homogeneous $SL_3(\mathbb{C})^*$–space $SL_3(\mathbb{C})^* \big/ H^{\langle\perp\rangle}$ (whose Poisson bracket is trivial).

Remark. It is worth stressing that this example — no matter how rephrased — could not be developed in the language of formal quantizations as a direct application of the construction in [5], for only strict quantizations were taken into account there.

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On the Integrality of Modular Symbols and Kato's Euler System for Elliptic Curves

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Abstract. Let $E/\mathbb{Q}$ be an elliptic curve. We investigate the denominator of the modular symbols attached to E. We show that one can change the curve in its isogeny class to make these denominators coprime to any given odd prime of semi-stable reduction. This has applications to the integrality of Kato's Euler system and the main conjecture in Iwasawa theory for elliptic curves.

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1 Introduction

Let $E/\mathbb{Q}$ be an elliptic curve. Integrating a Néron differential ω_E against all elements in $H_1\bigl(E(\mathbb{C}),\mathbb{Z}\bigr)$, we obtain the Néron lattice $\mathscr{L}_E$ of E in $\mathbb{C}$. For any $r\in\mathbb{Q}$, define $\lambda(r) = 2\pi i\int_\infty^r f(\tau)d\tau$ where f is the newform associated to the isogeny class of E. A theorem by Manin [12] and Drinfeld [7] shows that the values $\lambda(r)$ are commensurable with $\mathscr{L}_E$. In other words, if Ω_E^+ and Ω_E^- are the minimal absolute values of non-zero elements in $\mathscr{L}_E$ on the real and the imaginary axis respectively, then

$$\lambda(r) = 2\pi i\int_\infty^r f(\tau)d\tau = [r]_E^+\cdot\Omega_E^+ + [r]_E^-\cdot\Omega_E^-\cdot i$$

for two rational numbers $[r]_E^\pm$, which we will call the modular symbols of E.

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The first aim of this paper is to improve on the bound for the denominator of $[r]^{\pm}_{E}$ given by the Theorem of Manin and Drinfeld. It is not true in general that $[r]^{\pm}_{E}$ is an integer for all r. The only odd primes that can divide these denominators are those which divide the degree of an isogeny $E \to E'$ defined over $\mathbb{Q}$. Even by allowing to change the curve in the isogeny class, we can not always achieve that the modular symbols are integers; for instance 3 will be a denominator of $[r]^{\pm}_{E}$ for some $r \in \mathbb{Q}$ for all E of conductor 27. However the following theorem says that we may get rid of all odd primes p such that p^2 does not divides the conductor N of E.

THEOREM 1. *Let $E/\mathbb{Q}$ be an elliptic curve. Then there exists an elliptic curve $E_\bullet$, which is isogenous to E over $\mathbb{Q}$, such that $[r]^{\pm}_{E_\bullet}$ is a p-integer for all $r \in \mathbb{Q}$ and for all odd primes p for which E has semi-stable reduction.*

As stated here one could take $E_\bullet$ to be one of the curves in the isogeny class with maximal Néron lattice. However it is a consequence of Theorem 4, which is more precise and says that there is a curve $E_\bullet$ whose Néron lattice is contained in the lattice of all values of $\lambda(r)$ with index not divisible by any odd prime of semi-stable reduction.
As a direct consequence of this Theorem 1, one deduces that the algebraic part of the special values of the twisted L-series $L(E_\bullet, \chi, s)$ at $s = 1$ are p-adic integers for all Dirichlet characters χ and all odd semi-stable primes p. See Corollary 7.
The second part of this paper is devoted to another application of this theorem. Let p be an odd prime of semi-stable reduction. Kato has constructed in [10] an Euler system for the isogeny class of E. See Section 3 for details of the definitions. There are two sets of p-adic "zeta-elements": First, a set of integral zeta elements denoted by ${}_{c,d}z_m(\alpha)$ in the Galois cohomology of a lattice T_f canonically associated to f which provides upper bounds for Selmer groups. Secondly, a set of zeta elements denoted by z_γ which are linked to the p-adic L-functions. The latter are not known to be integral with respect to T_f. We will show in Proposition 8 that T_f is equal to the Tate module $T_pE_\bullet$ of the curve $E_\bullet$ in Theorem 1.
Let K_n be the n-th layer in the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Let $\mathbf{z} \in \varprojlim_n H^1(K_n, T_pE_\bullet) \otimes \mathbb{Q}_p$ be the zeta element that is sent to the p-adic L-function for $E_\bullet$ via the Coleman map.

THEOREM 2. *If the reduction is good at p, then $\mathbf{z}$ belongs to the integral Iwasawa cohomology $\varprojlim_n H^1(K_n, T_pE_\bullet)$.*

This is Theorem 13 in the text. Actually, the proof gives a more precise result. The global Iwasawa cohomology group $\mathbf{H}^1(T_pE)$ with restricted ramification turns out to be very often, but not always, a free module of rank 1 over the Iwasawa algebra of the $\mathbb{Z}_p$-extension. If it is free for $E = E_\bullet$ then the integrality of $\mathbf{z}$ is easily deduced; otherwise one can show that $\mathbf{H}^1(T_pE_\bullet)$ is at worst equal to the maximal ideal in the Iwasawa algebra and the integrality above follows then from the interpolation property of the p-adic L-function $L_p(E)$.

Another consequence of Theorem 1 concerns the main conjecture in Iwasawa theory for elliptic curves. We formulate it here for the full cyclotomic $\mathbb{Z}_p^\times$-extension.

Theorem 3. *Let E be an elliptic curve and p an odd prime of semi-stable reduction. Assume that $E[p]$ is reducible as a Galois module over $\mathbb{Q}$. Then the characteristic series of the dual of the Selmer group over the cyclotomic extension $\mathbb{Q}\bigl(\zeta_{p^\infty}\bigr)$ divides the ideal generated by the p-adic L-function $L_p(E)$ in the Iwasawa algebra $\Lambda = \mathbb{Z}_p\bigl[\bigl[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})\bigr]\bigr]$.*

Note that our assumptions in the theorem imply that the reduction of E at p is ordinary in the sense that E has either good ordinary or multiplicative reduction, because $E[p]$ is irreducible when E has supersingular reduction, see Proposition 12 in [22]. In the case when E has split multiplicative reduction, we can strengthen our theorem, see Theorem 16.

This theorem was proven by Kato in [10] in the case that the reduction is ordinary and the representation on the Tate module was surjective. The method of proof follows and generalises the incomplete proof in [30], where unfortunately the integrality issue had been overlooked.

For most good ordinary primes p for which $E[p]$ is irreducible the full main conjecture, asserting the equality rather than the divisibility in the above theorem, is now known thanks to the work of Skinner and Urban [25]. However their proof of the converse divisibility does not seem to extend easily to the reducible case.

Nonetheless, the above theorem has applications to the conjecture of Birch and Swinnerton-Dyer and to the explicit computations of Tate-Shafarevich groups as in [26]. The theorem also implies that all p-adic L-functions for elliptic curves at odd primes p of semi-stable ordinary reductions are integral elements in the Iwasawa algebra. See Corollary 18.

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2 The lattice of all modular symbols

Let E be an elliptic curve defined over $\mathbb{Q}$. In what follows p will always stand be an odd prime and we suppose that E does not have additive reduction at p. The only case for which the integrality of Kato's Euler system may not hold is when E admits an isogeny of degree p defined over $\mathbb{Q}$; so we may just as well assume that we are in this "reducible" case. All conclusions in this section and in the rest of the paper are still valid without this assumption, however they are not our original work but rather well-known results. Denote by N the conductor of E.

In the isogeny class of E there are two interesting elliptic curves. The first is the optimal curve E_0 with respect to the modular parametrisation from

the modular curve $X_0(N)$, which is also often called the strong Weil curve. The second is the optimal curve E_1 with respect to the parametrisation from $X_1(N)$. The definition of optimality is given in [28], for instance the map $H_1\bigl(X_0(N)(\mathbb{C}),\mathbb{Z}\bigr)\to H_1\bigl(E_0(\mathbb{C}),\mathbb{Z}\bigr)$ is surjective. Another interesting curve is the so-called minimal curve (see [28]), which is conjecturally equal to E_1, but we will not make use of it in this article. Recall that a cyclic isogeny $A\to A'$ defined over $\mathbb{Q}$ is étale (this is a slight abuse of notation, we should say more precisely that it extends to an étale isogeny on the Néron models over $\mathbb{Z}$) if the pull-back of a Néron differential of A' yields a Néron differential of A.
Let f be the newform of level N corresponding to the isogeny class of E. We write $\omega_f = 2\pi i f(\tau)d\tau = f(q)dq/q$ for the corresponding differential form on the modular curve $X_1(N)$. For any curve A in the isogeny class of E, we define the Néron lattice $\mathscr{L}_A$ to be the image of

$$\int \omega_A \colon H_1\bigl(A(\mathbb{C}),\mathbb{Z}\bigr)\to\mathbb{C}$$

where ω_A is a choice of a Néron differential. We denote by $\mathscr{L}_0$ and $\mathscr{L}_1$ the lattices $\mathscr{L}_{E_0}$ and $\mathscr{L}_{E_1}$ respectively. Then $\mathscr{L}_f$ is defined to be the lattice of all $\int_\gamma \omega_f$ where γ varies in $H_1\bigl(X_1(N),\mathbb{Z}\bigr)$. Finally, we define

$$\hat{\mathscr{L}}_f = \left\{ \int_\gamma \omega_f \;\middle|\; \gamma\in H_1\Bigl(X_1(N)(\mathbb{C}),\{\text{cusps}\},\mathbb{Z}\Bigr)\right\}.$$

obtained by integrating ω_f along all paths between cusps in $X_1(N)$. This is the lattice of all modular symbols attached to f. By the Theorem of Manin–Drinfeld $\hat{\mathscr{L}}_f$ is a lattice with $\hat{\mathscr{L}}_f\subset\mathscr{L}_f\,\mathbb{Q}$. In fact, we know that all the lattices above are commensurable and we view them now as $\mathbb{Z}$-modules inside $V=\mathscr{L}_1\otimes\mathbb{Q}$.

Theorem 4. *Let $E/\mathbb{Q}$ be an elliptic curve. Then there exists an elliptic curve $E_\bullet/\mathbb{Q}$ in the isogeny class of E whose lattice $\mathscr{L}_\bullet=\mathscr{L}_{E_\bullet}$ satisfies $\mathscr{L}_\bullet\otimes\mathbb{Z}_p = \hat{\mathscr{L}}_f\otimes\mathbb{Z}_p$ inside $V\otimes\mathbb{Q}_p$ for all odd primes p at which E has semi-stable reduction. Moreover the cyclic isogeny from E_1 to $E_\bullet$ is étale.*

Alternatively, we could also say that the index of $\mathscr{L}_\bullet\subset\hat{\mathscr{L}}_f$ is coprime to any odd prime of semi-stable reduction. We should also emphasise that the statement does not hold in general for primes p of additive reduction or for $p=2$. Counter-examples for these will be provided later. The proof will require some intermediate lemmas.

Lemma 5. *Let $A/\mathbb{Q}$ be an elliptic curve and let p be an odd prime. Suppose P is a point of exact order p in A, defined over an abelian extension of $\mathbb{Q}$ which is unramified at p. Then the isogeny with kernel generated by P is defined over $\mathbb{Q}$.*

Proof. Let G be the Galois group of $\mathbb{Q}\bigl(A[p]\bigr)$ over $\mathbb{Q}$. Let H be the subgroup corresponding to the field of definition $\mathbb{Q}(P)$ of P. Then H is a normal subgroup

$$\begin{tikzcd}
 & & \mathbb{Q}(A[p]) \arrow[dll, dash, "H"'] \arrow[d, dash, "H\cap S"'] \arrow[dr, dash, "S"] & \\
 & & \cdot \arrow[dll, dash] \arrow[dr, dash] & \\
\mathbb{Q}(P) \arrow[dr, dash] & & & \mathbb{Q}(\mu_p) \arrow[dll, dash] \\
 & \mathbb{Q} & &
\end{tikzcd}$$

of G with abelian quotient. In any basis of $A[p]$ with P as the first element, the group H is contained in $\left(\begin{smallmatrix}1&*\\0&*\end{smallmatrix}\right)$ when we view G as a subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$. Let $S = G\cap \mathrm{SL}_2(\mathbb{F}_p)$ be the kernel of the determinant $G \to \mathbb{F}_p^\times$. Hence $H\cap S$ is contained in the subgroup of matrices of the form $\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)$. So we have two cases to distinguish. Either $H\cap S$ is equal to the cyclic group of order p of all matrices of this form or it is trivial. But note first that the Weil pairing implies that $\mathbb{Q}(\mu_p)$ is contained in $\mathbb{Q}\bigl(A[p]\bigr)$. So G/S is isomorphic to $\mathbb{F}_p^\times$ via the determinant. Since $\mathbb{Q}(P)$ is unramified at p, it must be linearly disjoint from $\mathbb{Q}(\mu_p)$. For our groups, this means that $HS = G$. Hence $H/(H\cap S) = G/S = \mathbb{F}_p^\times$.

Case 1: $H\cap S$ is equal to the cyclic group of order p generated by $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. The above then implies that H is equal to the subgroup of all matrices $\left(\begin{smallmatrix}1&*\\0&*\end{smallmatrix}\right)$. Now G is contained in the normaliser of this group H inside $\mathrm{GL}_2(\mathbb{F}_p)$, which is easily seen to be equal to the Borel subgroup of matrices of the form $\left(\begin{smallmatrix}*&*\\0&*\end{smallmatrix}\right)$. In particular, the subgroup generated by P is fixed by G.

Case 2: H intersects S trivially. Then $\mathbb{Q}\bigl(A[P]\bigr)$ is the composition of $\mathbb{Q}(\mu_p)$ and $\mathbb{Q}(P)$. Hence G is the abelian group $H\times S$. Note that H is now a cyclic group of order $p-1$. Let h be a non-trivial element of $H \subset \bigl\{\left(\begin{smallmatrix}1&*\\0&*\end{smallmatrix}\right)\bigr\}$. It has two eigenvalues, one equal to 1 and the other λ must be different than 1 as otherwise h would belong to S. Let $Q\in A[p]$ be an eigenvector for h with eigenvalue λ and use the basis $\{P,Q\}$ for $A[p]$. For H to be an abelian subgroup of $\bigl\{\left(\begin{smallmatrix}1&*\\0&*\end{smallmatrix}\right)\bigr\}$ containing the element $h = \left(\begin{smallmatrix}1&0\\0&\lambda\end{smallmatrix}\right)$, it is necessary that H is contained in the diagonal matrices. Therefore H is the group of all matrices of the form $\left(\begin{smallmatrix}1&0\\0&*\end{smallmatrix}\right)$.

We know that S has to commute with H. It is easy to see that this implies that S is contained in the group of matrices of the form $\left(\begin{smallmatrix}a&0\\0&1/a\end{smallmatrix}\right)$. It follows that G is contained in the diagonal matrices. Once again the isogeny defined by P is fixed by G. $\square$

If A is an elliptic curve defined over $\mathbb{Q}$, we know by [2] that there is a non-constant morphism of curves $\varphi_A\colon X_0(N)\to A$ defined over $\mathbb{Q}$. We normalise it by requiring that it is of minimal degree and that the cusp ∞ maps to $O\in A(\mathbb{Q})$. It is well-defined up to composition with an automorphism of A.

Lemma 6. *Let $A/\mathbb{Q}$ be an elliptic curve and let p be an odd prime such that A has semi-stable reduction at p. Let $r \in \mathbb{Q}$ represent a cusp on $X_0(N)$ such that the image $\varphi_A(r)$ in $A(\bar{\mathbb{Q}})$ has order divisible by p. Let $P \in A(\bar{\mathbb{Q}})$ be a multiple of $\varphi_A(r)$ which has exact order p. Then the isogeny with kernel generated by P is étale and defined over $\mathbb{Q}$.*

Proof. Let D be the greatest common divisor of the denominator of r and N. Next, let d be the greatest common divisor of D and $\frac{N}{D}$. So by definition d is only divisible by primes of additive reduction and hence it is coprime to p. By the description of the Galois-action on cusps of $X_0(N)$ given in Theorem 1.3.1. in [27],we see that the cusp r on $X_0(N)$, and hence its image in $A(\bar{\mathbb{Q}})$, are defined over the cyclotomic field $K = \mathbb{Q}(\zeta_d)$. The previous Lemma 5 proves that the isogeny generated by P is defined over $\mathbb{Q}$. Since the kernel acquires a point over an extension which is unramified at p, it has to be étale. □

Proof of Theorem 4. The lattice $\hat{\mathscr{L}}_f$ is the set of all values of integrating $\omega_f = 2\pi i f(\tau) d\tau$ as τ runs along a geodesic from one cusp $r_1 \in \mathbb{Q}$ to another $r_2 \in \mathbb{Q}$ inside the upper half plane. So it is also the set of all $\int_\gamma \omega_f$ as γ varies in $H_1\bigl(X_0(N), \{\text{cusps}\}, \mathbb{Z}\bigr)$. We are allowed to switch here from $X_1(N)$ to $X_0(N)$ and to identify ω_f on both of them as the pullback of ω_f under $X_1(N) \to X_0(N)$ is again ω_f because it is determined by the q-expansion of f.
The Manin constant c_0 for the optimal curve E_0 is an integer such that $\varphi_0^*(\omega_0) = c_0 \cdot \omega_f$, where $\varphi_0 \colon X_0(N) \to E_0$ is the modular parametrisation of minimal degree and ω_0 is a Néron differential on E_0. One can choose φ_0 and ω_0 in such a way as to make $c_0 > 0$. It is known that c_0 is coprime to any odd prime for which E has semi-stable reduction. For this and more on the Manin constant we refer to [1]. From the description of optimality above, we can deduce that $c_0 \cdot \mathscr{L}_f = \mathscr{L}_0$ and hence that $c_0 \cdot \hat{\mathscr{L}}_f \supset \mathscr{L}_0$.
To start, we set A to be the optimal curve E_0. We shall successively replace A by one of its quotients by an étale kernel until we reach $E_\bullet$. Pick an odd semi-stable prime that divides the index i_A of $\mathscr{L}_A$ in $c_0 \cdot \hat{\mathscr{L}}_f$. The modular parametrisation $\varphi_A \colon X_0(N) \to A$ factors through E_0. The quotient $\bigl(c_0\hat{\mathscr{L}}_f\bigr)/\mathscr{L}_A$ is generated by the images $\varphi_A(r) \in A(\mathbb{C}) \cong \mathbb{C}/\mathscr{L}_A$ of all cusps r in $X_0(N)$. So we find a cusp r whose image in $A(\bar{\mathbb{Q}})$ has order divisible by p. We can now apply Lemma 6, which gives us an étale isogeny $A \to A'$ such that the index of $\mathscr{L}_{A'}$ in $c_0\hat{\mathscr{L}}_f$ is now $i_{A'} = i_A/p$. We replace now A by A' and repeat the procedure until the index i_A is coprime to all odd semi-stable primes. By the above mentioned property of c_0, we now have $\mathscr{L}_A \otimes \mathbb{Z}_p = \hat{\mathscr{L}}_f \otimes \mathbb{Z}_p$ for all odd semi-stable primes
By construction, A is now an étale quotient of E_0. We consider the isogeny $E_1 \to E_0 \to A$. The cyclic isogeny $E_1 \to E_0$ has a constant kernel and hence it is étale over $\mathbb{Z}[\frac{1}{2}]$, as explained in Remark 1.8 in [29]. If it is étale over $\mathbb{Z}$, we can set $E_\bullet = A$ and we are done. Otherwise, there is an isogeny $E_0 \to E_0'$ whose degree is a power of 2 such that the cyclic isogeny from E_1 to E_0' is étale. Since the degree of $E_0 \to A$ is odd by construction, there is an isogeny $A \to E_\bullet$

of the same degree as $E_0 \to E_0'$ such that $E_1 \to E_\bullet$ is étale. □

For any A in the isogeny class of E, we write Ω_A^+ for the smallest positive real element of $\mathscr{L}_A$ and Ω_A^- for the smallest absolute value of a purely imaginary element in $\mathscr{L}_A$. For any $r \in \mathbb{Q}$, the modular symbols $[r]^\pm \in \mathbb{Q}$ attached to A are defined by

$$[r]^+ = \frac{1}{\Omega_A^+}\operatorname{Re}\biggl(\int_r^\infty \omega_f\biggr) \qquad \text{and} \qquad [r]^- = \frac{1}{\Omega_A^-}\operatorname{Im}\biggl(\int_r^\infty \omega_f\biggr).$$

Then our theorem tells us that $[r]^\pm$ will have denominator coprime to any odd semi-stable prime for the curve $E_\bullet$. In particular, it is obvious from the construction (see [14]) of the p-adic L-function by modular symbols that it will be an integral power series in $\mathbb{Z}_p[\![T]\!]$ for ordinary primes p. However this also follows from Proposition 3.7 in [9] and the fact that $E_1 \to E_\bullet$ is étale.
A reformulation of the theorem is the following integrality statement.

Corollary 7. *Let E be an elliptic curve over $\mathbb{Q}$ and p an odd prime for which E has semi-stable reduction. Then there is a curve $E_\bullet$ which is isogenous to E over $\mathbb{Q}$ such that for all Dirichlet characters χ we have*

$$\frac{G(\chi)\cdot L(E_\bullet,\chi,1)}{\Omega_{E_\bullet}^+} \in \mathbb{Z}_p[\chi] \qquad \text{if } \chi(-1) = 1 \text{ or}$$
$$\frac{G(\chi)\cdot L(E_\bullet,\chi,1)}{i\,\Omega_{E_\bullet}^-} \in \mathbb{Z}_p[\chi] \qquad \text{if } \chi(-1) = -1$$

where $\mathbb{Z}_p[\chi]$ is the ring of integers in the extension of $\mathbb{Q}_p$ generated by the values of χ and $G(\chi)$ stands for the Gauss sum.

Proof. This follows from the formula of Birch, see formula (8.6) in [14]:

$$L(E,\chi,1) = \frac{1}{G(\bar\chi)} \sum_{a \bmod m} \bar\chi(a)\biggl(\int_{a/m}^\infty \omega_f\biggr)$$

where m is the conductor of χ. □

2.1 The semi-stable case

Let $E/\mathbb{Q}$ be an elliptic curve with semi-stable reduction at all primes. Hence N is square-free. So d in the proof of Lemma 6 is equal to 1 for all cusps and hence they are all defined over $\mathbb{Q}$. By Mazur's Theorem [13], we may obtain $E_\bullet$ satisfying $\hat{\mathscr{L}}_f \otimes \mathbb{Z}[\frac{1}{2}] = \mathscr{L}_\bullet \otimes \mathbb{Z}[\frac{1}{2}]$ by taking the quotient of E_0 only by at most a p-torsion point defined over $\mathbb{Q}$ for some $p = 3$, 5 or 7. In particular, if $E_0(\mathbb{Q})[3\cdot 5\cdot 7] = \{O\}$, then $E_\bullet = E_0$. If instead, there is a rational torsion point of odd order, then we might have to take the isogeny with kernel $E_0(\mathbb{Q})[p]$. Nonetheless the curve labelled 66c1 in [5] shows that in some examples we can have $E_\bullet = E_0$ even when E_0 has a rational 5-torsion point.

2.2 Examples

We can present here a few examples; in all of them we know that $c_0 = 1$. Throughout, we use the notations from Cremona's tables [5]. First, for the class 11a and $p = 5$, we find that E_1 =11a3, E_0 =11a1, and $E_\bullet$ =11a2 and the étale isogenies $E_1 \to E_0 \to E_\bullet$ are all of degree 5. To justify this, one has to note that $L(f,1) = \frac{1}{5}\Omega^+_{E_0}$ and so $[0]^+ = \frac{1}{5}$ for E_0. Hence the lattice $\hat{\mathscr{L}}_f$ has index at least 5 in $\mathscr{L}_0$.
For the class 17a, the curve E_0 =17a1 has Mordell-Weil group $E(\mathbb{Q}) = {}^{\mathbb{Z}}\!/_{4\mathbb{Z}}$. The optimal curve E_1 corresponds to a sublattice of index 4 in $\mathscr{L}_0$ and it is the minimal curve 17a4. It is easy to compute the modular symbols for f. Since $L(f,1) = \frac{1}{4}\Omega^+_0$, we find that $\hat{\mathscr{L}}_f$ has index at least 4 in $\mathscr{L}_0$. In fact, $\hat{\mathscr{L}}_f$ is the lattice $\frac{1}{2}\mathscr{L}_{17\text{a}3}$. This shows that the above lemma is not valid for $p = 2$.
In the class 91b, we find that E_0 and E_1 are equal to 91b1, which has 3-torsion points over $\mathbb{Q}$. It turns out that $E_\bullet$, which is equal to 91b2, has a 3-torsion point as well. So it is not true in general that $E_\bullet(\mathbb{Q})$ has no p-torsion even when it is different from E_0.
Now to elliptic curves, which are not semi-stable. The class 98a is the twist of 14a by -7. This time the lattice $\hat{\mathscr{L}}_f$ is equal to the lattice of 98a5, which has the same real period as E_0, but the imaginary period is divided by 9. Both E_0 and $E_\bullet$ have only a 2-torsion point defined over $\mathbb{Q}$. The two cyclic isogenies of degree 3 acquire a rational point in the kernel only over $\mathbb{Q}(\sqrt{-7})$.
For the curves 27a, which admit complex multiplication, we find that $\hat{\mathscr{L}}_f = \frac{1}{3}\mathscr{L}_0$. The same happens for 54a. However in both cases E does not have semi-stable reduction at $p = 3$. This shows that the lemma and theorem can not be extended to primes p with additive reduction.

3 Kato's Euler system

Let $E/\mathbb{Q}$ be an elliptic curve and p an odd prime. Suppose E has semi-stable reduction at p. Since we are mainly interested in the case when $E[p]$ is reducible, we may assume that the reduction at E is ordinary.
We now follow the notations and definitions in [10]. As before f is the newform of weight 2 and level N associated to the isogeny class of E. Define the $\mathbb{Q}_p$-vector space $V_{\mathbb{Q}_p}(f)$ as the largest quotient of $H^1_{\text{ét}}\bigl(\overline{Y_1(N)}, \mathbb{Q}_p\bigr)$ on which the Hecke operators act by multiplication with the coefficients of f. Further the image of $H^1_{\text{ét}}\bigl(\overline{Y_1(N)}, \mathbb{Z}_p\bigr)$ in $V_{\mathbb{Q}_p}(f)$ is a $\mathrm{Gal}\bigl(\bar{\mathbb{Q}}/\mathbb{Q}\bigr)$-stable lattice, denoted by $V_{\mathbb{Z}_p}(f)$.

Proposition 8. *We have an equality of* $\mathrm{Gal}\bigl(\bar{\mathbb{Q}}/\mathbb{Q}\bigr)$*-stable lattices* $V_{\mathbb{Z}_p}(f)(1) = T_pE_\bullet$ *inside* $V_{\mathbb{Q}_p}(f)(1)$.

Proof. We consider first the version with coefficients in $\mathbb{Z}$ rather than in $\mathbb{Z}_p$ as in 6.3 of [10]. We define $V_{\mathbb{Q}}(f)$ as the maximal quotient of $H^1\bigl(Y_1(N)(\mathbb{C}), \mathbb{Q}\bigr)$ and $V_{\mathbb{Z}}(f)$ as the image of $H^1\bigl(Y_1(N)(\mathbb{C}), \mathbb{Z}\bigr)$ inside $V_{\mathbb{Q}}(f)$. By Poincaré duality, we have

$$H^1\big(Y_1(N)(\mathbb{C}),\mathbb{Z}\big) \cong H_1\big(X_1(N)(\mathbb{C}),\{\text{cusps}\},\mathbb{Z}\big)$$

as in 4.7 in [10]. Now let $\varphi_1\colon X_1(N) \to E_1$ be the optimal modular parametrisation. The optimality implies that φ_1 induces a surjective map from $H_1\big(X_1(\mathbb{C}),\mathbb{Z}\big)$ to $H_1\big(E_1(\mathbb{C}),\mathbb{Z}\big)$. Hence we may identify $V_{\mathbb{Q}}(f)$ via φ_1 with $H_1\big(E_1(\mathbb{C}),\mathbb{Q}\big)$. Under this identification, the lattice $V_{\mathbb{Z}}(f)$ is mapped to the image of the relative homology $H_1\big(X_1(N)(\mathbb{C}),\{\text{cusps}\},\mathbb{Z}\big)$. It contains the lattice $H_1\big(E_1(\mathbb{C}),\mathbb{Z}\big)$. Through the map integrating against the Néron differential ω_1 of E_1, the lattice $V_{\mathbb{Z}}(f)$ is brought to $c_1\hat{\mathcal{L}}_f$ containing $\mathcal{L}_1$ where c_1 is the Manin constant of φ_1, i.e. the integer such that $\varphi_1^*(\omega_1) = c_1\omega_f$. Since c_1 is a p-adic unit by Proposition 3.3 in [9], our Theorem 4 shows that

$$V_{\mathbb{Z}}(f)\otimes\mathbb{Z}_p = H_1\big(E_\bullet(\mathbb{C}),\mathbb{Z}\big)\otimes\mathbb{Z}_p \quad \text{inside} \quad V_{\mathbb{Q}}(f)\otimes\mathbb{Q}_p = H_1\big(E_1(\mathbb{C}),\mathbb{Q}\big)\otimes\mathbb{Q}_p.$$

Following 8.3 in [10], we can identify $V_{\mathbb{Z}_p}(f)$ with $V_{\mathbb{Z}}(f)\otimes\mathbb{Z}_p$ through the comparison of Betti and étale cohomology. We identify again $V_{\mathbb{Q}_p}(f)$ with $H^1_{\text{ét}}\big(\overline{E_1},\mathbb{Q}_p\big)$ through φ_1 and we obtain that

$$V_{\mathbb{Z}_p}(f) = H^1_{\text{ét}}\big(\overline{E_\bullet},\mathbb{Z}_p\big) \cong T_pE_\bullet(-1) \quad \text{containing} \quad H^1_{\text{ét}}\big(\overline{E_1},\mathbb{Z}_p\big) \cong T_pE_1(-1)$$

at least as $\mathbb{Z}_p$-lattices inside $V_{\mathbb{Q}_p}(f)$. But the Galois action is the same on both $V_{\mathbb{Z}_p}(f)$ and $T_p(E_\bullet)(-1)$. $\square$

From now on we will denote this lattice in our Galois representation simply by $T = V_{\mathbb{Z}_p}(f)(1) = T_pE_\bullet$. Kato constructs in 8.1 in [10] two sets of p-adic zeta-elements in the Galois cohomology of T. First, let a and $A \geqslant 1$ be two integers. Then there is an element

$${}_{c,d}z_m\big(\tfrac{a}{A}\big) = {}_{c,d}z_m^{(p)}\big(f,1,1,a(A),\text{primes}(pA)\big) \in H^1_{\text{ét}}\big(\mathbb{Z}[\tfrac{1}{p},\zeta_m],T\big)$$

for all integers $m \geqslant 1$ and integers c,d coprime to $6pA$. They are linked to the modular symbol obtained from the path from $\frac{a}{A}$ to ∞ in the upper half plane. Also, ζ_m is a primitive m-th root of unity.

Secondly, for any $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, there are elements

$${}_{c,d}z_m(\alpha) = {}_{c,d}z_m^{(p)}\big(f,1,1,\alpha,\text{primes}(pN)\big) \in H^1_{\text{ét}}\big(\mathbb{Z}[\tfrac{1}{p},\zeta_m],T\big)$$

for any integer $m \geqslant 1$ and integers $c \equiv d \equiv 1 \pmod{N}$ coprime to $6pN$. They are linked to the image under α of the path from 0 to ∞ in the upper half plane.

The advantage of these integral elements (with respect to our lattice T) is that they form an Euler system (13.3 in [10]). Namely by fixing α, c and d as above, the elements $\big({}_{c,d}z_m(\alpha)\big)_m$ form an Euler system.

Out of the above elements for m being a power of p, Kato builds the zeta-elements that are linked to the p-adic L-functions. We denote by

$$\Lambda = \mathbb{Z}_p\Big[\Big[\mathrm{Gal}\big(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}\big)\Big]\Big] = \varprojlim_n \mathbb{Z}_p\Big[\mathrm{Gal}\big(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}\big)\Big]$$

the Iwasawa algebra of the cyclotomic $\mathbb{Z}_p^\times$-extension of $\mathbb{Q}$. Then we have the following finitely generated Λ-module

$$\mathbf{H}^1(T) := \varprojlim_n H^1_{\text{ét}}\bigl(\mathbb{Z}[\zeta_{p^n}, \tfrac{1}{p}], T\bigr) = \varprojlim_n H^1\bigl(G_\Sigma(\mathbb{Q}(\zeta_{p^n})), T\bigr),$$

where Σ is any set of primes containing the infinite places and those dividing pN and $G_\Sigma(K)$ is the Galois group of the maximal extension of K which is unramified outside Σ. See Section 3.4.1 in [17] for the independence on Σ. For each $\gamma \in T$, there is a

$$z_\gamma = z_\gamma^{(p)} \in \mathbf{H}^1(T) \otimes \mathbb{Q}_p = \varprojlim_n H^1_{\text{ét}}\bigl(\mathbb{Z}[\tfrac{1}{p}, \zeta_{p^n}], T\bigr) \otimes \mathbb{Q}_p\,.$$

In fact, they are defined in 13.9 in [10] as elements in the larger $\mathbf{H}^1(T) \otimes_\Lambda \operatorname{Frac}(\Lambda)$ as they are quotients of elements of the form ${}_{c,d}z_m(\alpha)$ by certain elements $\mu(c,d)$ in Λ. However Kato shows in 13.12 that they belong to the much smaller $\mathbf{H}^1(T) \otimes \mathbb{Q}_p$ by comparing them with elements of the form ${}_{c,d}z_{p^n}(\frac{a}{A})$. See also appendix A in [6] for more information about the division by $\mu(c,d)$.

3.1 Criteria for the Iwasawa cohomology to be free over the Iwasawa algebra

The Λ-module $\mathbf{H}^1(T)$ is torsion-free of rank 1 as shown in Theorem 12.4 in [10]. If $E[p]$ is irreducible, then Theorem 12.4.(3) shows that $\mathbf{H}^1(T)$ is free. In this section we gather further cases in which we can prove that $\mathbf{H}^1(T)$ is free or otherwise determine how far we are off from being free. When it is free one deduces that z_γ integral for all $\gamma \in T$. We will later turn back to this question in Section 3.3

Lemma 9. *Let p be an odd prime of semi-stable reduction. If the X_0-optimal curve E_0 has no rational p-torsion point, but the degree of the cyclic isogeny from E_0 to $E_\bullet$ is divisible by p, then $\mathbf{H}^1(T)$ is free of rank 1 over Λ.*

This lemma is essentially about curves that are not semi-stable. It applies to all twists of a semi-stable curve by a square-free $D \neq \pm p$. This follows from the fact that for semi-stable curves a result by Serre [24, Proposition 1] and [22, Proposition 21] shows that $E[p]$ is an extension of $\mathbb{Z}/p\mathbb{Z}$ by $\mu[p]$ or an extension of $\mu[p]$ by $\mathbb{Z}/p\mathbb{Z}$.
Conversely, if E_0 has a point of order $p > 2$ defined over $\mathbb{Q}$, then it has semi-stable reduction at all places, except for $p = 3$ when we could have fibres of type IV or IV*.

Proof. We claim that under our hypothesis, the Mordell-Weil group $E_\bullet\bigl(\mathbb{Q}(\zeta_p)\bigr)$ contains no p-torsion points. Let $\phi\colon A \to A'$ be a cyclic isogeny of degree p in the isogeny $E_0 \to E_\bullet$ and assume by induction that A has no torsion point defined over $\mathbb{Q}$. From the proof of Theorem 4, we know that $A[\phi]$ acquires rational points over $\mathbb{Q}(\zeta_d)$ with $d \mid N$ as in the proof of Lemma 6. In particular

p does not divide d and so $A[\phi]$ will not contain a rational point defined over $\mathbb{Q}(\zeta_p)$; neither will $A'[\hat{\phi}]$ as it is its Cartier dual. This means that the semi-simplification of $A[p]$ is the sum of two distinct characters with conductor divisible by a prime different from p. Hence A and A' both have no p-torsion point defined over $\mathbb{Q}(\zeta_p)$.

One way to prove the lemma is by adapting Kato's argument at the end of 13.8. The argument works as long as the twisted $\mathbb{F}_p(r)$ does not appear in $E[p]$ as a Galois sub-module. Instead we give a second proof here.

Let $\Gamma = \mathrm{Gal}\big(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}(\zeta_p)\big)$. Using the Tate spectral sequence [15, Theorem 2.1.11] we see that $\mathbf{H}^1(T)_\Gamma$ injects into $H^1\big(G_\Sigma(\mathbb{Q}(\zeta_p)), T\big)$ via the corestriction map. Now the torsion subgroup of the latter is equal to the torsion subgroup of $\varprojlim E\big(\mathbb{Q}(\zeta_p)\big)/p^n$, which is trivial if $E\big(\mathbb{Q}(\zeta_p)\big)$ has no p-torsion. Hence $\mathbf{H}^1(T)_\Gamma$ is a free $\mathbb{Z}_p$-module.

Choose an injection $\iota\colon \mathbf{H}^1(T) \to \Lambda$ with finite cokernel F. We deduce an exact sequence

$$0 \longrightarrow F^\Gamma \longrightarrow \mathbf{H}^1(T)_\Gamma \longrightarrow \Lambda_\Gamma \longrightarrow F_\Gamma \longrightarrow 0$$

Since $\mathbf{H}^1(T)_\Gamma$ is torsion-free, we obtain that $F^\Gamma = 0$. Since F is finite, F_Γ is of the same size. But by Nakayama's Lemma $F_\Gamma = 0$ implies that $F = 0$. Hence $\mathbf{H}^1(T)$ is Λ-free. □

We refine our analysis of $\mathbf{H}^1(T)$ now a bit for the remaining cases. Any Λ-module M comes equipped with an action by the group $\Delta = \mathrm{Gal}\big(\mathbb{Q}(\zeta_p)/\mathbb{Q}\big)$ and we split M up into the eigenspaces $M = \bigoplus_{i=0}^{p-2} M_i$ where Δ acts on $M_i = M(-i)^\Delta$ by the i-th power of the Teichmüller character. Now M_i is a $\Lambda(\Gamma) = \mathbb{Z}_p[\![\Gamma]\!]$-module.

Lemma 10. *Let $\phi\colon E \to E'$ be an isogeny whose kernel has a point of order p defined over $\mathbb{Q}$. Then $\mathbf{H}^1(T_pE)_i$ and $\mathbf{H}^1(T_pE')_i$ are free of rank 1 over $\Lambda(\Gamma)$ for all $1 < i \leqslant p-2$. Furthermore $\mathbf{H}^1(T_pE)_1$ and $\mathbf{H}^1(T_pE')_0$ are also free of rank 1. The remaining $\mathbf{H}^1(T_pE)_0$ and $\mathbf{H}^1(T_pE')_1$ are either free of rank 1 or there is an injection into $\Lambda(\Gamma)$ with image equal to the maximal ideal.*

Proof. We have two short exact sequence

$$0 \longrightarrow T_pE \xrightarrow{\ \phi\ } T_pE' \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

$$0 \longleftarrow \mu_p \longleftarrow T_pE \xleftarrow[\ \hat{\phi}\ ]{} T_pE' \longleftarrow 0$$

which induces two exact sequences

$$0 \longrightarrow \mathbf{H}^1(T_pE) \xrightarrow{\ \phi\ } \mathbf{H}^1(T_pE') \longrightarrow \mathbf{H}^1\big(\mathbb{Z}/p\mathbb{Z}\big) \qquad (*)$$

$$\mathbf{H}^1(\mu_p) \longleftarrow \mathbf{H}^1(T_pE) \xleftarrow[\ \hat{\phi}\ ]{} \mathbf{H}^1(T_pE') \longleftarrow 0.$$

Here the last terms are the projective limits as n tends to infinity of the groups $H^1\big(G_\Sigma(\mathbb{Q}(\zeta_{p^n})),\mathbb{Z}/p\mathbb{Z}\big)$ and of $H^1\big(G_\Sigma(\mathbb{Q}(\zeta_{p^n})),\mu[p]\big)$ respectively. Since $p = 3$, 5 or 7, the class group of $\mathbb{Q}(\zeta_{p^n})$ has no p-torsion and hence $H^1(G_\Sigma(\mathbb{Q}(\zeta_{p^n})),\mu[p])$ is the quotient of the global Σ-units by its p-th powers. Lemma 4.3.4 and Proposition 4.5.3 in [4] show that $\mathbf{H}^1(\mu[p]) = \mathbb{F}_p(1)\oplus\Lambda^+/p$ as a $\Lambda = \mathbb{Z}_p[\Delta][\![\Gamma]\!]$-module, where Λ^+ the part of Λ fixed by complex conjugation. Also we have $\mathbf{H}^1(\mathbb{Z}/p\mathbb{Z}) = \mathbf{H}^1(\mu[p])(-1) = \mathbb{F}_p\oplus\Lambda^-/p$. Because the composition of ϕ and $\hat\phi$ is the multiplication by p, the cokernels of the end maps of the two exact sequences (*) above have to be finite because $\mathbf{H}^1(T_pE)$ and $\mathbf{H}^1(T_pE')$ are known to be torsion-free Λ-modules of rank 1.

If i is not 0 or 1, then the argument in the proof of Lemma 9 applies to show that $\mathbf{H}^1(T_pE)_i$ and $\mathbf{H}^1(T_pE')_i$ are both free since the p-torsion subgroup of $E\big(\mathbb{Q}(\zeta_p)\big)$ and $E'\big(\mathbb{Q}(\zeta_p)\big)$ have trivial i-th eigenspace under the action of Δ.

Let now $i=0$ and set $A = \mathbf{H}^1(T_pE)_0$ and $B = \mathbf{H}^1(T_pE')_0$. In the case $i=1$, we would just swap the roles of A and B. The exact sequences (*) show that $\phi\colon A\to B$ has finite cokernel of size at most p and that $\hat\phi\colon B\to A$ has cokernel in $\Lambda(\Gamma)/p\cong\mathbb{F}_p[\![\Gamma]\!]$. Choose an injection $\iota\colon B\to\Lambda(\Gamma)$ with finite cokernel F. We now view B via ι and A via $\phi\circ\iota$ as ideals in $\Lambda(\Gamma)$ of finite index. The map $\hat\phi\colon B\to A$ becomes the multiplication by p.

Let I be the kernel of the map $\Lambda(\Gamma)\to\mathbb{Z}_p$ sending all elements of Γ to 1. Then we obtain the exact sequence

$$0\longrightarrow F^\Gamma\longrightarrow A/IA\longrightarrow \Lambda/I\longrightarrow F/IF\longrightarrow 0.$$

Again if $A/IA = A_\Gamma$ is $\mathbb{Z}_p$-free, then A is $\Lambda(\Gamma)$-free and since $A\to B$ has finite cokernel, then B has to be free, too. Assume therefore that A/IA is not free. We know that A/IA injects into $H^1\big(G_\Sigma(\mathbb{Q}),T_pE\big)$ whose torsion part is the p-primary part of $E(\mathbb{Q})$. Hence it is at most of order p. We conclude that F^Γ and F_Γ are both of order p under our assumption. Hence $A/IA\cong {}^{\mathbb{Z}}\!/_{p\mathbb{Z}}\oplus\mathbb{Z}_p$ and we can take $p+IA$ to be the generator of the free part. Let $a\in A$ be such that $a+IA$ is a generator of the torsion part. It must lie in I but not in IA. By Nakayama's Lemma p and a generate the ideal A. Consider now the exact sequence

$$0\longrightarrow p\Lambda(\Gamma)/pB\longrightarrow A/pB\longrightarrow A/p\Lambda(\Gamma)\longrightarrow 0$$

where the middle term is a finite index sub-$\Lambda(\Gamma)$-module of $\Lambda(\Gamma)/p$. But a such does not have any finite non-zero sub-modules. Hence $p\Lambda(\Gamma) = pB$ shows that B is $\Lambda(\Gamma)$-free of rank 1. Since the smaller ideal A has index p it has no choice but to be the maximal ideal of $\Lambda(\Gamma)$. □

Here is an example for which $\mathbf{H}^1(T_pE)_0$ is not free. The semi-stable isogeny class 11a contains three curves

$$E_1 = 11\mathrm{a}3\xrightarrow{\ \phi\ }E_0 = 11\mathrm{a}1\xrightarrow{\ \psi\ }E_\bullet = 11\mathrm{a}2$$

where the direction of the arrow is the isogeny with kernel ${}^{\mathbb{Z}}\!/_{p\mathbb{Z}}$ with $p=5$. While E_1 and E_0 have rational 5-torsion points, the Mordell-Weil group of $E_\bullet$

over $\mathbb{Q}$ is trivial. Hence by the proof of Lemma 9, we see that $\mathbf{H}^1(T_pE_\bullet)_0$ is $\Lambda(\Gamma)$-free. This lemma does not apply to E_0, however Lemma 10 does and shows that $\mathbf{H}^1(T_pE_0)_0$ is also $\Lambda(\Gamma)$-free. We will now show that $\mathbf{H}^1(T_pE_1)_0$ is not free.

For this we continue the first exact sequence in (*) as follows

$$\mathbf{H}^1(T_pE_1)_0 \xrightarrow{\phi} \mathbf{H}^1(T_pE_0)_0 \longrightarrow \mathbb{F}_p \longrightarrow \mathbf{H}^2(T_pE_1)_0 \xrightarrow{\phi_2} \mathbf{H}^2(T_pE_0)_0$$

where $\mathbf{H}^2(\cdot)$ stands for the projective limit of $H^2\big(G_\Sigma(\mathbb{Q}(\zeta_{p^n})),\cdot\big)$. Our aim is to show that ϕ_2 is injective. Let $Z_{v,i}$ be the projective limit of $H^2\big(\mathbb{Q}_v(\zeta_{p^n}),T_pE_i\big)_0$ as $n\to\infty$ and consider the localisation maps

$$\begin{tikzcd}
0 \arrow[r] & Y_1 \arrow[r] \arrow[d] & \mathbf{H}^2(T_pE_1)_0 \arrow[r] \arrow[d, "\phi_2"] & \bigoplus_{v\in\Sigma} Z_{v,1} \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & Y_0 \arrow[r] & \mathbf{H}^2(T_pE_0)_0 \arrow[r] & \bigoplus_{v\in\Sigma} Z_{v,0} \arrow[r] & {}
\end{tikzcd}$$

By global duality the kernels Y_1 and Y_0 are fine Selmer groups which we will properly define in Section 4; for our purpose here it is sufficient to say that they are both trivial in our example. To show that ϕ_2 is injective it is sufficient to show that $\phi\colon Z_{v,1}\to Z_{v,0}$ is injective for all $v\in\Sigma=\{5,11\}$. Local duality shows that $Z_{v,i}$ is dual to the p-primary part of the group of points of E_i over $\mathbb{Q}_v(\zeta_{p^\infty})^\Delta$. Hence we want to show that for all $v\in\ \{5,11\}$ the map

$$\hat{\phi}\colon E_0\big(\mathbb{Q}_v(\zeta_{p^\infty})^\Delta\big)[p^\infty]\to E_1\big(\mathbb{Q}_v(\zeta_{p^\infty})^\Delta\big)[p^\infty]$$

is surjective. First for $v=11$ where both curves have split multiplicative reduction; however the Tamagawa number for E_0 is 5 while it is 1 for E_1. We conclude that the p-primary part of $E\big(\mathbb{Q}_{11}(\zeta_{5^\infty})\big)$ is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$ for $E=E_0$ and it is equal to $\mathbb{Q}_p/\mathbb{Z}_p\oplus\mathbb{Z}/p\mathbb{Z}$ for $E=E_1$. The map $\hat{\phi}$ is easily seen to be surjective by looking at the 5-torsion points over $\mathbb{Q}_{11}$.

Next for $v=5$, where the reduction is good ordinary. Here the p-primary parts of both groups of local points are equal to ${}^{\mathbb{Z}}\!/_{5\mathbb{Z}}$. This follows from the fact that the formal group of these curves have torsion group isomorphic to μ_{p^∞} which has no Δ-fixed points and from the existence of the rational 5-torsion points over $\mathbb{Q}_5$.

This ends the proof that $\mathbf{H}^1(T_pE_1)_0$ is not free but equal to the maximal ideal as shown in Lemma 10. Note that the same argument won't work for ψ, because $\hat{\psi}$ is not surjective locally on the p-primary part neither at $v=5$ nor at $v=11$.

3.2 Link to the p-adic L-function

For any extension $K/\mathbb{Q}_p$, we write $H^1_f(K,T)$ for the Bloch-Kato group of local conditions. The quotient group $H^1_s(K,T)=H^1(K,T)/H^1_f(K,T)$ is in fact dual to $E_\bullet(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ by local Tate duality. We set $\mathbf{H}^1_s(T)$ to be the projective limit of $H^1_s\big(\mathbb{Q}_p(\zeta_{p^n}), T\big)$, which is a Λ-module of rank 1.

Perrin-Riou has constructed a Coleman map $\mathrm{Col}\colon \mathbf{H}^1_s(T) \to \Lambda$. Proposition 17.11 in [10] shows that the Coleman map $\mathrm{Col}\colon \mathbf{H}^1_s(T) \to \Lambda$ is injective and has finite cokernel if the reduction of E at p is good. The same proof also applies when the reduction is non-split multiplicative. Instead in the case when E has split multiplicative reduction, then Theorem 4.1 in [11] proves that the Coleman map $\mathrm{Col}\colon \mathbf{H}^1_s(T) \to \Lambda$ is injective and has image with finite index inside $I = \ker\big(\mathbb{1}\colon \Lambda \to \mathbb{Z}_p\big)$ where the map $\mathbb{1}$ sends all elements of the Galois group $\mathrm{Gal}\big(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}\big)$ to 1. Extend Col to an injective map $\mathrm{Col}\colon \mathbf{H}^1_s(T)\otimes\mathbb{Q}_p \to \Lambda \otimes \mathbb{Q}_p$.

Choose $\gamma \in T$ such that $\gamma = \gamma^+ + \gamma^-$ with $\gamma^\pm$ being $\mathbb{Z}_p$-generators of the subspaces $T^\pm$ on which the complex conjugation acts by ± 1. We now apply Theorem 16.6 in [10] with this "good choice" of γ and with the "good choice" of the Néron differential $\omega = \omega_{E_\bullet}$ in the terminology of 17.5. Consider the zeta element $\mathbf{z} = z_\gamma \in \mathbf{H}^1(T) \otimes \mathbb{Q}_p$. The theorem yields

$$\mathrm{Col}\big(\mathrm{loc}(\mathbf{z})\big) = L_p(E_\bullet) \in \Lambda,$$

where $\mathrm{loc}\colon \mathbf{H}^1(T) \otimes \mathbb{Q}_p \to \mathbf{H}^1_s(T) \otimes \mathbb{Q}_p$ is the localisation followed by the quotient map.

Let $Z_T = Z(f, T)$ be the Λ-module generated by z_γ in $\mathbf{H}^1(T) \otimes \mathbb{Q}_p$ and let Z be the Λ-submodule of $\mathbf{H}^1(T)$ generated by all $\big({}_{c,d}z_{p^n}(\alpha)\big)_n$ and $\big({}_{c,d}z_{p^n}(\tfrac{a}{A})\big)_n$ where c, d, a, A and α run over all permitted choices in the construction of these integral elements. Then Theorem 12.6 in [10] states that Z is contained in Z_T with finite index. Here it is crucial that we work with exactly the lattice $T = V_{\mathbb{Z}_p}(f)(1)$. Kato allows himself the flexibility of twists by the cyclotomic character and works with $V_{\mathbb{Z}_p}(f)(r)$; we only need $r = 1$ here.

Since $\mathbf{H}^1(T)$ is Λ-torsion-free, there is an injective Λ-morphism $\iota\colon \mathbf{H}^1(T) \to \Lambda$ with finite cokernel. The linear extension $\iota_{\mathbb{Q}}\colon \mathbf{H}^1(T) \otimes \mathbb{Q}_p \to \Lambda \otimes \mathbb{Q}_p$ sends Z_T to a sub-Λ-module J. This J contains the integral ideal $\iota(Z) \subset \Lambda$ with finite index. Hence J itself is an integral ideal in Λ. Write $\lambda = \iota_{\mathbb{Q}}(\mathbf{z}) \in J$.

LEMMA 11. *For any $k \geqslant 0$ such that $p^k Z_T \subset Z$, the index of $p^k\mathbf{z}$ in $\mathbf{H}^1(T)$, defined as*

$$I = \mathrm{ind}_\Lambda(p^k\mathbf{z}) = \Big\{ \psi\big(p^k\mathbf{z}\big) \;\Big|\; \psi \in \mathrm{Hom}_\Lambda(\mathbf{H}^1(T), \Lambda) \Big\},$$

satisfies $I_\mathfrak{p} = \lambda\Lambda_\mathfrak{p}$ for all height one prime ideals $\mathfrak{p}$ of Λ that do not contain p.

Proof. Let $\mathfrak{p} \not\ni p$ be prime ideal of Λ of height 1. Because ι has finite cokernel, we have $\mathbf{H}^1(T)_\mathfrak{p} = \Lambda_\mathfrak{p}$ via ι. Hence

$$\begin{aligned} I_\mathfrak{p} &= \Big\{ \psi\big(p^k\mathbf{z}\big) \;\Big|\; \psi \in \mathrm{Hom}_{\Lambda_\mathfrak{p}}(\mathbf{H}^1(T)_\mathfrak{p}, \Lambda_\mathfrak{p}) \Big\} \\ &= \Big\{ \tilde\psi\big(\iota(p^k\mathbf{z})\big) \;\Big|\; \tilde\psi \in \mathrm{Hom}_{\Lambda_\mathfrak{p}}(\Lambda_\mathfrak{p}, \Lambda_\mathfrak{p}) \Big\} \\ &= \iota(p^k\mathbf{z})\Lambda_\mathfrak{p} = p^k\lambda\Lambda_\mathfrak{p} = \lambda\Lambda_\mathfrak{p}. \end{aligned}$$

because p does not belong to $\mathfrak{p}$. □

3.3 INTEGRALITY OF z_γ

Recall first how Kato deduces the integrality of his second set of zeta-elements in the case $E[p]$ is irreducible.

LEMMA 12. *If* $\mathbf{H}^1(T)$ *is free over* Λ *then* $z_\gamma \in \mathbf{H}^1(T)$ *for all* $\gamma \in T$.

Proof. This is 13.14 in [10]: For every prime ideal $\mathfrak{p}$ of height 1 in Λ, we have $(Z_T)_\mathfrak{p} \subset \mathbf{H}^1(T)_\mathfrak{p}$ since Z has finite index in Z_T. Hence $Z_T \subset \mathbf{H}^1(T)$. □

We will concentrate here on one case that interests us most. Let $\mathbf{z}_0$ be the corestriction of $\mathbf{z}$ from $\mathbf{H}^1(T)$ to $\mathbf{H}^1(T)_0$, which is the limit $\varprojlim_n H^1\bigl(G_\Sigma(K_n), T\bigr)$ as K_n increases in the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$.

THEOREM 13. *Let* $E/\mathbb{Q}$ *be an elliptic curve and* p *an odd prime at which* E *has good reduction. Then* $\mathbf{z}_0$ *belongs to* $\mathbf{H}^1(T)_0$.

In other words $\mathbf{z}_0$ is integral with respect to the Tate module of $E_\bullet$.

Proof. First, we may apply the idea of the proof in Lemma 9, to conclude that $\mathbf{H} = \mathbf{H}^1(T)_0$ is free over $\Lambda(\Gamma)$ if $E_\bullet(\mathbb{Q})$ has no p-torsion point. If so the previous lemma shows that $\mathbf{z}_0$ lies in $\mathbf{H}$.
Assume now that $E_\bullet$ admits a rational p-torsion point. Let $\phi\colon E_\bullet \to E'$ be the isogeny whose kernel contains the rational p-torsion points. We apply Lemma 10 to see that either $\mathbf{H}$ is free or it injects into $\Lambda(\Gamma)$ with index p. As the former case is done with the previous lemma, we assume that we are in the latter. We know already that the Coleman map $\mathrm{Col}_0\colon \mathbf{H} \to \Lambda(\Gamma)$ is injective with finite cokernel. Now, since $\mathbf{H}$ is isomorphic to the maximal ideal, the image of Col_0 has to be equal to the maximal ideal of $\Lambda(\Gamma)$. Therefore if $\mathbf{z}_0$ is not integral, the image $\mathrm{Col}_0\bigl(\mathrm{loc}(\mathbf{z}_0)\bigr) = L_p(E_\bullet)_0 \in \Lambda_0 = \Lambda(\Gamma)$ must be a unit. However the interpolation property of the p-adic L-function tells us that

$$\mathbb{1}\bigl(L_p(E_\bullet)_0\bigr) = \bigl(1-\alpha^{-1}\bigr)^2 \cdot [0]^+_{E_\bullet}$$

where α is the unit root of the characteristic polynomial of Frobenius and the map $\mathbb{1}\colon \Lambda(\Gamma) \to \mathbb{Z}_p$ sends all elements of Γ to 1. Since we have a p-torsion point on the reduction of $E_\bullet$ to $\mathbb{F}_p$, the valuation of $1-\alpha^{-1}$ is 1. By construction of $E_\bullet$ the modular symbol $[0]^+_{E_\bullet}$ is a p-adic integer. Therefore the p-adic L-function cannot be a unit. Hence $\mathbf{z}_0$ is integral. □

4 THE FINE SELMER GROUP

Let E be an elliptic curve with a p-isogeny for an odd prime p. In this section, we do not need any condition on the type of reduction at p. We define the fine[2]

[2]This group is sometimes called the "strict" or "restricted" Selmer group.

Selmer group $\mathcal{R}\big(E/\mathbb{Q}(\zeta_{p^n})\big)$ as the kernel of the localisation map

$$H^1\Big(G_\Sigma\big(\mathbb{Q}(\zeta_{p^n})\big), E[p^\infty]\Big) \longrightarrow \bigoplus_{v\in\Sigma} H^1\Big(\mathbb{Q}_v(\zeta_{p^n}), E[p^\infty]\Big)$$

where the sum runs over all places v in $\mathbb{Q}(\zeta_{p^n})$ above those in Σ. It is independent of the choice of the finite set Σ as long as it contains p and all the places of bad reduction. By global duality it is dual to the kernel

$$H^2\Big(G_\Sigma\big(\mathbb{Q}(\zeta_{p^n})\big), T_pE\Big) \longrightarrow \bigoplus\nolimits_{v\in\Sigma} H^2\Big(\mathbb{Q}_v(\zeta_{p^n}), T_pE\Big).$$

The Pontryagin dual of the direct limit of the groups $\mathcal{R}(E/\mathbb{Q}(\zeta_{p^n}))$ will be denoted by $Y(E)$; it is a finitely generated Λ-module. Theorem 13.4.1 in [10] proves that $Y(E)$ is Λ-torsion.

LEMMA 14. *Let E be an elliptic curve and p an odd prime such that E admits an isogeny of degree p defined over $\mathbb{Q}$. Then the fine Selmer group $Y(E)$ is a finitely generated $\mathbb{Z}_p$-module.*

Proof. Let $\phi\colon E \to E'$ be an isogeny with cyclic kernel $E[\phi]$ of order p defined over $\mathbb{Q}$. The extension F of $\mathbb{Q}$ fixed by the kernel of $\rho_\phi\colon G_\Sigma\big(\mathbb{Q}\big) \to \mathrm{Aut}\big(E[\phi]\big)$ is a cyclic extension of degree dividing $p-1$. Let G be the Galois group of $K = F(\zeta_p)$ over $\mathbb{Q}(\zeta_p)$. Over the abelian field K, the curve admits a p-torsion point. We can therefore apply Corollary 3.6 in [3] (a consequence of the Theorem of Ferrero-Washington) to the dual $Y(E/K_\infty)$ of the Selmer group over the cyclotomic $\mathbb{Z}_p$-extension $K_\infty = K(\zeta_{p^\infty})$ of K. This proves that $Y(E/K_\infty)$ is a finitely generated $\mathbb{Z}_p$-module. Then we have the following diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & \widehat{Y(E/K_\infty)}^{\,\Delta} & \longrightarrow & H^1\big(G_\Sigma(K_\infty), E[p^\infty]\big)^\Delta \\
 & & \uparrow & & \uparrow \\
0 & \longrightarrow & \widehat{Y(E)} & \longrightarrow & H^1\big(G_\Sigma(\mathbb{Q}(\zeta_{p^\infty})), E[p^\infty]\big) \\
 & & & & \uparrow \\
 & & & & H^1\big(G, E(K_\infty)[p^\infty]\big)
\end{array}$$

and since the group G is of order prime to p, the kernel on the right is trivial. We deduce that the left hand side is injective, too, and hence that the dual map $Y(E/K_\infty) \to Y(E)$ is surjective. Therefore $Y(E)$ is a finitely generated $\mathbb{Z}_p$-module. □

For any torsion Λ-module M, we define the characteristic series $\mathrm{char}_\Lambda(M)$ as the product of the ideals $\mathfrak{p}^{l_\mathfrak{p}}$ where $l_\mathfrak{p} = \mathrm{length}_{\Lambda_\mathfrak{p}}(M_\mathfrak{p})$ as $\mathfrak{p}$ runs through all primes of height 1 in Λ.
Recall that we have defined $\lambda = \iota_{\mathbb{Q}}(\mathbf{z})$ as an element in $J \subset \Lambda$ just before Lemma 11.

Proposition 15. *Suppose E does not have additive reduction at p. Then the characteristic series* $\mathrm{char}_\Lambda\bigl(Y(E)\bigr)$ *divides* $\lambda\,\Lambda$.

Proof. We will first prove this proposition in the case E is the curve $E_\bullet$ in Theorem 4. With a sufficiently large choice of k, the element $p^k\cdot\mathbf{z}\in Z\cap\mathbf{H}^1(T)$ extends to an Euler system for T as in [21]. Since the representation ρ_p is not surjective, the Euler system argument gives us only a divisibility of the form

$$\mathrm{char}_\Lambda\bigl(Y(E)\bigr)\qquad\text{divides}\qquad J\cdot\mathrm{ind}_\Lambda\bigl(p^k\mathbf{z}\bigr)$$

for some ideal J of Λ which is a product of primes containing p, see Theorem 2.3.4 in [21] or Theorem 13.4 in [10]. By Lemma 11, we know that $\mathrm{ind}_\Lambda\bigl(p^k\mathbf{z}\bigr)=J'\lambda\,\Lambda$ for some ideal J' which is a product of primes containing p. The previous lemma shows that $\mathrm{char}_\Lambda(Y(E))$ is not divisible by any prime ideal containing p, so the proposition follows for $E_\bullet$.
Now an isogeny $E\to E_\bullet$ can only change the μ-invariants of the dual of the fine Selmer groups, i.e. only by ideals containing p, but the previous lemma shows that they are zero for all curves in the isogeny class. ☐

5 The first divisibility in the main conjecture

Let E be an elliptic curve defined $\mathbb{Q}$ such that $E[p]$ is reducible for some odd prime of semi-stable reduction. Recall that this implies that the reduction of E at p can not be good supersingular. The Selmer group E over $\mathbb{Q}(\zeta_{p^n})$ is defined as usual as the elements in $H^1\bigl(G_\Sigma(\mathbb{Q}(\zeta_{p^n})),E[p^\infty]\bigr)$ that are locally in the image of the points. It fits into the exact sequence

$$0\longrightarrow\mathcal{R}\bigl(E/\mathbb{Q}(\zeta_{p^n})\bigr)\longrightarrow\mathrm{Sel}\bigl(E/\mathbb{Q}(\zeta_{p^n})\bigr)\longrightarrow H^1\bigl(\mathbb{Q}_p(\zeta_{p^n}),E[p^\infty]\bigr).$$

We denote the dual of the limit of the Selmer group by $X(E)$; it is a finitely generated Λ-module. If the reduction is good ordinary, Theorem 17.4 in [10] shows that $X(E)$ is Λ-torsion. The same conclusion holds in general in our situation; see [11] for the split multiplicative case.

Theorem 16. *Let $E/\mathbb{Q}$ be an elliptic curve and let $p>2$ be a prime. Suppose that E has semi-stable reduction at p and that $E[p]$ is reducible as a $G_\mathbb{Q}$-module. Then* $\mathrm{char}_\Lambda\bigl(X(E)\bigr)$ *divides the ideal generated by $L_p(E)$. If the reduction of E is split multiplicative at p, then $I\cdot\mathrm{char}_\Lambda\bigl(X(E)\bigr)$ divides the ideal generated by $L_p(E)$, where I is the kernel of the homomorphism $\Lambda\to\mathbb{Z}_p$ that sends all elements of* $\mathrm{Gal}\bigl(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}\bigr)$ *to* 1.

The main conjecture asserts that the element $L_p(E)$ generates the characteristic ideal $\mathrm{char}_\Lambda\bigl(X(E)\bigr)$.

Lemma 17. *To prove Theorem 16 for E, it is sufficient to prove it for any one curve in the isogeny class of E.*

Proof. The fact that Theorem 16 is invariant under isogenies follows from the formula for the change of the μ-invariant under isogenies for the characteristic series by Perrin-Riou [16, Appendice] when compared to the change of the p-adic L-function. See in particular her Lemme on page 455. □

Proof of Theorem 16. By the previous Lemma 17, we may choose E to be the curve $E_\bullet$ in the isogeny class. Recall from Section 3.2 that the Coleman map $\mathrm{Col}\colon \mathbf{H}^1_s(T) \to \Lambda$ is injective and has image with finite index inside I in the multiplicative case and it has a finite cokernel in the other cases. In what follows we treat only the case when the reduction is not split multiplicative; otherwise one has to multiply with I where appropriate.
Rohrlich [20] has shown that $L_p(E)$ is non-zero and hence $\mathrm{loc}(\mathbf{z})$ is not torsion. Choose a k such that $p^k Z_T \subset Z$. Then the Λ-torsion module $\mathbf{H}^1_s(T)/p^k \,\mathrm{loc}(\mathbf{z})\Lambda$, which is equal to $\mathrm{Col}\bigl(\mathbf{H}^1_s(T)\bigr)/p^k L_p(E)\,\Lambda$, has characteristic series $p^k L_p(E)\Lambda$. The characteristic series of $\mathbf{H}^1(T)/p^k\mathbf{z}\Lambda$ is equal to the characteristic series of $\Lambda/\iota(p^k\mathbf{z})\Lambda$ and therefore equal to $p^k\lambda\Lambda$, where $\iota\mathbf{H}^1(T) \to \Lambda$ is an injective Λ-morphism with finite cokernel.
By global duality (see Proposition 1.3.2 in [18]), we have the following exact sequence

$$0 \longrightarrow \mathbf{H}^1(T) \longrightarrow \mathbf{H}^1_s(T) \longrightarrow X(E) \longrightarrow Y(E) \longrightarrow 0.$$

It induces an exact sequence of torsion Λ-modules

$$0 \longrightarrow \frac{\mathbf{H}^1(T)}{p^k\mathbf{z}\,\Lambda} \longrightarrow \frac{\mathbf{H}^1_s(T)}{p^k\mathbf{z}\Lambda} \longrightarrow X(E) \longrightarrow Y(E) \longrightarrow 0.$$

Using Proposition 15, we conclude that

$$\begin{aligned} \mathrm{char}_\Lambda\bigl(X(E)\bigr) &= \mathrm{char}_\Lambda\bigl(Y(E)\bigr)\cdot\bigl(p^k L_p(E)\Lambda\bigr)\cdot\bigl(p^k\lambda\,\Lambda\bigr)^{-1} \\ &\quad\text{divides } \lambda\cdot p^k L_p(E)\cdot p^{-k}\lambda^{-1}\Lambda = L_p(E)\Lambda. \end{aligned}$$

□

6 Consequences

Corollary 18. *The analytic p-adic L-function $L_p(E)$ belongs to Λ for all elliptic curves $E/\mathbb{Q}$ with semi-stable ordinary reduction at $p > 2$.*

The conclusion can certainly not be extended to the supersingular case since the p-adic L-functions in this case will never be integral. The supersingular case is well explained in [19] where it is shown how one can extract integral power series.

Corollary 19. *If $E/\mathbb{Q}$ is a semi-stable elliptic curve and p an odd prime where E has ordinary reduction, then $\mathrm{char}_\Lambda\bigl(X(E)\bigr)$, or $I\,\mathrm{char}_\Lambda\bigl(X(E)\bigr)$ in the split multiplicative case, divides the ideal generated by $L_p(E)$.*

Proof. By a Theorem of Serre ([24, Proposition 1] and [22, Proposition 21]), we know that the image of the representation $\bar{\rho}_p\colon G_{\mathbb{Q}} \to \mathrm{Aut}(E[p])$ is either the whole of $\mathrm{GL}_2(\mathbb{F}_p)$ or it is contained in a Borel subgroup. In the latter case the representation $\bar{\rho}_p$ is reducible and in the first case the representation $\rho_p\colon G_{\mathbb{Q}} \to \mathrm{Aut}(T_pE)$ is surjective by another result of Serre [23, Lemme 15] unless $p = 3$. Finally for $p = 3$ we use the following lemma to exclude that ρ_p is not surjective. □

Unfortunately, the hypothesis in Corollary 19 that E is semi-stable can not be dropped. For instance, there are curves $E/\mathbb{Q}$ such that $\bar{\rho}_p$ has its image in the normaliser of a non-split Cartan subgroup.

Lemma 20. *Let $p = 3$ and suppose p^2 does not divide the conductor N. If the residual representation $\bar{\rho}\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_p)$ is surjective then the p-adic representation $\rho\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}_p)$ is surjective, too.*

Proof. We make use of the explicit parametrisation of all these exotic cases by Elkies in [8]. Let $E/\mathbb{Q}$ be an elliptic curve such that ρ is not surjective, but $\bar{\rho}$ is. Then its j-invariant satisfies

$$j(E) = 1728 - \frac{27\,A(n:m)^2\,B(n:m)^2\,C(n:m)}{D(n:m)^9} \qquad \text{with}$$
$$\begin{aligned} A(n:m) &= n^6 + 6n^5m + 4n^3m^3 + 12n^2m^4 - 18nm^5 - 23m^6,\\ B(n:m) &= 7n^6 + 24n^5m + 18n^4m^2 - 26n^3m^3 - 33n^2m^4 + 18nm^5 + 28m^6,\\ C(n:m) &= 2n^3 - 3n^2m + 4m^3,\\ D(n:m) &= n^3 - 3nm^3 - m^3. \end{aligned}$$

for two coprime integers n and m. Note first that the denominator $D(n:m)$ in $j(E)$ is never divisible by 9, so $j(E)$ is a 3-adic integer.
With a bit more work one can see that $j(E) \equiv 2\cdot 3^3 \pmod{3^4}$: If $n \not\equiv m \pmod 3$, then $A(n:m) \equiv (n-m)^6 \equiv B(n:m) \pmod 3$, $C(n:m) \equiv 2(n-m)^3$ and $D(n:m) \equiv (n-m)^3 \pmod 3$ gives the result. For $n = m + 3k$, we can use $A(n:m) \equiv B(n:m) \equiv 3^2 \pmod{3^3}$, $C(n:m) \equiv 3 \pmod{3^2}$, and $D(n:m) \equiv 2\cdot 3 \pmod{3^2}$ to conclude.
Now suppose E is given by a Weierstrass equation minimal at 3. We may assume that it is of the form $y^2 = x^3 + a_2x^2 + a_4x + a_6$ with $a_2 \in \{-1, 0, +1\}$ and a_4, $a_6 \in \mathbb{Z}$. If $a_2 = \pm 1$, then

$$j(E) = 16\frac{-27a_4^3 + 27a_4^2 - 9a_4 + 1}{\Delta}$$

where Δ is the discriminant. However this is a contradiction with $j(E) \in 3^3\mathbb{Z}_3$. Hence $a_2 = 0$ and so

$$j(E) = 3^3 \cdot 2^6 \cdot \frac{a_4^3}{a_4^3 + 27a_6^2/4}$$

and we see that it is impossible that $j(E) \equiv 2 \cdot 3^3 \pmod{3^4}$ unless 3 divides a_4 and the discriminant $\Delta = 4a_4^3 + 27a_6^2$. Therefore E has bad reduction at 3. The fact that $j(E)$ is a 3-adic integer shows that the reduction is additive. $\square$

Finally, here is the usual application to the Birch and Swinnerton-Dyer conjecture.

Proposition 21. *Let E be an elliptic curve over $\mathbb{Q}$ such that $L(E,1) \neq 0$. Let c_v be the Tamagawa number of E at each finite place v and the number of components in $E(\mathbb{R})$ for $v = \infty$. Then*

$$\#Ш(E/\mathbb{Q}) \qquad \textit{divides} \qquad C \cdot \frac{L(E,1)}{\Omega_E^+} \cdot \frac{\bigl(\#E(\mathbb{Q})\bigr)^2}{\prod_v c_v}$$

where C is a rational number only divisible by 2, primes of additive reduction or primes for which the Galois representation on $E[p]$ is neither surjective nor contained in a Borel subgroup.

In particular, for semi-stable curve C is a power of 2. The methods in [26] can now be extended to the reducible case, too.

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