# Martingale Inequalities and Operator Space Structures on $L_p$

Gilles Pisier[1]

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Abstract. We describe a new operator space structure on $L_p$ when $p$ is an even integer and compare it with the one introduced in our previous work using complex interpolation. For the new structure, the Khintchine inequalities and Burkholder's martingale inequalities have a very natural form: the span of the Rademacher functions is completely isomorphic to the operator Hilbert space $OH$, and the square function of a martingale difference sequence $d_n$ is $\Sigma\ d_n \otimes \bar{d}_n$. Various inequalities from harmonic analysis are also considered in the same operator valued framework. Moreover, the new operator space structure also makes sense for non-commutative $L_p$-spaces associated to a trace with analogous results. When $p \to \infty$ and the trace is normalized, this gives us a tool to study the correspondence $E \mapsto \underline{E}$ defined as follows: if $E \subset B(H)$ is a completely isometric emdedding then $\underline{E}$ is defined so that $\underline{E} \subset CB(OH)$ is also one.

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Contents



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## Introduction

In probability theory and harmonic analysis, the classical inequalities for martingales in $L_p$ due to Don Burkholder, and also to Richard Gundy and Burgess Davis have had an invaluable impact, with multiple interaction with other fields. See [8] for a recent review.

In [26], a non-commutative version of Burkholder's martingale inequalities is given. This is valid in any non-commutative $L_p$-space say $L_p(\tau)$ (associated to a finite trace $\tau$ on a von Neumann algebra) for any $1 < p < \infty$. In particular this applies to martingales of the form $f_n = \sum_1^n \varepsilon_k \otimes x_k$ where $x_k \in L_p(\tau)$ and $(\varepsilon_n)$ is a standard random choice of signs $\varepsilon_n = \pm 1$ (equivalently we can think of $(\varepsilon_n)$ as the Rademacher functions on [0,1]). In that case, Burkholder's inequality reduces to Khintchine's inequality, for which the non-commutative case is due to Lust–Piquard ([20]).

In the classical setting, Khintchine's inequality expresses the fact that the closed span in $L_p$ of $\{\varepsilon_n\}$ is isomorphic to $\ell_2$ (as a Banach space). If $\{\varepsilon_n\}$ is replaced by a sequence $(g_n)$ of independent standard Gaussian random variables, the span in $L_p$ becomes isometric to $\ell_2$. In the recently developed theory of operator spaces, Lust–Piquard's non-commutative Khintchine inequalities can be interpreted (see [23, p. 108]) as saying that the span in $L_p$ of $[\varepsilon_n]$ (or $(g_n)$) is completely isomorphic to a Hilbertian operator space that we will denote here by $\mathcal{KH}_p$. The precise description of $\mathcal{KH}_p$ is not important for this paper, but for reference let us say merely that, for $2 < p < \infty$ (resp. $1 \le p < 2$), $\mathcal{KH}_p$ has the structure of intersection (resp. sum) of row and column spaces in the

Schatten class $S_p$. For the non-commutative Burkholder inequalities from [26], the situation is analogous: the relevant square function combines the two cases of "row" and "column" in analogy with the definition of $\mathcal{KH}_p$.
This result was a bit of a disappointment because there is a canonical notion of "operator Hilbert space," namely the space $OH$ from [22] and one would have expected in closer analogy to the classical case, that the span in $L_p$ of $[\varepsilon_n]$ (or $(g_n)$) should be completely isomorphic to $OH$. In the preceding references, the spaces $L_p$ (commutative or not) are always equipped with what we call their "natural" operator space structure defined using complex interpolation. Then the space $\mathcal{KH}_p$ is completely isomorphic to $OH$ only when $p = 2$.
In the present paper, we take a different route. We will equip $L_p$ with another o.s.s., hopefully still rather natural, but limited to $p$ equal to an even integer (for some of our results we even assume $p = 2^k$). Roughly we imitate the classical idea that $f \in L_4$ iff $|f|^2 \in L_2$ in order to define our new o.s.s. on $L_4$ and then we iterate the process to define the same for $L_6, L_8$ and so on. Thus given the space $L_p(\Omega, \mu)$ we can associate to it (assuming $p \in 2\mathbb{N}$ and $p \geq 2$) an operator space that we denote by $\Lambda_p(\Omega, \mu)$ that is isometric to the original $L_p(\Omega, \mu)$ as a Banach space.
It turns out that with this new structure a quite different picture emerges for Khintchine's (or more generally Burkholder's) inequalities. Indeed, we will prove that the span of $\{\varepsilon_n\}$ in $\Lambda_p(\Omega, \mu)$ is completely isomorphic to the space $OH$, (i.e. to $\ell_2$ equipped with the o.s.s. of $OH$). Similarly we will prove martingale inequalities involving a square function that is simply defined as $S = \sum d_n \otimes \bar{d}_n$ when $(d_n)$ is a martingale difference sequence.
We limit our treatment to $L_p$ for $p$ an even integer. Thus we stopped shy of making the obvious extensions: we can use duality to define $\Lambda_q$ for $1 < q < 2$ of the form $q = \frac{2n}{2n-1}$ for some integer $n > 1$ and then use complex interpolation to define $\Lambda_p$ in the remaining intermediate values of $p$'s or $q$'s. While this procedure makes perfectly good sense it is rather "unnatural" given that if $p(0), p(1)$ and $p(\theta)$ are even integers such that $p_0 < p_\theta < p_1$ and $p(\theta)^{-1} = (1-\theta)p(0)^{-1} + \theta p(1)^{-1}$, the space $\Lambda_{p(\theta)}$ does *not* coincide (in general) with $(\Lambda_{p(0)}, \Lambda_{p(1)})_\theta$. This happens for instance when $p(0) = 2, p(1) = \infty$ and $\theta = 1/2$, since the spaces $L_4$ and $\Lambda_4$ differ as operator spaces.
We will now review the contents of this paper. After some general background in §1, we explain in §2 some basic facts that will be used throughout the paper. The main point is that we use an ordering on $B(H) \otimes \overline{B(H)}$ denoted by $T \prec S$ that is such that

$$0 \prec T \prec S \Rightarrow \|T\| \leq \|S\|$$

where the norm is the minimal (or spatial) tensor product on $B(H) \otimes_{\min} \overline{B(H)}$, i.e. the norm induced by $B(H \otimes_2 \overline{H})$. As we explain in §2, it is convenient to abuse the notation and to extend the notation $T \prec S$ to pairs, $T, S$ in $B(H_1) \otimes \cdots \otimes B(H_{2n})$ when the collection $\{H_1, \ldots, H_{2n}\}$ can be permuted to be of the form $\{K_1, \ldots, K_n, \overline{K}_1, \ldots, \overline{K}_n\}$ so that $T, S$ can be identified with elements of $B(\mathcal{H}) \otimes \overline{B(\mathcal{H})}$ with $\mathcal{H} = K_1 \otimes \cdots \otimes K_n$.

In §3, we use the properties of this ordering to prove a version of Hölder's inequality that allows us to introduce, for each even integer $p$, a new operator space structure on the space $L_p(\Omega, \mathcal{A}, \mu)$ associated to a general measure space $(\Omega, \mathcal{A}, \mu)$. We follow the same route that we used for $p = 2$ in [22] to define the space $OH$ starting from Haagerup's Cauchy–Schwarz inequality. Suitable iterations of the latter leads to versions of Hölder's inequality in $L_4, L_6, L_8, \ldots$ from which a specific norm can be introduced on $B(H) \otimes L_p(\mu)$ that endows $L_p(\mu)$ with an operator space structure. We denote by $\Lambda_p(\Omega, \Sigma, \mu)$ the resulting operator space. It is natural to identify $\Lambda_1(\Omega, \Sigma, \mu)$ with $L_1(\mu)$ equipped with its maximal operator space structure in the Blecher-Paulsen sense (see e.g. [11, 25] for details on this).

Some of the calculations involving $\Lambda_p(\mu)$ are rather satisfactory, e.g. for any $f_j \in B(H_j) \otimes L_p(\mu)$, $j = 1, \ldots, p$ the pointwise product $L_p \times \cdots \times L_p \to L_1$ applied to $(f_1, \ldots, f_n)$ leads to an element denoted by $F = f_1 \dot{\otimes} \cdots \dot{\otimes} f_n$ in $B(H_1) \otimes \cdots \otimes B(H_n) \otimes L_1$, and if $\mathcal{H} = H_1 \otimes \cdots \otimes H_n$ we have

$$\|F\|_{B(\mathcal{H}) \otimes_{\min} \Lambda_1(\mu)} \leq \Pi \|f_j\|_{B(H_j) \otimes_{\min} \Lambda_p}.$$

In particular, if $q \leq p$ are even integers and if $\mu(\Omega) = 1$, the inclusion $\Lambda_p(\mu) \to \Lambda_q(\mu)$ is completely contractive. We also show that all conditional expectations are completely contractive on $\Lambda_p(\mu)$. When $p \to \infty$, we recover the usual operator space structure of $L_\infty(\mu)$ as the limit of those of $\Lambda_p(\mu)$.

In §4 we prove a version of Burkholder's square function inequalities for martingales in $\Lambda_p(\mu)$. If $(d_n)$ is a sequence of martingale differences in $B(H) \otimes L_p$ the relevant square function is $S = \Sigma d_n \dot{\otimes} \bar{d}_n$. Here we restrict to $p = 2^k$ for some $k$, but at least one side of the inequality is established for any even $p$ by a different argument in §13. We also prove an analogue for $\Lambda_p$ of the inequality due to Stein expressing that for any sequence $(f_n)$ in $L_p$ the $L_p$-norm of $(\sum |f_n|^2)^{1/2}$ dominates that of $(\sum |\mathbb{E}_n(f_n)|^2)^{1/2}$ for any $1 < p < \infty$.

In §5, we consider the conditioned square function $\sigma = \sum \mathbb{E}_{n-1}(d_n \dot{\otimes} \bar{d}_n)$ and we prove a version of the Burkholder–Rosenthal inequality adapted to $\Lambda_p(\mu)$. As can be expected, the preceding inequalities imply the complete boundedness of the multipliers called "martingale transforms". Not surprisingly, in §6 we can also prove similar results for the Hilbert transform, say on $\mathbb{T}$ or $\mathbb{R}$, using the well known Riesz-Cotlar trick.

In §7, we compare the "old" and the "new" o.s.s. on $L_p(\mu)$. We show that the (isometric) inclusion $L_p(\mu) \to \Lambda_p(\mu)$ is completely contractive, but its inverse is not completely bounded and we show that its c.b. norm in the $n$-dimensional case grows at least like $n^{\frac{1}{2}(\frac{1}{2} - \frac{1}{p})}$.

In §8, we turn to the non-commutative case. We introduce the space $\Lambda_p(\tau)$ associated to a non-commutative measure space $(\mathcal{M}, \tau)$. By this we mean a von Neumann algebra $\mathcal{M}$ equipped with a semi-finite faithful normal trace $\tau$.

In §9, we repeat the comparison made in §7. It turns out that the non-commutative case is significantly more intricate, mainly because the (joint) complete boundedness of the product map $L_p \times L_q \to L_r$ $(p^{-1} + q^{-1} = r^{-1})$ no

longer holds in general (I am grateful to Quanhua Xu for drawing my attention to this). This leads us to consider yet another operator space structure on $L_p(\tau)$ that we denote by $\mathcal{L}_p(\tau)$ for which it still holds (see Proposition 9.1). When $p \in 2\mathbb{N}$, we then extend the main result of §7 by showing (see Corollary 9.2) that the identity defines a completely contractive map $\mathcal{L}_p(\tau) \cap \mathcal{L}_p(\tau)^{op} \to \Lambda_p(\tau)$. In the commutative case $L_p(\tau)$ and $\mathcal{L}_p(\tau)$ are identical. We give estimates of the growth in $n$ of the c.b. norms of the maps $L_p(M_n, tr) \to \Lambda_p(M_n, tr)$ and $\Lambda_p(M_n, tr) \to L_p(M_n, tr)$ induced by the identity.
In §10, assuming $\tau(1) = 1$, we study the "limit" when $p \to \infty$ of the spaces $\Lambda_p(\mathcal{M}, \tau)$, that we denote by $\Lambda_\infty(\mathcal{M}, \tau)$. Surprisingly, we are able to identify the resulting operator space: Indeed, when $(\mathcal{M}, \tau)$ is $M_n$ equipped with its normalized trace then $\Lambda_\infty(\mathcal{M}, \tau)$ can be identified completely isometrically with $CB(OH_n, OH_n)$, i.e. the space of c.b. maps on the $n$-dimensional operator Hilbert space. More generally (see Theorem 10.3), when $\mathcal{M} \subset B(H)$, the o.s.s. of $\Lambda_\infty(\mathcal{M}, \tau)$ can be identified with the one induced by $CB(OH, OH)$, where by $OH$ we mean $H$ equipped with its unique self-dual structure in the sense of [25]. The verification of these facts leads us to several observations on the space $CB(OH, OH)$ that may be of independent interest. In particular, the latter space satisfies a curious identity (see (10.4)) that appears like an operator space analogue of the Gelfand axiom for $C^*$-algebras. Furthermore, to any operator space $E \subset B(H)$, we associate the operator space $\underline{E} \subset CB(OH)$ (equipped with the operator space structure induced by $CB(OH)$), and we show that if $F$ is another operator space, for any cb-map $u: E \to F$ we have

$$\|u: \underline{E} \to \underline{F}\|_{cb} \leq \|u: E \to F\|_{cb}.$$

In §12 we extend the Burkholder inequalities except that—for the moment—we can only prove the two sides of the martingale inequality for $p = 4$. Note however that the right hand side is established for all even $p$ in §13 by a different method based on the notion of $p$-orthogonality.
Nevertheless, in §11, using Buchholz's ideas in [7] we can prove versions of the non-commutative Khintchine inequalities for $\Lambda_p(\tau)$ with optimal constants for any even integer $p$. We may consider spin systems, free semi-circular (or "free-Gaussian") families, or the free generators of the free group in the associated free group factor. Returning to the commutative case this yields the Rademacher function case with optimal constants. The outcome is that the span of each of these sequences in $\Lambda_p$ is completely isomorphic to $OH$ and completely complemented.
In §14 we transplant the results of [13] (see also [14]) on non-commutative lacunary series to the setting of $\Lambda_p(\tau)$. We use the view point of [24] to abbreviate the presentation. Let $\Gamma$ be a discrete group. We will consider $\Lambda(p)$-sets in Rudin's sense inside $\Gamma$. The main point is that a certain class of $\Lambda(p)$-subsets of $\Gamma$ again spans a copy of $OH$ in the operator spaces $\Lambda_p(M, \tau)$ when $M$ is the von Neumann algebra of $\Gamma$. For our new o.s.s. the relevant notion of $\Lambda(p)$-set is slightly more general than the one needed in [13].

Lastly in the appendix §15 we include a discussion of elements that have $p$-th moments defined by pairings as in Buchholz's paper [7]. We simply translate in the abstract language of tensor products some very well known classical ideas on Wick products for Gaussian random variables. Our goal is to emphasize the similarity between the Gaussian case and the free or $q$-Gaussian analogues. We feel this appendix fits well with the extensive use of tensor products throughout the sections preceding it.
The proof of the initial non-commutative martingale inequalities of [26] is the main source of inspiration for the present results. We also make crucial use of our version for the space $\Lambda_p$ of Junge's "dual Doob" inequality from [15]. Although we take a divergent route, we should point out to the reader that the methods of [26] have been considerably improved in a series of important later works, such as [17, 18, 19], by M. Junge and Q. Xu, or [28, 29, 21] by N. Randrianantoanina and J. Parcet. See also [16, 30, 31, 32, 37] for progress related to Khintchine's inequalities. The reader is referred to these papers to get an idea of what the "main stream" on non-commutative martingale and Khintchine inequalities is about.

## 1 Background on operator spaces

In this section we summarize the Theory of Operator Spaces. We refer either to [11] or [25] for full details.
We recall that an operator space is just a closed subspace of the algebra $B(H)$ of all bounded operators on a Hilbert space $H$.
Given an operator space $E \subset H$, we denote by $M_n(E)$ the space of $n \times n$ matrices with entries in $E$ and we equip it with the norm induced by that of $M_n(B(H))$, i.e. by the operator norm on $H \oplus \cdots \oplus H$ ($n$ times). We denote by $E \otimes F$ the *algebraic* tensor product of two vector spaces $E, F$.
If $E \subset B(H)$ and $F \subset B(K)$ are operator spaces, we denote by $E \otimes_{\min} F$ the closure of $E \otimes F$ viewed as a subspace of $B(H \otimes_2 K)$. We denote by $\| \ \|_{\min}$ the norm induced by $B(H \otimes_2 K)$ on $E \otimes F$ or on its closure $E \otimes_{\min} F$. A linear map $u\colon\ E \to F$ between operator spaces is called completely bounded (c.b. in short) if the associated maps $u_n\colon\ M_n(E) \to M_n(F)$ defined by $u_n([x_{ij}]) = [u(x_{ij})]$ are bounded uniformly over $n$, and we define

$$\|u\|_{cb} = \sup_{n \geq 1} \|u_n\|.$$

We say that $u$ is completely isometric if $u_n$ is isometric for any $n \geq 1$ and that $u$ is a complete isomorphism if it is an isomorphism with c.b. inverse.
By a well known theorem due to Ruan (see [11, 25]), an operator space $E$ can be characterized up to complete isometry by the sequence of normed spaces $\{M_n(E) \mid n \geq 1\}$. The data of the sequence of norms on the spaces $M_n(E)$ ($n \geq 1$) constitutes the operator space structure (o.s.s. in short) on the vector space underlying $E$. Note that $M_n(E) = B(H_n) \otimes_{\min} E$ where $H_n$ denotes here the $n$-dimensional Hilbert space.

Actually, the knowledge of the o.s.s. on $E$ determines that of the norm on $B(H) \otimes E$ for any Hilbert space $H$. Therefore we (may and) will take the viewpoint that the o.s.s. on $E$ consists of the family of normed spaces (before completion)

$$(B(H) \otimes E, \| \cdot \|_{\min}),$$

where $H$ is an arbitrary Hilbert space. The reader should keep in mind that we may restrict either to $H = \ell_2$ or to $H = \ell_2^n$ with $n \geq 1$ allowed to vary arbitrarily. (If we fix $n = 1$ everywhere, the theory reduces to the ordinary Banach space theory.) To illustrate our viewpoint we note that

$$\|u\|_{cb} = \sup_H \|u_H \colon\ B(H) \otimes_{\min} E \to B(H) \otimes_{\min} F\|$$

where the mapping $u_H$ is the extension (by density) of $id \otimes u$.
The most important examples for this paper are the spaces $L_p(\mu)$ associated to a measure space $(\Omega, \mathcal{A}, \mu)$. Our starting point will be the 3 cases $p = \infty$, $p = 1$ and $p = 2$. For $p = \infty$, the relevant norm on $B(H) \otimes L_\infty(\mu)$ is the unique $C^*$-norm, easily described as follows: any $f$ in $B(H) \otimes L_\infty(\mu)$ determines a function $f\colon\ \Omega \to B(H)$ taking values in a finite dimensional subspace of $B(H)$ and we have

$$\|f\|_{B(H)\otimes_{\min} L_\infty(\mu)} = \operatorname*{ess\,sup}_{\omega\in\Omega} \|f(\omega)\|_{B(H)}. \tag{1.1}$$

For $p = 1$, the relevant norm on $B(H) \otimes L_1(\mu)$ is defined using operator space duality, but it can be explicitly written as follows

$$\|f\|_{B(H)\otimes_{\min} L_1(\mu)} = \sup \left\| \int f(t) \otimes g(t)\ d\mu(t) \right\|_{B(H\otimes_2 K)} \tag{1.2}$$

where the sup runs over all $g$ in the unit ball of $(B(K) \otimes L_\infty(\mu), \| \cdot \|_{\min})$ and over all possible $K$. Equivalently, we may restrict to $H = K = \ell_2$. This is consistent with the standard dual structure on the dual $E^*$ of an operator space. There is an embedding $E^* \subset B(\mathcal{H})$ such that the natural identification

$$B(H) \otimes E^* \longleftrightarrow B(E, B(H))$$

defines for any $H$ an isometric embedding

$$B(H) \otimes_{\min} E^* \subset CB(E, B(H)).$$

With this notion of duality we have $L_1(\mu)^* = L_\infty(\mu)$ completely isometrically. Moreover the inclusion $L_1(\mu) \subset L_\infty(\mu)^*$ is completely isometric, and this is precisely reflected by the formula (1.2).
To define the o.s.s. on $L_2(\mu)$, we will use the complex conjugate $\overline{H}$ of a Hilbert space $H$. Note that the map $x \to x^*$ defines an anti-isomorphism on $B(H)$. Since $\overline{B(H)} = B(\overline{H})$ (canonically), we may view $x \to x^*$ as a linear $*$-isomorphism from $\overline{B(H)} = B(\overline{H})$ to $B(H)^{op}$ where $B(H)^{op}$ is the same

$C^*$-algebra as $B(H)$ but with reversed product. Then for any $x_k \in B(H)$, $y_k \in B(K)$ we have

$$\left\|\sum x_k \otimes y_k\right\|_{\min} = \sup\left\{\left\|\left(\sum x_k \otimes y_k\right)(\xi)\right\|_{H\otimes_2 K} \mid \xi \in H \otimes K, \|\xi\|_{H\otimes_2 K} \le 1\right\}.$$

Let us denote by $S_2(H,K)$ the class of Hilbert–Schmidt operators from $H$ to $K$ with norm denoted by $\|\cdot\|_{S_2(H,K)}$ or more simply by $\| \ \|_2$. We may identify canonically $\xi \in H \otimes_2 K$ with an element $\hat{\xi}\colon\ K^* \to H$ with Hilbert–Schmidt norm $\|\hat{\xi}\|_2 = \|\xi\|_{H\otimes_2 K}$. Then for any $y \in B(K)$ let ${}^t y\colon\ K^* \to K^*$ denote the adjoint operator. We have then

$$\left\|\sum x_k \otimes y_k\right\|_{\min} = \sup\left\{\left\|\sum x_k \hat{\xi}\ {}^t y_k\right\|_2 \ \Big|\ \|\hat{\xi}\|_2 \le 1\right\}.$$

Using the identification $\overline{B(K)} = B(K)^{op}$ via $\underline{\bar{x} \to x^*}$, (let $x \to \bar{x}$ be the identity map on $B(K)$ viewed as a map from $B(K)$ to $\overline{B(K)}$) we find

$$\left\|\sum x_k \otimes \bar{y}_k\right\|_{B(H\otimes_2 \overline{K})} = \sup\left\{\left\|\sum x_k a y_k^*\right\|_2 \ \Big|\ a \in S_2(K,H),\ \|a\|_2 \le 1\right\}.$$

We now define the "natural" o.s.s. on the space $\ell_2$ according to [22]. This is defined by the following formula: for any $f$ in $B(H) \otimes \ell_2$, of the form $f = \sum_1^n x_k \otimes e_k$ (here $(e_k)$ denotes the canonical basis of $\ell_2$) we have

$$\|f\|_{B(H)\otimes_{\min}\ell_2} = \left\|\sum x_k \otimes \bar{x}_k\right\|^{1/2}_{B(H\otimes_2 \overline{H})}. \tag{1.3}$$

The resulting o.s. is called "the operator Hilbert space" and is denoted by $OH$. Actually, the same formula works just as well for any Hilbert space $\mathcal{H}$ with an orthonormal basis $(e_i)_{i\in I}$. The resulting o.s. will be denoted by $\mathcal{H}_{oh}$ (so that $OH$ is just another notation for $(\ell_2)_{oh}$).

In this paper our main interest will be the space $L_2(\mu)$. The relevant o.s.s. can then be described as follows: for any $f$ in $B(H) \otimes L_2(\mu)$ we have

$$\|f\|_{B(H)\otimes_{\min} L_2(\mu)_{oh}} = \left\|\int f(\omega) \otimes \overline{f(\omega)} d\mu(\omega)\right\|^{1/2}_{B(H\otimes_2 \overline{H})}. \tag{1.4}$$

It is not hard to see that this coincides with the definition (1.3) when $(e_n)$ is an orthonormal basis of $L_2(\mu)$.

We refer the reader to [22] for more information on the space $\mathcal{H}_{oh}$, in particular for the proof that this space is uniquely characterized by its self-duality in analogy with Hilbert spaces among Banach spaces. We note that $\mathcal{H}_{oh}$ and $H_{oh}$ are completely isometric iff the Hilbert spaces $\mathcal{H}$ and $H$ are isometric (i.e. of the same Hilbertian dimension).

The "natural o.s.s." on $L_p = L_p(\mu)$ is defined in [23] for $1 < p < \infty$ using complex interpolation. It is characterized by the following isometric identity: For any finite dimensional Hilbert space $H$

$$B(H) \otimes_{\min} L_p = (B(H) \otimes_{\min} L_\infty, B(H) \otimes_{\min} L_1)_{1/p}.$$

When $p = 2$ we recover the o.s.s. defined above for $L_2(\mu)_{oh}$.
We now turn to multilinear maps. Let $E_1, \ldots, E_m$ and $F$ be operator spaces. Consider an $m$-linear map $\varphi\colon\ E_1 \times \cdots \times E_m \to F$. Let $H_1, \ldots, H_m$ be Hilbert spaces. We set $B_j = B(H_j)$. By multilinear algebra we can associate to $\varphi$ an $m$-linear map

$$\widehat{\varphi}\colon\ B_1 \otimes E_1 \times \cdots \times B_m \otimes E_m \to B_1 \otimes \cdots \otimes B_m \otimes F$$

characterized by the property that $(\forall b_j \in B_j, \forall e_j \in E_j)$

$$\widehat{\varphi}(b_1 \otimes e_1, \cdots, b_m \otimes e_m) = b_1 \otimes \cdots \otimes b_m \otimes \varphi(e_1, \cdots, e_m).$$

We say that $\varphi$ is (jointly) completely bounded (c.b. in short) if $\widehat{\varphi}$ is bounded from

$$B_1 \otimes_{\min} E_1 \times \cdots \times B_m \otimes_{\min} E_m \quad \text{to} \quad B_1 \otimes_{\min} \cdots \otimes_{\min} B_m \otimes_{\min} F.$$

It is easy to see that we may reduce to the case when $H_1 = H_2 = \cdots = H_m = \ell_2$ (equivalently we could restrict to finite dimensional Hilbert spaces of arbitrary dimension). With this choice of $H_j$ we set

$$\|\varphi\|_{cb} = \|\widehat{\varphi}\|.$$

## 2 Preliminary results

We first recall Haagerup's version of the Cauchy–Schwarz inequality on which is based a lot of what follows.
Let $H, K$ be Hilbert spaces, $a_k \in B(H), b_k \in B(K)$ $(k = 1, \ldots, n)$. We have then

$$\left\|\sum a_k \otimes \bar{b}_k\right\|_{B(H \otimes_2 \overline{K})} \le \left\|\sum a_k \otimes \bar{a}_k\right\|^{1/2}_{B(H \otimes_2 \overline{H})} \left\|\sum b_k \otimes \bar{b}_k\right\|^{1/2}_{B(K \otimes_2 \overline{K})}$$

where $\otimes_2$ denotes the Hilbert space tensor product.
We will sometimes need the following reformulation: let $\mathcal{H}$ be another Hilbert space and for any $f \in \mathcal{H} \otimes B(H)$, say $f = \sum x_k \otimes a_k$, and $g \in \mathcal{H} \otimes B(K)$, say $g = \sum y_\ell \otimes b_\ell$ let $\langle\langle f, g\rangle\rangle \in B(H) \otimes \overline{B(K)}$ be defined by

$$\langle\langle f, g\rangle\rangle = \sum\nolimits_{k,\ell} \langle x_k, y_\ell\rangle a_k \otimes \bar{b}_\ell.$$

We have then

$$\|\langle\langle f, g\rangle\rangle\|_{B(H \otimes_2 \overline{K})} \le \|\langle\langle f, f\rangle\rangle\|^{1/2} \|\langle\langle g, g\rangle\rangle\|^{1/2}. \tag{2.1}$$

More generally for any finite sequences $(f_\alpha)_{1 \le \alpha \le N}$ in $\mathcal{H} \otimes B(H)$ and $(g_\alpha)_{1 \le \alpha \le N}$ in $\mathcal{H} \otimes B(K)$ we have

$$\left\|\sum\nolimits_\alpha \langle\langle f_\alpha, g_\alpha\rangle\rangle\right\|_{B(H \otimes_2 \overline{K})} \le \left\|\sum\nolimits_\alpha \langle\langle f_\alpha, f_\alpha\rangle\rangle\right\|^{1/2}_{B(H \otimes_2 \overline{H})} \left\|\sum\nolimits_\alpha \langle\langle g_\alpha, g_\alpha\rangle\rangle\right\|^{1/2}_{B(K \otimes_2 \overline{K})}. \tag{2.2}$$

It is convenient to also observe here that if $E_1, E_2$ are orthogonal subspaces of $\mathcal{H}$ and if $f_j \in E_j \otimes B(H)$ $(j = 1, 2)$ then

$$\langle\langle f_1 + f_2, f_1 + f_2\rangle\rangle = \langle\langle f_1, f_1\rangle\rangle + \langle\langle f_2, f_2\rangle\rangle. \tag{2.3}$$

We will use an order on $B(H) \otimes \overline{B(H)}$: Let $C_+$ be the set of all finite sums of the form $\sum a_k \otimes \bar{a}_k$. If $x, y$ are in $B(H) \otimes \overline{B(H)}$, we write $x \prec y$ (or $y \succ x$) if $y - x \in C_+$. In particular $x \succ 0$ means $x \in C_+$. Any element $x \in B(H) \otimes \overline{B(H)}$ defines a (finite rank) sequilinear form $\tilde{x}\colon\ B(H)^* \times B(H)^* \to \mathbb{C}$. More generally, for any complex vector space $E$, we may define similarly $x \succ 0$ for any $x \in E \otimes \bar{E}$.
The following criterion is easy to show by linear algebra.

Lemma 2.1. *Let $x \in B(H) \otimes \overline{B(H)}$. Then $x \in C_+$ iff $\tilde{x}$ is positive definite i.e. $\tilde{x}(\xi, \xi) \geq 0$ for any $\xi$ in $B(H)^*$. Moreover, this holds iff $\tilde{x}(\xi, \xi) \geq 0$ for any $\xi$ in the predual $B(H)_* \subset B(H)^*$ of $B(H)$. Lastly, if $H$ is separable, there is a countable subset $D \subset B(H)_*$ such that $x \succ 0$ iff $\tilde{x}(\xi, \xi) \geq 0$ for any $\xi$ in $D$.*

*Proof.* The first part is a general fact valid for any complex Banach space $E$ in place of $B(H)$: Assume $x \in E \otimes \bar{E}$, then $x \in F \otimes \bar{F}$ for some finite dimensional $F \subset E$, thus the equivalence in Lemma 2.1 just reduces to the classical spectral decomposition of a positive definite matrix. If $E$ is a dual space with a predual $E_* \subset E^*$, then $E_*$ is $\sigma(E^*, E)$-dense in $E^*$, so the condition $\tilde{x}(\xi, \xi) \geq 0$ will hold for any $\xi \in E^*$ if it does for any $\xi \in E_*$. When the predual $E_*$ is separable, the last assertion becomes immediate. □

*Remark* 2.2. Let $E$ be a complex Banach space. Let $[a_{ij}]$ be a complex $n \times n$ matrix, $x_j \in E$. Consider $x = \sum a_{ij} x_i \otimes \bar{x}_j \in E \otimes \bar{E}$. Then $x \succ 0$ if $[a_{ij}]$ is positive definite, and if the $x_j$'s are linearly independent, the converse also holds. Indeed, by the preceding argument, all we need to check is $\tilde{x}(\xi, \xi) = \sum a_{ij} \xi(x_i)\overline{\xi(x_j)} \geq 0$.

The importance of this ordering for us lies in the following fact:

Lemma 2.3. *If $x, y \in B(H) \otimes \overline{B(H)}$ and $0 \prec x \prec y$ then*

$$\|x\|_{\min} \leq \|y\|_{\min}$$

*where $\|x\|_{\min} = \|x\|_{B(H \otimes_2 \overline{H})}$.*

*Proof.* Let $x = \sum a_k \otimes \bar{a}_k$. Then the lemma is immediate from the identity

$$\left\|\sum a_k \otimes \bar{a}_k\right\| = \sup\left\{\left(\sum \|\xi a_k \eta\|_2^2\right)^{1/2} \;\middle|\; \|\xi\|_4 \leq 1, \|\eta\|_4 \leq 1\right\}$$

for which we refer to [22]. Indeed, if $d = \sum b_j \otimes \bar{b}_j$ and $y = x + d$ this identity applied to $x + d$ makes it clear that $\|x\| \leq \|x + d\| = \|y\|$. □

It will be convenient to extend our notation: Let $H_1, H_2, \dots, H_m$ be an $m$-tuple of Hilbert spaces. For any $k = 1, \dots, m$ we set

$$H_{m+k} = \overline{H}_k.$$

Let $\sigma$ be any permutation of $\{1, \dots, 2m\}$. For any element $x$ in $B(H_1) \otimes \cdots \otimes B(H_{2m})$ we denote by

$$\sigma \cdot x \in B(H_{\sigma(1)}) \otimes \cdots \otimes B(H_{\sigma(2m)})$$

the element obtained from $x$ by applying $\sigma$ to the factors, i.e. if $x = t_1 \otimes \cdots \otimes t_{2m}$ then $\sigma \cdot x = t_{\sigma(1)} \otimes \cdots \otimes t_{\sigma(2m)}$ and $x \to \sigma \cdot x$ is the linear extension of this map.
Let $\mathcal{H} = H_1 \otimes \cdots \otimes H_m$. Now if we are given a permutation $\sigma$ and $x, y$ in $B(H_{\sigma(1)}) \otimes \cdots \otimes B(H_{\sigma(2m)})$ we note that

$$\sigma^{-1} \cdot x, \sigma^{-1} \cdot y \in B(\mathcal{H}) \otimes \overline{B(\mathcal{H})}.$$

We will write (abusively) $x \prec y$ (or $y \succ x$) if we have

$$\sigma^{-1} \cdot x \prec \sigma^{-1} \cdot y.$$

Of course this order depends on $\sigma$ and although our notation does not keep track of that, we will need to remember $\sigma$, but hopefully no confusion should arise. While we will use various choices for $\sigma$, we never change our choice in the middle of a calculation, e.g. when adding two "positive" terms.
For instance we allow ourselves to write that $\forall a_k \in B(H)$ $\forall b_k \in B(K)$ we have

$$\sum a_k \otimes \bar{a}_k \otimes b_k \otimes \bar{b}_k \succ 0 \tag{2.4}$$

in $B(H \otimes \overline{H} \otimes K \otimes \overline{K})$, where implicitly we are referring to the permutation $\sigma$ that takes $H \otimes \overline{H} \otimes K \otimes \overline{K}$ to $H \otimes K \otimes \overline{H} \otimes \overline{K}$. In particular, we note that with this convention $\forall x, y \in B(H) \otimes \overline{B(H)}$ $\forall b \in B(K)$ we have

$$x \prec y \Rightarrow b \otimes x \otimes \bar{b} \prec b \otimes y \otimes \bar{b}. \tag{2.5}$$

Note that since the minimal tensor product is commutative, we still have, for any $x, y$ in $B(H_{\sigma(1)}) \otimes \cdots \otimes B(H_{\sigma(2m)})$, that

$$0 \prec x \prec y \Rightarrow \|x\|_{\min} \le \|y\|_{\min}. \tag{2.6}$$

From the obvious identity $(x, y \in B(H))$

$$(x+y) \otimes \overline{(x+y)} + (x-y) \otimes \overline{(x-y)} = 2(x \otimes \bar{x} + y \otimes \bar{y})$$

it follows that

$$(x+y) \otimes \overline{(x+y)} \prec 2(x \otimes \bar{x} + y \otimes \bar{y}). \tag{2.7}$$

Note that if we set $\Phi(x) = x \otimes \bar{x}$, then the preceding expresses the "order convexity" of this function:

$$\Phi((x+y)/2) \prec (\Phi(x) + \Phi(y))/2.$$

More generally, for any finite set $x_1, \ldots, x_n$ in $B(H)$ we have $\Phi(n^{-1}\sum_1^n x_k) \prec n^{-1}\sum_1^n \Phi(x_k)$, or

$$\left(\sum\nolimits_1^n x_k\right) \otimes \overline{\left(\sum\nolimits_1^n x_k\right)} \prec n \sum\nolimits_1^n x_k \otimes \bar{x}_k. \tag{2.8}$$

We need to record below several variants of the "order convexity" of $\Phi$.
From now on we assume that $H$ is a *separable* Hilbert space.
More generally, for any $x$ in $B(H) \otimes L_2(\Omega, \mathcal{A}, \mathbb{P})$ and any $\sigma$-subalgebra $\mathcal{B} \subset \mathcal{A}$, we may associate to $x \otimes \bar{x}$ the function $x \dot{\otimes} \bar{x}\colon\ \omega \to B(H) \otimes \overline{B(H)}$ defined by $x \dot{\otimes} \bar{x}(\omega) = x(\omega) \otimes \bar{x}(\omega)$. We have then almost surely

$$0 \prec \mathbb{E}^{\mathcal{B}}(x \dot{\otimes} \bar{x}), \tag{2.9}$$

and more precisely (again almost surely)

$$(\mathbb{E}^{\mathcal{B}} x) \dot{\otimes} \overline{(\mathbb{E}^{\mathcal{B}} x)} \prec \mathbb{E}^{\mathcal{B}}(x \dot{\otimes} \bar{x}). \tag{2.10}$$

Indeed, (2.9) follows from Lemma 2.1 (separable case) and the right hand side of (2.10) is equal to

$$(\mathbb{E}^{\mathcal{B}} x) \dot{\otimes} \overline{(\mathbb{E}^{\mathcal{B}} x)} + \mathbb{E}^{\mathcal{B}}(y \dot{\otimes} \bar{y}) \quad \text{where} \quad y = x - \mathbb{E}^{\mathcal{B}} x.$$

When $\mathcal{B}$ is the trivial algebra, we obtain

$$0 \prec \int x \dot{\otimes} \bar{x}, \tag{2.11}$$

and hence for any measurable subset $A \subset \Omega$

$$\int_A x \dot{\otimes} \bar{x}\ d\mathbb{P} \prec \int_\Omega x \dot{\otimes} \bar{x}\ d\mathbb{P}, \tag{2.12}$$

and also

$$(\mathbb{E} x) \otimes (\mathbb{E} \bar{x}) \prec \mathbb{E}(x \dot{\otimes} \bar{x}). \tag{2.13}$$

We need to observe that for any integer $m \geq 1$ we have

$$0 \prec x \Rightarrow 0 \prec x^{\otimes m} \tag{2.14}$$

and more generally

$$0 \prec x \prec y \Rightarrow 0 \prec x^{\otimes m} \prec y^{\otimes m}. \tag{2.15}$$

Furthermore, if $x_1, y_1 \in B(H_1) \otimes \overline{B(H_1)}$ and $x_2, y_2 \in B(H_2) \otimes \overline{B(H_2)}$

$$0 \prec x_1 \prec y_1 \text{ and } 0 \prec x_2 \prec y_2 \Rightarrow 0 \prec x_1 \otimes x_2 \prec y_1 \otimes y_2, \tag{2.16}$$

where the natural permutation is applied to $x_1 \otimes x_2$ and $y_1 \otimes y_2$, allowing to view them as elements of $B(H_1 \otimes H_2) \otimes \overline{B(H_1 \otimes H_2)}$.
Returning to (2.10), we note that it implies

$$((\mathbb{E}^{\mathcal{B}} x)\dot{\otimes}\overline{(\mathbb{E}^{\mathcal{B}} x)})^{\otimes 2} \prec (\mathbb{E}^{\mathcal{B}}(x\dot{\otimes}\bar{x}))^{\otimes 2} \prec \mathbb{E}^{\mathcal{B}}(x\dot{\otimes}\bar{x}\dot{\otimes}x\dot{\otimes}\bar{x}),$$

and hence

$$\|((\mathbb{E}^{\mathcal{B}} x)\dot{\otimes}\overline{(\mathbb{E}^{\mathcal{B}} x)})^{\otimes 2}\| \le \|\mathbb{E}^{\mathcal{B}}(x\dot{\otimes}\bar{x}\dot{\otimes}x\dot{\otimes}\bar{x})\|.$$

More generally, iterating this argument, we obtain for any integer $k \ge 1$

$$((\mathbb{E}^{\mathcal{B}} x)\dot{\otimes}\overline{(\mathbb{E}^{\mathcal{B}} x)})^{\otimes 2^k} \prec \mathbb{E}^{\mathcal{B}}((x\dot{\otimes}\bar{x})^{\otimes 2^k}). \tag{2.17}$$

In particular

$$((\mathbb{E} x)\dot{\otimes}\overline{(\mathbb{E} x)})^{\otimes 2^k} \prec \mathbb{E}((x\dot{\otimes}\bar{x})^{\otimes 2^k}), \tag{2.18}$$

and consequently for any finite sequence $x_1, \cdots, x_n \in B(H) \otimes L_2(\Omega, \mathcal{A}, \mathbb{P})$

$$\|\sum\nolimits_j ((\mathbb{E}^{\mathcal{B}} x_j)\dot{\otimes}\overline{(\mathbb{E}^{\mathcal{B}} x_j)})^{\otimes 2^k}\| \le \|\sum\nolimits_j \mathbb{E}^{\mathcal{B}}((x_j\dot{\otimes}\bar{x}_j)^{\otimes 2^k})\|. \tag{2.19}$$

In a somewhat different direction, for any measure $\mu$, let $f \in B(H) \otimes L_2(\mu)$, let $P$ be any orthogonal projection on $L_2(\mu)$ and let $g = (I \otimes P)(f)$. Then

$$0 \prec \int g\dot{\otimes}\bar{g}\, d\mu \prec \int f\dot{\otimes}\bar{f}\, d\mu. \tag{2.20}$$

Indeed, this is immediate by (2.3).

## 3 Definition of $\Lambda_{2m}$

Our definition of the operator space $\Lambda_{2m}(\mu)$ is based on the case $m = 1$, i.e. on the operator Hilbert space $OH$, studied at length in [22]. The latter is based on the already mentioned Cauchy–Schwarz inequality due to Haagerup as follows: Let $H, K$ be Hilbert spaces and let $a_k \in B(H)$, $b_k \in B(K)$

$$\left\|\sum a_k \otimes b_k\right\| \le \left\|\sum a_k \otimes \bar{a}_k\right\|^{1/2} \left\|\sum b_k \otimes \bar{b}_k\right\|^{1/2}. \tag{3.1}$$

This is usually stated with $\sum a_k \otimes \bar{b}_k$ on the left hand side, but since the right hand side is unchanged if we replace $b_k$ by $\bar{b}_k$ we may write this as well. It will be convenient for our exposition to use the functional version of (3.1) as follows: For any Hilbert spaces $H, K$ and any $f \in B(H) \otimes L_2(\mu)$ and $g \in B(K) \otimes L_2(\mu)$, we denote by $f\dot{\otimes}g$ the $B(H) \otimes B(K)$-valued function defined by

$$(f\dot{\otimes}g)(\omega) = f(\omega) \otimes g(\omega).$$

Of course, using the identity $B(H) \otimes \overline{B(H)} \simeq B(H) \otimes B(\overline{H})$, this extends the previously introduced notation for $f\dot{\otimes}\bar{f}\colon\ \Omega \to B(H) \otimes \overline{B(H)}$.

Similarly, given $n$ measurable functions $f_j\colon\ \Omega \to B(H_j)$, we denote by $f_1\dot{\otimes}\cdots\dot{\otimes}f_n$ the pointwise product viewed as a function with values in $B(H_1)\otimes\cdots\otimes B(H_n)$.
Note that if $f_j$ corresponds to an element in $B(H_j)\otimes L_{p_j}$ with $p_j>0$ such that $\sum p_j^{-1}=p^{-1}$, then by Hölder's inequality, $f_1\dot{\otimes}\cdots\dot{\otimes}f_n \in B(H_1)\otimes\cdots\otimes B(H_n)\otimes L_p$.
By (2.1) applied with $\mathcal{H}=L_2(\mu)$, we have

$$\left\|\int f\dot{\otimes}\bar{g}\ d\mu\right\|_{B(H\otimes\overline{K})} \le \left\|\int f\dot{\otimes}\bar{f}\ d\mu\right\|^{1/2}_{B(H\otimes\overline{H})}\left\|\int g\dot{\otimes}\bar{g}\ d\mu\right\|^{1/2}_{B(K\otimes\overline{K})}. \tag{3.2}$$

Replacing $\bar{g}$ by $g$, we obtain

$$\left\|\int f\dot{\otimes}g\ d\mu\right\|_{B(H\otimes K)} \le \left\|\int f\dot{\otimes}\bar{f}\ d\mu\right\|^{1/2}_{B(H\otimes\overline{H})}\left\|\int g\dot{\otimes}\bar{g}\ d\mu\right\|^{1/2}_{B(K\otimes\overline{K})}. \tag{3.3}$$

We note that this functional variant of (3.1) appears in unpublished work by Furman and Shalom (personal communication).
We will also invoke the following variant: for any $\psi\in B(\ell_2)\otimes L_\infty$ with norm $\|\psi\|_{\min}\le 1$ we have

$$\left\|\int f\dot{\otimes}g\dot{\otimes}\psi\ d\mu\right\|_{B(H\otimes K\otimes\ell_2)} \le \left\|\int f\dot{\otimes}\bar{f}\ d\mu\right\|^{1/2}_{B(H\otimes\overline{H})}\left\|\int g\dot{\otimes}\bar{g}\ d\mu\right\|^{1/2}_{B(K\otimes\overline{K})}. \tag{3.4}$$

This (which can be interpreted as saying that the product map $L_2\times L_2\to L_1$ is jointly completely contractive) can be verified rather easily using complex interpolation, see e.g. the proof of Lemma 7.1 below for a more detailed argument.
For simplicity in this section we abbreviate $L_p(\mu)$ or $L_p(\Omega,\mu)$ and we simply write $L_p$ instead.
We start by a version of Hölder's inequality adapted to our needs that follows easily from (3.3). The proof uses an iteration idea already appearing in [6].

Lemma 3.1. *Let $m\ge 1$ be any integer. Then for any $f_1,\ldots,f_{2m}$ in $B(H)\otimes L_{2m}$ we have $f_1\dot{\otimes}\cdots\dot{\otimes}f_{2m}\in B(H)^{\otimes 2m}\otimes L_1$ and*

$$\left\|\int f_1\dot{\otimes}\cdots\dot{\otimes}f_{2m}\ d\mu\right\| \le \prod_{k=1}^{2m}\left\|\int f_k^{\dot{\otimes}m}\dot{\otimes}\bar{f}_k^{\dot{\otimes}m}\ d\mu\right\|^{\frac{1}{2m}}, \tag{3.5}$$

*where we denote $f^{\dot{\otimes}m}=f\dot{\otimes}\cdots\dot{\otimes}f$ (m times).*

*Proof.* By homogeneity we may (and do) normalize and assume that

$$\left\|\int f_k^{\dot{\otimes}m}\dot{\otimes}\bar{f}_k^{\dot{\otimes}m}\ d\mu\right\|\le 1. \qquad \forall k=1,\ldots,2m$$

Let
$$C = \max\{\|\int g_1\dot\otimes\cdots\dot\otimes g_{2m}\ d\mu\|\}$$
where the maximum runs over all $g_k$ in the set $\{f_1,\ldots,f_{2m},\bar f_1,\ldots,\bar f_{2m}\}$.
It clearly suffices to prove $C\le 1$. In the interest of the reader, we first do the proof in the simplest case $m=2$. We have by (3.3)
$$\|\int f_1\dot\otimes\cdots\dot\otimes f_4 d\mu\| \le \|\int f_1\dot\otimes f_2\dot\otimes\overline{f_1\dot\otimes f_2}d\mu\|^{1/2}\|\int f_3\dot\otimes f_4\dot\otimes\overline{f_3\dot\otimes f_4}d\mu\|^{1/2}$$
which we may rewrite as
$$\|\int f_1\dot\otimes\cdots\dot\otimes f_4 d\mu\| \le \|\int f_1\dot\otimes\bar f_1\dot\otimes f_2\dot\otimes\bar f_2 d\mu\|^{1/2}\|\int f_3\dot\otimes\bar f_3\dot\otimes f_4\dot\otimes\bar f_4 d\mu\|^{1/2}, \tag{3.6}$$
and by (3.3) again we have
$$\|\int f_1\dot\otimes\bar f_1\dot\otimes f_2\dot\otimes\bar f_2 d\mu\|^{1/2}\le$$
$$\le \|\int f_1\dot\otimes\bar f_1\dot\otimes\bar f_1\dot\otimes f_1 d\mu\|^{1/2}\|\int f_2\dot\otimes\bar f_2\dot\otimes\bar f_2\dot\otimes f_2 d\mu\|^{1/2}\le 1$$
and similarly for the other factor in (3.6). Thus we obtain the announced inequality for $m=2$.
To check the general case, let us denote
$$I(f_1,\ldots,f_{2m}) = \left\|\int f_1\dot\otimes\cdots\dot\otimes f_{2m}\ d\mu\right\|.$$
By (3.3) we find
$$I(f_1,\ldots,f_{2m})\le (I(f_1,\ldots,f_m,\bar f_1,\ldots,\bar f_m)C)^{1/2}.$$
Note that $I(f_1,\ldots,f_{2m})$ is invariant under permutation of entries. Thus we have
$$I(f_1,\ldots,f_m,\bar f_1,\ldots,\bar f_m) = I(f_1,\bar f_1,f_2,\bar f_2,\ldots,f_m,\bar f_m).$$
Using (3.3) again we find
$$I(f_1,\ldots,f_m,\bar f_1,\ldots,\bar f_m)\le (I(f_1,\bar f_1,f_1,\bar f_1,f_2,\bar f_2,\ldots)C)^{1/2}$$
and continuing in this way we obtain
$$I(f_1,\ldots,f_{2m})\le I(f_1,\bar f_1,f_1,\bar f_1,\ldots,f_1,\bar f_1)^{\theta}C^{1-\theta}$$
where $0<\theta<1$ is equal to $2^{-K}$ with $K$ the number of iterations.
Since we assume $I(f_1,\bar f_1,\ldots,f_1,\bar f_1)\le 1$ we find
$$I(f_1,\ldots,f_{2m})\le C^{1-\theta}.$$

But we may replace $f_1, \ldots, f_{2m}$ by $g_1, \ldots, g_{2m}$ and the same argument gives us

$$I(g_1, \ldots, g_{2m}) \le C^{1-\theta}$$

and hence $C \le C^{1-\theta}$, from which $C \le 1$ follows immediately. $\square$

PROPOSITION 3.2. *Let $m \ge 1$ be an integer and $p = 2m$. There is an isometric embedding*

$$L_p(\mu) \subset B(\mathcal{H})$$

*so that for any $f \in B(H) \otimes L_p(\mu)$ and any $H$ we have*

$$\|f\|_{B(H\otimes\mathcal{H})} = \left\| \int f^{\dot\otimes m} \dot\otimes \bar{f}^{\dot\otimes m}\, d\mu \right\|^{\frac{1}{2m}}$$

*where we again denote $f^{\dot\otimes m} = f \dot\otimes \cdots \dot\otimes f$ ($m$ times).*

*Proof.* Let $B = B(H)$. With the notation in Lemma 3.1 we have

$$\left\| \int f^{\dot\otimes m} \dot\otimes \bar{f}^{\dot\otimes m}\, d\mu \right\|^{\frac{1}{2m}} = \max\{I(f, \bar{f}, g_2, \bar{g}_2, \ldots, g_m, \bar{g}_m)^{1/2}\} \tag{3.7}$$

where the supremum runs over all $g_2, \ldots, g_m$ in $B \otimes L_{2m}$ such that $\left\| \int g_k^{\dot\otimes m} \dot\otimes \bar{g}_k^{\dot\otimes m}\, d\mu \right\| \le 1$ for any $k = 1, \ldots, m$.
Indeed, it follows easily from (3.5) that the latter maximum is attained for the choice of $g_2 = \cdots = g_m = \lambda f$ with $\lambda = \left\| \int f^{\dot\otimes m} \dot\otimes \bar{f}^{\dot\otimes m}\, d\mu \right\|^{-1/2m}$. Thus we can proceed as in [22] for the case $m = 1$: We assume $H = \ell_2$ to fix ideas. Let $\mathcal{S}$ denote the collection of all $G = g_2 \dot\otimes \cdots \dot\otimes g_m$ where $(g_2, \ldots, g_m)$ runs over the set appearing in (3.7). Note that by (3.5) we know that for any $f$ in $B \otimes L_p$ we have $f \dot\otimes G \in B(H^{\otimes m}) \otimes L_2$. Then for any $G$ in $\mathcal{S}$ we introduce the linear map

$$u_G \colon\ L_p(\mu) \to B(H^{\otimes m}) \otimes_{\min} (L_2)_{oh}$$

defined by

$$u_G(f) = f \dot\otimes G.$$

Then (3.5) and (1.4) imply $\|u_G\| \le 1$. We then define the embedding

$$u \colon\ L_p(\mu) \to \bigoplus_{G \in \mathcal{S}} B(H^{\otimes m-1}) \otimes_{\min} (L_2)_{oh}$$

by setting

$$u(f) = \bigoplus_G u_G(f).$$

Then (3.7) gives us that for any $f$ in $B \otimes L_p$ we have

$$\|(id \otimes u)(f)\| = \left\| \int f^{\dot\otimes m} \dot\otimes \bar{f}^{\dot\otimes m}\, d\mu \right\|^{\frac{1}{2m}}.$$

Thus the embedding $u$ has the required properties. $\square$

Definition 3.3. We denote by $\Lambda_p(\mu)$ $(p = 2m)$ the operator space appearing in Proposition 3.2. Note that $\Lambda_p(\mu)$ is isometric to $L_p(\mu)$ and for any $H$ and for any $f \in B(H) \otimes \Lambda_p(\mu)$ we have

$$\|f\|_{B(H)\otimes_{\min}\Lambda_p(\mu)} = \left\|\int f^{\dot{\otimes} m} \dot{\otimes} \bar{f}^{\dot{\otimes} m}\, d\mu\right\|^{\frac{1}{2m}}. \tag{3.8}$$

Proposition 3.4. *Let $f$ be as in Definition 3.3 with $p = 2m$. Then*

$$\|f\|_{B(H)\otimes_{\min}\Lambda_p} = \|f^{\dot{\otimes} m}\|^{1/m}_{B(H^{\otimes m})\otimes_{\min} L_2} = \|f^{\dot{\otimes} m} \dot{\otimes} \bar{f}^{\dot{\otimes} m}\|^{\frac{1}{2m}}_{B(H^{\otimes m}\otimes \bar{H}^{\otimes m})\otimes_{\min} L_1} \tag{3.9}$$

*where $L_2$ and $L_1$ are equipped respectively with their oh and maximal operator space structure. So if we make the convention that $\Lambda_1 = L_1$ equipped with its maximal o.s.s. then we have*

$$\|f\|_{B(H)\otimes_{\min}\Lambda_{2m}} = \|(f\dot{\otimes}\bar{f})^{\dot{\otimes} m}\|^{\frac{1}{2m}}_{B((H\otimes\bar{H})^{\otimes m})\otimes_{\min}\Lambda_1}.$$

*Proof.* The first equality is immediate since it is easy to check that for any $g \in B(H) \otimes L_2$ we have

$$\|g\|_{\min} = \left\|\int g\dot{\otimes}\bar{g}\, d\mu\right\|^{1/2}_{B(H\otimes\overline{H})}.$$

For the second one, we use (1.2): for any $\varphi$ in $B(H) \otimes L_1$

$$\|\varphi\|_{B(H)\otimes_{\min} L_1} = \sup\left\{\left\|\int \varphi\dot{\otimes}\psi\, d\mu\right\|\right\}$$

where the supremum runs over all $\psi$ in $B(\ell_2) \otimes L_\infty$ with $\|\psi\|_{\min} \leq 1$. Applying this to $\varphi(t) = f(t)^{\otimes m} \otimes \overline{f(t)}^{\otimes m}$ we find using (3.4)

$$\|f^{\otimes m} \otimes \bar{f}^{\otimes m}\|_{B(H)\otimes_{\min} L_1} \leq \|f\|^{2m}_{B(H)\otimes_{\min}\Lambda_p},$$

and the choice of $\psi \equiv 1$ shows that this inequality is actually an equality. □

Notation: For any $f \in B(H) \otimes L_p(\mu)$ (or equivalently $f \in B(H) \otimes \Lambda_p(\mu)$), we denote

$$\|f\|_{(p)} = \|f\|_{B(H)\otimes_{\min}\Lambda_p(\mu)}.$$

The preceding Proposition shows that if $p = 2m$, we have

$$\|f\|_{(p)} = \|f^{\dot{\otimes} p/2}\|^{2/p}_{(2)} = \|(f\dot{\otimes}\bar{f})^{\dot{\otimes} p/2}\|^{1/p}_{(1)}.$$

Note that by (2.15), if $g, f \in B(H) \otimes L_p(\mu)$ and $p = 2m$

$$\big(g(\omega)\dot{\otimes}\bar{g}(\omega) \prec f(\omega)\dot{\otimes}\bar{f}(\omega) \quad \forall_{a.s}\ \omega \in \Omega\big) \Rightarrow \|g\|_{(p)} \leq \|f\|_{(p)}. \tag{3.10}$$

Corollary 3.5. *The conditional expectation $\mathbb{E}^{\mathcal{B}}$ is completely contractive on $\Lambda_p$ for any $p \in 2\mathbb{N}$.*

*Proof.* The case $p = 2$ is clear by the homogeneity of the spaces $OH$. Using (2.10) one easily passes from $p$ to $2p$. By induction, this proves the Corollary for any $p$ of the form $p = 2^m$. For $p$ equal to an arbitrary even integer, we need a different argument. Let $f \in B(H) \otimes \Lambda_p$ and $g = \mathbb{E}^{\mathcal{B}} f$. By the classical property of conditional expectations, we have

$$\|g\|_{(p)}^p = \| \int g \dot{\otimes} \bar{g} \dot{\otimes} g \dot{\otimes} \bar{g} \cdots d\mu \| = \| \int f \dot{\otimes} \bar{g} \dot{\otimes} g \dot{\otimes} \bar{g} \cdots d\mu \|$$

and hence by (3.5) $\|g\|_{(p)}^p \leq \|f\|_{(p)} \|g\|_{(p)}^{p-1}$ or equivalently $\|g\|_{(p)} \leq \|f\|_{(p)}$. □

With the above definition, Lemma 3.1 together with (3.4) immediately implies (see the beginning of §7 for the definition of jointly completely contractive multilinear maps):

Corollary 3.6. *Let $p = 2m$. The product mapping $(f_1, \ldots, f_{2m}) \mapsto f_1 \times \ldots \times f_{2m}$ is a jointly completely contractive multilinear map from $\Lambda_p(\mu)^{2m}$ to $\Lambda_1(\mu) = L_1(\mu)$.*

*Proof.* By Lemma 3.1 and (3.4) we have for any $\psi \in B(\ell_2) \otimes L_\infty$ with norm $\|\psi\|_{\min} \leq 1$

$$\| \int f_1 \dot{\otimes} \cdots \dot{\otimes} f_p \dot{\otimes} \psi \| \leq \prod_1^p \|f_j\|_{(p)}.$$

Taking the supremum over all such $\psi$ we obtain

$$\|f_1 \dot{\otimes} \cdots \dot{\otimes} f_p\|_{(1)} \leq \prod_1^p \|f_j\|_{(p)},$$

which is nothing but a reformulation of the assertion of this corollary. □

Corollary 3.7. *Let $p \geq q \geq 2$ be even integers. If $\mu$ is a probability, the inclusion $L_p(\mu) \subset L_q(\mu)$ is a complete contraction from $\Lambda_p(\mu)$ to $\Lambda_q(\mu)$.*

*Proof.* Take $p = 2m$, $q = 2n$ and $r = 2m - 2n$ with $n < m$. Let $f \in B(H) \otimes L_p(\mu)$ and $g \in B(H) \otimes L_p(\mu)$. Consider $f_1 \otimes \cdots \otimes f_{2m} = (f \otimes \bar{f})^n \otimes g^{\otimes r}$, and let us choose for $g$ the constant function that is identically equal to the identity operator on $H$. Then, using (3.8), (3.5) yields

$$\|f\|^q_{B(H) \otimes_{\min} \Lambda_q(\mu)} \leq \|f\|^q_{B(H) \otimes_{\min} \Lambda_p(\mu)} \times 1^{p-q}$$

and hence $\|f\|_{B(H) \otimes_{\min} \Lambda_q(\mu)} \leq \|f\|_{B(H) \otimes_{\min} \Lambda_p(\mu)}$. □

*Remark* 3.8. Let $f \in B(H)\otimes L_\infty(\mu)$. We may view $f$ as an essentially bounded $B(H)$-valued function. Then if $\mu$ is finite we have

$$\|f\|_{L_\infty(\mu;B(H))} = \lim_{p\to\infty} \|f\|_{(p)}. \tag{3.11}$$

Indeed, we may assume $\mu(\Omega) = 1$ and it clearly suffices to prove that for any measurable subset $A \subset \Omega$ with $\mu(A) > 0$ we have

$$\left\|\mu(A)^{-1}\int\limits_A f\ d\mu\right\| \le \lim_{p\to\infty} \|f\|_{(p)}. \tag{3.12}$$

To verify this let $f_A = \mu(A)^{-1}\int\limits_A f\ d\mu$. By (2.17) with $p = 2^k$ we have

$$\begin{aligned}\|f_A\| = \|(f_A\otimes \bar f_A)^{\otimes p}\|^{\frac{1}{2p}} &\le \left\|\mu(A)^{-1}\int\limits_A (f\dot\otimes\bar f)^{\dot\otimes p}\right\|^{\frac{1}{2p}}\\ &\le \mu(A)^{-\frac{1}{2p}}\|f\|_{(2p)}\end{aligned}$$

where we also use (2.12) and Lemma 2.3, and hence letting $p\to\infty$ we obtain (3.12).

## 4 Martingale inequalities in $\Lambda_p$

Our main result is the following one. This is an operator valued version of Burkholder's martingale inequalities. Although our inequality seems very different from the one appearing in [26], the method used to prove it is rather similar.

Throughout this section $H$ is an arbitrary Hilbert space and we set $B = B(H)$. We give ourselves a probability space $(\Omega,\mathcal{A},\mathbb{P})$ and we set $L_p = L_p(\Omega,\mathcal{A},\mathbb{P})$. We also give ourselves a filtration $(\mathcal{A}_n)_{n\ge0}$ on $(\Omega,\mathcal{A},\mathbb{P})$. We assume that $\mathcal{A}_0$ is trivial and that $\mathcal{A}_\infty = \sigma\big(\bigcup_{n\ge0}\mathcal{A}_n\big)$ is equal to $\mathcal{A}$.

We may view any $f$ in $B\otimes L_p$ as a $B$-valued random variable $f\colon\ \Omega\to B$. We denote $\mathbb{E}_n f = \mathbb{E}^{\mathcal{A}_n} f$, $d_0 = \mathbb{E}_0 f$ and $d_n = \mathbb{E}_n f - \mathbb{E}_{n-1} f$ for any $n \ge 1$. We will say that $f\in B\otimes L_p$ is a test function if it is $\mathcal{A}_N$-measurable for some $N\ge1$, or equivalently if $f$ can be written as a *finite* sum $f = \sum_0^\infty\ d_n$. We then denote

$$S(f) = \sum\nolimits_0^\infty d_n\dot\otimes\bar d_n.$$

We also denote

$$\sigma(f) = d_0\dot\otimes\bar d_0 + \sum\nolimits_1^\infty \mathbb{E}_{n-1}(d_n\dot\otimes\bar d_n).$$

We recall the notation

$$x^{\dot\otimes m} = x\dot\otimes\cdots\dot\otimes x$$

where $x$ is repeated $m$ times. Note that $\sigma(f)\in B\otimes\overline{B}\otimes L_{p/2}$ and also that

$$(d_n\dot\otimes\bar d_n)^{\dot\otimes p/2} \in (B\otimes\overline{B})^{\otimes p/2}\otimes L_1.$$

Theorem 4.1. *For any $p \geq 2$ of the form $p = 2^k$ for some $k \geq 1$ there are positive constants $C_1(p), C_2(p)$ such that for any test function $f$ in $B \otimes L_p$ we have*

$$C_1(p)^{-1}\|S(f)\|^{1/2}_{B\otimes_{\min}\overline{B}\otimes_{\min}\Lambda_{p/2}} \leq \|f\|_{B\otimes_{\min}\Lambda_p} \leq C_2(p)\|S(f)\|^{1/2}_{B\otimes_{\min}\overline{B}\otimes_{\min}\Lambda_{p/2}}. \tag{4.1}$$

*First part of the proof.* We start by the case $p = 4$. Let $f = \sum d_n$, $S = S(f)$ and $\sigma = \sigma(f)$. Then

$$f\dot{\otimes}\bar{f} = S + a + b$$

where $a = \sum d_n \dot{\otimes} \bar{f}_{n-1}$ and $b = \sum f_{n-1}\dot{\otimes}\bar{d}_n$. Let $g = a+b$ so that $f\dot{\otimes}\bar{f} - S = g$. Note that by (2.7) applied pointwise (or, say, alomst surely)

$$g\dot{\otimes}\bar{g} \prec 2(a\dot{\otimes}\bar{a} + b\dot{\otimes}\bar{b})$$

and hence $\mathbb{E}(g\dot{\otimes}\bar{g}) \prec 2\mathbb{E}(a\dot{\otimes}\bar{a}) + 2\mathbb{E}(b\dot{\otimes}\bar{b})$. By Lemma 2.3 we have

$$\|\mathbb{E}(g\dot{\otimes}\bar{g})\| \leq 2\|\mathbb{E}(a\dot{\otimes}\bar{a})\| + 2\mathbb{E}(b\dot{\otimes}\bar{b})\|. \tag{4.2}$$

Note that for any $f$

$$\|\mathbb{E}(f\dot{\otimes}\bar{f})\|^{1/2} = \|f\|_{B\otimes_{\min}\Lambda_2}.$$

By orthogonality $\mathbb{E}(a\dot{\otimes}\bar{a}) = \mathbb{E}(\sum d_n\dot{\otimes}\bar{f}_{n-1}\dot{\otimes}\bar{d}_n\dot{\otimes}f_{n-1})$. By (2.10) we have

$$\bar{f}_{n-1}\dot{\otimes}f_{n-1} \prec \mathbb{E}_{n-1}(\bar{f}\dot{\otimes}f)$$

therefore by (2.5)

$$d_n\dot{\otimes}\bar{d}_n\dot{\otimes}\bar{f}_{n-1}\dot{\otimes}f_{n-1} \prec d_n\dot{\otimes}\bar{d}_n\dot{\otimes}\mathbb{E}_{n-1}(\bar{f}\dot{\otimes}f)$$

and hence

$$\mathbb{E}(d_n\dot{\otimes}\bar{d}_n\dot{\otimes}\bar{f}_{n-1}\dot{\otimes}f_{n-1}) \prec \mathbb{E}(\mathbb{E}_{n-1}(d_n\dot{\otimes}\bar{d}_n)\dot{\otimes}\bar{f}\dot{\otimes}f)$$

which yields (after a suitable permutation) by Lemma 2.3

$$\|\mathbb{E}(a\dot{\otimes}\bar{a})\| \leq \|\mathbb{E}(\sigma\dot{\otimes}\bar{f}\dot{\otimes}f)\|,$$

and hence by Haagerup's Cauchy–Schwarz inequality

$$\|\mathbb{E}(a\dot{\otimes}\bar{a})\| \leq \|\mathbb{E}(\sigma\dot{\otimes}\bar{\sigma})\|^{1/2}\|\mathbb{E}(\bar{f}\dot{\otimes}f\dot{\otimes}f\dot{\otimes}\bar{f})\|^{1/2}. \tag{4.3}$$

Similarly we find

$$\|\mathbb{E}(b\dot{\otimes}\bar{b})\| \leq \|\mathbb{E}(\sigma\dot{\otimes}\bar{\sigma})\|^{1/2}\|\mathbb{E}(f\dot{\otimes}\bar{f}\dot{\otimes}\bar{f}\dot{\otimes}f)\|^{1/2}. \tag{4.4}$$

We now claim that

$$\|\mathbb{E}(\sigma\dot{\otimes}\bar{\sigma})\|^{1/2} \leq 2\|\mathbb{E}(S\dot{\otimes}\overline{S})\|^{1/2}.$$

Using this claim the conclusion is easy: By (4.2), (4.3), (4.4) we have

$$\|\mathbb{E}(g\dot{\otimes}\bar{g})\| \le 8\|\mathbb{E}(S\dot{\otimes}\overline{S})\|^{1/2}\|\mathbb{E}(f\dot{\otimes}\bar{f}\dot{\otimes}f\dot{\otimes}\bar{f})\|^{1/2}.$$

But now $g \to \|\mathbb{E}(g\dot{\otimes}\bar{g})\|^{1/2} = \|g\|_{B\otimes_{\min}L_2}$ is a norm so by its subadditivity, recalling $g = f\dot{\otimes}\bar{f} - S$, we have

$$\big|\|f\dot{\otimes}\bar{f}\|_{B\otimes_{\min}L_2} - \|S\|_{B\otimes_{\min}L_2}\big| \le \|g\|_{B\otimes_{\min}L_2} \le 8^{1/2}xy$$

where $x^2 = \|f\dot{\otimes}\bar{f}\|_{B_{\min}\otimes L_2}$ and $y^2 = \|S\|_{B\otimes_{\min}L_2}$. Equivalently, we have

$$|x^2 - y^2| \le 8^{1/2}xy$$

from which it immediately follows that

$$\max\left\{\frac{x}{y}, \frac{y}{x}\right\} \le \sqrt{2} + \sqrt{3}.$$

Thus, modulo our claim, we obtain the announced inequality (4.1) for $p = 4$. To prove the claim we note that it is a particular case of the "dual Doob inequality" appearing in the next Lemma. □

Lemma 4.2. *Let $\theta_1, \ldots, \theta_N$ be arbitrary in $B \otimes L_4$. Let $\alpha = \sum_1^N \mathbb{E}_n(\theta_n\dot{\otimes}\bar{\theta}_n)$ and $\beta = \sum_1^N \theta_n\dot{\otimes}\bar{\theta}_n$. Then*

$$\|\mathbb{E}(\alpha\dot{\otimes}\bar{\alpha})\|^{1/2} \le 2\|\mathbb{E}(\beta\dot{\otimes}\bar{\beta})\|^{1/2}.$$

*Proof.* Let $\alpha_n = \mathbb{E}_n(\theta_n\dot{\otimes}\bar{\theta}_n)$ and $\beta_n = \theta_n\dot{\otimes}\bar{\theta}_n$. Note that $\beta_n \succ 0$ and $\alpha_n \succ 0$. We have $\alpha\dot{\otimes}\bar{\alpha} = \sum_{n,k} \alpha_n\dot{\otimes}\bar{\alpha}_k$ and hence

$$\begin{aligned}
\mathbb{E}(\alpha\dot{\otimes}\bar{\alpha}) &= \mathbb{E}\left(\sum_{n\le k} \mathbb{E}_n\beta_n\dot{\otimes}\overline{\mathbb{E}_k\beta_k}\right) + \mathbb{E}\left(\sum_{n>k} \mathbb{E}_n\beta_n\dot{\otimes}\overline{\mathbb{E}_k\beta_k}\right)\\
&= \mathbb{E}\left(\sum\nolimits_{n\le k} (\mathbb{E}_n\beta_n)\dot{\otimes}\bar{\beta}_k\right) + \mathbb{E}\left(\sum\nolimits_{n>k} \beta_n\dot{\otimes}(\overline{\mathbb{E}_k\beta_k})\right)\\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Using a suitable permutation (as explained before (2.5)) we have by (2.5) or (2.16)

$$(\mathbb{E}_n\beta_n)\dot{\otimes}\bar{\beta}_k \succ 0 \quad \text{and} \quad \beta_n\dot{\otimes}(\overline{\mathbb{E}_k\beta_k}) \succ 0.$$

Therefore, using (2.9), it follows from Lemma 2.3 that

$$\|\mathrm{I}\| \le \left\|\sum\nolimits_{n,k} \mathbb{E}((\mathbb{E}_n\beta_n)\dot{\otimes}\bar{\beta}_k)\right\| = \|\mathbb{E}(\alpha\dot{\otimes}\bar{\beta})\|,$$

and hence by (3.3)

$$\|\mathrm{I}\| \le \|\mathbb{E}(\alpha\dot{\otimes}\bar{\alpha})\|^{1/2}\|\mathbb{E}(\beta\dot{\otimes}\bar{\beta})\|^{1/2}.$$

A similar bound holds for $\|\mathrm{II}\|$. Thus we obtain

$$\|\mathbb{E}(\alpha\dot{\otimes}\bar{\alpha})\| \le \|\mathrm{I}\| + \|\mathrm{II}\| \le 2\|\mathbb{E}(\alpha\dot{\otimes}\bar{\alpha})\|^{1/2}\|\mathbb{E}(\beta\dot{\otimes}\bar{\beta})\|^{1/2}.$$

After division by $\|\mathbb{E}(\alpha\dot{\otimes}\bar{\alpha})\|^{1/2}$ we find the inequality in Lemma 4.2. □

We will need to extend Lemma 4.2 as follows:

Lemma 4.3. *Let $m \ge 1$ be any integer. Let $\theta_1, \dots, \theta_N$ be arbitrary in $B \otimes L_{2m}$. Let $\alpha, \beta$ be as in the preceding lemma. Then*

$$\|\mathbb{E}(\alpha^{\dot{\otimes} m})\| \le m^m \|\mathbb{E}(\beta^{\dot{\otimes} m})\|. \tag{4.5}$$

*Proof.* Note that up to a permutation of factors $(\alpha)^{\dot{\otimes} 2}$ and $\alpha\dot{\otimes}\bar{\alpha}$ are the same. In any case $\|\mathbb{E}((\alpha)^{\dot{\otimes} 2})\| = \|\mathbb{E}(\alpha\dot{\otimes}\bar{\alpha})\|$, so the case $m = 2$ follows from the preceding lemma (and $m = 1$ is trivial).
We use the same notation as in the preceding proof. We can write

$$(\alpha)^{\dot{\otimes} m} = \sum_{n(1),\dots,n(m)} \alpha_{n(1)}\dot{\otimes}\alpha_{n(2)}\dot{\otimes}\cdots\dot{\otimes}\alpha_{n(m)}.$$

We can partition the set of $m$-tuples $n = (n(1), \dots, n(m))$ into subsets $S_1, \dots, S_m$ so that we have

$$n(1) = \max_j n(j), \qquad \forall n \in S_1$$
$$n(2) = \max_j n(j), \qquad \forall n \in S_2$$

and so on. Let then $T(j) = \mathbb{E}\left(\sum_{n \in S_j} \alpha_{n(1)}\dot{\otimes}\cdots\dot{\otimes}\bar{\alpha}_{n(m)}\right)$. Consider $j = 1$ for simplicity, arguing as in the preceding proof, we have

$$\begin{aligned} T(1) &= \sum_{n \in S_1} \mathbb{E}(\alpha_{n(1)}\dot{\otimes}\cdots\dot{\otimes}\alpha_{n(m)}) \\ &= \sum_{n \in S_1} \mathbb{E}(\beta_{n(1)}\dot{\otimes}\alpha_{n(2)}\cdots\dot{\otimes}\alpha_{n(m)}) \\ &\prec \sum_{n(1),\dots,n(m)} \mathbb{E}(\beta_{n(1)}\dot{\otimes}\alpha_{n(2)}\cdots\dot{\otimes}\alpha_{n(m)}) = \mathbb{E}(\beta\dot{\otimes}\alpha^{\dot{\otimes} m-1}) \end{aligned}$$

and hence by (3.5)

$$\|\mathbb{E}T(1)\| \le \|\mathbb{E}(\beta\dot{\otimes}\alpha^{\dot{\otimes} m-1})\| \le \|\mathbb{E}(\beta^{\dot{\otimes} m})\|^{\frac{1}{m}}\|\mathbb{E}(\alpha^{\dot{\otimes} m})\|^{\frac{m-1}{m}}.$$

A similar bound holds for each $T(j)$ $(j = 1, \dots, m)$. Thus we find

$$\|\mathbb{E}(\alpha^{\dot{\otimes} m})\| = \left\|\sum\nolimits_1^m \mathbb{E}T(j)\right\| \le m\|\mathbb{E}(\beta^{\dot{\otimes} m})\|^{\frac{1}{m}}\|\mathbb{E}(\alpha^{\dot{\otimes} m}\|^{\frac{m-1}{m}},$$

and again after a suitable division we obtain (4.5). □

*Proof of Theorem 4.1 in the dyadic case.* Assume that $d_n\dot{\otimes}\bar{d}_n$ is $\mathcal{A}_{n-1}$-measurable. Note that this holds in the dyadic case when $\Omega = \{-1,1\}^{\mathbb{N}}$ as well as for the filtration naturally associated to the Haar orthonormal system. In that case we can give a short proof of the following inequality for any $m \geq 1$

$$\|\mathbb{E}(S(a)^{\dot{\otimes}m})\|^{\frac{1}{m}} \leq m\|\mathbb{E}(S^{\dot{\otimes}2m})\|^{\frac{1}{2m}}\|\mathbb{E}((f\dot{\otimes}\bar{f})^{\dot{\otimes}2m})\|^{\frac{1}{2m}}, \tag{4.6}$$

and a similar bound for $\|\mathbb{E}(S(b)^{\dot{\otimes}m})\|^{\frac{1}{m}}$.
Indeed, recall $a = \sum d_n\dot{\otimes}\bar{f}_{n-1}$. Note that $S(a)$ is up to permutation the same as

$$S_1 = \sum d_n\dot{\otimes}\bar{f}_{n-1}\dot{\otimes}f_{n-1}\dot{\otimes}\bar{d}_n.$$

By (2.10) and (2.5), we have

$$S_1 \prec \sum d_n\dot{\otimes}\mathbb{E}_{n-1}(\bar{f}\dot{\otimes}f)\dot{\otimes}\bar{d}_n = \sum \mathbb{E}_{n-1}(d_n\dot{\otimes}\bar{f}\dot{\otimes}f\dot{\otimes}\bar{d}_n)$$

where the last equality holds because $d_n\dot{\otimes}\bar{d}_n$ is assumed $(n-1)$-measurable. By (2.15) this implies

$$S_1^{\dot{\otimes}m} \prec \left(\sum \mathbb{E}_{n-1}(d_n\dot{\otimes}\bar{f}\dot{\otimes}f\dot{\otimes}\bar{d}_n)\right)^{\dot{\otimes}m}$$

and hence by (4.5) (recall $S = S(f)$ and the permutation invariance of the norm)

$$\begin{aligned}\|\mathbb{E}(S_1^{\dot{\otimes}m})\| &\leq m^m \left\|\mathbb{E}\left(\left(\sum d_n\dot{\otimes}\bar{f}\dot{\otimes}f\dot{\otimes}\bar{d}_n\right)^{\dot{\otimes}m}\right)\right\| \\ &= m^m\|\mathbb{E}((S\dot{\otimes}\bar{f}\dot{\otimes}f)^{\dot{\otimes}m})\|\end{aligned}$$

and hence by Lemma 3.1

$$\leq m^m\|\mathbb{E}(S^{\dot{\otimes}2m})\|^{1/2}\|\mathbb{E}((f\dot{\otimes}\bar{f})^{\dot{\otimes}2m})\|^{1/2}.$$

Thus we obtain (4.6) as announced:

$$\|\mathbb{E}(S(a)^{\dot{\otimes}m})\| = \|\mathbb{E}(S_1^{\dot{\otimes}m})\| \leq m^m\|\mathbb{E}(S^{\dot{\otimes}2m})\|^{1/2}\|\mathbb{E}((f\dot{\otimes}\bar{f})^{\dot{\otimes}2m})\|^{1/2}.$$

The proof with $b$ in place of $a$ is identical.
Using (4.6) and the analogue for $b$, it is easy to show (assuming that $d_n\dot{\otimes}\bar{d}_n$ is $\mathcal{A}_{n-1}$-measurable) that the validity of (4.1) for $p = 2m$ implies its validity for $p = 4m$. Indeed, let us assume (4.1) for $p = 2m$ and let $C = C_2(2m)$. We find, recalling $f\dot{\otimes}\bar{f} - S = a + b$

$$\begin{aligned}\|f\dot{\otimes}\bar{f} - S\|_{B\otimes_{\min}\Lambda_{2m}} &\leq \|a\|_{B\otimes_{\min}\Lambda_{2m}} + \|b\|_{B\otimes_{\min}\Lambda_{2m}} \\ &\leq C(\|S(a)\|^{1/2}_{B\otimes_{\min}\Lambda_m} + \|S(b)\|^{1/2}_{B\otimes_{\min}\Lambda_m})\end{aligned}$$

and hence by (4.6)

$$\leq 2Cm^{1/2}\|S\|^{1/2}_{B\otimes_{\min}\Lambda_{2m}}\|f\dot{\otimes}\bar{f}\|^{1/2}_{B\otimes_{\min}\Lambda_{2m}}.$$

Therefore, again setting $x = \|f\dot{\otimes}\bar{f}\|_{B\otimes_{\min}\Lambda_{2m}}$ and $y = \|S\|_{B\otimes_{\min}\Lambda_{2m}}$ we find

$$|x-y| \leq 2Cm^{1/2}\sqrt{xy},$$

and we conclude as before that $x$ and $y$ are comparable, so that (4.1) holds for $p = 4m$. □

The next corollary is now immediate from the dyadic case. However, we will later show that it is valid for any $p$ in $2\mathbb{N}$ (see Corollary 11.3).

COROLLARY 4.4. *Assume $p \geq 2$ of the form $p = 2^k$ for some $k \geq 1$. If $\Omega = \{-1,+1\}^{\mathbb{N}}$, the closed span of the coordinates $(\varepsilon_n)$ (or equivalently, of the Rademacher functions on $\Omega = [0,1]$) in $\Lambda_p$ is completely isomorphic to the space $OH$, i.e. to $\ell_2$ equipped with the o.s.s. of OH. Moreover, the orthogonal projection $P$ onto it is c.b. on $\Lambda_p$.*

*Proof.* Let $f = \sum x_n\varepsilon_n$ $(x_n \in B(H))$. By (4.1) , $\|f\|_{(p)}$ is equivalent to $\|\sum x_n \otimes \bar{x}_n\|^{1/2}$ and the latter is equal to the norm of $\sum e_n \otimes x_n$ in $OH \otimes_{\min} B$ where $(e_n)$ is any orthonormal basis of $OH$. Thus the closed span of $(\varepsilon_n)$ in $\Lambda_p$ is isomorphic to $OH$. We skip the proof of the complementation because we give the details for that in the proof of Proposition 11.1 below. □

*Proof of the right hand side of* (4.1). We will use induction on $k$. The case $k = 1$ is clear. Assume that the right hand side of (4.1) holds for $p = m$, we will show it for $p = 2m$. With the preceding notation, recall $g = a + b$ and hence our assumption yields

$$\|g\|_{B\otimes\overline{B}\otimes\Lambda_m} \leq \|a\|_{B\otimes\overline{B}\otimes\Lambda_m} + \|b\|_{B\otimes\overline{B}\otimes\Lambda_m} \leq C_2(m)(\|S(a)\|^{1/2}_{\bullet} + \|S(b)\|^{1/2}_{\bullet}) \tag{4.7}$$

where the dot stands for $B \otimes_{\min} \overline{B} \otimes_{\min} \overline{B} \otimes_{\min} B \otimes_{\min} \Lambda_{m/2}$. Since by (2.7)

$$\bar{f}_{n-1}\dot{\otimes}f_{n-1} \prec 2(\bar{f}_n\dot{\otimes}f_n + \bar{d}_n\dot{\otimes}d_n)$$

we have using (2.5) and (2.6) (in a suitable permutation)

$$\|S(a)\|^{1/2}_{\bullet} = \left\|\sum d_n\dot{\otimes}\bar{f}_{n-1}\dot{\otimes}\bar{d}_n\dot{\otimes}f_{n-1}\right\|^{1/2}_{\bullet} \leq \mathrm{I} + \mathrm{II}$$

where

$$2^{-1/2}\mathrm{I} = \left\|\sum d_n\dot{\otimes}\bar{f}_n\dot{\otimes}\bar{d}_n\dot{\otimes}f_n\right\|^{\frac{1}{2}}_{\bullet} \quad \text{and} \quad 2^{-1/2}\mathrm{II} = \left\|\sum d_n\dot{\otimes}\bar{d}_n\dot{\otimes}\bar{d}_n\dot{\otimes}d_n\right\|^{\frac{1}{2}}_{\bullet}.$$

Note that for any $F$ in $B \otimes \overline{B} \otimes \overline{B} \otimes B \otimes \Lambda_{m/2}$ we have $\|F\|_{\bullet} = \|\mathbb{E}((F\dot{\otimes}\overline{F})^{\dot{\otimes}m/4})\|^{\frac{2}{m}}$.

Recall that, by (2.10), $f_n\dot\otimes\bar f_n \prec \mathbb{E}_n(f\dot\otimes\bar f)$. Thus we have by (2.5) (2.6) and (2.15)

$$2^{-1/2}I \le \left\|\mathbb{E}\left(\left(\sum d_n\dot\otimes\bar d_n\dot\otimes\mathbb{E}_n(f\dot\otimes\bar f)\right)^{\dot\otimes m/2}\right)\right\|^{\frac{1}{m}}$$
$$= \left\|\mathbb{E}\left(\left(\sum \mathbb{E}_n(d_n\dot\otimes\bar d_n\dot\otimes f\dot\otimes\bar f)\right)^{\dot\otimes m/2}\right)\right\|^{\frac{1}{m}}$$

and hence by (4.5) and (3.5) (or actually (3.3))

$$\le (m/2)^{1/2}\|\mathbb{E}((S\dot\otimes f\dot\otimes\bar f)^{\dot\otimes m/2})\|^{\frac{1}{m}}$$
$$\le (m/2)^{1/2}\|\mathbb{E}(S^{\dot\otimes m})\|^{\frac{1}{2m}}\|\mathbb{E}((f\dot\otimes\bar f)^{\dot\otimes m})\|^{\frac{1}{2m}}.$$

Moreover, recalling (2.4), we have obviously $0 \prec d_n\otimes\bar d_n\otimes\bar d_k\otimes d_k$ for all $n,k$ and hence $\sum d_n\dot\otimes\bar d_n\dot\otimes\bar d_n\dot\otimes d_n \prec S\dot\otimes\overline{S}$. Therefore, again by (2.6) and (2.15)

$$2^{-1/2}\mathrm{II} \le \|\mathbb{E}((S\dot\otimes\overline{S}\dot\otimes\overline{S}\dot\otimes S)^{m/4})\|^{1/m} = \|\mathbb{E}(S^{\dot\otimes m})\|^{\frac{1}{m}}.$$

Let $x = \|\mathbb{E}(S^{\dot\otimes m})\|^{\frac{1}{m}}$ and $y = \|\mathbb{E}(f\dot\otimes\bar f)^{\dot\otimes m}\|^{\frac{1}{m}}$. This yields

$$\|S(a)\|_{\bullet}^{\frac{1}{2}} \le \sqrt{m}\sqrt{xy} + \sqrt{2}x$$

and a similar bound for $S(b)$. Thus we obtain

$$\|g\|_{B\otimes\overline{B}\otimes\Lambda_m} \le 2C_2(m)(\sqrt{m}\sqrt{xy} + \sqrt{2}x).$$

Since $g = f\dot\otimes\bar f - S$ we have

$$\big|\|f\dot\otimes\bar f\|_{B\otimes\overline{B}\otimes\Lambda_m} - \|S\|_{B\otimes\overline{B}\otimes\Lambda_m}\big| \le \|g\|_{B\otimes\overline{B}\otimes\Lambda_m}$$

and hence we obtain

$$|y-x| \le 2C_2(m)(\sqrt{m}\sqrt{xy} + \sqrt{2}x)$$

From the latter it is clear that there is a constant $C_2(2m)$ such that

$$\sqrt{y} \le C_2(2m)\sqrt{x}$$

and this is the right hand side of (4.1) for $p = 2m$, □

To prove the general case of both sides of (4.1), the following Lemma will be crucial. We will use this only for $m = 1$, but the inductive argument curiously requires to prove it for all dyadic $m$.

Lemma 4.5. *Let $f \in B(H)\otimes L_{4mp}$ be a test function. As before we set $f_n = \mathbb{E}_n f$ and $d_n = f_n - f_{n-1}$ for all $n \ge 1$. Let $p = 2^k$ for some integer $k \ge 0$. Then, for any integer $m \ge 1$ of the form $m = 2^\ell$ for some $\ell \ge 0$, there is a constant $C = C(m,p)$ such that*

$$\left\|\sum\nolimits_1^\infty (d_n\dot\otimes\bar d_n)^{\dot\otimes m}\dot\otimes(f_{n-1}\dot\otimes\bar f_{n-1})^{\dot\otimes m}\right\|_{(p)} \le C\|S\|^m_{(2mp)}\|f\|^{2m}_{(4mp)}. \tag{4.8}$$

*Proof.* We use induction on $k$ starting from $p=1$. We may assume $d_0=0$ for simplicity. Let

$$I(m,p)=\|\sum_1^\infty (d_n\dot\otimes\bar d_n)^{\dot\otimes m}\dot\otimes(f_{n-1}\dot\otimes\bar f_{n-1})^{\dot\otimes m}\|_{(p)}.$$

By (2.17) and (2.5) (and by the self-adjointness of $\mathbb{E}_{n-1}$) we have

$$\begin{aligned} I(m,1) &\le \left\|\mathbb{E}\left(\sum\nolimits_1^\infty (d_n\dot\otimes\bar d_n)^{\dot\otimes m}\dot\otimes\mathbb{E}_{n-1}((f\dot\otimes\bar f)^{\dot\otimes m})\right)\right\| \\ &= \left\|\mathbb{E}\left(\sum\nolimits_1^\infty \mathbb{E}_{n-1}((d_n\dot\otimes\bar d_n)^{\dot\otimes m})\dot\otimes(f\dot\otimes\bar f)^{\dot\otimes m}\right)\right\|, \end{aligned}$$

and hence by (3.3)

$$\le \|\mathbb{E}(\sigma_m\dot\otimes\bar\sigma_m)\|^{1/2}\|\mathbb{E}((f\dot\otimes\bar f)^{\dot\otimes 2m})\|^{1/2}$$

where we have set

$$\sigma_m=\sum\nolimits_1^\infty \mathbb{E}_{n-1}((d_n\dot\otimes\bar d_n)^{\dot\otimes m}).$$

Note that by Lemma 4.2

$$\|\sigma_m\|_{(2)}\le 2\|\sum\nolimits_1^\infty (d_n\dot\otimes\bar d_n)^{\dot\otimes m}\|_{(2)}$$

but obviously (recalling (2.4)) $\sum(d_n\dot\otimes\bar d_n)^{\dot\otimes m}\prec(\sum d_n\dot\otimes\bar d_n)^{\dot\otimes m}$ and hence (recalling (2.15))

$$\left(\sum(d_n\dot\otimes\bar d_n)^{\dot\otimes m}\right)^{\dot\otimes 2}\prec S^{\dot\otimes 2m} \tag{4.9}$$

so we obtain

$$\|\mathbb{E}(\sigma_m\dot\otimes\bar\sigma_m)\|^{1/2}\le 2\|S\|_{(2m)}^m.$$

Thus we find

$$I(m,1)\le 2\|S\|_{(2m)}^m\|f\|_{(4m)}^{2m},$$

so that (4.8) holds for $p=1$ and any $m\ge 1$ with $C(m,1)=2$.
Let us now denote by $(4.8)_p$ the inequality (4.8) meant for a given fixed $p$ but for any $m\ge1$. We will show that for any $p\ge 2$

$$(4.8)_{p/2}\Rightarrow(4.8)_p.$$

Assuming that $m\ge1$ is fixed, let $x_n=(d_n\dot\otimes\bar d_n)^{\dot\otimes m}$ and $y_n=(f_{n-1}\dot\otimes\bar f_{n-1})^{\dot\otimes m}$. We write

$$\sum x_n\dot\otimes y_n=a+b$$

$$\text{with}\qquad a=\sum\mathbb{E}_{n-1}(x_n)\dot\otimes y_n\quad\text{and}\quad b=\sum(x_n-\mathbb{E}_{n-1}(x_n))\dot\otimes y_n.$$

We have $I(m,p) = \| \sum x_n \dot{\otimes} y_n \|_{(p)}$ and hence

$$I(m,p) \le \|a\|_{(p)} + \|b\|_{(p)}, \tag{4.10}$$

so it suffices to majorize $a$ and $b$ separately. We have by (2.17)

$$a \prec \sum \mathbb{E}_{n-1}(x_n) \dot{\otimes} \mathbb{E}_{n-1}((f \dot{\otimes} \bar{f})^{\dot{\otimes} m}) = \sum \mathbb{E}_{n-1}(\mathbb{E}_{n-1}(x_n) \dot{\otimes} (f \dot{\otimes} \bar{f})^{\dot{\otimes} m}))$$

and hence by (4.5) and (3.3)

$$\begin{aligned} \|a\|_{(p)} &\le p \left\| \sum \mathbb{E}_{n-1}(x_n) \dot{\otimes} (f \dot{\otimes} \bar{f})^{\dot{\otimes} m} \right\|_{(p)} \\ &\le p \|\sigma_m \dot{\otimes} (f \dot{\otimes} \bar{f})^{\dot{\otimes} m}\|_{(p)} \\ &= p \|\sigma_m^{\dot{\otimes} p} \dot{\otimes} (f \dot{\otimes} \bar{f})^{\dot{\otimes} mp}\|_{(1)}^{1/p} \\ &\le p \|\sigma_m^{\dot{\otimes} p}\|_{(2)}^{\frac{1}{p}} \|(f \dot{\otimes} \bar{f})^{\dot{\otimes} mp}\|_{(2)}^{\frac{1}{p}} \\ &= p \|\sigma_m\|_{(2p)} \|f\|_{(4mp)}^{2m}. \end{aligned}$$

But now by (4.5) again

$$\|\sigma_m\|_{(2p)} \le 2p \left\| \sum (d_n \dot{\otimes} \bar{d}_n)^{\dot{\otimes} m} \right\|_{(2p)}$$

and hence by (4.9)

$$\|\sigma_m\|_{(2p)} \le 2p \|S\|_{(2mp)}^m.$$

Thus we obtain

$$\|a\|_{(p)} \le p(2p) \|S\|_{(2mp)}^m \|f\|_{(4mp)}^{2m}. \tag{4.11}$$

We now turn to $b$. Note that since $y_n$ is "predictable" $\{(x_n - \mathbb{E}_{n-1}(x_n)) \dot{\otimes} y_n\}$ is a martingale difference sequence. We will apply the right hand side of (4.1) to $b$. Note that

$$S(b) \approx \sum (x_n - \mathbb{E}_{n-1}(x_n)) \dot{\otimes} \overline{(x_n - \mathbb{E}_{n-1}(x_n))} \dot{\otimes} y_n \dot{\otimes} \overline{y}_n,$$

and hence by (2.7)

$$\frac{1}{2} S(b) \prec \sum x_n \dot{\otimes} \overline{x}_n \dot{\otimes} y_n \dot{\otimes} \overline{y}_n + \sum \mathbb{E}_{n-1}(x_n) \dot{\otimes} \mathbb{E}_{n-1}(\overline{x}_n) \dot{\otimes} y_n \dot{\otimes} \overline{y}_n.$$

By (2.10) we get (since $y_n$ is predictable)

$$\frac{1}{2} S(b) \prec \sum x_n \dot{\otimes} \overline{x}_n \dot{\otimes} y_n \dot{\otimes} \overline{y}_n + \sum \mathbb{E}_{n-1}(x_n \dot{\otimes} \overline{x}_n \dot{\otimes} y_n \dot{\otimes} \overline{y}_n)$$

and hence

$$\frac{1}{2} \|S(b)\|_{(p/2)} \le \left\| \sum x_n \dot{\otimes} \overline{x}_n \dot{\otimes} y_n \dot{\otimes} \overline{y}_n \right\|_{(p/2)} + \left\| \sum \mathbb{E}_{n-1}(x_n \dot{\otimes} \overline{x}_n \dot{\otimes} y_n \dot{\otimes} \overline{y}_n) \right\|_{(p/2)}.$$

By (4.5) this yields

$$\|S(b)\|_{(p/2)} \leq 2(1+(p/2)) \left\| \sum x_n \dot{\otimes} \overline{x}_n \dot{\otimes} y_n \dot{\otimes} \overline{y}_n \right\|_{(p/2)}.$$

But since

$$\sum x_n \dot{\otimes} \overline{x}_n \dot{\otimes} y_n \dot{\otimes} \overline{y}_n \approx \sum (d_n \dot{\otimes} \bar{d}_n)^{\dot{\otimes} 2m} \dot{\otimes} (f_{n-1} \dot{\otimes} \bar{f}_{n-1})^{\dot{\otimes} 2m}$$

we may use the induction hypothesis $(4.8)_{p/2}$ (with $m$ replaced by $2m$) and we obtain

$$\|S(b)\|_{(p/2)} \leq 2(1+(p/2))C(2m,p/2)\|S\|^{2m}_{(2mp)}\|f\|^{4m}_{(4mp)}.$$

By the right hand side of $(4.1)_p$ we then find

$$\begin{aligned} \|b\|_{(p)} &\leq C_2(p)\|S(b)\|^{1/2}_{(p/2)} \\ &\leq C'(m,p)\|S\|^{m}_{(2mp)}\|f\|^{2m}_{(4mp)} \end{aligned}$$

for some constant $C'(m,p)$. Thus we conclude by (4.10) and (4.11)

$$I(m,p) \leq (p(2p) + C'(m,p))\|S\|^{m}_{(2mp)}\|f\|^{2m}_{4mp}.$$

In other words we obtain $(4.8)_p$. This completes the proof of $(4.8)_p$ for $p=2^k$ by induction on $k$. □

*Proof of Theorem 4.1 (General case).* We will show that $(4.1)_p \Rightarrow (4.1)_{2p}$. We again start from

$$f\dot{\otimes}\bar{f} - S = a + b$$

where $a = \sum d_n \dot{\otimes} \bar{f}_{n-1}$ and $b = \sum f_{n-1} \dot{\otimes} \bar{d}_n$. By the right hand side of $(4.1)_p$ we have

$$\|f\dot{\otimes}\bar{f} - S\|_{(p)} \leq 2C_2(p) \left\| \sum d_n \dot{\otimes} \bar{d}_n \dot{\otimes} f_{n-1} \dot{\otimes} \bar{f}_{n-1} \right\|^{1/2}_{(p/2)}$$

and hence by (4.8)

$$\leq 2C_2(p)C(1,p/2)^{1/2}\|S\|^{1/2}_{(p)}\|f\|_{(2p)}.$$

Thus we find a fortiori setting $C'' = 2C_2(p)C(1,p/2)^{1/2}$

$$\left| \|f\dot{\otimes}\bar{f}\|_{(p)} - \|S\|_{(p)} \right| \leq C''\|S\|^{1/2}_{(p)}\|f\|_{(2p)}.$$

Thus setting again $x = \|f\|_{(2p)}$, $y = \|S\|^{1/2}_{(p)}$ we find

$$|x^2 - y^2| \leq C''xy$$

and we conclude that $x$ and $y$ must be equivalent quantities, or equivalently that $(4.1)_{2p}$ holds. By induction this completes the proof. □

## 5 Burkholder-Rosenthal inequality

Let $2 < p < \infty$ be fixed. The usual form of the Burkholder-Rosenthal inequality expresses the equivalence, for scalar valued martingales, of $\| \sum d_n \|_p$ and

$$BR_\infty = \|\sigma\|_p + \| \sup |d_n| \|_p. \tag{5.1}$$

It is easy to deduce from that the equivalence of that same norm with

$$BR_q = \|\sigma\|_p + \left\| \left( \sum |d_n|^q \right)^{1/q} \right\|_p \tag{5.2}$$

for any $q$ such that $2 < q \leq \infty$.
Indeed, we have obviously $BR_\infty \leq BR_q$. Conversely, using (here $\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{\infty}$)

$$\left\| \left( \sum |d_n|^q \right)^{1/q} \right\|_p \leq \|S\|_p^{1-\theta} \| \sup |d_n| \|_p^\theta$$

and the equivalence $\|S\|_p \simeq \| \sum d_n \|_p$, one can easily deduce that there is a constant $C'$ such that

$$BR_q \leq C' \| \sum d_n \|_p^{1-\theta} BR_\infty^\theta.$$

Thus an inequality of the form

$$\left\| \sum d_n \right\|_p \leq C BR_\infty$$

implies "automatically"

$$BR_\infty \leq BR_q \leq C' C^{1-\theta} BR_\infty.$$

Similarly,

$$\left\| \sum d_n \right\|_p \leq C BR_q \Rightarrow \left\| \sum d_n \right\|_p \leq CC' \| \sum d_n \|_p^{1-\theta} BR_\infty^\theta \Rightarrow$$
$$\Rightarrow \left\| \sum d_n \right\|_p \leq (CC')^{1/\theta} BR_\infty.$$

Thus, modulo simple manipulations, the Burkholder-Rosenthal inequality reduces to the equivalence for some $q$ such that $2 < q \leq \infty$ of $\| \sum d_n \|_p$ and $BR_q$.
Note that the one sided inequality expressing that $BR_q = \|\sigma\|_p + \|(\sum |d_n|^q)^{1/q}\|_p$ is dominated by $\| \sum d_n \|_p$ reduces obviously to

$$\|\sigma\|_p \leq C \left\| \sum |d_n|^2 \right\|_{p/2}^{1/2}$$

that holds for $p \geq 2$ by Burkholder–Davis–Gundy dualization of Doob's inequality. Therefore, the novelty of the Burkholder-Rosenthal inequality is the fact that there is a constant $C'''$ such that

$$\|\sum d_n\|_p \leq C''' B R_q.$$

In the original Rosenthal inequality, restricted to sums of independent $d_n$'s, or in the non-commutative version of [17, 19], the value $q = p$ is the most interesting choice. In the inequalities below, for $p = 2^k \geq 4$, we will work with $q = 4$.

We will use the following extension of (3.3).

PROPOSITION 5.1. *For any integer $m \geq 1$ and finite sequences $(a_k)$, $(b_k)$ in $B(H) \otimes L_{2m}$ we have*

$$\left\|\mathbb{E}\left(\left(\sum a_k \dot{\otimes} b_k\right)^{\dot{\otimes} m}\right)\right\| \leq \left\|\mathbb{E}\left(\left(\sum a_k \dot{\otimes} \bar{a}_k\right)^{\dot{\otimes} m}\right)\right\|^{1/2} \left\|\mathbb{E}\left(\left(\sum b_k \dot{\otimes} \bar{b}_k\right)^{\dot{\otimes} m}\right)\right\|^{1/2}. \tag{5.3}$$

*More generally, consider finite sequences $(a_k^{(j)}), (b_k^{(j)})$ in $B(H) \otimes L_{2m}$ for $j = 1, \ldots, m$.*
*Let $T_j = \sum_k a_k^{(j)} \dot{\otimes} b_k^{(j)}$ and let $\alpha_j = \sum_k a_k^{(j)} \otimes \overline{a_k^{(j)}}$ and $\beta_j = \sum_k b_k^{(j)} \otimes \overline{b_k^{(j)}}$. We have then*

$$\|\mathbb{E}(T_1 \dot{\otimes} \cdots \dot{\otimes} T_m)\| \leq \|\mathbb{E}(\alpha_1 \dot{\otimes} \cdots \dot{\otimes} \alpha_m)\|^{1/2} \|\mathbb{E}(\beta_1 \otimes \cdots \otimes \beta_m)\|^{1/2}. \tag{5.4}$$

*Proof.* Up to permutation, $\mathbb{E}((\sum a_k \dot{\otimes} b_k)^{\dot{\otimes} m})$ is the same as

$$\mathbb{E}\left(\sum_{k(1),\ldots,k(m)} a_{k(1)} \dot{\otimes} \cdots \dot{\otimes} a_{k(m)} \dot{\otimes} b_{k(1)} \dot{\otimes} \cdots \dot{\otimes} b_{k(m)}\right).$$

Therefore, by (3.3) we have

$$\begin{aligned}
\left\|\mathbb{E}\left(\left(\sum a_k \dot{\otimes} b_k\right)^{\dot{\otimes} m}\right)\right\| &\leq \left\|\mathbb{E} \sum_{k(1),\ldots,k(m)} a_{k(1)} \ldots a_{k(m)} \dot{\otimes} \bar{a}_{k(1)} \ldots \bar{a}_{k(m)}\right\|^{\frac{1}{2}} \\
&\quad \times \left\|\mathbb{E} \sum_{k(1),\ldots,k(m)} b_{k(1)} \ldots b_{k(m)} \dot{\otimes} \bar{b}_{k(1)} \ldots \bar{b}_{k(m)}\right\|^{\frac{1}{2}} \\
&= \left\|\mathbb{E}\left(\left(\sum a_k \dot{\otimes} \bar{a}_k\right)^{\dot{\otimes} m}\right)\right\|^{\frac{1}{2}} \left\|\mathbb{E}\left(\left(\sum b_k \dot{\otimes} \bar{b}_k\right)^{\dot{\otimes} m}\right)\right\|^{\frac{1}{2}}.
\end{aligned}$$

Up to permutation $T_1 \dot{\otimes} \cdots \dot{\otimes} T_m$ is the same as

$$\sum_{k(1),\ldots,k(m)} a_{k(1)}^{(1)} \dot{\otimes} \cdots \dot{\otimes} a_{k(m)}^{(m)} \dot{\otimes} b_{k(1)}^{(1)} \dot{\otimes} \cdots \dot{\otimes} b_{k(m)}^{(m)},$$

which can be written as $\sum_k a_k \dot{\otimes} b_k$ with $k = (k(1), \ldots, k(m))$, $a_k = a^{(1)}_{k(1)} \dot{\otimes} \cdots \dot{\otimes} a^{(m)}_{k(m)}$ and $b_k = b^{(1)}_{k(1)} \dot{\otimes} \cdots \dot{\otimes} b^{(m)}_{k(m)}$. Therefore (5.4) follows from the $m = 1$ case of (5.3).

□

*Remark* 5.2. Let $H = \ell_2$ and $B = B(H)$. The preceding Proposition shows that

$$\left\| \sum a_k \dot{\otimes} \bar{a}_k \right\|^{\frac{1}{2}}_{(m)} = \sup \left\{ \left\| \sum a_k \dot{\otimes} b_k \right\|_{(m)} \right\} \tag{5.5}$$

where the supremum runs over the set $\mathcal{D}$ of all finite sequences $(b_k)$ in $B \otimes L_{2m}$ such that $\| \sum b_k \dot{\otimes} b_k \|_{(m)} \leq 1$. (Indeed the sup is attained for $b_k = \bar{a}_k$, suitably normalized.) Thus (5.5) allows us to define an o.s.s. on the space $L_{2m}(\Omega, \mu; \ell_2)$, corresponding to "$\Lambda_{2m}$ with values in $OH$". Indeed, we can proceed as before for $\Lambda_p$: we consider the subspace $E_0 \subset L_{2m} \otimes \ell_2$ formed of all finite sums $\sum a_k \otimes e_k$ $(a_k \in L_{2m})$ and we define

$$J\colon\ E_0 \longrightarrow \bigoplus_{(b_k) \in \mathcal{D}} (\Lambda_m \otimes_{\min} B)$$

by

$$J\left( \sum a_k \otimes e_k \right) = \bigoplus_{(b_k) \in \mathcal{D}} \sum a_k \dot{\otimes} b_k.$$

This produces an o.s.s. on $L_{2m}(\mu; \ell_2)$. It is easy to see that if $a \in L_{2m}$ is fixed in the unit sphere, the restriction of $J$ to $a \otimes \ell_2$ induces on $\ell_2$ the o.s.s. of $OH$ while if $x \in \ell_2$ is fixed in the unit sphere, restricting $J$ to $L_{2m} \otimes x$ induces on $L_{2m}$ the o.s.s. of $\Lambda_{2m}$.
Note in passing that, in sharp contrast with [23], except for the preceding special case, we do not have any reasonable definition to propose for the "vector valued" analogue of the $\Lambda_p$ spaces.

As a consequence we find an analogue of Stein's inequality (here we could obviously replace $\mathbb{E}_{n-1}$ by $\mathbb{E}_n$):

Corollary 5.3. *Let $x_n$ be an arbitrary finite sequence in $B(H) \otimes L_{4m}$. Let $v = \sum \mathbb{E}_{n-1}(x_n \dot{\otimes} \bar{x}_n) \dot{\otimes} \mathbb{E}_{n-1}(\bar{x}_n \dot{\otimes} x_n)$ and $\delta = \sum x_n \dot{\otimes} \bar{x}_n \dot{\otimes} \bar{x}_n \dot{\otimes} x_n$. Then for any integer $m \geq 1$*

$$\|\mathbb{E}(v^{\dot{\otimes} m})\| \leq m^m \|\mathbb{E}(\delta^{\dot{\otimes} m})\|. \tag{5.6}$$

*Proof.* Let $w = \sum \mathbb{E}_{n-1}(x_n \dot{\otimes} \bar{x}_n \dot{\otimes} \bar{x}_n \dot{\otimes} x_n)$. By (2.10) we have $v \prec w$. Then by (2.15), (2.11), Lemma 2.3 and (4.5), we have

$$\|\mathbb{E}(v^{\dot{\otimes} m})\| \leq \|\mathbb{E}(w^{\dot{\otimes} m})\| \leq m^m \|\mathbb{E}(\delta^{\dot{\otimes} m})\|.$$

□

LEMMA 5.4. *Let $p = 2^k \geq 4$ as before. Let $\delta = \sum d_n \dot{\otimes} \bar{d}_n \dot{\otimes} \bar{d}_n \dot{\otimes} d_n$. There is a constant $C_4(p)$ such that*

$$\|\mathbb{E}(S^{\dot{\otimes} p/2})\|^{1/p} \leq C_4(p)[\|\mathbb{E}(\sigma^{\dot{\otimes} p/2})\|^{\frac{1}{p}} + \|\mathbb{E}(\delta^{\dot{\otimes} p/4})\|^{\frac{1}{p}}]. \tag{5.7}$$

*Proof.* Note that

$$S - \sigma = \sum dc_n$$

where $c_n = d_n \dot{\otimes} \bar{d}_n - \mathbb{E}_{n-1}(d_n \dot{\otimes} \bar{d}_n)$. Thus by the right hand side of (4.1) we have

$$\|S - \sigma\|_{B \otimes_{\min} \overline{B} \otimes_{\min} \Lambda_{p/2}} \leq C_2(p/2) \|S(c)\|^{1/2}_{B \otimes_{\min} \overline{B} \otimes_{\min} \overline{B} \otimes_{\min} B \otimes_{\min} \Lambda_{p/4}}.$$

By (2.7)

$$\frac{1}{2} dc_n \dot{\otimes} d\bar{c}_n \prec d_n \dot{\otimes} \bar{d}_n \dot{\otimes} \bar{d}_n \dot{\otimes} d_n + \mathbb{E}_{n-1}(d_n \dot{\otimes} \bar{d}_n) \dot{\otimes} \mathbb{E}_{n-1}(\bar{d}_n \dot{\otimes} d_n)$$

therefore

$$\frac{1}{2} S(c) \prec \delta + v,$$

where we now set $v = \sum \mathbb{E}_{n-1}(d_n \dot{\otimes} \bar{d}_n) \dot{\otimes} \mathbb{E}_{n-1}(\bar{d}_n \dot{\otimes} d_n)$. Thus we find

$$\begin{aligned} \Big| \|S\|_{B \otimes_{\min} \overline{B} \otimes_{\min} \Lambda_{p/2}} - \|\sigma\|_{B \otimes_{\min} \overline{B} \otimes_{\min} \Lambda_{p/2}} \Big| &\leq \|S - \sigma\|_{B \otimes_{\min} \overline{B} \otimes_{\min} \Lambda_{p/2}} \\ &\leq C_2(p/2) \sqrt{2} (\|\delta\|_\bullet^{\frac{1}{2}} + \|v\|_\bullet^{\frac{1}{2}}) \end{aligned}$$

where $\| \ \|_\bullet$ is the norm in $B \otimes_{\min} \overline{B} \otimes_{\min} \overline{B} \otimes_{\min} B \otimes_{\min} \Lambda_{p/4}$. By (5.6) we have

$$\|v\|_\bullet \leq (p/4) \|\delta\|_\bullet$$

and hence

$$\|S\|_{B \otimes \overline{B} \otimes \Lambda_{p/2}} \leq \|\sigma\|_{B \otimes \overline{B} \otimes \Lambda_{p/2}} + \ C_2(p/2) \sqrt{2} (1 + (p/4)^{\frac{1}{2}}) \|\delta\|_\bullet^{\frac{1}{2}}.$$

Taking the square root of the last inequality we obtain (5.7). $\square$

We now give a version (corresponding to $BR_q$ with $q = 4$) for $\Lambda_p$ of the Burkholder-Rosenthal inequality :

THEOREM 5.5. *For any $p \geq 4$ of the form $p = 2^k$ for some $k \geq 1$ there are positive constants $C_1'(p)$, and $C_2'(p)$ such that for any test function $f$ in $B \otimes L_p$ we have*

$$C_1'(p)^{-1} [f]_p \leq \|f\|_{B \otimes_{\min} \Lambda_p} \leq C_2'(p) [f]_p \tag{5.8}$$

*where*

$$[f]_p = \|\sigma(f)\|^{1/2}_{B \otimes_{\min} \overline{B} \otimes_{\min} \Lambda_{p/2}} + \left\| \mathbb{E}(\sum d_n \dot{\otimes} \bar{d}_n \dot{\otimes} d_n \dot{\otimes} \bar{d}_n)^{\dot{\otimes} p/4} \right\|^{1/p}.$$

*Proof.* Note that $\overline{S}$ and $S$ are the same after a transposition of the two factors, thus the same is true for $S\dot{\otimes}\overline{S}$ and $S^{\dot{\otimes}2}$, and we have

$$\|\mathbb{E}(S\dot{\otimes}\overline{S})\| = \|\mathbb{E}(S^{\dot{\otimes}2})\|$$

and similarly for any even $m \geq 1$

$$\mathbb{E}((S\dot{\otimes}\overline{S})^{\dot{\otimes}m/2})\| = \|\mathbb{E}(S^{\dot{\otimes}m})\|.$$

Recall that in a suitable permutation we may write $0 \prec d_n\dot{\otimes}\bar{d}_n\dot{\otimes}\bar{d}_k\dot{\otimes}d_k$ for all $n, k$ and hence

$$\sum d_n\dot{\otimes}\bar{d}_n\dot{\otimes}\bar{d}_n\dot{\otimes}d_n \prec \sum\nolimits_{n,k} d_n\dot{\otimes}\bar{d}_n\dot{\otimes}\bar{d}_k\dot{\otimes}d_k = S\dot{\otimes}\overline{S},$$

and hence for any even integer $m$

$$(\sum d_n\dot{\otimes}\bar{d}_n\dot{\otimes}\bar{d}_n\dot{\otimes}d_n)^{\dot{\otimes}m/2} \prec S^{\dot{\otimes}m}.$$

Therefore

$$\left\|\mathbb{E}\left(\sum d_n\dot{\otimes}\bar{d}_n\dot{\otimes}d_n\dot{\otimes}\bar{d}_n\right)^{\dot{\otimes}m/2}\right\| \leq \|\mathbb{E}(S^{\dot{\otimes}m})\|. \tag{5.9}$$

Let $\sigma = \sigma(f)$. Now if $p = 2m$, (4.5) implies

$$\|\sigma\|^{1/2}_{B\otimes_{\min}\overline{B}\otimes_{\min}\Lambda_{p/2}} = \|\mathbb{E}(\sigma^{\dot{\otimes}m})\|^{\frac{1}{2m}} \leq m^{1/2}\|\mathbb{E}(S^{\dot{\otimes}m})\|^{\frac{1}{2m}}, \tag{5.10}$$

and hence by (5.9) and (5.10)

$$\begin{aligned}[f]_p &\leq (m^{1/2}+1)\|\mathbb{E}(S^{\dot{\otimes}m})\|^{1/2m} \\ &= (m^{1/2}+1)\|S\|^{1/2}_{B\otimes_{\min}\overline{B}\otimes_{\min}\Lambda_m}.\end{aligned}$$

Thus the left hand side of (5.8) follows from (4.1). Since the converse inequality follows from Lemma 5.4 and (4.1), this completes the proof. □

## 6 Hilbert transform

Consider the Hilbert transform on $L_p(\mathbb{T}, dm)$. We will show that this defines a completely bounded operator on $\Lambda_p(\mathbb{T}, m)$ again for $p \geq 2$ of the form $p = 2^k$ with $k \in \mathbb{N}$. The proof is modeled on Marcel Riesz's proof as presented in Zygmund's classical treatise on trigonometric series. One of the first references using this trick in a broader context is Cotlar's paper [10]. Let $f$ be a trigonometric polynomial with coefficients in $B(H)$, i.e. $f = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{int}$ with $\hat{f}\colon\ \mathbb{Z} \to B(H)$ finitely supported. The Hilbert transform $Tf$ is defined by

$$Tf = \sum\nolimits_{n\in\mathbb{Z}} \varphi(n)\hat{f}(n)e^{int} \tag{6.1}$$

where $\varphi(0) = 0$ and
$$\varphi(n) = -i \text{ sign}(n). \qquad \forall n \in \mathbb{Z}$$

Note that $T^2 = -id$ on the subspace $\{f \mid \hat{f}(0) = 0\}$. We will use the following classical identity valid for any pair $f, g$ of complex valued trigonometric polynomials
$$T(fg - (Tf)(Tg)) = fTg + (Tf)g. \tag{6.2}$$

This can be checked easily as a property of $\varphi$ since it reduces to the case $f = z^n$, $g = z^m$ $(n, m \in \mathbb{Z})$. A less pedestrian approach is to recall that if $f$ is *real valued*, $Tf$ is characterized as the unique real valued $v$, the "conjugate function", actually here also a trigonometric polynomial, such that $\hat{v}(0) = 0$ and $z \mapsto f(z) + iv(z)$ is the boundary value of an analytic function (actually a polynomial in $z$) inside the unit disc $D$. Then (6.2) boils down to the observation that since $(f + iTf)(g + iTg)$ is the product of two analytic functions on $D$, $f(Tg) + (Tf)g$ must be the "conjugate" of $fg - (Tf)(Tg)$. The complex case follows from the real one: for a complex valued $f$, we define $Tf = T(\Re(f)) + iT(\Im(f))$ and (6.2) remains valid. From (6.2) in the $\mathbb{C}$-valued case, it is immediate to deduce that for any pair $f, g$ of $B(H)$-valued trigonometric polynomials we have
$$T(f\dot{\otimes}g - (Tf)\dot{\otimes}(Tg)) = f\dot{\otimes}(Tg) + (Tf)\dot{\otimes}g \tag{6.3}$$

where (as before) the notation $f\dot{\otimes}g$ stands for the $B(H)\otimes B(H)$ valued function $z \to f(z) \otimes g(z)$ on $\mathbb{T}$, and where we still denote by $T$ the mapping (that should be denoted by $T \otimes I$) taking $f \otimes b$ $(f \in L_2, b \in B(H))$ to $(Tf) \otimes b$. Now, it is a simple exercise to check that for any such $f$
$$\overline{Tf} = T(\bar{f})$$

(this is an equality between two $\overline{B(H)}$ valued functions). Therefore we have also:
$$T(f \otimes \bar{g} - (Tf)\dot{\otimes}T(\bar{g})) = f \otimes T(\bar{g}) + Tf \otimes \bar{g}. \tag{6.4}$$

We can now apply the well known Riesz–Cotlar trick to our situation:

Theorem 6.1. *For any $p \geq 2$, of the form $p = 2^k$ with $k \in \mathbb{N}$, the Hilbert transform $T$ is a c.b. mapping on $\Lambda_p(\mathbb{T}, m)$.*

*Proof.* If we restrict (as we may) to functions such that $\hat{f}(0) = 0$, we have $T^2 = -id$ and hence (6.4) implies
$$Tf\dot{\otimes}T\bar{f} - f\dot{\otimes}\bar{f} = T(f\dot{\otimes}(\overline{Tf}) + (Tf)\dot{\otimes}\bar{f}). \tag{6.5}$$

We can then again use induction on $k$. Assume the result known for $p$, i.e. that there is a constant $C$ such that
$$\|Tf\|_{B(H)\otimes_{\min}\Lambda_p} \leq C\|f\|_{B(H)\otimes_{\min}\Lambda_p}.$$

We will prove that the same holds for $2p$ in place of $p$ (with a different constant). Let $B = B(H)$. By (6.5), we have

$$\|Tf\dot{\otimes}\overline{Tf} - f \otimes \bar{f}\|_{B\otimes_{\min}\overline{B}\otimes_{\min}\Lambda_p} \le 2C\|f \otimes \overline{Tf}\|_{B\otimes_{\min}\overline{B}\otimes\Lambda_p}.$$

By (3.5), this term is

$$\le 2C\|f \otimes \bar{f}\|^{1/2}_{B\otimes_{\min}\overline{B}\otimes_{\min}\Lambda_p}\|Tf \otimes \overline{Tf}\|^{1/2}_{B\otimes_{\min}\overline{B}\otimes\Lambda_p}.$$

We have

$$\|f\|^2_{B\otimes_{\min}\Lambda_{2p}} = \|f\dot{\otimes}\overline{f}\|_{B\otimes_{\min}\overline{B}\otimes_{\min}\Lambda_p}$$

and

$$\|Tf\|^2_{B\otimes_{\min}\Lambda_{2p}} = \|Tf\dot{\otimes}\overline{Tf}\|_{B\otimes_{\min}\overline{B}\otimes_{\min}\Lambda_p}.$$

Therefore, denoting this time $x = \|Tf\|_{B\otimes_{\min}\Lambda_{2p}}$ and $y = \|f\|_{B\otimes_{\min}\Lambda_{2p}}$, and using

$$|\ \|Tf\dot{\otimes}\overline{Tf}\|_{(p)} - \|f \otimes \bar{f}\|_{(p)}| \le \|Tf\dot{\otimes}\overline{Tf} - f \otimes \bar{f}\|_{(p)}$$

we find again

$$|x^2 - y^2| \le 2Cxy.$$

Thus we conclude that $x$ and $y$ are “equivalent,” completing the proof with $2p$ in place of $p$. $\square$

## 7 Comparison with $L_p$

Let $B = B(H)$ with (say) $H = \ell_2$. Let $E_1, \cdots, E_m$ and $G$ be operator spaces. Recall that an $m$-linear mapping

$$u:\ E_1 \times \cdots \times E_m \to G$$

is called (jointly) completely bounded (j.c.b. in short) if the associated $m$-linear mapping from

$$\hat{u}:\ (B \otimes_{\min} E_1) \times \cdots \times (B \otimes_{\min} E_m) \to B \otimes_{\min} \cdots \otimes_{\min} B \otimes_{\min} G$$

is bounded. We set $\|u\|_{cb} = \|\hat{u}\|$, and we say that $u$ is (jointly) completely contractive if $\|u\|_{cb} \le 1$. Note the obvious stability of these maps under composition: for instance if $F, L$ are operator spaces and if $v\colon\ G \times F \to L$ is bilinear and j.c.b. then the $(m+1)$-linear mapping $w;\ E_1 \times \cdots \times E_m \times F \to L$ defined by

$$w(x_1, \cdots, x_m, y) = v(u(x_1, \cdots, x_m), y)$$

is also j.c.b. with $\|w\|_{cb} \le \|u\|_{cb}\|v\|_{cb}$. Moreover, if in the above definition we replace $B$ by the space $K$ of compact operators on $\ell_2$, the definition and the value of $\|u\|_{cb}$ is unchanged. This allows to extend (following [23]) the complex interpolation theorem for multilinear mappings. In particular, we have

Lemma 7.1. *Let $1 \le p,q,r \le \infty$ be such that $1/p + 1/q = 1/r$. Then the pointwise product from $L_p \times L_q$ to $L_r$ is completely contractive. More generally, if $1 \le p_j \le \infty$ $(1 \le j \le N)$ are such that $\sum 1/p_j = 1/r$, the product map $L_{p_1} \times \cdots \times L_{p_N} \to L_r$ is completely contractive. In particular, if $p$ is any positive integer, the pointwise product $P_p$ from $L_p \times \cdots \times L_p$ (p-times) to $L_1$ is completely contractive.*

*Proof.* The three cases either $q = \infty, p = r$ or $p = \infty, q = r$ or $q = p', r = 1$ are obvious. By interpolation and then exchanging the roles of $p$ and $q$, this implies the general case. By the preceding remark, one can iterate and the second assertion becomes clear. □

Theorem 7.2. *Let $p = 2m$, $m \in \mathbb{N}$. The identity map $L_p \to \Lambda_p$ is completely contractive.*

*Proof.* By the preceding Lemma 7.1, $P_p\colon\ L_p \times \cdots \times L_p \to L_1$ is completely contractive. Therefore

$$\left\| \int f_1 \dot{\otimes} \cdots \dot{\otimes} f_{p/2} \dot{\otimes} \bar{f}_1 \dot{\otimes} \cdots \dot{\otimes} \bar{f}_{p/2} \right\| \le \left( \prod_1^{p/2} \|f_j\|_{B \otimes_{\min} L_p} \right)^2 .$$

Thus taking $f_1 = \cdots = f_{p/2}$ we get by (3.8)

$$\|f\|^p_{B \otimes_{\min} \Lambda_p} \le \|f\|^p_{B \otimes_{\min} L_p}$$

and we obtain $\|L_p \to \Lambda_p\|_{cb} = 1$. □

*Remark* 7.3. The preceding argument (together with Corollary 3.6) shows that the o.s.s. on $\Lambda_p$ is essentially the minimal one on $L_p$ such that $P_p\colon \Lambda_p \times \cdots \times \Lambda_p \to L_1$ is completely contractive. More precisely, assume $p \in \mathbb{N}$. Let $Q_p\colon\ L_p \times \cdots \times L_p \times \bar{L}_p \times \cdots \times \bar{L}_p$ (where $L_p$ and $\bar{L}_p$ are repeated $p/2$ times) be the $p$-linear mapping taking $(f_1, \cdots, f_{p/2}, \bar{g}_1, \cdots, \bar{g}_{p/2})$ to $\int f_1 \cdots f_{p/2} \bar{g}_1, \cdots, \bar{g}_{p/2} d\mu$. Then if $X_p$ is an o.s. isometric to $L_p$, such that $Q_p\colon X_p \times \cdots \times X_p \times \bar{X}_p \times \cdots \times \bar{X}_p \to L_1$ is completely contractive, the identity map $X_p \to \Lambda_p$ is completely contractive.
In the case of $L_p$ itself with its interpolated o.s.s. we could consider $P_p$ instead of $Q_p$ because the map $f \mapsto \bar{f}$ is a completely isometric antilinear isomorphism from $L_p$ to itself, and hence defines a completely isometric *linear* isomorphism from $L_p$ to $\bar{L}_p$. (This can be checked easily by interpolation starting from $p = \infty$ and-by duality-$p = 1$.)

This remark leads to:

Corollary 7.4. *For any integer $p \ge 1$, we have a completely contractive inclusion*

$$(\Lambda_p, L_\infty)_{1/2} \to \Lambda_{2p}.$$

*Proof.* Let $X_{2p} = (\Lambda_p, L_\infty)_{1/2}$. We will argue as in the proof of Lemma 7.1, applying complex interpolation to the product map $P_{2p}\ :\ (x_1, \cdots, x_{2p}) \mapsto x_1 \cdots x_{2p}$. Clearly, by Lemma 7.1, $P_{2p}$ is completely contractive both as a map from $(\Lambda_p)^p \times (\bar{L}_\infty)^p$ to $L_1$ and as one from $(L_\infty)^p \times (\bar{\Lambda}_p)^p$ to $L_1$. Therefore, by interpolation, $P_{2p}\colon X_{2p} \times \cdots \times X_{2p} \times \bar{X}_{2p} \times \cdots \times \bar{X}_{2p} \to L_1$ is completely contractive. The preceding remark (applied with $2p$ in place of $p$) then yields this corollary. □

*Remark* 7.5. We wish to compare here the operator spaces $L_p$ and $\Lambda_p$. We already know that they are different since the Khintchine inequalities lead to two different operator spaces in both cases, but we can give a more precise quantitative estimate.

Let us denote by $L_p^n$ the space $L_p(\Omega_n, \mu_n)$ when $\Omega_n = [1, \dots, n]$ and $\mu_n$ is the uniform probability measure on $\Omega$. We then set

$$\Lambda_p^n = \Lambda_p(\Omega_n, \mu_n).$$

We claim that for any even integer $p > 2$, there is $\delta_p > 0$ such that for any $n$ the identity map (denoted id) satisfies

$$\|id\colon\ \Lambda_p^n \to L_p^n\|_{cb} \geq \delta_p n^{\frac{1}{p}(\frac{1}{2}-\frac{1}{p})}.$$

To prove this, we will use an adaptation (with $\Lambda_p$ instead of $L_p$) of the results in [13, 24]. Indeed, by Corollary 14.2 below, using the classical "Rudin examples" of $\Lambda(p)$-sets, one can show that the space $\Lambda_p^n$ contains a subspace $E_n \subset \Lambda_p^n$ with $\dim E_n = d(n) \geq n^{2/p}$ and such that the inclusion $OH_{d(n)} \subset E_n$ satisfies $\|OH_{d(n)} \to E_n\|_{cb} \leq \chi_p$. Moreover, there is a projection $P_n\colon\ \Lambda_p^n \to E_n$ with $\|P_n\|_{cb} \leq \chi_p$. Here $p$ is an even integer $> 2$ and $\chi_p$ is a constant depending only on $p$. In addition, by [13], the same space $E_n$ considered in $L_p^n$ is (uniformly over $n$) completely isomorphic to $R_p(d(n)) \cap C_p(d(n))$ (intersection of row and column space in $S_p^{d(n)}$). In fact we use only the easy direction of this result, namely that $\|E_n \to R_p(d(n))\|_{cb} \leq 1$ and $\|E_n \to C_p(d(n))\|_{cb} \leq 1$.

It follows that there is a constant $\delta_p > 0$ such that if $id$ denotes the identity map we have

$$\|id\colon\ \Lambda_p^n \to L_p^n\|_{cb} \geq \delta_p \|OH_{d(n)} \to C_p(d(n))\|_{cb}.$$

Recall that (see [25, p. 219]) $\|OH(d) \to C_\infty(d)\|_{cb} = d^{1/4}$. Thus by interpolation, we have for any $d$ if $\frac{1}{p} = \frac{\theta}{2}$

$$\|C_p(d) \to C_\infty(d)\|_{cb} \leq \|OH(d) \to C_\infty(d)\|_{cb}^{\theta} = d^{\theta/4},$$

also

$$\|OH(d) \to C_p(d)\|_{cb} \|C_p(d) \to C_\infty(d)\|_{cb} \geq \|OH(d) \to C_\infty(d)\|_{cb},$$

therefore we find

$$\|OH(d) \to C_p(d)\|_{cb} \geq d^{1/4} d^{-\theta/4}$$

and we conclude for some $\delta'_p > 0$

$$\|id\colon\ \Lambda_p^n \to L_p^n\|_{cb} \geq \delta'_p (n^{2/p})^{\frac{1-\theta}{4}} = \delta'_p n^{\frac{\theta(1-\theta)}{4}}.$$

A similar argument applies to compare $\Lambda_p^n$ with either $\min(L_p^n)$ or $\max(L_p^n)$. Using the projections $P_n$, we easily deduce that for some constant $\chi'_p > 0$

$$\|\Lambda_p^n \to \max(L_p^n)\|_{cb} \geq \chi'_p \|OH_{d(n)} \to \max(\ell_2^{d(n)})\|_{cb}$$

and

$$\|\min(L_p^n) \to \Lambda_p^n\|_{cb} \geq \chi'_p \|\min(\ell_2^{d(n)}) \to OH_{d(n)}\|_{cb}.$$

But it is known (see [25, p. 220]) that for any $d$

$$\|OH_d \to \max(\ell_2^d)\|_{cb} = \|\min(\ell_2^d) \to OH_d\|_{cb} \simeq cd^{1/2}$$

where $c > 0$ is independent of $d$. Thus we obtain

$$\|\Lambda_p^n \to \max(L_p^n)\|_{cb} \geq c\chi'_p d(n)^{1/2} \simeq c' n^{1/p}$$

and similarly

$$\|\min(L_p^n) \to \Lambda_p^n\|_{cb} \geq c' n^{1/p}.$$

## 8 The non-commutative case

Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semi-finite faithful trace $\tau$, and let $L_p(\tau)$ be the associated "non-commutative" $L_p$-space. The preceding procedure works equally well in the non-commutative case, but requires a little more care. To define the o.s.s. on $L_p(\tau)$ that will be of interest to us we consider $f$ in $B(H) \otimes L_p(\tau)$ of the form $f = \sum_1^n b_k \otimes x_k$ and we define $f^* \in \overline{B(H)} \otimes L_p(\tau)$ by

$$f^* = \sum\nolimits_1^n \bar{b}_k \otimes x_k^*.$$

Consider $f = \sum_1^n b_k \otimes x_k \in B(H) \otimes L_p(\tau)$ as above and $g = \sum c_j \otimes y_j \in B(K) \otimes L_q(\tau)$ $(p, q \geq 1)$. We denote by $f \dot{\otimes} g \in B(H) \otimes B(K) \otimes L_r(\tau)$ $\big(r \geq 1, \frac{1}{r} = \frac{1}{p} + \frac{1}{q}\big)$ the element defined by

$$f \dot{\otimes} g = \sum\nolimits_{k,j} b_k \otimes c_j \otimes x_k y_j.$$

Given $f \in B(H) \otimes L_1(\tau)$ we denote $\hat{\tau} = id_{B(H)} \otimes \tau\colon\ B(H) \otimes L_1(\tau) \to B(H)$. More explicitly if $f$ is as above (here $p = 1$) we set

$$\hat{\tau}(f) = \sum b_k \tau(x_k),$$

and since the norm and the cb-norm coincide for linear forms, we have

$$\|\hat{\tau}(f)\| \leq \|f\|_{B \otimes_{\min} L_1(\tau)}.$$

Then, by the trace property, if $r = 1$, $\hat\tau(f\dot\otimes g)$ and $\hat\tau(g\dot\otimes f)$ are the same up to transposition of the two factors, and hence have the same minimal norm. More generally, given finite sequences $f_\ell \in B(H) \otimes L_p(\tau)$ as above and $g_\ell \in B(K) \otimes L_q(\tau)$ $(r \geq 1, 1/r = 1/p + 1/q)$, the same reasoning yields

$$\|\hat\tau(\sum\nolimits_\ell f_\ell\dot\otimes g_\ell)\| = \|\hat\tau(\sum\nolimits_\ell g_\ell\dot\otimes f_\ell)\|. \tag{8.1}$$

This identity (8.1) will considerably facilitate the generalization of most of the preceding proofs to the non-commutative case, in a rather easier fashion than for the corresponding steps in [26].

Now, Haagerup's version of the Cauchy–Schwarz inequality for the Hilbert space $\ell_2(L_2(\tau))$ becomes:

Lemma 8.1. *Let $f_k, g_k \in B \otimes L_2(\tau)$ $(k = 1, \ldots, N)$. Then*

$$\left\|\sum\nolimits_1^N \widehat{\tau}(f_k^*\dot\otimes g_k)\right\| \leq \left\|\sum \widehat{\tau}(f_k^*\dot\otimes f_k)\right\|^{1/2} \left\|\sum \widehat{\tau}\left(\sum g_k^*\dot\otimes g_k\right)\right\|^{1/2}, \tag{8.2}$$

*and actually this is valid when $\tau$ is any (not necessarily tracial) state on $M$.*

*Proof.* Let $\mathcal{H}$ be the Hilbert space obtained (after quotient and completion) from $M$ equipped with the scalar product $\langle x, y\rangle = \tau(y^*x)$. Then this lemma appears as a particular case of (2.2). □

We will use repeatedly the identification

$$\overline{B(H)} = B(\overline{H}).$$

The operator space $\Lambda_p(\tau)$ will be defined as isometric to $L_p(\tau)$ but with an o.s.s. such that for any $f$ in $B(H) \otimes L_p(\tau)$ ($p$ an even integer) we have

$$\|f\|_{B(H)\otimes_{\min}\Lambda_p(\tau)} = \|\hat\tau(f^*\dot\otimes f \otimes \cdots \dot\otimes f^*\dot\otimes f)\|^{\frac{1}{p}}_{B(\overline{H}\otimes_2 H\otimes_2\cdots\otimes_2\overline{H}\otimes_2 H)} \tag{8.3}$$

where $f^*\dot\otimes f$ and $\overline{H} \otimes_2 H$ are repeated $p/2$-times.

To prove that (8.3) really defines a norm (and an o.s.s.) on $B(H) \otimes L_p(\tau)$ we proceed exactly as in the commutative case by first establishing a Hölder type inequality:

Lemma 8.2. *Let $p \geq 2$ be an even integer. Consider $f_j \in B(H_j) \otimes L_p(\tau)$. Let*

$$\|f_j\|_{(p)} = \|\hat\tau(f_j^*\dot\otimes f_j\dot\otimes \cdots \dot\otimes f_j^*\dot\otimes f_j)\|^{1/p}_{B(\overline{H}_j\otimes H_j\otimes\cdots\otimes\overline{H}_j\otimes H_j)}$$

*where $f_j^*\dot\otimes f_j$ is repeated $p/2$ times. We have then*

$$\|\hat\tau(f_1\dot\otimes \cdots \dot\otimes f_p)\| \leq \prod_{j=1}^p \|f_j\|_{(p)}. \tag{8.4}$$

*Proof.* We will use repeatedly the fact that the minimal tensor product is commutative i.e. a permutation $\sigma$ of the factors induces a complete isometry (and actually a $*$-isomorphism) from $B(H_1 \otimes_2 \cdots \otimes H_n)$ to

$$B(H_{\sigma(1)} \otimes_2 \cdots \otimes_2 H_{\sigma(n)}).$$

Thus for any $x = \sum b_j^1 \otimes \cdots \otimes b_j^n \otimes x_j \in B(H_1) \otimes \cdots \otimes B(H_n) \otimes L_p(\tau)$, if we denote $\sigma[x] = \sum b_j^{\sigma(1)} \otimes \cdots \otimes b_j^{\sigma(n)} \otimes x_j$ we have

$$\|x\|_{\min} = \|\sigma[x]\|_{\min}.$$

Let $y = \sigma[x]$. To indicate that one can pass from $x$ to $y$ by a permutation, it will be convenient to write $x \approx y$.
Thus $x \approx y$ guarantees $\|x\|_{\min} = \|y\|_{\min}$. For example, let

$$f_1 \in B(H_1) \otimes L_p(\tau) \qquad f_2 \in B(H_2) \otimes L_p(\tau).$$

Then $\hat{\tau}((f_1 \dot{\otimes} f_2)^*) \approx \hat{\tau}(f_2^* \dot{\otimes} f_1^*)$ and hence

$$\|\hat{\tau}((f_1 \dot{\otimes} f_2)^*)\| = \|\hat{\tau}(f_2^* \dot{\otimes} f_1^*)\|.$$

Also using the trace property we have for any $f$ in $B(H) \otimes L_2(\tau)$

$$\hat{\tau}(f^* \dot{\otimes} f) \approx \hat{\tau}(f \dot{\otimes} f^*)$$

and hence

$$\|f\|_{(2)} = \|f^*\|_{(2)}. \tag{8.5}$$

More generally, for any $f_1, \ldots, f_p$ as before we have by (8.1)

$$\|\hat{\tau}(f_1 \dot{\otimes} \cdots \dot{\otimes} f_p)\|_{B(H_1 \otimes_2 \cdots \otimes_2 H_p)} = \|\hat{\tau}(f_p \dot{\otimes} f_1 \dot{\otimes} \cdots \dot{\otimes} f_{p-1})\|_{B(H_p \otimes_2 \cdots \otimes_2 H_{p-1})}. \tag{8.6}$$

In particular this gives us for any $j$

$$\|f_j\|_{(p)} = \|\hat{\tau}(f_j \dot{\otimes} f_j^* \dot{\otimes} \cdots \dot{\otimes} f_j \dot{\otimes} f_j^*)\|^{\frac{1}{p}} \tag{8.7}$$

or equivalently

$$\|f_j\|_{(p)} = \|f_j^*\|_{(p)}. \tag{8.8}$$

To prove the Lemma, we start with $p = 2$. In that case (8.4) reduces to (8.1). Let us denote by $(8.4)_p$ the inequality (8.4) for a given value of $p$. We will show

$$(8.4)_p \Rightarrow (8.4)_{2p}.$$

This covers only the case $p = 2^k$, but actually the argument used earlier for $L_p(\mu)$ (see Lemma 3.1) when $p$ is an even integer can be easily adapted to the case of $L_p(\tau)$ (note that the invariance of $I(f_1, \cdots, f_p) = \hat{\tau}(f_1 \dot{\otimes} \cdots \dot{\otimes} f_p)$ under cyclic permutations suffices to adapt this argument here). We leave the details to the reader at this point.

So assume $(8.4)_p$ proved for some integer $p \geq 2$. Consider

$$f_j \in B(H_j) \otimes L_{2p}(\tau) \qquad j = 1, \dots, 2p.$$

Let

$$g_j = f_{2j-1} \dot{\otimes} f_{2j} \in B(H_{2j-1} \otimes_2 H_{2j}) \otimes L_p(\tau) \qquad (j = 1, \dots, p).$$

By $(8.4)_p$ we have

$$\|\hat{\tau}(g_1 \dot{\otimes} \cdots \dot{\otimes} g_p)\| \leq \prod_1^p \|g_j\|_{(p)}. \tag{8.9}$$

Moreover using (8.6) we find

$$\|g_j\|_{(p)} = \|\hat{\tau}(f^*_{2j-1} \dot{\otimes} f_{2j-1} \dot{\otimes} f_{2j} \dot{\otimes} f^*_{2j} \dot{\otimes} \cdots)\|^{\frac{1}{p}}$$

where the preceding expression is repeated $p/2$ times.
By $(8.4)_p$ we have

$$\|g_j\|_{(p)} \leq \|f^*_{2j-1} \dot{\otimes} f_{2j-1}\|^{\frac{1}{2}}_{(p)} \|f_{2j} \dot{\otimes} f^*_{2j}\|^{\frac{1}{2}}_{(p)}$$

and hence by (8.7)

$$\leq \|f_{2j-1}\|_{(2p)} \|f_{2j}\|_{(2p)}.$$

Thus we find that (8.9) implies $(8.4)_{2p}$. □

We then have just like in the commutative case:

Theorem 8.3. *Let $p \geq 2$ be an even integer. The space $L_p(\tau)$ can be equipped with an o.s.s. so that denoting by $\Lambda_p(\tau)$ the resulting operator space we have for any $H$ and any $f$ in $B(H) \otimes L_p(\tau)$*

$$\|f\|_{B(H) \otimes_{\min} \Lambda_p(\tau)} = \|f\|_{(p)}.$$

*Proof.* We may reduce consideration to $H = \ell_2$ for simplicity of notation. We have then by (8.4)

$$\|f\|_{(p)} = \sup \|\hat{\tau}(f \dot{\otimes} f_2 \dot{\otimes} \cdots \dot{\otimes} f_p)\| \tag{8.10}$$

where the supremum runs over all $f_j$ in $B(H) \otimes L_p(\tau)$ $(2 \leq j \leq p)$ with $\|f_j\|_{(p)} \leq 1$. We then define for any $x$ in $L_p(\tau)$

$$J(x) = \oplus[\hat{\tau}(x \dot{\otimes} f_2 \dot{\otimes} \cdots \dot{\otimes} f_p)] \tag{8.11}$$

where the direct sum runs over all choices of $(f_j)$ $(j \geq 2)$ as before. Then (8.10) ensures that

$$\|f\|_{(p)} = \|(id_{B(H)} \otimes J)(f)\|_{\min}.$$

Thus $J$ defines an isometric embedding of $L_p(\tau)$ into some $B(\mathcal{H})$ (here $\mathcal{H}$ is a suitably "huge" direct sum) as in (8.11) so that the associated o.s.s. satisfies the desired property. □

By exactly the same argument as for Corollary 3.7 above, we have

Corollary 8.4. *Let $p \geq q \geq 2$ be even integers. If $\tau(1) = 1$, the inclusion $\Lambda_p(\tau) \subset \Lambda_q(\tau)$ is a complete contraction from $\Lambda_p(\tau)$ to $\Lambda_q(\tau)$.*

It is important for the sequel to observe that $0 \prec \widehat{\tau}(f^* \dot{\otimes} f)$ for any $f$ in $B \otimes L_2(\tau)$. This follows from a very general fact on sesquilinear forms.

Lemma 8.5. *Let $B$ and $E$ be complex vector spaces. Let $x \in (B \otimes E) \otimes (\overline{B \otimes E})$ be such that $x \succ 0$, meaning by this that $x$ can be written as a finite sum $x = \sum t_k \otimes \bar{t}_k$ with $t_k \in B \otimes E$. We will use the natural identification $\overline{B \otimes E} = \overline{B} \otimes \overline{E}$. Let $\varphi\colon\ E \otimes \overline{E} \to \mathbb{C}$ be a bilinear form (equivalently $\varphi$ defines a sesquilinear form on $E \times E$). Let $y = (\varphi \otimes \mathrm{id}_{B \otimes \overline{B}})(x) \in B \otimes \overline{B}$ (more precisely here $\varphi$ acts on the second and fourth factors, so, to indicate this, the notation $y = (\varphi)_{24}(x)$ would be less abusive). If $\varphi$ is positive definite (meaning that $\varphi(a \otimes \bar{a}) \geq 0$ $\forall a \in E$), then $y \succ 0$.*

*Proof.* Note that we may as well assume $B$ and $E$ finite dimensional. Consider then $t = \sum b_k \otimes a_k \in B \otimes E$, $\xi \in B^*$ and $s = (\xi \otimes \mathrm{id}_E)(t) \in E$. We have $(\xi \otimes \bar{\xi} \otimes \mathrm{id}_{E \otimes \overline{E}})(t \otimes \bar{t}) = s \otimes \bar{s} \succ 0$ and hence $(\xi \otimes \bar{\xi})(y) = \varphi(s \otimes \bar{s}) \geq 0$. By the proof of Lemma 2.1 we conclude that $y \succ 0$. $\square$

In particular, since $\tau(a^* a) = \tau(a a^*) \geq 0$ for any $a$ in $L_2(\tau)$, this implies:

Lemma 8.6. *For any $f$ in $B \otimes L_2(\tau)$, we have*

$$\widehat{\tau}(f^* \dot{\otimes} f) \succ 0 \quad (\textit{and } \widehat{\tau}(f \dot{\otimes} f^*) \succ 0).$$

*Remark* 8.7. By the classical property of conditional expectations, if $1 \leq p, p' \leq \infty$ are conjugate (i.e. $p' = p(p-1)^{-1}$) and if $T\colon\ L_p(\tau) \to L_p(\tau)$ is the conditional expectation with respect to a (von Neumann) subalgebra of $M$, then: $\forall x \in L_p(\tau)$ $\forall y \in L_{p'}(\tau)$ we have

$$\tau(T(x)y) = \tau(xT(y)) = \tau(T(x)T(y)).$$

Therefore for any $f \in B(H_1) \otimes L_p(\tau)$ and $g \in B(H_2) \otimes L_{p'}(\tau)$ we have:

$$\widehat{\tau}(T(f) \dot{\otimes} g) = \widehat{\tau}(f \dot{\otimes} T(g)) = \widehat{\tau}(T(f) \dot{\otimes} T(g)) \tag{8.12}$$

where we still denote abusively by $T$ the operator $I \otimes T$ acting either on $B(H_1) \otimes L_p(\tau)$ or on $B(H_2) \otimes L_{p'}(\tau)$. Moreover, it is easy to check that $T(f^*) = T(f)^*$ for any $f \in B \otimes L_p(\tau)$.

In the rest of this section we continue to abusively denote by $T$ the operator $I \otimes T$ on $B \otimes L_p(\tau)$.

Lemma 8.8. *Let $T\colon\ L_p(\tau) \to L_p(\tau)$ be the conditional expectation with respect to a von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}$. Let $p = 2m$ be an even integer. Then for any $f$ in $B \otimes L_p(\tau)$ we have*

$$\|Tf\|_{(p)} \leq \|f\|_{(p)}. \tag{8.13}$$

*Proof.* By (8.12), we have

$$\widehat{\tau}(T(f)\dot{\otimes}T(f)^*\dot{\otimes}\cdots\dot{\otimes}T(f)\dot{\otimes}T(f)^*) = \widehat{\tau}(f\dot{\otimes}T(f)^*\dot{\otimes}\cdots\dot{\otimes}T(f)\dot{\otimes}T(f)^*).$$

Indeed, just observe that if $g = T(f)^*\dot{\otimes}\cdots\dot{\otimes}T(f)\dot{\otimes}T(f)^*$ then $T(g) = g$. Therefore by (8.4) we have

$$\|T(f)\|_{(p)}^p \le \|f\|_{(p)}\|T(f)\|_{(p)}^{p-1}$$

and hence after a suitable division we obtain (8.13). □

*Remark* 8.9. In the preceding situation for any $f$ in $B \otimes L_2(\tau)$, let $f_0 = T(f)$ and $d_1 = f - T(f)$. We have then $T(f^*\dot{\otimes}f) = f_0^*\dot{\otimes}f_0 + T(d_1^*\dot{\otimes}d_1)$, and hence (since $\hat{\tau}T = \hat{\tau}$) $\hat{\tau}(f^*\dot{\otimes}f) = \hat{\tau}(f_0^*\dot{\otimes}f_0) + \hat{\tau}(d_1^*\dot{\otimes}d_1)$ Therefore, by Lemma 8.6, we have both $\hat{\tau}(f_0^*\dot{\otimes}f_0) \prec \hat{\tau}(f^*\dot{\otimes}f)$ and $\hat{\tau}(d_1^*\dot{\otimes}d_1) \prec \hat{\tau}(f^*\dot{\otimes}f)$.

By Corollary 8.4, assuming $\tau(1) = 1$, for all even integers $p \ge q \ge 2$, and any $f \in B \otimes \mathcal{M}$, we have $\|f\|_{(p)} \le \|f\|_{(q)}$, so that it is again natural to define

$$\|f\|_{(\infty)} = \lim_{p\to\infty} \|f\|_{(p)}.$$

This norm is clearly associated to a well defined o.s.s. on $\mathcal{M}$, so we are led to the following

Definition 8.10. Assume $\tau(1) = 1$. We will denote by $\Lambda_\infty(\mathcal{M},\tau)$ the Banach space $\mathcal{M}$ equipped with the o.s.s. determined by the identities

$$\forall f \in B \otimes \mathcal{M} \qquad \|f\|_{B\otimes_{\min}\Lambda_\infty(\mathcal{M},\tau)} = \|f\|_{(\infty)} = \sup_{p\in 2\mathbb{N}} \|f\|_{(p)}.$$

We warn the reader that in sharp contrast with the commutative case, in general $\Lambda_\infty(\mathcal{M},\tau)$ is not completely isometric to $\mathcal{M}$. See §10 below for more on this, including the case study of $\mathcal{M} = M_n$ equipped with its normalized trace.

## 9 Comparisons

We need to recall the definition of the "opposite" of an operator space $E \subset B(H)$. The "opposite" of $E$, denoted by $E^{op}$, is the same Banach space as $E$, but equipped with the following norms on $M_n(E)$. For any $(a_{ij})$ in $M_n(E)$ we define

$$\|(a_{ij})\|_{M_n(E^{op})} \stackrel{\text{def}}{=} \|(a_{ji})\|_{M_n(E)}.$$

Equivalently, $E^{op}$ can be defined as the operator space structure on $E$ for which the transposition: $x \to {}^t x \in B(H^*)$ defines a completely isometric embedding of $E^{op}$ into $B(H^*)$.

Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semi-finite faithful trace $\tau$, and let $L_p(\tau)$ be the associated "non-commutative" $L_p$-space. We need to recall the definition of the "natural" o.s.s. on $L_p(\tau)$ in the sense of

[23] (we follow the clarification in [25, p. 139] that is particularly important at this point). We set $L_\infty(\tau) = \mathcal{M}$. Of course we view $L_\infty(\tau)$ as an operator space completely isometric to $\mathcal{M}$. The space $L_1(\tau)$ is classically defined as the completion of $\{x \in \mathcal{M} \mid \tau(|x|) < \infty\}$ for the norm $x \mapsto \|x\|_1 = \tau(|x|)$. It can be identified isometrically with $\mathcal{M}_*$ via the mapping $x \mapsto \varphi_x$ defined by $\varphi_x(a) = \tau(xa)$. The space $\mathcal{M}_* \subset \mathcal{M}^*$ is equipped with the o.s.s. induced by the dual of the von Neumann algebra $\mathcal{M}$ (this duality uses Ruan's theorem, see e.g. [25, 11]). The "natural" o.s.s. on $L_1(\tau)$ is defined as the one transferred from the space $\mathcal{M}_*^{op}$ via the preceding isometric identification $x \mapsto \varphi_x$. In short we declare that $L_1(\tau) = \mathcal{M}_*^{op}$ completely isometrically. Then using complex interpolation, we define the "natural" o.s.s. on $L_p(\tau)$ $(1 < p < \infty)$ by the completely isometric identity $L_p(\tau) = (L_\infty(\tau), L_1(\tau))_{1/p}$.

For example, when $(\mathcal{M}, \tau) = (B(\ell_2), tr)$, the space $L_p(\tau)$ can be identified with the Schatten $p$-class. The column (resp. row) matrices in $B(\ell_2)$ form an operator space usually denoted by $C$ (resp. $R$). However, when considered as a subspace of $L_1(\tau)$ they are completely isometric to $R$ (resp. $C$), while when considered as subspaces of $L_2(\tau)$ they both are completely isometric to $OH$. The non-commutative case of §7 requires us to introduce yet another o.s.s. on $L_p(\tau)$.

We set again $\mathcal{L}_\infty(\tau) = \mathcal{M}$ but we set $\mathcal{L}_1(\tau) = \mathcal{M}_*$ (so that $\mathcal{L}_1(\tau) = L_1(\tau)^{op}$) and we denote by $\mathcal{L}_p(\tau)$ the operator space defined by

$$\mathcal{L}_p(\tau) = (\mathcal{L}_\infty(\tau), \mathcal{L}_1(\tau))_{1/p}.$$

The space $\mathcal{L}_p(\tau)$ is isometric to $L_p(\tau)$ but in the non-commutative case its o.s.s. is different. For instance if $(\mathcal{M}, \tau) = (B(\ell_2), tr)$, the column (resp. row) matrices in $\mathcal{L}_p(\tau)$ form an operator space that is completely isometric to $C$ (resp. $R$), for *all* $1 \le p \le \infty$. In sharp contrast, the o.s.s. of the subspace formed of the diagonal matrices is the same in $\mathcal{L}_p(\tau)$ or $L_p(\tau)$, and it can be identified completely isometrically with $\ell_p$ equipped with its natural o.s.s. . In particular, $\mathcal{L}_2(\tau)$ is isometric to the Hilbert-Schmidt class $S_2$, but the column (resp. row) matrices in $\mathcal{L}_2(\tau)$ are completely isometric to $C$ (resp. $R$) while the diagonal ones are completely isometric to $OH$.

PROPOSITION 9.1. *Let $1 \le p, q, r \le \infty$ be such that $r^{-1} = p^{-1} + q^{-1}$. The product mapping*

$$(x, y) \mapsto xy$$

*is (jointly) completely contractive from $\mathcal{L}_p(\tau) \times \mathcal{L}_q(\tau)$ to $\mathcal{L}_r(\tau)$.*

*Proof.* We start by the two cases $p = r = 1, q = \infty$ and $q = r = 1, p = \infty$. We need to show that $(x, \varphi_y) \mapsto \varphi_{xy}$ (resp. $(\varphi_x, y) \mapsto \varphi_{xy}$) are (jointly) completely contractive from $\mathcal{M} \times \mathcal{M}_*$ to $\mathcal{M}_*$ (resp. from $\mathcal{M}_* \times \mathcal{M}$ to $\mathcal{M}_*$). Consider $x = [x_{kl}] \in M_m(\mathcal{M})$ with $\|x\| \le 1$ and $y = [y_{ij}] \in M_n(\mathcal{M}_*)$ with $\|y\| \le 1$ (resp. $x = [x_{kl}] \in M_m(\mathcal{M}_*)$ with $\|x\| \le 1$ and $y = [y_{ij}] \in M_n(\mathcal{M})$ with $\|y\| \le 1$). It suffices to show that in both cases we have $\|[\varphi_{x_{kl} y_{ij}}]\|_{M_{mn}(\mathcal{M}_*)} \le 1$. Equivalently, we need to show that the map $u : \mathcal{M} \to M_n \otimes M_m$ defined by

$u(a) = \sum e_{ij} \otimes e_{kl}\ \tau(y_{ij} a x_{kl})$ satisfies $\|u\|_{cb} \le 1$.
Consider $v:\ \mathcal{M} \to M_m \otimes \mathcal{M}$ defined by $v(a) = \sum e_{kl} \otimes a x_{kl} = (I \otimes a)x$. Clearly $\|v\|_{cb} \le \|x\| \le 1$. Let $w:\ \mathcal{M} \to M_n$ defined by $w(a) = \sum e_{ij} \tau(y_{ij} a)$. Then $\|w\|_{cb} = \|y\|_{M_n(\mathcal{M}_*)} \le 1$. We have $(I \otimes w)v(a) = \sum e_{kl} \otimes e_{ij}\ \tau(y_{ij} a x_{kl})$ which is $u(a)$ up to permutation of the tensor product. Therefore $\|u\|_{cb} \le \|I \otimes w\|_{cb} \|v\|_{cb} \le \|w\|_{cb} \|v\|_{cb} \le 1$
(resp. let $V:\ \mathcal{M} \to M_n \otimes \mathcal{M}$ and $W:\ \mathcal{M} \to M_m$ be defined by $V(a) = \sum e_{ij} \otimes a y_{ij}$ and $W(a) = \sum e_{kl} \tau(x_{kl} a)$, then we have $u(a) = (I \otimes W)V(a)$ and we conclude similarly). □

Corollary 9.2. *For any $p \in 2\mathbb{N}$ and any $f \in B \otimes L_p(\tau)$ we have*

$$\|f\|_{(p)} = \|f\|_{B \otimes \Lambda_p(\tau)} \le \max\{\|f\|_{B \otimes_{\min} \mathcal{L}_p(\tau)}, \|f^*\|_{\bar{B} \otimes_{\min} \mathcal{L}_p(\tau)}\}.$$

*In other words, the identity defines a completely contractive map $\mathcal{L}_p(\tau) \cap \mathcal{L}_p(\tau)^{op} \to \Lambda_p(\tau)$ (where $\mathcal{L}_p(\tau) \cap \mathcal{L}_p(\tau)^{op}$ denotes the o.s.s. on $L_p(\tau)$ induced by the embedding $x \mapsto x \oplus x \in \mathcal{L}_p(\tau) \oplus \mathcal{L}_p(\tau)^{op}$.*

*Proof.* By iteration, the preceding statement implies that for any integer $N$ the product mapping is (jointly) completely contractive from $\mathcal{L}_{p_1}(\tau) \times \cdots \times \mathcal{L}_{p_N}(\tau)$ to $\mathcal{L}_r(\tau)$ when $1/r = \sum 1/p_j$. Equivalently, setting $B_1 = \cdots = B_N = B$, the mapping $(f_1, \cdots, f_N) \mapsto f_1 \dot{\otimes} \cdots \dot{\otimes} f_N$ is contractive from $B_1 \otimes_{\min} \mathcal{L}_{p_1}(\tau) \times \cdots \times B_N \otimes_{\min} \mathcal{L}_{p_N}(\tau)$ to $B_1 \otimes_{\min} \cdots \otimes_{\min} B_N \otimes_{\min} \mathcal{L}_r(\tau)$. A fortiori, when $r = 1$, $(f_1, \cdots, f_N) \mapsto \hat{\tau}(f_1 \dot{\otimes} \cdots \dot{\otimes} f_N)$ is contractive from $B_1 \otimes_{\min} \mathcal{L}_{p_1}(\tau) \times \cdots \times B_N \otimes_{\min} \mathcal{L}_{p_N}(\tau)$ to $B_1 \otimes_{\min} \cdots \otimes_{\min} B_N$. Therefore, if $p$ is an even integer, we have

$$\|f\|_{(p)}^p = \|\hat{\tau}(f^* \dot{\otimes} f \dot{\otimes} \cdots \dot{\otimes} f^* \dot{\otimes} f)\| \le \|f\|^{p/2}_{B \otimes_{\min} \mathcal{L}_p(\tau)} \|f^*\|^{p/2}_{\bar{B} \otimes_{\min} \mathcal{L}_p(\tau)}.$$

A fortiori we obtain the announced result. Note that $\bar{x} \mapsto x^*$ is a completely isometric *linear* isomorphism both from $\bar{\mathcal{M}}$ to $\mathcal{M}^{op}$ and from $\bar{\mathcal{M}}_*$ to $\mathcal{M}_*^{op}$, and hence also from $\overline{\mathcal{L}_p(\tau)}$ to $\mathcal{L}_p(\tau)^{op}$ for all $1 \le p \le \infty$. Therefore, if $f = \sum b_j \otimes x_j$ we have $\|f^*\|_{\bar{B} \otimes_{\min} \mathcal{L}_p(\tau)} = \|\sum \bar{b}_j \otimes x_j^*\|_{\bar{B} \otimes_{\min} \mathcal{L}_p(\tau)} = \|\sum \bar{b}_j \otimes \bar{x}_j\|_{\bar{B} \otimes_{\min} \overline{\mathcal{L}_p(\tau)^{op}}} = \|\sum b_j \otimes x_j\|_{B \otimes_{\min} \mathcal{L}_p(\tau)^{op}}$. Thus $\|f^*\|_{\bar{B} \otimes_{\min} \mathcal{L}_p(\tau)} = \|f\|_{B \otimes_{\min} \mathcal{L}_p(\tau)^{op}}$, whence the last assertion. □

We will now examine the particular case when $(\mathcal{M}, \tau) = (B(\ell_2), tr)$. Recall that $R$ (resp. $C$) is the subspace of $\mathcal{M} = B(\ell_2)$ formed by all row (resp. column) matrices. More generally, we denote by $R_p$ (resp. $C_p$) the operator space obtained by equipping $R$ (resp. $C$) with the o.s.s. induced by $L_p(\tau)$. We also denote by $R_p^n$ (resp. $C_p^n$) the $n$-dimensional version of $R_p$ (resp. $C_p$).
Similarly, we will denote by $\widetilde{R}_p$ (resp. $\widetilde{C}_p$) the operator space obtained by equipping $R$ (resp. $C$) with the o.s.s. induced by $\Lambda_p(\tau)$.
Furthermore, let $\widetilde{D}_p$ be the operator subspace of $\Lambda_p(\tau)$ formed of all the diagonal matrices. As a Banach space this is isometric to $\ell_p$, and it is easy to check that as an operator space $\widetilde{D}_p$ is completely isometric to the space $\lambda_p = \Lambda_p(\mathbb{N}, \mu)$

with $\mu$ equal to the counting measure on $\mathbb{N}$.
Let $b_j \in B$ $(j = 1, \cdots, n)$ and let $f = \sum b_j \otimes e_{1j} \in B \otimes R$ (resp. $g = \sum b_i \otimes e_{i1} \in B \otimes C$). Then $f \dot{\otimes} f^* = \sum b_j \otimes \bar{b}_j \otimes e_{11}$ (resp. $g^* \dot{\otimes} g = \sum \bar{b}_j \otimes b_j \otimes e_{11}$). Note that $\| \sum \bar{b}_j \otimes b_j \|^{1/2} = \| \sum b_j \otimes \bar{b}_j \|^{1/2}$. Therefore, viewing $f$ and $g$ as elements of $B \otimes \Lambda_p(\tau)$, for any $p \in 2\mathbb{N}$, we have

$$\|f\|_{(p)} = \|g\|_{(p)} = \| \sum b_j \otimes \bar{b}_j \|^{1/2}.$$

Thus we find:

Lemma 9.3. *The spaces $\widetilde{R}_p$ and $\widetilde{C}_p$ are both completely isometric to $OH$ for any $p \in 2\mathbb{N}$, while $\widetilde{D}_p$ is completely isometric to $\lambda_p$.*

Again let $b_j \in B$ $(j = 1, \cdots, n)$ and let $f = \sum b_j \otimes e_{1j}$. We have $\|f\|_{B \otimes L_p(\tau)} = \sup\{\| \sum b_j a b_j^* \|_p^{1/2} \mid \|a\|_p \leq 1\}$ (see [22, p. 83-84] or [36] for details). In case $b_j = e_{j1}$, this gives us $\|f\|_{B \otimes L_p(\tau)} = n^{1/2p}$. Therefore the natural inclusion $R_p^n \to R_2^n$ has c.b. norm $\geq n^{1/4 - 1/2p}$. Similarly, using instead $b_j = e_{1j}$, we find $\|f\|_{B \otimes L_p(\tau)} = n^{(1/2)(1-1/p)}$ and hence $\|R_2^n \to R_p^n\|_{cb} \geq n^{1/4 - 1/2p}$. This shows:

Lemma 9.4. *For any $p \in 2\mathbb{N}$ and any integer $n \geq 1$, the $n$-dimensional identity maps satisfy*

$$\|L_p(M_n, tr) \to \Lambda_p(M_n, tr)\|_{cb} \geq n^{1/4 - 1/2p}$$

*and*

$$\|\Lambda_p(M_n, tr) \to L_p(M_n, tr)\|_{cb} \geq n^{1/4 - 1/2p}.$$

## 10 Connection with CB maps on $OH$

Given a Hilbert space $H$ we denote by $OH$ the operator Hilbert space isometric to $H$, as defined in [22]. This means that whenever $(T_j)$ is an orthonormal basis of $OH$, for any finitely supported family $(b_j)$ in $B$ we have

$$\| \sum b_j \otimes T_j \| = \| \sum \bar{b}_j \otimes b_j \|^{1/2}. \tag{10.1}$$

Assume $\mathcal{M} \subset B(H)$ and $\tau(1) = 1$. We will compare the limit o.s.s. of $\Lambda_p(\mathcal{M}, \tau)$ when $p \to \infty$ to the one induced on $\mathcal{M}$ by $CB(OH)$ equipped with its usual operator space structure.
The latter can be described as follows (see e.g. [11]): Whenever $E, F$ are operator spaces the space $CB(E, F)$ of all c.b. maps from $E$ to $F$ is equipped with the (unique) o.s.s. determined by the isometric identity

$$\forall N \geq 1 \qquad M_N(CB(E, F)) = CB(E, M_N(F)).$$

More generally, we have an isometric embedding

$$B \otimes_{\min} CB(E, F) \subset CB(E, B \otimes_{\min} F). \tag{10.2}$$

If either $E$ or $F$ is finite dimensional, we may identify completely isometrically $CB(E,F)$ with $E^* \otimes_{\min} F$. When $E = F$, we denote simply $CB(E) = CB(E,E)$. Thus in particular $CB(OH_n)$ can be identified with $OH_n^* \otimes_{\min} OH_n$, or equivalently by the selfduality of $OH_n$, with $\overline{OH_n} \otimes_{\min} OH_n$ or $OH_n \otimes_{\min} \overline{OH_n}$. We first recall a well known fact.

Lemma 10.1. *Let $E, F, G$ be operator spaces. Let $B' = B(H')$ for some Hilbert space $H'$. Then for any $f = \sum b_j \otimes x_j \in B \otimes CB(F,G)$ and $g = \sum b'_k \otimes y_k \in B' \otimes CB(E,F)$ we have*

$$\|f \dot{\otimes} g\|_{B \otimes_{\min} B' \otimes_{\min} CB(E,G)} \le \|f\|_{B \otimes_{\min} CB(F,G)} \; \|g\|_{B' \otimes_{\min} CB(E,F)}, \qquad (10.3)$$

*where, as before, we denote $f \dot{\otimes} g = \sum_{j,k} b_j \otimes b'_k \otimes x_j y_k \in B \otimes B' \otimes CB(E,G)$. In other words, the composition $(x,y) \mapsto xy$ is (jointly) completely contractive from $CB(F,G) \times CB(E,F)$ to $CB(E,G)$.*

*Proof.* To prove (10.3), note that $f$ (resp. $g$) defines a c.b. map $\tilde{f} : F \to G \otimes_{\min} B$ (resp. $\tilde{g} : E \to F \otimes_{\min} B'$) and $\|f\|_{B \otimes_{\min} CB(F,G)} = \|\tilde{f}\|_{cb}$ (resp. $\|g\|_{B' \otimes_{\min} CB(E,F)} = \|\tilde{g}\|_{cb}$). Indeed, recall that, if we wish, $G \otimes_{\min} B$ can be identified with $B \otimes_{\min} G$. Similarly, $f \dot{\otimes} g$ defines a c.b. map $\Psi : E \to G \otimes_{\min} B \otimes_{\min} B'$ such that $\|f \dot{\otimes} g\|_{B \otimes_{\min} B' \otimes_{\min} CB(E,G)} = \|\Psi\|_{cb}$. But since $\Psi = (\tilde{f} \otimes Id_{B'}) \circ \tilde{g}$ we have $\|\Psi\|_{cb} \le \|\tilde{f}\|_{cb} \|\tilde{g}\|_{cb}$ and (10.3) follows. □

*Remark* 10.2. In particular, the preceding Lemma implies a fortiori that if $D, E, F, G$ are operator spaces and if $u \in CB(D,E)$ and $v \in CB(F,G)$ are fixed complete contractions, then the mapping $x \mapsto vxu$ is a complete contraction from $CB(E,F)$ to $CB(D,G)$. Indeed, the latter can be viewed as the restriction of the triple product map to $\mathbb{C}v \times CB(E,F) \times \mathbb{C}u$.

Theorem 10.3. *Let $(\mathcal{M}, \tau)$ be as before with $\mathcal{M} \subset B(H)$ and $\tau(1) = 1$. Let us denote by $\underline{\mathcal{M}}$ the operator space obtained by equipping $\mathcal{M}$ with the o.s.s. induced by $CB(OH)$. Then*

$$\Lambda_\infty(\mathcal{M}, \tau) = \underline{\mathcal{M}}$$

*completely isometrically.*

The proof of this Theorem will require some observations about the space $CB(OH)$ that may be of independent interest.
The following rather striking identity (10.4) appears as analogous to Gelfand's axiom (namely $\|x\|^2 = \|x^*x\|$) for $C^*$-algebras. It seems to express that $CB(OH)$ is an o.s. analogue of a $C^*$-algebra...

Theorem 10.4. *Let us denote simply by $\mathcal{B}$ the operator space $CB(OH)$. (Note that $\mathcal{B}$ is isometric to $B(H)$ as a Banach space.) For any $f \in B \otimes \mathcal{B}$ we have*

$$\|f\|^2_{B \otimes_{\min} \mathcal{B}} = \|f^* \dot{\otimes} f\|_{\bar{B} \otimes_{\min} B \otimes_{\min} \mathcal{B}} = \|f \dot{\otimes} f^*\|_{B \otimes_{\min} \bar{B} \otimes_{\min} \mathcal{B}}. \qquad (10.4)$$

*Moreover, we also have*

$$\|f^*\|_{\bar{B} \otimes_{\min} \mathcal{B}} = \|f\|_{B \otimes_{\min} \mathcal{B}}. \qquad (10.5)$$

*Proof.* Let $H_i \subset H$ be an increasing net of finite dimensional subspaces with dense union. Assuming $f = \sum b_j \otimes x_j$, let $f(i) = \sum b_j \otimes P_{H_i} x_{j|H_i} \in B \otimes CB(OH_i)$. Then, using the homogeneity of $OH$ in the sense of [22, p. 19] or [23], one checks that each side of either (10.4) or (10.5) is equal to the supremum over $i$ of the expression obtained after substituting $f_i$ for $f$. Thus it suffices to prove (10.4) or (10.5) when $\dim(H) < \infty$.
In that case, denoting by $T_j$ an orthonormal basis of $OH_n$, and using the identity $CB(OH_n) = OH_n \otimes_{\min} \overline{OH_n}$, we may write any $f \in B \otimes \mathcal{B}$ as $f = \sum b_{ij} \otimes T_i \otimes \bar{T}_j$ with $b_{ij} \in B$, and $\|f\|_{B \otimes \mathcal{B}} = \|\sum b_{ij} \otimes T_i \otimes \bar{T}_j\|_{B \otimes_{\min} OH_n \otimes_{\min} \overline{OH_n}}$. Using $\|x\| = \|\bar{x}\|$ for any operator $x$, and permuting the second and third factors, we have then obviously (the norm being the min-norm)

$$\|\sum b_{ij} \otimes T_i \otimes \bar{T}_j\| = \|\sum \bar{b}_{ij} \otimes \bar{T}_i \otimes T_j\| = \|\sum \bar{b}_{ij} \otimes T_j \otimes \bar{T}_i\| = \|\sum \bar{b}_{ji} \otimes T_i \otimes \bar{T}_j\|,$$

and this is clearly equivalent to (10.5).
Let $y_i = \sum_j b_{ij} \otimes \bar{T}_j$. By (10.1), we have $\|\sum b_{ij} \otimes T_i \otimes \bar{T}_j\|_{B \otimes_{\min} OH_n \otimes_{\min} \overline{OH_n}} = \|\sum y_i \otimes T_i\| = \|\sum \bar{y}_i \otimes y_i\|^{1/2} = \|\sum y_{jk} \otimes T_j \otimes \bar{T}_k\|^{1/2}$ where $y_{jk} = \sum_i \bar{b}_{ij} \otimes b_{ik}$. Using again the identity $CB(OH_n) = OH_n \otimes_{\min} \overline{OH_n}$ we find $f^* \dot{\otimes} f = \sum y_{jk} \otimes T_j \otimes \bar{T}_k$. Thus, we have $\|f\| = \|f^* \dot{\otimes} f\|^{1/2}$, and by (10.5) we obtain (10.4). □

*Remark* 10.5. Note that after iteration, for any $p = 2^m$ $(m \geq 1)$, (10.4) yields

$$\|f\|^p_{B \otimes_{\min} \mathcal{B}} = \|f^* \dot{\otimes} f \dot{\otimes} f^* \dot{\otimes} f \cdots\|_{\bar{B} \otimes_{\min} B \otimes_{\min} \cdots \otimes_{\min} \mathcal{B}}. \tag{10.6}$$

Corollary 10.6. *Let $H_I = \oplus_{i \in I} H_i$ be an orthogonal decomposition of a Hilbert space $H_I$. We have then a completely isometric embedding*

$$\oplus_{i \in I} CB(OH_i) \subset CB(OH_I).$$

*Proof.* Let $u : \oplus_{i \in I} CB(OH_i) \to CB(OH_I)$ denote this embedding. It is easy to reduce the proof to the finite case so we assume $|I| < \infty$. Since the coordinatewise inclusions and projections relative to $OH_I$ are all completely contractive, it is easy to check using Lemma 10.1 that $\|u\|_{cb} \leq |I| < \infty$. Consider now $f \in B \otimes_{\min} (\oplus_{i \in I} CB(OH_i))$ and let $g = (Id_B \otimes u)(f) \in B \otimes_{\min} CB(OH_I)$. We need to show that $\|g\| = \|f\|$. Note that $(Id_{\bar{B} \otimes B \otimes \cdots} \otimes u)(f^* \dot{\otimes} f)^{\dot{\otimes} m}) = (g^* \dot{\otimes} g)^{\dot{\otimes} m}$. Thus, by (10.3) and (10.4) we have for any integer $m$

$$\|g\| = \|(g^* \dot{\otimes} g)^{\dot{\otimes} m}\|^{1/2m} \leq (|I| \|(f^* \dot{\otimes} f)^{\dot{\otimes} m}\|)^{1/2m} = |I|^{1/2m} \|f\|$$

so that letting $m \to \infty$ we obtain $\|g\| \leq \|f\|$. Since the converse inequality follows easily from Remark 10.2 applied to the coordinate projections, we have equality. □

Theorem 10.7. *Let $E \subset B(H)$ be any operator space. Let us denote again by $\underline{E}$ the operator space obtained by inducing on $E$ the o.s.s. of $CB(OH)$. Let $F \subset B(K)$ be another operator space. Then for any $u \in CB(E, F)$ we have*

$$\|u\|_{CB(\underline{E}, \underline{F})} \leq \|u\|_{CB(E,F)}.$$

*In particular, if $u : E \to F$ is completely isometric, then $u : \underline{E} \to \underline{F}$ also is.*

*Proof.* We may clearly assume $F = B(K)$ and $\underline{F} = CB(OK)$ and by the same argument as in the preceding proof, we may assume $\dim(K) = n < \infty$. Assume $\|u\|_{CB(E,F)} \le 1$. Then $u$ extends to a c.b. map $\hat{u}: B(H) \to B(K)$ with the same cb-norm. Since $M_n(B(H)^*) = M_n(B(H)_*)^{**}$ isometrically (see e.g. [11, p. 75]) $\hat{u}$ is then a point norm limit of normal maps with cb-norm $\le 1$, so we may assume that $u$ is normal on $E = B(H)$. Then (see [34, p. 45]) there is a factorization of $u$ of the form $u(x) = V\rho(x)W$ with $\|V\| \le 1, \|W\| \le 1$ where $\rho$ is an "ampliation", i.e. $\rho$ takes its values in $B(\oplus_{i\in I} H_i)$ for some set $I$ with $H_i = H$ for all $i \in I$ and $\rho(x) = \oplus_{i\in I}\rho_i(x)$ with $\rho_i(x) = x$ for all $i \in I$. This reduces the Lemma to the case when $u$ is an ampliation and to the case when $u$ is of the form $u(x) = VxW$.
Let us first assume $u(x) = VxW$ with $V: H \to K$ and $W: K \to H$ of norm 1. By the homogeneity of $OH$ we know that the cb norm of $V: OH \to OK$ is 1, and similarly for $W: OK \to OH$. Then by Remark 10.2 $\|u\|_{CB(CB(OH),CB(OK))} \le 1$.
We now assume that $u$ is an ampliation i.e. $u = u_I$ where $u_I(x) = \oplus_{i\in I} u_i(x) \in B(\oplus_{i\in I} H_i)$ with $H_i = H$ and $u_i(x) = x$ for all $i \in I$. Let $H_I = \oplus_{i\in I} H_i$. By Corollary 10.6 $u_I$ is a complete isometry from $CB(OH)$ to $CB(OH_I)$. Since both multiplications and ampliations have been checked, the proof of the first assertion is complete. The second assertion is then immediate. □

Corollary 10.8. *Let $E \subset B(H)$ be any operator space. The o.s.s. of $\underline{E}$ (induced on $E$ by that of $CB(OH)$) is independent of the completely isometric embebdding $E \subset B(H)$, i.e. it depends only on the o.s.s. of $E$.*

*Proof of Theorem 10.3.* We first give a simple argument for the special case when $\mathcal{M} = M_n$ equipped with its normalized trace $\tau_n$. We will show that $\Lambda_\infty(M_n, \tau_n)$ can be identified completely isometrically with $CB(OH_n)$ for any $n \ge 1$. We first claim that the identity from $\overline{OH_n} \otimes OH_n$ to itself induces a mapping $V_n: \overline{OH_n} \otimes_h OH_n \to \overline{OH_n} \otimes_{\min} OH_n$ such that $\|V_n\|_{cb} \le 1$ and $\|V_n^{-1}\|_{cb} \le n^{1/2}$. By the minimality of the minimal tensor product the first assertion is obvious. To check the second one, recall the identity map on $n$-dimensional Hilbert space defines an isomorphism $u_n: R_n \to OH_n$ such that $\|u_n\|_{cb} = \|u_n^{-1}\|_{cb} = n^{1/4}$ (see e.g. [25, p. 219]). Therefore, we have a factorization of $V_n^{-1}$ as follows

$$\overline{OH_n} \otimes_{\min} OH_n \overset{Id\otimes u_n^{-1}}{\to} \overline{OH_n} \otimes_{\min} R_n = \overline{OH_n} \otimes_h R_n \overset{Id\otimes u_n}{\to} \overline{OH_n} \otimes_h OH_n,$$

where we used the identity $E \otimes_{\min} R_n = E \otimes_h R_n$ for which we refer e.g. to [25, p. 95]. From this follows $\|V_n^{-1}\|_{cb} \le \|u_n^{-1}\|_{cb}\|u_n\|_{cb} = n^{1/2}$.
More explicitly, recall that for any pair of Hilbert spaces $H, K$ we have (see [22, Cor. 2.12])

$$OH \otimes_h OK = O(H \otimes_2 K);$$

in particular, $\Lambda_2(M_n, tr)$ is the same as $L_2(M_n, tr) = \overline{OH_n} \otimes_h OH_n$. Therefore, for any $g \in B \otimes \Lambda_2(M_n, tr)$ we have

$$\|g\|_{B\otimes_{\min} CB(OH_n)} \le \|g\|_{B\otimes_{\min}\Lambda_2(M_n,tr)} \le n^{1/2}\|g\|_{B\otimes_{\min} CB(OH_n)}. \qquad (10.7)$$

Consider an even integer $p$ and $f \in B \otimes M_n$. Let $g = f^* \dot{\otimes} f \dot{\otimes} \cdots$ (here there are $p/2$ factors equal to $f^* \dot{\otimes} f$). We have by (10.4)

$$\|g\|^2_{\bar{B} \otimes_{\min} \cdots \otimes_{\min} CB(OH_n)} = \|g \dot{\otimes} g^*\|_{\bar{B} \otimes_{\min} \cdots \otimes_{\min} CB(OH_n)}.$$

We have also $g \dot{\otimes} g^* = f^* \dot{\otimes} f \dot{\otimes} \cdots$ (here there are $p$ such factors) and hence, assuming $p = 2^m$ for some $m$ and using (10.6), we find

$$\|g\|^2_{\bar{B} \otimes_{\min} \cdots \otimes_{\min} CB(OH_n)} = \|f\|^p_{B \otimes_{\min} CB(OH_n)}. \tag{10.8}$$

By definition of $\Lambda_p$ we have

$$\|f\|^p_{B \otimes_{\min} \Lambda_p(M_n, tr)} = \|g\|^2_{\Lambda_2(M_n, tr)}$$

and hence by (10.7)

$$\|g\|^2_{B \otimes_{\min} CB(OH_n)} \le \|f\|^p_{B \otimes_{\min} \Lambda_p(M_n, tr)} \le n \|g\|^2_{B \otimes_{\min} CB(OH_n)}.$$

Then by (10.8) we obtain

$$\|f\|^p_{B \otimes_{\min} CB(OH_n)} \le \|f\|^p_{B \otimes_{\min} \Lambda_p(M_n, tr)} \le n \|f\|^p_{B \otimes_{\min} CB(OH_n)},$$

and, if we take the $p$-th root and let $p \to \infty$ this yields

$$\|f\|_{B \otimes_{\min} CB(OH_n)} = \|f\|_{B \otimes_{\min} \Lambda_\infty(M_n, \tau_n)}.$$

We now consider the general case. Let $f \in B \otimes \mathcal{M}$. For any $p \in 2\mathbb{N}$ we have $\|f\|^p_{(p)} = \hat{\tau}(f^* \dot{\otimes} f \cdots)$. As a linear form on $\mathcal{M}$, $\tau$ has norm 1, and hence c.b. norm equal to 1 on $\underline{\mathcal{M}}$. Therefore

$$\|f\|^p_{(p)} = \|\hat{\tau}(f^* \dot{\otimes} f \cdots)\| \le \|f^* \dot{\otimes} f \cdots\|_{\bar{B} \otimes \cdots B \otimes \underline{\mathcal{M}}} = \|f^* \dot{\otimes} f \cdots\|_{\bar{B} \otimes \cdots B \otimes \mathcal{B}}$$

and by (10.6) (assuming $p = 2^m$)

$$\|f^* \dot{\otimes} f \cdots\|_{\bar{B} \otimes \cdots B \otimes \mathcal{B}} = \|f\|^p_{B \otimes_{\min} \mathcal{B}} = \|f\|^p_{B \otimes_{\min} \underline{\mathcal{M}}}.$$

Thus we obtain $\|f\|_{(p)} \le \|f\|_{B \otimes_{\min} \underline{\mathcal{M}}}$, and taking the supremum over $p$ yields

$$\|f\|_{B \otimes_{\min} \Lambda_\infty(\mathcal{M}, \tau)} \le \|f\|_{B \otimes_{\min} \underline{\mathcal{M}}}.$$

It remains to prove the converse inequality.
Consider $f \in B \otimes \mathcal{M}$. Let $F : OH \to OH \otimes_{\min} B$ be the associated c.b. map (as in the proof of Lemma 10.1). By Corollary 10.8 we may assume that $H = L_2(\tau)$ and that the inclusion $\mathcal{M} \subset B(L_2(\tau))$ is the usual realization of $\mathcal{M}$ acting on $L_2(\tau)$ by left multiplication.
Let $B'$ be another copy of $B$. Note that for any $\xi \in B' \otimes OH$ we have

$$\|\xi\|_{B' \otimes_{\min} OH} = \|\xi\|_{(2)}.$$

Moreover, if $\xi \in B' \otimes \mathcal{M} \subset B' \otimes OH$, then up to permutation of factors $(Id_{B'} \otimes F)(\xi) \approx f\dot{\otimes}\xi$. Since $\|F\|_{cb} = \|Id_{B'} \otimes F\|_{cb}$, the definition of the o.s.s of $CB(OH)$ (see (10.2) above) shows that

$$\|f\|_{B\otimes_{\min}\underline{\mathcal{M}}} = \|F\|_{cb} = \sup\{\|f\dot{\otimes}\xi\|_{(2)} \mid \xi \in B' \otimes \mathcal{M},\ \|\xi\|_{(2)} \leq 1\}.$$

Fix $\xi \in B' \otimes \mathcal{M}$ with $\|\xi\|_{(2)} \leq 1$. To complete the proof it suffices to show that

$$\|f\dot{\otimes}\xi\|_{(2)} \leq \sup_{p\in 2\mathbb{N}} \|f\|_{(p)} = \|f\|_{B\otimes_{\min}\Lambda_\infty(\mathcal{M},\tau)}.$$

To verify this, we claim that for any $p$ of the form $p = 2^m$ we have

$$\|f\dot{\otimes}\xi\|_{(2)} \leq \|\hat{\tau}((f^*\dot{\otimes}f)^{\dot{\otimes}p/2}\dot{\otimes}\xi\dot{\otimes}\xi^*)\|^{1/p}. \tag{10.9}$$

This is easy to check by induction on $m$. Indeed, by (8.3) (for $p = 2$), equality holds in the case $m = 1$ and if we assume our claim proved for a given value of $m$ then the Haagerup-Cauchy-Schwarz inequality (8.2) shows that it holds also for $m + 1$, because we may write (recall (8.1))

$$\|\hat{\tau}((f^*\dot{\otimes}f)^{\dot{\otimes}p/2}\dot{\otimes}\xi\dot{\otimes}\xi^*)\| \leq \|(f^*\dot{\otimes}f)^{\dot{\otimes}p/2}\dot{\otimes}\xi\|_{(2)}\|\xi^*\|_{(2)} = \\ = \|\hat{\tau}((f^*\dot{\otimes}f)^{\dot{\otimes}p}\dot{\otimes}\xi\dot{\otimes}\xi^*)\|^{1/2}\|\xi^*\|_{(2)},$$

and by (8.5) $\|\xi^*\|_{(2)} \leq 1$, so we obtain (10.9) with $2p$ in place of $p$.
We now use the claim to conclude: By (8.2) again (or by (8.4)) we have

$$\|\hat{\tau}((f^*\dot{\otimes}f)^{\dot{\otimes}p/2}\dot{\otimes}\xi\dot{\otimes}\xi^*)\|^{1/p} \leq \|f\|_{(2p)}\|\xi\dot{\otimes}\xi^*\|_{(2)}^{1/p}.$$

Now $\xi \in B' \otimes \mathcal{M}$ implies $\xi\dot{\otimes}\xi^* \in B' \otimes \bar{B}' \otimes \mathcal{M}$ and, since $\tau$ is finite, we have $\|\xi\dot{\otimes}\xi^*\|_{(2)} < \infty$, therefore $\|\xi\dot{\otimes}\xi^*\|_{(2)}^{1/p} \to 1$ when $p \to \infty$ and we deduce from (10.9)

$$\|f\dot{\otimes}\xi\|_{(2)} \leq \limsup_{p\to\infty} \|\hat{\tau}((f^*\dot{\otimes}f)^{\dot{\otimes}p/2}\dot{\otimes}\xi\dot{\otimes}\xi^*)\|^{1/p} \leq \limsup_{p\to\infty} \|f\|_{(2p)} = \sup_{p\in 2\mathbb{N}} \|f\|_{(p)},$$

which completes the proof. □

*Remark* 10.9. By the minimality of the min tensor product, we know that we have a completely contractive inclusion $OH^* \otimes_h OH \to OH^* \otimes_{\min} OH \subset CB(OH)$. Therefore, for any pair of sets $I, J$, in analogy with the inclusion of the Hilbert-Schmidt class into the bounded operators, we have a completely contractive inclusion

$$OH(I \times J) \to CB(OH(I), OH(J)).$$

## 11 Non-commutative Khintchine inequalities

We start by a fairly simple statement mimicking a classical commutative fact:

Proposition 11.1. *Let $p = 2n$. Let $\{x_k\}$ be a sequence in $L_p(\tau)$, such that, for some constant $C$, for any finite sum $f = \sum b_k \otimes x_k$ with coefficients $b_k$ in $B(H)$, we have*

$$\|f\|_{(p)} \le C \left\| \sum b_k \otimes \bar{b}_k \right\|^{1/2}. \tag{11.1}$$

*Assume moreover that $\{x_k\}$ is orthonormal in $L_2(\tau)$ and $\tau(1) = 1$. Then the closed span of $(x_k)$ in $\Lambda_p(\tau)$ is completely isomorphic to $OH$ and completely complemented in $\Lambda_p(\tau)$. More precisely the orthogonal projection $P$ onto this span satisfies $\|P\colon\ \Lambda_p \to \Lambda_p\|_{cb} \le C$.*

*Proof.* Let $P$ be the orthogonal projection on $\Lambda_2$ onto the span under consideration. For any $f \in B \otimes \Lambda_p$, let $h = (Id \otimes P)(f)$. By a well known fact (see [22, p. 19]), $P$ is completely contractive on $\Lambda_2$, so that $\|h\|_{(2)} \le \|f\|_{(2)}$. By Corollary 8.4, we have $\|f\|_{(2)} \le \|f\|_{(p)}$ and by our assumption $\|h\|_{(p)} \le C\|h\|_{(2)}$. Therefore $\|h\|_{(p)} \le C\|f\|_{(p)}$. Thus, the c.b. norm of $P$ acting from $\Lambda_p$ to itself is automatically $\le C$. Moreover, for any $h \in B \otimes \overline{\mathrm{span}}[x_k]$, we have $\|h\|_{(2)} \le \|h\|_{(p)} \le C\|h\|_{(2)}$, which shows that the span is completely isomorphic to $OH$. □

With the "natural" o.s.s. introduced in [23] the Khintchine inequalities for $1 < p < \infty$ are due to F. Lust-Piquard [20] . For $p$ an even integer, A. Buchholz [7] found a beautiful proof that yields optimal constants. His proof is valid for a much more general class of variables instead of the Rademacher functions. We will now follow his ideas to investigate the analogous question in the space $\Lambda_p$.

Let $P_2(2n)$ denote the set of all partitions of $[1, \ldots, 2n]$ onto subsets each with exactly 2 elements. So an element $\nu$ in $P_2$ can be described as a collection of disjoint pairs $\{k_i, j_i\}$ $(1 \le i \le n)$ with $k_i \ne j_i$ such that $\{1, \ldots, 2n\} = \{k_1, \ldots, k_n, j_1, \ldots, j_n\}$.

We call such a partition into pairs a 2-partition. Let $p = 2n$ be an even integer $\ge 2$. Following [6] we say that a sequence $\{x_k\}$ in $L_p(\tau)$, has $p$-th moments defined by pairings if there is a function $\psi\colon\ P_2(2n) \to \mathbb{C}$ defined on the set of 2-partitions of $[2n] = \{1, \ldots, 2n\}$ such that for any $k_1, \ldots, k_{2n}$ we have

$$\tau(x_{k_1} x_{k_2}^* x_{k_3} \ldots x_{k_{2n-1}} x_{k_{2n}}^*) = \sum_{\nu \sim (k_1, \ldots, k_{2n})} \psi(\nu)$$

where the notation $\nu \sim (k_1, \ldots, k_{2n})$ means that $k_i = k_j$ whenever the pair $\{i, j\}$ is a block of the partition $\nu$.

Note that, for each $k$, taking the $k_j$'s all equal to $k$, this implies

$$\tau(|x_k|^p) = \sum\nolimits_{\nu \in P_2(2n)} \psi(\nu). \tag{11.2}$$

Now let $E = \mathrm{span}[x_j]$ and $B = B(H)$. Consider $f \in B \otimes E$ of the form

$$f = \sum b_j \otimes x_j.$$

We have

$$\widehat{\tau}((f \dot{\otimes} f^*)^{\dot{\otimes}\, n}) = \sum_{k_1,\ldots,k_{2n}} \sum_{\nu \sim (k_1,\ldots,k_{2n})} \psi(\nu) b_{k_1} \otimes \bar{b}_{k_2} \otimes \cdots \otimes b_{k_{2n-1}} \otimes \bar{b}_{k_{2n}}.$$

Therefore

$$\begin{aligned} \|f\|_{(2n)}^{2n} &= \left\| \sum_{\nu \in P_2(2n)} \psi(\nu) \sum_{(k_1,\ldots,k_{2n}) \sim \nu} b_{k_1} \otimes \bar{b}_{k_2} \otimes \cdots \otimes \bar{b}_{k_{2n}} \right\| \\ &\le \sum_{\nu \in P_2(2n)} |\psi(\nu)| \left\| \sum_{(k_1,\ldots,k_{2n}) \sim \nu} b_{k_1} \otimes \bar{b}_{k_2} \otimes \cdots \otimes \bar{b}_{k_{2n}} \right\|. \end{aligned}$$

But now let

$$\Phi(\nu) = \sum_{(k_1,\ldots,k_{2n}) \sim \nu} b_{k_1} \otimes \bar{b}_{k_2} \otimes \cdots \otimes b_{k_{2n-1}} \otimes \bar{b}_{k_{2n}}.$$

Then up to permutation $\Phi(\nu)$ is equal to a product of $n$ terms of the form either $\sum b_k \otimes b_k$, $\sum \bar{b}_k \otimes \bar{b}_k$ or $\sum b_k \otimes \bar{b}_k$. Let $T_1, \cdots T_n$ be an enumeration of the latter terms. Since the permutation leaves the norm invariant, we have $\|\Phi(\nu)\| = \prod_1^n \|T_j\|$. By (3.1) $\|T_j\| \le \|\sum b_k \otimes \bar{b}_k\|$ for each $j$ (actually there is equality for terms the third kind), and hence

$$\|\Phi(\nu)\| \le \left\| \sum b_k \otimes \bar{b}_k \right\|^n$$

and we conclude that

$$\|f\|_{(2n)} \le \left( \sum_{\nu \in P_2(2n)} |\psi(\nu)| \right)^{1/2n} \left\| \sum b_k \otimes \bar{b}_k \right\|^{1/2}. \tag{11.3}$$

Moreover by (11.2) we know that if $\psi(\nu) \ge 0$ for all $\nu$, then the constant $\sum_{\nu \in P_2(2n)} |\psi(\nu)|$ is optimal. Recapitulating, we have proved:

Theorem 11.2. *Let $p = 2n$. Let $\{x_k\}$ be as above a sequence in $L_p(\tau)$, with $p$-th moments defined by pairings via a function $\psi\colon\ P_2(2n) \to \mathbb{C}$. Then for any finite sum $f = \sum b_k \otimes x_k$ $(b_k \in B(H))$, we have*

$$\|f\|_{(p)} \le C_{\psi,p} \left\| \sum b_k \otimes \bar{b}_k \right\|^{1/2}, \tag{11.4}$$

*where $C_{\psi,p} = \left( \sum_{\nu \in P_2(2n)} |\psi(\nu)| \right)^{1/2n}$. Moreover this constant is optimal if $\psi(\nu) \ge 0$ for all $\nu$.*

Buchholz applied the preceding statement to a $q$-Gaussian family with $q \in [-1,1]$. The latter have moments defined by pairings. When $q \in [0,1]$, the function $\psi$ is non-negative, so the constant $C_{\psi,p}$ is optimal and, by (11.2), we know $C_{\psi,p} = \|x_1\|_p$. In particular, we have:

COROLLARY 11.3. *Let $(x_k)$ be a sequence of independent Gaussian normal random variables on a probability space $(\Omega, \mathbb{P})$. Then the span of $(x_k)$ is completely isomorphic to $OH$ and is completely complemented in $\Lambda_p(\Omega, \mathbb{P})$ for every even integer $p$. Moreover,* (11.4) *holds with a constant $C_{\psi,p} = \|x_1\|_p$ that is $O(\sqrt{p})$ when $p \to \infty$.*

*Remark* 11.4. The preceding Corollary also holds when $(x_k)$ is a sequence $(\varepsilon_k)$ of independent symmetric $\pm 1$ valued variables (or equivalently for the Rademacher functions). We show this in Corollary 11.12 below, but here is a quick proof with a slightly worse constant. Let $(x_k)$ be independent Gaussian normal random variables and assume that $(\varepsilon_k)$ is independent from $(x_k)$. It is well known that $(x_k)$ has the same distribution as $(\varepsilon_k|x_k|)$. Let $\delta = \mathbb{E}(|x_k|) = 2/\sqrt{\pi}$. The conditional expectation $\mathcal{E}$ with respect to $(\varepsilon_k)$ satisfies $\mathcal{E}(\varepsilon_k|x_k|) = \delta\varepsilon_k$. Therefore $\delta \sum \varepsilon_k b_k = \mathcal{E}(\sum \varepsilon_k |x_k| b_k)$, and by (8.13), this implies

$$\delta \| \sum \varepsilon_k b_k \|_{(p)} \le \| \sum \varepsilon_k |x_k| b_k \|_{(p)} = \| \sum x_k b_k \|_{(p)}.$$

So we obtain the Rademacher case with a constant $\le \delta^{-1} C_{\psi,p}$ since Proposition 11.1 ensures the complete complementation.

The preceding result applies to $q$-Gaussian and in particular free semi-circular (or circular) elements, see [7] for details. We have then $C_{\psi,p} \le 2/\sqrt{1-|q|}$ for all even $p$.

In either the semi-circular ($q=0$) or the circular case, we have $C_{\psi,p} \le 2$ for all even $p$, and hence:

COROLLARY 11.5. *For any even integer $p$, the closed span of a free semi-circular (or circular) family, is completely isomorphic to $OH$ and completely complemented (by the orthogonal projection) in the space $\Lambda_p$ for the associated trace (on the free group factor). Moreover, the corresponding constants are bounded by 2 uniformly over $p$.*

COROLLARY 11.6. *Let $\mathcal{M}$ be the von Neumann algebra of the free group $\mathbb{F}_\infty$ with infinitely many generators $(g_k)$. For any $p = 2n$ and any finite sum $\tilde{f} = \sum b_k \otimes \lambda(g_k)$ ($b_k \in B(H)$), we have*

$$\left\| \sum b_k \otimes \bar{b}_k \right\|^{1/2} \le \|\tilde{f}\|_{(p)} \le 2 \left\| \sum b_k \otimes \bar{b}_k \right\|^{1/2}. \tag{11.5}$$

*More generally, let $W_d \subset \mathbb{F}_\infty$ denote the subset formed of the reduced words of length $d$. Then for any finitely supported function $b : W_d \to B$ we have*

$$\| \sum\nolimits_{t \in W_d} b(t) \otimes \lambda(t) \|_{B \otimes \underline{\mathcal{M}}} \le (d+1) \left\| \sum\nolimits_{t \in W_d} b(t) \otimes \bar{b}(t) \right\|^{1/2}. \tag{11.6}$$

*Proof.* The left hand side of (11.5) follows from Corollary 8.4 with $q = 2$ and the orthonormality of $(\lambda(g_k))$ in $L_2(\tau)$. By [5, Th. 2.8] the operator space spanned by $\{\lambda(t)\}_{t\in W_d}$ is completely isomorphic to the intersection $X$ of a family of $d+1$ operator spaces $X_i$, $0 \le i \le d$, with associated constant equal to $d+1$. On one hand, the space $X_0$ (resp. $X_d$) is completely isometric to $R$ (resp. $C$), the underlying respective Hilbert space being $\ell_2(W_d)$. On the other hand, when $0 < j < d$ the space $X_j$ is completely isometric to the subspace of $B(\ell_2(W_{d-j}), \ell_2(W_j))$ associated to matrices of the form $[a(st)]$ when $a$ is supported on $W_d$. Identifying each $W_i$ simply with $\mathbb{N}$ we see that $\underline{X_0}$ (resp. $\underline{X_d}$) is completely isometric to $OH(\mathbb{N})$, while $\underline{X_i}$ is completely isometric to the associated subspace of $CB(OH(\mathbb{N}))$. By Remark 10.9 we have a completely isometric inclusion $\underline{X_0} \to \underline{X_i}$ for any $0 \le i \le d$, therefore the intersection of the family $\underline{X_i}$ $0 \le i \le d$ is completely isometric to $OH$ with $H = \ell_2(W_d)$. Since by Corollary 10.6 we know that $\underline{X} = \cap_{0\le i\le d}\underline{X_i}$, (11.6) follows. □

*Remark* 11.7. A comparison with known results (see [7] for detailed references) shows that the limit of $\|\tilde{f}\|_{(p)}$ when $p \to \infty$ is *not* equivalent to $\|\tilde{f}\|_{B(H)\otimes_{\min}\mathcal{M}}$ (here $\mathcal{M}$ is the von Neumann algebra of the free group with infinitely many generators), in sharp contrast with (3.11) above.

More generally, let $L_p(\mathcal{N}, \varphi)$, or briefly $L_p(\mathcal{N}, \varphi)$, be another non-commutative (semi-finite) $L_p$-space. Consider $f_k \in B \otimes L_p(\varphi)$ and let

$$F = \sum f_k \otimes x_k \in B \otimes L_p(\varphi \times \tau)$$

where $\{x_k\}$ is as in Theorem 11.2. We have then:

Theorem 11.8. *Let $p = 2n$ and let $C = C_{\psi,p}$ be the constant appearing in* (11.4). *Then for any $F$ as above we have*

$$\|F\|_{(p)} \le C \max\left\{ \left\|\sum f_k \dot{\otimes} f_k^*\right\|_{(p/2)}^{1/2}, \left\|\sum f_k^* \dot{\otimes} f_k\right\|_{(p/2)}^{1/2} \right\}. \tag{11.7}$$

*Proof.* Repeating the steps of the proof of Theorem 11.2, all we need to do is majorize

$$\left\| \widehat{\varphi}\left( \sum_{(k_1,\ldots,k_{2n})\sim\nu} f_{k_1} \dot{\otimes} f_{k_2}^* \dot{\otimes} \cdots \dot{\otimes} f_{k_{2n}}^* \right) \right\|$$

by the right side of (11.7). This is established in Lemma 11.11 below that is a rather easy adaptation to our $\Lambda_p$-setting of [6, Lemma 2]. □

By the same argument as in Remark 11.4, the case of the free generators of the free group can be deduced from the "free-Gaussian" one. Indeed, let $(c_k)$ be a free circular family (sometimes called "complex free-Gaussian"). The polar decomposition $c_k = u_k|c_k|$, is such that the $*$-distribution of $(u_k)$ is identical to that of a free family of Haar unitaries in the sense of [35], or equivalently $(u_k)$ has the same $*$-distribution as that of the free generators $\lambda(g_k)$ in the von

Neumann algebra of the free group with infinitely many generators. Moreover, a simple calculation relative to the circular distribution yields $\|c_k\|_1 = 8/3\pi$. These observations lead us to :

Corollary 11.9. *With the same notation as in Corollary 11.6, let $\widetilde{F} = \sum f_k \otimes \lambda(g_k)$. We have then*

$$(3\pi/4)^{-1}\|\widetilde{F}\|_{(p)} \leq \max\left\{\left\|\sum f_k \dot{\otimes} f_k^*\right\|_{(p/2)}^{1/2}, \left\|\sum f_k^* \dot{\otimes} f_k\right\|_{(p/2)}^{1/2}\right\} \leq \|\widetilde{F}\|_{(p)} \tag{11.8}$$

*Proof.* Let $\tilde{\tau}$ denote the normalized trace on the von Neumann algebra of the free group with generators $(g_k)$. Let $\mathcal{E}$ denote the conditional expectation equal to the orthogonal projection from $L_2(\varphi \otimes \tilde{\tau})$ onto $L_2(\varphi) \otimes 1$. Then $\mathcal{E}(\widetilde{F} \dot{\otimes} \widetilde{F}^*) = \sum f_k^* \dot{\otimes} f_k$. Since $\|\widetilde{F}\|_{(p)}^2 = \|\widetilde{F} \dot{\otimes} \widetilde{F}^*\|_{p/2}$ and $\|\widetilde{F} \dot{\otimes} \widetilde{F}^*\|_{p/2} \geq \|\mathcal{E}(\widetilde{F} \dot{\otimes} \widetilde{F}^*)\|_{p/2}$ by (8.13), the right hand side follows. To prove the left hand side, consider $F = \sum f_k \otimes c_k = \sum f_k \otimes u_k|c_k|$ with $(c_k)$ free circular as above and note that by the preceding observations (this is similar to Remark 11.4) we have $\sum f_k \otimes u_k = (3\pi/8)(Id \otimes \mathcal{E}_1)(\sum f_k \otimes u_k|c_k|)$ where $\mathcal{E}_1$ denotes the conditional expectation from the von Neumann algebra generated by $\{c_k\}$ onto the one generated by $\{u_k\}$. Since $\{u_k\}$ and $\{\lambda(g_k)\}$ have identical $*$-moments, we find

$$\|\widetilde{F}\|_{(p)} = \|\sum f_k \otimes u_k\|_{(p)} \leq (3\pi/8)\|\sum f_k \otimes u_k|c_k|\|_{(p)} = (3\pi/8)\|\sum f_k \otimes c_k\|_{(p)}$$

and hence the left hand side of (11.8) follows from (11.7), recalling that $C \leq 2$ when $(x_k)$ is a free circular sequence. □

*Remark* 11.10. A more careful estimate probably yields the preceding Corollary with the constant 2 in place of $3\pi/4$.

Lemma 11.11. *With the preceding notation let*

$$S(\nu) = \sum_{(k_1,\ldots,k_{2n}) \sim \nu} f_{k_1} \dot{\otimes} f_{k_2}^* \dot{\otimes} \cdots \dot{\otimes} f_{k_{2n}}^*,$$

*and let $\nu_0'$ (resp. $\nu_0''$) denote the partition of $[1,\ldots,2n]$ into consecutive pairs of the form $\{1,2\}$, $\{3,4\}, \ldots$ (resp. $\{2n,1\}$, $\{2,3\}$, $\{4,5\},\ldots$). We have then*

$$\|\widehat{\varphi}(S(\nu))\| \leq \max\{\|\widehat{\varphi}(S(\nu_0'))\|, \|\widehat{\varphi}(S(\nu_0''))\|\}.$$

*Sketch of Proof.* We set

$$\mathcal{C} = \sup \|\widehat{\varphi}(S(\nu))\|$$

where the sup runs over all pair partitions $\nu$ in $P_2(2n)$. By the cyclicity of the trace (see (8.1)) we may assume that $k_1,\ldots,k_{2n}$ is such that for some $j$ with $n < j \leq 2n$, the pair $\{k_n, k_j\}$ is a block of our partition $\nu$. Let

$$F(\nu) = \sum_{k_1,\ldots,k_{2n} \sim \nu} f_{k_1} \dot{\otimes} f_{k_2}^* \dot{\otimes} \cdots \dot{\otimes} f_{k_{2n}}^*.$$

We may rewrite $F(\nu)$ as

$$F(\nu) = \sum\nolimits_\alpha \sum\nolimits_\beta a_{\alpha,\beta} \dot{\otimes} b_{\alpha,\beta} \tag{11.9}$$

where $\alpha$ represents the set of indices $k_j$ such that the pair containing $j$ is split by the partition $[1,\dots,n][n+1,\dots,2n]$, and $\beta$ represents the remaining indices, and the sum is restricted to $(k_1,\dots,k_{2n}) \sim \nu$. Since the indices in $\beta$ correspond to pairs of indices $\{k_i,k_j\}$ with $\{i,j\}$ included either in $[1,\dots,n]$ or in $[n+1,\dots,2n]$, we can rewrite the sum (11.9) as

$$F(\nu) = \sum\nolimits_\alpha \sum\nolimits_{\beta',\beta''} \alpha_{\alpha,\beta'} \dot{\otimes} b_{\alpha,\beta''}.$$

Then

$$S(\nu) = \sum\nolimits_\alpha x_\alpha \dot{\otimes} y_\alpha$$

with $x_\alpha = \sum_{\beta'} a_{\alpha,\beta'}$ and $y_\alpha = \sum_{\beta''} b_{\alpha,\beta''}$. By (8.2) we find

$$\|\widehat{\varphi}(S(\nu))\| \le \left\| \widehat{\varphi}\left(\sum\nolimits_\alpha x_\alpha \dot{\otimes} x_\alpha^*\right)\right\|^{1/2} \left\| \widehat{\varphi}\left(\sum\nolimits_\alpha y_\alpha^* \dot{\otimes} y_\alpha\right)\right\|^{1/2}.$$

But now $\sum_\alpha x_\alpha \otimes x_\alpha^*$ is a sum of the kind $S(\nu')$ for some $\nu'$ but for which we know (by our initial choice relative to the pair $\{n,j\}$) that the pair $\{n,n+1\}$ appears in $\nu'$. If we then iterate the argument in the style of [6] we end up with a number $0<\theta<1$ such that we have either

$$\|\widehat{\varphi}(S(\nu))\| \le (\mathcal{C}')^\theta \mathcal{C}^{1-\theta}$$

or

$$\|\widehat{\varphi}(S(\nu))\| \le (\mathcal{C}'')^\theta \mathcal{C}^{1-\theta}$$

where $\mathcal{C}' = \|\widehat{\varphi}(S(\nu_0'))\|$ and $\mathcal{C}'' = \|\widehat{\varphi}(S(\nu_0''))\|$. Thus we conclude that

$$\mathcal{C} \le (\max(\mathcal{C}',\mathcal{C}''))^\theta \mathcal{C}^{1-\theta}$$

and hence $\mathcal{C} \le \max(\mathcal{C}',\mathcal{C}'')$. Since $S(\nu_0') = (\sum f_k \dot{\otimes} f_k^*)^{\dot{\otimes} n}$ and $S(\nu_0'') = (\sum f_k^* \dot{\otimes} f_k)^{\dot{\otimes} n}$, this completes the proof. □

By a spin system we mean a system of anticommuting self-adjoint unitaries assumed realized over a non-commutative probability space $(M,\tau)$. In the $q$-Gaussian case with $q=-1$, Theorem 11.2 describes the closed span of a spin system in $\Lambda_p$, and exactly for the same reason as in [7] we obtain optimal constants for those.

Corollary 11.12. *If $(x_k)$ is a spin system, then* (11.4) *holds with the same optimal constant $C_{\psi,p}$ as in the Gaussian case. In particular, this constant grows like $\sqrt{p}$ when $p\to\infty$. Moreover, the same result holds for the (Rademacher) sequence $(\varepsilon_k)$, and again the Gaussian constant is optimal.*

*Proof.* Let $\psi(q)$ denote the function $\psi$ for a $q$-Gaussian system. By Bożejko and Speicher's results (see [7]) we have $\psi(q)(\nu) = q^{i(\nu)}$ where $i(\nu)$ is the crossing number of the the partition $\nu$. This implies $|\psi(q)(\nu)| = \psi(|q|)(\nu)$ and hence also $C_{\psi(q),p} = C_{\psi(|q|),p}$. In particular, any spin system $(x_k)$ satisfies (11.4) with the constant $C_{\psi(1),p}$, i.e. the same constant as in the Gaussian case. We now address the Rademacher case. Just as in [7] we use the fact that the sequences $(x_k \otimes x_k)$ and $(\varepsilon_k)$ have the same distribution. We then apply (11.7) to $\sum f_k \otimes x_k$ with $f_k = b_k \otimes x_k$. Recalling that the $x_k$'s are unitary, we find $\sum f_k^* \dot{\otimes} f_k = \sum \bar{b}_k \otimes b_k \otimes 1$ and $\sum f_k \dot{\otimes} f_k^* = \sum b_k \otimes \bar{b}_k \otimes 1$. This gives us $\| \sum b_k \otimes \varepsilon_k \|_{(p)} \le C \| \sum b_k \otimes \bar{b}_k \|^{1/2}$ where $C = C_{\psi(1),p}$ is the Gaussian constant. By the central limit theorem, the latter is optimal. □

In the rest of this section we turn to the span of an i.i.d. sequence of Gaussian random matrices of size $N \times N$ in $\Lambda_p$. We will use ideas from [12] and [6]. We analyze the dependence in $N$ using a concentration of measure argument. Let $\{g_{ij} \mid i, j \ge 1\}$ be a doubly indexed family of complex valued Gaussian random variables such that $\mathbb{E} g_{ij} = 0$ and $\mathbb{E}|g_{ij}|^2 = 1$. Let $Y^{(N)}$ be the random $N \times N$ matrix defined by

$$Y^{(N)}(i,j) = N^{-1/2} g_{ij}.$$

Let $Y_1^{(N)}, Y_2^{(N)}, \dots$ be an i.i.d. sequence of copies of $Y^{(N)}$ on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We will view $(Y_j^{(N)})_{j \ge 1}$ as a sequence in $L_p(\mathbb{P} \times \tau_N)$ where $\tau_N$ denotes the normalized trace on $M_N$.

By the Appendix §15, we know that, for any even $p \ge 2$, $(Y_j^{(N)})_{j \ge 1}$ has $p$-th moments defined by pairings via the function

$$Y^{(N)}(\nu) = \mathbb{E}\tau_N(Y^{(N)\nu})$$

where $Y^{(N)\nu} = Y_{k_1}^{(N)} Y_{k_2}^{(N)^*} \dots Y_{k_{p-1}}^{(N)} Y_{k_p}^{(N)^*}$ for $k = (k_j)$ such that $k_i = k_j$ if and only if $(i,j)$ belong to the same block of $\nu$. It is easy to see that the distribution of $Y^{(N)\nu}$ does not depend on the choice of such a $k$. Moreover, $\psi^{(N)}(\nu) \ge 0$ for any $\nu$ since $\psi^{(N)}(\nu)$ is a sum of terms of the form

$$\mathbb{E}\left( Y_{k_1}^{(N)}(i_1, j_1) \overline{Y_{k_2}^{(N)}(i_2, j_2)} \dots \overline{Y_{k_p}^{(N)}(i_p, j_p)} \right)$$

and, when $k \sim \nu$, these are either $= 0$ or $= N^{-p/2}(\mathbb{E}|g_{11}|^2)^{p/2} = N^{-p/2}$. Therefore we again have

$$\sum |\psi^{(N)}(\nu)| = \sum \psi^{(N)}(\nu) = \mathbb{E}\tau_N(|Y^{(N)}|^p).$$

By (11.4) we have:

Corollary 11.13. *Let $p = 2n$ and let $(b_k)$ be any finite sequence in $B = B(H)$. Let $f \in \sum b_k \otimes Y_k^{(N)} \in B \otimes L_p(\mathbb{P} \times \tau_N)$. We have*

$$\|f\|_{(p)} \le (\mathbb{E}\tau_N(|Y^{(N)}|^p))^{1/p} \left\| \sum b_k \otimes \bar{b}_k \right\|^{1/2}$$

*and this constant is optimal.*

## 12 Non-commutative martingale inequalities

In this section, we assume given a filtration $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots$ of von Neumann subalgebras of $\mathcal{M}$. We assume for simplicity that $\mathcal{M}$ coincides with the von Neumann algebra generated by $\cup\mathcal{M}_n$. We will denote again by $\mathbb{E}_n$ the conditional expectation with respect to $\mathcal{M}_n$. Then to any $f$ in $L_p(\tau)$ $(1 \le p < \infty)$ we can associate a martingale $(f_n)$ (defined by $f_n = \mathbb{E}_n(f)$) that converges in $L_p(\tau)$ to $f$. We will continue to denote $d_0 = f_0$ and $d_n = f_n - f_{n-1}$.
It is natural to expect that Theorems 4.1 and 5.5 will extend to the non-commutative case. However, at the time of this writing, we have completed this task only for $p = 4$. We also proved below (see Theorem 13.1) a one sided version of (4.1) using the notion of $p$-orthogonal sums.
Let $H_1, H_2$ be two Hilbert spaces. To lighten the notation in the rest of this section we set $B_1 = B(H_1)$ and $B_2 = B(H_2)$. It is useful to observe that for any $f_1 \in B_1 \otimes L_4(\tau)$ and $f_2 \in B_2 \otimes L_4(\tau)$ we have

$$\widehat{\tau}(f_1^* \dot{\otimes} f_1 \dot{\otimes} f_2^* \dot{\otimes} f_2) \approx \widehat{\tau}(f_1 \dot{\otimes} f_2^* \dot{\otimes} f_2 \dot{\otimes} f_1^*) \succ 0. \tag{12.1}$$

Indeed, the first sign $\approx$ is by the trace property while sign $\succ 0$ holds because $f_1 \dot{\otimes} f_2^* \dot{\otimes} f_2 \dot{\otimes} f_1^* \approx F \dot{\otimes} F^*$ with $F = f_1 \dot{\otimes} f_2^* \in B_1 \otimes \overline{B}_2 \otimes L_2(\tau)$. In the next lemma, we extend this observation to $\widehat{\tau}(f_1^* \dot{\otimes} f_1 \dot{\otimes} T(f_2^* \dot{\otimes} f_2))$ where $T\colon\ L_2(\tau) \to L_2(\tau)$ is a completely positive map (e.g. a conditional expectation). The reader can convince himself easily that the simplest case of maps of the form $T(x) = \sum a_k^* x a_k$ $(a_k \in \mathcal{M})$, follows immediately from (12.1).

Lemma 12.1. *With the preceding notation, let $\mathcal{B} = B_1 \otimes \overline{B}_2 = B(H_1 \otimes_2 \overline{H}_2)$. For any completely positive map $T\colon\ L_2(\tau) \to L_2(\tau)$, we have for any $f_1 \in B_1 \otimes L_4(\tau)$ and $f_2 \in B_2 \otimes L_4(\tau)$*

$$\widehat{\tau}(f_1^* \dot{\otimes} f_1 \dot{\otimes} T(f_2^* \dot{\otimes} f_2)) \succ 0$$

*where the latter element is identified with an element of $\mathcal{B} \otimes \overline{\mathcal{B}}$, via the permutation $\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix} \to \begin{pmatrix}2 & 3 & 1 & 4\end{pmatrix}$ of the tensorial factors that takes $\overline{B}_1 \otimes B_1 \otimes \overline{B}_2 \otimes B_2$ to $\mathcal{B} \otimes \overline{\mathcal{B}}$.*

*Remark* 12.2. Let $E = L_4(\tau) \otimes \overline{L_4(\tau)}$. Let $T\colon\ L_2(\tau) \to L_2(\tau)$ be a completely positive map, so that for any finite sequence $a_1, \ldots, a_n$ in $L_4(\tau)$ the matrix $[T(a_i^* a_j)]$ is in $L_2(M_n(\mathcal{M}))_+$. Let $\Phi\colon\ E \otimes \overline{E} \to \mathbb{C}$ be the bilinear form defined by

$$\Phi(a_1 \otimes \bar{b}_1 \otimes \overline{a_2 \otimes \bar{b}_2}) = \tau(a_2^* a_1 T(b_1^* b_2)).$$

We claim that $\Phi$ is positive definite on $E \otimes \overline{E}$, i.e. $\Phi(e \otimes \bar{e}) \ge 0$ for any $e$ in $E$. Indeed, if $e = \sum a_i \otimes \bar{b}_i \in E$ we have

$$\Phi(e \otimes \bar{e}) = \sum\nolimits_{ij} \tau(a_j^* a_i T(b_i^* b_j)),$$

so that, if $\tau_n$ denote the trace on $M_n(\mathcal{M})$, we have $\Phi(e \otimes \bar{e}) = \tau_n(\alpha\beta) = \tau_n(a^{1/2} \beta \alpha^{1/2}) \ge 0$ where $\beta_{ij} = T(b_i^* b_j)$ $\alpha_{ij} = a_i^* a_j$ and of course $\alpha \ge 0$ and $\beta \ge 0$. This proves our claim.

*Proof of Lemma 12.1.* Let $L_p = L_p(\tau)$. Consider $f_1 \otimes \bar{f}_2 \in B_1 \otimes L_4 \otimes \overline{B}_2 \otimes \bar{L}_4$. Let $g \in \mathcal{B} \otimes E$ be the element obtained by the natural permutation of factors from $f_1 \otimes \bar{f}_2$. Then an easy verification shows that

$$\widehat{\tau}(f_1^* \dot{\otimes} f_1 \dot{\otimes} T(f_2^* \dot{\otimes} f_2)) \approx (I \otimes \Phi)(g \otimes \bar{g})$$

or more precisely with the notation indicated in Lemma 8.5

$$\widehat{\tau}(f_1^* \dot{\otimes} f_1 \dot{\otimes} T(f_2^* \dot{\otimes} f_2)) \approx (\Phi)_{24}(g \otimes \bar{g})$$

so that Lemma 12.1 follows from Lemma 8.5 and the preceding Remark. □

PROPOSITION 12.3. *Let $(f_n)_{n \geq 0}$ be a martingale in $B \otimes L_4(\tau)$. Assume for simplicity $f = f_N$ for some $N \geq 0$. Let $g = f^* \dot{\otimes} f - \sum d_n^* \dot{\otimes} d_n$. We have then*

$$\|g\|_{(2)} \leq \|f\|_{(4)}(\|\sigma_r\|_{(2)}^{1/2} + \|\sigma_c\|_{(2)}^{1/2}). \tag{12.2}$$

*where*

$$\sigma_r = \sum \mathbb{E}_{n-1}(d_n \dot{\otimes} d_n^*) \quad \textit{and} \quad \sigma_c = \sum \mathbb{E}_{n-1}(d_n^* \dot{\otimes} d_n).$$

*Proof.* As usual we start by $g = x + y$ with $x = \sum d_n^* \dot{\otimes} f_{n-1}$ and $y = \sum f_{n-1}^* \dot{\otimes} d_n$, so that $\|g\|_{(2)} \leq \|x\|_{(2)} + \|y\|_{(2)}$. Then

$$\|x\|_{(2)}^2 = \|\widehat{\tau}(x \dot{\otimes} x^*)\| = \left\|\widehat{\tau}\left(\sum d_n^* \dot{\otimes} f_{n-1} \dot{\otimes} f_{n-1}^* \dot{\otimes} d_n\right)\right\|.$$

Let $\delta_n = f - f_{n-1}$. Note that since $\mathbb{E}_{n-1}(\delta_n) = 0$

$$\mathbb{E}_{n-1}(f \dot{\otimes} f^*) = f_{n-1} \dot{\otimes} f_{n-1}^* + \mathbb{E}_{n-1}(\delta_n \dot{\otimes} \delta_n^*)$$

and hence

$$\widehat{\tau}\left(\sum d_n^* \dot{\otimes} f_{n-1} \dot{\otimes} f_{n-1}^* \dot{\otimes} d_n\right) =$$
$$= \widehat{\tau}\left(\sum d_n^* \dot{\otimes} \mathbb{E}_{n-1}(f \dot{\otimes} f^*) \dot{\otimes} d_n\right) - \widehat{\tau}\left(\sum d_n^* \dot{\otimes} \mathbb{E}_{n-1}(\delta_n \dot{\otimes} \delta_n^*) \dot{\otimes} d_n\right),$$

By the trace property and by Lemma 12.1, these last three terms can all be viewed as $\succ 0$ in a suitable permutation of the factors. This shows by (2.6)

$$\left\|\widehat{\tau}\left(\sum d_n^* \dot{\otimes} f_{n-1} \dot{\otimes} f_{n-1}^* \dot{\otimes} d_n\right)\right\| \leq \left\|\widehat{\tau}\left(\sum d_n^* \dot{\otimes} \mathbb{E}_{n-1}(f \dot{\otimes} f^*) \dot{\otimes} d_n\right)\right\|.$$

Since

$$\widehat{\tau}\left(\sum d_n^* \dot{\otimes} \mathbb{E}_{n-1}(f \dot{\otimes} f^*) \dot{\otimes} d_n\right) \approx \widehat{\tau}\left(\mathbb{E}_{n-1}(f \dot{\otimes} f^*) \dot{\otimes} \sum d_n \dot{\otimes} d_n^*\right),$$

and since $\mathbb{E}_{n-1}$ is self-adjoint, we have

$$\left\|\widehat{\tau}\left(\sum d_n^* \dot{\otimes} \mathbb{E}_{n-1}(f \dot{\otimes} f^*) \dot{\otimes} d_n\right)\right\| = \left\|\widehat{\tau}\left(f \dot{\otimes} f^* \dot{\otimes} \sum \mathbb{E}_{n-1}(d_n \dot{\otimes} d_n^*)\right)\right\|,$$

thus we find

$$\|x\|_{(2)}^2 \le \|\widehat{\tau}(f\dot{\otimes} f^*\dot{\otimes}\sigma_r)\| \le \|f\|_{(4)}^2\|\sigma_r\|_{(2)}.$$

A similar reasoning leads to

$$\|y\|_{(2)}^2 \le \|f\|_{(4)}^2\|\sigma_c\|_{(2)},$$

so we conclude

$$\|g\|_{(2)} \le \|f\|_{(4)}(\|\sigma_r\|_{(2)}^{1/2} + \|\sigma_c\|_{(2)}^{1/2}). \qquad \square$$

To complete the case $p = 4$, we need to check the non-commutative extension of Lemma 4.2 as follows:

Lemma 12.4. *Let $\theta_n$ be any finite sequence in $B \otimes L_4(\tau)$, let $\beta_n = \theta_n^*\dot{\otimes}\theta_n$ and $\alpha_n = \mathbb{E}_n(\beta_n)$. Then*

$$\left\|\sum \alpha_n\right\|_{(2)} \le 2\left\|\sum \beta_n\right\|_{(2)}.$$

*Proof.* The proof is essentially the same as for Lemma 4.2. We just need to observe that if $n \le k$ we have

$$\widehat{\tau}(\alpha_n\dot{\otimes}\alpha_k) = \widehat{\tau}(\alpha_n\dot{\otimes}\beta_k),$$

but also by Lemma 12.1 (and the trace property)

$$\widehat{\tau}(\alpha_n\dot{\otimes}\beta_k) \approx \widehat{\tau}(\beta_k\dot{\otimes}\alpha_n) \succ 0, \qquad \forall n, k$$

so that again we have

$$\left\|\sum\nolimits_{n\le k}\widehat{\tau}(\alpha_n\dot{\otimes}\beta_k)\right\| \le \left\|\sum\nolimits_{n,k}\widehat{\tau}(\alpha_n\dot{\otimes}\beta_k)\right\| = \|\widehat{\tau}(\alpha\dot{\otimes}\beta)\| \le \|\alpha\|_{(2)}\|\beta\|_{(2)}$$

and similarly for $\|\sum_{n>k}\|$. $\square$

Let $S_r = \sum d_j\dot{\otimes}d_j^*$ abd $S_c = \sum d_j^*\dot{\otimes}d_j$. Applying this Lemma to Proposition 12.3, we find

$$\|g\|_{(2)} \le 2\sqrt{2}\|f\|_{(4)}\max\{\|S_r\|_{(2)}^{1/2}, \|S_c\|_{(2)}^{1/2}\}.$$

We then obtain by the same reasoning as for the commutative case:

Corollary 12.5. *There is a constant $C$ such that for any finite martingale $f_0, \ldots, f_N$ in $L_4(\tau)$ we have*

$$C^{-1}\max\left\{\left\|\sum d_j\dot{\otimes}d_j^*\right\|_{(2)}^{1/2}, \left\|\sum d_j^*\dot{\otimes}d_j\right\|_{(2)}^{1/2}\right\} \le \|f\|_{(4)} \le$$
$$\le C\max\left\{\left\|\sum d_j\dot{\otimes}d_j^*\right\|_{(2)}^{1/2}, \left\|\sum d_j^*\dot{\otimes}d_j\right\|_{(2)}^{1/2}\right\}.$$

*Remark* 12.6. We leave as an open problem whether the extension of the left hand side of Corollary 12.5 is valid for any even integer $p > 4$. Note however that the right hand side is proved below as a consequence of Theorem 13.1.

We will now extend Theorem 5.5 to the non-commutative case for $p = 4$. Given $f \in B \otimes L_4(\tau)$ let us denote

$$\|f\|_{[4]} = \max\{\|\hat{\tau}(\sum d_n \dot{\otimes} d_n^* \dot{\otimes} d_n \dot{\otimes} d_n^*)\|^{1/4}, \|\sigma_r\|_{(2)}^{1/2}, \|\sigma_c\|_{(2)}^{1/2}\}.$$

Corollary 12.7. *For any finite martingale $f_0, \ldots, f_N$ in $L_4(\tau)$ we have*

$$2C^{-1}\|f\|_{[4]} \le \|f\|_{(4)} \le 2C\|f\|_{[4]}, \tag{12.3}$$

*where $C$ is as in the preceding statement.*

*Proof.* By the preceding Corollary and by Lemma 12.4 we have $\max\{\|\sigma_r\|_{(2)}^{1/2}, \|\sigma_c\|_{(2)}^{1/2}\} \le 2C\|f\|_{(4)}$. Moreover, by (12.1) we have $\hat{\tau}(\sum d_n \dot{\otimes} d_n^* \dot{\otimes} d_n \dot{\otimes} d_n^*) \prec \hat{\tau}(\sum d_n \dot{\otimes} d_n^*) \dot{\otimes} (\sum d_j \dot{\otimes} d_j^*)^*)$, and hence $\|\hat{\tau}(\sum d_n \dot{\otimes} d_n^* \dot{\otimes} d_n \dot{\otimes} d_n^*)\|^{1/4} \le \left\|\sum d_j \dot{\otimes} d_j^*\right\|_{(2)}^{1/2}$. Therefore, the left hand side of (12.3) follows.
To prove the right hand side, we will use the preceding Corollary.
Let $x_n = d_n^* \dot{\otimes} d_n$ and $y_n = d_n \dot{\otimes} d_n^*$. We have

$$\sum x_n = \sigma_c + \sum \delta_n$$

where $\delta_n = x_n - \mathbb{E}_{n-1} x_n$. Then, by the triangle inequality $\|\sum x_n\|_{(2)} \le \|\sigma_c\|_{(2)} + \|\sum \delta_n\|_{(2)}$ and by the orthogonality of the martingale differences $(\delta_n)$ we have $\|\sum \delta_n\|_{(2)}^2 = \|\sum \hat{\tau}(\delta_n^* \dot{\otimes} \delta_n)\|$. But by Remark 8.9 we have $\hat{\tau}(\delta_n^* \dot{\otimes} \delta_n) \prec \hat{\tau}(x_n^* \dot{\otimes} x_n)$ and hence also $\sum \hat{\tau}(\delta_n^* \dot{\otimes} \delta_n) \prec \sum \hat{\tau}(x_n^* \dot{\otimes} x_n)$ from which follows by Lemma 2.3 that $\|\sum \hat{\tau}(\delta_n^* \dot{\otimes} \delta_n)\| \le \|\sum \hat{\tau}(x_n^* \dot{\otimes} x_n)\|$.
Recapitulating, we find $\|\sum x_n\|_{(2)} \le \|\sigma_c\|_{(2)} + \|\sum \hat{\tau}(x_n^* \dot{\otimes} x_n)\|^{1/2}$, and a fortiori

$$\|\sum x_n\|_{(2)}^{1/2} \le \|\sigma_c\|_{(2)}^{1/2} + \|\sum \hat{\tau}(x_n^* \dot{\otimes} x_n)\|^{1/4} \le 2\|f\|_{[4]}.$$

Since a similar argument applies to majorize $\|\sum y_n\|_{(2)}$, by Corollary 12.5 we obtain

$$C^{-1}\|f\|_{(4)} \le \max\{\|\sum x_n\|_{(2)}^{1/2}, \|\sum y_n\|_{(2)}^{1/2}\} \le 2\|f\|_{[4]}.$$

□

## 13 $p$-orthogonal sums

Let $L_p(\tau)$ be as before the "non-commutative" $L_p$-space associated to a von Neumann algebra equipped with a standard (= faithful, normal) semi-finite trace. (Of course, if $M$ is commutative, we recover the classical $L_p$ associated to a measure space.) Let $p \geq 2$ be an even integer. A family $d = (d_i)_{i \in I}$ is called $p$-orthogonal if, for any *injective* function $g\colon\ [1, 2, \ldots, p] \to I$ we have

$$\tau(d^*_{g(1)} d_{g(2)} d^*_{g(3)} d_{g(4)} \ldots d^*_{g(p-1)} d_{g(p)}) = 0.$$

Clearly, any martingale difference sequence is $p$-orthogonal, but the class of $p$-orthogonal sums is more general. In the commutative case, *i.e.* for classical random variables, this notion is very close to that of "multiplicative sequence" already considered in the literature, see the references in [24], on which this section is modeled.

By a natural extension, we will say that a sequence $(d_j)_{j \in I}$ in $B \otimes L_p(\tau)$ is $p$-orthogonal if for any *injective* function $g\colon\ [1, 2, \ldots, p] \to I$ as before we have

$$\widehat{\tau}(d^*_{g(1)} \dot{\otimes} d_{g(2)} \dot{\otimes} \ldots \dot{\otimes} d^*_{g(p-1)} \dot{\otimes} d_{g(p)}) = 0.$$

The method used in [24], that is based on a combinatorial formula involving the "Möbius function", is particularly easy to adapt to our setting where $\Lambda_p$ takes the place of $L_p$.

We will use crucially some well known ideas from the combinatorial theory of partitions, which can be found, for instance, in the book [1]. We denote by $P_n$ the lattice of all partitions of $[1, \ldots, n]$, equipped with the following order: we write $\sigma \leq \pi$ (or equivalently $\pi \geq \sigma$) when every "block" of the partition $\sigma$ is contained in some block of $\pi$. Let $\dot{0}$ and $\dot{1}$ be respectively the minimal and maximal elements in $P_n$, so that $\dot{0}$ is the partition into $n$ singletons and $\dot{1}$ the partition formed of the single set $\{1, \ldots, n\}$. We denote by $\nu(\pi)$ the number of blocks of $\pi$ (so that $\nu(\dot{0}) = n$ and $\nu(\dot{1}) = 1$).

For any $\pi$ in $P_n$ and any $i = 1, 2, \ldots, n$, we denote by $r_i(\pi)$ the number of blocks (possibly $= 0$) of $\pi$ of cardinality $i$. In particular, we have $\sum_1^n i r_i(\pi) = n$ and $\sum_1^n r_i(\pi) = \nu(\pi)$.

Given two partitions $\sigma, \pi$ in $P_n$ with $\sigma \leq \pi$ we denote by $\mu(\sigma, \pi)$ the Möbius function, which has the following fundamental property:

Let $V$ be a vector space. Consider two functions $\Phi\colon\ P_n \to V$ and $\Psi\colon\ P_n \to V$. If $\Psi(\sigma) = \sum\limits_{\pi \leq \sigma} \Phi(\pi)$, then $\Phi(\sigma) = \sum_{\pi \leq \sigma} \mu(\pi, \sigma) \Psi(\pi)$.

Essentially equivalently, if $\Psi(\sigma) = \sum_{\pi \geq \sigma} \Phi(\pi)$, then $\Phi(\sigma) = \sum_{\pi \geq \sigma} \mu(\sigma, \pi) \Psi(\pi)$.

In particular we have:

$$\forall\ \sigma \neq \dot{0} \quad \sum_{\dot{0} \leq \pi \leq \sigma} \mu(\pi, \sigma) = 0. \tag{13.1}$$

The last assertion follows from the above fundamental property applied with $\Phi$ equal to the delta function at $\dot{0}$ (i.e. $\Phi(\pi) = 0\ \forall\ \pi \neq \dot{0}$ and $\Phi(\dot{0}) = 1$) and $\Psi \equiv 1$.

We also recall Schützenberger's theorem (see [1]):

For any $\pi$ we have

$$\mu(\dot{0},\pi)=\prod_{i=1}^{n}[(-1)^{i-1}(i-1)!]^{r_i(\pi)},$$

and consequently

$$\sum_{\pi\in P_n}|\mu(\dot{0},\pi)|=n!.$$

We now apply these results to set the stage for the questions of interest to us. Let $E_1,\ldots,E_n,V$ be vector spaces equipped with a multilinear form (= a "product")

$$\varphi\colon\ E_1\times\cdots\times E_n\to V.$$

Let $I$ be a finite set. For each $k=1,2,\ldots,n$ and $i\in I$, we give ourselves elements $d_i(k)\in E_k$, and we form the sum

$$F_k=\sum_{i\in I}d_i(k).$$

Then we are interested in "computing" the quantity $\varphi(F_1,\ldots,F_n)$. We have obviously

$$\varphi(F_1,\ldots,F_n)=\sum_g\varphi(d_{g(1)}(1),\ldots,d_{g(n)}(n))$$

where the sum runs over all functions $g\colon\ [1,2,\ldots,n]\to I$. Let $\pi(g)$ be the partition associated to $g$, namely the partition obtained from $\bigcup\limits_{i\in I}g^{-1}(\{i\})$ after deletion of all the empty blocks. We can write

$$\varphi(F_1,\ldots,F_n)=\sum_{\sigma\in P_n}\Phi(\sigma)$$

where $\Phi(\sigma)=\sum_{g\colon\ \pi(g)=\sigma}\varphi(d_{g(1)}(1),\ldots,d_{g(n)}(n))$. Let $\Psi(\sigma)=\sum\limits_{\pi\geq\sigma}\Phi(\pi)$. Using (13.1) (with $\sigma,\pi$ exchanged), we have then:

$$\varphi(F_1,\ldots,F_n)=\Phi(\dot{0})+\sum_{\dot{0}<\sigma}\Phi(\sigma)=\Phi(\dot{0})+\sum_{\dot{0}<\sigma}\sum_{\pi\geq\sigma}\mu(\sigma,\pi)\Psi(\pi)\tag{13.2}$$

$$=\Phi(\dot{0})+\sum_{\dot{0}<\pi}\Psi(\pi)\cdot\sum_{\dot{0}<\sigma\leq\pi}\mu(\sigma,\pi)=\Phi(\dot{0})-\sum_{\dot{0}<\pi}\Psi(\pi)\mu(\dot{0},\pi).\tag{13.3}$$

Recapitulating, we found:

$$\varphi(F_1,\ldots,F_n)=\Phi(\dot{0})-\sum_{\dot{0}<\pi}\Psi(\pi)\mu(\dot{0},\pi)\tag{13.4}$$

where

$$\Phi(\dot{0}) = \sum_{g \text{ injective}} \varphi(d_{g(1)}(1), \ldots, d_{g(n)}(n))$$

and

$$\Psi(\pi) = \sum_{g:\ \pi(g) \geq \pi} \varphi(d_{g(1)}(1), \ldots, d_{g(n)}(n)).$$

Theorem 13.1. *Let $p = 2n$ be an even integer $> 2$. Then for any $p$-orthogonal finite sequence $(d_j)_{j \in I}$ in $B \otimes L_p(\tau)$ we have*

$$\left\| \sum d_j \right\|_{(p)} \leq (3\pi/2) p \max \left\{ \left\| \sum d_j \dot{\otimes} d_j^* \right\|_{(p/2)}^{1/2}, \left\| \sum d_j^* \dot{\otimes} d_j \right\|_{(p/2)}^{1/2} \right\}. \tag{13.5}$$

*Proof.* This proof is modeled on that in [24] so we will be deliberately sketchy. Let $f = \sum d_j$. We can write

$$\widehat{\tau}[(f^* \dot{\otimes} f)^{\dot{\otimes} n}] = - \sum_{\dot{0} < \pi} \mu(\dot{0}, \pi) \Psi(\pi)$$

where $\Phi$ and $\Psi$ are defined by

$$\Phi(\sigma) = \sum_{g:\ \pi(g) = \sigma} \widehat{\tau}(d_{g(1)}^* \dot{\otimes} d_{g(2)} \ldots \dot{\otimes} d_{g(p-1)}^* \dot{\otimes} d_{g(p)})$$

and $\Psi = \sum\limits_{\sigma \geq \pi} \Phi(\sigma)$, or equivalently,

$$\Psi(\pi) = \sum\nolimits_{g \sim \sigma} \widehat{\tau}(d_{g(1)}^* \dot{\otimes} \cdots \dot{\otimes} d_{g(p)})$$

where (as in §11) $g \sim \sigma$ means that $g(i) = g(j)$ whenever $i, j$ are in the same block of $\sigma$. Here the functions $\Phi$ and $\Psi$ take values in $\overline{B} \otimes B \otimes \cdots \otimes \overline{B} \otimes B$ where $\overline{B} \otimes B$ is repeated $n$-times .

Let $\alpha = 3\pi/4$ as in [24]. Arguing as in [24, p. 912] we see that it suffices to prove that

$$\|\Psi(\pi)\| \leq (\alpha \Delta)^{p - r_1(\pi)} \|f\|_{(p)}^{r_1(\pi)} \tag{13.6}$$

where $\Delta = \max \left\{ \left\| \sum d_j \dot{\otimes} d_j^* \right\|_{(p/2)}^{1/2}, \left\| \sum d_j^* \dot{\otimes} d_j \right\|_{(p/2)}^{1/2} \right\}$, and we recall that $r_1(\pi)$ is the number of singletons in $\pi$. Let $\mathbb{F}_I$ be the free group with generators $(g_j)_{j \in I}$, and let $\varphi$ be the normalized trace on the von Neumann algebra of $\mathbb{F}_I$. Let $f_k = \sum_{i \in I} d_i(k)$ be a finite sum in $B \otimes L_p(\tau)$, $k = 1, \ldots, p$. We denote by

$$\tilde{f}_k = \sum\nolimits_{i \in I} \lambda(g_i) \otimes d_i(k)$$

the corresponding sum in $L_p(\varphi \times \tau) \otimes B$. Note that by (11.8) we know that

$$\|\tilde{f}_k\|_{(p)} \leq (3\pi/4) \max \left\{ \left\| \sum d_j \dot{\otimes} d_j^* \right\|_{(p/2)}^{1/2}, \left\| \sum d_j^* \dot{\otimes} d_j \right\|_{(p/2)}^{1/2} \right\}. \tag{13.7}$$

Let $\pi$ be a partition of $[1,\dots,p]$. Let $B_1$ be the union of all the singletons in $\pi$ and let $B_2$ be the complement of $B_1$ in $[1,\dots,p]$. By the construction in the proof of [24, Sublemma 3.3] for a suitable discrete group $G$ there are $F_1,\dots,F_p$ in $L_p(\tau_G\times\tau)\otimes B$ such that $\|F_k\|_{(p)}=\|\tilde f_k\|_{(p)}$ for all $k$ in $B_2$, $\|F_k\|_{(p)}=\|f_k\|_{(p)}$ for all $k$ in $B_1$, and also

$$\widehat{\tau}\left(\sum_{\pi(g)\geq\pi} d_{g(1)}(1)\dot{\otimes}\cdots\dot{\otimes}d_{g(p)}(p)\right)=(\widehat{\tau_G\times\tau})[F_1\dot{\otimes}\cdots\dot{\otimes}F_p].$$

Then if we apply this to $d_j(k)=d_j^*$ if $k$ is odd and $d_j(k)=d_j$ if $k$ is even we find by (8.4)

$$\begin{aligned}\|\Psi(\pi)\|&=\|(\widehat{\tau_G\times\tau})(F_1\dot{\otimes}\cdots\dot{\otimes}F_p)\|\\&\leq\prod_{k=1}^{p}\|F_k\|_{(p)}\\&\leq\|f\|_{(p)}^{|B_1|}\cdot\prod_{k\in B_2}\|\tilde f_k\|_{(p)}\end{aligned}$$

and by (13.7) we obtain (13.6). □

Corollary 13.2. *Let $p$ be an even integer. Assume $\tau(1)=1$. Let $(f_j)$ be a $p$-orthogonal sequence in $L_p(\tau)$ that is orthonormal in $L_2(\tau)$. Consider a finite sequence $(b_j)$ with $b_j\in B$. We have then*

$$\|\sum b_j\otimes\bar b_j\|^{1/2}\leq\left\|\sum b_j\otimes f_j\right\|_{(p)}\leq(3\pi/2)p\|\sum b_j\otimes\bar b_j\|^{1/2}.$$

*Let $E_p$ denote the closed span of $(f_j)$ in $\Lambda_p(\tau)$. Then $E_p$ is completely isomorphic to $OH$, and moreover, the orthogonal projection $P$ from $L_2(\tau)$ onto the span of $(f_j)$ is c.b. on $\Lambda_p$ with c.b. norm at most $(3\pi/2)p$.*

*Proof.* The right hand side of the inequality follows from (13.5) since $(d_j)=(b_j\otimes f_j)$ is clearly $p$-orthogonal. By Corollary 8.4, the inclusion $\Lambda_p(\tau)\to\Lambda_2(\tau)=L_2(\tau)$ has c.b norm 1. Using this the left hand side follows. This shows that $E_p\simeq OH$. The projection $P$ can then be factorized as $\Lambda_p\to\Lambda_2\to E_2\to E_p$, which implies $\|P\colon\,\Lambda_p\to E_p\|_{cb}\leq(3\pi/2)p$, since the first two arrows are completely contractive. □

## 14 Lacunary Fourier series in $\Lambda_p$

In this section, we review the results of [13] and [24] with $\Lambda_p$ in place of $L_p$, and again we find the space $OH$ appearing in place of $R_p\cap C_p$. To save space, it will be convenient to adopt the general viewpoint in [24, §4], although this may seem obscure to a reader unfamiliar with [13].

Notation: Let $1=\sum_{k\in J}P_k$ be an orthogonal decomposition of the identity of $L_2(\tau)$ on a semi-finite "non-commutative" measure space $(M,\tau)$. Let $p=2n$ be an even integer $>2$. Let $(d_j)_{j\in I}$ be a finite family in $B\otimes L_p(\tau)$. We set $x^\omega=x^*$ if $n$ is odd and $x^\omega=x$ in $n$ is even.

Theorem 14.1. *Let $F$ be the set of all injective functions $g\colon\ [1,2,\dots,n]\to I$. For any $g$ in $F$, we let $x_g=d^*_{g(1)}\dot{\otimes}d_{g(2)}\dot{\otimes}d^*_{g(3)}\dots d^\omega_{g(n)}$. We define*

$$N(d)=\sup_{k\in J}\operatorname{card}\{g\in F\mid (P_k\otimes I)(x_g)\neq 0\}.$$

*Then*

$$\Big\|\sum_{j\in I}d_j\Big\|_{(p)}\le\left[(4N(d))^{1/p}+p\frac{9\pi}{8}\right]\max\left\{\Big\|\sum d_j\dot{\otimes}d_j^*\Big\|^{1/2}_{(p/2)},\Big\|\sum d_j^*\dot{\otimes}d_j\Big\|^{1/2}_{(p/2)}\right\}.$$

*Proof.* Since we follow closely the ideas in [13] and [24] we will merely sketch the proofs. We have

$$\|f\|^n_{(p)}=\|f^*\dot{\otimes}f\cdots\dot{\otimes}f^\omega\|_{(2)}.$$

Arguing as in [24, p. 919] we find

$$f^*\dot{\otimes}f\cdots\dot{\otimes}f^\omega=\Phi(\dot{0})-\sum_{\dot{0}<\pi\in P_n}\mu(\dot{0},\pi)\Psi(\pi)$$

with $\Phi(\sigma)=\sum_{\pi(g)=\sigma}x_g$ and

$$\Psi(\pi)=\sum\nolimits_{\sigma\le\pi}\Phi(\sigma).$$

Note that $\Phi(\dot{0})=\sum_{g\in F}x_g$. Using a suitable adaptation of [24, Sublemma 3.3] and replacing [24, (3.5)] by Corollary 11.9 above, we find:

$$\|\Psi(\pi)\|_{(2)}\le\|f\|^{r_1(\pi)}_{(p)}(\alpha\Delta)^{n-r_1(\pi)}$$

where

$$\Delta=\max\left\{\Big\|\sum d_j\dot{\otimes}d_j^*\Big\|^{1/2}_{(p/2)},\Big\|\sum d_j^*\dot{\otimes}d_j\Big\|^{1/2}_{(p/2)}\right\}.$$

Let $F_k=\{g\in F\mid (id\otimes P_k)x_g\neq 0\}$ and $\Phi_k=(id\otimes P_k)\Phi(\dot{0})$, so that

$$\Phi(\dot{0})=\sum\Phi_k\quad\text{and}\quad\Phi_k=\sum_{g\in F_k}x_g(k)$$

where $x_g(k)=(id\otimes P_k)(x_g)$. By (2.3) we have by "orthogonality" of $\Phi_k$

$$\|\Phi(\dot{0})\|^2_{(2)}=\left\|\widehat{\tau}\left(\sum\Phi_k\dot{\otimes}\Phi_k^*\right)\right\|.$$

Since $\text{card}(F_k) \le N(d)$, by (2.8) and Lemma 8.6 we have

$$\widehat{\tau}\left(\sum \Phi_k \dot{\otimes} \Phi_k^*\right) \prec N(d)\widehat{\tau}\left(\sum\nolimits_k \sum\nolimits_{g\in F_k} x_g(k)\dot{\otimes} x_g(k)^*\right) = \\ = N(d)\widehat{\tau}\left(\sum\nolimits_{g\in F} x_g \dot{\otimes} x_g^*\right).$$

Therefore, we find

$$\frac{1}{N(d)}\|\Phi(\dot{0})\|_{(2)}^2 \le \left\|\widehat{\tau}\left(\sum\nolimits_{g\in F} x_g \dot{\otimes} x_g^*\right)\right\| \le \\ \le \left\|\widehat{\tau}\left(\sum_{g(1),\dots,g(n)} d_{g(1)}^* \dot{\otimes}\cdots\dot{\otimes} d_{g(n)}^{\omega} \dot{\otimes} d_{g(n)}^{\omega *}\dots\dot{\otimes} d_{g(1)}\right)\right\|$$

and hence

$$\|\Phi(\dot{0})\|_{(2)}^2 \le N(d)\left\|\widehat{\tau}\left(\left(\sum d_j \dot{\otimes} d_j^*\right)^{\dot{\otimes} n}\right)\right\| = N(d)\left\|\sum d_j \dot{\otimes} d_j^*\right\|_{(p/2)}^{p/2}.$$

Thus we may conclude by the same reasoning as in [24, p. 920]. □

We can now reformulate the main result of [13] with $\Lambda_p$ in place of $L_p$:

COROLLARY 14.2. *Fix an even integer $p = 2n > 2$. Let $E \subset \Gamma$ be a subset of a discrete group $\Gamma$ with unit $e$. For any $\gamma$ in $\Gamma$ let $Z_p(\gamma, E)$ be the cardinality of the set of injective functions $g\colon [1,\dots,n] \to E$ such that*

$$\gamma = g(1)g(2)^{-1}g(3)\dots g(n)^w$$

*where $g^w = g^{-1}$ if $n$ is even and $g^w = g$ if $n$ is odd. We set*

$$Z(E) = \sup\{Z_p(\gamma, E) \mid \gamma \in \Gamma\}.$$

*Then for any finitely supported family $(b(t))_{t\in E}$ in $B = B(H)$ we have*

$$\left\|\sum\nolimits_{t\in E} \lambda(t)\otimes b(t)\right\|_{(p)} \le ((4Z(E))^{1/p} + (9\pi/8)p)\left\|\sum b(t)\otimes \overline{b(t)}\right\|^{1/2}. \quad (14.1)$$

*Proof.* Here $L_2(\tau)$ is the $L_2$-space associated to the usual trace on the von Neumann algebra associated to $\Gamma$. Any element in $L_2(\tau)$ has an orthonormal expansion in a "Fourier series" $x = \sum_{t\in\Gamma} x(t)\lambda(t)$, so we can apply Theorem 14.1 to this orthogonal decomposition (with $J = \Gamma$). Note that if $(t_j)$ are distinct elements in $E$ and if $d_j = \lambda(t_j)\otimes b(t_j)$ we have $N(d) \le Z(E)$. Lastly, we note that in the present situation, since $L_p(\varphi) = \mathbb{C}$, the term previously denoted by $\Delta$ coincides with

$$\left\|\sum b(t)\otimes \overline{b(t)}\right\|^{1/2}.$$ □

We have also a (one sided) version of the Littlewood–Paley inequality for $\Lambda_p$:

Corollary 14.3. *Consider a Fourier series of the form*

$$f = \sum\nolimits_{n>0} \hat{f}(n)e^{int}$$

*where* $n \to \hat{f}(n)$ *is a finitely supported* $B(H)$*-valued function. Let*

$$\Delta_n = \sum\nolimits_{2^n \le k < 2^{n+1}} \hat{f}(k)e^{ikt}$$

*and let*

$$\mathcal{S}(f) = \sum \Delta_n \dot{\otimes} \overline{\Delta_n}.$$

*There is an absolute constant* $C$ *such that for any even integer* $p \ge 2$

$$\|f\|_{(p)} \le C_p \|\mathcal{S}(f)\|_{(p/2)}^{1/2}.$$

*Proof.* It suffices to prove the inequality separately for the cases $f = \sum_m \Delta_{2m}$ and $f = \sum_m \Delta_{2m+1}$. But then each of these cases follows from Theorem 14.1 and elementary arithmetic involving lacunary sequences. □

It may be worthwhile to point out that in the commutative case, the following variant of Theorem 14.1 holds:

Theorem 14.4. *Consider the same situation as in Theorem 14.1 but with* $M$ *commutative so that* $L_2(\tau)$ *can be identified with* $L_2(\Omega, \mu)$*. Let* $y_g = d_{g(1)} \dot{\otimes} d_{g(2)} \dot{\otimes} \cdots \dot{\otimes} d_{g(n)}$ *and let*

$$N_+(d) = \sup_{k \in J} \operatorname{card}\{g \in F \mid (P_k \otimes I)(y_g) \ne 0\}.$$

*Then*

$$\left\| \sum d_j \right\|_{(p)} \le [(4N_+(d))^{1/p} + 9\pi/8] \left\| \sum d_j \dot{\otimes} \bar{d}_j \right\|_{(p/2)}^{1/2}.$$

*Proof.* We argue exactly as for Theorem 14.1 except that we start instead from

$$\|f\|_{(p)}^n = \|f \dot{\otimes} \cdots \dot{\otimes} f\|_{(2)}.$$ □

*Remark* 14.5. In particular, if $E \subset \Gamma$ is a subset of a *commutative* group, let $Z_{p+}(\gamma, E)$ be the cardinality of the set of injective $g\colon\ [1, \dots, n] \to E$ such that $\gamma = g(1)g(2)\dots g(n)$ and let $Z_+(E) = \sup\{Z_{p+}(\gamma, E) \mid \gamma \in \Gamma\} < \infty$. Then for any finitely supported family $(b(t))_{t \in E}$ in $B$ we have (14.1) with $Z_+(E)$ in place of $Z(E)$. Thus we obtain $OH$ also for the span of certain $\Lambda(p)$-sets originally considered by Rudin [33], which are *not* $\Lambda(p)_{cb}$-sets in the sense of [13].

## 15 Appendix

The goal of this appendix is to clarify the relation between "moments defined by pairings" used in §9 (following [6]) and the well known Wick formula. The latter (probably going back independently to Ito and Wick) was used by Ito in connection with multiple Wiener integrals and Wiener chaos. Although we reformulate them using tensor products, the results below are all well known.
We first consider the Gaussian case in a very general framework. Let $B$ be a real vector space. Let $X$ be a $B$-valued Gaussian variable. This means that for any $\mathbb{R}$-linear form $\xi \in B^*$, the real valued variable $\xi(X)$ is Gaussian with mean zero and variance equal to $\mathbb{E}\xi(X)^2$. If $B$ is a complex space, (e.g. if $B = \mathbb{C}$) we may view it a fortiori as a real one and the previous notion still makes sense.
Let $X = (X_1, \ldots, X_n)$ be a Gaussian variable with values in $B^n$. Then $X_1 \otimes \cdots \otimes X_n$ is a random variable with values in $B^{\otimes n}$. When $n$ is odd its mean vanishes. Let us assume that $X = (X_1, \ldots, X_n)$ is defined on $(\Omega, \mathcal{A}, \mathbb{P})$ and that $n$ is even.
Let $\pi$ be a partition of $[1, \ldots, n]$ into $K$ blocks. We will define a $B^{\otimes n}$-valued random variable $X^{\otimes\pi}$ on $(\Omega, \mathcal{A}, \mathbb{P})^{\otimes K}$ as follows: Assume that the blocks of $\pi$ have been enumerated as $\alpha_1, \ldots, \alpha_K$. We define $\widehat{\omega}_j$ for $j = 1, \ldots, n$ by setting $\widehat{\omega}_j = \omega_k$ if $j \in \alpha_k$. We then define

$$X^{\otimes\pi}(\omega_1, \ldots, \omega_K) = X_1(\widehat{\omega}_1) \otimes \cdots \otimes X_n(\widehat{\omega}_n).$$

Note that the distribution (and hence all the moments) of $X^{\otimes\pi}$ do not depend on the particular enumeration $(\alpha_1, \alpha_2, \ldots)$ chosen to define it. In particular, $\mathbb{E}(X^{\otimes\pi})$ depends only on $\pi$. We now may state

Proposition 15.1. *For any even integer $n$*

$$\mathbb{E}(X_1 \otimes \cdots \otimes X_n) = \sum\nolimits_{\nu \in P_2(n)} \mathbb{E}(X^{\otimes\nu}). \tag{15.1}$$

*Proof.* We will use the same trick as in [12, Prop. 1.5]) to deduce the formula from the rotational invariance of Gaussian distribution. Let $X(s) = (X_1(s), \ldots, X_n(s)$ be an i.i.d. sequence indexed by $s \in \mathbb{N}$ of copies of $X$. By the invariance of Gaussian distributions, the variable $\widehat{X}(s) = s^{-1/2}(X(1) + \cdots + X(s))$ has the same distribution as $X$. Therefore for any $s$, we have

$$\mathbb{E}(X_1 \otimes \cdots \otimes X_n) = \mathbb{E}(\widehat{X}_1(s) \otimes \cdots \otimes \widehat{X}_n(s))$$

and hence

$$\mathbb{E}(X_1 \otimes \cdots \otimes X_n) = \lim_{s \to \infty} \mathbb{E}(\widehat{X}_1(s) \otimes \cdots \otimes \widehat{X}_n(s)). \tag{15.2}$$

Let $E(s) = \mathbb{E}(\widehat{X}_1(s) \otimes \cdots \otimes \widehat{X}_n(s))$. We have

$$E(s) = s^{-n/2} \sum\nolimits_g \mathbb{E}(X_1(g(1)) \otimes \cdots \otimes X_n(g(n))),$$

where the sum runs over all functions $g\colon [1,\ldots,n] \to [1,\ldots,s]$. We claim that after elimination of all the (vanishing) odd terms and all the (asymptotically vanishing) terms such as

$$\frac{1}{s^2}\sum\nolimits_{t\le s} X_1(t)\otimes\cdots\otimes X_4(t)$$

we find

$$\lim\nolimits_{s\to\infty} E(s) = \sum\nolimits_{\nu\in P_2(n)} \mathbb{E}(X^{\otimes\nu}). \tag{15.3}$$

Indeed, if we let

$$t(g) = \mathbb{E}(X_1(g(1))\otimes\cdots\otimes X_n(g(n))$$

and if we denote by $\pi(g)$ the partition of $[1,\ldots,n]$ defined by $\bigcup_{k\le s} g^{-1}(k)$ we have (eliminating vanishing terms)

$$E(s) = s^{-n/2}\sum\nolimits_{g\in A_s} t(g)$$

where $A_s$ is the set of $g\colon [1,\ldots,n]\to[1,\ldots,s]$ such that $\pi(g)$ is a partition of $[1,\ldots,n]$ into blocks of even cardinality. For any such $\pi$, let

$$G_s(\pi) = \{g\in A_s \mid \pi(g)=\pi\}.$$

Let $B_s\subset A_s$ be the set of all $g$'s such that $\pi(g)$ is in $P_2(n)$ (i.e. is a partition into pairs) so that

$$B_s = \bigcup\nolimits_{\nu\in P_2(n)} G_s(\nu).$$

Note that for any $g$ in $G_s(\pi)$ we have

$$t(g) = \mathbb{E}(X^{\otimes\pi}).$$

Let $P_2'(n)$ denote the set of partitions $\pi$ of $[1,\ldots,n]$ into blocks of even cardinality. We have

$$E(s) = s^{-n/2}\sum\nolimits_{\pi\in P_2'(n)} |G_s(\pi)|\mathbb{E}(X^{\otimes\pi}).$$

Note that $P_2(n)\subset P_2'(n)$. Therefore

$$E(s) = \sum\nolimits_{\nu\in P_2(n)} s^{-n/2}|G_s(\nu)|\mathbb{E}(X^{\otimes\nu}) + \sum\nolimits_{\pi\in P_2'(n)\setminus P_2(n)} s^{-n/2}|G_s(\pi)|\mathbb{E}(X^{\otimes\pi}). \tag{15.4}$$

But now a simple counting argument shows that $|G_s(\nu)| = s(s-1)\ldots\left(s-\frac{n}{2}+1\right)\simeq s^{n/2}$ and hence $s^{-n/2}|G_s(\nu)|\to 1$, while for any $\pi$ in $P_2'(n)\setminus P_2(n)$ we have $s^{-n/2}|G_s(\pi)|\to 0$. Thus taking the limit when $s\to\infty$ in (15.4) yields our claim (15.3). By (15.2) this completes the proof. □

To emphasize the connection with the classical Wick formula of which (15.1) is but an abstract form, let us state:

COROLLARY 15.2. *Let $X = (X_j)$ be a Gaussian sequence of real valued random variables (i.e. all their linear combinations are Gaussian). Then*

$$\mathbb{E}(X_1 \cdots X_n) = \sum_\nu \prod \langle X_{k_j} X_{\ell_j} \rangle$$

*where the sum runs over all partitions $\nu$ of $[1, \ldots, n]$ into pairs, the product runs over all pairs $\{k_j, \ell_j\}$ $(j = 1 \cdots n/2)$ of $\nu$, and the scalar products are meant in $L_2$.*

COROLLARY 15.3. *For any even integer $p$ and any $N \geq 1$, any sequence $X = (X_j)$ of i.i.d. Gaussian variables with values in the space $M_N$ of $N \times N$ (complex) matrices has its $p$-th moments defined by pairings. (Here the moments are meant with respect to the functional $x \to \mathbb{E}\ \mathrm{tr}(x)$.)*

*Proof.* Consider the $\mathbb{R}$-linear map $\varphi$: $M_N \otimes \cdots \otimes M_N \to \mathbb{C}$ defined by

$$\varphi(x_1 \otimes \cdots \otimes x_p) = \mathrm{tr}(x_1 x_2^* \ldots x_{p-1} x_p^*).$$

Let $k = (k_1, \ldots, k_p)$ where $k_1, k_2, \ldots, k_p$ are positive integers. Let $Y_k = (X_{k_1}, \ldots, X_{k_p})$. Applying (15.2) to $X_{k_1} \otimes \cdots \otimes X_{k_p}$ we find

$$\mathbb{E}\varphi(X_{k_1} \otimes \cdots \otimes X_{k_p}) = \sum\nolimits_{\nu \in P_2(p)} \mathbb{E}\varphi(Y_k^{\otimes \nu}).$$

Since the variables $X_1, X_2, \ldots$ are assumed independent, we have $\mathbb{E}\varphi(Y_k^{\otimes \nu}) = 0$ except possibly when $k \sim \nu$ (indeed, if $\{i, j\}$ is a block of $\nu$ and $k_i \neq k_j$ then the entries of the $k_i$ factor of $Y_k^{\otimes \nu}$ are orthogonal to those of the $k_j$ factor and independent of all the other factors of $Y_k^{\otimes \nu}$). Moreover, since $X_1, X_2, \ldots$ all have the same distribution, it is easy to check that the distribution of $Y_k^{\otimes \nu}$ depends only on $\nu$ and not on $k$. Thus we obtain

$$\mathbb{E}\varphi(X_{k_1} \otimes \cdots \otimes X_{k_p}) = \sum\nolimits_{\nu \sim k} \psi(\nu)$$

with $\psi(\nu) = \mathbb{E}\varphi(Y^{\otimes \nu})$, with $Y = (X_1, \ldots, X_n)$. $\square$

*Remark* 15.4. The preceding result is used in [12] for the complex Gaussian random matrices $(Y_j^{(N)})_{j \geq 1}$ appearing in Corollary 11.13. In that case, since $Y^{(N)} \otimes Y^{(N)}$ has mean zero, there is an extra cancellation: $\psi(\nu) = 0$ for any partition $\nu \in P_2(n)$ admitting a block with two indices of the same parity.

More generally, the same proof shows

COROLLARY 15.5. *The preceding corollary is valid for any even integer $p$ for any i.i.d. Gaussian sequence with values in $L_p(M, \tau)$ for any non-commutative measure space $(M, \tau)$. (Here the moments are meant with respect to $x \mapsto \mathbb{E}\tau(x)$.)*

Using exactly the same method but replacing stochastic independence by freeness in the sense of [35] and Gaussian by semi-circular (or "free-Gaussian"), it is easy to extend the preceding Corollary to the free case. More generally, we can use the $q$-Fock space ($-1 \le q \le 1$) and the associated $q$-Gaussian variables described in [3, 4, 2].

Fix $q$ with $-1 \le q < 1$. Given a complex Hilbert space $\mathcal{H}$, we denote by $\mathcal{F}_q(\mathcal{H})$ the $q$-Fock space associated to $\mathcal{H}$. Let us assume that $\mathcal{H}$ is the complexification of a real Hilbert space $H$ so that $\mathcal{H} = H + iH$. For simplicity we assume $H = \ell_2(\mathbb{R})$. To any real Hilbert subspace $K \subset H$ we can associate (following [2]) a von Neumann algebra $\Gamma_q(K)$, so that we have natural embeddings $\Gamma_q(K_1) \subset \Gamma_q(K_2)$ when $K_1 \subset K_2$. Moreover $\Gamma_q(H)$ is equipped with a normalized trace (faithful and normal) denoted by $\tau_q$, that we may restrict to $\Gamma_q(K)$ to view the latter as a non-commutative probability space.

For any $h \in H$ we denote by $a^*(h)$ (resp. $a(h)$) the $q$-creation (resp. $q$-annihilation) operator on $\mathcal{F}_q(\mathcal{H})$ and we let $g_q(h) = a(h)+a^*(h)$. By definition, the von Neumann algebras $\Gamma_q(K)$ is generated by $\{g_q(h) \mid h \in K\}$. We will say that a family $X_1, \dots X_n$ in $\Gamma_q(H)$ is $q$-independent if there are mutually orthogonal real subspaces $K_j \subset H$ such that $X_j \in \Gamma_q(K_j)$.

Let $B = B(\ell_2)$. More generally, consider $x_1, \dots, x_n$ in $B \otimes \Gamma_q(H)$. We will say that $x_1, \dots, x_n$ are $q$-independent if there are $K_j$ as above such that $x_j \in B \otimes \Gamma_q(K_j)$ for all $j = 1, \dots, n$. The elements of $g_q(H) = \{g_q(h) \mid h \in H\}$ will be called $q$-Gaussian.

More generally, an element $x \in B \otimes \Gamma_q(H)$ will be called $q$-Gaussian if $x \in B \otimes g_q(H)$. The $q$-Gaussian elements satisfy an analogue (called "second quantization") of the rotational invariance of Gaussian distributions: For any $\mathbb{R}$-isometry $T\colon K \to H$ the families $\{g_q(t) \mid t \in K\}$ and $\{g_q(Tt) \mid t \in K\}$ have identical distributions. Here "same distribution" means equality of the moments of all non-commutative monomials. We will denote by $\widetilde{T}\colon B \otimes g_q(K) \to B \otimes g_q(H)$ the linear map taking $b \otimes g_q(t)$ to $b \otimes g_q(Tt)$ ($b \in B, t \in K$). In particular, if $x \in g_q(K)$ and if $K_1 = K_2 = \cdots = K_s = K$ we may use the isometry $u_s\colon K \to K_1 \oplus \cdots \oplus K_s \subset H$ defined by $u_s(x) = s^{-1/2}(x \oplus \cdots \oplus x)$ in order to define elements $x_j$ in $g_q(K_j)$ each with the same distribution as $x$ such that $x \overset{\text{dist}}{=} s^{-1/2}(x_1 + \cdots + x_s)$.

Let $x_1, \dots, x_n$ be any sequence in $B \otimes g_q(H)$ and let $\pi$ be a partition. For any block $\alpha$ of $\pi$ we give ourselves an isometry $u_\alpha\colon H \to H_\alpha \subset H$ where $H_\alpha$ are mutually orthogonal (real) Hilbert subspaces. Then we define a sequence $(y_1, \dots, y_n)$ in $B \otimes g_q(H)$ by setting

$$y_j = \widetilde{u}_\alpha x_j. \qquad \forall j \in \alpha$$

It is not hard to check that $\widehat{\tau}(y_1 \dot{\otimes} \cdots \dot{\otimes} y_n)$ depends only on $x = (x_1, \dots, x_n)$ and $\pi$ (and not on the $u_\alpha$'s). Therefore we may set

$$\widehat{\tau}(x^\pi) \overset{\text{def}}{=} \widehat{\tau}(y_1 \dot{\otimes} \cdots \dot{\otimes} y_n).$$

As before, by symmetry $\widehat{\tau}_q(x_1 \dot{\otimes} \cdots \dot{\otimes} x_n) = 0$ for all odd $n$.

The $q$-Gaussian analogue of (15.1) is as follows:

Proposition 15.6. *For any even integer $n$ and any $n$-tuple $x_1 \ldots, x_n$ in $B \otimes g_q(H)$ $(-1 \leq q < 1)$, we have*

$$\widehat{\tau}_q(x_1 \dot{\otimes} \cdots \dot{\otimes} x_n) = \sum\nolimits_{\nu \in P_2(n)} \widehat{\tau}_q(x^\nu).$$

*Proof.* With the preceding ingredients, this can be proved exactly as Proposition 15.1 above. □

In particular, replacing $B$ by $\mathbb{C}$, we find

Corollary 15.7. *Any $q$-Gaussian sequence $(x_j)$ in the sense of [2, Def. 3.3] with covariance equal to the identity matrix has $p$-th moment, defined by parings for any even integer $p$.*

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Gilles Pisier
Texas A&M University
College Station
TX 77843
U. S. A.
and
Université Paris VI
Equipe d'Analyse
Case 186
75252
Paris Cedex 05
France