Asymptotics of Spherical Superfunctions on Rank One Riemannian Symmetric Superspaces

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Abstract. We compute the Harish-Chandra c-function for a generic class of rank-one purely non-compact Riemannian symmetric superspaces $X = G/K$ in terms of Euler Γ functions, proving that it is meromorphic. Compared to the even case, the poles of the c-function are shifted into the right half-space. We derive the full asymptotic Harish-Chandra series expansion of the spherical superfunctions on X. In the case where the multiplicity of the simple root is an even negative number, they have a closed expression as Jacobi polynomials for an unusual choice of parameters.

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1. Introduction

Let $X = G/K$ be a Riemannian symmetric space. The mainstay of harmonic analysis on X is the study of the asymptotic behaviour of the spherical functions ϕ_λ [20, 23, 24]. These are K-invariant joint eigenfunctions of the G-invariant differential operators D on X, *viz.*

$$D\phi_\lambda = \Gamma(D)(\lambda)\phi_\lambda,$$

where $\Gamma(D)(\lambda)$ is the value of the Harish-Chandra homomorphism.

A basic observation is that $\phi_\lambda(a)$ admits an asympotic series expansion whose leading contribution is $c(\lambda)e^{(\lambda-\varrho)(a)}$, with ϱ the half sum of the positive roots. Here, $c(\lambda)$ is Harish-Chandra's famous c-function.

Moreover, $c(\lambda)$ admits an extension, as a meromorphic function, to the entire dual $\mathfrak{a}^*$ of the complex Cartan subspace, due to the Gindikin–Karpelevič formula

$$c(\lambda) = c_0 \prod_\alpha \frac{2^{\lambda_\alpha - 1}\Gamma(\lambda_\alpha)}{\Gamma\big(\frac{1}{2}(\lambda_\alpha + \frac{m_\alpha}{2} + 1)\big)\Gamma\big(\frac{1}{2}(\lambda_\alpha + \frac{m_\alpha}{2} + m_{2\alpha})\big)},$$

where the product is over all positive indivisible roots and $\lambda_\alpha := \frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$. This equation is basic in the proof of the fundamental theorems of harmonic analysis on Riemannian symmetric spaces: the support theorem for wave packets, the inversion, Paley–Wiener, Plancherel and Schwartz isomorphism theorems for the spherical Fourier transform.

Although the situation is more complicated for the Helgason–Fourier transform (for τ-spherical functions) and even more so for reductive non-Riemannian symmetric spaces, the basic philosophy of studying the asymptotics of spherical functions (or their replacements by more general Eisenstein integrals) remains valid, *cf.* Ref. [34].

In the present paper, we study the asymptotics of spherical functions defined on a supersymmetric generalisation of Riemannian symmetric spaces. These play a basic role in the so-called Efetov supersymmetry method of condensed matter physics. Indeed, as shown by Zirnbauer [39, 40], harmonic analysis on such superspaces can be used to give precise analytic expressions for the mean conductance of quasi-one dimensional disordered fermionic systems.

Remarkably, contrary to the classical case, there are discrete contributions in the Plancherel decomposition, and this is visible and relevant in the physics of the corresponding systems. Since the Plancherel measure is governed by the c-function, the failure of absolute continuity can already be observed from the location of the c-function zeroes.

Indeed, let (G, K) be one of the symmetric pairs listed in Table 4.1 below. Fix an indivisible restricted root α and h_0 with $\alpha(h_0) = 1$. Let $\varrho = \frac{1}{2}(m_\alpha\alpha + 2m_{2\alpha}\alpha)$ where m_α and $m_{2\alpha}$ are the multiplicities of α and 2α, respectively, and identify $\lambda \equiv \lambda(h_0)$. Our first main result is the following theorem.

Theorem A. *For every $\Re\lambda > 0$, the limit*

$$c(\lambda) = \lim_{t\longrightarrow\infty} \phi_\lambda(e^{th_0})e^{-t(\lambda-\varrho)}$$

exists. For some choice of $c_0 \equiv c_0(\varrho) \neq 0$, it is given by

$$c(\lambda) = c_0 \frac{2^{-\lambda}\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}\left(\lambda + \frac{m_\alpha}{2} + 1\right)\right)\Gamma\left(\frac{1}{2}\left(\lambda + \frac{m_\alpha}{2} + m_{2\alpha}\right)\right)} \tag{1.1}$$

if α is anisotropic, and if α is isotropic, then it is given by

$$c(\lambda) = c_0\lambda. \tag{1.2}$$

Formally, this result takes the same form as in the classical situation for the case of an anisotropic root α. However, for an isotropic root α (of multiplicity $m_\alpha = -2$), one would expect $c(\lambda) \simeq (\lambda - 1)$ from Equation (1.1) and the duplication formula. This differs from the true result in Equation (1.2) by the absence of a 'ϱ-shift'. This situation is similar for the Harish-Chandra homomorphism, see Ref. [1].

Moreover, in the present supersymmetric setting, m_α and hence ϱ may be arbitrarily large negative numbers, drastically changing the asymptotic behaviour of ϕ_λ. Moreover, due to the shift in the denominator in Equation (1.1), $c(\lambda)$ picks up zeroes in the right half plane. These lead to discrete contributions in the Plancherel formula, as we will show in a forthcoming paper.

The class of symmetric pairs considered in Theorem A is a choice of rank one pairs that is generic in the sense that every rank one subpair of a reductive symmetric pair (of even type) associated with a choice of indivisible root (even, odd, ot both) is generated by copies of the pairs listed in Table 4.1 below: An anisotropic root α such that 2α is a root corresponds to the $\mathfrak{gl}$ case; an anisotropic root such that 2α is not a root corresponds to the $\mathfrak{osp}$ case, even it is purely odd (here the parameter $p = 0$); finally, an isotropic root α corresponds to the $\mathfrak{gl}(1|1)$ case.

Choosing real forms of these symmetric pairs in such a way that the underlying symmetric spaces become Riemannian introduces extra conditions if one insists on taking real forms also of the odd part of the Lie superalgebra. However, these conditions are artificial from a physical point of view and moreover unnecessary for the analysis to go through. Therefore, we adopt the setting of '*cs* manifolds' invented by Joseph Bernstein, that is, of real manifolds with complex sheaves of superfunctions. In many respects, this theory is parallel to that of real supermanifolds; however, there are some caveats, and we carefully lay the foundations to help the reader navigate these impasses in Section 2.

As the statement of Theorem A suggests, the general form of c-function for a supersymmetric symmetric pair is not given by a simple-minded generalisation of the Gindikin–Karpelevič formula, since the contributions from isotropic roots are of a different form. Compare Ref. [6] for details.

The proof of Theorem A is somewhat more difficult than in the classical case. The reason lies again in the changed growth behaviour of the exponential $e^{t(\lambda-\varrho)}$: Upon parametrising the geodesic sphere at infinity of $X = G/K$ in

stereographical coordinates, the integrals in question exhibit a singular behaviour. One therefore has to make a careful choice of cutoffs in a stereographical atlas parametrised by the Weyl group and keep track of this in the limit of $t \longrightarrow \infty$.
In order for this to work, one needs to establish the Weyl group symmetry of the symmetric superfunctions ϕ_λ (Corollary 3.20). Although this appears to be quite innocent, it requires the extension of the usual integral formulæ for the Iwasawa and Bruhat decompositions to the 'parameter-dependent' setting of supermanifolds over a general base supermanifold. For this reason, we are obliged to develop some foundational material in Subsections 2.6 and 3.2.
Once one has an explicit formula for the c-function, Harish-Chandra's series expansion of the spherical function carries over. (This also relies on Corollary 3.20.) One has the following statement.

Theorem B. *Let (G, K) be such that the root α is even and $\Re\lambda > 0$, $\lambda \notin \frac{1}{2}\mathbb{Z}$. Then we have*

$$\phi_\lambda|_{A^+} = \sum_{w\in W_0} \Phi_{w\lambda}, \quad \Phi_{\pm\lambda}(e^{th_0}) = e^{t(\pm\lambda-\varrho)} \sum_{\ell=0}^{\infty} \gamma_\ell(\lambda) e^{-2\ell t},$$

where A^+ is the positive Weyl chamber, W_0 is the Weyl group, $\gamma_0(\pm\lambda) = c(\pm\lambda)$, and $\gamma_\ell(\lambda)$ follow a two-term recursion.

In most cases, $W_0 = \{\pm 1\}$, but for a suitable choice of parameters in case G is an orthosymplectic supergroup, it can be trivial. A similar situation occurs when α is an odd anisotropic root (Proposition 4.16). In this case, the complexification of G is $\mathrm{GL}(1|1, \mathbb{C})$.
Remarkably, when ϱ is a negative integer (as can happen in the $\mathfrak{osp}$ case), the above series expansion is *finite*, and ϕ_λ admits a simple closed expression in terms of Jacobi polynomials (Corollary 5.6) $P_n^{(a,b)}$. Here, the parameters are chosen such that $n = -\varrho$, so that in this case, the spherical superfunctions are exponential polynomials of a fixed degree independent of λ.

We end this introduction by a synopsis of the paper's contents. Section 2 is devoted to a brief collection of basic facts on *cs* manifolds. We highlight some points that to our knowledge are missing in the literature: Proposition 2.3 is a generalisation of Leites's theorem for morphisms from *cs* manifolds to complex manifolds. Subsection 2.4 contains an account of the exponential morphism for *cs* Lie supergroups. Subsection 2.5 explains *cs* forms of complex supergroups. Subsection 2.6 is a brief account of the basic theory of Berezin fibre integrals. In Subsection 2.7 a localization formula for Berezin integrals over superspheres is derived.
Section 3 introduces the main players of this article, the spherical superfunctions. It begins by setting the stage for symmetric superpairs in Subsection 3.1. In Subsection 3.2, we collect some integral formulæ necessary for the manipulation of the Harish-Chandra integral for ϕ_λ. Notably, we generalise the classical formula for the Haar measure on a Lie group in exponential coordinates in

Proposition 3.12, the nilpotent case being of particular importance to us. The definition of the spherical functions is stated in Subsection 3.3; here, we also show a basic symmetry property of the spherical functions (Corollary 3.20) that is important in the proof both of Theorem A (Theorem 4.2) and of Theorem B (Theorem 5.5).
Section 4 contains the statement and proof of Theorem A (Theorem 4.2). It is also the place where the main analytic problems have to be tackled and where the theory departs most from the classical cases (see the comments above). The theorem is stated in Subsection 4.1 and proved in Subsection 4.2. The proof proceeds case-by-case: the unitary case is contained in Subsubsection 4.2.1, the orthosymplectic one in Subsubsection 4.2.2, and the case of an anisotropic purely odd root (the '$\mathrm{GL}(1|1,\mathbb{C})$ case') in Subsection 4.2.3. The latter also contains the analytic expression for the spherical superfunctions ϕ_λ in this case.
The final Section 5 is devoted to the asymptotic series expansion of ϕ_λ, that is, the proof of Theorem B (Theorem 5.5). At this point, the main analytic difficulties have been overcome, so we can follow the ususal procedure of making a pertubation ansatz for the solutions of the eigenvalue equation for the radial part of the Laplacian on X, deducing a two-term recursion for the coefficients, and proving convergence *via* Gangolli estimates. The estimates miraculously also go through in cases of negative multiplicity; if the half sum ϱ of positive roots is a negative integer, the coefficients can even be computed explicitly, and the series terminates. In this case, we derive an expression for ϕ_λ in terms of Jacobi polynomials (Corollary 5.6) whose degree is fixed independent of λ.

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2. Basic facts on *cs* manifolds

In this section, we collect some basic facts on *cs* manifolds and *cs* Lie supergroups. As remarked above, this setting is necessary to cover the all the symmetric pairs we are interested in a Riemannian incarnation. We shall be suitably brief, including proofs only for those facts which so far have remained undocumented in the literature.

2.1. Basic definitions. We will work with *cs* manifolds and complex supermanifolds. The latter are covered by a wide literature, *e.g.* by Refs. [13,31,37]. The former, introduced by Joseph Bernstein, are covered to some degree in Ref. [15]. In many respects, they are similar to real supermanifolds, so one may follow the standard texts on that subject [15,30,37]. The differences that exist are quite subtle, and we will comment on these to the extent required in our applications. Specifically, we shall follow the conventions of Ref. [4].

In particular, we will work in the framework of $\mathbb{C}$-superspaces, that is, of locally superringed spaces over $\operatorname{Spec}\mathbb{C}$. Our notation will always be $X = (X_0, \mathcal{O}_X)$ for $\mathbb{C}$-superspaces, and $\varphi = (\varphi_0, \varphi^\sharp) : X \longrightarrow Y$ for morphisms, where we may consider $\varphi^\sharp$ as a sheaf map $\mathcal{O}_Y \longrightarrow \varphi_{0*}\mathcal{O}_X$ or $\varphi_0^{-1}\mathcal{O}_Y \to O_X$ by the fundamental adjunction of the direct and inverse image functors, see *e.g.* Refs. [11, I.4, Equation (5)] or [26, II.4, Theorem 4.8].

Somewhat abusing notation, we will also write X_0 for the reduced superspace associated with X, see [4, Construction 3.9]. There is a canonical closed embedding

$$j_{X_0} : X_0 \longrightarrow X$$

and for any $f \in \mathcal{O}_X(U)$, where $U \subseteq X_0$ is open, and any $x \in U$, we denote by $f(x) \coloneqq j_{X_0}^\sharp(f)(x) \in \varkappa(x) \coloneqq \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ the *value* of f at x. In the cases of interest to us, we will always have $\varkappa(x) = \mathbb{C}$.

We define the *cs* and complex affine superspaces as

$$\mathbb{A}^{p|q} \coloneqq (\mathbb{R}^p, \mathcal{C}^\infty_{\mathbb{R}^p} \otimes \textstyle\bigwedge(\mathbb{C}^q)^*), \quad \mathbb{A}^{p|q}_{hol} \coloneqq (\mathbb{C}^p, \mathcal{H}_{\mathbb{C}^p} \otimes \textstyle\bigwedge(\mathbb{C}^q)^*),$$

where $\mathcal{C}^\infty_{\mathbb{R}^p}$ is the sheaf of smooth complex-valued functions on $\mathbb{R}^p$ and $\mathcal{H}_{\mathbb{C}^p}$ is the sheaf of holomorphic functions on $\mathbb{C}^p$, and we consider the Euclidean topology. Then, by definition, a cs *manifold* is a $\mathbb{C}$-superspace locally isomorphic to $\mathbb{A}^{p|q}$, whereas a *complex supermanifold* is one locally isomorphic to $\mathbb{A}^{p|q}_{hol}$.

We are tempted speak of *cs* manifolds simply as "supermanifolds", but for the sake of convention, we will stick in this paper to the original appellation, which was introduced by Joseph Bernstein, *cf.* Ref. [15].

More generally, given two *cs* manifolds X and S, a cs *manifold over* S is a morphism $X \to S$ that is locally in X isomorphic to the projection of a direct product $S \times Y \to S$. In this case $\dim_S X \coloneqq \dim Y$ is called the *fibre dimension*. Usually, we will just write X/S without mentioning the structural morphism explicitly, denoting it by p_X where necessary. In particular, $\mathbb{A}^{p|q}_S :$ $= S \times \mathbb{A}^{p|q}$ is a *cs* manifold over S. A *morphism* $f : X/S \to Y/S$ *over* S is a morphism $f : X \to Y$ respecting the structural morphisms. Given an open embedding $X/S \to \mathbb{A}^{p|q}_S/S$ over S, the pullback of the standard coordinate functions on $\mathbb{A}^{p|q}$ is called a *fibre coordinate system* (over S). We recover the usual *cs* manifolds upon setting $S = *$, the terminal object of the category of $\mathbb{C}$-superspaces (*i.e.* the singleton space, together with the constant sheaf $\mathbb{C}$), but the relative point of will be important in Subsections 2.3, 2.6, 2.4, and 3.2. In Ref. [30], the corresponding notion for real supermanifolds is discussed under the name of *families*. In Ref. [4], which we follow, it is defined over base superspaces S more general than *cs* manifolds.

A coordinate-free view on affine superspaces will be useful. To that end, we define a *cs* vector space to be a real super-vector space $V = V_{\bar 0} \oplus V_{\bar 1}$ with a distinguished complex structure on $V_{\bar 1}$. We set

$$\mathbb{A}(V) \coloneqq \left(V_{\bar 0}, \mathcal{C}^\infty_{V_{\bar 0}} \otimes \textstyle\bigwedge V_{\bar 1}^*\right)$$

for any finite-dimensional *cs* vector space V. In particular, $\mathbb{A}^{p|q} = \mathbb{A}(\mathbb{R}^p \oplus \Pi\mathbb{C}^q)$. Similarly, for any finite-dimensional complex super-vector space V, we define

$$\mathbb{A}_{hol}(V) \coloneqq \big(V_{\bar 0}, \mathcal{H}_{V_{\bar 0}} \otimes \textstyle\bigwedge V_{\bar 1}^*\big),$$

so that $\mathbb{A}_{hol}^{p|q} = \mathbb{A}_{hol}(\mathbb{C}^{p|q})$, where we write $\mathbb{C}^{p|q} \coloneqq \mathbb{C}^p \oplus \Pi\mathbb{C}^q$.
We take note of the following trivial observation

$$V_{\bar 0} = 0 \implies \mathbb{A}(V) = \mathbb{A}_{hol}(V).$$

2.2. Morphisms to affine superspaces. One essential respect in which *cs* manifolds differ from real or complex supermanifolds is that morphisms to affine superspace are characterised in a subtly different fashion. Namely, for any $\mathbb{C}$-superspace X, denote by $\mathcal{O}_{X,\mathbb{R}}$ the sheaf whose local sections are the superfunctions f (*i.e.* local sections of $\mathcal{O}_X$) whose values $f(x)$ are real, for any $x \in X_0$ at which f is defined. Then we have the following generalisation of Leites's theorem on morphisms.

Proposition 2.1 ([4, Lemma 3.13, Corollary 4.15]). *Let T be a* cs *manifold and denote the standard coordinate system on $\mathbb{A}^{p|q}$ by (t_a). Then the natural map*

$$\mathbb{A}^{p|q}(T) \coloneqq \mathrm{Hom}(T, \mathbb{A}^{p|q}) \longrightarrow \Gamma\big(\mathcal{O}^p_{X,\mathbb{R},\bar 0} \times \mathcal{O}^q_{X,\bar 1}\big) : f \longmapsto (f^\sharp(t_a))$$

is a bijection. Here and in what follows, Hom *denotes morphisms of $\mathbb{C}$-superspaces and Γ denotes the global sections functor for sheaves.*

Expanding on the above conventions, we let, for any *cs* manifolds X and T, $X(T) \coloneqq \mathrm{Hom}(T, X)$ denote the set of morphisms $T \longrightarrow X$. We call any such morphism $x : T \longrightarrow X$ a T-valued point and write $x \in_T X$. For a morphism $f : X \to Y$ and $x \in_T X$, we denote by $f(x) \coloneqq f \circ x \in_T Y$. Given a T-valued point $x \in_T X$ where T/S is a *cs* manifold over S, we write $x \in_{T/S} X/S$ (or $x \in_{T/S} X$ for brevity) if x is over S as a morphism $T \to X$. We will use this suggestive notation constantly. Together with the Yoneda lemma, it provides a convenient way of defining and handling morphisms. We will be constantly using this point of view, so Proposition 2.1 is basic to our study.
A coordinate-free version of Proposition 2.1 can be formulated as follows: Following Ref. [4], we define

$$(V \otimes W)_{\bar 0} \coloneqq V_{\bar 0} \otimes_{\mathbb{R}} W_{\bar 0} \oplus V_{\bar 1} \otimes_{\mathbb{C}} W_{\bar 1}.$$

With this notation, the above proposition admits the following reformulation.

Corollary 2.2 ([4, Corollary 3.24]). *Let V be a finite-dimensional* cs *vector space and T a* cs *manifold. Then there is a natural bijection*

$$\mathbb{A}(V)(T) = \mathrm{Hom}(T, \mathbb{A}(V)) \longrightarrow \Gamma\big((\mathcal{O}_{T,\mathbb{R}} \otimes V)_{\bar 0}\big).$$

On the other hand, as the holomorphic version of Leites's theorem states, morphisms from a *complex* supermanifold T to $\mathbb{A}_{hol}^{p|q}$ constitute the same data as tuples of superfunctions on T *without any reality condition.* Remarkably, this carries over to the situation where T is a *cs* manifold, by the results of Ref. [4].

PROPOSITION 2.3. *Let V be a finite-dimensional complex super-vector space and T a* cs *manifold. Then there is a natural bijection*

$$\mathbb{A}_{hol}(V)(T) = \operatorname{Hom}(T, \mathbb{A}_{hol}(V)) \longrightarrow \Gamma\big((\mathcal{O}_T \otimes_{\mathbb{C}} V)_{\bar{0}}\big).$$

Proof. Broadly following Ref. [4], a $\mathbb{C}$-superspace T is called *holomorphically regular* if the above natural map is bijective for any V and any open subspace of T.

Thus, our claim is that any *cs* manifold is holomorphically regular. We will use the terminology of Ref. [4]. Since its use will be localized to this proof, we refer to that article for any undefined notions. Firstly, we may assume that $V = \mathbb{C}^n$, by Lemma 4.15 (*op. cit.*). Then $\mathbb{A}_{hol}(V) = \mathbb{A}^n_{hol}$ and $\mathbb{A}^n_{hol}(T) = \mathbb{A}^n_{hol}(T^{\bar{0}})$, where $T^{\bar{0}} \coloneqq (T_0, \mathcal{O}_{T,\bar{0}})$ is the even part of T, so it is sufficient to prove that $T^{\bar{0}}$ is holomorphically regular.

But by Proposition 5.24 (*op. cit.*), $T^{\bar{0}}$ (and hence, any open subspace thereof) admits a tidy embedding into some $\mathbb{A}^N$. In view of Proposition 5.11 and Lemma 5.2 (*op. cit.*), it is therefore sufficient to prove that $\mathbb{A}^N$ is holomorphically regular. By Lemma 5.2 (*op. cit.*) again, the natural map is injective for any open subspace of $\mathbb{A}^N$. Let us prove that it is also surjective.

So, let $U \subseteq \mathbb{R}^N$ be open and $f_1, \dots, f_n \in \mathcal{O}_{\mathbb{A}^N}(U) = \mathcal{C}^\infty(U, \mathbb{C})$. We may define $f_0 \coloneqq (f_1, \dots, f_n) : U \to \mathbb{C}^n$, and this map is smooth. Setting

$$f^\sharp_V(h) \coloneqq h \circ \big(f_0|_{f_0^{-1}(W)}\big), \quad h \in \mathcal{O}_{\mathbb{A}^n_{hol}}(W) = \mathcal{H}(W),$$

for any open $W \subseteq \mathbb{C}^n$, defines a morphism $f = (f_0, f^\sharp) : \mathbb{A}^N|_U \to \mathbb{A}^n_{hol}$ such that $f^\sharp(z_a) = f_a$, where (z_a) are the standard coordinates on $\mathbb{A}^n_{hol}$. □

For any finite-dimensional complex super-vector space V, we define a set-valued cofunctor $\mathbb{A}^{\mathbb{C}}(V)$ on the category of *cs* manifolds by setting

$$\mathbb{A}^{\mathbb{C}}(V)(T) \coloneqq \Gamma\big((\mathcal{O}_T \otimes_{\mathbb{C}} V)_{\bar{0}}\big)$$

on objects T, and by the obvious definition on morphisms. In these terms, Proposition 2.3 states that $\mathbb{A}^{\mathbb{C}}(V)$ is the restriction to the category of *cs* manifolds of the point functor of the complex supermanifold $\mathbb{A}_{hol}(V)$. (Of course, the restriction of the point functor of the *algebraic* affine superspace $\operatorname{Spec} S(V^*)$ to *cs* manifolds is also the same, but this is somewhat less remarkable.)

Proposition 2.3 has the following consequence, which will be important in the applications to supergroups below.

COROLLARY 2.4. *Let X and Y be complex supermanifolds and T a* cs *manifold. Then the natural map*

$$(X \times Y)(T) \longrightarrow X(T) \times Y(T)$$

is a bijection.

2.3. The tangent functor. Preparing for our discusson of the exponential morphism of *cs* Lie supergroups below, we introduce the 'total space' of the 'tangent bundle'. A subtlety for *cs* manifolds X is that such a 'total space' representing the (complex) tangent sheaf does not literally exist as a *cs* manifold. In this section, we introduce a replacement on the level of functors. We then single out a representable subfunctor, the 'real tangent bundle'. We will work with *cs* manifolds over an arbitrary base S. This will be used to the define the exponential morphism. It will also be important for the machinery of Berezinian fibre integration that we introduce in Subsection 2.6, on which our proof of the Weyl symmetry of the spherical superfunctions in Corollary 3.20 hinges.

Definition 2.5 (The tangent functor). Let X/S be a *cs* manifold over S. We let $\mathcal{T}_{X/S}$ be the sheaf of $p_{X,0}^{-1}\mathcal{O}_S$-linear superderivations of $\mathcal{O}_X$, and call this the *tangent sheaf over* S. Locally, it is spanned by fibre coordinate derivations and hence locally free. We define the *tangent functor* T_SX of X over S as the set-valued cofunctor on *cs* manifolds given on objects T/S by

$$(T_SX)(T/S) := \bigl\{(x,v) \bigm| x \in_{T/S} X, v \in \Gamma\bigl((x^*\mathcal{T}_{X/S})_{\bar 0}\bigr)\bigr\}$$

and by the obvious definition on morphisms. There are canonical morphisms of functors

$$0_X : X \longrightarrow T_SX : x \longmapsto (x,0), \quad \pi_{X/S} : T_SX \longrightarrow X : (x,v) \longmapsto x,$$

called the *zero section* and the *projection*, respectively. If $S = *$, we omit the corresponding subscripts.

The construction of T_SX is functorial: For any morphism $\varphi : X/S \to Y/S$ of *cs* manifolds over S, the tangent morphism $T_S\varphi : T_SX \to T_SY$ is defined by

$$(T_S\varphi)_T(x,v) := \bigl(\varphi(x), T_{x/S}\varphi(v)\bigr), \quad (T_{x/S}\varphi)(v)(f) := v\bigl(\varphi^\sharp(f)\bigr)$$

for all $f \in (\varphi_0^{-1}\mathcal{O}_Y)(U)$, $U \subseteq Y_0$ open. Here, we recall that $x^*\mathcal{T}_{X/S}$ is the sheaf of vector fields over S and along x.

Clearly, the functor $T_S : X \mapsto T_SX$ preserves fibre products over S, so $T_SS = S$ and $T* = *$, the point functor of the singleton space.

This definition of the tangent functor is compatible with the definition of the tangent spaces. If $x \in X_0$ is a point, considered as a morphism $* \longrightarrow X$, then the fibre product of functors $(T_SX)_x = * \times_X T_SX$ is given by

$$\begin{aligned}(T_SX)_x(T) &= (* \times_X T_SX)(T) = \Gamma\bigl((x_T^*\mathcal{T}_{X/S})_{\bar 0}\bigr)\\ &= \bigl\{\varphi \in \operatorname{Hom}_S(T \times \operatorname{Spec}\mathbb{D}_{\bar 0}, X) \bigm| \varphi|_{\varepsilon=0} = x_T\bigr\}\\ &= \mathbb{A}^{\mathbb{C}}(T_{x/S}X)(T).\end{aligned} \tag{2.1}$$

Here, x_T is the specialization of x, *i.e.* the composite $T \longrightarrow * \longrightarrow X$, and $T_{x/S}X$ is the tangent space of X over S at x—that is, the super-vector space over $\mathbb{C}$ whose homogeneous elements are the $\mathcal{O}_{S,p_0(x)}$-linear maps $v : \mathcal{O}_{X,x} \to \mathbb{C}$ such that

$$v(fg) = v(f)g + (-1)^{|f||v|} fv(g), \quad \forall f, g.$$

In other words, the fibre of $T_SX \longrightarrow X$ over x is the $\mathbb{C}$-affine superspace of $T_{x/S}X$. Moreover, $\operatorname{Spec}\mathbb{D}_{\bar 0} = (*, \mathbb{C}[\varepsilon]/(\varepsilon^2))$, and $(\cdot)|_{\varepsilon=0}$ denotes the canonical morphism $T \longrightarrow T \times \operatorname{Spec}\mathbb{D}_{\bar 0}$ that pulls back functions in $\mathcal{O}_T[\varepsilon]/(\varepsilon^2)$ by dropping the linear term in ε. The last of the equalities in Equation (2.1) amounts to the routine check that the map $\varphi^\sharp = x_T^\sharp + \varepsilon v$ is an algebra morphism if and only if v is a vector field over S along x_T.
Similarly, one checks that for any morphism $\varphi : X \to Y$, the action of $T\varphi$ on the fibre $(T_SX)_x$ is simply given by the application of the functor $\mathbb{A}^{\mathbb{C}}(\cdot)$ to the ordinary tangent map $T_{x/S}\varphi : T_xX \to T_{\varphi_0(y)}Y$.

Besides the (complex) tangent space $T_{x/S}X$ over S, we consider the *real tangent space*, defined as the *cs* vector space

$$T^{\mathbb{R}}_{x/S}X \coloneqq \big\{ v \in T_{x/S}X \;\big|\; v(\mathcal{O}_{X,\mathbb{R},\bar 0}) \subseteq \mathbb{R} \big\}$$

of real tangent vectors. More generally, we introduce the following.

Definition 2.6 (The real tangent bundle). The *real tangent sheaf* $\mathcal{T}^{\mathbb{R}}_{X/S}$ of X over S as the subsheaf of $\mathcal{T}_{X/S}$ whose local sections over an open set $U \subseteq X_0$ are those $v \in \mathcal{T}_{X/S}(U)$ such that

$$v(\mathcal{O}_{X,\mathbb{R},\bar 0}(V)) \subseteq \mathcal{O}_{X,\mathbb{R}}(V)$$

for all open subsets $V \subseteq U$.
If $(x_a) = (u, \xi)$ are local fibre coordinates of X over S defined on U, then

$$\mathcal{T}^{\mathbb{R}}_{X/S}\big|_U = \bigoplus_i \mathcal{O}_{X,\mathbb{R}} \frac{\partial}{\partial u_i} \oplus \bigoplus_j \mathcal{O}_X \frac{\partial}{\partial \xi_j},$$

where $\frac{\partial}{\partial u_i}$ and $\frac{\partial}{\partial \xi_j}$ are the fibre coordinate derivations introduced by (u, ξ). In this sense, $\mathcal{T}^{\mathbb{R}}_{X/S}$ is a 'locally free graded module over the ring extension $(\mathcal{O}_X, \mathcal{O}_{X,\mathbb{R}})$'.
We define the *real tangent functor* $T^{\mathbb{R}}_SX$ as the subfunctor of T_SX given on objects by

$$(T^{\mathbb{R}}_SX)(T/S) \coloneqq \big\{ (x, v) \in T_SX(T/S) \;\big|\; v \in \Gamma\big((x^*\mathcal{T}^{\mathbb{R}}_{X/S})_{\bar 0}\big) \big\}.$$

The condition on v in the last equation amounts to

$$v\big((x_0^{-1}\mathcal{O}_{X,\mathbb{R},\bar 0})(U)\big) \subseteq \mathcal{O}_{T,\mathbb{R}}(U)$$

for all open subsets $U \subseteq T_0$. Due to the local freeness of $\mathcal{T}^{\mathbb{R}}_{X/S}$, the functor $T^{\mathbb{R}}_SX$ is representable by a *cs* manifold, which is the total space of a vector bundle over X. This vector bundle, also denoted by $T^{\mathbb{R}}_SX$, is called the *real tangent bundle of* X.
By Proposition 2.1 or Corollary 2.2, its fibre at $x \in X_0$ is computed to be

$$(T^{\mathbb{R}}_SX)_x = \mathbb{A}\big(T^{\mathbb{R}}_{x/S}X\big),$$

the *cs* affine superspace associated with the *cs* vector space $T^{\mathbb{R}}_{x/S}X$.

2.4. The exponential morphism of a *cs* Lie supergroup. It is a well-known fact that there is an equivalence between the categories of Lie supergroups and supergroup pairs, *cf.* Ref. [13, 15, 28, 29]. This is true in the real and complex, smooth and analytical cases, and the arguments valid in these cases carry over unchanged to the setting of *cs* Lie supergroups.
However, for applications, in particular those to Harmonic Superanalysis we are concerned with in this paper, it useful to have a 'geometric' view on this equivalence. This uses the exponential morphism and the Campbell–Hausdorff series, neither of which has as yet been given a treatment in the literature in this setting.
The exponential morphism will also be used extensively in Section 4, for the derivation of the Iwawasa H-projection and the integration in exponential coordinates on the nilpotent Iwawasa N supergroup (on the basis of Proposition 3.12), both of which are primordial for the c-function asymptotics. Moreover, the *cs* Lie supergroups we will consider (see below for the definition) will be 'forms' of complex Lie supergroups. We will widely use their functors of points, and the avalability of a manageable expression thereof will be crucial. As we show below, in Proposition 2.14, such an expression can be derived by means of the exponential morphism. We begin with some basic definitions.

Definition 2.7. By definition, a cs *Lie supergroup* is a group object G in the category of *cs* manifolds.

Let G be a *cs* Lie supergroup. We set

$$\mathfrak{g} \coloneqq T_1G, \quad \mathfrak{g}_{\mathbb{R}} \coloneqq T_1^{\mathbb{R}}G,$$

where we write $T_1^{\mathbb{R}}G = T_{1/*}^{\mathbb{R}}G$ and otherwise use the notation from Subsection 2.3. We have

$$\mathfrak{g} = \big\{x \bigm| g = 1 + \varepsilon x_{\bar 0} + \tau x_{\bar 1} \in G(\operatorname{Spec}\mathbb{D})\big\}, \quad \mathbb{D} \coloneqq \mathbb{C}[\varepsilon|\tau]/(\varepsilon^2, \varepsilon\tau),$$

where we set $\operatorname{Spec}\mathbb{D} \coloneqq (*, \mathbb{D})$. Hence, for $x, y \in \mathfrak{g}$, we may define $[x, y]$ by

$$a = 1 + \varepsilon[x, y]_{\bar 0} + \tau[x, y]_{\bar 1}, \quad a \coloneqq (ghg^{-1}h^{-1})|_{\varepsilon=\varepsilon_1\varepsilon_2=\tau_1\tau_2, \tau=\tau_1\varepsilon_2=\varepsilon_1\tau_2},$$

where

$$g \coloneqq 1 + \varepsilon_1 x_{\bar 0} + \tau_1 x_{\bar 1}, \quad h \coloneqq 1 + \varepsilon_2 y_{\bar 0} + \tau_2 y_{\bar 1}.$$

With this bracket, $\mathfrak{g}$ acquires the structure of a complex Lie superalgebra. Moreover, we have $\mathfrak{g}_{\mathbb{R},\bar 0} = \mathfrak{g}_0$ and $\mathfrak{g}_{\bar 0} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathfrak{g}_0$ is the Lie algebra of the real Lie group G_0. Note that by Corollary 2.2, we have

$$\mathbb{A}(\mathfrak{g}_{\mathbb{R}})(\mathbb{A}^{0|q}) = T_1^{\mathbb{R}}((\Pi T^q)G)_0,$$

the Lie algebra of the real Lie group $((\Pi T)^q G)_0 = G(\mathbb{A}^{0|q})$.
When $\varphi : G \to H$ is a morphism of *cs* Lie supergroups, then $\varphi(1) = 1$, so that we have a map $d\varphi \coloneqq T_1\varphi : \mathfrak{g} = T_1G \to T_1H = \mathfrak{h}$. Using the definition of the bracket, it is immediate that $d\varphi$ is a morphism of complex Lie superalgebras. Moreover, we have $d\varphi(\mathfrak{g}_{\mathbb{R}}) \subseteq \mathfrak{h}_{\mathbb{R}}$.

We may now transfer the definition of the exponential morphism of a real Lie supergroup [21] to the setting of *cs* Lie supergroups. We shall be terse on the parts that are similar, highlighting only the use of the tangent functor.

PROPOSITION 2.8. *Let G be a* cs *Lie supergroup. There is a unique morphism*

$$\exp_G : \mathbb{A}(\mathfrak{g}_{\mathbb{R}}) \to G$$

of cs *manifolds whose action on $\mathbb{A}^{0|q}$-valued points is precisely the exponential map of the real Lie group $((\Pi T)^q G)_0 = G(\mathbb{A}^{0|q})$. The morphism $\exp_G$ is a local isomorphism in a neighbourhood of zero.*

Proof. The uniqueness is obvious, since the $\mathbb{A}^{0|q}$ form a set of generators for the category of *cs* manifolds.
For the existence, we construct a certain even vector field on $G \times \mathbb{A}(\mathfrak{g}_{\mathbb{R}})$. If m is the multiplication of G, we consider

$$0_G \times \iota : G \times \mathbb{A}^{\mathbb{C}}(\mathfrak{g}) \to TG \times TG, \quad Tm : TG \times TG = T(G \times G) \to TG,$$

where ι is the canonical morphism $\mathbb{A}^{\mathbb{C}}(\mathfrak{g}) = (TG)_1 \to TG$.
Let L denote their composite. Explicitly, on points, it is given by

$$L(g, v) = Tm(g, 1, 0, v) = \big(g, v_g\big), \quad v_g(f) \coloneqq (\mathrm{id} \otimes v)\big(m(g, \cdot)^\sharp(f)\big).$$

The inverse morphism $L^{-1} : TG \to G \times \mathbb{A}^{\mathbb{C}}(\mathfrak{g})$ is given on points by

$$L^{-1}(g, v) = (g, v_{g^{-1}}),$$

so that L is an isomorphism of functors.
Now, we define the morphism $\Theta : G \times \mathbb{A}(\mathfrak{g}_{\mathbb{R}}) \longrightarrow TG$ as the composite

$$G \times \mathbb{A}(\mathfrak{g}_{\mathbb{R}}) \longrightarrow G \times \mathbb{A}(\mathfrak{g}) \xrightarrow{\;L\;} TG.$$

By the Yoneda Lemma, Θ corresponds to a unique element (x, v) of

$$T(G)(G \times \mathbb{A}(\mathfrak{g}_{\mathbb{R}})).$$

It is clear that $\pi_G \circ \Theta = p_1 : G \times \mathbb{A}(\mathfrak{g}_{\mathbb{R}}) \to G$, so that $v \in \Gamma\big((p_1^* \mathcal{T}_G)_{\bar{0}}\big)$.
We may promote v to an even vector field on $G \times \mathbb{A}(\mathfrak{g}_{\mathbb{R}})$ over $\mathbb{A}(\mathfrak{g}_{\mathbb{R}})$. Arguing as in [21, Lemma 4.1], one sees that v is real and complete. Let

$$\gamma_v : \mathbb{A}^1 \times G \times \mathbb{A}(\mathfrak{g}_{\mathbb{R}}) \to G \times \mathbb{A}(\mathfrak{g}_{\mathbb{R}})$$

be its global flow. We define $\exp_G$ as the composite

$$\mathbb{A}(\mathfrak{g}_{\mathbb{R}}) \xrightarrow{(1,1)\times\mathrm{id}} \mathbb{A}^1 \times G \times \mathbb{A}(\mathfrak{g}_{\mathbb{R}}) \xrightarrow{\;\gamma_v\;} G \times \mathbb{A}(\mathfrak{g}_{\mathbb{R}}) \xrightarrow{\;p_1\;} G.$$

Using the naturality of the morphism L, we see that v corresponds on $G(\mathbb{A}^{0|q})$ to a similarly defined vector field that is known to generate the exponential flow [27]. This implies the claim. ∎

COROLLARY 2.9. *Let $\varphi : G \to H$ be a morphism of* cs *Lie supergroups. Then*

$$\exp_H \circ \mathbb{A}(d\varphi) = \varphi \circ \exp_G .$$

Proof. It is sufficient to check the equality on $\mathbb{A}^{0|n}$-valued points, where it is immediate from Proposition 2.8 and the classical theory [25]. □

The following is a form of the equivalence of supergroups and supergroup pairs, stating that *any supergroup G is G_0-equivariantly split.* (In fact, it gives the essential bijection on objects; the equivalence on morphisms then follows from Corollary 2.9.)

Corollary 2.10. *Let G be a* cs *Lie supergroup. Then the morphism*

$$G_0 \times \mathbb{A}(\mathfrak{g}_{\bar{1}}) \longrightarrow G : (g, x) \longmapsto g \exp_G x$$

is a G_0-equivariant isomorphism of cs *manifolds.*

Proof. Since $d\exp_G = \mathrm{id}_{\mathfrak{g}}$ by Proposition 2.8 and the morphism is G_0-equivariant, it has invertible tangent map at every point. The underlying map is id_{G_0}, so the claim follows from the inverse function theorem [13, 15, 30]. □

We end this subsection by a discussion of the *adjoint action.* This is the action $\mathrm{Ad}_G \coloneqq \mathrm{Ad}$ of G on $\mathbb{A}^{\mathbb{C}}(\mathfrak{g}) = (TG)_1$, given as follows: For $g \in_S G$ and $x \in (TG)_1(S)$, regard x as an element of $G(S \times \operatorname{Spec} \mathbb{D}_{\bar{0}})$ such that $x|_{\varepsilon=0} = 1_S$. (See the comments after Equation (2.1) for explanation.) Then

$$\mathrm{Ad}(g)(x) \coloneqq gxg^{-1} \in G(S \times \operatorname{Spec} \mathbb{D}_{\bar{0}}),$$

where g is specialized to $S \times \operatorname{Spec} \mathbb{D}_{\bar{0}}$ *via* the first projection. It follows that

$$\mathrm{Ad}(g)(x)|_{\varepsilon=0} = gg^{-1} = 1_S,$$

so $\mathrm{Ad}(g)(x) \in_S (TG)_1 = \mathbb{A}^{\mathbb{C}}(\mathfrak{g})$. Using the definitions, it is easy to check that for any morphism $\varphi : G \to H$ of *cs* Lie supergroups, we have

$$\mathrm{Ad}_H(\varphi(g))(\mathbb{A}^{\mathbb{C}}(d\varphi)(x))) = \mathbb{A}^{\mathbb{C}}(d\varphi)\big(\mathrm{Ad}_G(g)(x)\big)$$

for any $g \in_S G$, $x \in_S \mathbb{A}^{\mathbb{C}}(\mathfrak{g})$.

If we let $G_0 \times G_0$ act on $G_0 \times \mathbb{A}(\mathfrak{g}_{\bar{1}})$ by $(g_1, g_2)(g, x) = (g_1 g g_2^{-1}, \mathrm{Ad}(g_2)(x))$ for $(g_1, g_2) \in_T G_0 \times G_0$, then we see that the isomorphism stated in Corollary 2.10 is even $(G_0 \times G_0)$-equivariant.

Remarkably, the adjoint action of G passes to $\mathbb{A}(\mathfrak{g}_{\mathbb{R}})$. Namely, let $g \in_S G$ and $x \in_S \mathbb{A}(\mathfrak{g}_{\mathbb{R}})$. We need to check that for any $f \in \Gamma(\mathcal{O}_{G,\mathbb{R},\bar{0}})$, we have $y(f) \in \Gamma(\mathcal{O}_{S,\mathbb{R}})$ for $y \coloneqq \mathrm{Ad}(g)(x)$. By the definition, we have

$$y(f) = (\mathrm{id} \otimes x)(f(g(\cdot)g^{-1})),$$

where $\mathrm{id} \otimes x$ denotes the promotion of x to a vector field over S (which exists because of the local freeness of the tangent sheaf over S, *cf.* [30, Lemma 2.2.3]). Since $f(g(\cdot)g^{-1}) = (m(g, m(\cdot, g^{-1})))^\sharp(f)$ is an even and real-valued superfunction on $S \times G$, the assertion is immediate from the assumption on x.

Corollary 2.11. *Let G be a* cs *Lie supergroup. If $g \in_S G$ and $x \in_S \mathbb{A}(\mathfrak{g}_{\mathbb{R}})$, then*

$$\exp_G \mathrm{Ad}(g)(x) = g(\exp_G x)g^{-1}.$$

Proof. By the above, both sides of the equation are well-defined. Hence, it is sufficient to check the equality on $\mathbb{A}^{0|q}$-valued points, for every q. But on that level, it follows from Proposition 2.8 and the classical theory [25]. Alternatively, one may use a parameter version of Corollary 2.9. $\square$

Remark 2.12. Although $\mathfrak{g}_{\mathbb{R}}$ is in general not a Lie algebra, the bracket is well-defined on $\mathbb{A}(\mathfrak{g}_{\mathbb{R}})$. Indeed, $\mathfrak{g}$ has a homogeneous basis (e^a) contained in $\mathfrak{g}_{\mathbb{R}}$. Then $x, y \in \Gamma\big(\mathcal{O}_{T,\bar{1}} \otimes \mathfrak{g}_{\bar{1}}\big)$ admit representations $x = \sum_a x_a e^a$ and $y = \sum_a y_a e^a$, and

$$[x, y] = -\sum_{abc} x_a y_b C_c^{ab} e^c$$

where $C_c^{ab} \in \mathbb{C}$ are the structure constants of $\mathfrak{g}$. But $x_a y_b C_c^{ab}$ has the value zero, so that it is a section of $\mathcal{O}_{T,\mathbb{R},\bar{0}}$.

2.5. COMPLEX SUPERGROUPS AND *cs* FORMS. Our interest in *cs* supergroups comes from the fact that their are many '*cs* forms' of complex Lie supergroups, whereas there are comparatively few real forms. (See the examples we will be considering in Section 4 and beyond, and compare Ref. [15] for a more extensive list of reasons. In any case, any real Lie supergroup defines a *cs* Lie supergroup by the complexification of its structure sheaf.)
Since we will encounter many such *cs* forms, it will be useful to have a uniform description of their functors of points. We will derive such a description by the use of the exponential morphism.

DEFINITION 2.13 (Forms of complex supergroups in *cs* manifolds). Let $G_{\mathbb{C}}$ be a complex Lie supergroup with associated complex supergroup pair $(\mathfrak{g}, G_{\mathbb{C},0})$. Let G_0 be a real form of $G_{\mathbb{C},0}$, *i.e.* a closed subgroup whose Lie algebra is a real form of $\mathfrak{g}_{\bar{0}}$. The *cs* Lie supergroup G associated with $(\mathfrak{g}, G_0)$ is called a cs *form* of $G_{\mathbb{C}}$.

Any *cs* form G of a complex Lie supergroup $G_{\mathbb{C}}$ comes with a canonical morphism $G \longrightarrow G_{\mathbb{C}}$ of $\mathbb{C}$-superspaces. It can be given an expression in terms of the exponential morphism, as follows.
The real Lie supergroup $(G_{\mathbb{C}})_{\mathbb{R}}$ associated with $G_{\mathbb{C}}$ has an exponential morphism [21]. Since its differential is complex linear for the complex structure induced by $\mathfrak{g}$, it is holomorphic, and therefore induced by a unique morphism denoted by $\exp_{G_{\mathbb{C}}} : \mathbb{A}_{hol}(\mathfrak{g}) \to G_{\mathbb{C}}$.
By the above and Corollary 2.4, the canonical morphism $G \to G_{\mathbb{C}}$ is given on T-valued points as follows:

$$G \longrightarrow G_{\mathbb{C}} : g \exp_G(x) \longmapsto g \exp_{G_{\mathbb{C}}}(x), \quad g \in_T G_0, x \in_T \mathbb{A}(\mathfrak{g}_{\bar{1}}) = \mathbb{A}_{hol}(\mathfrak{g}_{\bar{1}})$$

Thus, the T-valued points of G can be characterised within $G_{\mathbb{C}}$ as follows.

PROPOSITION 2.14. *Let $G_{\mathbb{C}}$ be a complex Lie supergroup and G a* cs *form of $G_{\mathbb{C}}$. For any* cs *manifold T, the T-valued points of G are given by*

$$G(T) = \big\{ g \in_T G_{\mathbb{C}} \bigm| g_0 \in_{T_0} G_0 \big\}.$$

Proof. Express a T-valued point of $G_{\mathbb{C}}$ as $g\exp_{G_{\mathbb{C}}}(x)$ where $g\in_T G_{\mathbb{C},0}$, $x\in_T \mathbb{A}_{hol}(\mathfrak{g}_{\bar 1})$. By the above considerations, $g\exp_{G_{\mathbb{C}}}(x)\in G(T)$ if and only if $g\in_T G_0$.
Thus, we need to see that

$$g\in_T G_0 \iff g_0\in_{T_0} G_0.$$

Since $(\exp_{G_{\mathbb{C}}}(x))_0=1$, this will already be sufficient to prove the claim. Certainly, the left-hand statement implies that on the right-hand side.
On the other hand, assume that $g_0\in_{T_0} G_0$. This pins down the map underlying g. Local coordinates on G_0 can be chosen in such a way that they are the restriction of holomorphic local coordinates (z_a) on $G_{\mathbb{C},0}$. Then $g\in_T G_0$ translates to the requirement that $g^\sharp(z_a)$ be real-valued. Since $(g^\sharp(z_a))_0=z_a\circ g_0$, this is immediate by the assumption, thereby proving the claim. □

2.6. Integration on relative *cs* manifolds. Below, in Corollary 3.20, we prove the Weyl symmetry of the spherical superfunctions ϕ_λ by the use of integral geometry. Without this fact, we would not be able to prove convergence in the delicate c-function expansion, nor could we derive the Harish-Chandra series, so the result is absolutely essential.
In the proof of *loc. cit.*, we will need to handle integrals with parameters. A suitable formalism is that of fibre integrals over relative *cs* manifolds. We very briefly collect the basic definitions and facts to it set up.

Definition 2.15 (Relative Berezinian densities). Assume given a *cs* manifold X/Y over Y. For any fibre coordinate neighbourhood $U\subseteq X_0$, we let

$$\mathcal{B}er_{X/Y}|_U := \mathrm{Ber}\big(\mathcal{T}^*_{X/Y}(U)\big).$$

(Compare Ref. [31] for the definition of the Berezinian module of a free module.) This defines a locally free $\mathcal{O}_X$-module $\mathcal{B}er_{X/Y}$ with local basis of sections

$$Dx=D(u,\xi)=du_1\cdots du_p\frac{\partial}{\partial\xi_1}\cdots\frac{\partial}{\partial\xi_q}$$

of parity $\equiv q\,(2)$, for any local fibre coordinate system $x=(x_a)=(u,\xi)$. Twisting by the relative orientation sheaf, we obtain $|\mathcal{B}er|_{X/Y}:=or_{X_0/Y_0}\otimes_{\mathbb{Z}}\mathcal{B}er_{X/Y}$, with corresponding local basis of sections $|Dx|=|D(u,\xi)|$. The local sections of the latter sheaf are called *relative Berezinian densities.*

Definition 2.16 (Relative Berezin integral). Let X/Y be a *cs* manifold over Y. A retraction $r:X\to X_0$ (*i.e.* a left inverse of $j_{X_0}:X_0\to X$) is called a *retraction over* Y if there is a retraction $r_Y:p_X(X)\to p_X(X)_0$ such that $r_Y\circ p_X=p_{X_0}\circ r_X$. Here, $p_X(X)\subseteq Y$ is the open subspace of Y over the open set $p_{X,0}(X_0)\subseteq Y_0$.
Fix a retraction r of X over Y. A local system (u,ξ) of fibre coordinates is called r-*adapted* if $u=r^\sharp(u_0)$ for some fibre coordinate system u_0 of X_0/Y_0.
Let $\omega\in\Gamma(|\mathcal{B}er|_{X/Y})$. Then we may define

$$\fint_{Y}{}^{r}_{X}\,\omega\in\mathcal{O}_Y(p_{X,0}(X_0)),$$

the *relative Berezin integral* or *fibre integral* of ω, as follows.
Let (U_i) be a collection of open subspaces of X, such that $\operatorname{supp}\omega \subseteq \bigcup_i U_{i,0}$, there are r-adapted fibre coordinate systems (u^i, ξ^i) over Y defined on U_i, and there are r_Y-adapted coordinate systems (v^i, η^i) on $V_i := p_X(U_i)$, such that $(u^i, p_X^\sharp(v^i), \xi^i, p_X^\sharp(\eta^i))$ form coordinate systems of U_i. Let (χ^i) be a (not necessarily compactly supported) partition of unity on $\bigcup_i U_{i,0}$, subordinate to the cover.
We may expand

$$\omega|_{U_{i,0}} = |D(u^i, \xi^i)|\, f^i, \quad f^i = \sum_{I,J} \xi^I p_X^\sharp(\eta^J) r^\sharp(g^i_{IJ}), \quad g^i_{IJ} \in \mathcal{O}_{X_0}(U_{i,0}),$$

with I ranging through the subsets of $\{1, \dots, q\}$, $p|q := \dim_Y X$, and J ranging though the subsets of $\{1, \dots, n\}$, $m|n := \dim Y$.
With these data, we define

$$ {}_Y\!\!\fint_X^r \omega := \sum_{i,J} \eta^J p_Y^\sharp \Big({}_{Y_0}\!\!\fint_{X_0} |du_0^i|\, \chi_i g^i_{\mathbf{q}J} \Big),$$

whenever the integrals and the series converge absolutely.
Here, $\mathbf{q} = \{1, \dots, q\}$ and ${}_{Y_0}\!\fint_{X_0} \varpi$ denotes the function $y \longmapsto \int_{p_{X,0}^{-1}(y)} \varpi|_{p_{X,0}^{-1}(y)}$, defined on $p_{X,0}(X_0)$.

Remark 2.17. In Definition 2.16, the retraction r_Y is uniquely determined by r. In particular, not every retraction of X is over Y. For example, the retraction r on $\mathbb{A}^{2|2}$, given by $r(s) := (s_1 + s_2 s_3 s_4, s_2)$ for an T-valued point $s \in_T \mathbb{A}^{2|2}$, has no counterpart under the submersion $\psi \colon \mathbb{A}^{2|2} \longrightarrow \mathbb{A}^{1|2}, (s_1, s_2, s_3, s_4) \longmapsto (s_1, s_3, s_4)$.

Up to some computations in coordinates, the following fact is no harder to prove than the absolute situation where $Y = *$, *cf.* Ref. [3].

THEOREM 2.18. *The relative Berezin integral is well-defined independent of all choices and depends only on the choice of a retraction. In case the integrand* $\omega \in \Gamma(|\mathcal{B}er|_{X/Y})$ *is compactly supported in the fibres over* Y, i.e. $p_{X,0} : \operatorname{supp}\omega \to Y_0$ *is a proper map, then the integral is independent of the retraction.*

Recalling the definition of the direct image with proper supports from Ref. [11], the condition in the above theorem be succintly rephrased as follows:

$$\omega \in \Gamma(p_{X,0!}|\mathcal{B}er|_{X/Y}).$$

We will use this notation in the sequel.

COROLLARY 2.19. *Let* $\varphi : X'/Y \to X/Y$ *be a morphism of* cs *manifolds. Let* r *and* r' *be retractions of* X *and* X' *over* Y*, respectively. Assume that* $r \circ \varphi = \varphi_0 \circ r'$*. Then for any* $\omega \in \Gamma(|\mathcal{B}er|_{X/Y})$*, we have*

$$ {}_Y\!\!\fint_{X'}^{r'} \varphi^\sharp(\omega) = {}_Y\!\!\fint_X^r \omega$$

i.e. *both integrals exist if only one of them does, in which case they coincide.*

2.7. Integral localization in polar coordinates. It is known that under suitable symmetry assumptions, there are remarkable integral localization theorems for supermanifolds [16, 35]. Here, we show that for the special case of the supersphere, a precise form thereof can obtained by the use of polar coordinates.

We shall make extensive use of the formalism of T-valued points. Compare the remarks in Subsection 2.2; the reader may also consult Ref. [1, Appendix B].

Denote by $y = (y_a) = (v, \eta)$ the standard coordinates on $\mathbb{A}^{p|2q}$ and the Berezin–Lebesgue density by $|D\lambda| \coloneqq (-2\pi)^{-q}|Dy|$. The retraction associated with y is

$$r(x) \coloneqq x_{\bar 0} \coloneqq (x_1, \dots, x_p), \quad x \in \mathbb{A}^{p|2q}(T) = \Gamma\big(\mathcal{O}^p_{T,\bar 0,\mathbb{R}} \times \mathcal{O}^q_{T,\bar 1}\big).$$

One should avoid to confuse $x_{\bar 0} \in \mathbb{A}^p(T)$ with the underlying morphism $x_0 \in \mathbb{A}^p(T_0)$.

Furthermore, $\|\cdot\|^2 \colon \mathbb{A}^{p|2q} \to \mathbb{A}^1$ shall be given by

$$\|x\|^2 \coloneqq \sum_{i=1}^p x_i^2 + 2\sum_{j=1}^q x_{p+2j-1}x_{p+2j} \tag{2.2}$$

for $x \in_T \mathbb{A}^{p|2q}$. Using the positive square root, this yields

$$\|\cdot\| \coloneqq \sqrt{} \circ \|\cdot\|^2 \colon \mathbb{A}^{p|2q}_{\neq 0} \to \mathbb{A}^1_{>0},$$

where $\mathbb{A}^{p|2q}_{\neq 0} \coloneqq \mathbb{A}^{p|2q}|_{\mathbb{R}^p\setminus\{0\}}$ and similarly for the subscript "> 0".

Definition 2.20 (Rotationally invariant superfunctions). Let $f \in \Gamma(\mathcal{O}_{S\times\mathbb{A}^{p|2q}})$, where S is any *cs* manifold. In case $p > 0$, f will be called *rotationally invariant over* S if for some $\varepsilon > 0$, there exists an $f^\circ \in \mathcal{O}_{S\times\mathbb{A}^1}(S_0 \times (-\varepsilon, \infty))$ such that

$$f(s, x) = f^\circ(s, \|x\|), \quad (s, x) \in_T S \times \mathbb{A}^{p|2q}_{\neq 0}.$$

In case $p = 0$, $\|\cdot\|$ is not defined, so the definition of rotational invariance has to be modified as follows: f is called *rotationally invariant over* S if there exists $g \in \Gamma(\mathcal{O}_{S\times\mathbb{A}^1})$, with

$$f(s, x) = g(s, \|x\|^2), \quad (s, x) \in_T \mathbb{A}^{p|2q}.$$

In this case, we define $f^\circ(s, t) \coloneqq g(s, t^2)$.

Remark 2.21. In the case $p > 0$, the super function $f^\circ|_{S_0\times(0,\infty)}$ extends to a superfunction on $S \times \mathbb{A}^1$ such that $f(s, t) = f(s, -t)$. Such an extension is given by $f^\circ(s, t) \coloneqq f(s, te_1)$, where $t \in_T \mathbb{A}^1$ and e_1 is the first standard basis vector of $\mathbb{R}^p$.

Since f° is even in the second component, there is an extension g of $f^\circ \circ (\mathrm{id}_S \times \sqrt{})$ to $S \times \mathbb{A}^1$, and $f(s, x) = g(s, \|x\|^2)$ for all $(s, x) \in_T S \times \mathbb{A}^{p|2q}$.

Here and in what follows, when handling Berezin (fibre) integrals, we will use Theorem 2.18 and Corollary 2.19 implicitly.

PROPOSITION 2.22. *Let $f \in \Gamma(\mathcal{O}_{S\times\mathbb{A}^{p|2q}})$ be rotationally invariant over S. Then*

$$\int_{S\times\mathbb{A}^{p|2q}/S}^{r} |D\lambda(x)|\, f(s,x) = \begin{cases} \dfrac{\pi^{\frac{p-2q}{2}}(-1)^q}{\Gamma(\frac{p}{2})}\displaystyle\int_0^\infty dr\, r^{\frac{p}{2}-1}\partial_r^q f^\circ(s,\sqrt{r}), & p>0,\\ (-\pi)^{-q}\partial_{r=0}^q f^\circ(s,\sqrt{r}), & p=0,\end{cases}$$

in the sense that the integral exists if and only if the right-hand side exists, and in this case, they are equal.

Proof. Consider the superfunction g from above. Applying Taylor expansion yields

$$g(s,t+t') \equiv \sum_{k=1}^q \frac{1}{k!} t'^k \partial_{t'=0}^k g(s,t+t') \equiv \sum_{k=1}^q \frac{1}{k!} t'^k \partial_t^k g(s,t) \mod (t'^{q+1})$$

with $t = t' = \mathrm{id}_{\mathbb{A}^1}$. Hence, we have

$$f(s,y) = g(s,\|v\|^2+\|\eta\|^2) = \sum_{k=1}^q \frac{1}{k!}\partial_2^k g(s,\|v\|^2)\|\eta\|^{2k},$$

where $y = \mathrm{id}_{\mathbb{A}^{p|q}}$, $v = \mathrm{id}_{\mathbb{A}^p}$, $\eta = \mathrm{id}_{\mathbb{A}^{0|q}}$, and $(\partial_2 g)(s,t) \coloneqq \partial_t g(s,t)$.
The expression $\|\eta\|^{2k}$ contains $\eta_1\cdots\eta_{2q}$ if and only if $k=q$. In this case, it equals $2^q q!\eta_1\cdots\eta_{2q}$. For $p=0$, this means that the integral is $(-\pi)^{-q}\partial_{t=0}^q g(s,t)$, as claimed. Similarly, if $p>0$, then it takes the form

$$\int_{S\times\mathbb{A}^{p|2q}/S}^{r} |D\lambda(x)|\, f(s,x) = (-\pi)^{-q}\int_{\mathbb{R}^p} |dv_0|\, \partial_2^q g(s,\|v_0\|^2).$$

Applying polar coordinates for $p \geqslant 2$, we obtain for $C = 2\pi^{\frac{p-2q}{2}}(-1)^q\Gamma(\frac{p}{2})^{-1}$

$$= C\int_0^\infty dr\, r^{p-1}\partial_2^q g(s,r^2) = \frac{C}{2}\int_0^\infty dr\, r^{\frac{p}{2}-1}\partial_r^q g(s,r).$$

In case $p=1$, one obtains the same result by symmetry. □

We obtain the following localization formula.

COROLLARY 2.23. *Let $k \leqslant \min(\frac{p}{2},q)$ and $f \in \Gamma(\mathcal{O}_{S\times\mathbb{A}^{p|2q}})$ be rotationally invariant and compactly supported in the fibres over S. Then*

$$\int_{S\times\mathbb{A}^{p|2q}/S}^{r} |D\lambda(x)|\, f(s,x) = \int_{S\times\mathbb{A}^{p-2k|2q-2k}/S}^{r} |D\lambda(x)|\, f^\circ(s,x).$$

Proof. Use Proposition 2.22 and integration by parts for $k<\frac{p}{2}$. The fundamental theorem of calculus needs to be applied for $k=\frac{p}{2}$. □

Combining our results, the integral of rotationally invariant superfunctions takes the following form, which depends only on $p-2q$ and not on $p|2q$.

COROLLARY 2.24. *Let $f \in \Gamma(\mathcal{O}_{S\times\mathbb{A}^{p|2q}})$ be rotationally invariant and compactly supported along the fibres over S. Then*

$$\int_{S\times\mathbb{A}^{p|2q}/S}^{r} |D\lambda(x)|\, f(s,x) = \frac{2\pi^{\frac{p-2q}{2}}}{\Gamma(\frac{p-2q}{2})}\int_0^\infty dr\, r^{p-2q-1} f^\circ(s,r), \quad p-2q>0.$$

If $p - 2q \leqslant 0$, then the following two cases occur:

$$\begin{cases} (-\pi)^{\frac{p-2q}{2}} \partial_{r=0}^{\frac{2q-p}{2}} f^\circ(s, \sqrt{r}) = (-\pi)^{\frac{p-2q}{2}} \frac{(\frac{2q-p}{2})!}{(2q-p)!} \partial_{r=0}^{2q-p} f^\circ(s, r), & 2 \mid p - 2q \leqslant 0, \\ (-\pi)^{\frac{p-1-2q}{2}} \int_0^\infty dr\, r^{-\frac{1}{2}} \partial_r^{\frac{2q+1-p}{2}} f^\circ(s, \sqrt{r}), & 2 \nmid p - 2q < 0. \end{cases}$$

Proof. In view of Proposition 2.22 and Corollary 2.23, the only case that needs some consideration is that of $p = 0$. Here, Faà di Bruno's formula gives

$$\partial_{r=0}^{2q} f^\circ(s, r) = \partial_{r=0}^{2q} f^\circ(s, \sqrt{r^2}) = \sum_{k_1 + 2k_2 = 2q} \frac{(2q)!}{k_1! k_2!} \partial_{t=0}^{k_1+k_2} f^\circ(s, \sqrt{t})\, 0^{k_1} 1^{k_2}.$$

All summands except for $k_2 = q$ vanish. □

3. Symmetric superspaces and spherical superfunctions

In this section, we introduce our main objects, the *spherical superfunctions.* Before proceeding to the definition of the spherical superfunctions, we collect some ancillary facts concerning symmetric superpairs and integration formulæ.

3.1. Symmetric superpairs.

We review some facts on symmetric superpairs, referring to Refs. [1, 6] for omitted details.

DEFINITION 3.1 (Symmetric superpairs and Cartan decomposition). A symmetric superpair is a pair $(\mathfrak{g}, \theta)$, where $\mathfrak{g}$ be a complex Lie superalgebra and θ an involutive automorphism of $\mathfrak{g}$. The eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{k} := \ker(1 - \theta), \quad \mathfrak{p} := \ker(1 + \theta)$$

is called *Cartan decomposition* of $(\mathfrak{g}, \theta)$.
Let (G, K, θ) be given, where G is a *cs* Lie supergroup with Lie superalgebra $\mathfrak{g}$, θ is an involutive automorphism of G, and K is a closed subsupergroup, $\theta|_K = \mathrm{id}_K$, and its Lie superalgebra is $\mathfrak{k} = \ker(1 - \theta)$ (denoting the derivative of θ by the same letter). Then (G, K, θ) is called a symmetric supertriple, and the symmetric superpair $(\mathfrak{g}, \theta)$ is called the *infinitesimal superpair* associated with (G, K, θ).
Then (G, K, θ) is said to admit a *global Cartan decomposition* if the morphism

$$K \times \mathbb{A}(\mathfrak{p}_\mathbb{R}) \longrightarrow G : (k, x) \longmapsto ke^x \tag{3.1}$$

is an isomorphism of *cs* manifolds. Here, we write $e^x := \exp_G(x)$.

PROPOSITION 3.2. *A pair (G, K, θ) admits a global Cartan decomposition if and only if this is true for (G_0, K_0, θ_0).*

Proof. The morphism in (3.1) is a local isomorphism at $(1, z)$, for any $z \in \mathbb{A}(\mathfrak{p}_\mathbb{R})_0$, since its derivative is given by $\mathfrak{k} \times \mathfrak{p} \longrightarrow \mathfrak{g} : (y, x) \longmapsto y + x$. Since the morphism is K-equivariant, it is everywhere a local isomorphism. By the inverse function theorem [30], it is an isomorphism if and only if the underlying map is a bijection. □

Definition 3.3 (Reductive and even type conditions). The notions of a *reductive*, *strongly reductive*, or *even type* symmetric superpair $(\mathfrak{g},\theta)$ are defined in Ref. [1].
If $(\mathfrak{g},\theta)$ is the infinitesimal superpair of a symmetric supertriple (G,K,θ), then we accordingly apply these adjectives to (G,K,θ).

Definition 3.4. Let $(\mathfrak{g},\theta)$ be a reductive symmetric superpair of even type with even Cartan subspace $\mathfrak{a}$. Then

$$\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{a}\oplus\bigoplus_{\alpha\in\Sigma}\mathfrak{g}^\alpha,\quad \mathfrak{m}\coloneqq\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}),\quad \mathfrak{g}^\alpha\coloneqq\bigcap_{h\in\mathfrak{a}}\ker(\operatorname{ad}h-\alpha(h)),\tag{3.2}$$

where $\Sigma\subseteq\mathfrak{a}^*\setminus 0$, called the set of *restricted roots*, is the finite set defined by this equation. A root $\alpha\in\Sigma$ is called *even* and *odd*, if, respectively, $\mathfrak{g}_{\bar 0}^\alpha\neq 0$ and $\mathfrak{g}_{\bar 1}^\alpha\neq 0$. Note that roots may simultaneously be even and odd.
Given $h\in\mathfrak{a}_{\mathbb{R}}$ such that $\alpha(h)\in\mathbb{R}\setminus 0$ for all $\alpha\in\Sigma$, the subset

$$\Sigma^+\coloneqq\big\{\alpha\in\Sigma\bigm|\alpha(h)>0\big\}$$

is called a *positive system.* Roots $\alpha\in\Sigma^+$ for which $\frac{\alpha}{2}\notin\Sigma$ are called *indivisible.* We set $\varrho\coloneqq\frac12\sum_{\alpha\in\Sigma^+}m_\alpha\alpha$, where $m_\alpha\coloneqq\operatorname{sdim}\mathfrak{g}^\alpha=\dim\mathfrak{g}_{\bar 0}^\alpha-\dim\mathfrak{g}_{\bar 1}^\alpha$.
Fixing a positive system Σ^+, Equation (3.2) takes on the form

$$\mathfrak{g}=\bar{\mathfrak{n}}\oplus\mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n},\quad \mathfrak{n}\coloneqq\bigoplus_{\alpha\in\Sigma^+}\mathfrak{g}^\alpha,\quad \bar{\mathfrak{n}}\coloneqq\theta(\mathfrak{n})=\bigoplus_{\alpha\in\Sigma^+}\mathfrak{g}^{-\alpha},$$

the *Bruhat decomposition* of $\mathfrak{g}$. Since $\mathfrak{g}^\alpha\oplus\mathfrak{g}^{-\alpha}$ is θ-invariant, we have

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n},$$

the *Iwasawa decomposition* of $\mathfrak{g}$.

For the remainder of this subsection, let (G,K,θ) be a reductive symmetric supertriple of even type. Fix an even Cartan subspace and a positive system of roots. Denote by N, $\bar N$, and A the analytic subsupergroups of G corresponding to $\mathfrak{n}$, $\bar{\mathfrak{n}}$, and $\mathfrak{a}$, respectively. Here, the *analytic subsupergroup* of G corresponding to a subalgebra $\mathfrak{h}\subseteq\mathfrak{g}$ such that $\mathfrak{h}_{\bar 0,\mathbb{R}}\coloneqq\mathfrak{h}\cap\mathfrak{g}_{\bar 0,\mathbb{R}}$ is a real form of $\mathfrak{h}_{\bar 0}$ is defined to be the *cs* Lie supergroup associated with the supergroup pair $(\mathfrak{h},H_0)$, where H_0 is the analytic (*i.e.* , connected) subgroup of G_0 with the Lie algebra $\mathfrak{h}_{\bar 0,\mathbb{R}}$. Moreover, let M be the closed subsupergroup associated with the supergroup pair $(\mathfrak{m},M_0)$, where $M_0\coloneqq Z_{K_0}(\mathfrak{a})$.
The following proposition follows in the same way as Proposition 3.2.

Proposition 3.5 (Global Bruhat decomposition). *The morphism*

$$\bar N\times M\times A\times N\longmapsto G:(\bar n,m,a,n)\longmapsto\bar n man$$

is an open embedding if and only if this is true for the underlying map. In this case, we say that (G,K,θ) *admits a* global Bruhat decomposition*.*

For the following proposition, consult Refs. [1,6].

Proposition 3.6 (Global Iwasawa decomposition). *Both of the morphisms*

$$K \times A \times N \longrightarrow G : (k, a, n) \longmapsto kan,$$
$$N \times A \times K \longrightarrow G : (n, a, k) \longmapsto nak$$

are isomorphisms if and only if this is already true for one of the underlying maps. In this case, we say that (G, K, θ) *admits a* global Iwasawa decomposition.

Remark 3.7. Observe that M centralises $\mathfrak{a}$, since

$$\mathrm{Ad}(me^x)(h) = \mathrm{Ad}(m)\big(e^{\mathrm{ad}(x)}(h)\big) = h$$

for $m \in_S M_0$, $x \in_S \mathbb{A}(\mathfrak{m}_{\bar{1}})$, $h \in_S \mathbb{A}(\mathfrak{a}_{\mathbb{R}})$. Similarly, M normalises N and $\bar{N}$.

Let (G, K, θ) admit a global Iwasawa decomposition. We define morphisms

$$k, u \colon G \to K, \quad A, H \colon G \to \mathbb{A}(\mathfrak{a}_{\mathbb{R}}), \quad n, n_1 \colon G \to N$$

by requiring, for $g \in_S G$, that

$$g = k(g)e^{H(g)}n(g) = n_1(g)e^{A(g)}u(g). \tag{3.3}$$

Then

$$n_1(g) = n(g^{-1})^{-1}, \quad A(g) = -H(g^{-1}), \quad u(g) = k(g^{-1})^{-1}. \tag{3.4}$$

In view of Remark 3.7, H and k are right M-invariant, *i.e.*

$$H(gm) = H(g), \quad k(gm) = k(g), \quad g \in_S G, m \in_S M. \tag{3.5}$$

Moreover, we note for later reference that

$$H(gh) = H\big(gk(h)\big) + H(h), \quad k(gh) = k\big(gk(h)\big), \tag{3.6}$$

for all $g, h \in_S G$. This follows from a straightforward computation on points. The importance of the above decompositions is that they give rise to natural coordinate systems on certain homogeneous superspaces. Here and in the sequel, we will use quotients of *cs* Lie supergroups. They are defined in the same way as for real Lie supergroups, see Refs. [2, 13] for the latter. We will need the fact that for a closed *cs* Lie subsupergroup H of G, the sheaf of superfunctions on G/H is the direct image under the canonical projection $G_0 \longrightarrow G_0/H_0$ of the sheaf of H-invariant superfunctions on G.

Proposition 3.8. *Let* (G, K, θ) *admit a global Iwasawa decomposition. The morphism*

$$G \longrightarrow N \times A : g \mapsto \big(n_1(g), e^{A(g)}\big)$$

induces an isomorphism $G/K \longrightarrow N \times A$.

Let $Q = MAN$ be the closed subsupergroup of G generated by M, A, and N.

Proposition 3.9. *Let* (G, K, θ) *admit global Bruhat and Iwasawa decompositions. The composite*

$$G \xrightarrow{\;k\;} K \longrightarrow K/M$$

induces an isomorphism $G/Q \to K/M$. *The composite*

$$\bar{N} \longrightarrow G \longrightarrow G/Q \xrightarrow{\dot{k}} K/M$$

is an open embedding, which is also denoted by k.

3.2. Integral formulæ for supergroups and symmetric superspaces. In this subsection, we derive integral formulæ for the decompositions given above. These are crucial for the proof of the central Corollary 3.20 and therefore for the proof of the c-function asymptotics and the Harish-Chandra series. Moreover, the proof of *loc. cit.* hinges on the parameter-dependent version of these formulæ, which are therefore technically not a simple-minded copy of their classical relatives. We begin with some generalities, *cf.* Ref. [2].

Definition 3.10 (Invariant Berezinian densities). Let G be a *cs* Lie supergroup and $a\colon G \times X \to X$ an action. Consider the *cs* manifold $X_G := G \times X$ over G. A Berezinian density $\omega \in \Gamma(|\mathcal{B}er|_X)$ is said to be *G-invariant* if

$$(a, p_X)^\sharp\big(p_X^\sharp(\omega)\big) = p_X^\sharp(\omega).$$

Here, $p_X : X_G \to X$ is the projection.

The following is straightforward.

Lemma 3.11. *Let $\omega \in \Gamma(|\mathcal{B}er|_X)$ be a Berezinian density. Then ω is G-invariant if and only if, the following is true:*

$$\int_X \omega(x) f(gx) = \int_X \omega(x) f(x)$$

for any $g \in_S G$, any cs *manifold S, and any $f \in \Gamma_c(p_{X,0!}\mathcal{O}_{X_G})$.*

The G-superspace X is called *analytically unimodular* if there is a non-zero G-invariant Berezinian density. If the action is transitive and X is analytically unimodular, then $|\mathcal{B}er|_X$ has a global module basis, given by the choice of such a Berezinian density; moreover, it is unique up to a multiplicative constant [2]. Sufficient conditions for the analytic unimodularity of homogeneous G-superspaces are stated in [1, Proposition A.2].

Proposition 3.12. *Let G be analytically unimodular and $|Dg|$ a non-zero G-invariant Berezinian density. Let $U = -U \subseteq \mathbb{A}(\mathfrak{g}_{\mathbb{R}})_0$ be an open neighbourhood of 0 such that $\exp_G : \mathbb{A}(\mathfrak{g}_{\mathbb{R}})|_U \to G$ is an open embedding. Then*

$$\exp_G^\sharp(|Dg|) = |D\lambda|(x)\,|\mathrm{Ber}|\Big(\frac{1 - e^{-\operatorname{ad} x}}{\operatorname{ad} x}\Big)$$

on $\mathbb{A}(\mathfrak{g}_{\mathbb{R}})|_U$. Here, $|D\lambda|$ is an adequately normalized Berezin–Lebesgue density.

When G is nilpotent, then $\operatorname{str}\operatorname{ad} x = 0$ for the generic point x of $\mathbb{A}(\mathfrak{g}_{\mathbb{R}})$. Hence, by [1, Proposition A.2] and Proposition 3.12, the following is immediate.

Corollary 3.13. *Let G be a nilpotent* cs *Lie supergroup. Then G is analytically unimodular and*

$$\exp_G^\sharp(|Dg|) = |D\lambda|$$

for an appropriate normalization of the invariant density $|Dg|$ and the Berezin–Lebesgue density $|D\lambda|$.

Proof of Proposition 3.12. The claim follows along the lines of the classical result [23, Chapter I, § 2, Theorem 1.14].
If V is a sufficiently small neighbourhood of $0 \in \mathfrak{g}_0$, then the Campbell–Hausdorff morphism

$$C : \mathbb{A}(\mathfrak{g}_\mathbb{R})|_U \times \mathbb{A}(\mathfrak{g}_\mathbb{R})|_V \to \mathbb{A}(\mathfrak{g}_\mathbb{R}), \quad \exp_G C(x,y) = (\exp_G x)(\exp_G y)$$

is well-defined. We let

$$X \coloneqq \mathbb{A}(\mathfrak{g}_\mathbb{R})|_U \times \mathbb{A}(\mathfrak{g}_\mathbb{R})|_V, \quad S \coloneqq \mathbb{A}(\mathfrak{g}_\mathbb{R}),$$

and consider X as a *cs* manifold over S *via* p_1.
Set $\varphi \coloneqq (p_1, C)$ and let $\exp^\sharp(|Dg|) = |D\lambda|\rho$ for some $\rho \in \mathcal{O}_{\mathbb{A}(\mathfrak{g}_\mathbb{R})}(U)$. For any $f \in \Gamma_c(\mathcal{O}_{\mathbb{A}(\mathfrak{g}_\mathbb{R})}|_U)$, we have

$$\begin{aligned}\int_{\mathbb{A}(\mathfrak{g}_\mathbb{R})} |D\lambda|\,\rho f &= \int_G |Dg|\, f(\log g) = \int_G |Dg|\, f\big(\log(\exp_G(-x)g)\big)\\ &= \int_{\mathbb{A}(\mathfrak{g}_\mathbb{R})} |D\lambda(y)|\,\rho(y) f\big(C(-x,y)\big)\\ &= \int_{\mathbb{A}(\mathfrak{g}_\mathbb{R})} |D\lambda(y)|\,|\mathrm{Ber}|(\mathcal{T}_{\varphi/S})\rho(C(x,y))f(y).\end{aligned}$$

Here, x is the generic point of $T \coloneqq \mathbb{A}(\mathfrak{g}_\mathbb{R})|_U$ and the identity $y = C(x, C(-x,y))$ was applied. Since f was chosen arbitrarily, this implies

$$\rho(y) = |\mathrm{Ber}|\big(\mathcal{T}_{\varphi/S}\big)(x,y)\rho(C(x,y)), \quad y \in_T \mathfrak{g}|_V.$$

Setting $y = 0$ shows

$$\rho(x) = |\mathrm{Ber}|\big(\mathcal{T}_{\varphi/S}\big)(x,0)^{-1}\rho(0).$$

Hence, the claim will follow after suitable normalization and computing $\mathcal{T}_{\varphi/S}|_{y=0}$. For the latter, it suffices to prove

$$T_{(x,0)/S}(\varphi) = \frac{\operatorname{ad} x}{1 - e^{-\operatorname{ad} x}}$$

where we consider $\operatorname{ad} x$ as an $\mathcal{O}_T$-linear endomorphism of $\mathcal{O}_T \otimes \mathfrak{g}$. Interpreting vector fields as morphisms *via* dual numbers $\mathbb{D}_{\bar 0}$ (see Subsection 2.3) this reduces to an equation between morphisms, which can be checked on $\mathbb{A}^{0|q}$-valued points, and hence follows from the classical situation [25]. □

For the remainder of this subsection, let (G, K, θ) be a reductive symmetric supertriple of even type, such that a global Iwasawa decomposition exists. Further, assume that K_0 is compact. Then the $G \times G$-superspace G, the G-superspace G/K, and the K-superspace K/M are all analytically unimodular. We choose corresponding invariant Berezinian densities $|Dg|$, $|D\dot g|$, and $|D\dot k|$, as well as invariant densities $|Dn|$ and $|D\bar n| = \theta^\sharp(|Dn|)$ on N and $\bar N$, respectively, and a Haar density da on A. The normalization of these relative to each other is fixed by the Propositions 3.14, 3.15, and 3.17, which are simple generalizations of integral formulæ from Refs. [1, 6].

We will use without further mention the fact that for compactly supported integrands, the Berezin (fibre) integral is independent of the choice of retraction (Theorem 2.18) and thus admits coordinate transformations (Corollary 2.19).

PROPOSITION 3.14 ([1, Proposition 2.2]). *The pullback of the invariant Berezin density* $|D\dot{g}|$ via *the Iwasawa isomorphism is* $|Dk| \otimes da \otimes |Dn|$. *In particular,*

$$\int_G |Dg|\, f(s,g) = \int_K |Dk| \int_A da \int_N |Dn|\, f(s,kan) e^{2\varrho(\log a)}$$

for $f \in \Gamma_c(p_{1!}\mathcal{O}_{S\times G})$.

PROPOSITION 3.15 ([6, Lemma 4.2]). *Let* $f, h \in \Gamma_c(p_{1!}\mathcal{O}_{S\times K/M})$. *Then*

$$\int_K |Dk|\, f(s, k(g^{-1}k))h(s,k) = \int_K |Dk|\, f(s,k)h(s,k(gk))e^{-2\varrho(H(gk))}$$

for any $s \in_T S$ *and all* cs *manifolds* T.

COROLLARY 3.16. *Let* $f \in \Gamma(\mathcal{O}_{S\times K/M})$. *Then for any* $s \in_T S$, *we have*

$$\int_{K/M} |D\dot{k}|\, f\big(s, k(g^{-1}k)\big) = \int_{K/M} |D\dot{k}|\, f(s,k)e^{-2\varrho(H(gk))}.$$

Proof. In the classical case, this equation follows directly from Proposition 3.15. In the super setting, the volume of K may vanish, so one has to argue with greater care. Let $\chi \in \Gamma_c(\mathcal{O}_K)$, such that $\int_K |Dk'|\,\chi(k') = 1$. Writing $B = K/M$,

$$\begin{aligned}\int_B |D\dot{k}|\, f(s,k(g^{-1}k)) &= \int_K |Dk'|\,\chi(k') \int_B |D\dot{k}|\, f\big(s, k(g^{-1}k'k)\big)\\ &= \int_B |D\dot{k}| \int_K |Dk'|\,\chi(k'k^{-1}) f\big(s, k(g^{-1}k')\big),\end{aligned}$$

by the left-invariance of $|D\dot{k}|$ and the right-invariance of $|Dk'|$. By Proposition 3.15, this equals

$$= \int_B |D\dot{k}| \int_K |Dk'|\,\chi\big(k(gk')k^{-1}\big) f(s,k')e^{-2\varrho(H(gk'))},$$

so on applying the right-invariance of $|Dk'|$ and the left-invariance $|D\dot{k}|$ again, we obtain the equalities

$$\begin{aligned}&= \int_K |Dk'| \int_B |D\dot{k}|\,\chi\big(k(gk'k)k^{-1}\big) f(s,k'k)e^{-2\varrho(H(gk'k))}\\ &= \int_K |Dk'| \int_B |D\dot{k}|\,\chi\big(k(gk)k^{-1}k'\big) f(s,k)e^{-2\varrho(H(gk))}.\end{aligned}$$

Finally, the left-invariance of $|Dk'|$ gives

$$
\begin{aligned}
&= \int_B |D\dot{k}| \Big(\int_K |Dk'|\, \chi(k') \Big) f(s,k) e^{-2\varrho(H(gk))} \\
&= \int_B |D\dot{k}|\, f(s,k) e^{-2\varrho(H(gk))},
\end{aligned}
$$

which is the desired statement. □

We have the following fact from Ref. [6].

Proposition 3.17 ([6, Proposition 4.4]). *The pullback of the invariant Berezin density $|D\dot{k}|$ on K/M* via *the open embedding k from Proposition 3.9 is $|D\bar{n}|e^{-2\varrho(H(\bar{n}))}$.*

Remark 3.18. Note that the standard retraction on $\bar{N}$ is in general not compatible *via* k with any globally defined retraction on K/M. Hence, Theorem 2.18 does not apply, and the equality

$$
\int_{K/M} |D\dot{k}|\, f(s,k) = \int_{\bar{N}}^{r} |D\bar{n}|\, f\big(s, k(\bar{n})\big) e^{-2\varrho(H(\bar{n}))}
$$

will in general not hold for superfunctions $f \in \Gamma(\mathcal{O}_{S\times K/M})$, unless the intersection $\operatorname{supp} f$ with the image of $S \times \bar{N}$ is not compact along the fibres of p_1. Below, where we derive the leading asymptotics of the spherical superfunctions on G/K, we will resolve these issues by introducing an atlas of similar charts and cutting off the integrand in these charts by the choice of a partition of unity.

A similar problem occurs if we use polar coordinates on G/K, in fact, it is more severe, leading to singularities at the boundary of the Weyl chamber. In a forthcoming paper, where we treat the inversion formula for the spherical Fourier transform in rank one, we will discuss these singularities and the 'boundary terms' that they introduce at length.

3.3. Definition of the spherical superfunctions. Keeping the above assumptions, we can now define, for $\lambda \in \mathfrak{a}^*$, the *spherical superfunction* $\phi_\lambda \in \Gamma(\mathcal{O}_X)$, where $X \coloneqq G/K$, as follows. For $g \in_T G$, where T is any *cs* manifold, we let

$$
\phi_\lambda(g) \coloneqq \int_{K/M} |D\dot{k}|\, e^{(\lambda-\varrho)(H(gk))}. \tag{3.7}
$$

By Lemma 3.11, this defines a superfunction on X.

We will derive some alternative integral expressions for ϕ_λ in this subsection—in particular, we will obtain the symmetry property stated in Corollary 3.20, which will be essential for the Harish-Chandra series expansion of ϕ_λ. The statements we give here are essentially identical to the classical case, and their derivations are parallel to those given in Refs. [23, 24]. However, they are based on Corollary 3.16, whose proof was a little more subtle than classically, so we briefly give the details.

Proposition 3.19. *For any $\lambda \in \mathfrak{a}^*$ and $g, h \in_T G$, we have*

$$\phi_\lambda(hg^{-1}) = \int_{K/M} |D\dot{k}|\, e^{(\lambda-\varrho)(H(hk))} e^{(-\lambda-\varrho)(H(gk))}$$

Proof. On applying Equation (3.6) twice, we obtain

$$H(hg^{-1}k) = H\big(hk(g^{-1}k)\big) + H(g^{-1}k) = H\big(hk(g^{-1}k)\big) - H\big(gk(g^{-1}k)\big),$$

so that

$$\phi_\lambda(hg^{-1}) = \int_{K/M} |D\dot{k}|\, e^{(\lambda-\varrho)(H(hk(g^{-1}k)))} e^{-(\lambda-\varrho)(H(gk(g^{-1}k)))}.$$

Setting $S := G \times G$ and $f(g, h, k) := e^{(\lambda-\varrho)(H(hk))} e^{-(\lambda-\varrho)(H(gk))}$ in Corollary 3.16, we arrive at our claim. □

Setting $h = 1$ in Proposition 3.19, the following fact is immediate. It will be instrumental in deriving a series expansion for ϕ_λ.

Corollary 3.20. *For $\lambda \in \mathfrak{a}^*$ and $g \in_T G$, we have $\phi_\lambda(g^{-1}) = \phi_{-\lambda}(g)$.*

This allows us to derive an alternative expression for ϕ_λ, as follows: Consider the isomorphism $\tilde{\imath}_K : M\backslash K \to K/M$ induced by the inversion morphism i_K of K. The Berezinian density

$$|D\dot{k}_r| := \tilde{\imath}_K^\sharp(|D\dot{k}|),$$

where $|D\dot{k}|$ is the K-invariant Berezinian density on K/M, is right K-invariant. We have the following formula, which will be used in a subsequent paper.

Corollary 3.21. *For any $\lambda \in \mathfrak{a}^*$ and $g \in_T G$, we have*

$$\phi_\lambda(g) = \int_{M\backslash K} |D\dot{k}_r|\, e^{(\lambda+\varrho)(A(kg))}.$$

where $A(\cdot)$ is defined in Equation (3.3).

Proof. We have by the definition of $|D\dot{k}_r|$ and Equation (3.4)

$$\begin{aligned}\phi_\lambda(g) &= \int_{K/M} |D\dot{k}| e^{(\lambda-\varrho)(H(gk))} \\ &= \int_{K/M} |D\dot{k}| e^{(-\lambda+\varrho)(A(k^{-1}g^{-1}))} = \int_{M\backslash K} |D\dot{k}_r| e^{(-\lambda+\varrho)(A(kg^{-1}))},\end{aligned}$$

so the assertion follows from Corollary 3.20. □

4. The Harish-Chandra c-function

In this section, unless something else is stated, let (G, K, θ) be a reductive symmetric supertriple of even type admitting global Iwasawa and Bruhat decompositions. A fixed even Cartan subspace $\mathfrak{a}$ and positive system Σ^+ are understood, and in all the cases we will consider, the *rank* $\dim \mathfrak{a} - \dim \mathfrak{a} \cap \dim \mathfrak{z}(\mathfrak{g})$ will be one. The notation introduced in Subsection 3.1 will be used freely.

We will derive the leading asymptotics of the spherical superfunctions ϕ_λ. These are governed by the *Harish-Chandra c-function*. We compute these explicitly in terms of Euler Γ functions. While the general structure of these functions is similar to the even setting, the occurrence of negative root multiplicities leads to a shift in the location of poles and zeroes.

By a rank reduction procedure, this result has, in Ref. [6], been extended to arbitrary rank, on the basis of our results. In general, isotropic restricted roots occur, leading to terms in the general c-function formula that are not simply shifted versions of the 'purely even' c-function.

Moreover, contrary to the 'classical' setting of Riemannian symmetric spaces of the non-compact type, where the proof of convergence of the leading asymptotic term of ϕ_λ is a simple argument based on the dominated convergence theorem, in the super setting, this becomes a quite subtle issue. We need to cover the 'maximal boundary' $B = K/M$ by a whole atlas of Weyl-group related 'stereographical charts'. In the situations we consider (whose common feature is that they are of rank one), B is a 'geodesic supersphere at infinity', and this makes for two chart domains. These are essential, since the integrals on the individual domains are both divergent unless the integrand is cut off appropriately.

For the divergences in this expansion to cancel, the Weyl symmetry of the spherical functions (Corollary 3.20), derived above without any restriction on the rank, by the use of relative *cs* manifolds and fibre integrals, will be of central importance.

Based on the derivation of the leading asymptotics, we prove the existence of a full Harish-Chandra series expansion, similar to the even case.

We will show the existence of the c-function in three cases of rank one and compute it explicitly. Contrary to the classical situation, the proof of existence is by far the more difficult of these steps. Both (the proof of existence and the explicit determination) will be performed on a case-by-case basis.

4.1. Definition and statement of the main result.

Definition 4.1. Let $h_0 \in \mathfrak{a}$ such that $\alpha(h_0) > 0$ for all $\alpha \in \Sigma^+$. The *Harish-Chandra c-function* is defined as follows:

$$c(\lambda) \coloneqq \lim_{t\to\infty} e^{-(\lambda-\varrho)(th_0)}\phi_\lambda(e^{th_0}), \tag{4.1}$$

for $\lambda \in \mathfrak{a}^*$, $\Re\lambda(h_0) > 0$, provided that the limit exists.

Unless the *rank* of (G, K, θ), defined to be $\dim \mathfrak{a} - \dim \mathfrak{a} \cap \mathfrak{z}(\mathfrak{g})$, is one, it is not obvious *per se* that this definition is independent of the choice of h_0.

We will consider the cases stated in Table 4.1 below, where we abbreviate $\mathfrak{gl}(p|q, \mathbb{C})$ by $\mathfrak{gl}_{p|q}$, *etc.*, and allow arbitrary integers $p, q \geqslant 0$. We refer the reader to the following subsubsections for the precise definitions, in particular, regarding the *cs* supergoups G and K.

For any of these symmetric pairs, we have the following statement.

$\mathfrak{g}$	$\mathfrak{k}$	G	K
$\mathfrak{gl}_{2+p\|q}$	$\mathfrak{gl}_1 \times \mathfrak{gl}_{1+p\|q}$	$\mathrm{U}_{cs}(1,1+p\|q)$	$\mathrm{U}(1) \times \mathrm{U}_{cs}(1+p\|q)$
$\mathfrak{osp}_{2+p\|2q}$	$\mathfrak{osp}_{1+p\|2q}$	$\mathrm{SOSp}^+_{cs}(1,1+p\|2q)$	$\mathrm{SOSp}_{cs}(1+p\|2q)$
$\mathfrak{gl}_{1\|1} \times \mathfrak{gl}_{1\|1}$	$\mathfrak{gl}_{1\|1}$	$(\mathrm{GL} \times \mathrm{GL})^+_{\mathbb{C}\|\mathbb{R}}(1\|1)$	$\mathrm{UGL}^+_{\mathbb{R}}(1\|1)$

TABLE 1. Riemannian symmetric superspaces under consideration

THEOREM 4.2. *Let be (G,K) be one of the symmetric pairs listed in Table 4.1 and let α be a choice of positive indivisible restricted root. The c-function $c(\lambda)$ exists for $\Re\lambda > 0$. For some $c_0 \equiv c_0(\varrho) \neq 0$, it is given by*

$$c(\lambda) = c_0 \frac{2^{-\lambda}\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}\left(\lambda + \frac{m_\alpha}{2} + 1\right)\right)\Gamma\left(\frac{1}{2}\left(\lambda + \frac{m_\alpha}{2} + m_{2\alpha}\right)\right)}$$

if α is anisotropic, and if α is isotropic, then it is given by

$$c(\lambda) = c_0\lambda.$$

Here, we identify $\lambda \equiv \frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ if α is anisotropic and $\lambda \equiv \langle\lambda,\alpha\rangle$ otherwise.

In particular, up to the constant c_0, the value of the c-function $c(\lambda)$ is independent of the choice of h_0 with $\alpha(h_0) > 0$. After suitable normalisation of h_0, the constants c_0 depend on p and q only *via* ϱ, see below for details.

4.2. PROOF OF THEOREM 4.2. We now begin with the examination of the c-function for the Riemannian symmetric superspaces $X = G/K$ from Table 4.1, thereby proving Theorem 4.2 case-by-case. For the three individual cases, the formula is proved below in Theorem 4.8, Theorem 4.14, and Corollary 4.17, respectively.
We will constantly be using the functor of points, employing Proposition 2.1, Corollary 2.2 and Proposition 2.3 to compute it explicitly.

4.2.1. *The unitary case.* In the following, let $\mathfrak{g} \coloneqq \mathfrak{gl}(2+p|q,\mathbb{C})$. Moreover, we let $G \coloneqq \mathrm{U}_{cs}(1,1+p|q)$ denote the cs form of $\mathrm{GL}(2+p|q,\mathbb{C})$ corresponding to

$$G_0 \coloneqq \mathrm{U}(1,1+p) \times \mathrm{U}(q),$$

which is a the real form of the complex Lie group $G_{\mathbb{C},0} \coloneqq \mathrm{GL}(2+p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C})$.
An involution θ of G resp. of $\mathfrak{g}$ is given by $\theta(x) \coloneqq \sigma x \sigma$ where

$$\sigma \coloneqq \left(\begin{array}{c:c|c} -\mathbb{1}_1 & 0 & 0 \\ \hdashline 0 & \mathbb{1}_{1+p} & 0 \\ \hline 0 & 0 & \mathbb{1}_q \end{array}\right). \tag{4.2}$$

Under this involution, $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with

$$\mathfrak{k} = \left(\begin{array}{c:c|c} * & 0 & 0 \\ \hdashline 0 & * & * \\ \hline 0 & * & * \end{array}\right), \quad \mathfrak{p} = \left(\begin{array}{c:c|c} 0 & * & * \\ \hdashline * & 0 & 0 \\ \hline * & 0 & 0 \end{array}\right).$$

Here, dashed lines in the matrices indicate the action of the involution, whereas full lines signify the grading.
A non-degenerate invariant supersymmetric even bilinear form b on $\mathfrak{g}$ is given by $b(x,y) \coloneqq \mathrm{str}(xy)$. We let $\mathfrak{a}_{\mathbb{R}} \subseteq \mathfrak{p}_0$ be the subspace generated by the matrix

$$h_0 \coloneqq \left(\begin{array}{c:cc|c} 0 & 1 & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right). \tag{4.3}$$

Here and in what follows, the dimension of the rows and columns in all matrices will be 1, 1, p, and q, respectively.
As ungraded vector spaces, $\mathfrak{g} = \mathfrak{gl}(2+p+q,\mathbb{C})$, so since h_0 is even, the root decomposition is the same as in the classical case. Defining $\alpha \in \mathfrak{a}^*$ by $\alpha(h_0) = 1$, there are exactly four roots: $\pm\alpha, \pm 2\alpha$.
The general elements of $\mathfrak{g}^{\alpha}$ and $\mathfrak{g}^{-\alpha}$ are respectively

$$\left(\begin{array}{c:cc|c} 0 & 0 & B_{\bar 0} & B_{\bar 1} \\ \hdashline 0 & 0 & B_{\bar 0} & B_{\bar 1} \\ -C_{\bar 0} & C_{\bar 0} & 0 & 0 \\ \hline -C_{\bar 1} & C_{\bar 1} & 0 & 0 \end{array}\right), \quad \left(\begin{array}{c:cc|c} 0 & 0 & -B_{\bar 0} & -B_{\bar 1} \\ \hdashline 0 & 0 & B_{\bar 0} & B_{\bar 1} \\ C_{\bar 0} & C_{\bar 0} & 0 & 0 \\ \hline C_{\bar 1} & C_{\bar 1} & 0 & 0 \end{array}\right). \tag{4.4}$$

The general elements of $\mathfrak{g}^{2\alpha}$ and $\mathfrak{g}^{-2\alpha}$ are respectively

$$\left(\begin{array}{c:cc|c} -A & A & 0 & 0 \\ \hdashline -A & A & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \quad \left(\begin{array}{c:cc|c} -A & -A & 0 & 0 \\ \hdashline A & A & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right).$$

Moreover, $\mathfrak{m}$ consists of the following matrices:

$$\left(\begin{array}{c:cc|c} D & 0 & 0 & 0 \\ \hdashline 0 & D & 0 & 0 \\ 0 & 0 & E & F \\ \hline 0 & 0 & G & H \end{array}\right).$$

We let $\Sigma^+ \coloneqq \{\alpha, 2\alpha\}$, so that $\mathfrak{n} = \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{2\alpha}$ and $\bar{\mathfrak{n}} = \mathfrak{g}^{-\alpha} \oplus \mathfrak{g}^{-2\alpha}$. From the above, we see that

$$m_{\pm\alpha} = 2(p-q), \quad m_{\pm 2\alpha} = 1, \quad \varrho = (1+p-q)\alpha.$$

The analytic subsupergroup K of G with Lie superalgebra $\mathfrak{k}$ is $\mathrm{U}(1) \times \mathrm{U}_{cs}(1+p|q)$. The underlying Lie group is $K_0 = \mathrm{U}(1) \times \mathrm{U}(1+p) \times \mathrm{U}(q)$. Since the Riemannian symmetric space $G_0/K_0 = \mathrm{U}(1,1+p)/(\mathrm{U}(1) \times \mathrm{U}(1+p))$ is of non-compact type, G_0 admits global Iwasawa and Bruhat decompositions [22]. In view of Proposition 3.6 and Proposition 3.5, the same is true of G.
We parametrise $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ by setting $\tau \coloneqq \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 0 & \mathbb{1}_{p+q} \end{smallmatrix}\right)^t$ and

$$X(D,E,F) \coloneqq \tau \begin{pmatrix} D & E \\ F & 0 \end{pmatrix} \tau^t \sigma, \quad \bar{X}(D,E,F) \coloneqq \sigma\tau \begin{pmatrix} D & E \\ F & 0 \end{pmatrix} \tau^t, \tag{4.5}$$

so that

$$\mathfrak{n} = \{X(D,E,F) \mid D \in \mathbb{C}, E \in \mathbb{C}^{1\times p|q}, F \in \mathbb{C}^{p|q\times 1}\},$$
$$\bar{\mathfrak{n}} = \{\bar{X}(D,E,F) \mid D \in \mathbb{C}, E \in \mathbb{C}^{1\times p|q}, F \in \mathbb{C}^{p|q\times 1}\}.$$

Let $\mathfrak{g}_{\mathbb{R}} \coloneqq \mathfrak{g}_{\bar{0},\mathbb{R}} \oplus \mathfrak{g}_{\bar{1}}$, $\mathfrak{g}_{\bar{0},\mathbb{R}}$ being the Lie algebra of G_0, and $A_{\mathbb{R}} \coloneqq A \cap \mathfrak{g}_{\mathbb{R}}$ for any $A \subseteq \mathfrak{g}$. Then by Proposition 2.14, we have an isomorphism φ, given by

$$\varphi\colon \mathbb{A}^1 \times \mathbb{A}^{p|q} \times \mathbb{A}^{p|q} \longrightarrow \mathbb{A}(\bar{\mathfrak{n}}_{\mathbb{R}}),$$
$$(a,b,c) \longmapsto X\big(\tfrac{1}{2i}a, E(b,c), F(b,c)\big),$$

on T-valued points $b = b_{\bar{0}} + b_{\bar{1}}, c = c_{\bar{0}} + c_{\bar{1}} \in_T \mathbb{A}^{p|q}$, where we set

$$E(b,c) \coloneqq \begin{pmatrix} -(b_{\bar{0}} - ic_{\bar{0}})^t & -(b_{\bar{1},1} - c_{\bar{1},1}) & \cdots & -(b_{\bar{1},q} - c_{\bar{1},q}) \end{pmatrix},$$
$$F(b,c) \coloneqq \begin{pmatrix} (b_{\bar{0}} + ic_{\bar{0}})^t & b_{\bar{1},1} + c_{\bar{1},1} & \cdots & b_{\bar{1},q} + c_{\bar{1},q} \end{pmatrix}^t.$$

A similar statement holds for $\mathfrak{n}$.

In what follows, we will again make extensive use of the formalism of T-valued points, see above for explanations. For any *cs* manifold T, the T-valued points of N are of the form

$$\begin{aligned} n_{DEF} &\coloneqq e^{X(D,E,F)} = 1 + X(D,E,F) + \tfrac{1}{2}X(D,E,F)^2 \\ &= \mathbb{1}_{2+p+q} + X\big(D + \tfrac{1}{2}EF, E, F\big) \\ \bar{n}_{DEF} &\coloneqq e^{\bar{X}(D,E,F)} = \mathbb{1}_{2+p+q} + \bar{X}\big(D + \tfrac{1}{2}EF, E, F\big), \end{aligned}$$

with D, E, F constrained appropriately, since $\tau^t\sigma\tau = \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$. Obviously, we have the equality $\theta(n_{DEF}) = \bar{n}_{DEF}$.

Lemma 4.3. *The restriction of $H : G \longrightarrow \mathbb{A}(\mathfrak{a}_{\mathbb{R}})$ to $\bar{N}$ is given by*

$$H(\bar{n}) = \frac{1}{2}\log\big((1-BC)^2 - 4A^2\big)h_0, \quad \bar{n} = \bar{n}_{DEF} \in_T \bar{N}.$$

Proof. Formally, the calculations are the same as in the classical case, the only difference being that one has to work with generalised instead of ordinary points.

Write $\bar{n} = \bar{n}_{DEF} = ke^{th_0}n_{D'E'F'}$, so that $H(\bar{n}) = th_0$. Then

$$n_{(-A)(-B)(-C)}\bar{n}_{DEF} = \theta(\bar{n})^{-1}\bar{n} = \bar{n}_{(-D')(-E')(-F')}e^{2th_0}n_{D'E'F'}. \tag{4.6}$$

Let $v \coloneqq (1,1,0)$. The simple identities

$$v\sigma\tau = 0, \quad v\tau = \begin{pmatrix} 2 & 0 \end{pmatrix}$$

give

$$vn_{DEF} = \begin{pmatrix} 1 - 2D - EF & 1 + 2D + EF & 2E \end{pmatrix}, \quad v\bar{n}_{DEF} = v, \tag{4.7}$$

$$n_{DEF}v^t = v^t, \qquad \bar{n}_{DEF}v^t = \begin{pmatrix} -2D - EF & 1 + 2D + EF & 2F \end{pmatrix}^t. \tag{4.8}$$

Applying these on the right-hand side of Equation (4.6) gives

$$v\theta(\bar{n})^{-1}\bar{n}v^t = \cosh 2t + \sinh 2t = 2e^{2t}.$$

On the other hand, applying these on the left-hand side, we obtain

$$v\theta(\bar{n})^{-1}\bar{n}v^t = 2\big((1-EF)^2 - 4D^2\big).$$

This proves the lemma. $\square$

The classical approach to derive $c(\lambda)$ is to write

$$\phi_\lambda(g) = \int_{K/M} |D\dot{k}| e^{(\lambda-\varrho)(H(gk))} = \int_{\bar{N}} |D\bar{n}| e^{(\lambda-\varrho)(H(gk(\bar{n})))}.$$

However, if $q \neq 0$, this cannot be done, since the standard retraction on $\bar{N}$ does not extend to K/M. Since ϱ can be an arbitarily large negative multiple of α, the integral on the right-hand side does not even have to exist.

The next guess might be to interchange the limit in Equation (4.1) with the integral over K/M. However, this is not permitted, since $\exp(\lambda-\varrho)(H(e^{th_0}ke^{-th_0}))$ does admit a smooth limit, even in the classical case.

The solution we propose is to apply Proposition 3.17 after cutting off the integrand inside $k_0(\bar{N}_0)$. Thus, fix $\chi \in \mathcal{C}_c^\infty(0,\infty)$ such that $\chi = 1$ on a neighbourhood of 1, and define $\Xi \in \Gamma(\mathcal{O}_{K/M})$ by

$$\Xi(k(\bar{n})) \coloneqq \chi\big((1-EF)^2 - 4D^2\big), \quad \bar{n} = \bar{n}_{DEF} \in_T \bar{N}$$

on the open subspace $k(\bar{N}) \subseteq K/M$, and by zero otherwise. Clearly, this superfunction is well-defined and has compact support in the image $k(\bar{N})$ of $k\colon \bar{N} \to K/M$.

Then for any $w \in M_0' \coloneqq N_{K_0}(\mathfrak{a})$, we have

(4.9)

$$\phi_\lambda(g) = \int_{K/M} |D\dot{k}|\, e^{(\lambda-\varrho)(H(gk))}\Xi(k) + \int_{K/M} |D\dot{k}|\, e^{(\lambda-\varrho)H(gwk)}(1-\Xi)(wk)$$

by K-invariance of $|Dk|$. Denote the summands by $I_0^\lambda(g)$ and $I_\infty^\lambda(g)$, respectively.

Proposition 3.17 applies to the first of these two integrals. To do the same for the second, a good choice for w has to be made. The Weyl group $W_0 = M_0'/M_0 = \{\pm 1\}$; let $w_0 \in M_0'$ be a representative of the non-trivial element. By the Bruhat decomposition [22, Theorem 1.3], we have

$$(K_0/M_0) \setminus k_0(\bar{N}_0) = w_0 o, \quad o \coloneqq k(1) = 1M_0.$$

Since $1 - \Xi$ vanishes in a neighbourhood of o, this shows that the second integrand in Equation (4.9) has compact support inside $(K_0/M_0) \setminus \{w_0 o\} = k_0(\bar{N}_0)$.

On applying Theorem 2.18, Corollary 2.19, Equation (3.6), and Proposition 3.17, we obtain

$$\begin{aligned} I_0^\lambda(g) &= \int_{\bar{N}} |D\bar{n}| e^{(\lambda-\varrho)(H(g\bar{n}))} e^{-(\lambda+\varrho)(H(\bar{n}))}\Xi\big(k(\bar{n})\big) \\ I_\infty^\lambda(g) &= \int_{\bar{N}} |D\bar{n}| e^{(\lambda-\varrho)(H(gw_0\bar{n}))} e^{-(\lambda+\varrho)(H(\bar{n}))}(1-\Xi)\big(w_0 k(\bar{n})\big). \end{aligned} \tag{4.10}$$

For the following considerations, let $g \in_T A$. Moreover, we will identify $\mathfrak{a}$ with $\mathbb{C}$ *via* $1 \longmapsto h_0$ and $\mathfrak{a}^*$ with $\mathbb{C}$ *via* $\lambda \longmapsto \lambda(h_0)$. It is obvious from Corollary 2.11 that

$$e^{th_0}\bar{n}_{DEF}e^{-th_0} = \exp_G e^{t\,\mathrm{ad}\,h_0}(\bar{X}(D,E,F)) = \bar{n}_{e^{-2t}D,e^{-t}E,e^{-t}F},$$

and thus, we have for $\bar{n} = \bar{n}_{DEF}$:

$$H(e^{th_0}\bar{n}e^{-th_0}) = \tfrac{1}{2}\log\bigl((1-e^{-2t}EF)^2 - 4e^{-4t}D^2\bigr), \tag{4.11}$$

$$\begin{aligned} H(e^{th_0}w_0\bar{n}e^{-th_0}) &= H(e^{-th_0}\bar{n}e^{th_0}) - 2t \\ &= \tfrac{1}{2}\log\bigl((e^{-2t}-EF)^2 - 4D^2\bigr), \end{aligned} \tag{4.12}$$

$$H(e^{th_0}k(\bar{n})e^{-th_0}) = \tfrac{1}{2}\log\left(\frac{(1-e^{-2t}EF)^2 - 4e^{-4t}D^2}{(1-EF)^2-4D^2}\right), \tag{4.13}$$

$$H(e^{th_0}w_0k(\bar{n})e^{-th_0}) = \tfrac{1}{2}\log\left(\frac{(e^{-2t}-EF)^2 - 4D^2}{(1-EF)^2-4D^2}\right), \tag{4.14}$$

where in the third and fourth line, the identity $H(ak(g)a^{-1}) = H(aga^{-1}) - H(g)$, which follows from Equation (3.6), was applied.
Using Equation (4.13) and Corollary 2.11, one sees that

$$\chi\bigl(e^{-2H(a_tk(\bar{n})a_t^{-1})}\bigr) = \chi\left(\frac{(1-EF)^2-4D^2}{(1-e^{-2t}EF)^2-4e^{-4t}D^2}\right) \longrightarrow \Xi(k(\bar{n})) \quad (t\to\infty)$$

for $a_t = e^{th_0}$ and $\bar{n} = \bar{n}_{DEF}$. The convergence is uniform with derivatives on compact sets in $\bar{n}$. In particular, it holds with $k(\bar{n})$ replaced by the generic point k of K. On applying Equation (4.12), we see that

$$\begin{aligned} \Xi(w_0k(\bar{n})) &= \lim_{t\to\infty}\chi\bigl(e^{-2H(a_tw_0k(\bar{n})a_t^{-1})}\bigr) \\ &= \lim_{t\to\infty}\chi\left(\frac{(1-EF)^2-4D^2}{(e^{-2t}-EF)^2-4D^2}\right) = \chi\left(\frac{(1-EF)^2-4D^2}{(EF)^2-4D^2}\right) \end{aligned}$$

We now apply the coordinates introduced by φ. We write (s,y) for the coordinates and for the T-valued points of $\mathbb{A}^1\times\mathbb{A}^{2p|2q}$, where $y = (b\ c)$. In terms of the norm squared function $\|\cdot\|^2$ defined in Equation (2.2), we have

$$-4D^2 = s^2, \quad -EF = \|y\|^2, \quad \bar{X}(D,E,F) = \varphi(s,y).$$

By Corollary 3.13 and since the isomorphism φ is linear, the pullback of $|D\dot{k}|$ *via* $\exp\circ\varphi$ is a Berezin–Lebesgue density, which, by taking $|D\dot{k}|$ to be normalised adequately, may be assumed to be that from Subsection 2.7. We will denote the latter by $|D\mu|$ instead of $|D\lambda|$ to avoid confusion with the parameter λ.

Therefore, it is immediate that Equation (4.10) takes the form

(4.15)
$$I_0^\lambda(e^{th_0})e^{-t(\lambda-\varrho)} = \int_{\mathbb{A}^{1+2p|2q}} |D\mu| \, \frac{\psi(1,e^{-t}y,e^{-2t}s)^{\frac{1}{2}(\lambda-\varrho)}}{\psi(1,y,s)^{\frac{1}{2}(\lambda+\varrho)}} \chi\big(\psi(1,y,s)\big)$$
$$I_\infty^\lambda(e^{th_0})e^{-t(\lambda-\varrho)} = \int_{\mathbb{A}^{1+2p|2q}} |D\mu| \, \frac{\psi(e^{-2t},y,s)^{\frac{1}{2}(\lambda-\varrho)}}{\psi(1,y,s)^{\frac{1}{2}(\lambda+\varrho)}} (1-\chi)\left(\frac{\psi(1,y,s)}{\psi(0,y,s)}\right)$$

for ψ defined by

$$\psi(c,y,s) \coloneqq (c+\|y\|^2)^2 + s^2.$$

Since the integrands above are rotationally invariant over $\mathbb{A}^1$, Corollary 2.24 applies. (Here, y is the variable in which we rotate, and s is the base parameter.) In particular, ϕ_λ only depends on m_α and $m_{2\alpha}$. Therefore, if $m_\alpha > 0$, we may assume that $q = 0$ and use the well-known formula for the c-function in the classical case [23, Chapter IV, Theorem 6.4] to arrive at the following conclusion.

Proposition 4.4. *Let $m_\alpha > 0$ and $\Re\lambda > 0$. Then $c(\lambda)$ exists and*

$$c(\lambda) = c_0 \frac{2^{-\lambda}\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}\left(\lambda + \frac{m_\alpha}{2} + 1\right)\right)\Gamma\left(\frac{1}{2}\left(\lambda + \frac{m_\alpha}{2} + m_{2\alpha}\right)\right)}$$

for some constant c_0.

In case $m_\alpha \leqslant 0$, we apply Proposition 2.22. Again, the rotation variable is y and the parameter is s. If f is one of the integrands in Equation (4.15), then f° is determined by replacing ψ with ψ°, defined by

$$\psi^\circ(c,r,s) \coloneqq (c+r)^2 + s^2. \tag{4.16}$$

We obtain the following intermediate result.

Lemma 4.5. *Let $m_\alpha \leqslant 0$ and $\Re\lambda > 0$. Then*

$$\begin{aligned} c(\lambda) &= C_0 \int_0^\infty ds \, \partial_{r=0}^{-\frac{m_\alpha}{2}} \psi^\circ(1,r,s)^{-(\lambda+\varrho)/2} \chi\big(\psi^\circ(1,r,s)\big) \\ &\quad + C_0 \int_0^\infty ds \, \partial_{r=0}^{-\frac{m_\alpha}{2}} \frac{\psi^\circ(0,r,s)^{(\lambda-\varrho)/2}}{\psi^\circ(1,r,s)^{(\lambda+\varrho)/2}} (1-\chi)\left(\frac{\psi^\circ(1,r,s)}{\psi^\circ(0,r,s)}\right) \end{aligned} \tag{4.17}$$

and $c(\lambda)$ is meromorphic.

Proof. The integrands have compact support, so we may interchange $\lim_{t\to\infty}$ with the integral. The same can be done for ∂_λ. □

In order to finally derive $c(\lambda)$, the function χ needs to be removed. The result is the following integral expression:

Proposition 4.6. *If $m_\alpha \leqslant 0$, and $\Re\lambda > 0$, then $c(\lambda)$ exists, and we have*

$$c(\lambda) = C_0 \int_0^\infty ds \, \partial_{r=0}^{1-\varrho} \big((1+r)^2 + s^2\big)^{-(\lambda+\varrho)/2}. \tag{4.18}$$

In the *proof*, the following estimate will be used repeatedly.

LEMMA 4.7. *For all $r, s \geqslant 0$, $z \in \mathbb{C}$, and $k \in \mathbb{N}$, we have*

$$\left|\partial_r^k\big((1+r)^2+s^2\big)^{z/2}\right| < c_k\big((1+r)^2+s^2\big)^{(\Re z-k)/2},$$

where c_k is some z-dependent constant independent of r, s.

Proof. As one can see by induction,

$$\partial_r^k\big((1+r)^2+s^2\big)^{z/2} = p_k\left(\frac{1+r}{s}\right)s^k\big((1+r)^2+s^2\big)^{(z-2k)/2},$$

where p_k is a polynomial of order at most k. Since $\lim_{t\to\infty}(t^2+1)^{-\frac{k}{2}}p_k(t)$ exists, there is a constant c_k such that $|(t^2+1)^{-\frac{k}{2}}p_k(t)| < c_k$ for all $t \geqslant 0$. Taking $t = s^{-1}(1+r)$, this implies

$$\left|\partial_r^k\big((1+r)^2+s^2\big)^{z/2}\right| < c_k\big((1+r)^2+s^2\big)^{(\Re z-k)/2}. \qquad \square$$

Proof of Proposition 4.6. Firstly, note that $-\frac{m_\alpha}{2} = 1-\varrho$. Denote the second integral in Equation (4.17) by $c_{II}(\lambda)$. Since $\psi^\circ(c, ut, st) = t^2\psi^\circ(ct^{-1}, u, s)$, the substitution $r = su$ in the derivative yields

(4.19)

$$\begin{aligned}
C_0^{-1}c_{II}(\lambda) &= \int_0^\infty ds\,\partial_{u=0}^{1-\varrho}s^{\varrho-1}\frac{\psi^\circ(0,su,s)^{(\lambda-\varrho)/2}}{\psi^\circ(1,su,s)^{(\lambda+\varrho)/2}}(1-\chi)\left(\frac{\psi^\circ(1,su,su)}{\psi^\circ(0,su,s)}\right)\\
&= \int_0^\infty ds\,\partial_{u=0}^{1-\varrho}s^{\lambda-1}\frac{\psi^\circ(0,u,1)^{(\lambda-\varrho)/2}}{\psi^\circ(1,su,s)^{(\lambda+\varrho)/2}}(1-\chi)\left(\frac{\psi^\circ(s^{-1},u,1)}{\psi^\circ(0,u,1)}\right)
\end{aligned}$$

The next step is to exchange $\int ds$ and ∂_u^q. Since only the limit $u \to 0$ is of interest, u may be assumed to be small.

Let $\varepsilon < 1$ be small enough for $(1-\chi) = 0$ on the interval $[1, 1+4\varepsilon^2]$. Since

$$\frac{\psi^\circ(s^{-1},u,1)}{\psi^\circ(0,u,1)} = \frac{(s^{-1}+u)^2+1}{u^2+1} \leqslant 1+4\varepsilon^2, \quad \forall s\,:\, s\varepsilon > 1,$$

the function

$$[0,\varepsilon]\times(0,\infty) \longrightarrow [0,1] : (u,s) \longmapsto (1-\chi)\left(\frac{(s^{-1}+u)^2+1}{u^2+1}\right)$$

is of compact support. Therefore, $\partial_{u=0}^{1-\varrho}$ and $\int_\delta^\infty ds$ can be exchanged for any $\delta > 0$.

Now choose $0 < \delta < 1$ sufficiently small, such that

$$(1-\chi)\left(\frac{(s^{-1}+u)^2+1}{u^2+1}\right) = 1, \quad \forall s \in (0,\delta).$$

By Lemma 4.7, we have, for $u \leqslant \varepsilon$, $s \leqslant \delta$, and $k \leqslant q$, that

$$\begin{aligned}
\left|s^{\lambda-1}\partial_u^k\psi^\circ(1,su,s)^{-(\lambda+\varrho)/2}\right| &< c_k s^{\Re\lambda-1+k}\psi^\circ(1,su,s)^{-(\Re\lambda+\varrho+k)/2}\\
&\leqslant c_k 5^{-\varrho/2}s^{\Re\lambda-1}.
\end{aligned}$$

Since $s^{\Re\lambda-1}$ is integrable on $[0,\delta]$, we may also exchange $\int_0^\delta ds$ and $\partial_{u=0}^{1-\varrho}$.

Therefore, the right-hand side of Equation (4.19) equals

$$\partial_{u=0}^{1-\varrho}\int_0^\infty ds\, s^{\lambda-1}\frac{\psi^\circ(0,u,1)^{(\lambda-\varrho)/2}}{\psi^\circ(1,su,s)^{(\lambda+\varrho)/2}}(1-\chi)\Big(\frac{\psi^\circ(s^{-1},u,1)}{\psi^\circ(0,u,1)}\Big)$$
$$=\partial_{u=0}^{1-\varrho}\int_0^\infty \frac{ds}{s}\frac{\psi^\circ(0,\sqrt{s}u,\sqrt{s})^{(\lambda-\varrho)/2}}{\psi^\circ(1/\sqrt{s},\sqrt{s}u,\sqrt{s})^{(\lambda+\varrho)/2}}(1-\chi)\Big(\frac{\psi^\circ(1/\sqrt{s},\sqrt{s}u,\sqrt{s})}{\psi^\circ(0,\sqrt{s}u,\sqrt{s})}\Big).$$

We substitute $t^{-1}=\psi^\circ(0,\sqrt{s}u,\sqrt{s})=s(u^2+1)$ in the integral. Since $s^{-1}ds$ and $\psi^\circ(1/\sqrt{s},\sqrt{s}u,\sqrt{s})=s^{-1}+2u+s(u^2+1)$ are invariant under this substitution, this leads to

$$\partial_{u=0}^{1-\varrho}\int_0^\infty \frac{dt}{t}\frac{t^{-(\lambda-\varrho)/2}}{\psi^\circ(1/\sqrt{t},\sqrt{t}u,\sqrt{t})^{(\lambda+\varrho)/2}}(1-\chi)\big(t\psi^\circ(1/\sqrt{t},\sqrt{t}u,\sqrt{t})\big),$$
$$=\partial_{u=0}^{1-\varrho}\int_0^\infty dt\, t^{\varrho-1}\psi^\circ(1,tu,t)^{-(\lambda+\varrho)/2}(1-\chi)(\psi^\circ(1,tu,t)).$$

Again, the derivatives and the integral have to be exchanged. Possibly after shrinking ε, one may assume that

$$(1-\chi)((1+tu)^2+t^2)=0,\quad \forall t,u:0<t,u\leqslant\varepsilon.$$

Therefore, it suffices to consider the integral $\int_\varepsilon^\infty dt$. Clearly, $\int_\varepsilon^R dt$ and $\partial_{u=0}^{1-\varrho}$ may be exchanged, where R is large enough to arrange for

$$(1-\chi)((1+tu)^2+t^2)=0,\quad \forall t:t\geqslant R.$$

On applying Lemma 4.7 once again, we see that

$$\begin{aligned}\big|\partial_u^k t^{\varrho-1}\psi^\circ(1,tu,t)^{-(\lambda+\varrho)/2}\big| &< c_k t^{\varrho-1+k}\psi^\circ(1,tu,t)^{-(\Re\lambda+\varrho+k)/2}\\ &\leqslant c_k 5^{-\varrho/2}t^{-\Re\lambda-1},\end{aligned}$$

which is integrable over (R,∞) for $\Re\lambda>0$.
Thus, $\int_R^\infty dt$ and $\partial_{u=0}^{1-\varrho}$ can also be exchanged. Therefore, we find

$$\begin{aligned}c_{II}(\lambda)&=C_0\int_0^\infty dt\,\partial_{u=0}^{1-\varrho}t^{\varrho-1}\psi^\circ(1,tu,t)^{-(\lambda+\varrho)/2}(1-\chi)(\psi^\circ(1,tu,t))\\ &=C_0\int_0^\infty dt\,\partial_{r=0}^{1-\varrho}\psi^\circ(1,r,t)^{-(\lambda+\varrho)/2}(1-\chi)(\psi^\circ(1,r,t))\end{aligned}$$

upon substituting $r=tu$ in the derivative. Up to a replacement of χ by $1-\chi$, this is equal to the first integral in Equation (4.17), and so, the assertion follows. □

We now arrive at our conclusion in the unitary case.

Theorem 4.8. *The c-function $c(\lambda)$ for the symmetric pair of $G=\mathrm{U}_{cs}(1,1+p|2q)$ and $K=\mathrm{U}(1)\times\mathrm{U}_{cs}(1+p|q)$ exists for $\Re\lambda>0$. Explicitly, it is given by*

$$c(\lambda)=c_0\frac{2^{-\lambda}\Gamma(\lambda)}{\Gamma\big(\frac12\big(\lambda+\frac{m_\alpha}{2}+1\big)\big)\Gamma\big(\frac12\big(\lambda+\frac{m_\alpha}{2}+m_{2\alpha}\big)\big)},\quad c_0\equiv c_0(\varrho)\neq0.$$

Proof. The case of $m_\alpha > 0$ is the content of Proposition 4.4, so let $m_\alpha \leqslant 0$. Since $c(\lambda)$ is meromorphic, we may assume that $\Re\lambda > -\varrho$. Proposition 4.6 gives

$$C_0^{-1}c(\lambda) = \int_0^\infty ds\, \partial_{r=0}^{1-\varrho}\big((1+r)^2+s^2\big)^{-\frac{\lambda+\varrho}{2}} = \partial_{r=0}^{1-\varrho}\int_0^\infty ds\,\big((1+r)^2+s^2\big)^{-\frac{\lambda+\varrho}{2}}.$$

Here, integral and derivative may be exchanged due to Lemma 4.7, since

$$\begin{aligned}\left|\partial_r^k((1+r)^2+s^2)^{-(\lambda+\varrho)/2}\right| &< c_k((1+r)^2+s^2)^{-(\Re\lambda+\varrho+k)/2}\\ &\leqslant c_k(1+s^2)^{-(\Re\lambda+\varrho)/2}.\end{aligned}$$

for $k \leqslant 1-\varrho$. This is integrable by assumption.
Substitution with $s = (1+r)\sqrt{t}$ yields

$$\begin{aligned}c(\lambda) &= \frac{C_0}{2}\partial_{r=0}^{1-\varrho}(1+r)^{-\lambda-\varrho+1}\int_0^\infty dt\, t^{-\frac12}\big(1+t\big)^{-(\lambda+\varrho)/2}\\ &= \frac{C_0(1-\varrho)!(-1)^{1-\varrho}}{2}\binom{\lambda-1}{1-\varrho}\frac{\Gamma\big(\frac12\big)\Gamma\big(\frac12(\lambda+\varrho-1)\big)}{\Gamma\big(\frac12(\lambda+\varrho)\big)}\\ &= \frac{C_0\sqrt{\pi}(-1)^{1-\varrho}}{2}\frac{\Gamma(\lambda)\Gamma\big(\frac12(\lambda+\varrho-1)\big)}{\Gamma(\lambda+\varrho-1)\Gamma\big(\frac12(\lambda+\varrho)\big)} = C_0(-2)^{1-\varrho}\pi\frac{2^{-\lambda}\Gamma(\lambda)}{\Gamma\big(\frac12(\lambda+\varrho)\big)^2}.\end{aligned}$$

Here, we have applied the integral formula for the Euler beta function, the identity $\binom{-x}{k} = (-1)^k\binom{x+k-1}{k}$, and the duplication formula

$$\Gamma(z) = \frac{1}{\sqrt{\pi}}2^{z-1}\Gamma\big(\tfrac12 z\big)\Gamma\big(\tfrac12(z+1)\big).$$

The claim now follows from $m_{2\alpha} = 1$ and $m_\alpha = 2(\varrho-1)$. $\square$

4.2.2. *The ortho-symplectic case.* Let $\mathfrak{g} := \mathfrak{osp}(1,1+p|2q,\mathbb{C})$ be the complex Lie subsuperalgebra of $\mathfrak{gl}(2+p|2q,\mathbb{C})$, given by

$$\mathfrak{osp}(1,1+p|2q,\mathbb{C}) = \big\{x\in\mathfrak{gl}(2+p|2q,\mathbb{C})\;\big|\; x^{st^3}J + Jx = 0\big\},$$

where

$$\begin{pmatrix}R & S\\ T & V\end{pmatrix}^{st^3} := \begin{pmatrix}R^t & T^t\\ -S^t & V^t\end{pmatrix}$$

and

$$J = \left(\begin{array}{c:c|c} -\mathbb{1}_1 & 0 & 0\\ \hdashline 0 & \mathbb{1}_{1+p} & 0\\ \hline 0 & 0 & J_q\end{array}\right). \tag{4.20}$$

Here J_q denotes the $2q\times 2q$ matrix with q copies of the 2×2 matrix $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ on the diagonal. Therefore, $\mathfrak{g}$ consists of the matrices of the form

$$\left(\begin{array}{c:c|c} 0 & X_{12} & X_{13}\\ \hdashline X_{12}^t & X_{22} & X_{23}\\ \hline J_qX_{13}^t & -J_qX_{23}^t & X_{33}\end{array}\right),\quad X_{22}\in\mathfrak{so}(1+p,\mathbb{C}), X_{33}\in\mathfrak{sp}(2q,\mathbb{C}), \tag{4.21}$$

with no further restrictions on the other entries. We let

$$\mathfrak{g}_{\mathbb{R}} \coloneqq \mathfrak{g}_{\bar{0},\mathbb{R}} \oplus \mathfrak{g}_{\bar{1}}, \quad \mathfrak{g}_{\bar{0},\mathbb{R}} \coloneqq \mathfrak{so}(1+p,\mathbb{R}) \times \mathfrak{usp}(2q),$$

where $\mathfrak{usp}(2q)$ is the compact form of $\mathfrak{sp}(2q,\mathbb{C})$, given by $\mathfrak{usp}(2q) \coloneqq \mathfrak{u}(2q) \cap \mathfrak{sp}(2q,\mathbb{C})$.

Let $G = \mathrm{SOSp}^+_{cs}(1,1+p|2q)$ be the cs form of $\mathrm{OSp}(2+p|2q,\mathbb{C})$ given by the underlying Lie group $\mathrm{SO}(1,1+p,\mathbb{R}) \times \mathrm{USp}(2q)$. Here, $\mathrm{USp}(2q) \coloneqq \mathrm{U}(2q) \cap \mathrm{Sp}(2q,\mathbb{C})$.

The involution θ is the restriction of the one considered in the previous subsubsection. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding eigenspace decomposition. The analytic subsupergroup K corresponding to $\mathfrak{k}$ is $K = \mathrm{SOSp}_{cs}(1+p|2q)$, with underlying Lie group $\mathrm{SO}(1+p,\mathbb{R}) \times \mathrm{USp}(2q)$ and Lie cs algebra $\mathfrak{k}$. For $p=0$, K is the semi-direct product $\mathrm{USp}(2q) \ltimes \mathbb{A}^{0|2q}$, where $\mathbb{A}^{0|2q}$ is given the additive supergroup structure of the super-vector space $\mathbb{C}^{0|2q}$.

Let $\mathfrak{a} \subseteq \mathfrak{p}_{\bar{0}}$ be the even Cartan subalgebra generated by the element h_0 defined in Equation (4.3). Since σ and J commute, the restricted root space decomposition for $\mathfrak{g}$ is obtained by restricting that for $\mathfrak{gl}$ from Equation (4.4) to $\mathfrak{osp}(1,1+p|2q,\mathbb{C})$.

Only two restricted roots remain: α and $-\alpha$, where $\alpha(h_0)=1$. The general elements of $\mathfrak{g}^{\alpha}$ and $\mathfrak{g}^{-\alpha}$ are respectively

$$\left(\begin{array}{c:cc|c} 0 & 0 & B_{\bar{0}} & B_{\bar{1}} \\ \hdashline 0 & 0 & B_{\bar{0}} & B_{\bar{1}} \\ B_{\bar{0}}^t & -B_{\bar{0}}^t & 0 & 0 \\ \hline J_q B_{\bar{1}}^t & -J_q B_{\bar{1}}^t & 0 & 0 \end{array}\right), \quad \left(\begin{array}{c:cc|c} 0 & 0 & B_{\bar{0}} & B_{\bar{1}} \\ \hdashline 0 & 0 & -B_{\bar{0}} & -B_{\bar{1}} \\ B_{\bar{0}}^t & B_{\bar{0}}^t & 0 & 0 \\ \hline J_q B_{\bar{1}}^t & J_q B_{\bar{1}}^t & 0 & 0 \end{array}\right). \tag{4.22}$$

The general element of $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ is

$$\left(\begin{array}{c:cc|c} 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 0 & 0 & E & F \\ \hline 0 & 0 & -J_q G^t & H \end{array}\right).$$

We take $\Sigma^+ \coloneqq \{\alpha\}$ as our positive system, hence $\mathfrak{n} = \mathfrak{g}^{\alpha}$ and $\bar{\mathfrak{n}} = \mathfrak{g}^{-\alpha}$. From Equation (4.22), we have $m_\alpha = p - 2q$, so $\varrho = \frac{1}{2}(p-2q)\alpha$.

Using the notation from Equation (4.5), we set

$$X(F) \coloneqq X(0,E,F), \quad \bar{X}(F) \coloneqq \bar{X}(0,E,F), \quad E \coloneqq \begin{pmatrix} -F_{\bar{0}}^t & -F_{\bar{1}}^t J_q \end{pmatrix}.$$

Since the Iwasawa decomposition of $\mathfrak{g}$ is the restriction of that for $\mathfrak{gl}$ considered above, we see that

$$\mathfrak{n} = \big\{ X(F) \bigm| F \in \mathbb{C}^{p|2q\times 1} \big\}, \quad \bar{\mathfrak{n}} = \big\{ \bar{X}(E) \bigm| E \in \mathbb{C}^{p|2q\times 1} \big\}.$$

Then by Proposition 2.14, we have an isomorphism φ, given by

$$\varphi : \mathbb{A}^{p|2q} \longrightarrow \mathbb{A}(\bar{\mathfrak{n}}_{\mathbb{R}}) : y \longmapsto \bar{X}(y),$$

and similarly for $\mathfrak{n}$. The T-valued points of N and $\bar{N}$ are

$$n_y \coloneqq \exp_G X(y) = \mathbb{1}_{2+p+2q} + X(y), \quad \bar{n}_b \coloneqq \mathbb{1}_{2+p+2q} + \bar{X}(y),$$

where $y \in_T \mathbb{A}^{p|2q}$ and T is any *cs* manifold.
As in the previous subsubsection, we will identify $\mathfrak{a}$ and $\mathfrak{a}^*$ with $\mathbb{C}$ *via* the bases h_0 and α. The Iwasawa decompositions for $\mathfrak{gl}$ and $\mathfrak{osp}$ are compatible, so that we have the following lemma.

Lemma 4.9. *The restriction to $\bar{N}$ of the morphism $H : \bar{N} \to \mathbb{A}(\mathfrak{a}_{\mathbb{R}})$ is given by*

$$H(\bar{n}) = \log(1 + \|y\|^2), \quad \bar{n} = \bar{n}_y, \quad y \in_T \mathbb{A}^{p|2q},$$

for any cs *manifold T, where $\|\cdot\|^2$ is defined in Equation* (2.2)*.*

Proof. Lemma 4.3 applies, since $\bar{n}_y = \bar{n}_{0yb}$ where $b = -(y_0^t \; y_1^t J_q)$. We obtain

$$H(\bar{n}) = \frac{1}{2}\log\big((1-by)^2\big) = \log(1-by) = \log(1+\|y\|^2),$$

by the definition of the norm squared function. □

Equations (4.13) and (4.14) specialise to

$$\begin{aligned} H\big(e^{th_0}k(\bar{n}_y)e^{-th_0}\big) &= \log\left(\frac{1+e^{-2t}\|y\|^2}{1+\|y\|^2}\right), \\ H\big(e^{th_0}w_0k(\bar{n}_y)e^{-th_0}\big) &= \log\left(\frac{e^{-2t}+\|y\|^2}{1+\|y\|^2}\right), \end{aligned}$$

where $w_0 \in M_0'$ again denotes a representative of the non-trivial Weyl group element.
Note that this only makes sense if $p > 0$. Otherwise, the root α is not even and the Weyl group would be trivial in this case. Therefore, we assume $p > 0$ for the time being, postponing the case of $p = 0$ to the end of this subsubsection.
Let χ be as before and define $\Xi \in \Gamma(\mathcal{O}_{K/M})$ by

$$\Xi\big(k(\bar{n}_y)\big) \coloneqq \chi(1+\|y\|^2)$$

for $\bar{n}_y \in_T \bar{N}$. As functions on K/M, both Ξ and $k \mapsto (1-\Xi)(w_0k)$ have compact support inside $k_0(\bar{N}_0)$, and

$$(1-\Xi)\big(w_0k(\bar{n}_y)\big) = (1-\chi)\left(\frac{1+\|y\|^2}{\|y\|^2}\right).$$

In analogy with Equations (4.9), (4.10), and (4.15), we have by Theorem 2.18 and Corollary 2.19 that

(4.23)
$$
\begin{aligned}
\phi_\lambda(g) &= I_0^\lambda(g) + I_\infty(g),\\
I_0^\lambda(g) &:= \int_{K/M} |D\dot{k}|\, e^{(\lambda-\varrho)(H(gk))}\Xi(k),\\
I_\infty^\lambda(g) &:= \int_{K/M} |D\dot{k}|\, e^{(\lambda-\varrho)H(gwk)}(1-\Xi)(w_0k),\\
I_0^\lambda(e^{th_0})e^{-t(\lambda-\varrho)} &= \int_{\mathbb{A}^{p|2q}} |D\mu(y)| \frac{(1+e^{-2t}\|y\|^2)^{\lambda-\varrho}}{(1+\|y\|^2)^{\lambda+\varrho}}\chi(1+\|y\|^2),\\
I_\infty^\lambda(e^{th_0})e^{-t(\lambda-\varrho)} &= \int_{\mathbb{A}^{p|2q}} |D\mu(y)| \frac{(e^{-2t}+\|y\|^2)^{\lambda-\varrho}}{(1+\|y\|^2)^{\lambda+\varrho}}(1-\chi)\left(\frac{1+\|y\|^2}{\|y\|^2}\right).
\end{aligned}
$$

Here, again, $|D\mu|$ denotes the Berezin–Lebesgue density on $\mathbb{A}^{p|2q}$.
By Corollary 2.23, ϕ_λ and therefore $c(\lambda)$ only depends on m_α and not on p and q separately in this case, too. Therefore, for $m_\alpha > 0$, [23, Chapter IV, Theorem 6.4] implies the following.

Proposition 4.10. *Let $m_\alpha > 0$. Then $c(\lambda)$ exists for $\Re\lambda > 0$, and we have*

$$
c(\lambda) = c_0 \frac{2^{-\lambda}\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}(\lambda+\frac{m_\alpha}{2}+1)\right)\Gamma\left(\frac{1}{2}(\lambda+\frac{m_\alpha}{2}+m_{2\alpha})\right)}, \quad c_0 \equiv c_0(\varrho) \neq 0.
$$

For $2\varrho = m_\alpha \leqslant 0$, we get a similar expression.

Proposition 4.11. *Let $m_\alpha \leqslant 0$ and $p > 0$. Then $c(\lambda)$ exists for $\Re\lambda > 0$, and*

$$
c(\lambda) = c_0' \frac{\Gamma(\lambda)}{\Gamma(\lambda+\varrho)}, \quad c_0' \equiv c_0'(\varrho).
$$

Proof. Following Corollary 2.24, we distinguish between the even and odd cases. Firstly, assume that m_α be even. In this case, the integrals in Equation (4.23) reduce to a pointwise derivative, which may be exchanged with limits to arrive at

$$
c(\lambda) = C\partial_{r=0}^{-\varrho}\left((1+r)^{-(\lambda+\varrho)}\chi(1+r) + r^{\lambda-\varrho}(1+r)^{-(\lambda+\varrho)}(1-\chi)\left(\frac{1+r}{r}\right)\right).
$$

for some constant C. As functions of r, $\chi(1+r)$ and $(1-\chi)(r^{-1}(1+r))$ are constant near zero. Furthermore, $\partial_{r=0}^k r^{\lambda-\varrho} = 0$ for $k \leqslant -\varrho$, so that

$$
c(\lambda) = Cq!\binom{-(\lambda+\varrho)}{-\varrho} = C(-1)^\varrho q!\binom{\lambda-1}{-\varrho} = C(-1)^\varrho \frac{\Gamma(\lambda)}{\Gamma(\lambda+\varrho)}.
$$

Now, assume that $m_\alpha < 0$ is odd. By Lemma 4.7, we may exchange limits and integrals, to obtain

(4.24)
$$\begin{aligned}c(\lambda) = C \int_0^\infty dr\, r^{-\frac12}\partial_r^{\frac12-\varrho}\big((1+r)^{-(\lambda+\varrho)}\chi(1+r)\big)\\ + C\int_0^\infty dr\, r^{-\frac12}\partial_r^{\frac12-\varrho}\big(r^{\lambda-\varrho}(1+r)^{-(\lambda+\varrho)}(1-\chi)\big(r^{-1}(1+r)\big)\big).\end{aligned}$$

Denote the two integrals by $c_I(\lambda)$ and $c_{II}(\lambda)$, respectively. As in the proof of Proposition 4.6, we will rewrite these in order to show that the respective contributions of the cutoff function χ cancel.

This will be done by integrating by parts. Recall that $(1-\chi)(r^{-1}(1+r)) = 1$ for small r. Therefore, for $\Re\lambda > 0$ and $k \leqslant \frac12 - \varrho$, we have

$$\begin{aligned}&\lim_{r\to 0}(\partial_r^{k-1} r^{-\frac12})\partial_r^{-\varrho-\frac12-k}\big(r^{\lambda-\varrho}(1+r)^{-(\lambda+\varrho)}(1-\chi)\big(r^{-1}(1+r)\big)\big)\\ &= \lim_{r\to 0} r^{\frac12-k}\sum\nolimits_{\ell=0}^{\frac12-\varrho-k} c_\ell r^{\lambda-\varrho-(\frac12-\varrho-k-\ell)}(1+r)^{-(\lambda-\varrho)-\ell}\\ &= \lim_{r\to 0}\sum\nolimits_{\ell=0}^{\frac12-\varrho-k} c_\ell r^{\lambda+\ell}(1+r)^{-(\lambda-\varrho)-\ell} = 0\end{aligned}$$

Therefore, no boundary terms occur upon integrating by parts, and

$$\begin{aligned}c_{II}(\lambda) &= \big(\tfrac12-\varrho\big)!\binom{-\varrho}{\frac12-\varrho}\int_0^\infty \frac{dr}{r}\, r^\lambda(1+r)^{-(\lambda+\varrho)}(1-\chi)\big(r^{-1}(1+r)\big)\\ &= \big(\tfrac12-\varrho\big)!\binom{-\varrho}{\frac12-\varrho}\int_0^\infty \frac{ds}{s}\, s^{-\lambda}\big(s^{-1}(1+s)\big)^{-(\lambda+\varrho)}(1-\chi)(1+s)\\ &= \big(\tfrac12-\varrho\big)!\binom{-\varrho}{\frac12-\varrho}\int_0^\infty \frac{ds}{s}\, s^{\varrho}(1+s)^{-(\lambda+\varrho)}(1-\chi)(1+s).\end{aligned}$$

where $r^{\varrho-1}r^{\lambda-\varrho} = r^{\lambda-1}$ and the substitution $s = r^{-1}$ were applied.

Since similarly, for $k \leqslant \frac12 - \varrho$, we have as $r \to \infty$

$$\partial_r^{k-1} r^{-\frac12}\partial_r^{\frac12-\varrho-k}\big((1+r)^{-(\lambda+\varrho)}(1-\chi)(1+r)\big) \simeq r^{\frac12-k}(1+r)^{-\lambda-\frac12+k} \longrightarrow 0,$$

we may integrate by parts to find

$$\begin{aligned}&\int_0^\infty dr\, r^{-\frac12}\partial_r^{\frac12-\varrho}\big((1+r)^{-(\lambda+\varrho)}(1-\chi)(1+r)\big)\\ &\qquad= \big(\tfrac12-\varrho\big)!\binom{-\varrho}{\frac12-\varrho}\int_0^\infty \frac{ds}{s}\, s^{\varrho}(1+s)^{-(\lambda+\varrho)}(1-\chi)(1+s),\end{aligned}$$

since $(1-\chi)(1+r)$ vanishes for r small and equals 1 for r large.

Combining $c_I(\lambda)$ and $c_{II}(\lambda)$ leads to

$$c(\lambda) = C\int_0^\infty dr\, r^{-\frac12}\partial_r^{\frac12-\varrho}(1+r)^{-(\lambda+\varrho)} = C(-1)^{-\varrho-\frac12}\sqrt{\pi}\frac{\Gamma(\lambda)}{\Gamma(\lambda+\varrho)}$$

for $\Re\lambda > -\varrho$, as in the proof of Theorem 4.8. Moreover, Equation (4.24) shows that $c(\lambda)$ is holomorphic in λ for $\Re\lambda > 0$, since the integrands are sufficiently bounded. Therefore, the case of $0 < \Re\lambda \leqslant -\varrho$ follows by analytic continuation. □

The case where $p = 0$ is still open. This situation is interesting, in that there are only purely odd restricted roots (which, however, are anisotropic). Fortunately, one can evaluate ϕ_λ explicitly in this case.

Proposition 4.12. *If $p = 0$, then ϕ_λ is for some constant c_1 given by*

$$\phi_\lambda(e^{th_0}) = c_1 e^{\lambda t} \sum_{k=0}^{-\varrho} \binom{\lambda-\varrho}{k}\binom{-\lambda-\varrho}{-\varrho-k} e^{(-\varrho-2k)t}$$

Proof. The *cs* manifolds K/M and $\bar{N}$ admit only one retraction, since the underlying spaces are trivial. Hence, we may by Theorem 2.18 and Corollary 2.19 pull back the defining integral of ϕ_λ and apply Equation (3.6), to obtain

$$\begin{aligned}
\phi_\lambda(e^{th_0})e^{-t(\lambda-\varrho)} &= \int_{\bar{N}} |D\bar{n}|\, e^{(\lambda-\varrho)(H(e^{th_0}\bar{n}e^{-th_0}))} e^{-(\lambda+\varrho)(H(\bar{n}))} \\
&= \int_{\mathbb{A}^{0|2q}} |D\mu(y)|\, (1+e^{-2t}\|y\|^2)^{\lambda-\varrho}(1+\|y\|^2)^{-(\lambda+\varrho)} \\
&\simeq \partial_{r=0}^q (1+e^{-2t}r)^{\lambda-\varrho}(1+r)^{-(\lambda+\varrho)} \\
&\simeq q! \sum\nolimits_{k=0}^q \binom{\lambda-\varrho}{k}\binom{-\lambda-\varrho}{q-k} e^{-2kt}.
\end{aligned}$$

This proves the claim. □

Corollary 4.13. *If $p = 0$, the $c(\lambda)$ exists for $\Re\lambda > 0$ and*

$$c(\lambda) = c_0' \frac{\Gamma(\lambda)}{\Gamma(\lambda+\varrho)}, \quad c_0' \equiv c_0'(\varrho).$$

Using the duplication formula, Proposition 4.10, Proposition 4.11, and Corollary 4.13 combine to the following result.

Theorem 4.14. *The c-function $c(\lambda)$ for the symmetric pair of the* cs *Lie supergroups $G = \mathrm{SOSp}^+_{cs}(1,1+p|2q)$ and $K = \mathrm{SOSp}_{cs}(1+p|2q)$ exists for $\Re\lambda > 0$. Explicitly,*

$$c(\lambda) = c_0 \frac{2^{-\lambda}\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}\left(\lambda+\frac{m_\alpha}{2}+1\right)\right)\Gamma\left(\frac{1}{2}\left(\lambda+\frac{m_\alpha}{2}+m_{2\alpha}\right)\right)}, \quad c_0 \equiv c_0(\varrho) \neq 0.$$

4.2.3. *The case of* $\mathrm{GL}(1|1)$. Let $\mathfrak{g} \coloneqq \mathfrak{gl}(1|1,\mathbb{C}) \times \mathfrak{gl}(1|1,\mathbb{C})$. Then $\mathfrak{g}_{\bar{0}}$ is Abelian. We will write the elements of $\mathfrak{g}$ as double matrices of the form

$$\left(\begin{array}{cc|cc} A & B & E & F \\ C & D & G & H \end{array}\right).$$

Let $\mathfrak{g}_{\mathbb{R}} \coloneqq \mathfrak{g}_{\bar{0},\mathbb{R}} \oplus \mathfrak{g}_{\bar{1}}$ be given by requiring $\mathfrak{g}_{\bar{0},\mathbb{R}}$ to be the Lie algebra of the Lie group G_0 whose general elements are

$$\left(\begin{array}{cc|cc} z & 0 & \bar{z}^{-1} & 0 \\ 0 & r & 0 & s \end{array}\right), \quad z \in \mathbb{C}^\times, r, s > 0.$$

The *cs* form G of $G_{\mathbb{C}} = \mathrm{GL}(1|1,\mathbb{C}) \times \mathrm{GL}(1|1,\mathbb{C})$ defined by G_0 will be denoted by $(\mathrm{GL}\times\mathrm{GL})^+_{\mathbb{C}|\mathbb{R}}(1|1)$. An involution θ on $\mathfrak{g}$ can be defined by

$$\theta\left(\begin{array}{cc|cc} A & B & E & F \\ C & D & G & H \end{array}\right) = \left(\begin{array}{cc|cc} E & F & A & B \\ G & H & C & D \end{array}\right).$$

The general elements of the eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$ are respectively

$$\left(\begin{array}{cc|cc} A & B & A & B \\ C & D & C & D \end{array}\right), \quad \left(\begin{array}{cc|cc} A & B & -A & -B \\ C & D & -C & -D \end{array}\right).$$

Let K be the analytic *cs* Lie subgroup of G with Lie *cs* algebra $\mathfrak{k}$, denoted by $K = \mathrm{UGL}^+_{\mathbb{R}}(1|1)$. The general element of K_0 is

$$\left(\begin{array}{cc|cc} z & 0 & z & 0 \\ 0 & r & 0 & r \end{array}\right), \quad z \in \mathrm{U}(1), r > 0.$$

Note that although K_0 is not compact, its adjoint image in $\mathrm{GL}(\mathfrak{p}_{\bar 0,\mathbb{R}})$ is. This explains why there is no contradiction in the fact that the negative of the supertrace form is actually positive on $\mathfrak{p}_{\bar 0,\mathbb{R}}$, so that (G_0, K_0) is a Riemannian symmetric pair.

The only choice of Cartan subalgebra is $\mathfrak{a} := \mathfrak{p}_{\bar 0}$, since $\mathfrak{p}_{\bar 0}$ is Abelian and consists of semisimple elements. Let

$$h_1 := \left(\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad h_2 := \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right)$$

be a basis of $\mathfrak{a}$. An easy calculation shows that there are only the roots $\pm\alpha$, with α defined by

$$\alpha(a_1 h_1 + a_2 h_2) := a_1 - a_2.$$

We have $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) = \mathfrak{k}$. The general elements of $\mathfrak{g}^{\alpha}$ and $\mathfrak{g}^{-\alpha}$ are respectively

$$\left(\begin{array}{cc|cc} 0 & B & 0 & 0 \\ 0 & 0 & G & 0 \end{array}\right), \quad \left(\begin{array}{cc|cc} 0 & 0 & 0 & F \\ C & 0 & 0 & 0 \end{array}\right).$$

We let $\Sigma^+ := \{\alpha\}$ be our positive system, so $\mathfrak{n} = \mathfrak{g}^{\alpha}$, $\bar{\mathfrak{n}} = \mathfrak{g}^{-\alpha}$, $m_\alpha = -2$, and $\varrho = -\alpha$. Since G_0 admits a global Iwasawa decomposition, this is also the case for G. Indeed, every element in G_0 can be decomposed uniquely as

$$\left(\begin{array}{cc|cc} z & 0 & \bar z^{-1} & 0 \\ 0 & r & 0 & s \end{array}\right) = \left(\begin{array}{cc|cc} \frac{z}{|z|} & 0 & * & * \\ 0 & \sqrt{rs} & * & * \end{array}\right)\left(\begin{array}{cc|cc} \log|z| & 0 & * & * \\ 0 & \frac{1}{2}\log(\frac{r}{s}) & * & * \end{array}\right),$$

where the first factor is in K_0 and the second in A_0. Note that $N_0 = 1$.

For any *cs* manifold T, the T-valued points of N and $\bar N$ are of the form

$$n_{B,G} = \left(\begin{array}{cc|cc} 1 & B & 1 & 0 \\ 0 & 1 & G & 1 \end{array}\right), \ \bar n_{C,F} = \left(\begin{array}{cc|cc} 1 & 0 & 1 & F \\ C & 1 & 0 & 1 \end{array}\right), \ B, C, F, G \in \Gamma(\mathcal{O}_{T,\bar 1}).$$

Lemma 4.15. *The restriction to $\bar N$ of the morphism $H : G \to \mathbb{A}(\mathfrak{a}_{\mathbb{R}})$ is given by*

$$H(\bar n) = \tfrac{1}{2} CF h^+, \quad \bar n = \bar n_{C,F} \in_T \bar N,$$

where we set $h^{\pm} := h_1 \pm h_2$.

Proof. Again, $H\colon \bar{N} \longrightarrow \mathfrak{a}$ will be derived from the equation

$$\theta(\bar{n}_{C,F})^{-1}\bar{n}_{C,F} = \theta(n_{B,G})^{-1}\exp(2H(\bar{n}_{C,F}))n_{B,G}, \qquad \bar{n}_{C,F} \in_T \bar{N}, n_{B,G} \in_T N.$$

The left-hand side equals

$$\left(\begin{array}{cc|cc} 1 & -F & 1 & 0 \\ 0 & 1 & -C & 1 \end{array}\right)\left(\begin{array}{cc|cc} 1 & 0 & 1 & F \\ C & 1 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|cc} 1-FC & -F & 1 & F \\ C & 1 & -C & 1-CF \end{array}\right).$$

Upon writing $H(\bar{n}_{C,F}) = h_1t_1 + h_2t_2$ for some $t_1, t_2 \in \Gamma(\mathcal{O}_{T,\bar{0}})$, the right-hand side takes the form

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & -B \\ -G & 1 & 0 & 1 \end{array}\right)\left(\begin{array}{cc|cc} e^{2t_1} & 0 & e^{-2t_1} & 0 \\ 0 & e^{2t_2} & 0 & e^{-2t_2} \end{array}\right)\left(\begin{array}{cc|cc} 1 & B & 1 & 0 \\ 0 & 1 & G & 1 \end{array}\right)$$
$$= \left(\begin{array}{cc|cc} e^{2t_1} & Be^{2t_1} & e^{-2t_1} - BGe^{-2t_2} & -Be^{-2t_2} \\ -Ge^{2t_1} & e^{2t_2} - GBe^{2t_1} & Ge^{-2t_2} & e^{-2t_2} \end{array}\right),$$

Therefore, $e^{2t_1} = 1 - FC$ and $e^{-2t_2} = 1 - CF = 1 + FC$, so that

$$t_1 = t_2 = -\tfrac{1}{2}FC = \tfrac{1}{2}CF,$$

which proves the lemma. $\square$

We find the following formula for ϕ_λ.

Proposition 4.16. *For a suitable normalisation of the invariant density on K/M, we have for any $\lambda \in \mathfrak{a}^*$*

$$\phi_\lambda(e^h) = -\lambda(h^+)e^{\lambda(h)}\sinh\alpha(h), \quad h \in_T \mathbb{A}(\mathfrak{a}_\mathbb{R}).$$

Remarkably, the function ϕ_λ is not invariant under the symmetry $h \longmapsto -h$. This is due to the fact that there are no even restricted roots in the case under consideration, so that the even Weyl group is trivial.

Proof of Proposition 4.16. We introduce coordinates ξ_1, ξ_2 on $\bar{N}$ by

$$\xi_1(\bar{n}_{C,F}) = C, \quad \xi_2(\bar{n}_{C,F}) = F.$$

In terms of these coordinates, we may take $|D\bar{n}| = |D\xi| = |D(\xi_1, \xi_2)|$.
The spaces K/M and $\bar{N}$ being of purely odd dimension, they both admit only one retraction. Thus, we may by Theorem 2.18 and Corollary 2.19 pull back the defining integral of ϕ_λ, leading to

$$\phi_\lambda(e^h)e^{-(\lambda-\varrho)(h)} = \int_{\bar{N}} |D\bar{n}|e^{(\lambda-\rho)(H(e^h\bar{n}e^{-h}))}e^{-(\lambda+\varrho)(H(\bar{n}))} \tag{4.25}$$

for $h \in_T \mathbb{A}(\mathfrak{a}_\mathbb{R})$, by Proposition 3.17 and Equation (3.6).
Clearly, we have by Corollary 2.11 that

$$e^h\bar{n}_{C,F}e^{-h} = \bar{n}_{e^{-2\alpha(h)}C, e^{-2\alpha(h)F}},$$

and $\varrho(h^+) = -\alpha(h^+) = 0$. Furthermore, Lemma 4.15 gives

$$e^{\lambda(H(\bar{n}_{C,F}))} = e^{\frac{1}{2}\lambda(h^+)CF} = 1 + \tfrac{1}{2}\lambda(h^+)CF.$$

Inserting these findings into Equation (4.25), we arrive at

$$\begin{aligned}\phi_\lambda(e^h)e^{-(\lambda-\varrho)(h)} &= \int_{\bar N}|D\xi|\,\bigl(1+\tfrac12\lambda(h^+)e^{-2\alpha(h)}\xi_1\xi_2\bigr)\,\bigl(1-\tfrac12\lambda(h^+)\xi_1\xi_2\bigr)\\ &= \tfrac12\lambda(h^+)(e^{-2\alpha(h)}-1) = -\lambda(h^+)e^{\varrho(h)}\sinh\alpha(h),\end{aligned}$$

proving the assertion. □

We immediately obtain the following expression for the c-function.

COROLLARY 4.17. *Let $\lambda \in \mathfrak{a}^*$ be arbitrary. Using $h_0 = c_+h^+ + c_-h^-$ in Equation* (4.1)*, $c(\lambda)$ exists if and only if $c_- = 0$ or $\Re c_- > 0$, and then*

$$c(\lambda) = \begin{cases} 0, & c_- = 0,\\ -\frac12\lambda(h^+), & \Re c_- > 0.\end{cases}$$

Proof. Since $\alpha(h^+) = 0$ and $\alpha(h^-) = 2$, we have

$$e^{\varrho(th_0)}\sinh\alpha(th_0) = e^{-2c_-t}\sinh 2c_-t = \tfrac12\bigl(1-e^{-4c_-t}\bigr).$$

Thus, the limit for $t\to\infty$ exists if and only if $c_- = 0$ or $\Re c_- > 0$. In these cases, it equals 0 and $\frac12$, respectively. □

Thus, in particular, $c(\lambda)$ can be defined independently of a choice of h_0 with $\alpha(h_0) = 2c_- > 0$. Since $\alpha(h) = \frac12\langle h^+,h\rangle$, the above result for $c(\lambda)$ can be rewritten in the form stated in Theorem 4.2.

5. DETERMINATION OF THE SPHERICAL FUNCTIONS

In this subsection, we derive a convergent series expansion for the function ϕ_λ. The standard procedure to do so is to solve a differential equation by a pertubation ansatz, *cf.* [23, Chapter IV, 5]. We follow the same procedure.
In the following, suppose (G,K) is one of the two following symmetric pairs

$$\begin{aligned}&\bigl(\mathrm{U}_{cs}(1,1+p|2q),\mathrm{U}(1)\times\mathrm{U}_{cs}(1+p|2q)\bigr),\\ &\bigl(\mathrm{SOSp}^+_{cs}(1,1+p|2q),\mathrm{SOSp}_{cs}(1+p|2q)\bigr).\end{aligned}\tag{5.1}$$

For $(G,K) = \bigl((\mathrm{GL}\times\mathrm{GL})^+_{\mathbb{C}|\mathbb{R}}(1|1),\mathrm{UGL}^+_{\mathbb{R}}(1|1)\bigr)$, a closed expression for ϕ_λ was derived above in Proposition 4.16. Moreover, in the orthosymplectic case, a closed expression was obtained for $p=0$, in Proposition 4.12. This section aims to remove such restrictions.
Let $h_0\in\mathfrak{a}_{\mathbb{R}}$ be determined uniquely by $\alpha(h_0) = 1$. By the use of the bases h_0 and α of $\mathfrak{a}$ and $\mathfrak{a}^*$, respectively, we will identify these spaces with $\mathbb{C}$ where convenient.

PROPOSITION 5.1. *Let $\Delta(L)$ be the differential operator on A given by*

$$\Delta(L)(f)(e^{th_0}) := \bigl(\partial_t^2 + (m_\alpha\coth(t) + 2m_{2\alpha}\coth(2t))\partial_t\bigr)f(e^{th_0}).$$

Then, for any $\lambda\in\mathfrak{a}^*$, ϕ_λ is an eigenfunction of $\Delta(L)$. More precisely,

$$\Delta(L)\phi_\lambda = (\lambda^2-\varrho^2)\phi_\lambda.\tag{5.2}$$

Proof. One follows the standard procedure, showing that $\Delta(L)$ is the N-radial part of the Laplacian on G/K, *cf.* [20, 4.2]. This can also be seen by a computation in $\mathfrak{U}(\mathfrak{g})$ along the lines of the proof of [38, Proposition 9.1.2.11]. The eigenvalue equation is derived as in Ref. [1]. □

Since $e^{(\lambda-\varrho)t}$ is an eigenfunction of $\partial_t(\partial_t + 2\varrho)$ for the correct eigenvalue, we make the following perturbation ansatz:

$$\Phi_\lambda(e^{th_0}) = e^{(\lambda-\varrho)t}\sum\nolimits_{\ell=0}^{\infty}\gamma_\ell(\lambda)e^{-2\ell t}, \quad e^{th_0}\in A^+, \tag{5.3}$$

where A^+ is the open subset of $A_0 = A$ defined by $A^+ := \exp((0,\infty)h_0)$. For $\varepsilon > 0$, we denote by A^+_ε the subset of A_0 defined by $A^+_\varepsilon := \exp([\varepsilon,\infty)h_0)$.

Proposition 5.2. *Let $\lambda \in \mathfrak{a}^*$, $\lambda \equiv \lambda(h_0) \notin \mathbb{N}$. Assuming that the series Φ_λ in Equation* (5.3) *converges to a solution of Equation* (5.2)*, absolutely on A^+_ε for any $\varepsilon > 0$, the coefficients $\gamma_\ell(\lambda)$ are determined by the choice of $\gamma_0(\lambda)$ and the relation*

$$\begin{aligned}\ell(\ell-\lambda)\gamma_\ell(\lambda) &= \frac{m_\alpha}{2}\big(\varrho-\lambda+2(\ell-1)\big)\gamma_{\ell-1}(\lambda)\\ &\quad+\big(\varrho-\lambda+\ell-2\big)\big(\varrho+\ell-2\big)\gamma_{\ell-2}(\lambda).\end{aligned}\tag{5.4}$$

Conversely, define $\gamma_\ell(\lambda)$ by the above equation. Then the series Φ_λ converges to a solution of Equation (5.2)*, absolutely on A^+_ε for all $\varepsilon > 0$.*

Proof. In view of Proposition 5.1, we may follow the standard procedure from Ref. [23, Chapter IV, § 5]. We obtain the recursion relation

$$\ell(\ell-\lambda)\gamma_\ell(\lambda) = \sum_{n=1}^{2}\frac{nm_{n\alpha}}{2}\sum_{k\geqslant 1,\ell\geqslant nk}\big(\varrho-\lambda+2(\ell-nk)\big)\gamma_{\ell-nk}(\lambda).\tag{5.5}$$

On the right-hand side, the contribution of the three terms corresponding to $(n,k) = (1,1),(1,2),(2,1)$ is

$$\frac{m_\alpha}{2}\big(\varrho-\lambda+2(\ell-1)\big)\gamma_{\ell-1}(\lambda) + \varrho(\varrho-\lambda+2(\ell-2))\gamma_{\ell-2}(\lambda).$$

On applying the recursion relation to the remaining terms, we see that the remainder amounts to

$$(\ell-2)(\ell-2-\lambda)\gamma_{\ell-2}(\lambda).$$

Thus, we obtain Equation (5.4).
For the convergence, we follow the procedure of [23, Chapter IV, §5, Lemma 5.3]: Since we have

$$\frac{\ell(\ell-\lambda)}{\ell^2}\longrightarrow 1, \quad \frac{\varrho-\lambda+2\ell}{\ell+1}\longrightarrow 2$$

for $\ell\to\infty$, there are constants $0 < c < \frac{1}{2}$ and $2 < C < \infty$ such that

$$|\varrho-\lambda+2\ell| \leqslant C(\ell+1), \quad |\ell(\ell-\lambda)| \geqslant 2c\ell^2$$

for all $\ell \in \mathbb{N}$. For any $\varepsilon > 0$, there is some $\ell_0 \in \mathbb{N}$, $\ell_0 \geqslant 2$, such that

$$\sum_{n=1}^{2} \frac{n m_{n\alpha}}{4} \big(\coth\big(\tfrac{nt}{2}\big) - 1\big) \leqslant \frac{c\ell_0}{C}, \quad \forall t : t \geqslant \varepsilon.$$

Choose $K = K_\varepsilon > 0$ such that

$$|\gamma_\ell(\lambda)| \leqslant K e^{\ell t}, \quad \forall \ell, t : \ell \leqslant \ell_0, t \geqslant \varepsilon.$$

We claim that the 'Gangolli estimate' $|\gamma_\ell(\lambda)| \leqslant K e^{\ell t}$ holds for all $\ell \in \mathbb{N}$ and $t \geqslant \varepsilon$. Indeed, assume that this holds for all $\ell' < \ell_1$, where $\ell_1 > \ell_0$. Then Equation (5.5) and the assumption imply

$$\begin{aligned} |\gamma_{\ell_1}(\lambda)| &\leqslant \frac{C(\ell_1 + 1)}{2c\ell_1^2} \sum_{n=1}^{2} \frac{n|m_{n\alpha}|}{2} \sum_{k \geqslant 1, \ell \geqslant nk} |\gamma_{\ell - nk}(\lambda)| \\ &\leqslant \frac{K e^{\ell t} C}{\ell_0 c} \sum_{n=1}^{2} \frac{n|m_{n\alpha}|}{4} \big(\coth\big(\tfrac{nt}{2}\big) - 1\big) \leqslant K e^{\ell t} \end{aligned}$$

because $\ell_0(\ell_1 + 1) \leqslant 2\ell_1^2$ and $\coth(\frac{t}{2}) - 1 = 2\sum_{k=1}^{\infty} e^{-kt}$. This proves our claim. In particular, we have

$$\sum_{\ell=0}^{\infty} \big|\gamma_\ell(\lambda) e^{-2\ell t}\big| \leqslant K \sum_{\ell=0}^{\infty} e^{-\ell t} \leqslant \frac{K}{1 - e^{-\varepsilon}} < \infty$$

and the series converges absolutely on A_ε^+. This justifies the term-by-term differentiation and implies that the limit Φ_λ is a solution of Equation (5.2). $\square$

Remark 5.3. When $m_{2\alpha} = 0$ (*i.e.* in the orthosymplectic case), the two-term recursion for $\gamma_\ell(\lambda)$ simplifies, and we easily obtain the following closed expression

$$\begin{aligned} \gamma_\ell(\lambda) &= \gamma_0(\lambda) \prod_{m=0}^{\ell-1} \frac{(m + \varrho)(m + \varrho - \lambda)}{(m + 1)(m + 1 - \lambda)} \\ &= \gamma_0(\lambda) c(-\lambda)(-1)^l \binom{-\varrho}{\ell} \frac{-\lambda}{(\ell - \lambda) c(\ell - \lambda)}. \end{aligned} \tag{5.6}$$

Using this formula, the convergence of the Harish-Chandra series follows from a simple-minded application of the ratio test.

From now on, will make the choice

$$\gamma_0(\lambda) \coloneqq c(\lambda). \tag{5.7}$$

Contrary to the classical situation of Riemannian symmetric spaces of non-compact type, the Harish-Chandra series may be finite.

COROLLARY 5.4. *If $G = \mathrm{SOSp}_{cs}(1, 1 + p|2q)$ and $m_\alpha = p - 2q \leq 0$ is even, then the series Φ_λ is finite. More precisely, we have*

$$\Phi_\lambda(e^{th_0}) = c_1 e^{(\lambda - \varrho)t} \sum_{\ell=0}^{-\varrho} \binom{\lambda - \varrho}{\ell} \binom{-\lambda - \varrho}{-\varrho - \ell} e^{-2\ell t}.$$

for some constant c_1. In particular, Φ_λ is well-defined on A.

Proof. In view of Equation (5.6), we have $\gamma_\ell(\lambda) = 0$ for $\ell > -\varrho$. Recall from Proposition 4.11 and Corollary 4.13 that

$$c(\lambda) = c_0' \frac{\Gamma(\lambda)}{\Gamma(\lambda+\varrho)} = c_0' \prod_{k=1}^{-\varrho} (\lambda - k). \tag{5.8}$$

Inserting $\gamma_0(\lambda) = c(\lambda)$ into Equation (5.6), we obtain

$$\begin{aligned} \gamma_\ell(\lambda) &= c_0'(-1)^\ell \binom{-\varrho}{\ell} \prod_{k=\ell+1}^{-\varrho} (\lambda - k) \prod_{m=0}^{\ell-1} (\lambda - \varrho - m) \\ &= c_0'(-\varrho)! \binom{-\lambda-\varrho}{-\varrho-\ell} \binom{\lambda-\varrho}{\ell}. \end{aligned} \tag{5.9}$$

By the definition of Φ_λ, this proves the claim. □

Combining the above results, we arrive at the following expression for the spherical superfunctions.

Theorem 5.5. *Let (G,K) be one of the symmetric pairs in Equation* (5.1). *For $\Re\lambda > 0$ and $\lambda \notin \frac{1}{2}\mathbb{Z}$, we have*

$$\phi_\lambda|_{A^+} = \sum_{w\in W_0} \Phi_{w\lambda},$$

where Φ_λ is defined by Equations (5.3) *and* (5.7)*. Moreover, $W_0 = \{\pm 1\}$ unless $G = \mathrm{SOSp}_{cs}(1,1|2q)$, in which case $W_0 = \{1\}$.*

Proof. Firstly, Equation (4.1), Theorem 4.2, and Proposition 5.2 lead to

$$\lim_{t\to\infty} \phi_\lambda(e^{th_0}) e^{-(\lambda-\varrho)t} = c(\lambda) = \lim_{t\to\infty} \Phi_\lambda(e^{th_0}) e^{-(\lambda-\varrho)t}. \tag{5.10}$$

Therefore, in case W_0 is trivial, then Proposition 4.12 and Corollary 5.4 combine to prove $\phi_\lambda = \Phi_\lambda$.

It remains to prove the assertion in case W_0 is non-trivial. We observe that by Proposition 5.2 again, the functions $\Phi_{\pm\lambda}$ are solutions of Equation (5.2). This is a differential equation of order two and $\Phi_{\pm\lambda}$ are visibly linearly independent, so

$$\phi_\lambda = b_1\Phi_\lambda + b_2\Phi_{-\lambda}, \quad b_1, b_2 \in \mathbb{C}.$$

Let $w_0 \in M_0'$ be a representative of the non-trivial Weyl group element. Using Corollary 3.20 and the $K\times K$-invariance of ϕ_λ, we derive for any $a \in A^+$ that

$$\phi_\lambda(a) = \phi_{-\lambda}(a^{-1}) = \phi_{-\lambda}(waw^{-1}) = \phi_{-\lambda}(a).$$

This implies $b_1 = b_2$. Using Equation (5.10), and

$$\lim_{t\to\infty} \Phi_{-\lambda}(e^{th_0}) e^{-(\lambda-\varrho)t} = 0,$$

which is derived easily from Proposition 5.2, we conclude that $b_1 = 1$. □

The result of Theorem 5.5 can be given a more explicit form in the cases where the Harish-Chandra series is finite.

COROLLARY 5.6. *Let $G = \mathrm{SOSp}_{cs}(1, 1 + p|2q)$ and $m_\alpha = p - 2q \leq 0$ be even. Then*

$$\phi_\lambda(e^{th_0}) = c_1 e^{(\lambda-\varrho)t} P_{-\varrho}^{(-\lambda,2\varrho-1)}(1 - 2e^{-2t}),$$

where the $P_n^{(a,b)}$ denote the Jacobi polynomials.

Proof. By [18, §10.8, (16)], we have

$$P_n^{(a,b)}(x) = \binom{n+a}{n} {}_2F_1\Big(-n, 1 + a + b + n; a + 1; \frac{x-1}{2}\Big),$$

where ${}_2F_1$ is the Gaussian hypergeometric function. Inserting the hypergeometric series, we obtain easily that

$$P_n^{(a,b)}(x) = \frac{(a+n)!}{n!(a+b+n)!} \sum_{\ell=0}^{n} \binom{n}{\ell} \frac{(a+b+n+\ell)!}{(a+\ell)!} \Big(\frac{x-1}{2}\Big)^\ell,$$

where we write $y! = \Gamma(y + 1)$. Let

$$n \coloneqq -\varrho = q - \frac{p}{2}, \quad a \coloneqq -\lambda, \quad b \coloneqq 2\varrho - 1 = p - 2q - 1.$$

On applying Equation (5.9) and Corollary 5.4, we obtain

$$\Phi_\lambda(e^{th_0}) = c'_0(-1)^{-\varrho} P_n^{(a,b)}(x)$$

for $x = 1 - 2e^{-2t}$, c'_0 denoting the constant from Equation (5.8). □

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