Szpiro's Small Points Conjecture for Cyclic Covers

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Abstract. Let X be a smooth, projective and geometrically connected curve of genus at least two, defined over a number field. In 1984, Szpiro conjectured that X has a "small point". In this paper we prove that if X is a cyclic cover of prime degree of the projective line, then X has infinitely many "small points". In particular, we establish the first cases of Szpiro's small points conjecture, including the genus two case and the hyperelliptic case. The proofs use Arakelov theory for arithmetic surfaces and the theory of logarithmic forms.

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1 Introduction

Let X be a smooth, projective and geometrically connected curve of genus at least two, defined over a number field. In 1984, Szpiro [Szp85a] conjectured that X has a "small point", where a "small point" is an algebraic point of the curve X with "height" bounded from above in a certain way. We refer to Section 2 for a precise formulation of Szpiro's small points conjecture. Szpiro proved that his conjecture implies an "effective Mordell conjecture". In other words, Szpiro's remarkable approach shows that to bound the height of all rational points of any curve X, it suffices to produce for any curve X at least one "small point". The small points conjecture was studied in Szpiro's influential seminars [Szp85b, Szp90a], see also Szpiro's articles [Szp86, Szp90b].
The results of this paper are as follows. Let $\mathcal{C}$ be the set of curves X as above which are cyclic covers of prime degree of the projective line. For example, if X has genus two or is hyperelliptic, then $X \in \mathcal{C}$. Our first result (see Theorem 3.1) gives that any $X \in \mathcal{C}$ has infinitely many "small points". In particular, we establish the first cases of Szpiro's small points conjecture. Furthermore, we

improve Theorem 3.1 for hyperelliptic curves (see Theorem 3.2) and for genus two curves (see Theorem 3.3) in the sense that we produce "smaller points" on such curves. To give the reader a more concrete idea of our results we now state a special case of Theorem 3.3. If the curve X has genus two and is defined over $\mathbb{Q}$, then X has infinitely many algebraic points x that satisfy

$$\max\big(h_{NT}(x), h(x)\big) \leq (10 \prod p)^{10^6}$$

with the product taken over all bad reduction primes p of X. Here h_{NT} is the Néron-Tate height and h is the stable Arakelov height, see Section 2. We also give in Proposition 3.4 and Proposition 5.3 versions of the above theorems with "exponentially smaller points". However these versions are either conditional on the *abc* conjecture, or they depend on lower bounds for Faltings' delta invariant which are not known to be effective.

We remark that "effective Siegel or Mordell" applications of our completely explicit results require hyperbolic curves which admit Kodaira-Paršin type constructions with all fibers in $\mathcal{C}$. There exists no hyperbolic curve for which Kodaira's construction (see [Szp85b, p.266]) is of this form and we are not aware of a hyperbolic curve for which Paršin's construction (see Paršin [Par68], or [Szp85b, p.268]) has all fibers in $\mathcal{C}$. Therefore our results have at the time of writing no "effective Siegel or Mordell" applications. However, there is the hope that new and more suitable Kodaira-Paršin type constructions will be discovered. For example, recently Levin [Lev12] gave a new Paršin type construction for integral points on hyperelliptic curves with all fibers in $\mathcal{C}$.

We next describe the main ingredients for our proofs. Let X be a smooth, projective and geometrically connected curve of genus $g \geq 2$, defined over a number field K. On using fundamental results of Arakelov [Ara74], Faltings [Fal84] and Zhang [Zha92], we show that for any $\epsilon > 0$ there exist infinitely many algebraic points x of X that satisfy

$$h_{NT}(x) \leq 2g(g-1)h(x) \leq g \cdot e(X) + \epsilon. \tag{1}$$

Here $e(X)$ is a stable Arakelov self-intersection number, see (7). Thus, to produce "small points" of X, it suffices to estimate $e(X)$ effectively in terms of K, S and g, for S the set of finite places of K where X has bad reduction.

The proof of Theorem 3.1 uses properties of the Belyi degree $\deg_B(X)$ of X, which is defined in (8). From [Jav14, Thm 1.1.1] we obtain

$$e(X) \leq 10^8 \deg_B(X)^5 g. \tag{2}$$

To control $\deg_B(X)$ we use an effective version of Belyi's theorem [Bel79], worked out by Khadjavi in [Kha02]. We deduce an explicit upper bound for $\deg_B(X)$ in terms of K, g and the heights $h(\lambda)$ of the cross-ratios λ of four (geometric) branch points of a finite morphism $X \to \mathbb{P}^1_K$. To estimate $h(\lambda)$ we assume that $X \in \mathcal{C}$ and then we combine [dJR11, Prop 2.1] of de Jong-Rémond with [vK14, Prop 6.1 (ii)]; here we mention that the latter result is based on the

theory of logarithmic forms and the former result generalizes ideas of Paršin [Par72] and Oort [Oor74]. This leads to an explicit upper bound, in terms of K, S and g, for $\deg_B(X)$, then for $e(X)$ by using (2), and then finally for $h_{NT}(x)$ and $h(x)$ by applying (1).
To discuss the proof of Theorem 3.3 we assume that $g = 2$. It was shown in [vK14, Prop 5.1 (v)] that Faltings' delta invariant $\delta(X_{\mathbb{C}})$ of a compact connected Riemann surface $X_{\mathbb{C}}$ of genus two satisfies $-186 \leq \delta(X_{\mathbb{C}})$. Then the Noether formula for arithmetic surfaces, due to Faltings [Fal84] and Moret-Bailly [MB89], leads to the following explicit inequality

$$e(X) \leq 12h_F(J) + 201. \tag{3}$$

Here $h_F(J)$ is the stable Faltings height (see [Fal83, p.354]) of the Jacobian $J = \mathrm{Pic}^0(X)$ of X. The inequalities in [vK14, Prop 4.1 (i), Prop 6.1 (ii)] slightly refine a method developed by Paršin [Par72], Oort [Oor74] and de Jong-Rémond [dJR11]. On combining these inequalities, we deduce an explicit upper bound, in terms of K, S and g, for $h_F(J)$, then for $e(X)$ by using (3), and then finally for $h_{NT}(x)$ and $h(x)$ by applying (1).
To prove Theorem 3.2 we use the strategy of proof of Theorem 3.3. In particular, we combine this strategy with a formula of de Jong [dJ09, Thm 4.3] and we use in addition the explicit estimate for the hyperelliptic discriminant modular form which was established in [vK14, Lem 5.4].
The plan of this paper is as follows. In Section 2 we discuss Szpiro's small points conjecture and its variation which involves the Néron-Tate height. Then in Section 3 we state our theorems, and we give Proposition 3.4 which is conditional on the *abc* conjecture. In Section 4 we collect results from Arakelov theory for arithmetic surfaces. We also give an upper bound for the Belyi degree of X. In Section 5 we consider curves $X \in \mathcal{C}$ and we prove Theorem 3.1. We also establish Proposition 5.3. It refines our theorems, with the disadvantage inherent that it involves a lower bound for Faltings' delta invariant which is not known to be effective. In Section 6 we first show Lemma 6.1 and Lemma 6.2. They give explicit results for certain (Arakelov) invariants of hyperelliptic curves which may be of independent interest. Then we use these lemmas to prove Theorem 3.2. Finally, in Section 7, we give a proof of Theorem 3.3.

Notation

Throughout this paper we shall use the following notations and conventions. Let K be a number field. We denote by $\bar{K}$ a fixed algebraic closure of K. If L is a field extension of K, then we write $[L : K]$ for the relative degree of L over K. By a curve X over K we mean a smooth, projective and geometrically connected curve $X \to \mathrm{Spec}(K)$. For any finite place v of K, we write N_v for the number of elements in the residue field of v and we let $v(\mathfrak{a})$ be the order of v in a fractional ideal $\mathfrak{a}$ of K. We denote by $|S|$ the cardinality of an arbitrary set S. Finally, by log we mean the principal value of the natural logarithm and we define the product taken over the empty set as 1.

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2 Small points conjectures

In this section we state and discuss Szpiro's small points conjecture, and its variation which involves the Néron-Tate height. Let K be a number field, let $g \geq 2$ be an integer and let X be a curve over K of genus g.
We take an algebraic point $x \in X(\bar{K})$. The classical result of Deligne-Mumford gives a finite extension L of K such that X_L has semi-stable reduction over B and such that $x \in X(L)$, where B is the spectrum of the ring of integers of L. We denote by $\omega_{\mathcal{X}/B}$ the relative dualizing sheaf of the minimal regular model $\mathcal{X}$ of X_L over B. Let $\omega = (\omega_{\mathcal{X}/B}, \|\cdot\|)$ with $\|\cdot\|$ the Arakelov metric, let $(\cdot,\cdot)$ be the intersection product of Arakelov divisors on $\mathcal{X}$ and identify ω and x with the corresponding Arakelov divisors on $\mathcal{X}$, see [Fal84, Section 2] for definitions. Then the stable Arakelov height $h(x)$ of x is the real number defined by

$$[L:\mathbb{Q}]h(x) = (\omega, x). \tag{4}$$

The factor $[L:\mathbb{Q}]$ and the semi-stability of $\mathcal{X}$ assure that the definition of $h(x)$ does not depend on the choice of a field L with the above properties.
Let S be a set of finite places of K. We say that a constant c, depending on some data $(\mathcal{D})$, is effective if one can in principle explicitly determine the real number c provided that $(\mathcal{D})$ is given. In 1984, Szpiro [Szp85a, p.101] formulated his small points conjecture in terms of the Arakelov self-intersection (x, x) of x. However, Arakelov's adjunction formula $(\omega, x) = -(x, x)$ in [Ara74] shows that Szpiro's small points conjecture coincides with the following conjecture.

Conjecture (*sp*). *There exists an effective constant c, depending only on K, S and g, with the following property. Suppose that X is a curve over K of genus g, with set of bad reduction places S. If X has semi-stable reduction over the ring of integers of K, then there exists $x \in X(\bar{K})$ that satisfies $h(x) \leq c$.*

It is known (see Szpiro [Szp85a, p.101]) that this conjecture implies an "effective Mordell conjecture". Further, Szpiro established a rather strong geometric analogue of Conjecture (*sp*) in [Szp81]. We also mention that if Conjecture (*sp*) holds, then it holds without the semi-stable assumption. Indeed, on combining results of Grothendieck-Raynaud [GR72, Proposition 4.7] and Serre-Tate [ST68, Theorem 1] with Dedekind's discriminant theorem, one obtains a finite field extension M of K such that X_M has semi-stable reduction over the ring of integers of M and such that $[M:K]$ and the relative discriminant of M over K are effectively controlled in terms of K, S and g.

In the Grothendieck Festschrift, Szpiro [Szp90b] formulated another version of Conjecture (*sp*). This formulation involves the Néron-Tate height

$$h_{NT}(x) \tag{5}$$

of $x \in X(\bar{K})$ which is defined as the Néron-Tate height of the divisor class $(2g-2)x - \Omega^1$ in the Jacobian $\mathrm{Pic}^0(X_L)$ of X_L for L as above. Here Ω^1 denotes the sheaf of differential one-forms of the curve X_L over L. We now give a version of Szpiro's "Conjecture des deux petits points pour Néron-Tate" which was stated by Szpiro in [Szp90b, p.244].

Conjecture $(sp)^*$. *Any curve X over K of genus g has two distinct points $x_i \in X(\bar{K})$, $i = 1, 2$, that satisfy $h_{NT}(x_i) \leq c$, where c is an effective constant which depends only on K, g and the geometry of the bad reduction of X.*

It follows for example from Lemma 4.5 below that Conjecture (*sp*) implies Conjecture $(sp)^*$. An unconditional proof of the converse implication seems to be difficult. However, we point out that Szpiro's arguments in [Szp90b, p.244] combined with Moret-Bailly's proof of [MB90, Théorème 5.1] show that Conjecture $(sp)^*$ still implies an "effective Mordell conjecture".
We mention that the conjecture in [Szp90b, p.244] describes the constant c in Conjecture $(sp)^*$ more precisely. See also [JvK13, Section 2] for a discussion of the possible shape of the constants in Conjectures (*sp*) and $(sp)^*$. For further discussions and conjectures related to the small points conjecture, we refer the reader to the works of Paršin [Par88] and Moret-Bailly [MB90].

3 Statements of results

In this section we state and discuss Theorems 3.1, 3.2 and 3.3. We also give Proposition 3.4 which relies on the *abc* conjecture.
To state our results we need to introduce some notation. Let $g \geq 2$ be an integer, let K be a number field and let S be a set of finite places of K. We denote by D_K the absolute value of the discriminant of K over $\mathbb{Q}$ and we write $d = [K : \mathbb{Q}]$ for the degree of K over $\mathbb{Q}$. To measure K, S and g we use

$$D_K,\ d,\ N_S = \prod_{v \in S} N_v,\ g \ \text{ and } \ \nu = d(5g)^5. \tag{6}$$

We mention that the only purpose of introducing ν is to simplify the exposition. Let h be the stable Arakelov height and let h_{NT} be the Néron-Tate height. These heights are defined in (4) and (5) respectively. Let $\mathcal{C} = \mathcal{C}(K)$ be the set of curves X over K of genus ≥ 2 such that there is a finite morphism $X \to \mathbb{P}^1_K$ of prime degree which is geometrically a cyclic cover. Our first theorem establishes in particular Conjectures (*sp*) and $(sp)^*$ for all curves $X \in \mathcal{C}$.

Theorem 3.1. *Let X be a curve over K of genus g, with set of bad reduction places S. If $X \in \mathcal{C}$ then there exist infinitely many $x \in X(\bar{K})$ that satisfy*

$$\log \max\big(h_{NT}(x), h(x)\big) \leq \nu^{d\nu}(N_S D_K)^{\nu}.$$

Next, we present two results which improve Theorem 3.1 in certain cases. We say that a curve X over K of genus g is a hyperelliptic curve over K if there is a finite morphism $X \to \mathbb{P}^1_K$ of degree two. For example, any genus two curve over K is a hyperelliptic curve over K. We obtain in particular the following theorem for hyperelliptic curves over K.

Theorem 3.2. *Suppose that X is a hyperelliptic curve over K of genus g, with set of bad reduction places S and set of Weierstrass points $\mathcal{W}$. Then it holds*

$$\sum_{x \in \mathcal{W}} h_{NT}(x) \leq \nu^{8^g d\nu}(N_S D_K)^{\nu}.$$

The Néron-Tate height h_{NT} is non-negative, and any hyperelliptic curve over K of genus g has exactly $2g+2$ Weierstrass points. Thus Theorem 3.2 gives another proof of Conjecture $(sp)^*$ for all hyperelliptic curves over K of genus g. Our next result refines Theorem 3.1 in the special case of genus two curves.

Theorem 3.3. *Suppose that X is a curve over K of genus two, with set of bad reduction places S. Then there are infinitely many $x \in X(\bar{K})$ that satisfy*

$$\max\big(h_{NT}(x), h(x)\big) \leq \nu^{2d\nu}(N_S D_K)^{\nu}, \quad \nu = 10^5 d.$$

To state Proposition 3.4 we need to recall the *abc*-conjecture of Masser-Oesterlé [Mas02] over number fields. For any non-zero triple $\alpha, \beta, \gamma \in K$ we denote by $H(\alpha, \beta, \gamma)$ the usual absolute multiplicative Weil height of the corresponding point in $\mathbb{P}^2(K)$, see [BG06, 1.5.4]. We define the height function $H_K = H^d$ and we write $S_K(\alpha, \beta, \gamma) = \prod N_v^{e_v}$ with the product extended over all finite places v of K such that $v(\alpha)$, $v(\beta)$ and $v(\gamma)$ are not all equal, where $e_v = v(p)$ for p the residue characteristic of v. We mention that Masser [Mas02] added the ramification index e_v in the definition of the support S_K to obtain a natural behaviour of S_K with respect to finite field extensions.

Conjecture (*abc*). *For any integer $n \geq 1$, and any real $r, \epsilon > 1$, there is a constant c, which depends only on n, r, ϵ, such that if K is a number field of degree $[K : \mathbb{Q}] \leq n$, and $\alpha, \beta, \gamma \in K$ are non-zero and satisfy $\alpha + \beta = \gamma$, then $H_K(\alpha, \beta, \gamma) \leq c S_K(\alpha, \beta, \gamma)^r D_K^{\epsilon}$.*

The following proposition is conditional on Conjecture (*abc*). It improves exponentially, in terms of N_S and D_K, the inequalities in our theorems. We put $u(g) = 8^{(11g)^3 8^g}$ and now we can state Proposition 3.4.

Proposition 3.4. *Let $r, \epsilon > 1$ be real numbers and write*

$$\Omega = (r + \frac{\epsilon}{d}) \log N_S + \frac{\epsilon}{d} \log D_K.$$

Suppose that Conjecture (abc) holds for $n = 24g^4 d, r, \epsilon$ with the constant c. Then there exist effective constants c_1, c_2, c_3, depending only on c, r, ϵ, d and g, such that for any curve X over K of genus g, with set of bad reduction places S, the following statements hold.

(i) If $X \in \mathcal{C}$, then there are infinitely many $x \in X(\bar{K})$ that satisfy

$$\log \max\big(h_{NT}(x), h(x)\big) \leq \nu^\nu \Omega + c_1.$$

(ii) Suppose that X is a hyperelliptic curve over K, with set of Weierstrass points $\mathcal{W}$. Then it holds

$$\sum_{x \in \mathcal{W}} h_{NT}(x) \leq u(g)(3g-1)(8g+4)\Omega + c_2.$$

(iii) If $g = 2$, then there are infinitely many $x \in X(\bar{K})$ that satisfy

$$\frac{1}{4} h_{NT}(x) \leq h(x) \leq 6u(2)\Omega + c_3.$$

We remark that the factor $u(g)$ appearing in Proposition 3.4 is not optimal. Its origin shall be explained after Proposition 5.3. We also point out that Proposition 3.4 only requires the validity of (*abc*) for some fixed n, r, ϵ, instead of all n, r, ϵ, and (*abc*) with fixed n, r, ϵ is often called "weak *abc* conjecture". Further we mention that Elkies [Elk91] used Belyi's theorem to show that (effective) (*abc*) implies (effective) Mordell. However, it is not clear if Conjecture (*sp*) or Conjecture $(sp)^*$ follows from the effective version of the Mordell conjecture which Elkies deduces from an effective version of (*abc*).
In general, we conducted some effort to obtain constants reasonably close to the best that can be acquired with the present method of proof. However, to simplify the form of our inequalities we freely rounded off several of the numbers appearing in our estimates.

4 Self-intersection, Belyi degree and heights

In this section we first give two lemmas which describe properties of the Belyi degree, and then we collect results from Arakelov theory for arithmetic surfaces. We also prove a lemma which was used in Section 2.
Throughout this section we denote by X a curve of genus $g \geq 2$, defined over a number field K. Suppose that L, ω and $(\cdot,\cdot)$ are as in Section 2. Then the stable self-intersection $e(X)$ of ω is the real number defined by

$$[L : \mathbb{Q}]e(X) = (\omega, \omega). \tag{7}$$

We observe that this definition does not depend on any choices. Let D be the set of degrees of finite morphisms $X_{\bar{K}} \to \mathbb{P}^1_{\bar{K}}$ which are unramified outside $0, 1, \infty$. Belyi's theorem [Bel79] shows that D is non-empty, and then the Belyi degree $\deg_B(X)$ of X is defined by

$$\deg_B(X) = \min D. \tag{8}$$

Our proof of Theorem 3.1 uses two fundamental properties of the Belyi degree. We now state the first of these properties in the following lemma.

Lemma 4.1. *It holds* $e(X) \leq 10^8 \deg_B(X)^5 g$.

Proof. The statement is proven in [Jav14, Theorem 1.1.1]. $\square$

The next lemma gives the second property. It is a consequence of an effective version of Belyi's theorem [Bel79] worked out by Khadjavi [Kha02]. We denote by $H(\alpha)$ the usual absolute multiplicative Weil height of $\alpha \in \mathbb{P}^1(\bar{K})$, defined in [BG06, 1.5.4]. Then we define the height H_Λ of a subset Λ of $\mathbb{P}^1(\bar{K})$ by $H_\Lambda = \sup\{H(\lambda), \lambda \in \Lambda\}$.

Lemma 4.2. *If* $\varphi : X \to \mathbb{P}^1_K$ *is a finite morphism, with set of (geometric) branch points* $\Lambda \subset \mathbb{P}^1(\bar{K})$ *and if* $m = 4[K : \mathbb{Q}](\deg(\varphi) + g - 1)^2$*, then*

$$\deg_B(X) \leq (4mH_\Lambda)^{9m^3 2^{m-2} m!} \deg(\varphi).$$

Proof. The absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ of $\bar{\mathbb{Q}}$ over $\mathbb{Q}$ acts in the usual way on $\mathbb{P}^1(\bar{\mathbb{Q}}) \cong \mathbb{P}^1(\bar{K})$. Let $\Lambda' = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \cdot \Lambda$ be the image of Λ under this action. The classical result of Hurwitz [Liu02, Theorem 7.4.16] implies that $[K(\lambda) : K]$ and $|\Lambda|$ are at most $2g - 2 + 2\deg(\varphi)$ for $K(\lambda)$ the field of definition of $\lambda \in \Lambda$. This gives $|\Lambda'| \leq m$, and the Galois invariance [BG06, 1.5.17] of H shows $H_\Lambda = H_{\Lambda'}$. Then an application of [Kha02, Theorem 1.1] with the Galois stable set Λ' gives a finite morphism $\psi : \mathbb{P}^1_{\bar{K}} \to \mathbb{P}^1_{\bar{K}}$ with the following properties. The morphism ψ is unramified outside $0, 1, \infty$, with $\psi(\Lambda) \subseteq \{0, 1, \infty\}$ and

$$\deg(\psi) \leq (4mH_\Lambda)^{9m^3 2^{m-2} m!}.$$

We observe that the composition $\psi \circ \varphi : X_{\bar{K}} \to \mathbb{P}^1_{\bar{K}}$ is unramified outside $0, 1, \infty$. This shows $\deg_B(X) \leq \deg(\psi)\deg(\varphi)$ and then the displayed inequality implies Lemma 4.2. $\square$

The bound in Khadjavi's [Kha02, Theorem 1.1], and thus Lemma 4.2, can be improved in special cases. See for example Liţcanu [Liţ04].
Let h be the stable Arakelov height defined in (4). To obtain infinitely many small points we shall use the following lemma which relies on a fundamental result of Zhang [Zha92].

Lemma 4.3. *Suppose that* $\epsilon > 0$ *is a real number. Then there exist infinitely many points* $x \in X(\bar{K})$ *that satisfy*

$$2(g-1)h(x) \leq e(X) + \epsilon.$$

Proof. It follows for example from Lemma 4.4 (i) below that any point $x \in X(\bar{K})$ satisfies $h(x) \geq 0$. Then we see that [Zha92, Theorem 6.3] implies the statement of Lemma 4.3. $\square$

We remark that Moret-Bailly showed that for any real number $\epsilon > 0$, there exists an algebraic point $x \in X(\bar{K})$ which satisfies $4(g-1)h(x) \leq e(X) + \epsilon$.

For details we refer the reader to the proof of [MB90, Proposition 3.4] which uses in particular Faltings' result [Fal84, Corollary, p.406].

The following lemma is a direct consequence of classical results of Arakelov [Ara74], Faltings [Fal84] and Moret-Bailly [MB89]. To state this lemma we need to introduce more notation. For any embedding $\sigma : K \hookrightarrow \mathbb{C}$, we denote by X_σ the compact connected Riemann surface which corresponds to the base change of X to $\mathbb{C}$ with respect to σ. Let $\delta(X_\sigma)$ be Faltings' delta invariant of X_σ, defined in [Fal84, p.402]. We denote by $M_g(\mathbb{C})$ the moduli space of smooth, projective and connected curves over $\mathbb{C}$ of genus g. Faltings' delta invariant, viewed as a function $M_g(\mathbb{C}) \to \mathbb{R}$, has a minimum which we denote by

$$c_\delta(g). \tag{9}$$

If $g \geq 3$, then effective lower bounds for $c_\delta(g)$ are not known. Let $h_F(J)$ be the stable Faltings height [Fal83, p.354] of the Jacobian $J = \mathrm{Pic}^0(X)$ of X, and let h_{NT} be the Néron-Tate height defined in (5). We now can state the lemma.

Lemma 4.4. *The following statements hold.*

(i) Any $x \in X(\bar{K})$ satisfies $h_{NT}(x) \leq 2g(g-1)h(x)$.

(ii) It holds $e(X) + c_\delta(g) \leq 12h_F(J) + 4g\log(2\pi)$.

Proof. To show (i) we take $x \in X(\bar{K})$. Let L, $\mathcal{X} \to B$, ω and $(\cdot,\cdot)$ be as in (4). We identify x with the corresponding Arakelov divisor on $\mathcal{X}$. Let Φ be the (unique) vertical $\mathbb{Q}$-Cartier divisor on $\mathcal{X}$ such that the supports of Φ and $x(B)$ are disjoint and such that any irreducible component Γ of any fiber of $\mathcal{X} \to B$ satisfies $((2g-2)x - \omega + \Phi, \Gamma) = 0$. Szpiro [Szp85c, p.276] observed that the adjunction formula in [Ara74] together with [Fal84, Theorem 4.c)] leads to

$$2h_{NT}(x) = -e(X) + 4g(g-1)h(x) + \frac{1}{[L:\mathbb{Q}]}(\Phi,\Phi).$$

Further, [Fal84, Theorem 5.a)] provides that $-e(X) \leq 0$. Therefore the inequality $(\Phi,\Phi) \leq 0$ implies assertion (i).

We now prove (ii). If $v \in B$ is closed, then δ_v denotes the number of singular points of the geometric special fiber of $\mathcal{X}$ over v. The (logarithmic) stable discriminant $\Delta(X)$ of X is the real number defined by

$$[L:\mathbb{Q}]\Delta(X) = \sum \delta_v \log N_v \tag{10}$$

with the sum taken over all closed points v of B. Then we see that Moret-Bailly's refinement [MB89, Théorème 2.5] of the Noether formula [Fal84, Theorem 6] implies the following formula

$$12h_F(J) = \Delta(X) + e(X) - 4g\log(2\pi) + \frac{1}{[L:\mathbb{Q}]}\sum \delta(X_\sigma)$$

with the sum taken over all embeddings $\sigma : L \hookrightarrow \mathbb{C}$. Therefore the estimates $\Delta(X) \geq 0$ and $\sum \delta(X_\sigma) \geq [L:\mathbb{Q}]c_\delta(g)$ prove assertion (ii) and this completes the proof of Lemma 4.4. ∎

The above results lead to the following useful lemma.

Lemma 4.5. *Suppose that $\epsilon > 0$ is a real number and assume that $x_0 \in X(\bar{K})$. Then there exist infinitely many $x \in X(\bar{K})$ that satisfy*

$$h_{NT}(x) \leq 4g^2(g-1)h(x_0) + \epsilon.$$

Proof. On combining Lemma 4.3 with Lemma 4.4 (i), we see that there exist infinitely many points $x \in X(\bar{K})$ that satisfy

$$h_{NT}(x) \leq g \cdot e(X) + \epsilon. \tag{11}$$

Further [Fal84, Theorem 5.b)] gives that any $x_0 \in X(\bar{K})$ satisfies the inequality $e(X) \leq 4g(g-1)h(x_0)$ and thus (11) implies Lemma 4.5. □

We conclude this section by the following remarks. Let S be the set of finite places of K where X has bad reduction. Suppose that there exists a finite morphism $\varphi : X \to \mathbb{P}^1_K$, with $\deg(\varphi)$ and H_Λ effectively bounded in terms of K, S and g, where H_Λ is the height of the set Λ of (geometric) branch points of φ. Then Lemma 4.1, Lemma 4.2 and Lemma 4.3 show that X has infinitely many "small points", and therefore X satisfies in particular Szpiro's small points conjecture. Similarly, if $\deg_B(X)$ is effectively bounded in terms of K, S and g, then Lemma 4.1 and Lemma 4.3 show that X has infinitely many "small points". For example, if the base change of X to $\mathbb{C}$, with respect to some embedding $K \hookrightarrow \mathbb{C}$, is a classical congruence modular curve, then a result of Zograf in [Zog91] gives $\deg_B(X) \leq 128(g+1)$.

5 Cyclic covers of prime degree

In this section we prove Theorem 3.1 and Proposition 3.4 (i). We also give Proposition 5.3 which may be of independent interest. It improves certain aspects of the inequalities in Theorem 3.1 and Proposition 3.4 (i). However, Proposition 5.3 has the disadvantage inherent that it now involves a constant which is not known to be effective.

Let X be a curve over a number field K of genus $g \geq 2$. We denote by S the set of finite places of K where X has bad reduction. Further we write $\mathcal{C} = \mathcal{C}(K)$ for the set of cyclic covers of prime degree which was introduced in Section 3. In this section we assume throughout that $X \in \mathcal{C}$.

We now give two lemmas which will be used in our proof of Theorem 3.1. To state and prove these lemmas we have to introduce some notation. Our assumption that $X \in \mathcal{C}$ provides a finite morphism $\varphi : X \to \mathbb{P}^1_K$ which is geometrically a cyclic cover of prime degree. Let q be the degree of φ and let L be a finite extension of K. We denote by $U = S(L,q)$ the set of places of L which divide q or a place in S. Let $\mathcal{O}_U^\times$ be the U-units in L and let $h(\alpha)$ be the usual absolute logarithmic Weil height of $\alpha \in L$, defined in [BG06, 1.6.1]. We write $\mu_U = \sup(h(\lambda), \lambda \in \mathcal{O}_U^\times \text{ and } 1-\lambda \in \mathcal{O}_U^\times)$. Let $\mathcal{R}$ be the set of field extensions L of K such that L is the compositum of the fields of definition of four distinct (geometric) ramification points of φ. We define

$$\mu_X = \sup(1, \mu_{S(L,q)}) \tag{12}$$

with the supremum taken over all fields $L \in \mathcal{R}$. Let $\deg_B(X)$ be the Belyi degree of X, defined in (8), and let d be the degree of K over $\mathbb{Q}$. We recall that $\nu = d(5g)^5$ and now we can state the following lemma.

LEMMA 5.1. *It holds* $\log \deg_B(X) \le \nu^{\nu/2}\mu_X$.

To prove this lemma we shall combine the estimate for $\deg_B(X)$ in Lemma 4.2 with the following observation of Paršin in [Par72]: The cross-ratios of the branch points, of a hyperelliptic map of a genus two curve X over K, are solutions of certain $S(L,2)$-unit equations. Oort [Oor74, Lemma 2.1] and de Jong-Rémond [dJR11, Proposition 2.1] generalized Paršin's idea to hyperelliptic curves and to cyclic covers of prime degree.

Proof of Lemma 5.1. We take three distinct (geometric) ramification points of φ. Let M be the compositum of their fields of definition. On composing φ with a suitable automorphism of $\mathbb{P}^1_M$, we get a finite morphism $X_M \to \mathbb{P}^1_M$ of degree q such that $\{0,1,\infty\} \subset \Lambda$, where Λ is the set of (geometric) branch points of φ. Let H_Λ be the height of Λ, defined in Section 4. To prove the inequality

$$\log H_\Lambda \le \mu_X,$$

we may and do take $\lambda \in \Lambda$ with $\lambda \neq 0,1,\infty$. We write $U = S(K(\lambda), q)$. From [dJR11, Proposition 2.1] we deduce that the cross-ratios $\lambda = \mathrm{cr}(\infty,0,1,\lambda)$ and $1-\lambda = \mathrm{cr}(\infty,1,0,\lambda)$ are U-units in $K(\lambda)$. This implies that $h(\lambda) \le \mu_X$, since $K(\lambda) \subseteq L$ for some $L \in \mathcal{R}$. Hence we obtain $\log H_\Lambda \le \mu_X$ as desired. Next, we observe that the ramification indexes of $\varphi_{\bar{K}} : X_{\bar{K}} \to \mathbb{P}^1_{\bar{K}}$ are in $\{1,q\}$, and $\mathrm{Gal}(\bar{K}/K)$ acts on the (geometric) ramification points of φ. Therefore Hurwitz leads to $q \le 2g+1$ and $[M:\mathbb{Q}] \le 15dg^3$. Then an application of Lemma 4.2 with $X_M \to \mathbb{P}^1_M$ gives an upper bound for $\deg_B(X)$ which together with the displayed inequality implies Lemma 5.1. □

We remark that if $L \in \mathcal{R}$, then Hurwitz (see the proof of Lemma 5.1) leads to $q \le 2g+1$ and $[L:K] \le 24g^4$, and [dJR11, Lemme 2.1] of de Jong-Rémond implies that L is unramified outside $S(K,q)$.
Next, we go into number theory and we give an upper bound for μ_X in terms of the quantities N_S, ν, d and D_K which are defined in (6).

LEMMA 5.2. *The following statements hold.*

(i) It holds $\mu_X \le \nu^{d\nu/8}(N_S D_K)^\nu$.

(ii) Let $r, \epsilon > 1$ *be real numbers. Suppose* (abc) *holds for* $n = 24g^4 d, r, \epsilon$ *with the constant* c. *Then there exists an effective constant* c^**, depending only on* c, r, ϵ, d *and* g*, such that*

$$d\mu_X \le (dr+\epsilon)\log N_S + \epsilon \log D_K + c^*.$$

To prove (i) we use [vK14, Proposition 6.1 (ii)]. It is based on the theory of logarithmic forms and we refer to the monograph of Baker-Wüstholz [BW07] in which the state of the art of this theory is exposed.

Proof of Lemma 5.2. We take $L \in \mathcal{R}$, and we write $U = S(L, q)$, $T = S(K, q)$ and $l = [L : K]$. Then we observe that $N_T = \prod_{v \in T} N_v$ satisfies $N_T \leq q^d N_S$. Furthermore, the remark after the proof of Lemma 5.1 gives that $q \leq 2g + 1$ and $l \leq 24g^4$, and that L is unramified outside T. We now apply [vK14, Proposition 6.1] with $U = U$, $T = T$ and $S = T$, where the symbols U, T, S on the left hand side of these equalities denote the sets in [vK14, Proposition 6.1]. In particular, [vK14, Proposition 6.1 (ii)] leads to $\mu_U \leq \nu^{d\nu/8}(D_K N_S)^\nu$ which implies statement (i).
To show statement (ii) we take real numbers $r, \epsilon > 1$. We may and do assume that Conjecture (abc) holds for $n = 24g^4 d, r, \epsilon$ with the constant c. Then Conjecture (abc) holds in particular for $n' = ld, r, \epsilon$ with the same constant c, since $n' \leq n$. Then [vK14, Proposition 6.1 (iii)] gives

$$d\mu_U \leq (dr + \epsilon) \log N_T + \epsilon \log D_K + \epsilon dt \log l - \frac{\epsilon}{l} \log N_T + \frac{1}{l} \log c$$

for $t = |T|$. From [vK14, Lemma 6.3] we get that $\epsilon dt \log l - \frac{\epsilon}{l} \log N_T$ is bounded from above by an effective constant, which depends only on ϵ, d and g. Hence we deduce statement (ii) and Lemma 5.2. □

We recall that h denotes the stable Arakelov height and that h_{NT} denotes the Néron-Tate height, defined in (4) and (5) respectively. We now prove Theorem 3.1 and Proposition 3.4 (i) simultaneously.

Proof of Theorem 3.1 and Proposition 3.4 (i). On combining Lemma 4.1, Lemma 4.3, Lemma 4.4 and Lemma 5.1, we obtain infinitely many $x \in X(\bar{K})$ that satisfy $\log \max\big(h_{NT}(x), h(x)\big) \leq 6\nu^{\nu/2}\mu_X$. Therefore we see that Lemma 5.2 (i) and Lemma 5.2 (ii) imply Theorem 3.1 (i) and Proposition 3.4 (i) respectively. □

The remaining of this section is devoted to the following Proposition 5.3. Let $c_\delta(g)$ be the minimum of Faltings' delta invariant on $M_g(\mathbb{C})$, defined in (9). We recall that if $g \geq 3$, then effective lower bounds for $c_\delta(g)$ in terms of g are not known. Put $u(g) = 8^{(11g)^3 8^g}$ and now we can state the following result.

PROPOSITION 5.3. *The following statements hold.*

(i) *There are infinitely many $x \in X(\bar{K})$ that satisfy*

$$h(x) \leq \nu^{8^g d\nu}(D_K N_S)^\nu - c_\delta(g).$$

(ii) *Let $r, \epsilon > 1$ be real numbers. Suppose that Conjecture (abc) holds for $n = 24g^4 d, r, \epsilon$ with the constant c. Then there exists an effective constant*

c_1', depending only on c, r, ϵ, d and g, with the property that there are infinitely many points $x \in X(\bar{K})$ which satisfy

$$h(x) \leq 6\frac{u(g)}{g-1}\big((r+\frac{\epsilon}{d})\log N_S + \frac{\epsilon}{d}\log D_K\big) + c_1' - \frac{c_\delta(g)}{2g-2}.$$

We observe that these (ineffective) inequalities improve exponentially, in terms of N_S and D_K, the estimates provided by Theorem 3.1 and Proposition 3.4 (i). On using Lemma 4.4 one can formulate Proposition 5.3 also in terms of h_{NT}. Further we remark that the factor $u(g)$ comes from explicit height comparisons of Rémond [Rém10] which rely inter alia on results of Bost-David-Pazuki [Paz12]. In fact, Rémond's explicit height comparisons hold for arbitrary curves over K, and in our special case where $X \in \mathcal{C}$ it seems possible to improve these height comparisons, and thus $u(g)$, up to a certain extent.
Our proof of Proposition 5.3 uses in particular the following tools. We combine Lemma 5.2 with [vK14, Proposition 4.1 (i)]. This slightly refines the method developed by Paršin [Par72], Oort [Oor74] and de Jong-Rémond [dJR11]. We also use Lemma 4.3 which is based on a theorem of Zhang [Zha92], and Lemma 4.4 (ii) which relies inter alia on Moret-Bailly's refinement [MB89] of the Noether formula in Faltings' article [Fal84].

Proof of Proposition 5.3. We denote by $h_F(J)$ the stable Faltings height of the Jacobian $J = \mathrm{Pic}^0(X)$ of X. The remark given below [vK14, Proposition 4 (i)] provides the following explicit inequality

$$h_F(J) \leq u(g)\mu_X. \tag{13}$$

Then Lemma 4.4 (ii) shows $e(X) \leq 12u(g)\mu_X - c_\delta(g) + 4g\log(2\pi)$ for $e(X)$ as in (7). Hence Lemmas 4.3 and 5.2 imply Proposition 5.3. $\square$

We conclude this section by the following remarks. Let $h_F(J)$ be as above, let $\Delta(X)$ be the stable discriminant of X defined in (10) and let $e(X)$ be as in (7). Further, we define the quantity $\delta(X) = \frac{1}{d}\sum \delta(X_\sigma)$ with the sum taken over all embeddings $\sigma : K \hookrightarrow \mathbb{C}$, where $\delta(X_\sigma)$ is defined in Section 4. Then it holds

$$\log\max(e(X), \delta(X), h_F(J), \Delta(X)) \leq \nu^{d\nu}(D_K N_S)^\nu. \tag{14}$$

Indeed, [Jav14, Theorem 1.1.1] gives that $e(X)$, $\delta(X)$, $h_F(J)$ and $\Delta(X)$ are at most $10^9 g^2 \deg_B(X)^5$, and then Lemmas 5.1 and 5.2 prove the displayed inequality. We mention that de Jong-Rémond [dJR11, Theorem 1.2] provides an estimate for $h_F(J)$ which is better than (14). Further [vK14, Theorem 3.2] gives an upper bound for $\Delta(X)$ which is exponentially better than (14). However, [vK14, Theorem 3.2] involves a constant, depending at most on g, which is only known to be effective for hyperelliptic curves X over K. We note that [dJR11, Theorem 1.2] and [vK14, Theorem 3.2] both depend inter alia on the above mentioned explicit height comparisons of Rémond in [Rém10], and such height comparisons are not used in our proof of (14).

Finally we point out that the method of this paper using the Belyi degree gives in addition the following generalizations: Theorem 3.1, Proposition 3.4 (i) and inequality (14) hold more generally for any curve Y over K which admits a finite étale morphism to some $X \in \mathcal{C}$. Indeed on using that Y is an étale cover of X we deduce from Hurwitz that $\deg_B(Y)$ is explicitly bounded in terms of $\deg_B(X)$ and the genus of Y, and then the above arguments prove the claimed generalization. Here we applied in addition [LL99, Corollary 4.10] which gives that Y has bad reduction at a finite place v of K if X has bad reduction at v.

6 Hyperelliptic curves

In this section we prove Theorem 3.2. We also show two lemmas which may be of independent interest. They provide explicit results for certain (Arakelov) invariants of hyperelliptic curves. Throughout this section we denote by X a hyperelliptic curve of genus $g \geq 2$, defined over a number field K.
As before, we denote by X_σ the compact connected Riemann surface corresponding to the base change of X to $\mathbb{C}$ with respect to an embedding $\sigma : K \hookrightarrow \mathbb{C}$. Let $T(X_\sigma)$ be the invariant of de Jong. It is the norm of a canonical isomorphism between certain line bundles on X_σ and we refer to [dJ05, Definition 4.2] for a precise definition of $T(X_\sigma)$.

Lemma 6.1. *It holds* $-36g^3 \leq \log T(X_\sigma)$.

To prove this lemma we use de Jong's formula [dJ05, Theorem 4.7]. It expresses $T(X_\sigma)$ in terms of a certain hyperelliptic discriminant modular form, which we then estimate by using the explicit inequality in [vK14, Lemma 5.4].

Proof of Lemma 6.1. We begin to state the formula for $T(X_\sigma)$ in [dJ05, Theorem 4.7]. Let $\mathfrak{H}_g$ be the Siegel upper half plane of complex symmetric $g \times g$ matrices with positive definite imaginary part. We denote by Δ_g the hyperelliptic discriminant modular form on $\mathfrak{H}_g$, defined in [vK14, Section 5]. Since X is hyperelliptic, there exists a finite morphism $\varphi : X_\sigma \to \mathbb{P}^1_{\mathbb{C}}$ of degree two. Let $H_1(X_\sigma, \mathbb{Z})$ be the first homology group of X_σ with coefficients in $\mathbb{Z}$. On following Mumford [Mum07, Chapter IIIa] we construct a canonical symplectic basis of $H_1(X_\sigma, \mathbb{Z})$ with respect to a fixed ordering of the $2g+2$ branch points of φ. Then [dJ05, Theorem 4.7] provides a basis of the global sections of the sheaf of differentials on X_σ with the following property: Integration of this basis over the canonical symplectic basis of $H_1(X_\sigma, \mathbb{Z})$ gives a period matrix $\tau_\sigma \in \mathfrak{H}_g$ that satisfies

$$T(X_\sigma) = (2\pi)^{-2g} \left|\Delta_g(\tau_\sigma)\det(\mathrm{im}(\tau_\sigma))^{2a}\right|^{-(3g-1)/(8bg)},$$

where $a = \binom{2g+1}{g+1}$ and $b = \binom{2g}{g+1}$. Furthermore, [vK14, Lemma 5.4] gives an effective constant k_1, depending only on g, such that

$$\left|\Delta_g(\tau_\sigma)\det(\mathrm{im}(\tau_\sigma))^{2a}\right| \leq k_1.$$

The effective constant k_1 is explicitly computed in [vK14, (15)], and then the displayed formula for $T(X_\sigma)$ leads to an inequality as stated in Lemma 6.1. $\square$

Let $h_F(J)$ be the stable Faltings height of the Jacobian $J = \mathrm{Pic}^0(X)$ of X and let h_{NT} be the Néron-Tate height which is defined in (5).

Lemma 6.2. *If $\mathcal{W}$ denotes the set of Weierstrass points of X, then*

$$\sum_{x\in\mathcal{W}} h_{NT}(x) \leq (3g-1)(8g+4)h_F(J) + 293g^5.$$

To prove Lemma 6.2 we use de Jong's formula [dJ09, Theorem 4.3]. This formula involves inter alia $h_F(J)$ and $\sum_{x\in\mathcal{W}} h_{NT}(x)$, and an analytic term related to $T(X_\sigma)$ which we control by Lemma 6.1.

Proof of Lemma 6.2. To state the formula in [dJ09, Theorem 4.3] we introduce quantities A_1, A_2 and A_3. Let L be a finite field extension of K such that X_L has semi-stable reduction over the spectrum B of the ring of integers of L and such that all Weierstrass points of X are in $X(L)$. We define

$$A_1 = -4g(2g-1)(g+1)\log(2\pi) + \frac{8g^2}{[L:\mathbb{Q}]}\sum \log T(X_\sigma)$$

with the sum taken over all embeddings $\sigma : L \hookrightarrow \mathbb{C}$. Let $\mathcal{X} \to B$, $(\cdot,\cdot)$ and ω be as in (4). We denote by E the residual divisor on $\mathcal{X}$ defined in [dJ09, p.286]. Let $\Delta(X)$ be the stable discriminant of X in (10). Then we take

$$A_2 = (2g-1)(g+1)\Delta(X) + \frac{4}{[L:\mathbb{Q}]}(E,\omega).$$

For any section $x \in \mathcal{X}(B)$ of $\mathcal{X} \to B$, we denote by Φ_x the (unique) vertical $\mathbb{Q}$-Cartier divisor on $\mathcal{X}$ with the following properties: The supports of Φ_x and $x(B)$ are disjoint, and any irreducible component Γ of any fiber of $\mathcal{X} \to B$ satisfies $((2g-2)x - \omega + \Phi_x, \Gamma) = 0$. We write

$$A_3 = \frac{1}{[L:\mathbb{Q}]g(g-1)}\sum_{x\in\mathcal{W}} -(\Phi_x,\Phi_x)n_x$$

with n_x the multiplicity of x in the Weierstrass divisor W on $\mathcal{X}$, where W is defined in [dJ09, p.286]. Then [dJ09, Theorem 4.3] gives

$$(3g-1)(8g+4)h_F(J) = A_1 + A_2 + A_3 + \frac{2}{g(g-1)}\sum_{x\in\mathcal{W}} h_{NT}(x)n_x.$$

We now estimate the quantities A_1, A_2 and A_3 from below. To deal with A_1 we use Lemma 6.1. It gives $-293g^5 \leq A_1$. The divisor E on $\mathcal{X}$ is vertical and effective, and our minimal $\mathcal{X}$ does not contain any exceptional curves. This implies that $(E,\omega) \geq 0$. Then $\Delta(X) \geq 0$ and $-(\phi_x,\phi_x) \geq 0$ show that A_2 and A_3 are both non-negative. Furthermore, [dJ09, Lemma 3.2] gives that $n_x = g(g-1)/2$. Thus we see that the above displayed formula and the lower bounds for A_1, A_2 and A_3 imply an inequality as claimed in Lemma 6.2. $\square$

On using the inequality given in Lemma 6.2 we now prove Theorem 3.2 and Proposition 3.4 (ii) simultaneously.

Proof of Theorem 3.2 and Proposition 3.4 (ii). Since X is a hyperelliptic curve over K, there exists a finite morphism $X \to \mathbb{P}^1_K$ which is geometrically a cyclic cover of prime degree two. Then, on combining Lemma 6.2, Lemma 5.2 and inequality (13), we deduce Theorem 3.2 and Proposition 3.4 (ii). □

Let $x \in \mathcal{W}$. We remark that the arguments of Burnol [Bur92, Theorem B] imply an upper bound for $h_{NT}(x)$ in terms of certain Arakelov invariants of X. However, it turns out that the bound for $h_{NT}(x)$ in Lemma 6.2 leads to a better inequality in Theorem 3.2.

7 Genus two curves

In this section we prove Theorem 3.3 and Proposition 3.4 (iii). Let $c_\delta(2)$ be the minimum of Faltings' delta invariant on $M_2(\mathbb{C})$, see (9). The following lemma was established in [vK14, Proposition 5.1 (v)].

Lemma 7.1. *It holds* $-186 \leq c_\delta(2)$.

We now combine Lemma 7.1 with the arguments used in the proof of Proposition 5.3 in order to prove Theorem 3.3 and Proposition 3.4 (iii).

Proof of Theorem 3.3 and Proposition 3.4 (iii). Let X be a genus two curve which is defined over a number field K. It is a hyperelliptic curve over K by [Liu02, Proposition 7.4.9]. Thus there exists a finite morphism $X \to \mathbb{P}^1_K$ which is geometrically a cyclic cover of prime degree two. As before, we denote by h and h_{NT} the stable Arakelov height and the Néron-Tate height respectively, defined in (4) and (5). Then Lemma 4.3, Lemma 4.4, inequality (13) and Lemma 7.1 give infinitely many points $x \in X(\bar{K})$ that satisfy

$$\frac{1}{4}h_{NT}(x) \leq h(x) \leq 6u(2)\mu_X + 101$$

for μ_X as in (12) and $u(2)$ as in Proposition 3.4. Therefore the upper bounds for μ_X, given in Lemma 5.2 (i) and Lemma 5.2 (ii), imply Theorem 3.3 and Proposition 3.4 (iii) respectively. □

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