Motivic Homology of Semiabelian Varieties

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Abstract. We generalize some classical results on the Chow group of an abelian variety to semiabelian varieties and to motivic (co)homology, using a result of Ancona–Enright-Ward–Huber [1] on a decomposition of the motive of a semiabelian variety in Voevodsky's category of motives.

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1 Introduction

Let k be a field. Let G be a semiabelian variety over k. By definition, G is an extension of an abelian variety A of dimension g by a torus T of rank r, i.e. there is the following exact sequence of smooth group schemes over k:

$$1 \to T \to G \to A \to 1. \tag{1.1}$$

We say that G is of rank r if the rank of T is r. A semiabelian variety of rank zero means an abelian variety.
In this paper, we study structure of motivic homology $H_p(G, \mathbb{Q}(q))$ (cf. Notation) and Bloch's higher Chow group $\mathrm{CH}^q(G, s; \mathbb{Q})$ ([6]). For an abelian variety A, Beauville [2] proved the following theorem (cf. [9, 18]):

Theorem 1.1 (Beauville [2]). *Let A be an abelian variety of dimension g over a field k. Let q be an integer with $0 \le q \le g$. For an integer i, let $\mathrm{CH}^q(G;\mathbb{Q})^{(i)}$ denote the subspace of the Chow group $\mathrm{CH}^q(A;\mathbb{Q})$:*

$$\mathrm{CH}^q(A;\mathbb{Q})^{(i)} := \{\alpha \in \mathrm{CH}^q(A;\mathbb{Q}) \mid n_A^*\alpha = n^i\alpha \textit{ for all } n\}.$$

Here n_A^* is the pull-back along the map $A \xrightarrow{\times n} A$ of multiplication by an integer n. Then $\mathrm{CH}^q(A;\mathbb{Q})^{(i)}$ is zero for $i \notin [q, g+q]$ and the Chow group $\mathrm{CH}^q(A;\mathbb{Q})$ is decomposed as follows: for $0 \le q \le g$,

$$\mathrm{CH}^q(A;\mathbb{Q}) = \bigoplus_{i=q}^{g+q} \mathrm{CH}^q(A;\mathbb{Q})^{(i)}.$$

Note that the index is different from the one of Beauville [2] (his notation $\mathrm{CH}^p_s(A)$ corresponds to our $\mathrm{CH}^p(A;\mathbb{Q})^{(2p-s)}$).
Bloch [5] studied an iterated Pontryagin product I_A^{*r} of the kernel I_A of the degree map $\mathrm{CH}_0(A) \to \mathbb{Z}$ for an abelian variety A over an algebraically closed field, and he proved the following theorem:

Theorem 1.2 (Bloch [5]). *Let A be an abelian variety of dimension g over an algebraically closed field. Then $I_A^{*i} = 0$ for $i > g$.*

For a smooth k-scheme X, motivic cohomology $H^p(X, R(q))$ (cf. Notation) is isomorphic to its higher Chow groups for any coefficients R ([26]): for any $p, q \in \mathbb{Z}$,

$$H^p(X, R(q)) \simeq \mathrm{CH}^q(X, 2q-p; R). \tag{1.2}$$

By Poincaré duality (cf. [11]), for a smooth proper scheme X of pure dimension d, motivic homology agrees with motivic cohomology:

$$H_p(X, R(q)) \simeq H^{2d-p}(X, R(d-q)). \tag{1.3}$$

In this paper, we generalize Theorem 1.1 and Theorem 1.2 to semiabelian varieties and to motivic homology and higher Chow groups.
Let G be a semiabelian variety over k. For an integer n, let n_G denote the map $G \xrightarrow{\times n} G$ of multiplication by n. For an integer i, we define the subgroup $\mathrm{CH}^q(G, s;\mathbb{Q})^{(i)}$ of the higher Chow group $\mathrm{CH}^q(G, s;\mathbb{Q})$ as follows (cf. Notation):

$$\mathrm{CH}^q(G, s;\mathbb{Q})^{(i)} := \{\alpha \in \mathrm{CH}^q(G, s;\mathbb{Q}) \mid n_G^*\alpha = n^i\alpha \text{ for all } n\}.$$

Here n_G^* is the pull-back along n_G. Similarly, for an integer i, we define the subgroup $H_p(G, \mathbb{Q}(q))_{(i)}$ of the motivic homology $H_p(G, \mathbb{Q}(q))$ as follows:

$$H_p(G, \mathbb{Q}(q))_{(i)} := \{\alpha \in H_p(G, \mathbb{Q}(q)) \mid n_{G*}\alpha = n^i\alpha \text{ for all } n\}.$$

Here n_{G*} denotes the push-forward map along n_G.

Theorem 1.3 (Corollary 3.5). *Let G be a semiabelian variety over a perfect field k, which is an extension of an abelian variety of dimension g by a torus of rank r.*

(1) *We have the following natural decomposition*

$$\mathrm{CH}^q(G,s;\mathbb{Q})=\bigoplus_{i=j_0}^{j_1}\mathrm{CH}^q(G,s;\mathbb{Q})^{(i)}$$

where $j_0=\max\{0,q-s\}$ *and* $j_1=\min\{2g+r,g+q\}$.

In the special case where $s=0$ *and* $q\geq 0$, *the decomposition is*

$$\mathrm{CH}^q(G;\mathbb{Q})=\bigoplus_{i=q}^{g+q}\mathrm{CH}^q(G;\mathbb{Q})^{(i)}.$$

(2) *We have the following natural decomposition*

$$H_p(G,\mathbb{Q}(q))=\bigoplus_{i=i_0}^{i_1}H_p(G,\mathbb{Q}(q))_{(i)}$$

where $i_0=\max\{0,q\}$ *and* $i_1=\min\{2g+r,g+r+p-q\}$.

In the special case where $q=0$ *and* $p\leq g$, *the decomposition is*

$$H_p(G,\mathbb{Q})=\bigoplus_{i=0}^{g+r+p}H_p(G,\mathbb{Q})_{(i)}$$

Remark 1.4. (a) For an abelian variety, Theorem 1.3 (1) and (2) are the same by the isomorphism (1.3). The special case in Theorem 1.3 (1) for abelian varieties is Beauville's result (Theorem 1.1).
(b) Let notation be as in Theorem 1.3. Let $q\in[0,g]$ be an integer. Beauville [2] has conjectured that for $i>2q$,

$$\mathrm{CH}^q(A;\mathbb{Q})^{(i)}=0.$$

Beauville's conjecture implies that $H_{2q-r}(G,\mathbb{Q}(q))_{(i)}=0$ for $i<2q-r$ and that $\mathrm{CH}^q(G;\mathbb{Q})^{(i)}=0$ for $i>2q$ (see Theorem 3.1 (4)).

We denote the 0-th Suslin homology of G with $\mathbb{Z}$-coefficient by $H_0^S(G,\mathbb{Z})$. Using Theorem 1.3 (2) in case that $p=q=0$ and results on Rojtman's theorem (see Theorem 4.3) by Spiess–Szamuely [22] and Geisser [13], we obtain Bloch's result for semiabelian varieties and 0-th Suslin homology:

Corollary 1.5 (Corollary 4.1). *Let G be a semiabelian variety over k, which is an extension of an abelian variety of dimension g by a torus of rank r. Let I_G be the kernel of the degree map* $\deg: H_0^S(G,\mathbb{Z})\to\mathbb{Z}$. *Then an iterated Pontryagin product I_G^{*i} is torsion for $i>g+r$.*
In particular, if k is an algebraically closed field, then $I_G^{*i}=0$ for $i>g+r$.

Remark 1.6. The subgroups $H_0(G,\mathbb{Q})_{(i)}$ have a description in terms of K-groups attached to the semiabelian variety G (see Proposition 4.8).

Remark 1.7. One can remove the assumption that a base field k is perfect in the above theorems, by a result of Cisinski-Déglise on a comparison of Beilinson motives $\mathbf{DM}_{\mathfrak{b}}$ (see [7, §14]) with Voevodsky motives: If k is not perfect, then we take the perfect closure k' of k. By [7, Proposition 2.1.9, Theorem 14.3.3], the pull-back functor

$$\mathbf{DM}_{\mathfrak{b}}(k) \to \mathbf{DM}_{\mathfrak{b}}(k')$$

is an equivalence. By [7, Theorem 16.1.4], the above categories of Beilinson motives are equivalent to $\mathbf{DM}(k,\mathbb{Q})$ and $\mathbf{DM}(k',\mathbb{Q})$ respectively.

Theorem 1.3 is deduced form a result on vanishing of motivic (co)homology (Theorem 3.1) and a description of the motive $\mathrm{M}(G)$ in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}(k,\mathbb{Q})$ in terms of symmetric products:

$$\varphi_G : \mathrm{M}(G) \xrightarrow{\simeq} \mathrm{Sym}\big(\mathrm{M}_1(G)\big)$$

which is due to Ancona–Enright-Ward–Huber [1] (see Theorem 2.5, Corollary 2.7). Here $\mathrm{M}_1(G)$ denotes the complex consisting of the homotopy invariant Nisnevich sheaf $\widetilde{G} := \mathrm{Hom}_{\mathrm{Sch}/k}(-,G)\otimes\mathbb{Q}$ with transfers concentrated in degree zero (see Definition 2.1).

There are two key ingredients in the proof of vanishing of motivic (co)homology. One is an exact triangle which relates G to a semiabelian variety of rank $\mathrm{rk}(G)-1$ (Lemma 3.9, Lemma 3.10). The exact triangle allows us to use induction on rank of G. Another is to interpret the Weil-Barsotti formula for an abelian variety in terms of motives in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}$. Using this interpretation, one can generalize Beauville's result to higher Chow groups, i.e. Theorem 1.3 for an abelian variety.

This paper is organized as follows: In Section 2, we recall the result (Theorem 2.5) on the description of the motive $\mathrm{M}(G)$ of a semiabelian variety G in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}(k,\mathbb{Q})$. We give a consequence of the description which is a decomposition of motivic (co)homology of semiabelian varieties.

In Section 3, we state and prove our main result on the vanishing of motivic (co)homology (Theorem 3.1) and a consequence of the main result (Theorem 1.3). We first prove the main result in case of a torus and an abelian variety. Lastly, we prove the general case by induction, using key triangles (Lemma 3.9, Lemma 3.10) in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}$. Lemma 3.10 is proved by applying a result on a filtration on symmetric products ([1, Proposition C.3.4]) to the category $\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q})$ of Nisnevich sheaves of $\mathbb{Q}$-modules with transfers. To apply the result, we need the exactness of the tensor product in $\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q})$. The exactness is proved in Appendix A of this paper (Proposition A.1).

In Section 4, we consider 0-th Suslin homology and prove Corollary 1.5 (Corollary 4.1). We also give a description of rational 0-th Suslin homology of a

semiabelian variety G in terms of K-groups attached to G (Proposition 4.8, Remark 4.9).
In Appendix A, we make some remarks about the tensor product in the category $\mathbf{ShT}_{\mathrm{Nis}}(k, \mathbb{Q})$. The main part is to show the exactness of the tensor product (Proposition A.1).

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Notation

1. Let k be a perfect field. For a commutative ring R, let $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}(k, R)$ denote Voevodsky's tensor triangulated category of effective motives over k with R-coefficients (cf. [25]). In case $R = \mathbb{Q}$, we simply write $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}$ for $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}(k, \mathbb{Q})$.

2. Let X be a smooth scheme of finite type over k. Let R be a commutative ring. We denote the motive of X in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}(k, R)$ by $\mathrm{M}(X)$. For any integers p and q, we denote the motivic (co)homology of X with R-coefficients by

$$H_p(X, R(q)) := \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}(k,R)}(R(q)[p], \mathrm{M}(X));$$
$$H^p(X, R(q)) := \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}(k,R)}(\mathrm{M}(X), R(q)[p]).$$

3. For an integer $i \geq 0$, we define the subgroup $H_p(G, \mathbb{Q}(q))^{(i)}$ (resp. $H^p(G, \mathbb{Q}(q))^{(i)}$) of $H_p(G, \mathbb{Q}(q))$ (resp. $H^p(G, \mathbb{Q}(q))$)) as follows:

$$H_p(G, \mathbb{Q}(q))_{(i)} := \{\alpha \in H_p(G, \mathbb{Q}(q)) \mid n_{G*}\alpha = n^i\alpha \text{ for all } n\},$$
$$H^p(G, \mathbb{Q}(q))^{(i)} := \{\alpha \in H_p(G, \mathbb{Q}(q)) \mid n_G^*\alpha = n^i\alpha \text{ for all } n\}.$$

Here n_G denotes the map $G \xrightarrow{\times n} G$ of multiplication by an integer n.

4. For a positive integer n, let Σ_n denote the group of permutations on n letters. Let $\mathcal{C}$ be a idempotent complete $\mathbb{Q}$-linear symmetric tensor category, and let M be an object of $\mathcal{C}$. Then we have a representation

$\Sigma_n \to \mathrm{End}(M^{\otimes n})$, and we let σ_M denote the endomorphism of M corresponding to $\sigma \in \Sigma_n$. We define s^n_M as follows:

$$s^n_M := \begin{cases} \mathrm{id}_{\mathbb{1}} & \text{if } n = 0, \\ \dfrac{1}{n!} \displaystyle\sum_{\sigma \in \Sigma_n} \sigma_M & \text{if } n > 0. \end{cases}$$

Here $\mathbb{1}$ denotes the unit object of $\mathcal{C}$. Then one can easily see that s^n_M is idempotent. We define the n-th symmetric product $\mathrm{Sym}^n M$ of M as follows:

$$\mathrm{Sym}^n M := \mathrm{Im}(s^n_M).$$

This image exists since $\mathcal{C}$ is idempotent complete. We denote the canonical projection $M^{\otimes n} \to \mathrm{Sym}^n M$ by s^n_M, and the canonical embedding $\mathrm{Sym}^n M \to M^{\otimes n}$ by ι^n_M.

5. Let $F \overset{f}{\hookrightarrow} G$ be a subobject in $\mathcal{C}$. We define a subobject $\mathrm{Fil}^F_i \mathrm{Sym}^n(G)$ of $\mathrm{Sym}^n(G)$ as follows: for an integer i with $0 \leq i \leq n$,

$$\mathrm{Fil}^F_i \mathrm{Sym}^n(G) := \\ \mathrm{Im}\big(s^n_G \circ (f^{\otimes i} \otimes \mathrm{id}_G^{\otimes(n-i)}) : F^{\otimes i} \otimes G^{\otimes(n-i)} \to G^{\otimes n} \to \mathrm{Sym}^n(G)\big).$$

For $i > n$, put $\mathrm{Fil}^F_i \mathrm{Sym}^n(G) = 0$. By definition, there is an inclusion map $\mathrm{Fil}^F_{i+1}\mathrm{Sym}^n(G) \to \mathrm{Fil}^F_i \mathrm{Sym}^n(G)$.

2 The motive of a semiabelian variety

2.1 The isomorphism φ_G

We recall here a recent result on the decomposition of the motive of semiabelian varieties in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}(k, \mathbb{Q})$, which is due to Enright-Ward [10]/Ancona–Enright-Ward–Huber [1].

Definition 2.1. (1) Let G be a semiabelian variety over k. We denote by $\underline{G}$ the étale sheaf $\mathrm{Hom}_{\mathrm{Sch}/k}(-, G)$ associated to G. Then the étale sheaf $\underline{G}$ has a canonical structure of an étale sheaf with transfers, and furthermore $\underline{G}$ is homotopy invariant ([22, proof of Lemma 3.2] [3, Lemma 1.3.2]).

We denote by $\widetilde{G}$ the étale sheaf of $\mathbb{Q}$-modulus with transfer attached to G, i.e., $\widetilde{G}(S) = \underline{G}(S) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Hom}_{\mathrm{Sch}/k}(S, G) \otimes_{\mathbb{Z}} \mathbb{Q}$ for any smooth scheme S.

(2) For a homotopy invariant Nisnevich sheaf with transfers F, we denote by $F[0]$ the complex consisting of F concentrated in degree one.

Following [1, 10], for a semiabelian variety G, we define

$$\mathrm{M}_1(G) := \widetilde{G}[0] \quad \text{in } \mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}.$$

Construction of φ_G: According to Spiess–Szamuely [22], we have morphism $\mathbb{Z}_{\mathrm{tr}}(G) \to \underline{G}$ of étale sheaves with transfers, and therefore there is a morphism in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}$:

$$a_G : \mathrm{M}(G) \to \mathrm{M}_1(G).$$

Using this map, we define $\varphi^i_G : \mathrm{M}(G) \to \mathrm{Sym}^i(\mathrm{M}_1(G))$ to be the composite morphism in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}$:

$$\mathrm{M}(G) \xrightarrow{\mathrm{M}(\Delta^i_G)} \mathrm{M}(G)^{\otimes i} \xrightarrow{a_G^{\otimes i}} \mathrm{M}_1(G)^{\otimes i} \xrightarrow{s^i_{\mathrm{M}_1(G)}} \mathrm{Sym}^i(\mathrm{M}_1(G)).$$

Proposition 2.2 ([1, Proposition 5.1.1],[10, Lemma 5.7.4]). *Let G be a semiabelian variety over k, which is an extension of an abelian variety of dimension g by a torus of rank r. Then $\mathrm{Sym}^i(\mathrm{M}_1(G))$ is zero in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}$ for $i > 2g + r$. In particular, $\mathrm{Sym}^{sg+r}(\mathrm{M}_1(G)) \simeq \Lambda(g+r)[2g+r]$ with a tensor-invertible Artin motive Λ. If the torus part of G is split, then $\Lambda = \mathbb{Q}$ (the unit motive).*

Proposition 2.2 for a torus is proved by Biglari [4, Proposition 2.3]. For the general case, by Lemma 3.10, we may assume that G is an abelian variety A. The statement for A is essentially deduced from the oddly finite-dimensionality for the Chow motive $h^1(A)$([17], [18]).

Let $\mathrm{Sym}(\mathrm{M}_1(G))$ denote the direct sum $\bigoplus_{i\geq 0} \mathrm{Sym}^i(\mathrm{M}_1(G))$. Here the sum is finite from Proposition 2.2. Then we define

$$\varphi_G := \bigoplus \varphi^i_G : \mathrm{M}(G) \to \mathrm{Sym}(\mathrm{M}_1(G)). \tag{2.1}$$

The motive $\mathrm{M}(G)$ has a canonical Hopf algebra structure defined by morphisms of schemes:

- the multiplication by the group law $m_G : G \times G \to G$;
- the comultiplication by the diagonal map $\Delta_G : G \to G \times G$;
- the antipodal map by the inverse on G;
- the unity by the neutral element;
- the counit by the structure map $G \to \mathrm{Spec}(k)$.

Definition 2.3. We define a Hopf algebra structure on $\mathrm{Sym}(\mathrm{M}_1(G))$ as follows:

(multiplication) For any $i, j \geq 0$,

$$\frac{(i+j)!}{i!j!} s^{i+j}_{\mathrm{M}_1(G)} \circ (\iota^i_{\mathrm{M}_1(G)} \otimes \iota^j_{\mathrm{M}_1(G)}) : \\ \mathrm{Sym}^i(\mathrm{M}_1(G)) \otimes \mathrm{Sym}^j(\mathrm{M}_1(G)) \to \mathrm{Sym}^{i+j}(\mathrm{M}_1(G));$$

(COMULTIPLICATION) For any $i, j \geq 0$,

$$(s^i_{\mathrm{M}_1(G)} \otimes s^j_{\mathrm{M}_1(G)}) \circ \iota^{i+j}_{\mathrm{M}_1(G)} : \\ \mathrm{Sym}^{i+j}(\mathrm{M}_1(G)) \to \mathrm{Sym}^i(\mathrm{M}_1(G)) \otimes \mathrm{Sym}^j(\mathrm{M}_1(G)).$$

The antipodal map, the unity and the counit are induced by the inverse map, the unit map and the structure map of $\mathrm{M}_1(G)$.

REMARK 2.4. The definition of the bialgebra structure on Sym in the above definition is not the standard one. The coefficient $\frac{(i+j)!}{i!j!}$ is usually used for the comultiplication. The reasons why we use the above definition are to fit into the classical result on motivic decomposition of Chow motive of abelian varieties (cf. [18]) and to make the map (2.1) to be an isomorphism of Hopf algebra objects in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}$. For details see [1].

THEOREM 2.5 (Ancona–Enright-Ward–Huber [1, Theorem 7.1.1]). *Let G be a semiabelian variety over a perfect field k and let $\mathrm{M}(G)$ be the motive of G in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}(k, \mathbb{Q})$. Then there exists a natural isomorphism of Hopf algebras in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}(k, \mathbb{Q})$:*

$$\varphi_G : \mathrm{M}(G) \xrightarrow{\simeq} \mathrm{Sym}\big(\mathrm{M}_1(G)\big).$$

In particular, the following diagrams commute:

$$\begin{tikzcd} \mathrm{M}(G) \otimes \mathrm{M}(G) \arrow[r, "\mathrm{M}(m_G)"] \arrow[d, "\varphi_G \otimes \varphi_G"', "\simeq"] & \mathrm{M}(G) \arrow[d, "\simeq"', "\varphi_G"] \\ \mathrm{Sym}\big(\mathrm{M}_1(G)\big) \otimes \mathrm{Sym}\big(\mathrm{M}_1(G)\big) \arrow[r] & \mathrm{Sym}\big(\mathrm{M}_1(G)\big); \end{tikzcd} \tag{2.2}$$

$$\begin{tikzcd} \mathrm{M}(G) \arrow[r, "\mathrm{M}(\Delta_G)"] \arrow[d, "\simeq"', "\varphi_G"] & \mathrm{M}(G) \otimes \mathrm{M}(G) \arrow[d, "\varphi_G \otimes \varphi_G"', "\simeq"] \\ \mathrm{Sym}\big(\mathrm{M}_1(G)\big) \arrow[r] & \mathrm{Sym}\big(\mathrm{M}_1(G)\big) \otimes \mathrm{Sym}\big(\mathrm{M}_1(G)\big). \end{tikzcd} \tag{2.3}$$

2.2 DECOMPOSITION OF MOTIVIC (CO)HOMOLOGY OF A SEMIABELIAN VARIETY

We give a decomposition of motivic (co)homology and a relationship between the decomposition and product structure, which is a consequence of Theorem 2.5. We first introduce product on motivic (co)homology.

DEFINITION 2.6. (1) Let a and b be elements in $H_p(G, \mathbb{Q}(q))$ and $H_{p'}(G, \mathbb{Q}(q'))$ respectively. Then we define *the Pontryagin product* $a * b$ of a and b to be the image of (a, b) under the morphism

$$H_p(G, \mathbb{Q}(q)) \times H_{p'}(G, \mathbb{Q}(q')) \to H_{p+p'}(G \times G, \mathbb{Q}(q + q')) \\ \overset{m_{G*}}{\to} H_{p+p'}(G, \mathbb{Q}(q + q')). \tag{2.4}$$

Here the first map is induced by the tensor structure in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}$. For any subgroups $F \subset H_p(G, \mathbb{Q}(q))$ and $G \subset H_{p'}(G, \mathbb{Q}(q'))$, we write $F * G$ for the subgroup generated by the image of $F \times G$ under the morphism (2.4).

(2) For $a \in H^p(G, \mathbb{Q}(q))$ and $b \in H^{p'}(G, \mathbb{Q}(q'))$, we define *the cup product* $a \cup b$ to be the image of (a, b) under the morphism

$$\begin{aligned} H^p(G, \mathbb{Q}(q)) \times H^{p'}(G, \mathbb{Q}(q')) \to & H^{p+p'}(G \times G, \mathbb{Q}(q+q')) \\ & \overset{\Delta_G^*}{\to} H^{p+p'}(G, \mathbb{Q}(q+q')). \end{aligned} \tag{2.5}$$

For any subgroups $F \subset H^p(G, \mathbb{Q}(q))$ and $G \subset H^{p'}(G, \mathbb{Q}(q'))$, we write $F \cup G$ for the subgroup generated by the image of $F \times G$ under the morphism (2.5).

Corollary 2.7. *Let G be a semiabelian variety over k, which is an extension of an abelian variety of dimension g by a torus of rank r. Let p, q be integers. Then we have the following natural decomposition of motivic (co)homology of G:*

$$H_p(G, \mathbb{Q}(q)) = \bigoplus_{i=0}^{2g+r} H_p(G, \mathbb{Q}(q))_{(i)}, \quad H^p(G, \mathbb{Q}(q)) = \bigoplus_{i=0}^{2g+r} H^p(G, \mathbb{Q}(q))^{(i)}$$

which satisfies

$$\begin{aligned} H_p(G, \mathbb{Q}(q))_{(i)} * H_{p'}(G, \mathbb{Q}(q'))_{(i')} &\subset H_{p+p'}(G, \mathbb{Q}(q+q'))_{(i+i')}, \\ H^p(G, \mathbb{Q}(q))^{(i)} \cup H^{p'}(G, \mathbb{Q}(q'))^{(i')} &\subset H^{p+p'}(G, \mathbb{Q}(q+q'))^{(i+i')}. \end{aligned}$$

Proof. The map n_G on $\mathrm{Sym}^i(\mathrm{M}_1(G))$ is $n^i \cdot \mathrm{id}_{\mathrm{Sym}^i(\mathrm{M}_1(G))}$. Thus we have

$$\begin{aligned} H_p(G, \mathbb{Q}(q))_{(i)} &= \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}}(\mathbb{Q}(q)[p], \mathrm{Sym}^i(\mathrm{M}_1(G))), \\ H^p(G, \mathbb{Q}(q))^{(i)} &= \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}}(\mathrm{Sym}^i(\mathrm{M}_1(G)), \mathbb{Q}(q)[p]). \end{aligned}$$

The assertion follows from this description and Theorem 2.5. $\square$

3 Main result

In this section, we state the main result on the vanishing of motivic (co)homology of a semiabelian variety and its consequences.

Theorem 3.1. *Let G be a semiabelian variety over k, which is an extension of an abelian variety of dimension g by a torus of rank r. Let p, q be integers. Then*

(1) *$H_p(G, \mathbb{Q}(q))$ vanishes in the following cases*

  (a) *$p < q$;*

(b) $g + r < q$;

(c) $p - 2q + r < 0$;

(d) $q = g + r$ *and* $p - 2q + r \geq 1$;

(e) $q = g + r - 1$ *and* $p - 2q + r \geq 2$.

(2) *Let* $i_0 = \max\{0, q\}$ *and* $i_1 = \min\{2g+r, g+r+p-q\}$. *Then* $H_p(G, \mathbb{Q}(q))_{(i)}$ *vanishes for* $i \notin [i_0, i_1]$.

(3) *Let* $s \geq 0$ *be an integer. Let* $j_0 = \max\{0, q-s\}$ *and* $j_1 = \min\{2g+r, g+q\}$. *Then* $\mathrm{CH}^q(G, s; \mathbb{Q})^{(i)}$ *vanishes for* $i \notin [j_0, j_1]$.

(4) *Beauville's conjecture implies that* $H_{2q-r}(G, \mathbb{Q}(q))_{(i)} = 0$ *for* $i < 2q - r$ *and that* $\mathrm{CH}^q(G; \mathbb{Q})^{(i)} = 0$ *for* $i > 2q$.

REMARK 3.2. By the definition of higher Chow groups and a computation of codimension one cycles (see Bloch [6]), we know the following result on vanishing of higher Chow groups:

THEOREM 3.3. *For any smooth scheme* X *of dimension* d *over a field* k *and for any abelian group* R, *we have* $\mathrm{CH}^i(X, s; R) = 0$ *in the cases*:
(1) $s < 0$; (2) $i \notin [0, d+s]$; (3) $i = 0$ *and* $s \geq 1$; (4) $i = 1$ *and* $s \geq 2$.

REMARK 3.4. For an abelian variety A of dimension g, by Poincaré duality ([11, 25]), we have

$$\mathrm{M}(A) \simeq \mathrm{M}(A)^*(g)[2g] \quad \text{in } \mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}.$$

Here $*$ denotes the dual in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}$. Thus we have an isomorphism (cf. (1.3))

$$H_p(A, \mathbb{Q}(q)) \simeq H^{2g-p}(A, \mathbb{Q}(g-q)) \simeq \mathrm{CH}^{g-q}(A, p-2q; \mathbb{Q}).$$

The last isomorphism is the isomorphism (1.2) in the introduction. Furthermore, for $0 \leq i \leq 2g$, the isomorphism induces an isomorphism

$$H_p(A, \mathbb{Q}(q))_{(i)} \simeq H^{2g-p}(A, \mathbb{Q}(g-q))^{(2g-i)} \simeq \mathrm{CH}^{g-q}(A, p-2q; \mathbb{Q})^{(2g-i)}.$$

Therefore Theorem 3.1 (1) is the same as Theorem 3.3, and Theorem 3.1(2) and (3) are the same.

The following corollary immediately follows from Corollary 2.7 and Theorem 3.1.

COROLLARY 3.5 (Theorem 1.3). *Let notation be as in Theorem 3.1. Then the decompositions in Corollary 2.7 are*

$$H_p(G, \mathbb{Q}(q)) = \bigoplus_{i=i_0}^{i_1} H_p(G, \mathbb{Q}(q))_{(i)}, \quad \mathrm{CH}^q(G, s; \mathbb{Q}) = \bigoplus_{i=j_0}^{j_1} \mathrm{CH}^q(G, s; \mathbb{Q})^{(i)}.$$

3.1 Proof of main result

Let G be a semiabelian variety over k, which is an extension of an abelian variety of dimension g by a torus of rank r.

3.1.1 Reduction to the case of an algebraically closed base field

Let L/k be a finite extension. Let G_L denote the scalar extension $G \otimes_k L$ and let $f_{L/k} : G_L \to G$ be the projection. Then the composition of the following homomorphisms is the multiplication by the degree $[L : k]$:

$$H_p(G, \mathbb{Q}(q)) \xrightarrow{f^*_{L/k}} H_p(G_L, \mathbb{Q}(q)) \xrightarrow{f_{L/k*}} H_p(G, \mathbb{Q}(q)).$$

Hence we obtain that the push- forward map ${f_{L/k}}^*$ is injective, and that there is a natural injection

$$H_p(G, \mathbb{Q}(q)) \hookrightarrow \operatorname*{colim}_{L/k} H_p(G_L, \mathbb{Q}(q)).$$

By a result of Ivorra [14, Proposition 4.16], we have a bijection

$$\operatorname*{colim}_{L/k} H_p(G_L, \mathbb{Q}(q)) \simeq H_p(G_{\bar{k}}, \mathbb{Q}(q)).$$

Here $\bar{k}$ denotes an algebraic closure of k. Thus we have an injection

$$H_p(G, \mathbb{Q}(q)) \hookrightarrow H_p(G_{\bar{k}}, \mathbb{Q}(q)).$$

Note that a similar assertion holds for $H_p(G, \mathbb{Q}(q))_{(i)}$, $\mathrm{CH}^q(G, s; \mathbb{Q})$ and $\mathrm{CH}^q(G, s; \mathbb{Q})^{(i)}$. Therefore we may assume that k is algebraic closed.

3.1.2 The case of a torus

Let $G = T$ be a torus of rank r. By the above argument, we may assume that $T = \mathbb{G}_m^r$. In this case, we know that

$$\mathrm{M}(\mathbb{G}_m^r) \simeq \bigoplus_{i=0}^{r} \big(\mathbb{Q}(i)[i]\big)^{\oplus c_i},$$

where $c_i = \binom{r}{i}$ denotes the binomial coefficient. Thus, by isomorphism 2.1, we obtain that for $0 \le i \le r$,

$$H_p(T, \mathbb{Q}(q))_{(i)} \simeq \mathrm{CH}^{i-q}(k, i + p - 2q; \mathbb{Q})^{\oplus c_i}. \tag{3.1}$$

From this equation and Theorem 3.3, one can easily obtain the following proposition which induces Theorem 3.1 for a torus.

Proposition 3.6. *Let T be the torus $\mathbb{G}_m^r$ of rank r. Let p, q be integers. Let i be an integer with $0 \le i \le r$. Then $H_p(T, \mathbb{Q}(q))_{(i)}$ vanishes in the following cases*: (1) $q > p$; (2) $q > i$; (3) $i < 2q - p$; (4) $q = i$ *and* $p - 2q + i \ge 1$; (5) $q = i - 1$ *and* $p - 2q + i \ge 2$.

3.1.3 The case of abelian varieties

Let G be an abelian variety A of dimension g over an algebraically closed field k. By Remark 3.4, it enough to prove Theorem 3.1 (2).

Lemma 3.7. *Let $\hat{A}$ be the dual abelian variety of A. There is an isomorphism in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}$*

$$\mathrm{M}_1(\hat{A}) \simeq (\mathrm{M}_1(A))^*(1)[2].$$

Here $*$ denotes the dual in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}$.

Proof. By the Weil-Barsotti formula, we have an isomorphism

$$\hat{A} \simeq \underline{\mathrm{Ext}}^1(A, \mathbb{G}_m).$$

The assertion follows from a result of Barbieri-Viale–Kahn ([3, §4]) on a comparison between Cartier dual of 1-motives and motivic dual of corresponding motives. □

Remark 3.8. Lemma 3.7 also follows from the Weil-Barsotti formula in Chow motives and Voevodsky's functor ψ from the category of non-effective Chow motives to the category $\mathbf{DM}_{\mathrm{gm}}$ of geometric motives deduced from [25, Proposition 2.1.4]. We used here the fact that the functor commutes with duality as a tensor functor, and Proposition 4.3.3 in [1] that ψ sends the Chow motive $h^1(A)$ to $\mathrm{M}_1(A)$.

Proof of Theorem 3.1 (2). By Lemma 3.7 and replacing $(A, \hat{A})$ by $(\hat{A}, A)$, for an integer $i \geq 0$, we have an isomorphism

$$\mathrm{Sym}^i(\mathrm{M}_1(A)) \simeq \left(\mathrm{Sym}^i(\mathrm{M}_1(\hat{A}))\right)^*(i)[2i].$$

¿From this isomorphism, we have

$$\begin{aligned} H_p(A, \mathbb{Q}(q))_{(i)} &= \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}}(\mathbb{Q}(q)[p], \mathrm{Sym}^i(\mathrm{M}_1(A))) \\ &= \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}}(\mathbb{Q}(q)[p], (\mathrm{Sym}^i(\mathrm{M}_1(\hat{A})))^*(i)[2i]) \\ &= \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}}(\mathrm{Sym}^i(\mathrm{M}_1(A)), \mathbb{Q}(i-q)[2i-p]) \\ &= \mathrm{CH}^{i-q}(\hat{A}, p-2q; \mathbb{Q})^{(i)} \\ &\subset \mathrm{CH}^{i-q}(\hat{A}, p-2q; \mathbb{Q}). \end{aligned}$$

¿From Theorem 3.3, $\mathrm{CH}^{i-q}(\hat{A}, p-2q; \mathbb{Q}) = 0$ for $i - q \notin [0, g+p-2q]$. Thus $H_p(A, \mathbb{Q}(q))_{(i)} = 0$ for $i \notin [i_0, i_1]$, since $\mathrm{Sym}^i(\mathrm{M}_1(A)) = 0$ for $i \notin [0, 2g]$ by Proposition 2.2. □

3.1.4 The general case

We prove Theorem 3.1 by induction on rank of a semiabelian variety G. Following [10], we first give some triangles in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}$ which allow us to use induction on the rank of a semiabelian variety.
We first give an exact sequence of smooth group schemes. Let G be a semiabelian variety of rank r over an algebraically closed field k. Then there is an exact sequence of smooth group schemes over k of the following form:

$$1 \to \mathbb{G}_m^r \xrightarrow{i} G \xrightarrow{f} A \to 1.$$

Then, following [10, Section 5.2], we consider the cokernel H of the composite map

$$\mathbb{G}_m \to \mathbb{G}_m^r \xrightarrow{i} G,$$

where the first map is the inclusion to the first factor of $\mathbb{G}_m^r$. Then H is a semiabelian variety of rank $r-1$ over k which fits into

$$1 \to \mathbb{G}_m \to G \to H \to 1, \tag{3.2}$$
$$1 \to \mathbb{G}_m^{r-1} \xrightarrow{i} H \xrightarrow{\bar{f}} A \to 1.$$

Lemma 3.9 ([1, 10]). *Let notation as above. Then we have the following exact triangles in* $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}$:

$$\mathrm{M}(H)(1)[1] \to \mathrm{M}(G) \to \mathrm{M}(H) \to \mathrm{M}(H)(1)[2].$$

Proof. From (3.2), we may regard G as a $\mathbb{G}_m$-torsor on H. Let E be the line bundle over H associated to the $\mathbb{G}_m$-torsor G. Let $s : H \to E$ be the zero section. By the Gysin triangle attached to E and $s(H)$, we have an exact triangle in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}$

$$\mathrm{M}(E - s(H)) \to \mathrm{M}(E) \to \mathrm{M}(s(H))(1)[2] \to \mathrm{M}(E - s(H))[1]$$

Since $E - s(H)$ is isomorphic to G and $\mathrm{M}(E) \simeq \mathrm{M}(H) \simeq \mathrm{M}(s(H))$, we get the desired triangle after shifting. $\square$

Lemma 3.10 ([1, 10]). *Let notation as above. Then we have the following exact triangles in* $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}$:

$$\begin{aligned}\mathrm{Sym}^{n-1}(\mathrm{M}_1(H))(1)[1] &\to \mathrm{Sym}^n(\mathrm{M}_1(G)) \\ &\to \mathrm{Sym}^n(\mathrm{M}_1(H)) \to \mathrm{Sym}^{n-1}(\mathrm{M}_1(H))(1)[2].\end{aligned}$$

Proof. By Corollary A.3, there is an isomorphism in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}$

$$\mathrm{Sym}^n(\mathrm{M}_1(G)) \simeq \mathrm{Sym}^n(\widetilde{G})[0]. \tag{3.3}$$

For the definition of $\widetilde{G}$ and a complex $F[0]$, see Definition 2.1. By the isomorphism (3.3), we may work in the category $\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q})$ of Nisnevich sheaves of $\mathbb{Q}$-modules with transfers. We have an exact sequence in $\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q})$:

$$1 \to \widetilde{\mathbb{G}}_m \to \widetilde{G} \to \widetilde{H} \to 1.$$

Proposition A.1 allows us to apply [1, Proposition C.3.4] to the above exact sequence in $\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q})$. Then we have exact triangles for $i \geq 0$,

$$\begin{aligned}\mathrm{Fil}^{\widetilde{\mathbb{G}}_m}_{i+1}\mathrm{Sym}^n(\widetilde{G})[0] &\to \mathrm{Fil}^{\widetilde{\mathbb{G}}_m}_{i}\mathrm{Sym}^n(\widetilde{G})[0]\\ &\to \mathrm{Sym}^i(\mathrm{M}_1(\mathbb{G}_m)) \otimes \mathrm{Sym}^{n-i}(\mathrm{M}_1(H)) \to \mathrm{Fil}^{\widetilde{\mathbb{G}}_m}_{i+1}\mathrm{Sym}^n(\widetilde{G})[1].\end{aligned}$$

For the definition of $\mathrm{Fil}^{\widetilde{\mathbb{G}}_m}_{i}\mathrm{Sym}^n(\widetilde{G})$, see Notation. We know that $\mathrm{M}_1(\mathbb{G}_m) \simeq \widetilde{\mathbb{G}}_m[0] \simeq \mathbb{Q}(1)[1]$ in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}$. Therefore, by [4, Proposition 2.3], $\mathrm{Sym}^n(\mathrm{M}_1(\mathbb{G}_m)) \simeq \mathrm{Alt}^n(\mathbb{Q})(n)[n] = 0$ for $n \geq 2$ in $\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}$. Thus we have

$$\mathrm{Fil}^{\widetilde{\mathbb{G}}_m}_{i}\mathrm{Sym}^n(\widetilde{G})[0] \simeq \begin{cases}\mathrm{Sym}^n(\mathrm{M}_1(G)) & \text{if } i=0,\\ \mathrm{Sym}^{n-1}(\mathrm{M}_1(H))(1)[1] & \text{if } i=1,\\ 0 & \text{otherwise.}\end{cases}$$

Hence we have the desired triangle. □

Proof of Theorem 3.1. By Lemma 3.9, we have an exact sequence

$$\cdots \to H_{p-1}(H,\mathbb{Q}(q-1)) \to H_p(G,\mathbb{Q}(q)) \to H_p(H,\mathbb{Q}(q)) \to \cdots.$$

By Lemma 3.10, we also have exact sequences

$$\cdots \to H_{p-1}(H,\mathbb{Q}(q-1))_{(i-1)} \to H_p(G,\mathbb{Q}(q))_{(i)} \to H_p(H,\mathbb{Q}(q))_{(i)} \to \cdots,$$

$$\begin{aligned}\cdots \leftarrow \mathrm{CH}^{q-1}(H,2q-p-1;\mathbb{Q})^{(i-1)} &\leftarrow \mathrm{CH}^q(G,2q-p;\mathbb{Q})^{(i)}\\ &\leftarrow \mathrm{CH}^q(H,2q-p;\mathbb{Q})^{(i)} \leftarrow \cdots.\end{aligned}$$

By these exact sequences, the assertion for a semiabelian variety G follows from the induction hypothesis and the assertion for abelian varieties. □

4 0-th Suslin homology

We consider here 0-th Suslin homology $H^S_0(G,\mathbb{Z})$ of a semiabelian variety G, and prove Corollary 1.5 (Corollary 4.1). We also give a description of rational 0-th Suslin homology $H^S_0(G,\mathbb{Q})$ in terms of K-groups (Proposition 4.8), using a result of Kahn–Yamazaki [16] (see Theorem 4.5).

4.1 Bloch's result for a semiabelian variety

Corollary 4.1 (Corollary 1.5). *Let G be a semiabelian variety over k, which is an extension of an abelian variety of dimension g by a torus of rank r. Let I_G be the kernel of the degree map $H_0^S(G,\mathbb{Z}) \to \mathbb{Z}$. Then an iterated Pontryagin product I_G^{*i} is torsion for $i > g+r$.*
In particular, if k is an algebraically closed field, then $I_G^{*i} = 0$ for $i > g+r$.

Proof. ¿From Theorem 3.5, we have

$$I_G \otimes \mathbb{Q} = \bigoplus_{n=1}^{g+r} H_0(G,\mathbb{Q})_{(n)}.$$

By compatibility of this decomposition with Pontryagin product (Corollary 2.7),

$$I_G^{*i} \otimes \mathbb{Q} \subset \bigoplus_{n=i}^{g+r} H_0(G,\mathbb{Q})_{(n)}.$$

Thus by Theorem 3.1 (2), we obtain that $I_G^{*i} \otimes \mathbb{Q} = 0$ for $i > g+r$.
For second assertion, we assume that k is algebraically closed. Then I_G is generated by cycles of the form $[a] - [0_G]$, where $a, 0_G \in G(k)$ and 0_G is the identity element of G. Thus I_G^{*2} is generated by cycles of the form

$$[a+b] - [a] - [b] - [0_G]$$

for $a, b \in G(k)$. Let alb_G be the albanese map from $I_G \to G(k)$. Then it is easily seen that

$$\mathrm{alb}_G(I_G^{*2}) = 0_G.$$

Since $I_G^{*i} \subset I_G^{*2}$ for $i \geq 2$, we have

$$\mathrm{alb}_G(I_G^{*i}) = 0 \quad \text{for } i \geq 2. \tag{4.1}$$

We claim that G has a smooth compactification. The following argument is attributed by J.-L. Colliot-Thélène and M. Brion (see also [15, p. 13]): Let $0 \to T \to G \to A \to 0$ be the canonical decomposition of G. Then we may assume $T = \mathbb{G}_m^r$ since k is algebraically closed. Let $Y := (\mathbb{P}^1)^r$ be a smooth compactification of T. Then the bundle $G \times^T Y$ associated to the T-torsor $G \to A$ exists, which is a smooth compactification of G. Now the second assertion follows from the first assertion, (4.1) and Theorem 4.3 below. $\square$

Remark 4.2. A smooth comapctification of a semiabelian variety over any field exists, since the case of a torus is shown in [8] and the argument in the proof of Corollary 4.1 works.

Theorem 4.3. *Let X be a smooth, quasi-projective scheme of finite type over an algebraically closed field k of characteristic $p \geq 0$. Assume that X has a smooth compactification. Then the albanese map*

$$\mathrm{alb}_X : H_0^S(X,\mathbb{Z})^0 \to Alb_X(k).$$

from the degree-0-part of Suslin homology to the k-valued points of the Albanese variety Alb_X of X induces an isomorphism on torsion groups.

Remark 4.4. For an open variety of a smooth projective variety, the theorem is proved by Spieß–Szamuely [22, Theorem 1.1] (prime-to-p-part) and Geisser [13, Theorem 1.1] (p-part).

4.2 A description of rational Suslin homology

We first briefly recall the definition of K-groups attached to semiabelian varieties. Let $F_1, \dots, F_r$ be homotopy invariant Nisnevich sheaves with transfers. We then define the K-group $K(k; F_1 \dots, F_r)$ to be the quotient group

$$\Big(\bigoplus_{L/k:\text{finite}} F_1(L) \otimes \cdots \otimes F_r(L) \Big) / R \tag{4.2}$$

where R is a subgroup whose elements corresponds to Projection formula and Weil reciprocity. For example, if $F_i = \underline{\mathbb{G}_m}$ (cf. Definition 2.1 (1)) for $i = 1, \dots, r$, then we have the r-th Milnor K-group

$$K(k; \underline{\mathbb{G}_m}, \dots, \underline{\mathbb{G}_m}) = K_r^M(k).$$

For the precise definition, see [21, 16].

Theorem 4.5 (Kahn–Yamazaki [16]). *Let $F_1 \dots, F_r$ be homotopy invariant Nisnevich sheaves with transfers . Then there is an isomorphism*

$$K(k; F_1 \dots, F_r) \simeq \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}(k,\mathbb{Z})}\big(\mathbb{Z}, F_1[0] \otimes \cdots \otimes F_r[0]\big).$$

Definition 4.6. (1) For a semiabelian variety G over a perfect field k, let $K_i(k; G)_{\mathbb{Q}}$ denotes the K-group $K(k; \underline{G}, \dots, \underline{G}) \otimes \mathbb{Q}$ attached to i copies of the homotopy invariant Nisnevich sheaf with transfers $\underline{G}$ attached to G (cf. Definition 2.1 (1)).
(2) For $a_i \in \underline{G}(L)$, let $\{a_1, \dots, a_i\}_{L/k}$ denote the element of $K_i(k; G)_{\mathbb{Q}}$ represented by $a_1 \otimes \cdots \otimes a_i$. We define an action of the permutation group Σ_i on $K_i(k; G)$ as follows: for $\sigma \in \Sigma_i$ and $a_i \in \underline{G}(L)$,

$$\sigma(\{a_1, \dots, a_i\}_{L/k}) = \{a_{\sigma(1)}, \dots, a_{\sigma(i)}\}_{L/k}.$$

Then we define $S_i(k; G)_{\mathbb{Q}}$ to be the image of an idempotent map $s^i := \dfrac{1}{i!} \displaystyle\sum_{\sigma \in \Sigma_i} \sigma$:

$$S_i(k; G)_{\mathbb{Q}} := \mathrm{Im}(s^i : K_i(k; G)_{\mathbb{Q}} \to K_i(k; G)_{\mathbb{Q}}).$$

Example 4.7. For $G = \mathbb{G}_m$ and an integer $i \geq 0$, we have

$$K_i(k; G) \simeq K_i^M(k).$$

Here $K_i^M(k)$ is the i-th Milnor K-group of k.

Proposition 4.8 (Gazaki [12](for an abelian variety)). *Let G be a semiabelian variety over k, which is an extension of an abelian variety by a torus of rank r. Let $H_0^S(G,\mathbb{Q})_{(i)}$ denote $H_0(G,\mathbb{Q}(0))_{(i)}$. Then for any $0 \le i \le r$, we have*

$$H_0^S(G,\mathbb{Q})_{(i)} \simeq S_i(k;G)_{\mathbb{Q}}.$$

In particular,

$$H_0^S(G,\mathbb{Q}) \simeq \bigoplus_{i=0}^{g+r} S_i(k;G)_{\mathbb{Q}}$$

Proof. ¿From Theorem 4.5, we have the following commutative diagram

$$\begin{tikzcd} K_i(k;G)_{\mathbb{Q}} \arrow[r, equal] \arrow[d, "s^i"] & \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}}(\mathbb{Q}, \mathrm{M}_1(G)^{\otimes i}) \arrow[r, equal] \arrow[d, "s^i_{\mathrm{M}_1(G)}"] & \big((\widetilde{G})^{\otimes^i_{\mathrm{HI}}}\big)(k) \arrow[d, "s^i_{\widetilde{G}}"] \\ K_i(k;G)_{\mathbb{Q}} \arrow[r, equal] & \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}}(\mathbb{Q}, \mathrm{M}_1(G)^{\otimes i}) \arrow[r, equal] & \big((\widetilde{G})^{\otimes^i_{\mathrm{HI}}}\big)(k). \end{tikzcd}$$

Here $\widetilde{G}$ denotes the Nisnevich sheaf of $\mathbb{Q}$-modules with transfers attached to G (Definition 2.1 (1)). Since $H_0^S(G,\mathbb{Q})^{(i)} = \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-,\mathrm{Nis}}}(\mathbb{Q}, \mathrm{Sym}^i(\mathrm{M}_1(G)))$, the assertion follows from this diagram. □

Remark 4.9. Let notation be as in Propostion 4.8. Let $F^nH_0^S(G)$ denote $\bigoplus_{i=n}^{g+r} S_i(k;G)_{\mathbb{Q}}$. Then the filtration $F^\bullet H_0^S(G)$ on $H_0^S(G,\mathbb{Q})$ satisfies the following:

(a) $F^1H_0^S(G) = \mathrm{Ker}(\deg : H_0^S(G) \to \mathbb{Q}) = I_{G,\mathbb{Q}}$;

(b) $F^2H_0^S(G) = \mathrm{Ker}\big(\mathrm{alb}_{G/k} : F^1H_0^S(G) \to G(k) \otimes \mathbb{Q}\big)$. This map is induced by the albanese map $G \to Alb_G$;

(c) $F^nH_0^S(G) * F^mH_0^S(G) \subset F^{n+m}H_0^S(G)$. Here $*$ denote the Pontryagin product;

(d) $(F^1H_0^S(G))^{*n} = 0$ for $n > g+r$ (see Corollary 4.1).

Bloch [5] studied a filtration on the Chow group $\mathrm{CH}_0(A)$ of an abelian variety over an algebraically closed field, which is defined by iterated Pontryagin product I_A^{*r} of the kernel I_A of the degree map $\mathrm{CH}_0(A) \to \mathbb{Z}$. A similar filtration on $\mathrm{CH}_0(A)$ for an abelian variety over a field is studied by Gazaki [12], using K-groups attached to A.

A Remarks on tensor product on $\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q})$

We make here two remarks about the tensor product $\otimes_{\mathrm{ShT}}$ on $\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q})$ One is that the tensor product is exact (Proposition A.1). A key of a proof of the exactness are results of Suslin–Voevodsky [23, 24] and Cisinski–Déglise

[7]. Another is to give an example of étale sheaves with transfers for which étale sheafification of presheaf tensor product is not isomorphic to the tensor product $\otimes_{\mathrm{ShT}}$.

Let us fix notation:

- $\mathcal{S}\mathbf{ch}_k$: the category of separated schemes of finite type over k;
- $\mathcal{S}\mathbf{m}_k$: the subcategory of $\mathcal{S}\mathbf{ch}_k$ of smooth k-schemes;
- $\mathcal{S}\mathbf{ch}_k^{\mathrm{cor}}$: the category of separated schemes of finite type over k with morphism finite correspondences (which are given by universally integral relative cycles ([24]));
- $\mathcal{S}\mathbf{m}_k^{\mathrm{cor}}$: the subcategory of $\mathcal{S}\mathbf{ch}_k^{\mathrm{cor}}$ of smooth k-schemes;
- $\mathbf{PST}(k,\mathbb{Q})$: the category of presheaves of $\mathbb{Q}$-modules on $\mathcal{S}\mathbf{m}_k^{\mathrm{cor}}$;
- $\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q})$: the category of Nisnevich sheaves of $\mathbb{Q}$-modules on $\mathcal{S}\mathbf{m}_k^{\mathrm{cor}}$;
- $\underline{\mathbf{Sh}}_{\mathrm{Nis}}(k,\mathbb{Q})$: the category of Nisnevich sheaves of $\mathbb{Q}$-modules on $\mathcal{S}\mathbf{ch}_k$;
- $\underline{\mathbf{ShT}}_{\mathrm{Nis}}(k,\mathbb{Q})$: the category of Nisnevich sheaves of $\mathbb{Q}$-modules on $\mathcal{S}\mathbf{ch}_k^{\mathrm{cor}}$;
- $\underline{\mathbf{Sh}}_{\mathrm{qfh}}(k,\mathbb{Q})$: the category of qfh-sheaves of $\mathbb{Q}$-modules on $\mathcal{S}\mathbf{ch}_k$.

We call an object of $\mathbf{PST}(k,\mathbb{Q})$ (resp. $\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q})$) a presheaf (resp. Nisnevich) with transfers. We call an object of $\underline{\mathbf{ShT}}_{\mathrm{Nis}}(k,\mathbb{Q})$ a generalized Nisnevich sheaf with transfers.

A.1 Exactness of the tensor product on $\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q})$

We recall the tensor product on **PST**: For $X \in \mathcal{S}\mathbf{m}_k$, $L(X)$ denotes the representable presheaf with transfers. We first define the tensor product $L(X)\otimes_{\mathrm{PST}} L(Y)$ as

$$L(X)\otimes_{\mathrm{PST}} L(Y) := L(X\times Y).$$

Let F be presheaf with transfers. We have a canonical projective resolution $\mathcal{L}(F)\to F$ of F of the following form

$$\cdots \to \bigoplus_j L(Y_j) \to \bigoplus_i L(X_i) \to F \to 0$$

Then for presheaves with transfers F and G, the tensor product $F\otimes_{\mathrm{PST}} G$ of F and G is defined to be

$$H_0\big(\mathrm{Tot}(\mathcal{L}(F)\otimes_{\mathrm{PST}}\mathcal{L}(G))\big).$$

For Nisnevich sheaves with transfers F and G, the tensor product $F\otimes_{\mathrm{ShT}} G$ is given by

$$F\otimes_{\mathrm{ShT}} G = \big(F\otimes_{\mathrm{PST}} G\big)_{\mathrm{Nis}}.$$

PROPOSITION A.1. *The bifunctor* $\otimes_{\mathrm{ShT}}$ *on* $\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q})$ *is exact.*

REMARK A.2. (1) The tensor product on $\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Z})$ is right exact by construction.
(2) If F is a presheaf of $\mathbb{Q}$-modules with transfers, then $F_{\mathrm{Nis}} = F_{\text{ét}}$ ([19, Corollary 14.22] [20, Ex. 1.21]). Therefore, Proposition A.1 holds for $\mathbf{ShT}_{\text{ét}}(k,\mathbb{Q})$.

¿From Proposition A.1 and the definition of tensor product on the derived category $D^-(\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q}))$ of $\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q})$, we obtain the following:

COROLLARY A.3. *Let F and G be Nisnevich sheaves of $\mathbb{Q}$-modules with transfers. Let $F[0]$ denote the complex consisting of F concentrated in degree zero. Then*
(1) $F[0] \otimes G[0]$ *is isomorphic to* $(F \otimes_{\mathrm{ShT}} G)[0]$ *in* $D^-(\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q}))$.
(2) $\mathrm{Sym}^i(F[0])$ *is isomorphic to* $\mathrm{Sym}^i(F)[0]$ *in* $D^-(\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q}))$.

We recall results of Suslin–Voevodsky [23, Corollary 6.6, Theorem 6.7][24, Theorem 4.2.12] (cf. Cisinski–Déglise [7, Theorem 10.5.5]) to prove Proposition A.1.

THEOREM A.4. *Let $X \in \mathcal{S}\mathbf{ch}_k$ be a separated scheme of finite type over k. Let $\underline{L}(X)_{\mathbb{Q}} \in \underline{\mathbf{ShT}}_{\mathrm{Nis}}(k,\mathbb{Q})$ denote the generalized Nisnevich sheaf with transfers represented by X. Then $\underline{L}(X)_{\mathbb{Q}}$ is a qfh-sheaf.*
Furthermore, let $\mathbb{Q}_{\mathrm{qfh}}(X) \in \underline{\mathbf{Sh}}_{\mathrm{qfh}}(k,\mathbb{Q})$ denote the qfh-sheaf represented by X. Then $\underline{L}(X)_{\mathbb{Q}}$ is isomorphic to the qfh-sheaf $\mathbb{Q}_{\mathrm{qfh}}(X)$.

By this theorem, for any qfh-sheaf $F \in \underline{\mathbf{Sh}}_{\mathrm{qfh}}(k,\mathbb{Q})$, Cisinski–Déglise [7, §10] defined a generalized Nisnevich sheaf with transfers $\rho(F)$ as follows: for any $X \in \mathcal{S}\mathbf{ch}_k$,
$$\rho(F)(X) := \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{qfh}}(k,\mathbb{Q})}(\underline{L}(X)_{\mathbb{Q}}, F).$$
Thus we have a functor
$$\rho : \underline{\mathbf{Sh}}_{\mathrm{qfh}}(k,\mathbb{Q}) \longrightarrow \underline{\mathbf{ShT}}_{\mathrm{Nis}}(k,\mathbb{Q}).$$
Now we have a functor
$$\psi : \underline{\mathbf{Sh}}_{\mathrm{qfh}}(k,\mathbb{Q}) \xrightarrow{\rho} \underline{\mathbf{ShT}}_{\mathrm{Nis}}(k,\mathbb{Q}) \xrightarrow{\iota^*} \mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q}).$$
where the second functor is the pull-back with respect to $\iota : \mathcal{S}\mathbf{m}_k^{\mathrm{cor}} \to \mathcal{S}\mathbf{ch}_k^{\mathrm{cor}}$. By [7, Lemma 10.4.6], the functor ι^* admits a left adjoint $\iota_!$ such that $\iota_!(L(X)) = \underline{L}(X)$ for any $X \in \mathcal{S}\mathbf{m}_k$. Thus we also have a functor
$$\phi : \mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q}) \xrightarrow{\iota_!} \underline{\mathbf{ShT}}_{\mathrm{Nis}}(k,\mathbb{Q}) \to \underline{\mathbf{Sh}}_{\mathrm{Nis}}(k,\mathbb{Q}) \longrightarrow \underline{\mathbf{Sh}}_{\mathrm{qfh}}(k,\mathbb{Q})$$
where the second one is the forgetful functor and the third one is the qfh-sheafification. Cisinski-Déglise showed the following properties of the functors ψ, ϕ:

Proposition A.5 ([7, Proposition 10.5.14]). *The following holds:*

(1) *For any smooth k- scheme X, $\psi(\mathbb{Q}_{\mathrm{qfh}}(X)) \simeq L(X)_{\mathbb{Q}}$;*

(2) *The functor ψ admits the left adjoint ϕ;*

(3) *For any smooth k-scheme X, $\phi(L(X)_{\mathbb{Q}}) \simeq \mathbb{Q}_{\mathrm{qfh}}(X)$;*

(4) *The functor ψ is exact and preserves colimits*;

(5) *The functor ϕ is exact, fully faithful and preserves colimits.*

Proof of Proposition A.1. Since the tensor product on $\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q})$ is right exact (cf. Remark A.2), it suffices to show that the tensor product is left exact. Thus, we need to show that for any injection $F_1 \to F_2$ in $\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q})$ and any objects G, H of $\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q})$, the map

$$\mathrm{Hom}_{\mathbf{ShT}_{\mathrm{Nis}}}(H, F_1 \otimes_{\mathrm{ShT}} G) \to \mathrm{Hom}_{\mathbf{ShT}_{\mathrm{Nis}}}(H, F_2 \otimes_{\mathrm{ShT}} G)$$

is injective. Since the functor ϕ is fully faithful by Proposition A.5 (5), we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q})}(H, F_1 \otimes_{\mathrm{ShT}} G) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q})}(H, F_2 \otimes_{\mathrm{ShT}} G) \arrow[d, equal] \\
\mathrm{Hom}_{\underline{\mathbf{Sh}}_{\mathrm{qfh}}(k,\mathbb{Q})}(\phi(H), \phi(F_1 \otimes_{\mathrm{ShT}} G)) \arrow[r] & \mathrm{Hom}_{\underline{\mathbf{Sh}}_{\mathrm{qfh}}(k,\mathbb{Q})}(\phi(H), \phi(F_2 \otimes_{\mathrm{ShT}} G))
\end{tikzcd}$$

Let us construct a natural isomorphism

$$\phi(F \otimes_{\mathrm{ShT}} G) = \phi(F) \otimes \phi(G). \tag{$*$}$$

Granting this, the bottom horizontal map is

$$\mathrm{Hom}_{\underline{\mathbf{Sh}}_{\mathrm{qfh}}(k,\mathbb{Q})}(\phi(H), \phi(F_1) \otimes \phi(G)) \to \mathrm{Hom}_{\underline{\mathbf{Sh}}_{\mathrm{qfh}}(k,\mathbb{Q})}(\phi(H), \phi(F_2) \otimes \phi(G)).$$

But, this map is injective because the functor ϕ is exact by Proposition A.5 (5) and the tensor on $\underline{\mathbf{Sh}}_{\mathrm{qfh}}(k,\mathbb{Q})$ is exact. Now our task is to construct the natural isomorphism ($*$).

Let $F, G \in \mathbf{ShT}_{\mathrm{Nis}}(k,\mathbb{Q})$ be Nisnevich sheaves with transfers. In case where $F = L(X)_{\mathbb{Q}}$ and $G = L(Y)_{\mathbb{Q}}$, we have

$$\begin{aligned}
\phi(L(X)_{\mathbb{Q}} \otimes_{\mathrm{ShT}} L(Y)_{\mathbb{Q}}) &= \phi(L(X \times Y)_{\mathbb{Q}}) \\
&\simeq \mathbb{Q}_{\mathrm{qfh}}(X \times Y) \\
&= \mathbb{Q}_{\mathrm{qfh}}(X) \otimes \mathbb{Q}_{\mathrm{qfh}}(Y) \simeq \phi(L(X)_{\mathbb{Q}}) \otimes \phi(L(Y)_{\mathbb{Q}}).
\end{aligned}$$

In general case, let $\mathcal{L}_1(F)$ (resp. $\mathcal{L}_1(G)$) be the presentation of F (resp. G) truncated above degree one. Then by the right exactness of $\otimes_{\mathrm{ShT}}$, the truncation of $\mathrm{Tot}(\mathcal{L}_1(F) \otimes_{\mathrm{ShT}} \mathcal{L}_1(G))$ is a presentation of $F \otimes_{\mathrm{ShT}} G$. Thus from the

the above special case and the exactness of ϕ, we obtain that

$$\begin{aligned}\phi(F \otimes_{\mathrm{ShT}} G) &= \phi(H_0(\mathrm{Tot}(\mathcal{L}_1(F) \otimes_{\mathrm{ShT}} \mathcal{L}_1(G)))) \\ &\simeq H_0(\mathrm{Tot}(\phi(\mathcal{L}_1(F)) \otimes \phi(\mathcal{L}_1(G)))) \\ &= \phi(F) \otimes \phi(G).\end{aligned}$$

□

A.2 An example

We prove that in general, the étale sheafification of a presheaf tensor $L(X) \otimes L(Y)$ can not be isomorphic to $L(X \times_k Y)$, even after tensoring with $\mathbb{Q}$.
We have a map

$$\phi : L(X) \otimes L(Y) \to L(X \times_k Y).$$

and consider the étale stalks at a strictly Henselian local scheme S , which is

$$\phi_S : \mathrm{Cor}(S, X) \otimes \mathrm{Cor}(S, Y) \to \mathrm{Cor}(S, X \times_k Y), \quad Z \otimes W \mapsto [Z \times_S W].$$

The claim in the beginning of this section follows from Proposition A.6 below for a strictly Henselian discrete valuation ring R.

Proposition A.6. *Let $X = Y = \mathbb{A}^1_k$. Let R be a discrete valuation ring over k and let $S := \mathrm{Spec}(R)$. Then ϕ_S is not surjective, even after tensoring with $\mathbb{Q}$.*

To show Proposition A.6, we consider some concrete cycles: Write $X = \mathrm{Spec}(k[x])$ and $Y = \mathrm{Spec}(k[y])$. Let t be a uniformizer of R. We define $f(x) \in R[x]$ as

$$f(x) = \begin{cases} x^2 - t & \text{if } \mathrm{char}(k) \neq 2, \\ x^3 - t & \text{if } \mathrm{char}(k) = 2. \end{cases}$$

and define cycles to be

$$Z := \mathrm{Spec}(R[x]/(f(x))) \in \mathrm{Cor}(S, X) \,\text{and}\, W := \mathrm{Spec}(R[y]/(f(y))) \in \mathrm{Cor}(S, Y).$$

Then we have

$$T := Z \times_S W = \mathrm{Spec}(R[x, y]/(f(x), f(y))).$$

And we define T_1 and T_2 as follows: in case $\mathrm{char}(k) \neq 2$,

$$T_1 := \mathrm{Spec}\big(R[x, y]/(f(x), x - y)\big),\ T_2 := \mathrm{Spec}\big(R[x, y]/(f(x), x + y)\big);$$

in case $\mathrm{char}(k) = 2$,

$$T_1 := \mathrm{Spec}\big(R[x, y]/(f(x), x - y)\big),\ T_2 := \mathrm{Spec}\big(R[x, y]/(f(x), x^2 + xy + y^2)\big).$$

Then, one easily sees that

$$\phi_S(Z \otimes W) = [T] = T_1 + T_2 \in \mathrm{Cor}(S, X \times_k Y).$$

Lemma A.7. *Let notation be as in above. The only pair (α, β) of integral cycles $\alpha \in \mathrm{Cor}(S, X)$ and $\beta \in \mathrm{Cor}(S, Y)$ such that $\phi_S(\alpha \otimes \beta)$ contains T_1(or T_2) is (Z, W).*

Proof. Integral cycles α and β are given by monic irreducible polynomials $g(x) \in R[x]$ and $h(y) \in R[y]$ respectively, i.e.

$$\alpha = \mathrm{Spec}(R[x]/(g(x))) \subset S \times X, \quad \beta = \mathrm{Spec}(R[y]/(h(y))) \subset S \times Y.$$

Put

$$\gamma := \alpha \times_S \beta = \mathrm{Spec}(R[x, y]/(g(x), h(y))) \subset S \times X \times Y.$$

Suppose that our cycle $T_1 \subset \gamma$ is a irreducible component.
Let p, q denote the projections from $S \times X \times Y$ to $S \times X$ and $S \times Y$ respectively. Then,

$$p(T_1) = Z, \quad q(T_1) = W.$$

Hence the images of γ along p and q contain Z and W respectively, i.e.

$$Z \subset \mathrm{Spec}(R[x]/(g(x))) \quad \text{and} \quad W \subset \mathrm{Spec}(R[y]/(h(y))).$$

These inclusions imply that

$$g(x) = a(x)f(x) \quad \text{and} \quad h(y) = b(y)f(y).$$

Since f, g and h are monic and irreducible, we have

$$g(x) = f(x), \quad h(y) = f(y).$$

Hence $\alpha = Z, \beta = W$.
The above argument works also for T_2. □

Proof of Proposition A.6. Assume that ϕ_S is surjective. Then we have an element $x := \sum n_{ij}(\alpha_i \otimes \beta_j) \in \mathrm{Cor}(S, X) \otimes \mathrm{Cor}(S, Y)$ such that $\phi_S(x) = T_1$. For some component $\alpha_i \otimes \beta_j$ of x, $\phi_S(\alpha_i \otimes \beta_j)$ contains T_1. By Lemma A.7, such $\alpha_i \otimes \beta_j$ is $Z \otimes W$ only. Therefore we have

$$x = m(Z \otimes W) + \sum n_{ij}(\alpha_i \otimes \beta_j),$$

where $m \geq 1$ and the sum is taken over all i, j such that $(\alpha_i, \beta_j) \neq (Z, W)$. Then

$$T_1 = \phi_S(x) = m(T_1 + T_2) + \sum n_{ij}\phi_S(\alpha_i \otimes \beta_j).$$

Hence

$$(1 - m)T_1 - mT_2 = -\sum n_{ij}\phi_S(\alpha_i \otimes \beta_j).$$

But this equality cannot be happen by Lemma A.7. Thus ϕ_S is not surjective. Since the above argument works even if the coefficients m, n_{ij} are rational numbers, ϕ_S is not surjective even after tensoring with $\mathbb{Q}$. □

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