The Pullbacks of Principal Coactions

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Abstract. We prove that the class of principal coactions is closed under one-surjective pullbacks in an appropriate category of algebras equipped with left and right coactions. This allows us to handle cases of C^*-algebras lacking two different non-trivial ideals. It also allows us to go beyond the category of comodule algebras. As an example of the former, we carry out an index computation for noncommutative line bundles over the standard Podleś sphere using the Mayer-Vietoris-type arguments afforded by a one-surjective pullback presentation of the C^*-algebra of this quantum sphere. To instantiate the latter, we define a family of coalgebraic noncommutative deformations of the U(1)-principal bundle $\mathrm{S}^7 \to \mathbb{C}\mathrm{P}^3$.

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Contents

1. Introduction and preliminaries

The idea of decomposing a complicated object into simpler pieces and connecting data is a fundamental computational principle throughout mathematics. In the case of (co)homology theory, it yields the Mayer-Vietoris long exact sequence whose significance and usefulness can hardly be overestimated. The categorical underpinning of all this are pullback diagrams: in a given category they give a rigorous meaning to putting together two objects over a third one.

In the classical setting of spaces, the freeness of a group action is a local property: if it is free on covering pieces, it is free everywhere. Slightly more generally, if we take the pushout of two equivariant maps $X_{12} \to X_1$ and $X_{12} \to X_2$ between spaces equipped with free actions, then the induced action on the pushout space is again free provided that at least one of the pushout maps is injective. Our aim is to work out a general noncommutative version of this simple fact. Let us explain our generalization step by step.

First, let us assume that all spaces are compact Hausdorff and all maps are continuous. Then we can easily dualize the aforementioned equivariant pushout to an equivariant pullback of homomorphisms of commutative unital C^*-algebras of all continuous complex-valued functions on compact Hausdorff spaces. Our assumption that at least one of the pushout maps is injective becomes now an assumption that at least one of the pullback maps is surjective. The right action $X \times G \to X$ of a topological group G translates into a representation of G in the automorphism group of the C^*-algebra $C(X)$ of functions on X:

$$\alpha\colon G \ni g \longmapsto \alpha_g \in \mathrm{Aut}(C(X)), \quad (\alpha_g f)(x) := f(xg). \tag{1.1}$$

Replacing $C(X)$ by an arbitrary unital C^*-algebra still allows us to consider one-surjective equivariant pullback diagrams of G-C^*-algebras.

Next, let G be a compact Hausdorff topological group acting by automorphisms on a unital C^*-algebra A. Then to define the freeness of such an action we need to dualize it to coaction:

$$\delta\colon A \longrightarrow C(G, A) = A \mathbin{\bar{\otimes}} C(G), \quad (\delta(a))(g) := \alpha_g a. \tag{1.2}$$

Here $C(G, A)$ is the C^*-algebra of all norm continuous functions from G to A naturally identified with the complete tensor product C^*-algebra (e.g. see [33, Corollary T.6.17]). Now we can replace $C(G)$ by the C^*-algebra of a compact quantum group [34, 36], still consider one-surjective equivariant pullback diagrams of C^*-algebras, and define freeness as a density condition [13].

Furthermore, we can define the Peter-Weyl functor [2] from the category of unital C^*-algebras equipped with an action of a compact quantum group (i.e. a coaction of its C^*-algebra) to the category of comodule algebras over the Hopf algebra of regular functions on the compact quantum group [36]. The main theorem of [2] states that the aforementioned density condition defining the freeness of an action of a compact quantum group on a unital C^*-algebra is, via the Peter-Weyl functor, equivalent to the principality of the coaction of the Hopf algebra of the compact quantum group. Thus the algebraic condition of principality of an appropriate comodule algebra encodes the analytical freeness condition of a compact quantum group action on a unital C^*-algebra. In the commutative case, the latter is equivalent to the freeness of a continuous compact Hausdorff group action on a compact Hausdorff space.

However, the category of principal comodule algebras, despite being very ample and enjoyable, does not encompass all interesting examples coming from quantizations along Poisson structures. A way to obtain a quantum group is by deforming a Poisson-Lie group along its Poisson structure. It is well known that Poisson-Lie groups admit few Poisson-Lie subgroups, so that it is important to consider coisotropic subgroups of Poisson-Lie groups. But the deformation along the Poisson structure of the natural action of a coisotropic subgroup on its Poisson-Lie group leads to a coaction of a coalgebra rather than a Hopf algebra [9, 20]. Such examples motivated the development of general theory of principal coactions of coalgebras on algebras [5–7]. The setting of our paper is based on this theory.

Finally, recall that coactions of discrete groups are defined as coactions of their group-ring Hopf algebras, and it is very easy to understand the principality of such coactions. Indeed, let P be a comodule algebra over a group ring $k\Gamma$. This is equivalent to P being graded by Γ:

$$P = \bigoplus_{\gamma\in\Gamma} P_\gamma, \quad P_\gamma := \{p \in P \mid \delta(p) = p \otimes \gamma\}, \quad \forall\, \gamma,\, \gamma' \in \Gamma \,:\, P_\gamma P_{\gamma'} \subseteq P_{\gamma\gamma'}.$$

The coaction of $k\Gamma$ on P is principal if and only if P is *strongly graded* by Γ [30], i.e.

$$\forall\, \gamma,\, \gamma' \in \Gamma \,:\, P_\gamma P_{\gamma'} = P_{\gamma\gamma'}. \tag{1.3}$$

The goal of this paper is to prove a general pullback theorem for principal coactions that significantly generalizes the main result of [15] restricted to comodule algebras and pullbacks of surjections. More precisely, our main result is that the pullback of principal coactions over morphisms of which at least one is surjective is again a principal coaction:

Theorem 2.2. *Let C be a coalgebra, and let P be the pullback of C-equivariant unital algebra homomorphisms $\pi_1 : P_1 \to P_{12}$ and $\pi_2 : P_2 \to P_{12}$. Then, granted minor technical requirements, if π_1 or π_2 is surjective and both coactions $P_1 \to P_1 \otimes C$ and $P_2 \to P_2 \otimes C$ are principal, then also the induced coaction $P \to P \otimes C$ is principal.*

It may be viewed as a non-linear version of the Milnor construction yielding an odd-to-even connecting homomorphism in algebraic K-theory [23]. Indeed, linearizing our pullback theorem with the help of a corepresentation gives precisely the odd-to-even construction of a projective module defining the connecting homomorphism in K-theory.

We apply this new result in two cases. In the first case, we keep the comodule-algebra setting but take a one-surjective pullback diagram (only one of the defining morphisms is surjective). In the second case, we proceed the other way round, that is we take a pullback diagram given by two surjections but take coactions that are not algebra homomorphisms.

The pullback picture of the standard quantum Hopf fibration gives us our first example. It provides a new way of computing the index pairing for the associated quantum Hopf line bundles (cf. [32]). This index pairing was computed in [14] using a noncommutative index formula, and re-derived in [25]. Here we give yet another method to compute it. This simple example shows the need to generalize from two-surjective to one-surjective pullback diagrams, and the pullback method of index computation seems attractive due to its inherent simplicity.

To obtain our second example, we first show how the piecewise structure [15] of a noncommutative join construction [11] allows one to define a certain class of piecewise principal coactions. Although this class of examples can also be handled by earlier methods, it definitely shows that there are interesting piecewise principal coactions that are not algebra homomorphisms. To obtain a concrete example, we take Pflaum's instanton bundle $\mathrm{S}^7_q \to \mathrm{S}^4_q$ [26] as the noncommutative join of $\mathrm{SU}_q(2)$, and turn it into the coalgebraic quantum principal bundle $\mathrm{S}^7_q \to \mathbb{C}\mathrm{P}^3_{q,s}$. We do it with the help of the canonical surjections $\pi : \mathcal{O}(\mathrm{SU}_q(2)) \to \mathcal{O}(\mathrm{SU}_q(2))/J_{q,s}$ determined by the coideals right ideals $J_{q,s} := (\mathcal{O}(\mathrm{S}^2_{q,s}) \cap \ker\varepsilon)\mathcal{O}(\mathrm{SU}_q(2))$, where $\mathrm{S}^2_{q,s}$ is a generic Podleś quantum sphere [27] and $\ker\varepsilon$ is the kernel of the counit map.

The paper is organized as follows. First, to make our exposition self-contained and to establish notation, we recall fundamental concepts that we use later on. The key Section 2 is devoted to the general pullback theorem for principal coactions of coalgebras on algebras, Section 3 is on deriving the index pairing for quantum Hopf line bundles as a corollary to the pullback presentation of the standard Hopf fibration of $\mathrm{SU}_q(2)$, and the final Section 4 presents new examples of piecewise principal coactions that go beyond Hopf-Galois theory.
Throughout the paper, we work with algebras and coalgebras over a field. The unadorned tensor product stands for the algebraic tensor product over this field. We employ the Heyneman-Sweedler-type notation (with the summation symbol suppressed) for the comultiplication $\Delta(c) =: c_{(1)} \otimes c_{(2)} \in C \otimes C$ and for coactions $\Delta_V(v) =: v_{(0)} \otimes v_{(1)} \in V \otimes C$, ${}_V\Delta(v) =: v_{(-1)} \otimes v_{(0)} \in C \otimes V$.

The convolution product of two linear maps from a coalgebra to an algebra is denoted by $*$: $(f*g)(c) := f(c_{(1)})g(c_{(2)})$. The set of natural numbers includes 0, that is $\mathbb{N} := \{0, 1, 2, \ldots\}$.

1.1. PULLBACK DIAGRAMS AND FIBRE PRODUCTS. The purpose of this section is to collect some elementary facts about fibre products. We consider the category of vector spaces as it will be the ambient category for all our pullback diagrams. Let $\pi_1 : A_1 \to A_{12}$ and $\pi_2 : A_2 \to A_{12}$ be linear maps. The *fibre product* of these maps is defined by

$$A_1 \underset{(\pi_1,\pi_2)}{\times} A_2 := \{(a_1, a_2) \in A_1 \times A_2 \mid \pi_1(a_1) = \pi_2(a_2)\}\,. \tag{1.4}$$

Together with the canonical projections

$$\mathrm{pr}_1 : A_1 \underset{(\pi_1,\pi_2)}{\times} A_2 \longrightarrow A_1, \qquad \mathrm{pr}_2 : A_1 \underset{(\pi_1,\pi_2)}{\times} A_2 \longrightarrow A_2 \tag{1.5}$$

it forms a universal construction completing the initially-given pair of linear maps into the following commutative diagram:

$$\begin{tikzcd} A_1 \underset{(\pi_1,\pi_2)}{\times} A_2 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & A_2 \arrow[d, "\pi_2"'] \\ A_1 \arrow[r, "\pi_1"] & A_{12}\,. \end{tikzcd} \tag{1.6}$$

Such diagrams are called *pullback diagrams*, and fibre products are often referred to as pullbacks.

Next, if $\pi_1 : A_1 \to A_{12}$ and $\pi_2 : A_2 \to A_{12}$ are morphisms of *-algebras, then the fibre product $A_1 \times_{(\pi_1,\pi_2)} A_2$ is a *-subalgebra of $A_1 \times A_2$. Furthermore, if we consider the pullback diagram (1.6) in the category of (unital) C^*-algebras, then $A_1 \times_{(\pi_1,\pi_2)} A_2$ with its componentwise multiplication and *-structure is a (unital) C^*-algebra. Much the same, if B is an algebra and $\pi_1 : A_1 \to A_{12}$ and $\pi_2 : A_2 \to A_{12}$ are morphisms of left B-modules, then the fibre product $A_1 \times_{(\pi_1,\pi_2)} A_2$ is a left B-module via the componentwise left action $b.(a_1, a_2) := (b.a_1, b.a_2)$.

1.2. ODD-TO-EVEN CONNECTING HOMOMORPHISM IN K-THEORY. Consider a pullback diagram

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ A_1 \arrow[dr, "\pi_1"'] & & A_2 \arrow[dl, "\pi_2"] \\ & A_{12} & \end{tikzcd} \tag{1.7}$$

in the category of unital algebras, and assume that one of the defining morphisms (here we choose π_1) is surjective. Then there exists a long exact sequence in algebraic K-theory [23]

$$\cdots \longrightarrow \mathrm{K}_1(A_{12}) \xrightarrow{\text{odd-to-even}} \mathrm{K}_0(A) \longrightarrow \mathrm{K}_0(A_1 \oplus A_2) \longrightarrow \mathrm{K}_0(A_{12}). \tag{1.8}$$

The mapping $\mathrm{K}_1(A_{12}) \xrightarrow{\text{odd-to-even}} \mathrm{K}_0(A)$ is obtained as follows. First, given left A_i-modules E_i, $i = 1, 2$, we obtain left A_{12}-modules $\pi_{i*}E_i$ defined by $A_{12} \otimes_{A_i} E_i$. Since A_{12} is unital, there are canonical morphisms

$$\pi_{i*} : E_i \longrightarrow \pi_{i*}E_i\,, \quad \pi_{i*}(e) = 1 \otimes_{A_i} e. \tag{1.9}$$

The modules E_i and $\pi_{i*}E_i$ can also be considered as left modules over the fibre product algebra A via the left actions given by $a.e_i := \mathrm{pr}_i(a).e_i$, for $e_i \in E_i$, and $a.f_i := \pi_i(\mathrm{pr}_i(a)).f_i$, for $f_i \in \pi_{i*}E_i$. Assume now that $h : \pi_{1*}E_1 \to \pi_{2*}E_2$ is a morphism of left A_{12}-modules. Then $h \circ \pi_{1*} : E_1 \to \pi_{2*}E_2$ and $\pi_{2*} : E_2 \to \pi_{2*}E_2$ can be lifted to morphisms of left A-modules, and we can consider their pullback diagram in the category of left A-modules:

(1.10)
$$\begin{tikzcd}
 & E_1 \underset{(h\circ\pi_{1*},\pi_{2*})}{\times} E_2 \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\
E_1 \arrow[d, "\pi_{1*}"'] & & E_2 \arrow[d, "\pi_{2*}"] \\
\pi_{1*}E_1 \arrow[rr, "h"'] & & \pi_{2*}E_2\,.
\end{tikzcd}$$

In [23, Section 2], it is proven in detail that, if E_i is a finitely generated projective module over A_i, $i = 1, 2$, and h is an isomorphism, then the fibre product $M := E_1 \times_{(h\circ\pi_{1*}\,,\,\pi_{2*})} E_2$ is a finitely generated projective A-module. Furthermore, up to isomorphism, every finitely generated projective module over A has this form, and the A_i-modules E_i and $\mathrm{pr}_{i*}M := A_i \otimes_A M$, $i = 1, 2$, are naturally isomorphic. In particular, if $E_1 \cong A_1^n$ and $E_2 \cong A_2^n$, the isomorphism $h : \pi_{1*}E_1 \to \pi_{2*}E_2$ is given by an invertible matrix $U \in \mathrm{GL}_n(A_{12})$. Using the canonical embedding $\mathrm{GL}_n(A_{12}) \subseteq \mathrm{GL}_\infty(A_{12})$, we get a map

$$\mathrm{GL}_\infty(A_{12}) \ni U \longmapsto M \in \mathrm{Proj}(A) \tag{1.11}$$

given by the pullback diagram

(1.12)
$$\begin{tikzcd}
 & M \arrow[dl] \arrow[dr] & \\
A_1^n \arrow[dr, "\pi_1"'] & & A_2^n \arrow[dl, "\pi_2"] \\
 & A_{12}^n \overset{U}{\cong} A_{12}^n\,. &
\end{tikzcd}$$

This map induces an odd-to-even connecting homomorphism on the level of both algebraic [23] and C^*-algebraic [17] K-theory. An explicit description of the module M is as follows. Assume that $\pi_1 : A_1 \to A_{12}$ is surjective. Then there exist liftings $c, d \in \mathrm{Mat}_n(A_1)$ such that evaluating π_1 on c and d componentwise yields U^{-1} and U respectively. Applying [12, Theorem 2.1] to our situation yields $E_1 \times_{(h\circ\pi_{1*},\,\pi_{2*})} E_2 \cong A^{2n}p$, where

$$p := \begin{pmatrix} (c(2-dc)d, 1) & (c(2-dc)(1-dc), 0) \\ ((1-dc)d, 0) & ((1-dc)^2, 0) \end{pmatrix} \in \mathrm{Mat}_{2n}(A). \tag{1.13}$$

1.3. Principal coactions and associated projective modules. Recall first the general definition of an entwining structure. Let C be a coalgebra with comultiplication Δ and counit ε, and let A be an algebra with multiplication m and unit η. A linear map

$$\psi : C \otimes A \longrightarrow A \otimes C \tag{1.14}$$

is called an *entwining structure* if and only if it is unital, counital, and distributive with respect to both the multiplication and comultiplication:

(1.15)
$$\psi \circ (\mathrm{id} \otimes m) = (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \psi) \circ (\psi \otimes \mathrm{id}), \qquad \psi \circ (\mathrm{id} \otimes \eta) = (\eta \otimes \mathrm{id}) \circ \mathrm{flip},$$

(1.16)
$$(\mathrm{id} \otimes \Delta) \circ \psi = (\psi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \psi) \circ (\Delta \otimes \mathrm{id}), \qquad (\mathrm{id} \otimes \varepsilon) \circ \psi = \mathrm{flip} \circ (\varepsilon \otimes \mathrm{id}).$$

If ψ is an entwining of a coalgebra C and an algebra A, and M is a right C-comodule and a right A-module, we call M an *entwined module* [4] when it satisfies the compatibility condition

$$(ma)_{(0)} \otimes (ma)_{(1)} = m_{(0)}\psi(m_{(1)} \otimes a). \tag{1.17}$$

Next, let P be an algebra equipped with a coaction $\Delta_P : P \to P \otimes C$ of a coalgebra C. Define the coaction-invariant subalgebra of P by

$$B := P^{\mathrm{co}C} := \{b \in P \mid \Delta_P(bp) = b\Delta_P(p) \text{ for all } p \in P\}. \tag{1.18}$$

We call the inclusion $B \subseteq P$ a *C-extension.* We call it a *coalgebra-Galois C-extension* when the canonical left P-module right C-comodule map

$$\mathsf{can} : P \underset{B}{\otimes} P \longrightarrow P \otimes C, \quad p \underset{B}{\otimes} p' \longmapsto p\Delta_P(p'), \tag{1.19}$$

is bijective [5]. Note that the bijectivity of can allows us to define the so-called translation map

$$\tau : C \longrightarrow P \underset{B}{\otimes} P, \quad \tau(c) := \mathsf{can}^{-1}(1 \otimes c). \tag{1.20}$$

Moreover, every coalgebra-Galois C-extension comes naturally equipped with a unique entwining structure that makes P a (P, C)-entwined module in the sense of (1.17). It is called the canonical entwining structure [5], and is very useful in calculations or further constructions. Explicitly, it can be written as:

$$\psi(c\otimes p) := \mathsf{can}(\mathsf{can}^{-1}(1\otimes c)p). \tag{1.21}$$

An algebra P with a right C-coaction Δ_P is said to be *e-coaugmented* if and only if there exists a group-like element $e \in C$ such that $\Delta_P(1) = 1\otimes e$. We call the C-extension $B := P^{\mathrm{co}C} \subseteq P$ e-coaugmented. (Much in the same way, one defines the coaugmentation of left coactions.) For the e-coaugmented coalgebra-Galois C-extensions, one can show that the coaction-invariant subalgebra defined in (1.18) can be expressed as

$$P^{\mathrm{co}C} = \{p\in P \mid \Delta_P(p) = p\otimes e\}. \tag{1.22}$$

Indeed, Formula (1.21) allows us to express the right coaction in terms of the entwining

$$\Delta_P(p) = \psi(e\otimes p), \tag{1.23}$$

and Equation (1.15) yields the right-in-left inclusion. The opposite inclusion is obvious.

Next, if ψ is invertible, one can use (1.16) to show that the formula

$${}_P\Delta(p) := \psi^{-1}(p\otimes e) \tag{1.24}$$

defines a left coaction ${}_P\Delta : P \to C\otimes P$. We define the left coaction-invariant subalgebra ${}^{\mathrm{co}C}P$ as in (1.18), and derive the left-sided version of (1.21). Hence, for any e-coaugmented coalgebra-Galois C-extension with *invertible canonical entwining*, the right coaction-invariant subalgebra coincides with the left coaction-invariant subalgebra:

$$P^{\mathrm{co}C} = \{p\in P \mid \Delta_P(p) = p\otimes e\} = \{p\in P \mid {}_P\Delta(p) = e\otimes p\} = {}^{\mathrm{co}C}P. \tag{1.25}$$

Finally, we need to assume one more condition on C-extensions to obtain a suitable definition of a principal coaction: *equivariant projectivity.* It is a pivotal property that guarantees the projectivity of associated modules, and thus leads to index pairings between K-theory and K-homology. Putting together the aforementioned four conditions, we say that a coalgebra C-extension $B\subseteq P$ is *principal* (or simply that P is principal) [6] iff:

(i) The canonical map $\mathsf{can} : P\otimes_B P \to P\otimes C$, $p\otimes_B p' \mapsto p\Delta_P(p')$, is bijective (Galois condition).
(ii) The right coaction is e-coaugmented for some group-like $e\in C$, i.e. $\Delta_P(1) = 1\otimes e$.
(iii) The canonical entwining $\psi : C\otimes P\to P\otimes C$, $c\otimes p\mapsto \mathsf{can}(\mathsf{can}^{-1}(1\otimes c)p)$, is bijective.
(iv) The algebra P is C-equivariantly projective as a left B-module, i.e. there exists a left-B-linear and right-C-colinear splitting of the multiplication map $B\otimes P\to P$.

In the framework of coalgebra extensions, the role of connections on principal bundles is played by strong connections [6]. Let P be an algebra and both a left and right e-coaugmented C-comodule. (Note that the left and right coactions need not commute.) A *strong connection* is a linear map $\ell : C \to P \otimes P$ satisfying

$$\begin{aligned} &\widetilde{\mathsf{can}} \circ \ell = 1 \otimes \mathrm{id}, \qquad (\mathrm{id} \otimes \Delta_P) \circ \ell = (\ell \otimes \mathrm{id}) \circ \Delta, \\ &({}_P\Delta \otimes \mathrm{id}) \circ \ell = (\mathrm{id} \otimes \ell) \circ \Delta, \qquad \ell(e) = 1 \otimes 1. \end{aligned} \tag{1.26}$$

Here $\widetilde{\mathsf{can}} : P \otimes P \to P \otimes C$ is the lifting of can to $P \otimes P$. Assuming that there exists an invertible entwining $\psi : C \otimes P \to P \otimes C$ making P an entwined module, the first three equations of (1.26) read in the Heyneman-Sweedler-type notation $c \mapsto \ell(c)^{\langle 1\rangle} \otimes \ell(c)^{\langle 2\rangle}$ as follows:

$$\begin{aligned} \ell(c)^{\langle 1\rangle}\psi(e \otimes \ell(c)^{\langle 2\rangle}) &= \ell(c)^{\langle 1\rangle}\ell(c)^{\langle 2\rangle}{}_{(0)} \otimes \ell(c)^{\langle 2\rangle}{}_{(1)} \\ &= 1 \otimes c\,, \end{aligned} \tag{1.27}$$

$$\begin{aligned} \ell(c)^{\langle 1\rangle} \otimes \psi(e \otimes \ell(c)^{\langle 2\rangle}) &= \ell(c)^{\langle 1\rangle} \otimes \ell(c)^{\langle 2\rangle}{}_{(0)} \otimes \ell(c)^{\langle 2\rangle}{}_{(1)} \\ &= \ell(c_{(1)})^{\langle 1\rangle} \otimes \ell(c_{(1)})^{\langle 2\rangle} \otimes c_{(2)}\,, \end{aligned} \tag{1.28}$$

$$\begin{aligned} \psi^{-1}(\ell(c)^{\langle 1\rangle} \otimes e) \otimes \ell(c)^{\langle 2\rangle} &= \ell(c)^{\langle 1\rangle}{}_{(-1)} \otimes \ell(c)^{\langle 1\rangle}{}_{(0)} \otimes \ell(c)^{\langle 2\rangle} \\ &= c_{(1)} \otimes \ell(c_{(2)})^{\langle 1\rangle} \otimes \ell(c_{(2)})^{\langle 2\rangle}\,. \end{aligned} \tag{1.29}$$

Applying $\mathrm{id} \otimes \varepsilon$ to (1.27) yields a useful formula

$$\ell(c)^{\langle 1\rangle}\ell(c)^{\langle 2\rangle} = \varepsilon(c). \tag{1.30}$$

It is worthwhile to observe the left-right symmetry of principal extensions. We already noted (see (1.25)) the equality of the left and right coaction-invariant subalgebras. Now let us define the left canonical map as

$$\mathsf{can}_L : P \underset{B}{\otimes} P \ni p \otimes q \longmapsto p_{(-1)} \otimes p_{(0)}q \in C \otimes P. \tag{1.31}$$

One can check that it is related to the right canonical map can by the formula [7]

$$\psi \circ \mathsf{can}_L = \mathsf{can}. \tag{1.32}$$

Also, if ℓ is a strong connection and $\widetilde{\mathsf{can}}_L := (\mathrm{id} \otimes m) \circ ({}_P\Delta \otimes \mathrm{id})$ is the lifted left canonical map, then $\widetilde{\mathsf{can}}_L \circ \ell = \mathrm{id} \otimes 1$. Hence

$$c \otimes p \longmapsto \ell(c)^{\langle 1\rangle} \otimes \ell(c)^{\langle 2\rangle}p \tag{1.33}$$

is a splitting of $\widetilde{\mathsf{can}}_L$ just as

$$p \otimes c \longmapsto p\ell(c)^{\langle 1\rangle} \otimes \ell(c)^{\langle 2\rangle} \tag{1.34}$$

is a splitting of $\widetilde{\mathsf{can}}$.

Lemma 1.1. *Let P be an object in the category ${}^C_e\mathbf{Alg}^C_e$ of all unital algebras with e-coaugmented left and right C-coactions. Assume that there exists an invertible entwining $\psi : C \otimes P \to P \otimes C$ making P an entwined module. Then, if P admits a strong connection ℓ, it is principal.*

Proof. Following [6], first we argue that

$$\sigma : P \ni p \longmapsto p_{(0)}\ell(p_{(1)})^{\langle 1\rangle} \otimes \ell(p_{(1)})^{\langle 2\rangle} \in B \otimes P \tag{1.35}$$

is a left-B-linear splitting of the multiplication map. Indeed, $m\circ\sigma = \mathrm{id}$ because of (1.30), and the calculation

$$\psi(e \otimes p_{(0)}\ell(p_{(1)})^{\langle 1\rangle}) \otimes \ell(p_{(1)})^{\langle 2\rangle} = p_{(0)}\ell(p_{(1)})^{\langle 1\rangle} \otimes e \otimes \ell(p_{(1)})^{\langle 2\rangle}, \tag{1.36}$$

obtained using (1.15), proves that $\sigma(P) \subseteq B \otimes P$. This splitting is evidently right C-colinear, so that its existence proves the equivariant projectivity.

Next, let us check that the formula

$$\mathsf{can}^{-1} : P \otimes C \longrightarrow P \underset{B}{\otimes} P, \quad p \otimes c \longmapsto p\ell(c)^{\langle 1\rangle} \underset{B}{\otimes} \ell(c)^{\langle 2\rangle}, \tag{1.37}$$

defines the inverse of the canonical map can, so that the coaction of C is Galois. It follows from (1.27) that

$$\mathsf{can}(\mathsf{can}^{-1}(p \otimes c)) = p\ell(c)^{\langle 1\rangle}\ell(c)^{\langle 2\rangle}{}_{(0)} \otimes \ell(c)^{\langle 2\rangle}{}_{(1)} = p \otimes c\,. \tag{1.38}$$

On the other hand, taking advantage of (1.30) and (1.35), we see that

$$\mathsf{can}^{-1}(\mathsf{can}(p\underset{B}{\otimes}q)) = pq_{(0)}\ell(q_{(1)})^{\langle 1\rangle}\underset{B}{\otimes}\ell(q_{(1)})^{\langle 2\rangle} = p\underset{B}{\otimes}q_{(0)}\ell(q_{(1)})^{\langle 1\rangle}\ell(q_{(1)})^{\langle 2\rangle} = p\underset{B}{\otimes}q.$$

Thus the conditions (i) and (iv) of the principality of a C-extension are satisfied. Finally, Condition (ii) is simply assumed, and Condition (iii) follows from the uniqueness of an entwining that makes P an entwined module. □

Note that, if there exists a strong connection ℓ, then (1.37) yields

$$\tau(c) = \ell(c)^{\langle 1\rangle} \underset{B}{\otimes} \ell(c)^{\langle 2\rangle}. \tag{1.39}$$

In the Heyneman-Sweedler-type notation, we write $\tau(c) = \tau(c)^{[1]} \otimes_B \tau(c)^{[2]}$. Then the canonical entwining reads

$$\begin{aligned}\psi(c \otimes p) &= \tau(c)^{[1]}(\tau(c)^{[2]}p)_{(0)} \otimes (\tau(c)^{[2]}p)_{(1)}\\ &= \ell(c)^{\langle 1\rangle}(\ell(c)^{\langle 2\rangle}p)_{(0)} \otimes (\ell(c)^{\langle 2\rangle}p)_{(1)}.\end{aligned} \tag{1.40}$$

Remark 1.2. In [6], there is the converse statement: if P is principal, it admits a strong connection. Thus, among all extensions satisfying the assumptions of Lemma 1.1, principal extensions can be characterized as these that admit a strong connection.

Recall now that classical principal bundles can be viewed as functors transforming finite-dimensional vector spaces into associated vector bundles. Analogously, one can prove that a principal C-extension $B \subseteq P$ defines a functor from the category of finite-dimensional left C-comodules into the category of finitely generated projective left B-modules [6]. Explicitly, if V is a left C-comodule with coaction ${}_V\Delta$, this functor assigns to it the cotensor product

(1.41) $$P \underset{C}{\Box} V := \{\textstyle\sum_i p_i \otimes v_i \in P \otimes V \mid \sum_i \Delta_P(p_i) \otimes v_i = \sum_i p_i \otimes {}_V\Delta(v_i)\}.$$

In particular, if $g \in C$ is a group-like element, the formula ${}_{\mathbb{C}}\Delta(1) := g \otimes 1$ defines a one-dimensional corepresentation, and

(1.42) $$P \underset{C}{\Box} \mathbb{C} = \{p \in P \mid \Delta_P(p) = p \otimes g\} =: P_g$$

can be viewed as a noncommutative associated complex line bundle. Then the general formula for computing an idempotent E_g of the associated module P_g out of a corepresentation and a strong connection becomes a very simple special case of [6, Theorem 3.1]:

(1.43)
$$P_g \cong B^n E_g\,, \quad (E_g)_{i,j=1}^n := \left(g_i^R g_j^L\right)_{i,j=1}^n, \quad \ell(g) =: \sum_{k=1}^n g_k^L \otimes g_k^R \in P_{g^{-1}} \otimes P_g\,.$$

A fundamental special case of principal extensions is provided by *principal comodule algebras*. One assumes then that $C = H$ is a Hopf algebra with bijective antipode S, the canonical map is bijective, and P is an H-equivariantly projective left B-module. This brings us in touch with compact quantum groups. Assume that $\bar{H}$ is the C^*-algebra of a compact quantum group in the sense of Woronowicz [34,36], and H is its dense Hopf *-subalgebra spanned by the matrix coefficients of the irreducible unitary corepresentations. Let $\bar{P}$ be a unital C^*-algebra and $\delta : \bar{P} \to \bar{P} \bar{\otimes} \bar{H}$ an injective C^*-algebraic right coaction of $\bar{H}$ on $\bar{P}$. (See [1, Definition 0.2] for a general definition and [3, Definition 1] for the special case of compact quantum groups.) Here $\bar{\otimes}$ denotes the minimal C^*-completion of the algebraic tensor product $\bar{P} \otimes \bar{H}$.

To extend Woronowicz's Peter-Weyl theory [36] from compact quantum groups to compact quantum principal bundles, one defines [2] the subalgebra $\mathcal{P}_\delta(\bar{P}) \subseteq \bar{P}$ of elements for which the coaction lands in $\bar{P} \otimes H$, i.e.

(1.44) $$\mathcal{P}_\delta(\bar{P}) := \{p \in \bar{P} \mid \delta(p) \in \bar{P} \otimes H\}.$$

One easily checks that it is an H-comodule algebra. We call $\mathcal{P}_\delta(\bar{P})$ the *Peter-Weyl comodule algebra* associated to the C^*-coaction δ. It follows from results of [3] and [28] that $\mathcal{P}_\delta(\bar{P})$ is dense in $\bar{P}$. Also, it is straightforward to verify [2] that the operation $\bar{P} \mapsto \mathcal{P}_\delta(\bar{P})$ gives a functor commuting with taking fibre products (pullbacks), and that $\mathcal{P}_\delta(\bar{P})^{\mathrm{co}H}$ coincides with the C^*-algebra $\bar{P}^{\mathrm{co}\bar{H}}$.

Finally, let us recall that, for a compact Hausdorff topological group G and a unital C^*-algebra A, we can use Formula (1.2) to translate a right $C(G)$-coaction into a G-action. Thus we can use the terminology of right $C(G)$-comodule C^*-algebras and G-C^*-algebras synonymously. It is important to bear in mind that the Peter-Weyl functor maps G-equivariant

*-homomorphisms to colinear homomorphisms of right $\mathcal{O}(G)$-comodule algebras [2].

1.4. STANDARD HOPF FIBRATION OF QUANTUM SU(2). The standard quantum Hopf fibration is given by an action of U(1) on the quantum group $\mathrm{SU}_q(2)$, $q \in (0, 1)$. The coordinate ring of $\mathcal{O}(\mathrm{SU}_q(2))$ is generated by α, β, γ, δ with relations

$$\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma, \quad \beta\gamma = \gamma\beta, \tag{1.45}$$

$$\alpha\delta - q\beta\gamma = 1, \quad \delta\alpha - q^{-1}\beta\gamma = 1, \tag{1.46}$$

and involution $\alpha^* := \delta$, $\beta^* := -q\gamma$. It is a Hopf *-algebra with comultiplication Δ, counit ε, and antipode S given by

$$\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma, \quad \Delta(\beta) = \alpha \otimes \beta + \beta \otimes \delta, \tag{1.47}$$

$$\Delta(\gamma) = \gamma \otimes \alpha + \delta \otimes \gamma, \quad \Delta(\delta) = \gamma \otimes \beta + \delta \otimes \delta, \tag{1.48}$$

$$\varepsilon(\alpha) = \varepsilon(\delta) = 1, \quad \varepsilon(\beta) = \varepsilon(\gamma) = 0, \tag{1.49}$$

$$S(\alpha) = \delta, \quad S(\beta) = -q^{-1}\beta, \quad S(\gamma) = -q\gamma, \quad S(\delta) = \alpha. \tag{1.50}$$

Let $\mathcal{O}(\mathrm{U}(1))$ denote the commutative and cocommutative Peter-Weyl Hopf *-algebra of U(1), and let u stand for its unitary group-like generator. Note that the counit ε and the antipode S satisfy $\varepsilon(u) = 1$ and $S(u) = u^*$. There is a Hopf *-algebra surjection

$$\pi : \mathcal{O}(\mathrm{SU}_q(2)) \longrightarrow \mathcal{O}(\mathrm{U}(1)), \quad \pi(\alpha) := u, \quad \pi(\delta) := u^{-1}, \quad \pi(\beta) := 0 =: \pi(\gamma). \tag{1.51}$$

Setting $\Delta_R := (\mathrm{id} \otimes \pi) \circ \Delta$, we obtain a right $\mathcal{O}(\mathrm{U}(1))$-coaction on $\mathcal{O}(\mathrm{SU}_q(2))$. On generators, the coaction reads

$$\Delta_R(\alpha) = \alpha \otimes u, \quad \Delta_R(\beta) = \beta \otimes u^{-1}, \quad \Delta_R(\gamma) = \gamma \otimes u, \quad \Delta_R(\delta) = \delta \otimes u^{-1}. \tag{1.52}$$

The *-subalgebra of coaction invariants defines the coordinate ring of the standard Podleś quantum sphere:

$$\mathcal{O}(\mathrm{S}^2_q) := \mathcal{O}(\mathrm{SU}_q(2))^{\mathrm{co}\mathcal{O}(\mathrm{U}(1))} = \{a \in \mathcal{O}(\mathrm{SU}_q(2)) \mid \Delta_R(a) = a \otimes 1\}. \tag{1.53}$$

One can prove (see [27]) that $\mathcal{O}(\mathrm{S}^2_q)$ is isomorphic to the *-algebra generated by B and self-adjoint A satisfying the relations

$$AB = q^2 BA, \quad B^*B = A - A^2, \quad BB^* = q^2 A - q^4 A^2. \tag{1.54}$$

An isomorphism is explicitly given by the formulas $A = -q^{-1}\beta\gamma$ and $B = -\beta\alpha$. The irreducible Hilbert space representations of $\mathcal{O}(\mathrm{S}^2_q)$ are given by

$$\rho_0(A) = \rho_0(B) = 0, \quad \rho_0(1) = 1 \quad \text{on } \mathscr{H} = \mathbb{C}, \tag{1.55}$$

$$\rho_+(A)e_n = q^{2n}e_n, \quad \rho_+(B)e_n = q^n(1 - q^{2n})^{1/2}e_{n-1} \quad \text{on } \mathscr{H} = \ell_2(\mathbb{N}), \tag{1.56}$$

where $\{e_n \mid n = 0, 1, \ldots\}$ is an orthonormal basis of $\ell_2(\mathbb{N})$.

Let $C(\mathrm{S}^2_q)$ denote the universal C^*-algebra generated by A and B satisfying (1.54). From the above representations, it follows that $C(\mathrm{S}^2_q)$ is the minimal unitalization of $\mathcal{K}(\ell_2(\mathbb{N}))$, that is,

$$(1.57) \qquad C(\mathrm{S}^2_q) \cong \mathcal{K}(\ell_2(\mathbb{N})) \oplus \mathbb{C} \subseteq \mathcal{B}(\ell_2(\mathbb{N})),$$
$$(k+\alpha)(k'+\alpha') = (kk' + \alpha' k + \alpha k') + \alpha\alpha', \quad k, k' \in \mathcal{K}(\ell_2(\mathbb{N})), \ \alpha, \alpha' \in \mathbb{C}.$$

Here $\mathcal{K}(\ell_2(\mathbb{N}))$ and $\mathcal{B}(\ell_2(\mathbb{N}))$ denote the C^*-algebras of compact and bounded operators respectively on the Hilbert space $\ell_2(\mathbb{N})$. The isomorphism (1.57) implies that $\mathrm{K}_0(C(\mathrm{S}^2_q)) \cong \mathbb{Z} \oplus \mathbb{Z}$, where one generator of K-theory is given by the class of the unit $1 \in C(\mathrm{S}^2_q)$, and the other by the class of the one-dimensional projection onto $\mathbb{C}e_0 \subseteq \ell_2(\mathbb{N})$.

Furthermore, $\mathrm{K}^0(C(\mathrm{S}^2_q)) \cong \mathbb{Z} \oplus \mathbb{Z}$. We identify one generator of K-homology with the class of the pair of representations $[(\mathrm{id}, \varepsilon)]$, where $\mathrm{id}(k+\alpha) = k + \alpha$ and $\varepsilon(k+\alpha) = \alpha$ for all $k \in \mathcal{K}(\ell_2(\mathbb{N}))$ and $\alpha \in \mathbb{C}$. The other generator can be given by the class of the pair of representations $[(\varepsilon, \varepsilon_0)]$ with the (non-unital) representation ε_0 of $\mathcal{K}(\ell_2(\mathbb{N})) \oplus \mathbb{C}$ defined by $\varepsilon_0(k+\alpha) = \alpha \mathrm{S}\mathrm{S}^*$, where

$$(1.58) \qquad \mathrm{S} : \ell_2(\mathbb{N}) \longrightarrow \ell_2(\mathbb{N}), \quad \mathrm{S}e_n = e_{n+1},$$

denotes the unilateral shift on $\ell_2(\mathbb{N})$. (See [22] for a detailed treatment of the K-homology and K-theory of Podleś quantum spheres.)

We shall also consider the coordinate ring of the quantum disc $\mathcal{O}(\mathrm{D}_q)$ generated by z and z^* with the relation

$$(1.59) \qquad z^*z - q^2 zz^* = 1 - q^2.$$

Its bounded irreducible Hilbert space representations are given by

$$(1.60) \qquad \mu_\theta(z) = \mathrm{e}^{\mathrm{i}\theta} \quad \text{on } \mathscr{H} = \mathbb{C}, \quad \theta \in [0, 2\pi),$$
$$(1.61) \qquad \mu(z)e_n = (1 - q^{2(n+1)})^{1/2} e_{n+1} \quad \text{on } \mathscr{H} = \ell_2(\mathbb{N}).$$

It has been shown in [18] that the universal C^*-algebra of $\mathcal{O}(\mathrm{D}_q)$ is isomorphic to the Toeplitz algebra given as the C^*-algebra generated by the unilateral shift S of Equation (1.58). The representation μ defines then an embedding of $\mathcal{O}(\mathrm{D}_q)$ into $\mathcal{T}$.

Let $C(\mathrm{U}(1))$ denote the C^*-algebra of continuous functions on the unit circle S^1, and let u be its unitary generator. The Toeplitz algebra gives rise to the following short exact sequence of C^*-algebras:

$$(1.62) \qquad 0 \longrightarrow \mathcal{K}(\ell_2(\mathbb{N})) \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(\mathrm{U}(1)) \longrightarrow 0.$$

Here the so-called symbol map $\sigma : \mathcal{T} \to C(\mathrm{U}(1))$ is given by $\sigma(\mathrm{S}) := u$. Since $\mathrm{S} - \mu(z)$ belongs to $\mathcal{K}(\ell_2(\mathbb{N}))$, it follows in particular that $\sigma(\mu(z)) = u$.

Now let us consider the associated quantum line bundles as finitely generated projective modules. They are defined by the one-dimensional corepresentations $\mathbb{C} \ni 1 \mapsto u^N \otimes 1$, $N \in \mathbb{Z}$, as cotensor products (1.42):

$$M_N := \{p \in \mathcal{O}(\mathrm{SU}_q(2)) \mid \Delta_R(p) = p \otimes u^N\}. \tag{1.63}$$

Since Δ_R is a morphism of algebras, M_N is an $\mathcal{O}(\mathrm{S}^2_q)$-bimodule. Our next step is to determine explicitly projections describing these projective modules.

For $l \in \frac{1}{2}\mathbb{N}$ and $i, j = -l, -l+1, \ldots, l$, let $t^l_{i,j}$ denote the matrix elements of the irreducible unitary corepresentations of $\mathcal{O}(\mathrm{SU}_q(2))$, so that we have

$$\Delta(t^l_{i,j}) = \sum_{k=-l}^{l} t^l_{i,k} \otimes t^l_{k,j}\,, \qquad \sum_{k=-l}^{l} t^{l*}_{k,i}\, t^l_{k,j} = \sum_{k=-l}^{l} t^l_{i,k}\, t^{l*}_{j,k} = \delta_{ij}\,. \tag{1.64}$$

By the Peter-Weyl theorem for compact quantum groups [35],

$$\mathcal{O}(\mathrm{SU}_q(2)) = \bigoplus_{l \in \frac{1}{2}\mathbb{N}} \bigoplus_{i,j=-l}^{l} \mathbb{C}\, t^l_{i,j}\,. \tag{1.65}$$

From the explicit description of $t^l_{i,j}$ [19, Section 4.2.4] and the definition of Δ_R, it follows that $\Delta_R(t^l_{i,j}) = t^l_{i,j} \otimes u^{-2j}$, whence $t^l_{i,-j} \in M_{2j}$. It can be shown [16, 29] that $t^{|j|}_{i,-j}$, $i = -|j|, \ldots, |j|$ generate M_{2j} as a left $\mathcal{O}(\mathrm{S}^2_q)$-module and $M_{2j} \cong \mathcal{O}(\mathrm{S}^2_q)^{2|j|+1} E_{2j}$ for all $j \in \frac{1}{2}\mathbb{Z}$, where

$$E_{2j} := \begin{pmatrix} t^{|j|}_{-|j|,-j} \\ \vdots \\ t^{|j|}_{|j|,-j} \end{pmatrix} \begin{pmatrix} t^{|j|*}_{-|j|,-j}\,, & \cdots\,, & t^{|j|*}_{|j|,-j} \end{pmatrix} \in \mathrm{Mat}_{2|j|+1}(\mathcal{O}(\mathrm{S}^2_q)). \tag{1.66}$$

It is clear that $E^*_{2j} = E_{2j}$, and it follows from (1.64) that $E^2_{2j} = E_{2j}$. Hence E_{2j} is a projection.

2. Principality of one-surjective pullbacks

We begin by defining an ambient category for pullback diagrams appearing in the second part of this section. Let P be a unital algebra equipped with both a right coaction $\Delta_P : P \to P \otimes C$ and a left coaction ${}_P\Delta : P \to C \otimes P$ of the same coalgebra C. We do *not* assume that these coactions commute, but we do assume that they are coaugmented by the same group-like element $e \in C$, i.e. $\Delta_P(1) = 1 \otimes e$ and ${}_P\Delta(1) = e \otimes 1$. For a fixed coalgebra C and a group-like $e \in C$, we consider the category ${}^C_e\mathbf{Alg}^C_e$ of all such unital algebras with e-coaugmented left and right C-coactions. Here morphisms are bicolinear algebra homomorphisms.

Since we work over a field, this category is evidently closed under any pullbacks. If $\pi_1 : P_1 \to P_{12}$ and $\pi_2 : P_2 \to P_{12}$ are morphisms in ${}^C_e\mathbf{Alg}^C_e$, then the fibre product algebra $P := P_1 \times_{(\pi_1,\pi_2)} P_2$ becomes a right C-comodule via

$$\Delta_P(p,q) := (p_{(0)},0)\otimes p_{(1)} + (0,q_{(0)})\otimes q_{(1)}, \tag{2.1}$$

and a left C-comodule via

$$_P\Delta(p,q) := p_{(-1)}\otimes (p_{(0)},0) + q_{(-1)}\otimes (0,q_{(0)}). \tag{2.2}$$

Also, it is clear that $\Delta_P(1,1) = (1,1)\otimes e$ and $_P\Delta(1,1) = e\otimes(1,1)$.

2.1. Principality of images and preimages. In the following lemma, we prove that any surjective morphism in ${}^C_e\mathbf{Alg}^C_e$ whose domain is a principal extension can be split by a left-colinear map and by a right-colinear map (not necessarily by a bicolinear map). Note that the first part of the lemma is proved much in the same way as in the Hopf-Galois case [15, Lemma 3.1]:

Lemma 2.1. *Let $\pi : P \to Q$ be a surjective morphism in the category ${}^C_e\mathbf{Alg}^C_e$ of unital algebras with e-coaugmented left and right C-coactions. If P is principal, then:*

(i) *The induced map $\pi^{\mathrm{co}C} : P^{\mathrm{co}C} \to Q^{\mathrm{co}C}$ is surjective.*
(ii) *There exists a unital right-C-colinear splitting of π.*
(iii) *There exists a unital left-C-colinear splitting of π.*
(iv) *Q is principal.*

Furthermore, if $Q' \in {}^C_e\mathbf{Alg}^C_e$, $Q' \subseteq Q$, is principal, then so is $\pi^{-1}(Q')$.

Proof. It follows from the right colinearity and surjectivity of π that $\pi(P^{\mathrm{co}C}) \subseteq Q^{\mathrm{co}C}$. To prove the converse inclusion, we take advantage of the left-$P^{\mathrm{co}C}$-linear retraction of the inclusion $P^{\mathrm{co}C} \subseteq P$ that was used to prove [6, Theorem 2.5(3)]:

$$\sigma_\varphi : P \longrightarrow P^{\mathrm{co}C}, \quad \sigma_\varphi(p) := p_{(0)}\ell(p_{(1)})^{\langle 1\rangle}\varphi(\ell(p_{(1)})^{\langle 2\rangle})\,. \tag{2.3}$$

Here ℓ is a strong connection on P and φ is any unital linear functional on P. It follows from (1.35) that $\sigma_\varphi(p) \in P^{\mathrm{co}C}$. If $\pi(p) \in Q^{\mathrm{co}C}$, then $\sigma_\varphi(p)$ is a desired element of $P^{\mathrm{co}C}$ that is mapped by π to $\pi(p)$. Indeed, since

$$\pi(p_{(0)})\otimes p_{(1)} = \pi(p)_{(0)}\otimes \pi(p)_{(1)} = \pi(p)\otimes e, \tag{2.4}$$

using the unitality of π, φ, and $\ell(e) = 1\otimes 1$, we compute

$$\pi(\sigma_\varphi(p)) = \pi(p_{(0)})\pi(\ell(p_{(1)})^{\langle 1\rangle})\varphi(\ell(p_{(1)})^{\langle 2\rangle}) = \pi(p). \tag{2.5}$$

To show the second assertion, let us choose any unital k-linear splitting of $\pi\restriction_{P^{\mathrm{co}C}}$ and denote it by $\alpha^{\mathrm{co}C}$. We want to prove that the formula

$$\alpha_R(q) := \alpha^{\mathrm{co}C}(q_{(0)}\pi(\ell(q_{(1)})^{\langle 1\rangle}))\ell(q_{(1)})^{\langle 2\rangle} \tag{2.6}$$

defines a unital right-colinear splitting of π. Since π is surjective, we can write $q=\pi(p)$. Then, using properties of π, we obtain:

$$
\begin{aligned}
q_{(0)}\,\pi(\ell(q_{(1)})^{\langle 1\rangle})\otimes \ell(q_{(1)})^{\langle 2\rangle} &= \pi(p)_{(0)}\,\pi(\ell(\pi(p)_{(1)})^{\langle 1\rangle})\otimes \ell(\pi(p)_{(1)})^{\langle 2\rangle}\\
&= \pi(p_{(0)})\,\pi(\ell(p_{(1)})^{\langle 1\rangle})\otimes \ell(p_{(1)})^{\langle 2\rangle}\\
&= \pi(p_{(0)}\ell(p_{(1)})^{\langle 1\rangle})\otimes \ell(p_{(1)})^{\langle 2\rangle}. \qquad (2.7)
\end{aligned}
$$

Now it follows from (1.35) that the above tensor is in $Q^{\mathrm{co}C}\otimes P$. Hence α_R is well defined. It is straightforward to verify that α_R is unital, right colinear, and splits π. (Note that, since $q\in Q^{\mathrm{co}C}$ implies $q_{(0)}\otimes q_{(1)}=q\otimes e$, we have $\alpha^{\mathrm{co}C}=\alpha_R\restriction_{Q^{\mathrm{co}C}}$.) The third assertion can be proven in an analogous manner.

To prove (iv), we first show that the inverse of the canonical map can_Q : $Q\otimes_{Q^{\mathrm{co}C}}Q\to Q\otimes C$ (see (1.19)) is given by

$$
\mathsf{can}_Q^{-1}: Q\otimes C\longrightarrow Q\underset{Q^{\mathrm{co}C}}{\otimes}Q,\quad q\otimes c\longmapsto q\pi(\ell(c)^{\langle 1\rangle})\underset{Q^{\mathrm{co}C}}{\otimes}\pi(\ell(c)^{\langle 2\rangle}). \qquad (2.8)
$$

Using the properties of π and ℓ, we get

$$
\begin{aligned}
(\mathsf{can}_Q\circ\mathsf{can}_Q^{-1})\,(\pi(p)\otimes c) &= \mathsf{can}_Q\Big(\pi(p\ell(c)^{\langle 1\rangle})\underset{Q^{\mathrm{co}C}}{\otimes}\pi(\ell(c)^{\langle 2\rangle})\Big)\\
&= \pi\left(p\,\ell(c)^{\langle 1\rangle}\,\ell(c)^{\langle 2\rangle}{}_{(0)}\right)\otimes \ell(c)^{\langle 2\rangle}{}_{(1)}\\
&= \pi(p)\otimes c. \qquad (2.9)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(\mathsf{can}_Q^{-1}\circ\mathsf{can}_Q)\Big(\pi(p)\underset{Q^{\mathrm{co}C}}{\otimes}\pi(p')\Big) &= \mathsf{can}_Q^{-1}\Big(\pi(pp'_{(0)})\otimes p'_{(1)}\Big)\\
&= \pi(pp'_{(0)}\ell(p'_{(1)})^{\langle 1\rangle})\underset{Q^{\mathrm{co}C}}{\otimes}\pi(\ell(p'_{(1)})^{\langle 2\rangle})\\
&= \pi(p)\underset{Q^{\mathrm{co}C}}{\otimes}\pi(p'_{(0)}\ell(p'_{(1)})^{\langle 1\rangle}\ell(p'_{(1)})^{\langle 2\rangle})\\
&= \pi(p)\underset{Q^{\mathrm{co}C}}{\otimes}\pi(p'). \qquad (2.10)
\end{aligned}
$$

Here we used the fact that $\pi(p'_{(0)}\ell(p'_{(1)})^{\langle 1\rangle})\otimes\ell(p'_{(1)})^{\langle 2\rangle}\in Q^{\mathrm{co}C}\otimes P$. Hence the extension $Q^{\mathrm{co}C}\subseteq Q$ is Galois, and we have the canonical entwining $\psi_Q: C\otimes Q\to Q\otimes C$.

Our next aim is to prove that ψ_Q is bijective. We know by assumption that the canonical entwining $\psi_P: C\otimes P\to P\otimes C$ is invertible. To determine its inverse, recall that the left and right coactions are given by $\psi_P^{-1}(p\otimes e)$ and

$\psi_P(e\otimes p)$ respectively. Then apply (1.15) to compute

$$
\begin{aligned}
\psi_P\Big((p\ell(c)^{\langle 1\rangle})_{(-1)}\otimes(p\ell(c)^{\langle 1\rangle})_{(0)}\,\ell(c)^{\langle 2\rangle}\Big) &= p\ell(c)^{\langle 1\rangle}\psi_P\Big(e\otimes\ell(c)^{\langle 2\rangle}\Big)\\
&= p\ell(c)^{\langle 1\rangle}\ell(c)^{\langle 2\rangle}{}_{(0)}\otimes\ell(c)^{\langle 2\rangle}{}_{(1)}\\
&= \ p\otimes c. \qquad (2.11)
\end{aligned}
$$

Hence $\psi_P^{-1}(p\otimes c)=(p\ell(c)^{\langle 1\rangle})_{(-1)}\otimes(p\ell(c)^{\langle 1\rangle})_{(0)}\,\ell(c)^{\langle 2\rangle}$. On the other hand,

$$
\begin{aligned}
\psi_Q(c\otimes\pi(p)) &= \pi\big(\ell(c)^{\langle 1\rangle}\big)\big(\pi(\ell(c)^{\langle 2\rangle})\,\pi(p)\big)_{(0)}\otimes\big(\pi(\ell(c)^{\langle 2\rangle})\,\pi(p)\big)_{(1)}\\
&= \pi\big(\ell(c)^{\langle 1\rangle}\big)\,\pi\big((\ell(c)^{\langle 2\rangle}\,p)_{(0)}\big)\otimes(\ell(c)^{\langle 2\rangle}\,p)_{(1)}\\
&= (\pi\otimes\mathrm{id})\big(\psi_P(c\otimes p)\big), \qquad (2.12)
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{id}\otimes\pi)\big(\psi_P^{-1}(p\otimes c)\big) &= (p\,\ell(c)^{\langle 1\rangle})_{(-1)}\otimes\pi\big((p\,\ell(c)^{\langle 1\rangle})_{(0)}\big)\,\pi\big(\ell(c)^{\langle 2\rangle}\big)\\
&= \big(\pi(p\,\ell(c)^{\langle 1\rangle})\big)_{(-1)}\otimes\big(\pi(p\,\ell(c)^{\langle 1\rangle})\big)_{(0)}\,\pi(\ell(c)^{\langle 2\rangle})\\
&= {}_Q\Delta\big(\pi(p)\,\pi(\ell(c)^{\langle 1\rangle})\big)\,\pi(\ell(c)^{\langle 2\rangle}). \qquad (2.13)
\end{aligned}
$$

The second part of the above computation implies that the assignment

$$
\psi_Q^{-1}: Q\otimes C\longrightarrow C\otimes Q,\quad \pi(p)\otimes c\longmapsto(\mathrm{id}\otimes\pi)(\psi_P^{-1}(p\otimes c)), \qquad (2.14)
$$

is well defined. Now it follows from the first part that ψ_Q^{-1} is the inverse of ψ_Q:

$$
\begin{aligned}
\psi_Q\Big(\psi_Q^{-1}\big(\pi(p)\otimes c\big)\Big) &= \psi_Q\Big((\mathrm{id}\otimes\pi)\big(\psi_P^{-1}(p\otimes c)\big)\Big)\\
&= (\pi\otimes\mathrm{id})\Big(\psi_P\big(\psi_P^{-1}(p\otimes c)\big)\Big)=\pi(p)\otimes c, \qquad (2.15)
\end{aligned}
$$

$$
\begin{aligned}
\psi_Q^{-1}\Big(\psi_Q\big(c\otimes\pi(p)\big)\Big) &= \psi_Q^{-1}\Big((\pi\otimes\mathrm{id})\big(\psi_P(c\otimes p)\big)\Big)\\
&= (\mathrm{id}\otimes\pi)\Big(\psi_P^{-1}\big(\psi_P(c\otimes p)\big)\Big)=c\otimes\pi(p). \qquad (2.16)
\end{aligned}
$$

On the other hand, we observe that $(\pi\otimes\pi)\circ\ell$ is a strong connection on Q. Combining it with the just-proven existence of a bijective entwining that makes Q an entwined module, we can apply Lemma 1.1 to conclude the proof of (iv).

To prove the final statement of the lemma, note first that $\pi^{-1}(Q')\in{}^{C}_{e}\mathbf{Alg}^{C}_{e}$. Next, observe that, if $\ell': C\to Q'\otimes Q'$ is a strong connection on Q', then it is also a strong connection on Q. Now, it follows from (1.40) that for any $q\in Q'$

$$
\psi_Q(c\otimes q)=\ell'(c)^{\langle 1\rangle}\Big(\ell'(c)^{\langle 2\rangle}\,q\Big)_{(0)}\otimes\Big(\ell'(c)^{\langle 2\rangle}\,q\Big)_{(1)}\in Q'\otimes C. \qquad (2.17)
$$

In much the same way, it follows from the Q-analog of the formula following (2.11) that $\psi_Q^{-1}(Q'\otimes C)\subseteq C\otimes Q'$. Hence to see that ψ_P and ψ_P^{-1} restrict to $\pi^{-1}(Q')$, we can apply (2.12) and (2.14) respectively.

A key step now is to construct a strong connection on $\pi^{-1}(Q')$. Let α_R and α_L be, respectively, right and left colinear unital splittings of π. Their existence is guaranteed by the already proven (*ii*) and (*iii*). The map

$$(\alpha_L\otimes\alpha_R)\circ\ell' : C\longrightarrow \pi^{-1}(Q)\otimes\pi^{-1}(Q) \tag{2.18}$$

is bicolinear and satisfies

$$\alpha_L(\ell'(e)^{\langle 1\rangle})\otimes\alpha_R(\ell'(e)^{\langle 2\rangle})=1\otimes 1. \tag{2.19}$$

However, it is possible that

$$1\otimes c-\big(\widetilde{\mathsf{can}}\circ(\alpha_L\otimes\alpha_R)\circ\ell'\big)(c)=1\otimes c-\alpha_L(\ell'(c_{(1)})^{\langle 1\rangle})\,\alpha_R(\ell'(c_{(1)})^{\langle 2\rangle})\otimes c_{(2)}\neq 0.$$

To solve this problem, we apply to it the splitting of the lifted canonical map given by a strong connection ℓ (see (1.34)), and add to $(\alpha_L\otimes\alpha_R)\circ\ell'$:

$$\ell_R(c):=(\alpha_L\otimes\alpha_R)(\ell'(c))+\ell(c)-\alpha_L(\ell'(c_{(1)})^{\langle 1\rangle})\,\alpha_R(\ell'(c_{(1)})^{\langle 2\rangle})\,\ell(c_{(2)}^{\langle 1\rangle})\otimes\ell(c_{(2)}^{\langle 2\rangle}). \tag{2.20}$$

Now $\widetilde{\mathsf{can}}\circ\ell_R=1\otimes\mathrm{id}$, as needed. Also, $\ell_R(e)=1\otimes 1$ and $((\pi\otimes\mathrm{id})\circ\ell_R)(C)\subseteq Q'\otimes P$. The right colinearity of ℓ_R is clear. To check the left colinearity of ℓ_R, using the fact that P is a ψ_P entwined and e-coaugmented module, we show that $(m_P\circ(\alpha_L\otimes\alpha_R)\circ\ell')*\ell$ is left colinear. (Here m_P is the multiplication of P.) First we note that

$$({}_P\Delta\otimes\mathrm{id})\circ\big((m_P\circ(\alpha_L\otimes\alpha_R)\circ\ell')*\ell\big)=\big(\mathrm{id}\otimes(m_P\circ(\alpha_L\otimes\alpha_R)\circ\ell')*\ell\big)\circ\Delta \tag{2.21}$$

is equivalent to

$$\begin{aligned}&\alpha_L(\ell'(c_{(1)})^{\langle 1\rangle})\,\alpha_R(\ell'(c_{(1)})^{\langle 2\rangle})\,\ell(c_{(2)})^{\langle 1\rangle}\otimes e\otimes\ell(c_{(2)})^{\langle 2\rangle}\\ &\quad=\psi_P\Big(c_{(1)}\otimes\alpha_L(\ell'(c_{(2)})^{\langle 1\rangle})\,\alpha_R(\ell'(c_{(2)})^{\langle 2\rangle})\,\ell(c_{(3)})^{\langle 1\rangle}\Big)\otimes\ell(c_{(3)})^{\langle 2\rangle}.\end{aligned} \tag{2.22}$$

Since $c_{(1)}\otimes\alpha_L(\ell'(c_{(2)})^{\langle 1\rangle})\otimes\ell'(c_{(2)})^{\langle 2\rangle}=\psi_P^{-1}\big(\alpha_L(\ell'(c)^{\langle 1\rangle})\otimes e\big)\otimes\ell'(c)^{\langle 2\rangle}$, we obtain

$$\begin{aligned}&\psi_P\Big(c_{(1)}\otimes\alpha_L(\ell'(c_{(2)})^{\langle 1\rangle})\,\alpha_R(\ell'(c_{(2)})^{\langle 2\rangle})\,\ell(c_{(3)})^{\langle 1\rangle}\Big)\otimes\ell(c_{(3)})^{\langle 2\rangle}\\
&\quad=\alpha_L(\ell'(c_{(1)})^{\langle 1\rangle})\,\psi_P\Big(e\otimes\alpha_R(\ell'(c_{(1)})^{\langle 2\rangle})\,\ell(c_{(2)})^{\langle 1\rangle}\Big)\otimes\ell(c_{(2)})^{\langle 2\rangle}\\
&\quad=\alpha_L(\ell'(c_{(1)})^{\langle 1\rangle})\,\alpha_R(\ell'(c_{(1)})^{\langle 2\rangle})\,\psi_P\Big(c_{(2)}\otimes\ell(c_{(3)})^{\langle 1\rangle}\Big)\otimes\ell(c_{(3)})^{\langle 2\rangle}\\
&\quad=\alpha_L(\ell'(c_{(1)})^{\langle 1\rangle})\,\alpha_R(\ell'(c_{(1)})^{\langle 2\rangle})\,\psi_P\Big(\psi_P^{-1}\big(\ell(c_{(2)})^{\langle 1\rangle}\otimes e\big)\Big)\otimes\ell(c_{(2)})^{\langle 2\rangle}\\
&\quad=\alpha_L(\ell'(c_{(1)})^{\langle 1\rangle})\,\alpha_R(\ell'(c_{(1)})^{\langle 2\rangle})\,\ell(c_{(2)})^{\langle 1\rangle}\otimes e\otimes\ell(c_{(2)})^{\langle 2\rangle}.\end{aligned} \tag{2.23}$$

Hence ℓ_R is a strong connection with the property $\ell_R(C)\subseteq\pi^{-1}(Q')\otimes P$. In a similar manner, we construct a strong connection ℓ_L with the property $\ell_L(C)\subset P\otimes\pi^{-1}(Q')$. Now we need to apply the splitting of the lifted left canonical map given by ℓ (see (1.33)) to derive the formula

$$\ell_L:=(\alpha_L\otimes\alpha_R)\circ\ell'+\ell-\ell*(m_P\circ(\alpha_L\otimes\alpha_R)\circ\ell'). \tag{2.24}$$

It is clear that $\ell_L(e) = 1\otimes 1$ and $\ell_L(C) \subseteq P\otimes\pi^{-1}(Q')$. A computation similar to (2.23) shows the right colinearity of ℓ_L. Since furthermore $\psi_P(1\otimes c) = c\otimes 1$ for any $c\in C$ and $\widetilde{\mathsf{can}} = \psi_P\circ\widetilde{\mathsf{can}}_L$, we obtain

$$\widetilde{\mathsf{can}}(\ell_L(c)) = \psi_P\big(\widetilde{\mathsf{can}}_L(\ell(c))\big) = \psi_P(c\otimes 1) = 1\otimes c. \tag{2.25}$$

Hence ℓ_L is a desired strong connection. Plugging it into (2.20) instead of ℓ, we get a strong connection

$$\ell_{LR} = (\alpha_L\otimes\alpha_R)\circ\ell' + \ell_L - (m_p\circ(\alpha_L\otimes\alpha_R)\circ\ell')*\ell_L \tag{2.26}$$

with the property $\ell_{LR}\subseteq\pi^{-1}(Q')\otimes\pi^{-1}(Q')$. Applying now Lemma 1.1 ends the proof of this lemma. □

2.2. The one-surjective pullbacks of principal coactions are principal. Our goal now is to show that the subcategory of principal extensions is closed under one-surjective pullbacks. Here the right coaction is the coaction defining a principal extension and the left coaction is the one defined by the inverse of the canonical entwining (see (1.24)). With this structure, principal extensions form a full subcategory of ${}^{C}_{e}\mathbf{Alg}^{C}_{e}$. The following theorem is the main result of this paper generalizing the theorem of [15] on the pullback of surjections of principal comodule algebras:

Theorem 2.2. *Let C be a coalgebra, $e\in C$ a group-like element, and P the pullback of $\pi_1 : P_1\to P_{12}$ and $\pi_2 : P_2\to P_{12}$ in the category ${}^{C}_{e}\mathbf{Alg}^{C}_{e}$ of unital algebras with e-coaugmented left and right C-coactions. If π_1 or π_2 is surjective and both P_1 and P_2 are principal e-coaugmented C-extensions, then also P is a principal e-coaugmented C-extension.*

Proof. Without loss of generality, we assume that π_1 is surjective. We first show that P inherits an entwined structure from P_1 and P_2.

Lemma 2.3. *Let ψ_1 and ψ_2 denote the entwining structures of P_1 and P_2 respectively. Then P is an entwined module with an invertible entwining structure*

$$\psi = \psi_1\circ(\mathrm{id}\otimes\mathrm{pr}_1) + \psi_2\circ(\mathrm{id}\otimes\mathrm{pr}_2). \tag{2.27}$$

Here pr_1 *and* pr_2 *are morphisms of the pullback diagram as in* (1.6).

Proof. Our strategy is to construct a bijective map

$$\tilde\psi : C\otimes(P_1\times P_2)\longrightarrow(P_1\times P_2)\otimes C, \tag{2.28}$$

and to show that it restricts to a bijective entwining on $C\otimes P$. We put

$$\tilde\psi := \psi_1\circ(\mathrm{id}\otimes\tilde{\mathrm{pr}}_1) + \psi_2\circ(\mathrm{id}\otimes\tilde{\mathrm{pr}}_2). \tag{2.29}$$

The symbols $\tilde{\mathrm{pr}}_1$ and $\tilde{\mathrm{pr}}_2$ stand for respective componentwise projections. Their restrictions to P yield pr_1 and pr_2. It is easy to check that the inverse of $\tilde\psi$ is given by

$$\tilde\psi^{-1} = \psi_1^{-1}\circ(\tilde{\mathrm{pr}}_1\otimes\mathrm{id}) + \psi_2^{-1}\circ(\tilde{\mathrm{pr}}_2\otimes\mathrm{id}). \tag{2.30}$$

To show that $\tilde{\psi}(C\otimes P)\subseteq P\otimes C$ and $\tilde{\psi}^{-1}(P\otimes C)\subseteq C\otimes P$, we note first that P_{12} and $\pi_2(P_2)$ are principal by Lemma 2.1(iv). Consequently, their canonical entwinings ψ_{12} and $\psi_{\pi_2(P_2)}$ are bijective. Furthermore, arguing as in the proof of Lemma 2.1, we see that $\psi_{\pi_2(P_2)}=\psi_{12}{\restriction}_{C\otimes\pi_2(P_2)}$ and $\psi^{-1}_{\pi_2(P_2)}=\psi_{12}^{-1}{\restriction}_{\pi_2(P_2)\otimes C}$. An advantage of having both summands in terms of ψ_{12} is that we can apply (2.12) to compute

$$\begin{aligned}
&\big((\pi_1\circ\tilde{\mathrm{pr}}_1-\pi_2\circ\tilde{\mathrm{pr}}_2)\otimes\mathrm{id}\big)\circ\tilde{\psi}\\
&\qquad=(\pi_1\circ\tilde{\mathrm{pr}}_1\otimes\mathrm{id})\circ\psi_1\circ(\mathrm{id}\otimes\tilde{\mathrm{pr}}_1)-(\pi_2\circ\tilde{\mathrm{pr}}_2\otimes\mathrm{id})\circ\psi_1\circ(\mathrm{id}\otimes\tilde{\mathrm{pr}}_1)\\
&\qquad\quad+(\pi_1\circ\tilde{\mathrm{pr}}_1\otimes\mathrm{id})\circ\psi_2\circ(\mathrm{id}\otimes\tilde{\mathrm{pr}}_2)-(\pi_2\circ\tilde{\mathrm{pr}}_2\otimes\mathrm{id})\circ\psi_2\circ(\mathrm{id}\otimes\tilde{\mathrm{pr}}_2)\\
&\qquad=(\pi_1\otimes\mathrm{id})\circ\psi_1\circ(\mathrm{id}\otimes\tilde{\mathrm{pr}}_1)-(\pi_2\otimes\mathrm{id})\circ\psi_2\circ(\mathrm{id}\otimes\tilde{\mathrm{pr}}_2)\\
&\qquad=\psi_{12}\circ(\mathrm{id}\otimes\pi_1)\circ(\mathrm{id}\otimes\tilde{\mathrm{pr}}_1)-\psi_{\pi_2(P_2)}\circ(\mathrm{id}\otimes\pi_2)\circ(\mathrm{id}\otimes\tilde{\mathrm{pr}}_2)\\
&\qquad=\psi_{12}\circ\big(\mathrm{id}\otimes(\pi_1\circ\tilde{\mathrm{pr}}_1-\pi_2\circ\tilde{\mathrm{pr}}_2)\big).
\end{aligned}\tag{2.31}$$

Hence $\tilde{\psi}(C\otimes P)\subseteq P\otimes C$. In much the same way, using (2.14) instead of (2.12), we show that $\tilde{\psi}^{-1}(P\otimes C)\subseteq C\otimes P$.

It remains to verify that the bijection $\psi=\tilde{\psi}{\restriction}_{C\otimes P}$ is an entwining that makes P an entwined module. The former is proven by checking directly (1.15) and (1.16). The latter follows from the fact that P_1 and P_2 are, respectively, ψ_1 and ψ_2 entwined modules:

$$\begin{aligned}
\Delta_P(pq)&=\Delta_{P_1}(\mathrm{pr}_1(p)\mathrm{pr}_1(q))+\Delta_{P_2}(\mathrm{pr}_2(p)\mathrm{pr}_2(q))\\
&=\mathrm{pr}_1(p_{(0)})\psi_1(p_{(1)}\otimes\mathrm{pr}_1(q))+\mathrm{pr}_2(p_{(0)})\psi_2(p_{(1)}\otimes\mathrm{pr}_2(q))\\
&=\big(\mathrm{pr}_1(p_{(0)})+\mathrm{pr}_2(p_{(0)})\big)\big(\psi_1(p_{(1)}\otimes\mathrm{pr}_1(q))+\psi_2(p_{(1)}\otimes\mathrm{pr}_2(q))\big)\\
&=p_{(0)}\psi(p_{(1)}\otimes q).
\end{aligned}\tag{2.32}$$

This proves the lemma. □

Let α_L^1 and α_R^1 be a unital left-colinear splitting and a unital right-colinear splitting of π_1, respectively. Also, let α_R^2 be a right-colinear splitting of π_2 viewed as a map onto $\pi_2(P_2)$. Such maps exist by Lemma 2.1. Furthermore, by [6, Lemma 2.2], since P_1 and P_2 are principal, they admit strong connections ℓ_1 and ℓ_2 respectively. For brevity, let us introduce the notation

$$\begin{gathered}
\alpha_L^{12}:=\alpha_L^1\circ\pi_2,\quad \alpha_R^{12}:=\alpha_R^1\circ\pi_2,\\
\alpha_R^{21}:=\alpha_R^2\circ\pi_1{\restriction}_{\pi_1^{-1}(\pi_2(P_2))},\quad L:=m_{P_1}\circ(\alpha_L^{12}\otimes\alpha_R^{12})\circ\ell_2\,,
\end{gathered}\tag{2.33}$$

where m_{P_1} is the multiplication of P_1. The situation is illustrated in the following diagram:

(2.34)
$$
\begin{tikzcd}
C \arrow[dd, "L"] & & P \arrow[ddll, "\mathrm{pr}_1"'] \arrow[ddrr, "\mathrm{pr}_2"] & & \\
& & \pi_1^{-1}(\pi_2(P_2)) \arrow[dll, hook] \arrow[drr, "\alpha_R^{21}"] & & \\
P_1 \arrow[ddrr, "\pi_1"] & & & & P_2 \arrow[llll, shift right, "\alpha_L^{12}"'] \arrow[llll, shift left, "\alpha_R^{12}"] \arrow[dll, shift left, "\pi_2"] \arrow[ddll, "\pi_2"] \\
& & \pi_2(P_2) \arrow[urr, "\alpha_R^{2}"] \arrow[d, hook] & & \\
& & P_{12} \arrow[uull, shift right, "\alpha_R^{1}"'] \arrow[uull, shift left, "\alpha_L^{1}"] & &
\end{tikzcd}
\quad .
$$

Our proof hinges on constructing a strong connection on P out of strong connections on P_1 and P_2. Roughly speaking, the basic idea is to take a strong connection on P_2, induce a strong connection on the the common part P_{12}, and prolongate it to P_1. To this end, we check that $(\alpha_L^{12}+\mathrm{id})\otimes(\alpha_R^{12}+\mathrm{id})$ is a unital bicolinear map from $P_2\otimes P_2$ to $P\otimes P$. Therefore, as a first approximation for constructing a strong connection on P, we choose the formula

$$\ell_I := \big((\alpha_L^{12}+\mathrm{id})\otimes(\alpha_R^{12}+\mathrm{id})\big)\circ \ell_2. \tag{2.35}$$

It is a bicolinear map from C to $P\otimes P$ satisfying $\ell_I(e)=1\otimes 1$ as needed.

However, it does not split the lifted canonical map:

$$
\begin{aligned}
&(\widetilde{\mathsf{can}}\circ \ell_I)(c) - 1\otimes c\\
&\qquad = \alpha_L^{12}(\ell_2(c)^{\langle 1\rangle})\,\alpha_R^{12}(\ell_2(c)^{\langle 2\rangle})_{(0)}\otimes \alpha_R^{12}(\ell_2(c)^{\langle 2\rangle})_{(1)}\\
&\qquad\quad + \ell_2(c)^{\langle 1\rangle}\ell_2(c)^{\langle 2\rangle}{}_{(0)}\otimes \ell_2(c)^{\langle 2\rangle}{}_{(1)} - 1\otimes c\\
&\qquad = \alpha_L^{12}(\ell_2(c_{(1)})^{\langle 1\rangle})\,\alpha_R^{12}(\ell_2(c_{(1)})^{\langle 2\rangle})\otimes c_{(2)} + (0,1)\otimes c - 1\otimes c\\
&\qquad = L(c_{(1)})\otimes c_{(2)} - (1,0)\otimes c \;\in\; P_1\otimes C.
\end{aligned}
\tag{2.36}
$$

We correct it by adding to $\ell_I(c)$ the splitting of the lifted canonical map on $P_1\otimes P_1$ afforded by ℓ_1 and applied to $(1,0)\otimes c - L(c_{(1)})\otimes c_{(2)}$:

$$
\begin{aligned}
\ell_{II}(c) &= \ell_I(c) + \ell_1(c)^{\langle 1\rangle}\otimes \ell_1(c)^{\langle 2\rangle} - L(c_{(1)})\,\ell_1(c_{(2)})^{\langle 1\rangle}\otimes \ell_1(c_{(2)})^{\langle 2\rangle}\\
&= (\ell_I + \ell_1 - L*\ell_1)(c).
\end{aligned}
\tag{2.37}
$$

The above approximation to a strong connection on P is clearly right colinear. Using the fact that P_1 is a ψ_1-entwined and e-coaugmented module, we follow the lines of (2.21)–(2.23) to show that $L*\ell_1$ is left colinear. Hence ℓ_{II} is bicolinear. It also satisfies $\ell_{II}(e) = 1 \otimes 1$.

The price we pay for having $\ell_{II}(c)^{\langle 1\rangle}\,\ell_{II}(c)^{\langle 2\rangle}{}_{(0)} \otimes \ell_{II}(c)^{\langle 2\rangle}{}_{(1)} = 1 \otimes c$ is that the image of ℓ_{II} is no longer in $P \otimes P$. The troublesome term $\ell_1 - L * \ell_1$ takes values in $P \otimes P_1$. Now one would like to compose $\mathrm{id} \otimes (\mathrm{id} + \alpha_R^{21})$ with $\ell_1 - L * \ell_1$ to force it taking values in $P \otimes P$. However, since α_R^{21} is defined only on $\pi_1^{-1}(\pi_2(P_2))$, we need to replace an arbitrary strong connection ℓ_1 by a strong connection taking values in $P_1 \otimes \pi_1^{-1}(\pi_2(P_2))$. Such a strong connection is provided for us by (2.24):

$$\tilde{\ell}_1 := (\alpha_L^{12} \otimes \alpha_R^{12}) \circ \ell_2 + \ell_1 - \ell_1 * L. \tag{2.38}$$

Inserting $\tilde{\ell}_1$ in place of ℓ_1 allows us to apply the correction map $\mathrm{id} \otimes (\mathrm{id} + \alpha_R^{21})$ to obtain

$$\ell_{III} = \ell_I + \big(\mathrm{id} \otimes (\mathrm{id} + \alpha_R^{21})\big) \circ (\tilde{\ell}_1 - L * \tilde{\ell}_1). \tag{2.39}$$

To end the proof, let us check that ℓ_{III} is indeed a strong connection on P. First, since $\ell_I(C) \subseteq P \otimes P$ and $(\mathrm{id} + \alpha_R^{21})(\pi_1^{-1}(\pi_2(P_2))) \subseteq P$, we conclude that ℓ_{III} takes values in $P \otimes P$. Next, it is bicolinear because α_R^{21} is right colinear. Also, it is clearly unital. To verify that ℓ_{III} splits the canonical map, first we note that $\widetilde{\mathsf{can}} \circ (\mathrm{id} \otimes \alpha_R^{21}) \circ (\tilde{\ell}_1 - L * \tilde{\ell}_1) = 0$ because $m_{P_1 \times P_2}(p_1 \otimes p_2) = 0$ for all $p_1 \in P_1$ and $p_2 \in P_2$. Combining this with the fact that $\widetilde{\mathsf{can}} \circ (\ell_1' - L * \ell_1')$ does not depend on the choice of a strong connection ℓ_1', we infer that $\widetilde{\mathsf{can}} \circ \ell_{III} = \widetilde{\mathsf{can}} \circ \ell_{II} = 1 \otimes \mathrm{id}$. Thus ℓ_{III} is a strong connection on P as desired. Combining this fact with Lemma 2.3 and Lemma 1.1 proves the theorem. □

Putting together the formulas in the proof of Theorem 2.2 , we obtain the following strong connection on P:

$$\begin{aligned} \ell \;=\; & \big((\alpha_L^{12} + \mathrm{id}) \otimes (\alpha_R^{12} + \mathrm{id})\big) \circ \ell_2 \\ & + (\eta_1 \circ \varepsilon - L) * \big((\mathrm{id} \otimes (\mathrm{id} + \alpha_R^{21})) \circ (\ell_1 - \ell_1 * L + (\alpha_L^{12} \otimes \alpha_R^{12}) \circ \ell_2)\big). \end{aligned} \tag{2.40}$$

3. The pullback picture of the standard quantum Hopf fibration

Recall that the classical Hopf fibration is a U(1)-principal bundle given by the maps

$$\begin{aligned} &\pi : \mathrm{S}^3 := \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} \longrightarrow \mathrm{S}^2 \cong \mathbb{C}\mathrm{P}^1, \\ &\pi((z_1, z_2)) := [(z_1 : z_2)], \end{aligned} \tag{3.1}$$

$$\mathrm{S}^3 \times \mathrm{U}(1) \longrightarrow \mathrm{S}^3, \quad (z_1, z_2) \triangleleft u := (z_1 u, z_2 u). \tag{3.2}$$

To unravel the structure of this non-trivial fibration, we split S^3 into two disjoint parts:

$$\mathrm{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 < 1, |z_2|^2 = 1 - |z_1|^2\} \;\cup\; \{(z_1, 0) \,|\, |z_1| = 1\}. \tag{3.3}$$

Note that both sets are invariant under the U(1)-action. The second set is U(1), and first set is U(1)-equivariantly homeomorphic to the interior of the solid torus $\mathrm{D}\times\mathrm{U}(1)$ equipped with the diagonal action. (Here $\mathrm{D}=\{z\in\mathbb{C}\,|\,|z|\leq 1\}$.) By an appropriate $\mathrm{U}(1)$-equivariant gluing of the boundary torus of $\mathrm{D}\times\mathrm{U}(1)$ with $\mathrm{U}(1)$, we recover S^3 with its $\mathrm{U}(1)$-action:

(3.4)
$$\phi_1(z,v):=(z,v\sqrt{1-|z|^2})\qquad \phi_2(u):=(u,0)$$
$$\begin{tikzcd}
& \mathrm{S}^3 & \\
\mathrm{D}\times\mathrm{U}(1) \arrow[ur, "\phi_1"] & & \mathrm{U}(1) \arrow[ul, "\phi_2"'] \\
& \mathrm{U}(1)\times\mathrm{U}(1) \arrow[ul, "(\iota,\mathrm{id})"] \arrow[ur, "\mathrm{pr}_1"'] &
\end{tikzcd}$$
$$(\iota,\mathrm{id})(u,v):=(u,v)\qquad \mathrm{pr}_1(u,v):=u.$$

However, to view $\mathrm{D}\times\mathrm{U}(1)$ as a trivial $\mathrm{U}(1)$-principal bundle, we need to gauge the diagonal action to the action on the right slot. This is achieved with the help of the following homeomorphism intertwining these two actions:

$$\Psi:\mathrm{D}\times\mathrm{U}(1)\longrightarrow\mathrm{D}\times\mathrm{U}(1),\quad \Psi(x,v):=(xv,v),\quad \Psi(x,vu)=\Psi(x,v)\triangleleft u.$$

Let us denote the restriction of Ψ to $\mathrm{U}(1)\times\mathrm{U}(1)$ by the same symbol. Now we can extend the above pushout diagram to the commutative diagram

(3.5)
$$\begin{tikzcd}
& & \mathrm{S}^3 & & \\
\mathrm{D}\times\mathrm{U}(1) \arrow[r, "\Psi"] & \mathrm{D}\times\mathrm{U}(1) \arrow[ur, "\phi_1"] & & \mathrm{U}(1) \arrow[ul, "\phi_2"'] & \mathrm{U}(1) \arrow[l, "\mathrm{id}"'] \\
& & \mathrm{U}(1)\times\mathrm{U}(1) \arrow[ul, "(\iota,\mathrm{id})"] \arrow[ur, "\mathrm{pr}_1"'] & & \\
& & \mathrm{U}(1)\times\mathrm{U}(1)\,, \arrow[uull, "(\iota,\mathrm{id})"] \arrow[u, "\Psi"] \arrow[uurr, "m"'] & &
\end{tikzcd}$$

where m is the multiplication map. The outer diagram is again a pushout diagram of $\mathrm{U}(1)$-spaces, but now its defining $\mathrm{U}(1)$-spaces are trivial $\mathrm{U}(1)$-principal bundles. It is the outer pushout diagram that we shall use to analyse a noncommutative deformation of the Hopf fibration.

3.1. Pullback comodule algebra. We consider the tensor products $P_1:=\mathcal{T}\otimes\mathcal{O}(\mathrm{U}(1))$, $P_2:=\mathbb{C}\otimes\mathcal{O}(\mathrm{U}(1))=\mathcal{O}(\mathrm{U}(1))$ and $P_{12}:=C(\mathrm{U}(1))\otimes\mathcal{O}(\mathrm{U}(1))$. These algebras are right $\mathcal{O}(\mathrm{U}(1))$-comodule algebras for the coaction $x\otimes u^N\mapsto x\otimes u^N\otimes u^N$, $N\in\mathbb{Z}$. Moreover, P_1 and P_2 are trivially principal with strong connections $\ell_i:\mathcal{O}(\mathrm{U}(1))\to P_i\otimes P_i$ given by

$$\ell_i(u^N):=(1\otimes u^{N*})\otimes(1\otimes u^N),\quad i=1,2. \tag{3.6}$$

To construct a pullback of P_1 and P_2, we define the following morphisms of right $\mathcal{O}(\mathrm{U}(1))$-comodule algebras:

$$\pi_1 : \mathcal{T} \otimes \mathcal{O}(\mathrm{U}(1)) \longrightarrow C(\mathrm{U}(1)) \otimes \mathcal{O}(\mathrm{U}(1)), \quad \pi_1(t \otimes w) := \sigma(t) \otimes w, \tag{3.7}$$

$$\pi_2 : \mathcal{O}(\mathrm{U}(1)) \longrightarrow C(\mathrm{U}(1)) \otimes \mathcal{O}(\mathrm{U}(1)), \quad \pi_2(w) := \Delta(w). \tag{3.8}$$

Then the fibre product $P := \mathcal{T} \otimes \mathcal{O}(\mathrm{U}(1)) \times_{(\pi_1,\pi_2)} \mathcal{O}(\mathrm{U}(1))$ defined by the pullback diagram

$$\begin{tikzcd}[column sep=tiny] & \mathcal{T} \otimes \mathcal{O}(\mathrm{U}(1)) \underset{(\pi_1,\pi_2)}{\times} \mathcal{O}(\mathrm{U}(1)) \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ \mathcal{T} \otimes \mathcal{O}(\mathrm{U}(1)) \arrow[dr, "\pi_1"'] & & \mathcal{O}(\mathrm{U}(1)) \arrow[dl, "\pi_2"] \\ & C(\mathrm{U}(1)) \otimes \mathcal{O}(\mathrm{U}(1)) & \end{tikzcd} \tag{3.9}$$

is a right $\mathcal{O}(\mathrm{U}(1))$-comodule algebra. By Proposition 2.2, it is principal.

Furthermore, define unital, respectively, left-colinear and right-colinear splittings of π_1 by

$$\alpha^1_L(f \otimes u^N) := T_f \otimes u^N =: \alpha^1_R(f \otimes u^N), \quad N \in \mathbb{Z}. \tag{3.10}$$

Here $f \in C(\mathrm{U}(1))$ and T_f denotes the Toeplitz operator with symbol f. In particular, $T_{u^N} = \mathrm{S}^N$ and $T_{u^{*N}} = \mathrm{S}^{*N}$. A right-colinear splitting of the map $\pi_2 : \mathcal{O}(\mathrm{U}(1)) \to \pi_2(\mathcal{O}(\mathrm{U}(1)))$ is given by

$$\alpha^2_R(u^N \otimes u^N) := u^N, \quad N \in \mathbb{Z}. \tag{3.11}$$

Inserting the definitions of α^1_L, α^i_R and ℓ_i, $i = 1, 2$, into (2.33) and (2.40), we obtain the following strong connection on P:

$$\ell(u^N) = (\mathrm{S}^{*N} \otimes u^{*N}, u^{*N}) \otimes (\mathrm{S}^N \otimes u^N, u^N), \tag{3.12}$$

$$\begin{aligned} \ell(u^{*N}) &= (\mathrm{S}^N \otimes u^N, u^N) \otimes (\mathrm{S}^{*N} \otimes u^{*N}, u^{*N}) \\ &\quad + ((1 - \mathrm{S}^N\mathrm{S}^{*N}) \otimes u^N, 0) \otimes ((1 - \mathrm{S}^N\mathrm{S}^{*N}) \otimes u^{*N}, 0), \quad N \in \mathbb{N}. \end{aligned} \tag{3.13}$$

Note next that, by construction, we have

$$P = \Big\{ \textstyle\sum_k (t_k \otimes u^k, \alpha_k u^k) \in \big(\mathcal{T} \otimes \mathcal{O}(\mathrm{U}(1))\big) \times \mathcal{O}(\mathrm{U}(1)) \;\Big|\; \sigma(t_k) = \alpha_k u^k \Big\}, \tag{3.14}$$

where $\alpha_k \in \mathbb{C}$. For ${}_{\mathbb{C}}\Delta(1) := u^N \otimes 1$, let

$$L_N := P \underset{\mathcal{O}(\mathrm{U}(1))}{\Box} \mathbb{C} = \{p \in P \mid \Delta_P(p) = p \otimes u^N\}. \tag{3.15}$$

Then $L_0 = P^{\mathrm{co}\mathcal{O}(\mathrm{U}(1))}$, each L_N is a left $P^{\mathrm{co}\mathcal{O}(\mathrm{U}(1))}$-module and $P = \bigoplus_{N\in\mathbb{Z}} L_N$. From

$$\Delta_P\big(\sum_k (t_k\otimes u^k, \alpha_k u^k)\big) = \sum_k (t_k\otimes u^k, \alpha_k u^k)\otimes u^k, \tag{3.16}$$

it follows that

$$L_N = \Big\{ (t\otimes u^N, \alpha u^N) \in \big(\mathcal{T}\otimes\mathcal{O}(\mathrm{U}(1))\big)\times\mathcal{O}(\mathrm{U}(1)) \;\Big|\; \sigma(t) = \alpha u^N \Big\}. \tag{3.17}$$

The next proposition shows that $L_0 \cong \mathcal{T}\times_{(\sigma,1)}\mathbb{C}$ is isomorphic to the C^*-algebra of the standard Podleś sphere and that

$$L_N \cong \mathcal{T}\times_{(u^{-N}\sigma,1)}\mathbb{C}, \tag{3.18}$$

where $\mathcal{T}\times_{(u^N\sigma,1)}\mathbb{C}$ is given by the pullback diagram

$$\begin{tikzcd} & \mathcal{T}\underset{(u^{-N}\sigma,1)}{\times}\mathbb{C} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ \mathcal{T} \arrow[d, "\sigma"'] & & \mathbb{C} \arrow[d, "\alpha\mapsto\alpha 1"] \\ C(\mathrm{U}(1)) \arrow[rr, "f\mapsto u^{-N}f"'] & & C(\mathrm{U}(1)). \end{tikzcd} \tag{3.19}$$

Proposition 3.1. *The fibre product $\mathcal{T}\times_{(\sigma,1)}\mathbb{C}$ is isomorphic to the C^*-algebra $C(\mathrm{S}^2_q)$, and L_N is isomorphic to $\mathcal{T}\times_{(u^{-N}\sigma,1)}\mathbb{C}$ as a left $C(\mathrm{S}^2_q)$-module with respect to the left $C(\mathrm{S}^2_q)$-action on $\mathcal{T}\times_{(u^{-N}\sigma,1)}\mathbb{C}$ given by $(t,\alpha)\cdot(h,\beta) := (th,\alpha\beta)$.*

Proof. For $N=0$, the mappings $\mathcal{T}\ni t\mapsto\sigma(t)\in C(\mathrm{U}(1))$ and $\mathbb{C}\ni\alpha\mapsto\alpha 1\in C(\mathrm{U}(1))$ are morphisms of C^*-algebras, so that $\mathcal{T}\times_{(\sigma,1)}\mathbb{C}$ is a C^*-algebra. Next, recall that $C(\mathrm{S}^2_q)\cong\mathcal{K}(\ell_2(\mathbb{N}))\oplus\mathbb{C}$ (see (1.57)), and define

$$\phi : \mathcal{T}\underset{(\sigma,1)}{\times}\mathbb{C} \longrightarrow \mathcal{K}(\ell_2(\mathbb{N}))\oplus\mathbb{C}, \quad \phi(t,\alpha) := t, \tag{3.20}$$

$$\phi^{-1} : \mathcal{K}(\ell_2(\mathbb{N}))\oplus\mathbb{C} \longrightarrow \mathcal{T}\underset{(\sigma,1)}{\times}\mathbb{C}, \quad \phi^{-1}(k+\alpha) := (k+\alpha,\alpha). \tag{3.21}$$

Clearly, $\phi : \mathcal{T}\times_{(\sigma,1)}\mathbb{C}\to\mathcal{B}(\ell_2(\mathbb{N}))$ is a morphism of C^*-algebras. Since $\phi(t,\alpha) = (t-\alpha)+\alpha$, and $\sigma(t-\alpha)=0$ by the pullback diagram (3.19), it follows from the short exact sequence (1.62) that $t-\alpha\in\mathcal{K}(\ell_2(\mathbb{N}))$. Hence $\phi(t,\alpha)\in\mathcal{K}(\ell_2(\mathbb{N}))\oplus\mathbb{C}$. One easily sees that ϕ^{-1} is the inverse of ϕ, so that $\mathcal{T}\times_{(\sigma,1)}\mathbb{C}\cong\mathcal{K}(\ell_2(\mathbb{N}))\oplus\mathbb{C}$.

The fact that $\mathcal{T}\times_{(u^{-N}\sigma,1)}\mathbb{C}$ with the given $C(\mathrm{S}^2_q)$-action is a left $C(\mathrm{S}^2_q)$-module follows from the discussion preceding the pullback diagram (1.10) with the free rank one modules $E_1=\mathcal{T}$, $E_2=\mathbb{C}$ and $\pi_{1*}E_1=\pi_{2*}E_2=C(\mathrm{U}(1))$. Obviously,

$L_N \ni (t \otimes u^N, \alpha u^N) \mapsto (t, \alpha) \in \mathcal{T} \times_{(u^{-N}\sigma,1)} \mathbb{C}$ defines an isomorphism of left $C(\mathrm{S}^2_q)$-modules. $\square$

3.2. EQUIVALENCE OF THE PULLBACK AND STANDARD CONSTRUCTIONS. Let us view U(1) as a compact quantum group. We consider its C^*-algebra $C(\mathrm{U}(1))$ of all continuous function together with the obvious coproduct, counit and antipode given by $\Delta(f)(x,y) = f(xy)$, $\varepsilon(f) = f(1)$ and $S(f)(x) = f(x^{-1})$, respectively. Furthermore, let $\bar{\otimes}$ stand for the completed tensor product of C^*-algebras. In our case it is unique because of the nuclearity of the involved C^*-algebras.

Now let $\pi_2 : C(\mathrm{U}(1)) \to C(\mathrm{U}(1)) \bar{\otimes} C(\mathrm{U}(1))$ be given by the coproduct, i.e. $\pi_2(f)(x,y) := (\Delta f)(x,y) = f(xy)$, and let σ denote the symbol map $\mathcal{T} \to C(\mathrm{U}(1))$. Then $\bar{P} := \mathcal{T} \bar{\otimes} C(\mathrm{U}(1)) \times_{(\pi_1,\pi_2)} C(\mathrm{U}(1))$ is defined by the pullback diagram

$$\begin{tikzcd}[column sep=small]
& \mathcal{T} \bar{\otimes} C(\mathrm{U}(1)) \underset{(\pi_1,\pi_2)}{\times} C(\mathrm{U}(1)) \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\
\mathcal{T} \bar{\otimes} C(\mathrm{U}(1)) \arrow[dr, "\pi_1 = \sigma\otimes\mathrm{id}"'] & & C(\mathrm{U}(1)) \arrow[dl, "\pi_2 = \Delta"] \\
& C(\mathrm{U}(1)) \bar{\otimes} C(\mathrm{U}(1)) \,. &
\end{tikzcd} \tag{3.22}$$

With the $C(\mathrm{U}(1))$-coaction given by the coproduct Δ on the right tensor factor $C(\mathrm{U}(1))$, π_1 and π_2 are morphisms in the category of right $C(\mathrm{U}(1))$-comodule C^*-algebras. Equivalently, we can view this diagram as a diagram in the category of U(1)-C^*-algebras (see Section 1.3). Therefore, $\bar{P}$ inherits the structure of a right U(1)-C^*-algebra. Using the counit $\varepsilon : C(\mathrm{U}(1)) \to \mathbb{C}$ and the fact that the Peter-Weyl functor commutes with taking pullbacks, we easily conclude that the Peter-Weyl comodule algebra $\mathcal{P}_\Delta(\bar{P}) = \mathcal{T} \otimes \mathcal{O}(\mathrm{U}(1)) \times_{(\pi_1,\pi_2)} \mathcal{O}(\mathrm{U}(1))$, so that $\mathcal{P}_\Delta(\bar{P})$ is the comodule algebra P of Section 3.1.

Consider next the *-representation of $\mathcal{O}(\mathrm{SU}_q(2))$ on $\ell_2(\mathbb{N})$ given by [31]

$$\begin{gathered}\rho(\alpha)e_n := (1-q^{2n})^{1/2} e_{n-1}, \quad \rho(\beta)e_n := -q^{n+1} e_n, \\ \rho(\gamma)e_n := q^n e_n, \quad \rho(\delta)e_n := (1-q^{2(n+1)})^{1/2} e_{n+1}.\end{gathered} \tag{3.23}$$

Note that $\rho(\beta)$, $\rho(\gamma) \in \mathcal{K}(\ell_2(\mathbb{N}))$. Comparing ρ with the representation μ of $\mathcal{O}(\mathrm{D}_q)$ from (1.61), one readily sees that $\rho(\mathcal{O}(\mathrm{SU}_q(2))) \subseteq \mathcal{T}$. Furthermore, the symbol map σ yields $\sigma(\rho(\beta)) = \sigma(\rho(\gamma)) = 0$. Using an appropriate diagonal compact operator k, we also obtain

$$\begin{aligned} &\sigma(\rho(\alpha)) = \sigma(\rho(\alpha) - \mathrm{S}^*) + \sigma(\mathrm{S}^*) = \sigma(k\mathrm{S}^*) + \sigma(\mathrm{S}^*) = u^{-1}, \\ &\sigma(\rho(\delta)) = \sigma(\rho(\alpha))^* = u. \end{aligned} \tag{3.24}$$

Thus we obtain a U(1)-equivariant *-algebra homomorphism $\mathcal{O}(\mathrm{SU}_q(2)) \xrightarrow{\iota} P$ by setting

$$\iota(\alpha) := (\rho(\alpha) \otimes u, u), \quad \iota(\gamma) := (\rho(\gamma) \otimes u, 0). \tag{3.25}$$

One easily checks that the image of a Poincaré-Birkhoff-Witt basis of $\mathcal{O}(\mathrm{SU}_q(2))$ remains linearly independent, so that ι is injective, and we can consider $\mathcal{O}(\mathrm{SU}_q(2))$ as a subalgebra of P. In particular, we have $\iota(M_N) \subseteq L_N$ as left $\mathcal{O}(\mathrm{S}^2_q)$-modules. (See Section 1.4 and Section 3.1 for the definitions of M_N and L_N respectively.)

The main objective of this section is to establish a U(1)-C^*-algebra isomorphism between $C(\mathrm{SU}_q(2))$ and $\bar{P}$. The universal C^*-algebra $C(\mathrm{SU}_q(2))$ of $\mathcal{O}(\mathrm{SU}_q(2))$ has been studied in [21] and [35]. Here we shall use the fact from [21, Corollary 2.3] that there is a faithful *-representation $\hat{\rho}$ of $C(\mathrm{SU}_q(2))$ on the Hilbert space $\ell_2(\mathbb{N})\bar{\otimes}\ell_2(\mathbb{Z})$ given by

$$\hat{\rho}(\alpha)(e_n \otimes b_k) := (1-q^{2n})^{1/2} e_{n-1} \otimes b_{k-1}, \quad \hat{\rho}(\gamma)(e_n \otimes b_k) := q^n e_n \otimes b_{k-1}, \tag{3.26}$$

where $\{e_n\}_{n\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ denote the standard bases of $\ell_2(\mathbb{N})$ and $\ell_2(\mathbb{Z})$ respectively. To compare (3.26) with [21, Corollary 2.3], one has to apply the unitary transformation

$$T : \ell_2(\mathbb{N})\bar{\otimes}\ell_2(\mathbb{Z}) \longrightarrow \ell_2(\mathbb{N})\bar{\otimes}\ell_2(\mathbb{Z}), \quad T(e_n \otimes b_k) := e_n \otimes b_{k-n}. \tag{3.27}$$

A right $C(\mathrm{U}(1))$-coaction on $C(\mathrm{SU}_q(2))$ is given by $(\mathrm{id}\otimes\bar{\pi})\circ\Delta$, where Δ denotes the coproduct of the compact quantum group $C(\mathrm{SU}_q(2))$ and $\bar{\pi}$ is the extension of the Hopf *-algebra surjection $\pi : \mathcal{O}(\mathrm{SU}_q(2)) \to \mathcal{O}(\mathrm{U}(1))$ defined in (1.51) to $C(\mathrm{SU}_q(2))$. Using the faithfulness of $\hat{\rho}$, we can transfer $\bar{\pi}$ to $\hat{\rho}(C(\mathrm{SU}_q(2)))$. In [21], it is shown that $\bar{\pi}$ gives rise to the short exact sequence of C^*-algebras:

$$0 \longrightarrow \mathcal{K}(\ell_2(\mathbb{N}))\bar{\otimes}C(\mathrm{U}(1)) \hookrightarrow \hat{\rho}(C(\mathrm{SU}_q(2))) \xrightarrow{\bar{\pi}} C(\mathrm{U}(1)) \longrightarrow 0\,. \tag{3.28}$$

Here $C(\mathrm{U}(1))$ is naturally identified with the operator algebra on $\ell_2(\mathbb{Z})$ generated by the unilateral down-shift.

Theorem 3.2. *The* U(1)-C^**-algebras* $C(\mathrm{SU}_q(2))$ *and* $\bar{P}$ *are isomorphic.*

Proof. First note that $\ker(\mathrm{pr}_1) = \{(0,y) \in \bar{P} \,|\, \pi_2(y) = \Delta(y) = 0\} = 0$. Hence we can identify $\bar{P}$ with the image of pr_1 in $\mathcal{T}\bar{\otimes}C(\mathrm{U}(1))$. We will prove the theorem by applying the Five Lemma to the following commutative diagram of U(1)-C^*-algebras:

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{K}(\ell_2(\mathbb{N}))\bar{\otimes}C(\mathrm{U}(1)) \arrow[r, hook] \arrow[d, "\mathrm{id}"] & \hat{\rho}(C(\mathrm{SU}_q(2))) \arrow[r, "\bar{\pi}"] \arrow[d, "\tau"] & C(\mathrm{U}(1)) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathcal{K}(\ell_2(\mathbb{N}))\bar{\otimes}C(\mathrm{U}(1)) \arrow[r, hook] & \mathrm{pr}_1(\bar{P}) \arrow[r, "\omega"] & C(\mathrm{U}(1)) \arrow[r] & 0\,.
\end{tikzcd} \tag{3.29}
$$

To define τ, recall that we realize $C(\mathrm{U}(1))$ as a concrete C^*-algebra of bounded operators on $\ell_2(\mathbb{Z})$ by setting $u(b_k) = b_{k-1}$. Then

$$
\begin{aligned}
\hat{\rho}(\alpha) &= \rho(\alpha)\otimes u = \mathrm{pr}_1(\rho(\alpha)\otimes u, u) \in \mathrm{pr}_1(\bar{P}),\\
\hat{\rho}(\gamma) &= \rho(\gamma)\otimes u = \mathrm{pr}_1(\rho(\gamma)\otimes u, 0) \in \mathrm{pr}_1(\bar{P}).
\end{aligned} \tag{3.30}
$$

Since $C(\mathrm{SU}_q(2))$ is generated by α and γ, we take τ to be the inclusion $\hat{\rho}(C(\mathrm{SU}_q(2))) \subset \mathrm{pr}_1(\bar{P})$. Next, we define the U(1)-C^*-algebra homomorphism ω by

$$
\omega : \mathrm{pr}_1(\bar{P}) \longrightarrow C(\mathrm{U}(1)), \quad \omega := (\varepsilon\circ\sigma)\otimes\mathrm{id}. \tag{3.31}
$$

The surjectivity of ω follows from $u^k = \omega(\rho(\alpha^k)\otimes u^k)$ and $u^{-k} = \omega(\rho(\alpha^{*k})\otimes u^{*k})$ for all $k\in\mathbb{N}$.

To prove the exactness of the lower row, note that

$$
\mathcal{K}(\ell_2(\mathbb{N}))\bar{\otimes}C(\mathrm{U}(1)) = \ker(\sigma)\bar{\otimes}C(\mathrm{U}(1)) \subseteq \ker(\omega). \tag{3.32}
$$

Now, let $f \in \mathrm{pr}_1(\bar{P}) \setminus \ker(\sigma)\bar{\otimes}C(\mathrm{U}(1))$. Then $(\sigma\otimes\mathrm{id})(f) \neq 0$. By the commutativity of Diagram (3.22), there exists a non-zero element $g\in C(\mathrm{U}(1))$ such that $(\sigma\otimes\mathrm{id})(f) = \Delta(g)$. Hence $\omega(f) = (\varepsilon\otimes\mathrm{id})\circ\Delta(g) = g \neq 0$, which proves that $\ker(\omega) = \mathcal{K}(\ell_2(\mathbb{N}))\bar{\otimes}C(\mathrm{U}(1))$.

It remains to show that Diagram (3.29) is commutative. This is clear for the left part since τ is just the inclusion. The commutativity of the right part follows from

$$
\omega\Big(\tau\big(\hat{\rho}(\alpha)\big)\Big) = \varepsilon\Big(\sigma\big(\rho(\alpha)\big)\Big)\otimes u = \varepsilon(u)u = u = \bar{\pi}(\hat{\rho}(\alpha)), \tag{3.33}
$$

$$
\omega\Big(\tau\big(\hat{\rho}(\gamma)\big)\Big) = \varepsilon\Big(\sigma\big(\rho(\gamma)\big)\Big)\otimes u = 0 = \bar{\pi}(\hat{\rho}(\gamma)), \tag{3.34}
$$

since α and γ generate $C(\mathrm{SU}_q(2))$. Therefore, by the Five Lemma, τ is an isomorphism of U(1)-C^*-algebras. □

By the paragraph below Diagram (3.22), we conclude from Theorem 3.2 that the Peter-Weyl comodule algebra $\mathcal{P}_\Delta(C(\mathrm{SU}_q(2)))$ and the comodule algebra P are isomorphic. We use this isomorphism to identify associated projective modules. For $N\in\mathbb{Z}$ and the left $\mathcal{O}(\mathrm{U}(1))$-coaction on $\mathbb{C}$ given by ${}_{\mathbb{C}}\Delta(1) := u^N\otimes 1$, we define a "completed" version of M_N (see (1.63)):

$$
\begin{aligned}
\bar{M}_N :=&\, \mathcal{P}_\Delta(C(\mathrm{SU}_q(2))) \underset{\mathcal{O}(\mathrm{U}(1))}{\Box} \mathbb{C}\\
=&\{p\in\mathcal{P}_\Delta(C(\mathrm{SU}_q(2))) \mid ((\mathrm{id}\otimes\bar{\pi})\circ\Delta)(p) = p\otimes u^N\}.
\end{aligned} \tag{3.35}
$$

Now it follows from Equation (3.15) that $\bar{M}_N \cong L_N$. Applying the same arguments as at the end of Section 1.4, we infer that $\bar{M}_N \cong C(\mathrm{S}^2_q)^{N+1}E_N$, with E_N being the projection matrix of Equation (1.66). Taking advantage of these isomorphisms of modules, we prove:

Lemma 3.3. *Identifying $C(\mathrm{S}^2_q)$ with $\mathcal{K}(\ell_2(\mathbb{N}))\oplus\mathbb{C}$, we obtain the following isomorphisms of left $C(\mathrm{S}^2_q)$-modules:*

$$C(\mathrm{S}^2_q)^{N+1}E_N \cong C(\mathrm{S}^2_q)p_N, \qquad p_N := \mathrm{S}^N\mathrm{S}^{N*}, \qquad N\geq 0, \tag{3.36}$$

$$C(\mathrm{S}^2_q)^{|N|+1}E_N \cong C(\mathrm{S}^2_q)^2 p_N, \quad p_N := \begin{pmatrix} 1 & 0\\ 0 & 1-\mathrm{S}^{|N|}\mathrm{S}^{|N|*}\end{pmatrix}, \quad N<0. \tag{3.37}$$

Proof. We apply (1.43) to construct projections P_N, $N\in\mathbb{Z}$, from the strong connection given in (3.12) and (3.13). For $N<0$, we obtain

$$(P_N)_{11} = (\mathrm{S}^{*|N|}\otimes u^N, u^N)(\mathrm{S}^{|N|}\otimes u^{|N|}, u^{|N|}) = (1\otimes 1, 1), \tag{3.38}$$

$$(P_N)_{12} = (P_N)^*_{21} = (\mathrm{S}^{*|N|}\otimes u^N, u^N)((1-\mathrm{S}^{|N|}\mathrm{S}^{*|N|})\otimes u^{|N|}, 0) = 0, \tag{3.39}$$

$$\begin{aligned}(P_N)_{22} &= ((1-\mathrm{S}^{|N|}\mathrm{S}^{*|N|})\otimes u^N, 0)((1-\mathrm{S}^{|N|}\mathrm{S}^{*|N|})\otimes u^{|N|}, 0)\\ &= ((1-\mathrm{S}^{|N|}\mathrm{S}^{*|N|})\otimes 1, 0).\end{aligned} \tag{3.40}$$

Analogously, for $N\geq 0$, we get

$$(P_N)_{11} = (\mathrm{S}^N\otimes u^N, u^N)(\mathrm{S}^{*N}\otimes u^{*N}, u^{*N}) = (\mathrm{S}^N\mathrm{S}^{*N}\otimes 1, 1). \tag{3.41}$$

Finally, applying the isomorphism (3.20) componentwise to P_N, $N\in\mathbb{Z}$, yields the result. □

The projections p_N of Lemma 3.3 can also be obtained from the odd-to-even construction in Section 1.2. First, let $N<0$. Since $L_N \cong \mathcal{T}\times_{(u^{-N}\sigma,1)}\mathbb{C}$ (see (3.18)), we can apply Formula (1.13) by taking $E_1=\mathcal{T}$, $E_2=\mathbb{C}$, $\pi_{1*}E_1 = \pi_{2*}E_2 = C(\mathrm{U}(1))$, and choosing h in (1.10) to be the isomorphism given by the multiplication with $u^{|N|}$. As the symbol map σ applied to S is u (see (1.62)), we can lift $u^{|N|}$ and its inverse $u^{-|N|}$ to $\mathrm{S}^{|N|}$ and $\mathrm{S}^{|N|*}$ respectively. Inserting $c=\mathrm{S}^{|N|*}$ and $d=\mathrm{S}^{|N|}$ into (1.13) gives $\mathcal{T}\times_{(u^{-N}\sigma,1)}\mathbb{C}\cong(\mathcal{T}\times_{(\sigma,1)}\mathbb{C})^2 Q_N$, where

$$Q_N = \begin{pmatrix} (1,1) & (0,0)\\ (0,0) & (1-\mathrm{S}^{|N|}\mathrm{S}^{|N|*},0)\end{pmatrix}. \tag{3.42}$$

Finally, applying the isomorphism (3.20) yields the projection in (3.37). Similarly, for $N\geq 0$, we insert $c=\mathrm{S}^N$ and $d=\mathrm{S}^{N*}$ into (1.13). Since $\mathrm{S}^{N*}\mathrm{S}^N=1$, we obtain $\mathcal{T}\times_{(u^{-N}\sigma,1)}\mathbb{C}\cong(\mathcal{T}\times_{(\sigma,1)}\mathbb{C})^2Q_N$ with

$$Q_N = \begin{pmatrix} (\mathrm{S}^N\mathrm{S}^{N*},1) & (0,0)\\ (0,0) & (0,0)\end{pmatrix}, \tag{3.43}$$

which is equivalent to $\mathcal{T}\times_{(u^{-N}\sigma,1)}\mathbb{C}\cong C(\mathrm{S}^2_q)\mathrm{S}^N\mathrm{S}^{N*}$.

3.3. Index pairing. Recall that for a C^*-algebra A, a projection $p \in \mathrm{Mat}_n(A)$, and *-representations ρ_+ and ρ_- of A on a Hilbert space $\mathscr{H}$ such that $[(\rho_+, \rho_-)] \in \mathrm{K}^0(A)$ (e.g. see [10, Chapter 4]), one has the following: if the operator $\mathrm{Tr}_{\mathrm{Mat}_n}(\rho_+ - \rho_-)(p)$ is trace class, then the formula

$$\langle [(\rho_+, \rho_-)], [p] \rangle = \mathrm{Tr}_{\mathscr{H}}(\mathrm{Tr}_{\mathrm{Mat}_n}(\rho_+ - \rho_-)(p)) \tag{3.44}$$

yields a pairing between the K-homology group $\mathrm{K}^0(A)$ and the K-theory group $\mathrm{K}_0(A)$.

In this section, we compute the pairing between the K_0-classes of the projective $C(\mathrm{S}^2_q)$-modules describing quantum line bundles and two generators of $\mathrm{K}^0(A)$. By Lemma 3.3, we can take the projections p_N as representatives of respective K_0-classes. Their simple form makes it very easy to compute the index pairing.

Theorem 3.4. *Let $\bar{M}_N$ be the associated modules of* (3.35)*, and let* $[(\mathrm{id}, \varepsilon)]$ *and* $[(\varepsilon, \varepsilon_0)]$ *denote the generators of* $\mathrm{K}^0(C(\mathrm{S}^2_q))$ *given in Section 1.4. Then, for all* $N \in \mathbb{Z}$,

$$\langle [(\varepsilon, \varepsilon_0)], [\bar{M}_N] \rangle = 1, \qquad \langle [(\mathrm{id}, \varepsilon)], [\bar{M}_N] \rangle = -N. \tag{3.45}$$

Proof. Let $N \geq 0$. Then $p_N = \mathrm{S}^N \mathrm{S}^{N*} = (\mathrm{S}^N \mathrm{S}^{N*} - 1) + 1$, so that $\varepsilon(p_N) = 1$ and $\varepsilon_0(p_N) = \mathrm{SS}^*$. Furthermore, since for any $N \in \mathbb{N}\backslash\{0\}$, the image of the projection $1 - \mathrm{S}^N \mathrm{S}^{N*}$ is $\mathrm{span}\{e_0, \ldots, e_{N-1}\} \subset \ell_2(\mathbb{N})$, the projection $1 - \mathrm{S}^N \mathrm{S}^{N*}$ is trace class. Moreover, with the help of Lemma 3.3 and Formula (3.44), it implies that

$$\langle [(\varepsilon, \varepsilon_0)], [\bar{M}_N] \rangle = \mathrm{Tr}_{\ell_2(\mathbb{N})}(\varepsilon - \varepsilon_0)(p_N) = \mathrm{Tr}_{\ell_2(\mathbb{N})}(1 - \mathrm{SS}^*) = 1, \tag{3.46}$$

$$\langle [(\mathrm{id}, \varepsilon)], [\bar{M}_N] \rangle = \mathrm{Tr}_{\ell_2(\mathbb{N})}(\mathrm{id} - \varepsilon)(p_N) = \mathrm{Tr}_{\ell_2(\mathbb{N})}(\mathrm{S}^N \mathrm{S}^{N*} - 1) = -N. \tag{3.47}$$

For $N < 0$, we have $\mathrm{Tr}_{\mathrm{Mat}_2}(p_N) = 2 - \mathrm{S}^{|N|} \mathrm{S}^{|N|*} = 2 - p_{|N|}$. Combining this with the above index pairing for $p_{|N|}$, the formulas $(\varepsilon - \varepsilon_0)(2) = 2(1 - \mathrm{SS}^*)$ and $(\mathrm{id} - \varepsilon)(2) = 0$, and (3.44), we obtain

$$\begin{aligned} \langle [(\varepsilon, \varepsilon_0)], [\bar{M}_N] \rangle &= \mathrm{Tr}_{\ell_2(\mathbb{N})}(\varepsilon - \varepsilon_0)(2 - p_{|N|}) = \mathrm{Tr}_{\ell_2(\mathbb{N})}(1 - \mathrm{SS}^*) \\ &= 1, \end{aligned} \tag{3.48}$$

$$\begin{aligned} \langle [(\mathrm{id}, \varepsilon)], [\bar{M}_N] \rangle &= \mathrm{Tr}_{\ell_2(\mathbb{N})}(\mathrm{id} - \varepsilon)(2 - p_{|N|}) = \mathrm{Tr}_{\ell_2(\mathbb{N})}(1 - \mathrm{S}^{|N|} \mathrm{S}^{|N|*}) \\ &= -N. \end{aligned} \tag{3.49}$$

This completes the proof. □

The above theorem agrees with the classical situation. Indeed, the pairing $\langle [(\varepsilon, \varepsilon_0)], [\bar{M}_N] \rangle$ yields the rank of the line bundles, and $\langle [(\mathrm{id}, \varepsilon)], [\bar{M}_N] \rangle$ computes the winding number of the map $u^{-N} : \mathrm{S}^1 \to \mathrm{S}^1$.

4. Examples of piecewise principal coalgebra coactions

We begin by recalling the piecewise structure [15] of a noncommutative join construction proposed in [11]. Then we specify it to $\mathrm{SU}_q(2)$ to obtain a quantum instanton bundle $\mathrm{S}^7_q \to \mathrm{S}^4_q$ [26] as a piecewise trivial principal comodule algebra. A key step is then to replace the Hopf algebra $\mathcal{O}(\mathrm{SU}_q(2))$ by the quotient of $\mathcal{O}(\mathrm{SU}_q(2))$ by a coideal right ideal $(\mathcal{O}(\mathrm{S}^2_{q,s}) \cap \ker\varepsilon)\mathcal{O}(\mathrm{SU}_q(2))$ provided by a generic Podleś quantum sphere $\mathrm{S}^2_{q,s}$, $s \neq 0$ [27]. The quotient coalgebra is isomorphic with $\mathcal{O}(\mathrm{U}(1))$ [24]. Applying our main theorem, we will prove that the induced right coaction of $\mathcal{O}(\mathrm{U}(1))$ is principal.

4.1. Piecewise principal coactions from a noncommutative join. Let $\bar{H}$ be the C^*-algebra of a compact quantum group and H its Peter-Weyl Hopf algebra [34, 36]. We take the algebra of norm continuous functions $C([a,b],\bar{H})$ from a closed interval $[a,b]$ to the C^*-algebra $\bar{H}$, and define

$$P_1 := \{f \in C([0,\tfrac{1}{2}],\bar{H}) \otimes H \mid f(0) \in \Delta(H)\}, \tag{4.1}$$

$$P_2 := \{f \in C([\tfrac{1}{2},1],\bar{H}) \otimes H \mid f(1) \in \mathbb{C} \otimes H\}. \tag{4.2}$$

Here we identify elements of $C([a,b],\bar{H})\otimes H$ with functions $[a,b] \to \bar{H}\otimes H$. The P_i's are right H-comodule algebras for the coaction $\Delta_{P_i} = \mathrm{id}_{C([a_i,b_i],\bar{H})} \otimes \Delta$, where Δ stands for the coproduct of H. The subalgebras of coaction invariants can be identified with

$$B_1 := \{f \in C([0,\tfrac{1}{2}],\bar{H}) \mid f(0) \in \mathbb{C}\},$$
$$B_2 := \{f \in C([\tfrac{1}{2},1],\bar{H}) \mid f(1) \in \mathbb{C}\}.$$

The comodule algebra P_2 is evidently the same as $B_2 \otimes H$. Unlike P_2, the comodule algebra P_1 does not coincide with $B_1 \otimes H$. However, there is a cleaving map $j : H \to P_1$ given by $j(h)(t) := \big(t \mapsto h_{(1)}\big) \otimes h_{(2)}$, that is $j(h)(t) := \Delta(h)$ for all $t \in [0,\frac{1}{2}]$. Since j is an algebra homomorphism, it identifies the comodule algebra P_1 with a smash product $B_1\#H$.

Now one can define the noncommutative join of $\bar{H}$ as the pullback right H-comodule algebra

$$P := \{(p,q) \in P_1 \oplus P_2 \mid \pi_1(p) = \pi_2(q)\} \tag{4.3}$$

given by the evaluation maps

$$\pi_1 := \mathrm{ev}_{\frac{1}{2}} \otimes \mathrm{id} : P_1 \to P_{12} := \bar{H} \otimes H, \quad \pi_2 := \mathrm{ev}_{\frac{1}{2}} \otimes \mathrm{id} : P_2 \to P_{12} := \bar{H} \otimes H,$$

where ev_t is defined by the evaluation of functions of $C([a,b],\bar{H})$ at $t \in [a,b]$.

Our next goal is to replace H by a quotient coalgebra without losing principality. Using [7, Example 2.29], it is straightforward to verify the following lemma.

Lemma 4.1. *Let H be a Hopf algebra with bijective antipode, let $\Delta_P : P \to P \otimes H$ be a coaction making P a right H-comodule algebra, and let J be a* *coideal right ideal of H. Then $C := H/J$ is a coalgebra coacting on P via $\rho_R := (\mathrm{id}\otimes\pi)\circ\Delta_P$, where $\pi : H \to C$ the canonical surjection, and the formula*

$$\Psi : C\otimes P \ni \bar{\pi}(h)\otimes p \longmapsto p_{(0)}\otimes\pi(hp_{(1)}) \in P\otimes C \tag{4.4}$$

defines a bijective entwining making P an entwined module. The inverse of Ψ is given by

$$\Psi^{-1}(p\otimes\pi(h)) = \pi(hS^{-1}(p_{(1)}))\otimes p_{(0)}, \tag{4.5}$$

and defines a left coaction on P via

$$\rho_L : P \ni p \longmapsto \Psi^{-1}(p\otimes\pi(1)) = \pi(S^{-1}(p_{(1)}))\otimes p_{(0)} \in C\otimes P. \tag{4.6}$$

Lemma 4.2. *Let P be a principal H-comodule algebra for $\Delta_P : P \to P\otimes H$. Also, let J be a coideal right ideal of H defining a coalgebra $C := H/J$, let $\rho_R := (\mathrm{id}\otimes\pi)\circ\Delta_P$, with $\pi : H \to C$ the canonical surjection, be its right coaction on P, and let $i : C \to H$ be a unital (i.e., $i(\pi(1)) = 1$) C-bicolinear (for the coactions $\Delta_H := (\mathrm{id}\otimes\pi)\circ\Delta$ and ${}_H\Delta := (\pi\otimes\mathrm{id})\circ\Delta$) splitting (i.e., $\pi\circ i = \mathrm{id}$). Then P is principal for the coaction ρ_R.*

Proof. Let $\ell : H \to P\otimes P$ be a strong connection on P. One can easily check that $\ell\circ i : C \to P\otimes P$ is a strong connection on P for the right coaction $\rho_R := (\mathrm{id}\otimes\pi)\circ\Delta_P$ and the left coaction $\rho_L := (\pi\otimes\mathrm{id})\circ{}_P\Delta$, where ${}_P\Delta(p) := S^{-1}(p_{(1)})\otimes p_{(0)}$ gives the left H-coaction on P viewed as a principal H-comodule algebra. Furthermore, it follows from Lemma 4.1 that

$$\Psi : C\otimes P \ni \pi(h)\otimes p \longmapsto p_{(0)}\otimes\pi(hp_{(1)}) \in P\otimes C \tag{4.7}$$

is a bijective entwining making P an entwined module. Therefore, since $\rho_R(1) = 1\otimes\pi(1)$, $\rho_L(1) = \pi(1)\otimes 1$, and $\rho_L(p) = \Psi^{-1}(p\otimes\pi(1))$ for all $p\in P$ by (4.6), the principality of P for the C-coaction ρ_R follows from Lemma 1.1. □

Combining Lemma 4.2 with Theorem 2.2 yields the following result.

Theorem 4.3. *Let $\bar{H}$ be the C^*-algebra of a compact quantum group, H its Peter-Weyl Hopf algebra, J a coideal right ideal of H and $\pi : H \to C := H/J$ the canonical surjection. Also, let*

$$P_1 := \{f\in C([0,\tfrac{1}{2}],\bar{H})\otimes H \,|\, (\mathrm{ev}_0\otimes\mathrm{id})(f)\in\Delta(H)\}, \tag{4.8}$$

$$P_2 := \{f\in C([\tfrac{1}{2},1],\bar{H})\otimes H \,|\, (\mathrm{ev}_1\otimes\mathrm{id})(f)\in\mathbb{C}\otimes H\}, \tag{4.9}$$

be right and left C-comodules for the right and left coactions

$$\rho_R^i := (\mathrm{id}\otimes\pi)\circ\Delta_{P_i},\quad \rho_L^i := (\pi\otimes\mathrm{id})\circ{}_{P_i}\Delta,\quad i=1,2, \tag{4.10}$$

respectively. Here $\Delta_{P_i} := \mathrm{id}\otimes\Delta$ and ${}_{P_i}\Delta := (S^{-1}\otimes\mathrm{id})\circ\mathrm{flip}\circ\Delta_{P_i}$. Then, if there exists a unital bicolinear splitting $i : C\to H$ of $\pi : H\to C$, the pullback algebra and $\pi(1)$-coaugmented C-comodule

$$P := \big\{(p_1,p_2)\in P_1\times P_2 \mid (\mathrm{ev}_{\frac{1}{2}}\otimes\mathrm{id})(p_1) = (\mathrm{ev}_{\frac{1}{2}}\otimes\mathrm{id})(p_2)\big\} \tag{4.11}$$

is principal.

First observe that, if both of π_1 and π_2 defining the pullback diagram (2.34) are surjective, then (2.40) simplifies to

$$\begin{aligned}(4.12)\qquad \ell &= \big((\alpha_L^{12}+\mathrm{id})\otimes(\alpha_R^{12}+\mathrm{id})\big)\circ\ell_2\\ &\quad+(\eta_1\circ\varepsilon-m_{P_1}\circ(\alpha_L^{12}\otimes\alpha_R^{12})\circ\ell_2)*\big((\mathrm{id}\otimes(\mathrm{id}+\alpha_R^{21}))\circ\ell_1\big).\end{aligned}$$

Indeed, since now α_R^{21} is defined on the whole comodule P_1, a special connection $\tilde{\ell}_1$ constructed in (2.39) can be replaced by any strong connection ℓ_1 on P_1. (Note that specializing (4.12) to comodule algebras coincides with what was obtained in [15].)

Next, specializing to the setting of Theorem 4.3, observe that the formulas

$$(4.13)\qquad \alpha^1 : P_{12}\longrightarrow P_1,\quad \alpha^1(\bar{h}\otimes h) := 2t\bar{h}\otimes h+(1-2t)\bar{\varepsilon}(\bar{h})h_{(1)}\otimes h_{(2)},$$

$$(4.14)\qquad \alpha^2 : P_{12}\longrightarrow P_2,\quad \alpha^2(\bar{h}\otimes h) := 2(1-t)\bar{h}\otimes h+(2t-1)\bar{\varepsilon}(\bar{h})\otimes h,$$

where $\bar{\varepsilon}$ is any unital linear functional on $\bar{H}$, define unital C-bicolinear splittings of π_1 and π_2 respectively. Hence we can take $\alpha_L^{12}=\alpha_R^{12}=\alpha^1\circ\pi_2$ and $\alpha_R^{21}=\alpha^2\circ\pi_1$. Combining this with the fact that a cleaving map j defines a strong connection via $\ell := (j^{-1}\otimes j)\circ\Delta$, we obtain very explicit formulae for strong connections on P_1 and P_2:

$$(4.15)\qquad \ell_1 := (j_1^{-1}\otimes j_1)\circ\Delta\circ i,\quad \ell_2 := (j_2^{-1}\otimes j_2)\circ\Delta\circ i.$$

Here $j_1 : H\to P_1$, $j_1(h) := (t\mapsto h_{(1)})\otimes h_{(2)}$, and $j_2 : H\to P_2$, $j_2(h) := 1\otimes h$ are cleaving maps for P_1 and P_2 respectively.

4.2. Quantum complex projective spaces $\mathbb{C}\mathrm{P}^3_{q,s}$. Finally, we specify $\bar{H}$ to be $C(\mathrm{SU}_q(2))$, $H=\mathcal{O}(\mathrm{SU}_q(2))$, $J=(\mathcal{O}(\mathrm{S}^2_{q,s})\cap\ker\varepsilon)\mathcal{O}(\mathrm{SU}_q(2))$, and $\bar{\varepsilon} : C(\mathrm{SU}_q(2))\to C$ to be the counit. Here $\mathcal{O}(\mathrm{S}^2_{q,s})$ stands for the coordinate algebra of a Podleś quantum sphere $\mathrm{S}^2_{q,s}$, $s\in[0,1]$, [8]. (Note that the case $s=0$ brings us to the comodule-algebra setting.) The most interesting part of this structure is the unital bicolinear splitting of $\pi : \mathcal{O}(\mathrm{SU}_q(2))\to\mathcal{O}(\mathrm{SU}_q(2))/J$ given by [8, Proposition 6.3].

All of this defines a family of noncommutative deformations of the U(1)-principal bundle $\mathrm{S}^7\to\mathbb{C}\mathrm{P}^3$. More precisely, we obtain deformations of a U(1)-principal action on

$$(4.16)\qquad \mathrm{S}^7 := \big\{(z_1,z_2,z_3,z_4)\in\mathbb{C}^4\;\big|\;|z_1|^2+|z_2|^2+|z_3|^2+|z_4|^2=1\big\}$$

given by

$$(4.17)\qquad (z_1,z_2,z_3,z_4)\mathrm{e}^{\mathrm{i}\varphi}=(z_1\mathrm{e}^{\mathrm{i}\varphi},z_2\mathrm{e}^{-\mathrm{i}\varphi},z_3\mathrm{e}^{\mathrm{i}\varphi},z_4\mathrm{e}^{-\mathrm{i}\varphi}).$$

However, this is isomorphic to the diagonal action of U(1) on S^7, so that the quotient space is again $\mathbb{C}\mathrm{P}^3$. Thus out of Pflaum's S^7 we obtain a family of quantum projective spaces $\mathbb{C}\mathrm{P}^3_{q,s}$. A very explicit Mayer-Vietoris-type formula for a strong connection on $\mathrm{S}^7_q\to\mathbb{C}\mathrm{P}^3_{q,s}$ should allow us to study the K-theoretic aspects of the tautological line bundle over $\mathbb{CP}^3_{q,s}$, but this is beyond the scope of the present paper.

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