A Remark on the Gradient Map

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Abstract. For a Hamiltonian action of a compact group U of isometries on a compact Kähler manifold Z and a compatible subgroup G of $U^{\mathbb{C}}$, we prove that for any closed G–invariant subset $Y \subset Z$ the image of the gradient map $\mu_{\mathfrak{p}}(Y)$ is independent of the choice of the invariant Kähler form ω in its cohomology class $[\omega]$.

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1. Introduction

Let (Z, ω) be a compact Kähler manifold and let U be a compact connected semisimple Lie group such that $U^{\mathbb{C}}$ acts holomorphically on Z, U preserves ω and there is a momentum map $\mu : Z \to \mathfrak{u}^*$. Let $G \subset U^{\mathbb{C}}$ be a *compatible* subgroup. By this we mean a subgroup which is compatible with the Cartan involution Θ of $U^{\mathbb{C}}$ which defines U, i.e. if $\mathfrak{p} = \mathfrak{g} \cap i\mathfrak{u}$ and $K = U \cap G$, then $G = K \cdot \exp \mathfrak{p}$. Let $\mu_{\mathfrak{p}} : Z \to \mathfrak{p}$ be the associated *gradient map* (see [4, 5] or section 2).
In this note we prove the following.

Theorem 1. *Let $Y \subset Z$ be a closed G-stable subset. Then up to translation the set $\mu_{\mathfrak{p}}(Y)$ is independent of the choice of the invariant Kähler form ω in the cohomology class $[\omega]$.*

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Since Z is compact and G is compatible there is a stratification of Z analogous to the Kirwan stratification, see [4]. This gives a stratification of any closed G–invariant subset Y of Z, by intersecting the strata in Z with Y. It follows from Theorem 1 that when the momentum map is properly normalized (see Lemma 2) this stratification does not depend on the choice of ω in its cohomology class. When Z is a projective manifold and ω is the pull-back of a Fubini-Study form via an equivariant embedding of Z in $\mathbb{P}^N$, Kirwan [6, §12] proved that the stratification in terms of a properly normalized μ can be defined purely in terms of algebraic geometry. In the present note we give a proof of this fact for a general compact Kähler manifold Z in the more general setting of *gradient* maps for actions of compatible subgroups on closed G–invariant subsets of Z. Another consequence of the above is the following. Assume that Z is a projective manifold and that $[\omega]$ is an integral class. Let $Y \subset Z$ be a closed G-invariant real semi-algebraic subset whose real algebraic Zariski closure is irreducible. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal subalgebra and let $\mathfrak{a}_+$ be a closed Weyl chamber in $\mathfrak{a}$. Then $A(Y)_+ := \mu_{\mathfrak{p}}(Y) \cap \mathfrak{a}_+$ is convex (see [2], which deals with the case when ω is the restriction of a Fubini-Study metric).

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2. Background

Let (Z, ω) be a compact Kähler manifold and let U be a compact Lie group. Assume that U acts on Z by holomorphic Kähler isometries. Since Z is compact the U-action extends to a holomorphic action of the complexified group $U^{\mathbb{C}}$. Assume also that there is a momentum map $\mu : Z \to \mathfrak{u}^* \cong \mathfrak{u}$, where $\mathfrak{u}^*$ is identified with $\mathfrak{u}$ using a fixed U-invariant scalar product on $\mathfrak{u}$ that we denote by $\langle\ ,\ \rangle$. We also denote by $\langle\ ,\ \rangle$ the scalar product on $i\mathfrak{u}$ such that multiplication by i is an isometry of $\mathfrak{u}$ onto $i\mathfrak{u}$. If $\xi \in \mathfrak{u}$ we denote by ξ_Z the fundamental vector field on Z and we let $\mu^\xi \in C^\infty(Z)$ be the function $\mu^\xi(z) := \langle \mu(z), \xi \rangle$. That μ is the momentum map means that it is U-equivariant and that $d\mu^\xi = i_{\xi_Z}\omega$.

For a closed subgroup $G \subset U^{\mathbb{C}}$ let $K := G \cap U$ and $\mathfrak{p} := \mathfrak{g} \cap i\mathfrak{u}$. The group G is called *compatible* if $G = K \cdot \exp \mathfrak{p}$ [4, 5]. In the following we fix a compatible subgroup $G \subset U^{\mathbb{C}}$. If $z \in Z$, let $\mu_{\mathfrak{p}}(z) \in \mathfrak{p}$ denote $-i$ times the component of $\mu(z)$ in the direction of $i\mathfrak{p}$. In other words we require that $\langle \mu_{\mathfrak{p}}(z), \beta \rangle = -\langle \mu(z), i\beta \rangle$ for any $\beta \in \mathfrak{p}$. The map

$$\mu_{\mathfrak{p}} : Z \to \mathfrak{p}$$

is called the *gradient map* (see [3]) or *restricted momentum map.* Let $\mu_{\mathfrak{p}}^\beta \in C^\infty(Z)$ be the function $\mu_{\mathfrak{p}}^\beta(z) = \langle \mu_{\mathfrak{p}}(z), \beta \rangle = \mu^{-i\beta}(z)$. Let $(\ ,\ )$ be the Kähler metric associated to ω, i.e. $(v, w) = \omega(v, Jw)$. Then β_Z is the gradient of $\mu_{\mathfrak{p}}^\beta$ with respect to $(\ ,\ )$.

Example 1. (1) For any compact subgroup $K \subset U$, both K and its complexification $G = K^{\mathbb{C}}$ are compatible. In particular $G = U^{\mathbb{C}}$ is a compatible subgroup. (2) If G is a real form of $U^{\mathbb{C}}$, then G is compatible. (3) For any $\xi \in i\mathfrak{u}$, the subgroup $G = \exp(\mathbb{R}\xi)$ is compatible.

Next we recall the Stratification Theorem for actions of compatible subgroups. Given a maximal subalgebra $\mathfrak{a} \subset \mathfrak{p}$ and a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ define

$$\eta_{\mathfrak{p}} : X \to \mathbb{R} \qquad \eta_{\mathfrak{p}}(x) := \frac{1}{2}||\mu_{\mathfrak{p}}(x)||^2$$
$$C_{\mathfrak{p}} := \mathrm{Crit}(\eta_{\mathfrak{p}}) \qquad \mathcal{B}_{\mathfrak{p}} := \mu_{\mathfrak{p}}(C_{\mathfrak{p}}) \qquad \mathcal{B}_{\mathfrak{p}}^+ := \mathcal{B}_{\mathfrak{p}} \cap \mathfrak{a}^+$$
$$X(\mu) := \{x \in X : \overline{G \cdot x} \cap \mu_{\mathfrak{p}}^{-1}(0) \neq \emptyset\}$$

where X is a compact G-invariant subset of Z. Points lying in $X(\mu)$ are called *semistable*. Using semistability and the function $\eta_{\mathfrak{p}}$ one can define a stratification of X in the following way, see [6] and [4]. For $\beta \in \mathcal{B}_{\mathfrak{p}}^+$ set

$$X_{||\beta||^2} := \{x \in X : \overline{\exp(\mathbb{R}\beta) \cdot x} \cap (\mu^{\beta})^{-1}(||\beta||^2) \neq \emptyset\}$$
$$X^{\beta} := \{x \in X : \beta_X(x) = 0\}$$
$$X^{\beta}_{||\beta||^2} := X^{\beta} \cap X_{||\beta||^2}$$
$$X^{\beta+}_{||\beta||^2} := \{x \in X_{||\beta||^2} : \lim_{t \to -\infty} \exp(t\beta) \cdot x \text{ exists and it lies in } X^{\beta}_{||\beta||^2}\}$$
$$G^{\beta+} := \{g \in G : \text{the limit } \lim_{t \to -\infty} \exp(t\beta) g \exp(-t\beta) \text{ exists in } G\}.$$

Set also

$$G^{\beta} := \{g \in G : \mathrm{Ad}\, g(\beta) = \beta\} \qquad \mathfrak{p}^{\beta} := \{\xi \in \mathfrak{p} : [\xi, \beta] = 0\}.$$

The group $G^{\beta} = K^{\beta} \cdot \exp(\mathfrak{p}^{\beta})$ is a compatible subgroup of $U^{\mathbb{C}}$ and the set $X^{\beta+}_{||\beta||^2}$ is $G^{\beta+}$-invariant. Denote by $\mu_{\mathfrak{p}^{\beta}}$ the composition of $\mu_{\mathfrak{p}}$ with the orthogonal projection $\mathfrak{p} \to \mathfrak{p}^{\beta}$. Then $\mu_{\mathfrak{p}^{\beta}}$ is a gradient map for the G^{β}-action on $X^{\beta+}_{||\beta||^2}$. We set $\widehat{\mu_{\mathfrak{p}^{\beta}}} := \mu_{\mathfrak{p}^{\beta}} - \beta$. Since β lies in the center of $\mathfrak{g}^{\beta}$ and since G^{β} is a compatible subgroup of $(U^{\beta})^{\mathbb{C}} = (U^{\mathbb{C}})^{\beta}$, it is a gradient map too. We let $S^{\beta+}$ denote the set of G^{β}-semistable points in $X^{\beta+}_{||\beta||^2}$ with respect to $\widehat{\mu_{\mathfrak{p}^{\beta}}}$, i.e.

$$S^{\beta+} := \{x \in X^{\beta+}_{||\beta||^2} : \overline{G^{\beta} \cdot x} \cap \mu_{\mathfrak{p}^{\beta}}^{-1}(\beta) \neq \emptyset\}.$$

The set $S^{\beta+}$ coincides with the set of semistable points of the group G^{β} in $X^{\beta+}_{||\beta||^2}$ after shifting. By definition the β-stratum is given by $S_{\beta} := G \cdot S^{\beta+}$.

Stratification Theorem. (See [4, Thm. 7.3]) *Assume that X is a compact G-invariant subset of Z. Then $\mathcal{B}_{\mathfrak{p}}^+$ is finite and*

$$X = \bigsqcup_{\beta \in \mathcal{B}_{\mathfrak{p}}^+} S_{\beta}.$$

Moreover

$$\overline{S_\beta} \subset S_\beta \cup \bigcup_{||\gamma||>||\beta||} S_\gamma.$$

3. Proof of Theorem 1

For a U-invariant function f on Z we set

$$\tilde{\omega} := \omega + dd^c f$$

where $d^c f := -2J^* df$. Since Z is compact and U acts by holomorphic transformations, any U-invariant Kähler form $\tilde{\omega}$ in the Kähler class $[\omega]$ can be written in this way. Since pluriharmonic functions on Z are constant, the function f is unique up to a constant.

Lemma 2. *If $\mu : Z \to \mathfrak{u}$ is a momentum map for the U-action on Z with respect to ω, then the map $\tilde{\mu} : Z \to \mathfrak{u}$ defined by*

$$\tilde{\mu}^\xi := \mu^\xi - d^c f(\xi_Z) \tag{3}$$

is a momentum map for the U-action on Z with respect to $\tilde{\omega}$.

Proof. That $\tilde{\mu}$ is a momentum map follows from Cartan formula using that $L_{\xi_Z} d^c f = d^c L_{\xi_Z} f = 0$. This in turn follows from the assumption that the action of U is holomorphic and f is U-invariant. $\square$

A more precise version of Theorem 1 is the following.

Theorem 4. *For any closed G-stable subset $Y \subset Z$ we have $\mu_{\mathfrak{p}}(Y) = \tilde{\mu}_{\mathfrak{p}}(Y)$.*

Proof. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal subalgebra and set $A := \exp \mathfrak{a}$. The group A is a compatible subgroup. Let $\mu_{\mathfrak{a}} : Z \to \mathfrak{a}$ be the restricted gradient map. Any connected subgroup $B \subset A$ is compatible. Given such a B, set $Z^{(B)} := \{z \in Z : A_z = B\}$. A connected component S of $Z^{(B)}$ will be called an A-stratum of type $\mathfrak{b}$. For a given S let C denote the connected component of Z^B containing S. Then C is a complex submanifold of Z and the Slice Theorem (see Theorem 14.10 and 14.21 in [3] or Theorem 2.2 in [2]) applied to the A-action on C shows that S is open and dense in C.
Let A^c be the Zariski closure of A in $U^{\mathbb{C}}$. The group A^c is a compatible subgroup of $U^{\mathbb{C}}$, $A^c \cap U = T$ is a torus and $A^c = T \exp(i\mathfrak{t})$, where $\mathfrak{t}$ denotes the Lie algebra of T. Moreover $\overline{S}$ is A^c-stable [2, Lemma 3.3 (1)]. Denote by $\mu_{\mathfrak{t}} : Z \longrightarrow \mathfrak{t}$ the momentum map obtained by projecting $\mu : Z \longrightarrow \mathfrak{u}$ to $\mathfrak{t}$, and denote by $\Pi : i\mathfrak{t} \to \mathfrak{a}$ the orthogonal projection. Then $\mu_{\mathfrak{a}} = \Pi \circ i\mu_{\mathfrak{t}}$ and $\mu_{\mathfrak{a}}(\overline{S}) = \Pi(i\mu_{\mathfrak{t}}(\overline{S}))$. By the convexity theorem of Atiyah-Guillemin-Sternberg $\mu_{\mathfrak{t}}(\overline{S})$ is a convex polytope and its vertices are images of points fixed by A^c. It follows that $\mu_{\mathfrak{a}}(\overline{S})$ is a convex polytope as well. Since Π is linear, any vertex of $\mu_{\mathfrak{a}}(\overline{S})$ is the projection of at least one vertex of $i\mu_{\mathfrak{t}}(\overline{S})$. Therefore $\mu_{\mathfrak{a}}(\overline{S})$ is the convex hull of $\mu_{\mathfrak{a}}(\overline{S}^A)$. Now we use Lemma 2: if $x \in \overline{S}^A$, then $\xi_Z(x) = 0$, so $\tilde{\mu}^\xi(x) = \mu^\xi(x)$, for any $\xi \in \mathfrak{a}$. Therefore $\tilde{\mu}_{\mathfrak{a}}(x) = \mu_{\mathfrak{a}}(x)$ for every A-fixed point

x. It follows that both $\mu_{\mathfrak{a}}(\overline{S})$ and the affine subspace spanned by $\mu_{\mathfrak{a}}(S)$ do not depend on the choice of the Kähler form ω.
Let Σ be the collection of affine hyperplanes of $\mathfrak{a}$ that are affine hulls of $\mu_{\mathfrak{a}}(\overline{S})$ for some A-stratum S. Set $P := \mu_{\mathfrak{a}}(Z)$ and

$$P_0 := P - \bigcup_{H \in \Sigma} P \cap H.$$

(This construction is similar to the one in [2, §§4-5].) The set P_0 is an open subset of $\mathfrak{a}$. Let $C(P_0)$ denote the set of its connected components. This is a finite set. For $\gamma \in C(P_0)$ let $P(\gamma)$ be the closure of the connected component γ. Then $P(\gamma)$ is a convex polytope. Since both P and the hyperplanes H are independent of ω, also the polytopes $P(\gamma)$ do not depend on ω. By [2, Corollary 5.8]

$$\mu_{\mathfrak{p}}(Y) \cap \mathfrak{a} = \bigcup_{\gamma \in F(\omega)} P(\gamma),$$

where $F(\omega) \subset \Gamma$ is some subset of $C(P_0)$. One can join ω to $\tilde{\omega}$ continuously, e.g. by $\omega_t := \omega + tdd^c f$. Then $\tilde{\mu}_t := \mu - td^c f(\cdot_Z)$ also depends continuously on t. So $P(\gamma) \subset \mu_{\mathfrak{p}}(Y) \cap \mathfrak{a}$ if and only if $P(\gamma) \subset \mu_{t,\mathfrak{p}}(Y) \cap \mathfrak{a}$. Therefore $F(\omega_t)$ is independent of t. Thus $\mu_{\mathfrak{p}}(Y) \cap \mathfrak{a} = \tilde{\mu}_{\mathfrak{p}}(Y) \cap \mathfrak{a}$. Since $\mu_{\mathfrak{p}}(Y) = K(\mu_{\mathfrak{p}}(Y) \cap \mathfrak{a})$ this implies $\mu_{\mathfrak{p}}(Y) = \tilde{\mu}_{\mathfrak{p}}(Y)$. □

Corollary 5. *Assume that Z is connected and let ω and $\tilde{\omega}$ be two cohomologous Kähler forms with momentum maps μ and $\tilde{\mu}$ respectively as in Lemma 2. Then $\tilde{\mu}$ is the unique momentum map such that $\mu(Z) = \tilde{\mu}(Z)$.*

Proof. Since two momentum maps with respect to $\tilde{\omega}$ differ by addition of an element of the center of $\mathfrak{u}$, it is clear that there is at most one such map with the image equal to $\mu(Z)$. To complete the proof it is therefore enough to check that $\tilde{\mu}(Z) = \mu(Z)$. This is a special case of the previous theorem. □

Theorem 6. *Let ω and $\tilde{\omega}$ be two cohomologous Kähler forms on Z, with momentum maps μ and $\tilde{\mu}$ respectively as in Lemma 2. Then the set $\mathcal{B}_{\mathfrak{p}}^+$ is the same for both momentum maps and the two stratifications of X coincide.*

Proof. By [4, Corollary 7.6]

$$\mathcal{B}_{\mathfrak{p}} = \{\beta \in \mathfrak{p} : \text{ there exists } x \in X : \frac{||\beta||^2}{2} = \inf_{G \cdot x} \eta_{\mathfrak{p}} \text{ and } \beta \in \mu_{\mathfrak{p}}(\overline{G \cdot x})\}. \quad (7)$$

Moreover for $\beta \in \mathcal{B}_{\mathfrak{p}}$

$$S_\beta = \{x \in X : \frac{||\beta||^2}{2} = \inf_{G \cdot x} \eta_{\mathfrak{p}} \text{ and } \beta \in \mu_{\mathfrak{p}}(\overline{G \cdot x})\}. \quad (8)$$

For any point $x \in X$, the set $\overline{G \cdot x}$ is closed and G-invariant. Hence by Theorem 4 $\mu_{\mathfrak{p}}(\overline{G \cdot x}) = \tilde{\mu}_{\mathfrak{p}}(\overline{G \cdot x})$. From this it follows that $\inf_{G \cdot x} \eta_{\mathfrak{p}} = \inf_{G \cdot x} \tilde{\eta}_{\mathfrak{p}}$, where $\tilde{\eta}_{\mathfrak{p}} := ||\tilde{\mu}_{\mathfrak{p}}||^2/2$. The result follows from (7) and (8). □

¿From the above we obtain the following generalization.

Corollary 9. *If Z is a complex projective manifold, U is a compact connected semisimple Lie group acting on Z, ω is a U-invariant Hodge metric and $Y \subset Z$ is a closed G-invariant real semi-algebraic subset whose real algebraic Zariski closure is irreducible, then $A(Y)_+$ is convex. Moreover if G is semisimple, then $X(\mu)$ is dense (if it is nonempty).*

Proof. By assumption there is a very ample line bundle $L \to Z$ such that $[\omega] = 2\pi\, \mathrm{c}_1(L)/m$ for an interger $m > 0$. Let ω_{FS} be a U-invariant Fubini-Study metric on $\mathbb{P}(H^0(Z, L)^*)$. Let μ_{FS} be the momentum map with respect to $\omega_{FS}|_Z$. In [2] the convexity theorem has been proved for μ_{FS}. A rescaling in the symplectic form yields a corresponding rescaling in the momentum map. Therefore the convexity theorem also holds for the momentum map $\tilde{\mu}$ relative to the symplectic form $\tilde{\omega} := \omega_{FS}/m$. So it holds also for μ, since $\mu_{\mathfrak{p}}(Y) = \tilde{\mu}_{\mathfrak{p}}(Y)$ by Theorem 4. The proof of the last statement is similar: see [2] and Corollary 5. □

Corollary 10. *Under the same assumptions, any local minimum of $||\mu_{\mathfrak{p}}||^2$ is a global minimum.*

Proof. This follows since $||\mu_{\mathfrak{p}}||^2$ is K-invariant and $\mu(Z)_+$ is a convex subset of $\mathfrak{a}_+$. □

Corollary 11. *If ω and ω' are cohomologous Kähler forms on Z with momentum maps μ and $\tilde{\mu}$ as in Lemma 2, then $X(\mu) = X(\tilde{\mu})$.*

Proof. It is enough to observe that $X(\mu) = S_0$. □

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