Fano Threefolds with 2-Torus Action

A picture book

Hendrik Süss

Received: December 16, 2013

Communicated by Thomas Peternell

Abstract. Following the work of Altmann and Hausen we give a combinatorial description for smooth Fano threefolds admitting a 2-torus action. We show that a whole variety of properties and invariants can be read off from this description. As an application we prove and disprove the existence of Kähler-Einstein metrics for some of these Fano threefolds, calculate their Cox rings and some of their toric canonical degenerations.

2010 Mathematics Subject Classification: Primary 14L30; Secondary 14J45, 32Q20

Keywords and Phrases: torus action, moment polytope, Fano variety, Kahler-Einstein metric, Cox ring

1 Introduction

Knowing the fan or the polytope of a projective toric variety does not only give a concrete construction of the variety but also enables us to explore its properties and to easily calculate a whole bunch of invariants. In low dimensions a close look on the corresponding pictures is enough to obtain detailed information about the variety. Loosely speaking we can even prove statements by drawing pictures.

In [AH06] and [AHS08] the description of toric varieties by cones and fans was generalized to the case of an arbitrary T-variety, i.e. a variety X endowed with the effective action of an algebraic torus T. This general description consists of combinatorial data living on the Chow quotient $X/\!\!/T$. A series of papers by different authors followed this approach to derive properties and calculate invariants out of the combinatorial data. This works especially well for the case

of complexity-one torus action, i.e. when the torus has one dimension less then the variety and the Chow quotient is just a curve.
The aim of this paper is to make these results available for Fano threefolds with an action of a two-dimensional torus. More precisely, we are going to state the needed combinatorial description for a list of those Fano threefolds and relate it to the classification of Mori-Mukai [MM86]. By the mentioned preliminary results we can immediately derive a lot of additional information about these Fano varieties.
We want to highlight two concrete applications. First, by [HS10] we are able to calculate the Cox rings of these T-varieties. The knowledge of the Cox ring may give further inside in the geometry of the variety. For example the automorphism group can be studied by the methods from [AHHL12].
As a second application we consider the question of the existence of Kähler-Einstein metrics on Fano manifolds. In general this is a hard question. There are some known obstructions such as the vanishing of the Futaki character [Fut83]. There is also a sufficient criterion due to Tian [Tia87]. Recently it was shown that there is, indeed, a completely algebraic formulation of the Kähler-Einstein property by the means of so called K-stability [CDS14, Tia12]. Nevertheless, for a given Fano variety it's still hard to check this stability condition directly.
Again for toric Fano varieties the Kähler-Einstein property can be easily reformulated in terms of the defining polytope. By a result of Wang and Zhu a toric Fano variety is Kähler-Einstein if and only if the Futaki character vanishes [WZ04]. Moreover, the Futaki character can be easily calculated as the barycenter of the polytope corresponding to the toric Fano manifold [Mab87]. We are going to generalize the last result for our situation of complexity-one torus actions in order to show that the Futaki character does not vanish for some of the threefolds, and hence disprove the existence of Kähler-Einstein metrics for them. We are using an application of Tian's criterion from [Süß13] to prove the existence of Kähler-Einstein metrics for three of the remaining ones.
Our main conclusions drawn from the given combinatorial description are summarized in the following theorem.

Theorem 1.1. *The Fano threefolds Q, 2.24*, 2.29–2.32, 3.8*, 3.10*, 3.18–3.24, 4.4, 4.5, 4.7 and 4.8 from Mori's and Mukai's classification admit a 2-torus action.*
The moment polytopes together with their Duistermaat-Heckman measures are given in Section 5. The Cox rings, Futaki characters $F(X)$ as well as the existence of a Kähler-Einstein metric can be found in the following table.

*Only one variety of the family is known to admit a 2-torus action

No.	*Cox ring*	*F(X)*	*Kähler-Einstein*
Q	$k[\underline{T}]/(T_1T_2+T_3T_4+T_5^2)$	0	*yes*
2.24^*	$k[\underline{T}]/(T_1T_2^2+T_3T_4^2+T_5T_6^2)$	0	*yes*
2.29	$k[\underline{T}]/(T_1T_2^2T_3+T_4T_5+T_6^2)$	0	*?*
2.30	$k[\underline{T},S_1]/(T_1T_2+T_3T_4+T_5^2)$	$\binom{0}{-2}$	*no*
2.31	$k[\underline{T}]/(T_1T_2+T_3T_4+T_5T_6^2)$	$\binom{-4/3}{-4/3}$	*no*
2.32	$k[\underline{T}]/(T_1T_2+T_3T_4+T_5T_6)$	0	*yes*
3.8^*	$k[\underline{T},S_1]/(T_1T_2^2+T_3T_4^2+T_5T_6^2)$	$\binom{5/3}{-5/6}$	*no*
3.10^*	$k[\underline{T}]/(T_1T_2^2T_3+T_4T_5^2T_6+T_7^2)$	0	*yes*
3.18	$k[\underline{T},S_1]/(T_1T_2^2T_3+T_4T_5+T_6^2)$	$\binom{0}{-7/8}$	*no*
3.19	$k[\underline{T},S_1,S_2]/(T_1T_2+T_3T_4+T_5^2)$	0	*?*
3.20	$k[\underline{T}]/(T_1T_2+T_3T_4+T_5T_6^2T_7)$	0	*?*
3.21	$k[\underline{T},S_1]/(T_1T_2^2+T_3T_4^2+T_5T_6)$	$\binom{7/8}{7/8}$	*no*
3.22	$k[\underline{T},S_1,S_2]/(T_1T_2+T_3T_4+T_5^2)$	$\binom{0}{-2/3}$	*no*
3.23	$k[\underline{T},S_1]/(T_1T_2+T_3T_4+T_5T_6^2)$	$\binom{-13/12}{-53/24}$	*no*
3.24	$k[\underline{T},S_1]/(T_1T_2+T_3T_4+T_5T_6)$	$\binom{-2/3}{-4/3}$	*no*
4.4	$k[\underline{T},S_1,S_2]/(T_1T_2^2T_3+T_4T_5+T_6^2)$	0	*?*
4.5	$k[\underline{T},S_1,S_2]/(T_1T_2^2+T_3T_4^2+T_5T_6)$	$\binom{5/24}{5/24}$	*no*
4.7	$k[\underline{T},S_1,S_2]/(T_1T_2+T_3T_4+T_5T_6)$	0	*?*
4.8	$k[\underline{T},S_1,S_2]/(T_1T_2+T_3T_4+T_5T_6)$	$\binom{0}{-13/12}$	*no*

The paper is organized as follows. In Section 2 we recall the combinatorial description of complexity-one torus actions from [AIP$^+$11]. In Section 3 we describe equivariant polarizations and give a characterization of the Fano property. Moreover, we recall how to calculate the Cox ring from the combinatorial description and we state a simple formula for the Futaki character. In Section 4 we show how to find toric degenerations for our T-varieties. Finally in Section 5 we state the list with the combinatorial descriptions for all the considered Fano varieties, which in turn leads to the proof of Theorem 1.1. The appendix provides the proof for the formula for the Futaki character.

Acknowledgments

I would like to thank Ivan Arzhantsev, Nathan Ilten and Johan Martens for their helpful comments and conversations.

2 Combinatorial description of torus actions of complexity one

F-divisors

We consider a variety $X/_{\mathbb{C}}$ with an effective action of an algebraic torus $T \cong (\mathbb{C}^*)^r$. Then X is called a T-variety of complexity $(\dim X - \dim T)$. In the following we restrict ourselves to the case of rational T-varieties of complexity one. Such objects where described and studied earlier in [KKMSD73, Tim08]. We are using the more general approach of [AH06] which describes T-varieties of arbitrary complexity, but restrict it to the case of rational complexity-one T-varieties. Our general references are [AH06, AHS08, AIP$^+$11].

We follow the standard terminology for toric varieties. The character lattice of the torus T is denoted by M and we set $N := M^*$. For the associated vector spaces we write $M_\mathbb{Q}$ and $N_\mathbb{Q}$ or $M_\mathbb{R}$ and $N_\mathbb{R}$, respectively.

For a polyhedron $\Delta \subset N_\mathbb{Q}$ we consider its tail (or recession) cone $\operatorname{tail}\Delta$ defined by $\operatorname{tail}\Delta := \{v \in N_\mathbb{Q} \mid \Delta + \mathbb{Q}_{\geq 0} \cdot v = \Delta\}$. For a complete polyhedral subdivision Ξ of $N_\mathbb{Q}$ the set of tail cones has the structure of a fan, which is called the tail fan of Ξ and will be denoted by $\operatorname{tail}\Xi$. Now, consider a pair

$$\mathcal{S} = \left(\sum_{P \in \mathbb{P}^1} \mathcal{S}_P \otimes P, \quad \deg \mathcal{S} \right)$$

Here, $\sum_P \mathcal{S}_P \otimes P$ is just a formal sum and $\mathcal{S}_P$ are complete polyhedral subdivisions of $N_\mathbb{Q}$ with some common tail fan denoted by $\operatorname{tail}\mathcal{S}$. For the moment the degree $\deg \mathcal{S}$ is just a subset of $N_\mathbb{Q}$. The subdivisions $\mathcal{S}_P$ are called *slices* of $\mathcal{S}$. We assume that there are only finitely many *non-trivial* slices, i.e. those which differ from the tail fan. Note, that for every full-dimensional $\sigma \in \operatorname{tail}(\mathcal{S})$ there is a unique polyhedron Δ_P^σ in $\mathcal{S}_P$ with $\operatorname{tail}(\Delta_P^\sigma) = \sigma$.

Definition 2.1. A pair as above is called a (complete) *f-divisor* if for any full-dimensional $\sigma \in \operatorname{tail}(\mathcal{S})$ we have either $\deg \mathcal{S} \cap \sigma = \emptyset$ or

$$\sum_P \Delta_P^\sigma = \deg \mathcal{S} \cap \sigma \subsetneq \sigma.$$

A (complete) f-divisor $\mathcal{S}$ as above corresponds to a rational complete T-variety $X(\mathcal{S})$ of complexity one, see [AIP$^+$11, Section 5.3]. It comes with a rational quotient map $\pi : X(\mathcal{S}) \dashrightarrow \mathbb{P}^1$.

Example 2.2 (Fano threefold 3.10). We consider the f-divisor $\mathcal{S}$ with three non-trivial slices and the degree given in Figure 1. It is straight forward to check, that the condition of Definition 2.1 is indeed fulfilled.

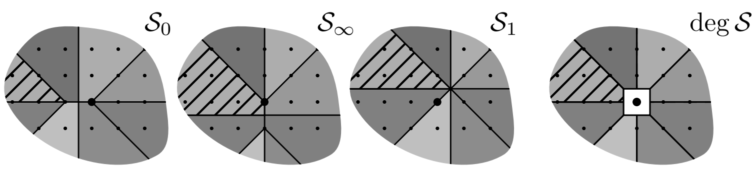


Figure 1: f-divisor of the Fano threefold 3.10.

As we will see later, this f-divisor corresponds to a blow up of the quadric threefold in two disjoint conics, i.e. it is an element of the family 3.10 from the classification of Mori and Mukai.

By [LS10, Proposition 5.1, Theorem 5.3] we have the following criterion for the smoothness of $X(\mathcal{S})$.

Theorem 2.3. *The T-variety $X(\mathcal{S})$ is smooth if and only if for every cone $\sigma \in \operatorname{tail}\mathcal{S}$ one of the following two conditions is fulfilled*

i. if $\sigma \cap \deg\mathcal{S} = \emptyset$ then cone $\mathbb{Q}_{\geq 0} \cdot (\Delta \times \{1\} + \sigma \times \{0\})$ is regular, for every P and every maximal polyhedron $\Delta \subset \mathcal{S}_P$ with tail cone σ.

ii. if σ is maximal and $\sigma \cap \deg\mathcal{S} \neq \emptyset$ then for at most two points $P = Q, R$ the polyhedron Δ_P^σ is not a lattice translate of σ and the cone

$$\mathbb{Q}_{\geq 0} \cdot \big(\sum_{P \neq Q} \Delta_P^\sigma \times \{1\} \;\cup\; \sigma \times \{0\} \;\cup\; \Delta_Q^\sigma \times \{-1\} \big)$$

is regular.

Example 2.4 (Fano threefold 3.10 (continued)). We consider again the f-divisor from Example 2.2. There is no maximal polyhedron Δ with $\operatorname{tail}\Delta \cap \deg\mathcal{S} = \emptyset$. Look at the shaded polyhedra in Figure 1. They have tail cone $\sigma = \mathbb{Q}_{\geq 0} \cdot (-1, 0) + \mathbb{Q}_{\geq 0} \cdot (-1, 1)$. Only Δ_1^σ and Δ_∞^σ are not a lattice translate of σ and we obtain

$$\begin{aligned} &\mathbb{Q}_{\geq 0} \cdot \big((\Delta_1^\sigma + {\scriptstyle (-1,0)}) \times \{1\} \cup \sigma \times \{0\} \cup \Delta_\infty^\sigma \times \{-1\} \big) \\ &= \mathbb{Q}_{\geq 0} \cdot (-1, 1, 2) \,+\, \mathbb{Q}_{\geq 0} \cdot (0, 0, -1) \,+\, \mathbb{Q}_{\geq 0} \cdot (0, -1, -2), \end{aligned}$$

which is regular cone. Similarly for the other cones from the tail fan we get the same result. Hence, by Theorem 2.3 we obtain the smoothness of $X(\mathcal{S})$.

Symmetries

In [AIP$^+$11, Section 5.3] there is also a characterization of f-divisors giving rise to isomorphic T-varieties. We consider isomorphisms $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ and $F : N \to N'$. Having this we define

$$F(\varphi^*\mathcal{S}) := \left(\sum_P (F(\mathcal{S}_{\varphi^{-1}(P)})) \otimes P, \quad F(\deg\mathcal{S}) \right).$$

Definition 2.5. Two f-divisors $\mathcal{S}$, $\mathcal{S}'$ on $\mathbb{P}^1$ are called equivalent if $\deg\mathcal{S} = \deg\mathcal{S}'$ and there are lattice points $v_P \in N$ with $\sum_P v_P = 0$, such that $\mathcal{S}_P + v_P = \mathcal{S}'_P$. In this situation we write $\mathcal{S} \sim \mathcal{S}'$.

Proposition 2.6. *$\mathcal{S}$ and $\mathcal{S}'$ lead to isomorphic T-varieties if and only if there are isomorphisms $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ and $F : N \to N'$, such that $F(\varphi^*\mathcal{S}) \sim \mathcal{S}'$.*

By considering the case $\mathcal{S} = \mathcal{S}'$ we even obtain a description of the group of equivariant automorphisms $\operatorname{Aut}_T(X)$. Therefore, we consider the set $\operatorname{Aut}(\mathcal{S})$ of pairs (F, φ) as above, such that $F(\varphi^*\mathcal{S}) \sim \mathcal{S}$ holds. This set becomes a group

via $(F, \varphi) \circ (F', \varphi') := (F \circ F',\ \varphi' \circ \varphi)$. Note, that there is a natural action of $\mathrm{Aut}(\mathcal{S})$ on the characters M and on $\mathbb{P}^1$, by considering only the first or only the second component of the pairs, respectively.
Now, we have the following proposition, due to [AIP$^+$11, Section 5.3].

Proposition 2.7. *The group* $\mathrm{Aut}(\mathcal{S})$ *is isomorphic to* $\mathrm{Aut}_T(X(\mathcal{S}))/T$.

Moreover, this isomorphism preserves the natural action of $\mathrm{Aut}_T(X(\mathcal{S}))/T$ on M by conjugation. Hence, we will identify $\mathrm{Aut}_T(X(\mathcal{S}))/T$ and $\mathrm{Aut}(\mathcal{S})$.

Definition 2.8. A T-variety of complexity one is called *symmetric* if there is a finite subgroup $G \subset \mathrm{Aut}_T(X(\mathcal{S}))$, with $M^G = \{0\}$.

Lemma 2.9. *Consider a complete rational variety X with torus action of complexity ≤ 1. Let T be the maximal torus of the automorphisms. Then* $\mathrm{Aut}_T(X)/T$ *is finite and* $\mathrm{Aut}_T(X)$ *is reductive.*

Proof. If X is a toric variety the statement is well known. Hence, we may consider the case of a complexity-one torus action. We consider the f-divisor $\mathcal{S}$ with $X = X(\mathcal{S})$ with non-trivial slices $\mathcal{S}_{P_1}, \ldots, \mathcal{S}_{P_\ell}$. All other slices are assumed to be just lattice translates of the tail fan. Assume that $\ell = 2$ then we may assume, that $P_1 = 0$ and $P_2 = \infty$. Now, every element $\varphi \in \mathbb{C}^* \subset \mathrm{Aut}(\mathbb{P}^1)$ will give rise to a pair $(\mathrm{id}, \varphi) \in \mathrm{Aut}(\mathcal{S})$. Hence, T was not a maximal torus and X is just a toric variety with an restricted torus action. Further on we may assume that $\ell \geq 3$.
Consider an element (F, φ) of $\mathrm{Aut}(\mathcal{S})$. Since, F has to be an automorphism of the tail fan, there are only finitely many choices for F. Since, $\ell \geq 3$ the automorphism φ of $\mathbb{P}^1$ is induced by a permutation of $P_1, \ldots, P_\ell$. Hence, there are only finitely many choices of for φ, as well.
Since, X is complete $\mathrm{Aut}_T(X)$ is an algebraic group and we have a Levi decomposition $\mathrm{Aut}_T(X) = H \ltimes R_u$, where H is reductive and R_u is the unipotent radical. We may assume that $T \subset H$. Hence, we obtain $\mathrm{Aut}_T(X)/T \cong H/T \times R_u$. By the finiteness of $\mathrm{Aut}_T(X)/T$ we conclude that R_u is trivial and $\mathrm{Aut}_T(X)$ is reductive. □

Proposition 2.10. *Consider an f-divisor $\mathcal{S}$ on $\mathbb{P}^1$ with at least three non-trivial slices. Then $X = X(\mathcal{S})$ is symmetric if and only if $M^{\mathrm{Aut}(\mathcal{S})} = 0$ holds.*

Proof. One direction is obvious. Now, assume that $M^{\mathrm{Aut}(S)} = 0$. By Lemma 2.9 we know that $\mathrm{Aut}(\mathcal{S}) = \mathrm{Aut}_T(X)/T$ is finite and $\mathrm{Aut}_T(X)$ is reductive. Now by the results of [Tit66] we can lift $\mathrm{Aut}(\mathcal{S})$ to a finite subgroup $G \subset \mathrm{Aut}_T(X)$, where G is not necessarily isomorphic to $\mathrm{Aut}(\mathcal{S})$. Now, we also have $M^G = 0$. □

The notion of symmetry for toric varieties goes back to [BS99] it was generalized in [Süß13] to give a sufficient criterion for the existence of a Kähler-Einstein metric on $X(\mathcal{S})$. For a vertex v in $\mathcal{S}_P$ let us denote by $\mu(v)$ the minimal natural number, such that $\mu(v) \cdot v$ is a lattice point. We call $\mu(v)$ the *multiplicity* of v.

For $P \in \mathbb{P}^1$ we set $\mu(P) = \max\{\mu(v) \mid v \in \mathcal{S}_P^{(0)}\}$, where $\mathcal{S}_P^{(0)}$ denotes the set of vertices in $\mathcal{S}_P$.

Theorem 2.11. *Let $X = X(\mathcal{S})$ be a symmetric smooth Fano T-variety of complexity one, given by an f-divisor $\mathcal{S}$. If one of the following conditions is fulfilled:*

i. there are three points $P_1, P_2, P_3 \in \mathbb{P}^1$ such that $\mu(P_i) > 1$ for $i = 1, 2, 3$,

ii. there are two points as in (i) which are swapped by an element of $\operatorname{Aut}(\mathcal{S})$*,*

iii. $\operatorname{Aut}(\mathcal{S})$ *acts fixed-point-free on $\mathbb{P}^1$,*

then X is Kähler-Einstein.

Proof. This is only a reformulation of [Süß13, Theorem 1.1]. The condition that T acts on $\pi^{-1}(P)$ with disconnected stabilizers is replaced by our condition $\mu(P) > 1$. But this is equivalent by [HS10, Proposition 4.11]. □

Now, we make use of Proposition 2.10 to check the symmetry of a variety $X(\mathcal{S})$.

Example 2.12 (Fano threefold 3.10 (continued)). We reconsider the f-divisor and variety from Example 2.2. By setting $v_0 = (-1, 0)$, $v_1 = (-1, 0)$, $v_\infty = (1, 1)$ and $v_P = 0$ for all other points $P \in \mathbb{P}^1$ we see that $\mathcal{S} \sim -\mathcal{S}$. Hence, the pair $(-\operatorname{id}_N, \operatorname{id}_{\mathbb{P}^1})$ is an element of $\operatorname{Aut}(\mathcal{S})$. Hence, $X(\mathcal{S})$ is symmetric, since $-\operatorname{id}_N$ has only 0 as a fixed point. Moreover, in the slices $\mathcal{S}_0, \mathcal{S}_\infty, \mathcal{S}_1$ there are vertices of multiplicity 2. By Theorem 2.11 (i) this shows, that this Fano threefolds is actually Kähler-Einstein.

3 Torus invariant divisors

In this section we recall the results of [PS11] concerning Weil and Carier divisors on T-varieties, see also [AIP+11, IS10].

The class group

On a T-variety $X(\mathcal{S})$ we find two different types of torus invariant prime divisors. Every vertex v in $\mathcal{S}_P$ correspond to a so called *vertical prime divisors* $V_v := V_{P.v}$ which projects to the point $P \in \mathbb{P}^1$, via the quotient map π. We denote the set of all vertices in $\mathcal{S}_P$ by $\mathcal{S}_P^{(0)}$ and the disjoint union $\coprod_P \mathcal{S}_P^{(0)}$ by $\mathcal{S}^{(0)}$. As before, for a vertex $v \in \mathcal{S}_P^{(0)}$ we denote by $\mu(v)$ the smallest positive integer μ such that $\mu \cdot v \in N$. *Horizontal prime divisors H_ρ* project surjectively onto $\mathbb{P}^1$ via the quotient map π. They correspond to the rays ρ in $\operatorname{tail} \mathcal{S}$ which do not intersect $\deg \mathcal{S}$. We denote the set of all such rays by $(\operatorname{tail} \mathcal{S})^\times$.

Theorem 3.1 ([PS11, Cor. 3.15]). *The divisor class group of $X(\mathcal{S})$ is isomorphic to*

$$\bigoplus_{\rho \in (\operatorname{tail} \mathcal{S})^\times} \mathbb{Z} \cdot H_\rho \oplus \bigoplus_{v \in \mathcal{S}^{(0)}} \mathbb{Z} \cdot V_v$$

modulo the relations

$$\begin{aligned} \sum_{v \in \mathcal{S}_P^{(0)}} \mu(v) V_v &= \sum_{v \in \mathcal{S}_Q^{(0)}} \mu(v) V_v, \\ 0 &= \sum_\rho \langle u, \rho \rangle H_\rho + \sum_{P,v} \mu(v) \langle u, v \rangle V_v \,. \end{aligned}$$

where $P, Q \in \mathbb{P}^1$ and $u \in M$.

Corollary 3.2. *The class group of X has rank*

$$1 + \sum_P (\# \mathcal{S}_P^{(0)} - 1) + \# (\operatorname{tail} \mathcal{S})^\times - \dim N.$$

Example 3.3 (Fano threefold 3.10 (continued)). We are coming back to our T-variety from Example 2.2. Since every ray in $\operatorname{tail} \mathcal{S}$ intersects the degree, by Theorem 3.1 the divisor class group of X is generated by the seven vertical divisors

$$\begin{array}{lll} D_1 = V_{0,(0,0)}, & D_2 = V_{0,(-1/2,0)}, & D_3 = V_{0,(-1,0)}, \\ D_4 = V_{\infty,(0,0)}, & D_5 = V_{\infty,(0,-1/2)}, & D_6 = V_{\infty,(0,-1)}, \\ & D_7 = V_{1,(1/2,1/2)}. & \end{array}$$

The relations are given by the rows of the following matrix

$$M = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 & 1 & -2 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 \end{pmatrix}$$

Calculating the Smith normal form $D = P \cdot M \cdot Q$ gives

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad P = I_4, \quad Q = \begin{pmatrix} 0 & 0 & -1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Hence, $\operatorname{Cl}(X) \cong \mathbb{Z}^3$. Where the identification is given by the last three columns of Q, i.e. $[D_1], [D_3] \mapsto (-1, 1, 1)$, $[D_2] \mapsto (1, 0, 0)$, $[D_4] \mapsto (0, 0, 1)$, $[D_5] \mapsto (0, 1, 0)$, $[D_6] \mapsto (0, 0, 1)$, $[D_7] \mapsto (0, 1, 1)$.

Ample divisors

We consider a piecewise affine concave function $\Psi : \Box \to \operatorname{Div}_{\mathbb{R}} \mathbb{P}^1$, defined on a lattice polytope $\Box \subset M_{\mathbb{R}}$. It induces corresponding piecewise affine and concave functions $\Psi_P : \Box \to \mathbb{R}$ via $\Psi(u) = \sum_{P \in \mathbb{P}^1} \Psi_P(u) \cdot P$. Using this notation we are going to define the crucial object of this section.

Definition 3.4. A *divisorial polytope* is a continuous piecewise affine concave function $\Psi : \Box \to \operatorname{Div}_{\mathbb{R}} \mathbb{P}^1$, such that

i. for every u in the interior of $\Box$, $\deg \Psi(u) > 0$,

ii. for every P the graph of Ψ_P is integral, i.e. has its vertices in $M \times \mathbb{Z}$.

We define the *support function* of Ψ to be the family $\Psi^* = \{\Psi^*_P\}_{P \in \mathbb{P}^1}$ of piecewise affine concave functions $\Psi^*_P : N_{\mathbb{R}} \to \mathbb{R}$ with

$$\Psi^*_P := \min_{u \in \Box_h} (\langle u, v \rangle - \Psi_P(u)).$$

We call $\operatorname{lin}(\Psi^*) := \min\langle \Box, \cdot \rangle$ the linear part of Ψ^*. It is a piecewise linear function on the normal fan of $\Box$.

Proposition 3.5 ([AIP+11, Theorem 63], [IS10, Theorem 3.2]). *There is a one-to-one correspondence between invariant ample divisors D on $X(\mathcal{S})$ and divisorial polytopes $\Psi : \Box \to \operatorname{Div}_{\mathbb{R}} \mathbb{P}^1$, with*

i. Ψ^*_P induces the subdivision $\mathcal{S}_P$, i.e. the maximal polyhedra of $\mathcal{S}_P$ are the regions of affine linearity of Ψ^*_P,

ii. $\deg \Psi(u) = 0$ holds for a point $u \in \Box_D$ iff $\langle u, v \rangle = \min\langle \Box, v \rangle$ for some $v \in \deg \mathcal{S}$.

Moreover, by [PS11, Corollary 3.19, Theorem 3.12] Ψ as in Proposition 3.5 corresponds to an anti-canonical divisor on $X = X(\mathcal{S})$, if there exists an integral divisor $K_{\mathbb{P}^1} = \sum a_P \cdot P$ of degree -2 on $\mathbb{P}^1$, such that

$$\Psi^*_P(v) = -a_P - 1 + 1/\mu(v), \quad \operatorname{lin}(\Psi^*)(v_\rho) = 1 \tag{1}$$

for every vertex $v \in \mathcal{S}_P$ and every ray ρ of $(\operatorname{tail} \mathcal{S})^\times$. In particular, X will be Fano. In this case we say that Ψ is a divisorial polytope corresponding to the Fano T-variety X.

Remark 3.6. By Proposition 3.5 (i) we have a one-to-one correspondence between facets of the graph of Ψ_P and vertices of $\mathcal{S}_P$. A facet of the graph of Ψ_P projects to a region $U \subset \Box$ where Ψ_P is affine linear, i.e. $\Psi_P|_U = \langle \cdot, v \rangle + c$ for some $v \in N_{\mathbb{R}}$. The corresponding vertex of $\mathcal{S}_P$ is exactly v and we have $\Psi^*(v) = c$. Taking this into account we can reformulate condition (1) as follows.

i. For every facet F of the graph of $\Psi_P + a_P + 1$ there is a vertex v such that $\langle \, \cdot \, , \mu(v)(v, 1) \rangle \equiv 1$ on F. In particular, F has lattice distance 1 from the origin. Moreover, we have $\Psi_P \equiv -a_P$ if $\mathcal{S}_P$ is trivial.

ii. the facets F of $\Box$ with $\deg \Psi|_F \not\equiv 0$ have lattice distance 1 from the origin.

Definition 3.7. The *volume* of a divisorial polytope is defined, by

$$\operatorname{vol} \Psi := \int_{\Box} \deg \Psi \, d\mu.$$

Here, we are integrating with respect to the Euclidean measure μ induced by the inclusion $M \subset M_{\mathbb{R}}$.

Theorem 3.8 ([PS11, Proposition 3.31]). *If D is ample on a T-variety $X = X(\mathcal{S})$ of dimension n, then its top self intersection number is given by*

$$(D)^n = n! \operatorname{vol} \Psi_D.$$

Example 3.9 (Fano threefold 3.10 (continued)). In Figure 2 the divisorial polytope $\Psi : \square \to \operatorname{Div}_{\mathbb{R}} \mathbb{P}^1$ for the canonical divisor is sketched. More precisely we give the non-trivial concave functions Ψ_P by drawing $\square$ and the regions of affine linearity of Ψ_P and giving the values of Ψ at the vertices of these regions.

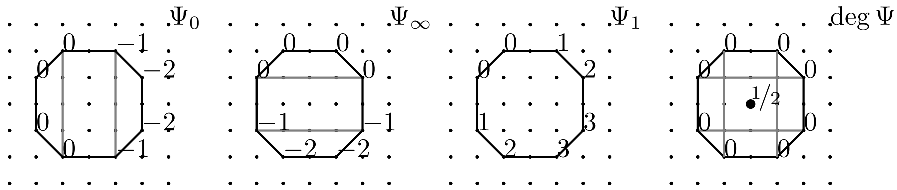


Figure 2: A divisorial polytope for the threefold 3.10.

We have $\deg \Psi = 0$ on the boundary of $\square$ and $\deg \mathcal{S}$ intersects all tail cones. Hence, condition (ii) of Proposition 3.5 is fulfilled. We will now check that the Ψ induces the subdivisions $\mathcal{S}_P$ and fulfills (1), for $K_{\mathbb{P}^1} = -2 \cdot [1]$. Calculating Ψ^* gives

$$\begin{aligned}
\Psi_0^* = \min\big(\, &\langle \tbinom{2}{1}, \cdot \rangle + 2,\ \langle \tbinom{1}{2}, \cdot \rangle + 1,\ \langle \tbinom{-1}{2}, \cdot \rangle + 0,\ \langle \tbinom{-2}{1}, \cdot \rangle + 0,\\
&\langle \tbinom{2}{-1}, \cdot \rangle + 2,\ \langle \tbinom{1}{-2}, \cdot \rangle + 1,\ \langle \tbinom{-1}{-2}, \cdot \rangle + 0,\ \langle \tbinom{-2}{-1}, \cdot \rangle + 0\big).\\
\Psi_\infty^* = \min\big(\, &\langle \tbinom{2}{1}, \cdot \rangle + 0,\ \langle \tbinom{1}{2}, \cdot \rangle + 0,\ \langle \tbinom{-1}{2}, \cdot \rangle + 0,\ \langle \tbinom{-2}{1}, \cdot \rangle + 0,\\
&\langle \tbinom{2}{-1}, \cdot \rangle + 1,\ \langle \tbinom{1}{-2}, \cdot \rangle + 2,\ \langle \tbinom{-1}{-2}, \cdot \rangle + 2,\ \langle \tbinom{-2}{-1}, \cdot \rangle + 1\big).\\
\Psi_1^* = \min\big(\, &\langle \tbinom{2}{1}, \cdot \rangle - 2,\ \langle \tbinom{1}{2}, \cdot \rangle - 1,\ \langle \tbinom{-1}{2}, \cdot \rangle + 0,\ \langle \tbinom{-2}{1}, \cdot \rangle + 0,\\
&\langle \tbinom{2}{-1}, \cdot \rangle - 3,\ \langle \tbinom{1}{-2}, \cdot \rangle - 3,\ \langle \tbinom{-1}{-2}, \cdot \rangle - 2,\ \langle \tbinom{-2}{-1}, \cdot \rangle - 1\big).
\end{aligned}$$

It's not hard to check, that Ψ_P^* is affine linear exactly on the maximal polyhedra of $\mathcal{S}_P$. Hence, by Proposition 3.5 it corresponds to an ample divisor D on $X(\mathcal{S})$. Moreover, we obtain

$$\begin{array}{rll}
\Psi_0^*(0,0) = 0 & h_0(-1/2, 0) = 1/2 & h_0(-1,0) = 0\\
\Psi_\infty^*(0,0) = 0 & h_\infty(0,-1/2) = 1/2 & h_\infty(0,-1) = 0\\
\Psi_1^*(1/2, 1/2) = 1 & &
\end{array}$$

Hence, (1) is fulfilled for our choice $K_{\mathbb{P}^1} = -2 \cdot [1]$ and D is an anti-canonical divisor.

By elementary calculations for $\operatorname{vol} \Psi$ we get

$$\operatorname{vol} \Psi = \int_{\square} \deg \Psi = \frac{13}{3}.$$

Putting together the observations of Example 2.2, 3.3 and 3.9 we conclude that X is a smooth Fano threefold of Picard rank 3 and Fano degree 26. By [MM86] this is a blow up of the quadric threefold in two disjoint conics.

3.1 Moment map and Duistermaat-Heckman measure

In this section we give an interpretation of divisorial polytopes in terms of moment maps.

The polarisation of a T-variety X by $L = \mathcal{O}(D)$, for an ample invariant divisor D, induces a symplectic structure on X by pulling back the Fubini-Study form of the corresponding torus equivariant embedding into projective space (at least for a suitable multiple of D). The action of the algebraic torus action on X induces a Hamiltonian action of the contained (maximal) compact torus T_c on the corresponding symplectic manifold. This action comes with a moment polytope P and a moment map $\phi : X \to P$. Now, the push forward of the canonical symplectic measure on X via the moment map gives a measure on P. This measure is called the Duistermaat-Heckman measure and the corresponding continuous density function f is called the Duistermaat-Heckman function. It is known to be piecewise polynomial and the degree corresponds to the complexity of the torus action, cf. [DH82].

The global sections L decompose in homogeneous components of weight $u \in M$. By [PS11, Proposition 3.23] we may express such a homogeneous component in terms of the corresponding divisorial polytope Ψ_D:

$$H^0\big(X, L\big)_u \cong \begin{cases} H^0\big(\mathbb{P}^1, \mathcal{O}(\Psi_D(u))\big) & , u \in \Box_D \\ 0 & , \text{else} \end{cases} \tag{2}$$

From this one can deduce that $\Box = \Box_D$ equals the moment polytope P, see [AIP$^+$11, Section 14.7]. More generally for powers if L we get by [IS10, Prop. 3.1]

$$H^0\big(X, L^{\otimes k}\big)_{k\cdot u} = H^0\big(\mathbb{P}^1, \mathcal{O}(k \cdot \Psi_D(u))\big) \tag{3}$$

for $u \in \Box \cap \frac{1}{k} \cdot M$.

One obtains the value $f(0)$ as the volume of the symplectic reduction of the moment fiber $\phi^{-1}(0)$. By the Kempf-Ness Theorem the symplectic reduction of $\phi^{-1}(0)$ by T_c is the same as the GIT quotient $Y = X/\!\!/^L T = \operatorname{Proj} \bigoplus_k H^0(X, L^{\otimes k})^T$. The GIT quotient comes together with a polarization by a $\mathbb{Q}$-line bundle $\mathcal{O}_Y(1)$, given by the invariant sections of L. This polarization is compatible with the symplectic structure on $\phi^{-1}(0)/T_c$. Hence, we have $\operatorname{vol} \phi^{-1}(0)/T_c = (\mathcal{O}_Y(1))^{\dim Y}$.

If 0 is in the interior of $\Box$ we get $Y = X/\!\!/^L T = \mathbb{P}^1$. Moreover, by (2) we have $\mathcal{O}_Y(1) = \mathcal{O}(\Psi(0))$ and we get $f(0) = \deg(\Psi(0))$. By shifting the linearization of L (and therefore the moment map) by $u \in \Box \cap M$ we get $f(u) = \deg(\Psi(u))$ for all lattice points in $\Box$. By (3) we get the same for rational points in $\Box$ and eventually by continuity for all points.

Altogether we obtain

PROPOSITION 3.10. *The Duistermaat-Heckman function corresponding to the polarization given by an invariant ample divisor D with divisorial polytope Ψ_D is given by* $\deg \Psi : \square_D \to \mathbb{R}$.

The other way around, we may interpret the divisorial polytope Ψ_D as a decomposition of the Duistermaat-Heckman function corresponding to the polarization of X by L. This shows the relation between our divisorial polytopes and the description of Hamiltonian torus actions given by Karshon and Tolman in [KT03, KT14].

THE COX RING

For a normal variety with a class group isomorphic to $\mathbb{Z}^r$ the Cox ring is defined as

$$\operatorname{Cox}(X) = \bigoplus_{\alpha \in \mathbb{Z}^r} H^0\left(X, \mathcal{O}(\textstyle\sum_i \alpha_i D_i)\right),$$

with $D_1, \ldots, D_r$ being a basis of $\operatorname{Cl}(X)$. For class groups which are not free the definition is little bit more involved. Since smooth Fano varieties always have free class group, we don't have to discuss this here.
In [HS10, Theorem 1.2] the Cox ring of a T-variety was calculated in terms of the combinatorial data. Due to [AIP+11] we have the following reformulation of this theorem in our language.

THEOREM 3.11 ([AIP+11, Theorem 40]).

$$\operatorname{Cox}(X) = \frac{\mathbb{C}\left[T_v, S_\rho \mid v \in \mathcal{S}^{(0)}, \rho \in (\operatorname{tail} \mathcal{S})^\times\right]}{\langle T^{\mu(0)} + cT^{\mu(\infty)} + T^{\mu(c)} \mid c \in \mathbb{C}^* \rangle},$$

where $T^{\mu(P)} = \prod_{v \in \mathcal{S}_P^{(0)}} T_v^{\mu(v)}$.
The $\operatorname{Cl}(X)$*-grading is given by* $\deg(T_v) = [V_v]$, $\deg(S_\rho) = [H_\rho]$.

EXAMPLE 3.12 (Fano threefold 3.10 (continued)). For our f-divisor from Example 2.2 we get the Cox ring

$$\mathbb{C}[T_1, \ldots, T_7]/\langle T_1 T_2^2 T_3 + T_4 T_5^2 T_6 + T_7 \rangle.$$

The grading is given by $\deg(T_i) = [D_i]$, with the notation of Example 3.3. Using the identification $\operatorname{Cl} X \cong \mathbb{Z}^3$ from Example 3.3 we obtain the weight matrix

$$\begin{array}{c} \begin{array}{ccccccc} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 \end{array} \\ \begin{pmatrix} -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

The Futaki character

In [Fut83] Futaki introduced the invariant as an obstruction for the existence of Kähler-Einstein metrics on Fano manifolds. Hence, for a Kähler-Einstein Fano manifold the Futaki character has to vanish.

Donaldson gave a an algebraic redefinition of this invariant in [Don02], which we are using in the following. Consider a $\mathbb{C}^*$-action λ on a normal variety X of dimension n and an invariant Cartier divisor D. Now, let $l_k = \dim H^0(X, \mathcal{O}(kD))$ and w_k the total weight of the $\mathbb{C}^*$-action on $H^0(X, \mathcal{O}(kD))$, i.e. $w_k = \sum_i i \cdot \dim H^0(X, \mathcal{O}(kD))_i$. Then l_k and w_k obtain expansions

$$\begin{aligned} l_k &= a_0 k^n + a_1 k^{n-1} + O(k^{n-2}),\\ w_k &= b_0 k^{n+1} + a_1 k^n + O(k^{n-1}). \end{aligned}$$

Definition 3.13. The Futaki invariant of this $\mathbb{C}^*$-action is defined as

$$F_D(\lambda) = 2 \cdot \frac{a_1 b_0 - a_0 b_1}{a_0}.$$

Remark 3.14. Note, that we choose a different scaling and sign as Donaldson for the invariant. More precisely his invariant equals $-\frac{1}{2a_0} F_D$ in our notation.

By plugging in one-parameter subgroups of an acting torus we obtain a linear map $F_D : N \to \mathbb{R}$, hence an element of $M_{\mathbb{R}}$. If X is Fano we set $F(X) := F_{(-K_X)}$.

For a toric variety Donaldson in [Don02] gives an easy formula to actually calculate the Futaki character. Let Δ be the polytope corresponding to an ample divisor D on a toric variety. Then the Futaki character is given by

$$F_D(v) \;=\; \int_{\partial\Delta} v \; d\mu_1 \;-\; \frac{\operatorname{vol}_1 \partial\Delta}{\operatorname{vol}\Delta} \cdot \int_{\Delta} v \; d\mu. \tag{4}$$

Here, we are integrating over Δ with the Euclidean measure μ induced by the lattice $M \subset M_{\mathbb{R}}$. For integrating over the boundary we are using the *facet measure* μ_1, which is induced by $(\mathbb{R} \cdot \delta) \cap M$ for every facet $\delta \prec \Delta$. The corresponding *facet volume* is denoted by vol_1 and the *facet barycenter* by bc_1. Note, that up to scaling by $(\operatorname{vol}_1 \partial\Delta)$, for the Futaki character we obtain the difference of the barycenters of Δ and its boundary:

$$F_D = (\operatorname{vol}_1 \partial\Delta) \cdot (\operatorname{bc}_1(\partial\Delta) - \operatorname{bc}(\Delta)). \tag{5}$$

It's easy to see that for the standard polytope of the canonical divisor $\Delta = \{u \mid \forall_{\rho\in\Sigma^{(1)}} : \langle u, n_\rho\rangle \geq -1\}$ the formula simplifies to $F(X) = \operatorname{vol}\Delta \cdot \operatorname{bc}\Delta$, i.e. the Futaki character is up to positive scaling given by the barycenter of the polytope. This was already observed by Mabuchi [Mab87]. For generalizing the last result to our case we define the *barycenter* $\overline{\operatorname{bc}}(\Psi)$ of a divisorial polytope $\Psi : \Box \to \operatorname{Div}_{\mathbb{R}} \mathbb{P}^1$ by

$$\langle \overline{\operatorname{bc}}(\Psi), v\rangle = \int_{\Box_h} v \cdot \deg \Psi \; d\mu.$$

This gives the same as the projection to $M_{\mathbb{R}}$ of the barycenter of

$$\Delta := \Delta(\deg \Psi) := \{(u,x) \mid u \in \Box,\ 0 \le x \le \deg(u)\}.$$

Theorem 3.15. *For a Fano T-variety X and a corresponding divisorial polytope Ψ we have*

$$F(X) = \operatorname{vol}\Psi \cdot \overline{\operatorname{bc}}\Psi.$$

We postpone the proof to the appendix.

Remark 3.16. By Proposition 3.10 we may interpret the Futaki character as barycenter of the moment polytope with respect to the Duistermaat-Heckman measure $(\deg \Psi) \cdot \mu$, see also [Mab87, Theorem 9.2.3] for closely related results.

Example 3.17. We would like to compute the Futaki invariant of the Fano threefold 3.23. Its divisorial polytope Ψ is given by Figure 3. We can read off $\Delta(\deg \Psi)$ directly from the rightmost picture. The barycenter of this poly-

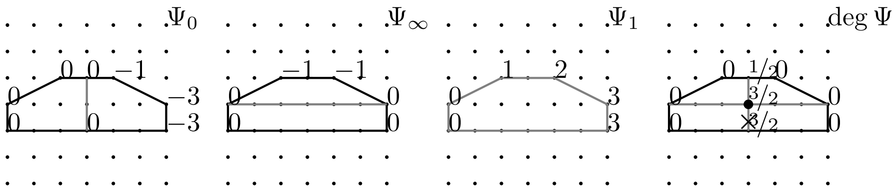


Figure 3: Divisorial polytope of 3.23

tope is $(0, -9/40, 37/80)^t$ and the volume $20/3$. Note, that we have $\operatorname{vol}\Psi = \operatorname{vol}\Delta(\deg\Psi)$. Hence, by applying Theorem 3.15 we get $F(X) = \binom{0}{-3/2} \neq 0$. In particular, there does not exists a Kähler-Einstein metric on X.

4 Toric degenerations

We are using the notation and results of [IV12] to describe degenerations to toric varieties via f-divisors.

First note, that the torus action of a T-variety $X(\mathcal{S})$ is not necessarily maximal. Consider an f-divisor $\mathcal{S}$ with two non-trivial slices $\mathcal{S}_0$ and $\mathcal{S}_\infty$. Remark 1.8 in [IV12] shows that the T-action on $X(\mathcal{S})$ extends to the action of a torus T' of full dimension. Hence, X is essentially toric with a fan Σ as follows: We embed the polyhedra of $\mathcal{S}_0$ in height 1 of $N_{\mathbb{Q}} \times \mathbb{Q}$ and consider the fan Σ_0 consisting of the cones over these polyhedra. We are doing the same with $\mathcal{S}_\infty$ embedded in height -1 and obtain Σ_∞. Now, we obtain Σ from $\Sigma_0 \cup \Sigma_\infty$ by joining cones which intersect in a common facet of the form $\sigma \times \{0\}$, such that $\sigma \cap \deg \mathcal{S} \neq \emptyset$.

Example 4.1. Consider the f-divisor pictured in Figure 4 having two non-trivial slice $\mathcal{S}_0$ and $\mathcal{S}_\infty$. The fan Σ_0 from above consists of five maximal cones:

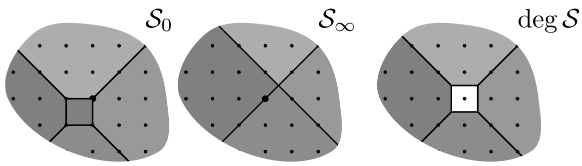


Figure 4: F-divisor of a downgraded toric variety.

$$
\begin{aligned}
\sigma_0 &= \operatorname{pos}\big((0,0,1),(-1,0,1),(-1,-1,1),(0,-1,1)\big)\\
\sigma_1 &= \operatorname{pos}\big((0,0,1),(-1,0,1),(1,1,0),(-1,1,0)\big)\\
\sigma_2 &= \operatorname{pos}\big((0,0,1),(0,-1,1),(1,1,0),(1,-1,0)\big)\\
\sigma_3 &= \operatorname{pos}\big((-1,0,1),(-1,-1,1),(-1,1,0),(-1,-1,0)\big)\\
\sigma_4 &= \operatorname{pos}\big((-1,-1,1),(0,-1,1),(1,-1,0),(-1,-1,0)\big).
\end{aligned}
$$

The fan Σ_∞ consists of the four cones

$$
\begin{aligned}
\tau_1 &= \operatorname{pos}\big((1,1,-2),(1,1,0),(-1,1,0)\big)\\
\tau_2 &= \operatorname{pos}\big((1,1,-2),(1,1,0),(1,-1,0)\big)\\
\tau_3 &= \operatorname{pos}\big((1,1,-2),(-1,1,0),(-1,-1,0)\big)\\
\tau_4 &= \operatorname{pos}\big((1,1,-2),(1,-1,0),(-1,-1,0)\big).
\end{aligned}
$$

We have to join four pairs of cones (σ_i, τ_i), $i = 1, \ldots, 4$ and obtain the fan Σ consisting of the five maximal cones

$$
\begin{aligned}
\delta_0 &= \sigma_0\\
\delta_1 &= \operatorname{pos}\big((0,0,1),(-1,0,1),(1,1,-2)\big)\\
\delta_2 &= \operatorname{pos}\big((0,0,1),(0,-1,1),(1,1,-2)\big)\\
\delta_3 &= \operatorname{pos}\big((-1,0,1),(-1,-1,1),(1,1,-2)\big)\\
\delta_4 &= \operatorname{pos}\big((-1,-1,1),(0,-1,1),(1,1,-2)\big).
\end{aligned}
$$

This is the fan of the projective cone over the quadric surface. Moreover, it is the face fan of the (reflexive) polytope

$$
\operatorname{conv}\big((0,0,1),(-1,0,1),(-1,-1,1),(0,-1,1),(1,1,-2)\big)
$$

which has ID 544395 in the classification of canonical toric Fano threefolds from [Kas10][†].

Now, we consider a polyhedral subdivision Ξ. A Minkowski-decomposition of Ξ consist of polyhedral subdivisions $\Delta = \Delta^1 + \ldots + \Delta^r$ for every $\Delta \in \Xi$ such that

[†] see `http://grdb.lboro.ac.uk/search/toricf3c?ID=544395`

i. $(\Delta \cap \nabla)^i = \Delta^i \cap \nabla^i$, for $\Delta, \nabla \in \Xi$ and $\Delta \cap \nabla \neq \emptyset$,

ii. For $I \subset \{1, \ldots, r\}$, $\mathcal{J} \subset \mathcal{I} \subset \Xi$ we have

$$\sum_{i \in I} \bigcap_{\Delta \in \mathcal{I}} \Delta^i \prec \sum_{i \in I} \bigcap_{\Delta \in \mathcal{J}} \Delta^i$$

We obtain polyhedral subdivisions $\Xi^i := \{\Delta^i \mid \Delta \in \Xi\}$. By abuse of notation we will also write $\Xi = \Xi^1 + \ldots + \Xi^r$ for this situation. Such a decomposition is called *admissible* if for every vertex v of Ξ there is at most one of the corresponding vertices $v_i \in \Xi_i$ with $v = \sum_i v_i$ which is not a lattice point.
We may start with an f-divisor $\mathcal{S}$ with non-trivial slices $\Xi_0 = \mathcal{S}_{P_0}, \ldots, \Xi_\ell = \mathcal{S}_{P_\ell}$ and an admissible Minkowski decomposition $\Xi_0 = \Xi_{\ell+1} + \ldots + \Xi_{\ell+r}$. As described in [IV12, Sections 2 and 4] this data gives rise to a deformation of $X(\mathcal{S})$. A general fiber of this deformation corresponds to an f-divisor $\mathcal{S}'$ with non-trivial slices being exactly

$$\mathcal{S}'_{P'_1} = \Xi_1, \ \ldots \ , \ \mathcal{S}'_{P'_{\ell+r}} = \Xi_{\ell+r}.$$

For some distinct points $P'_1, \ldots P'_{\ell+r} \in \mathbb{P}^1$.
We now reverse the above procedure in order to obtain toric degenerations. Let's start with an f-divisor and assume that all non-trivial slices are contained in $\{\mathcal{S}_{P_1}, \ldots \mathcal{S}_{P_r}, \mathcal{S}_\infty\}$ and the first r of them form an admissible Minkowski decomposition of some subdivision Ξ

$$\Xi = \mathcal{S}_{P_1} + \ldots + \mathcal{S}_{P_r}.$$

Then $X(\mathcal{S})$ is a deformation of the T-variety corresponding to the f-divisor

$$(\Xi \otimes [0] + \mathcal{S}_\infty \otimes [\infty], \deg \mathcal{S})$$

which describes a subtorus action on a toric variety as we have seen above.

Example 4.2. We consider the f-divisor with slices $\mathcal{S}_0$, $\mathcal{S}_\infty$, $\mathcal{S}_1$ and degree sketched in Figure 5. The corresponding variety turns out to be the quadric threefold.

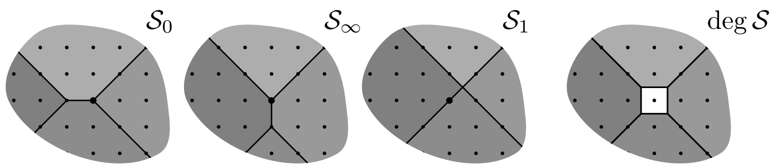


Figure 5: F-divisor of the quadric threefold

Note, that $\mathcal{S}_0$ and $\mathcal{S}_\infty$ form an admissible Minkowski decomposition, which is sketched in Figure 6.

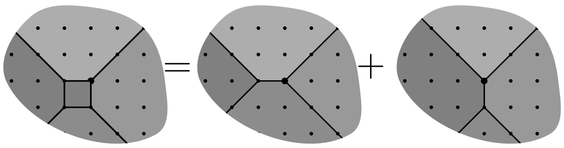


Figure 6: An admissible Minkowski decomposition

Hence, X degenerates to the toric variety from Example 4.1. We find another toric degeneration to the canonical toric Fano variety with ID 547378 in [Kas10] by adding up $\Xi = \mathcal{S}_0 + \mathcal{S}_1$. Adding up $\Xi = \mathcal{S}_0 + \mathcal{S}_\infty + \mathcal{S}_1$ will give a toric degeneration to a non-canonical toric Fano variety.

Remark 4.3. Note, that we do not get all toric degeneration with this method, but only those which are equivariant with respect to the T-action on X.

5 Proof by pictures

Aim of this section is to proof Theorem 1.1. For every variety from the list in Theorem 1.1 we will state the f-divisor $\mathcal{S}$ and a divisorial polytope Ψ, which corresponds to a canonical divisor. To verify the proof for a particular threefold one has to check first, that the f-divisor and divisorial polytope actually correspond to the given threefold. This is done by carrying out the following procedure.

- A.1 Applying Theorem 2.3: checking the smoothness of $X(\mathcal{S})$ by checking the regularity of certain cones (Example 2.2).
- A.2 Calculating the Picard rank of $X(\mathcal{S})$ by Corollary 3.2.
- A.3 Checking that Ψ corresponds to a canonical divisor (Example 3.9).
- A.4 Calculating the Fano degree of $X(\mathcal{S})$ by Theorem 3.8 (Example 3.9).

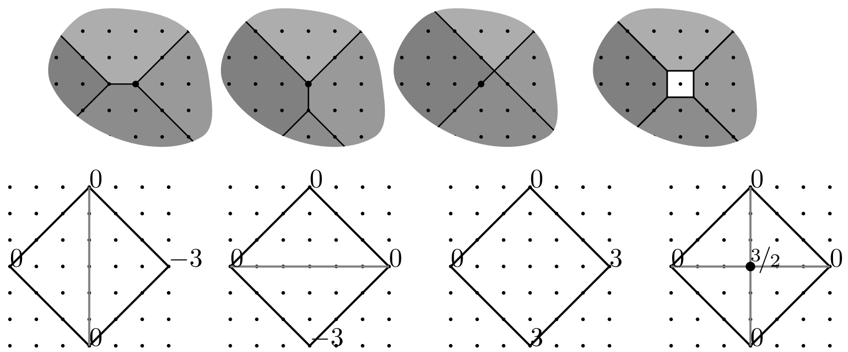
Now, by the classification in [MM86] we know that the given f-divisor describes a Fano threefold in the stated family. Hence, we constructed a Fano threefold with 2-torus action within this family. To check the other statement of Theorem 1.1 we have to apply additional steps

- B.1 Applying Theorem 3.11 to calculate the Cox ring (Example 3.12).
- B.2 Applying Theorem 3.15 to calculate the Futaki invariant (Example 3.17).
- B.3 Checking symmetry as in Example 2.12.
- B.4 Checking if $F(X) = 0$ and if possible apply Theorem 2.11 to check for the Kähler-Einstein property (Examples 2.12, 3.17).
- B.5 "Adding up slices" to obtain toric degenerations (Example 4.2).

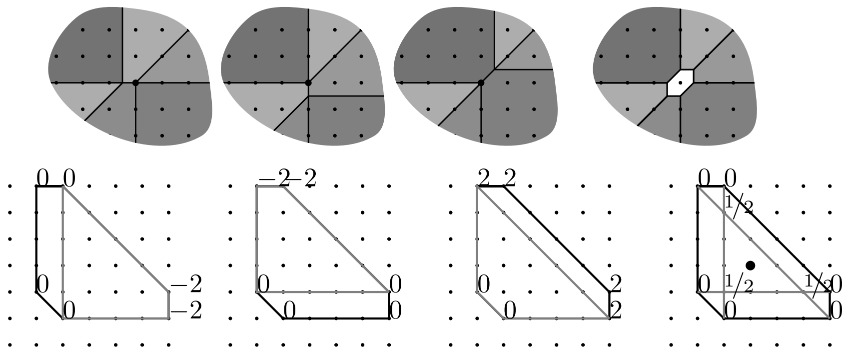
Beside the given examples we leave it to the reader to carry out the verification procedure for the given f-divisors.

We give some hints how to interpret the pictures and information given below. As in the examples we first state the non-trivial slice $\mathcal{S}_0$, $\mathcal{S}_\infty$, $\mathcal{S}_1$ and $\deg \mathcal{S}$ of an f-divisor $\mathcal{S}$. Next, we state the divisorial polytope of the canonical divisor by giving Ψ_0, Ψ_∞ and Ψ_1 (all other Ψ_P are assumed to be trivial). For convenience of the reader we state $\deg \Psi$, as well. In the pictures $\bullet$ marks the origin and $\times$ the Futaki character (if it differs from the origin). If there exists equivariant degenerations to canonical toric varieties then we state the corresponding IDs for the Fano polytopes in the *Graded Ring Database* [BDK+13].

Name :	Q	Quadric threefold			
Picard rank:	1	Fano Degree:	54	symm.:	yes
Futaki-char.:	0	Degenerations:	544395, 547378		
Cox ring: $\mathbb{C}[T_1,\dots,T_5]/\langle T_1T_2+T_3T_4+T_5^2\rangle$			$\deg(\underline{T})=(1,1,1,1,1)$		

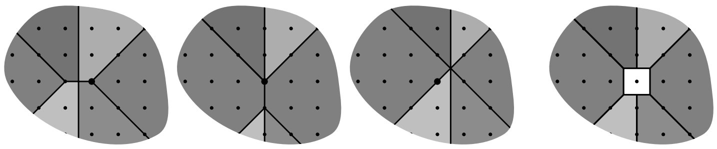


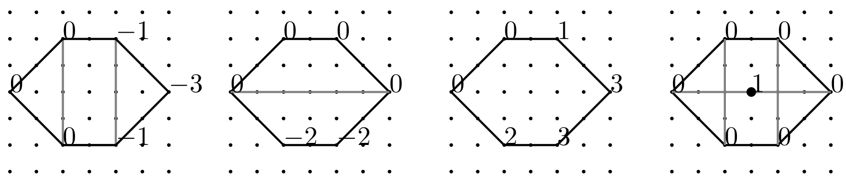
Name :	2.24	Divisor of bidegree $(1,2)$ in $\mathbb{P}^2\times\mathbb{P}^2$			
Picard rank:	2	Fano Degree:	54	symm.:	yes
Futaki-char.:	0	Degenerations:	—		
Cox ring: $\mathbb{C}[T_1,\dots,T_6]/\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle$			$\deg(\underline{T})=\left(\begin{smallmatrix}1&0&1&0&1&0\\0&1&0&1&0&1\end{smallmatrix}\right)$		



Name :	2.29 Blowup of Q in a conic				
Picard rank:	2	Fano Degree:	40	symm.:	yes
Futaki-char.:	0	Degenerations:	430083, 544339		

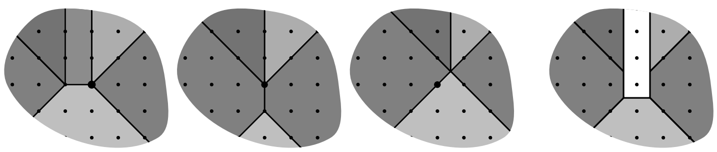
Cox ring: $\mathbb{C}[T_1,\dots,T_6]/\langle T_1T_2^2T_3+T_4T_5+T_5^2\rangle$ $\qquad \begin{pmatrix} -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$

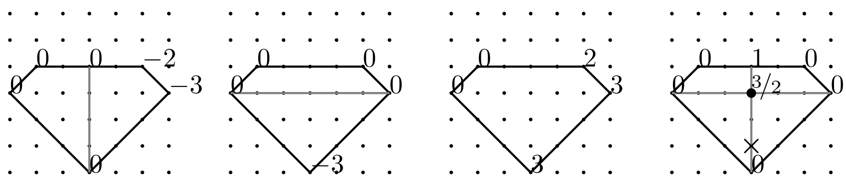




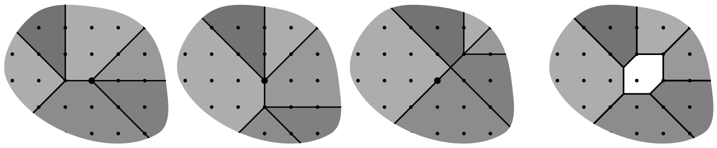
Name :	2.30 Blow up of Q in a point				
Picard rank:	2	Fano Degree:	46	symm.:	no
Futaki-char.:	$\binom{0}{-2}$	Degenerations:	520157, 544343		

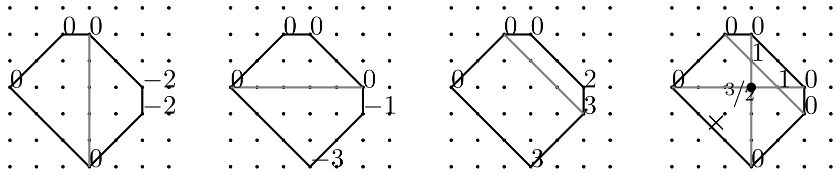
Cox ring: $\mathbb{C}[T_1,\dots,T_6]/\langle T_1T_2+T_3T_4+T_5^2\rangle$ $\qquad \begin{pmatrix} 1 & 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{pmatrix}$



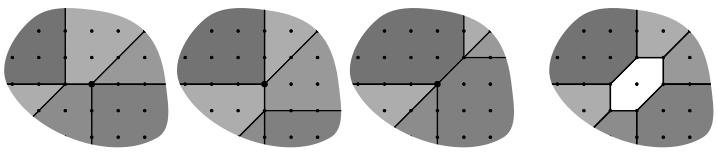


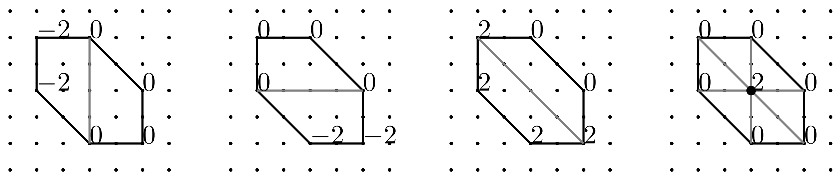
Name :	2.31	Blow up of Q in a line			
Picard rank:	2	Fano Degree:	46	symm.:	no
Futaki-char.:	$-\binom{4/3}{4/3}$	Degenerations:	520058, 520159		
Cox ring: $\mathbb{C}[T_1,\dots,T_6]/\langle T_1T_2+T_3T_4+T_5T_6^2\rangle$			$\left(\begin{smallmatrix}0&1&1&0&-1&1\\1&0&0&1&1&0\end{smallmatrix}\right)$		





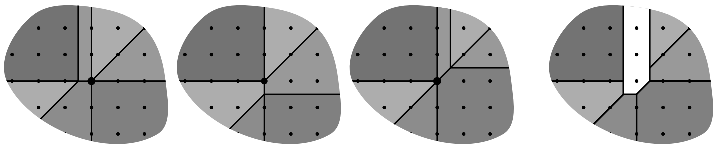
Name :	W or 2.32	Divisor of bidegree $(1,1)$ in $\mathbb{P}^2\times\mathbb{P}^2$			
Picard rank:	2	Fano Degree:	46	symm.:	yes
Futaki-char.:	0	Degenerations:	520058, 520159		
Cox ring: $\mathbb{C}[T_1,\dots,T_6]/\langle T_1T_2+T_3T_4+T_5T_6\rangle$			$\deg(\underline{T})=\left(\begin{smallmatrix}1&0&1&0&1&0\\0&1&0&1&0&1\end{smallmatrix}\right)$		

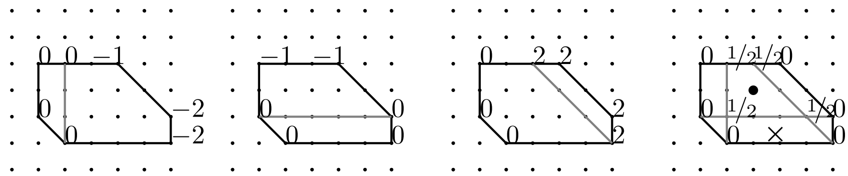




NAME :	3.8	Blowup of 2.24 in a line			
PICARD RANK:	3	FANO DEGREE:	24	SYMM.:	no
FUTAKI-CHAR.:	$\binom{5/6}{-5/3}$	DEGENERATIONS:	—		

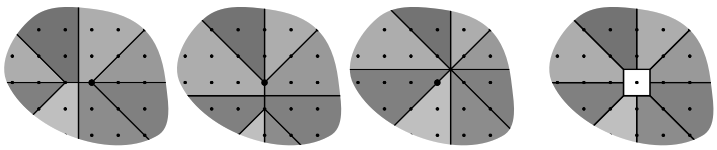
COX RING: $\mathbb{C}[T_1,\dots,T_7]/\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle$ $\quad \deg(\underline{T})=\begin{pmatrix}1&1&1&1&1&1&0\\0&1&2&0&2&0&0\\0&0&-2&1&0&0&1\end{pmatrix}$

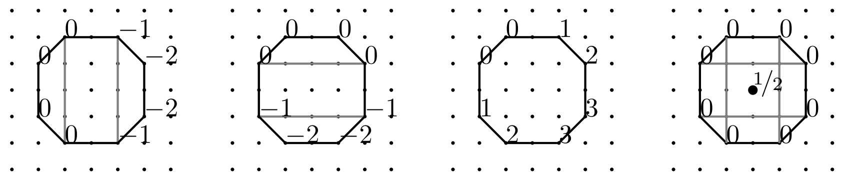




NAME :	3.10	Blowup of Q in two disjoint invariant conics			
PICARD RANK:	3	FANO DEGREE:	26	SYMM.:	yes
FUTAKI-CHAR.:	0	DEGENERATIONS:	—		

COX RING: $\mathbb{C}[T_1,\dots,T_7]/\langle T_1T_2^2T_3+T_4T_5^2T_6+T_7^2\rangle$ $\quad \deg(\underline{T})=\begin{pmatrix}0&1&0&0&1&0&1\\1&0&1&0&1&0&1\\1&0&1&1&0&1&1\end{pmatrix}$



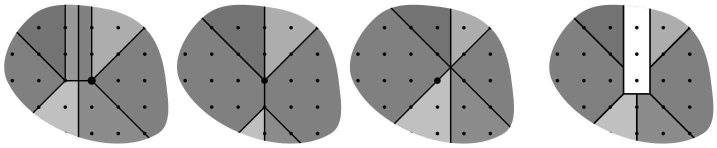


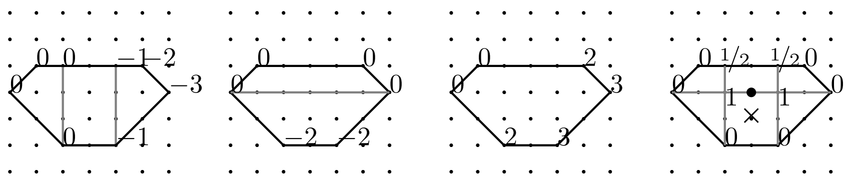
Name :	3.18	Blowup of Q in a point and a conic

Picard rank: 3 Fano Degree: 36 symm.: yes

Futaki-char.: $\binom{0}{-7/8}$ Degenerations: 255687, 519918

Cox ring: $\mathbb{C}[T_1,\ldots,T_7]/\langle T_1T_2^2T_3+T_4T_5+T_6^2\rangle$ $\deg(\underline{T})=\begin{pmatrix}0&1&0&2&0&1&-1\\1&0&1&2&0&1&-1\\1&0&1&1&1&1&0\end{pmatrix}$



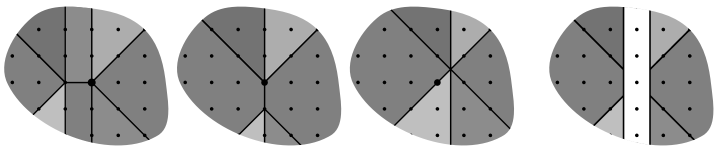


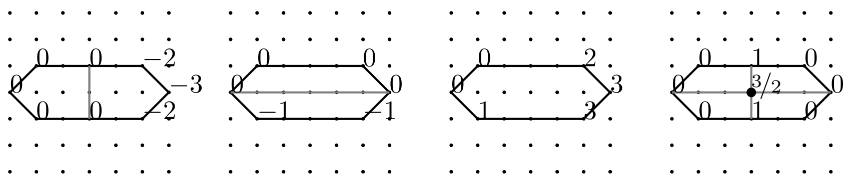
Name :	3.19	Blowup of Q in two points

Picard rank: 3 Fano Degree: 38 symm.: yes

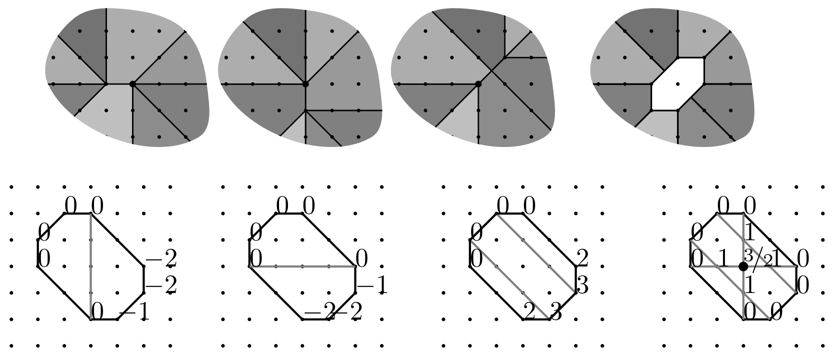
Futaki-char.: 0 Degenerations: 430426, 519917

Cox ring: $\mathbb{C}[T_1,\ldots,T_7]/\langle T_1T_2+T_3T_4+T_5^2\rangle$ $\deg(\underline{T})=\begin{pmatrix}1&1&2&0&1&-1&0\\0&0&-1&1&0&1&0\\0&0&0&0&0&1&1\end{pmatrix}$

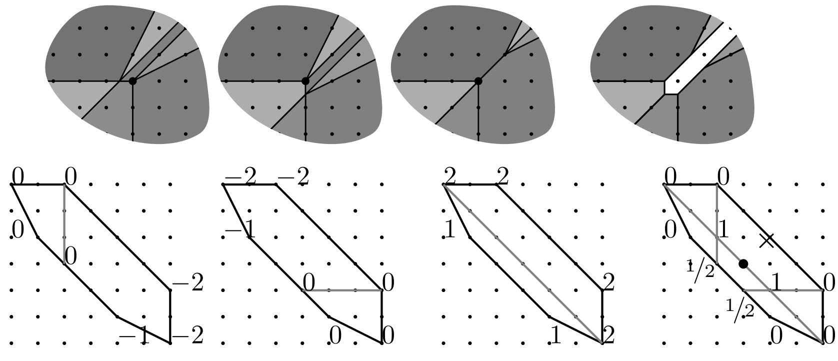




Name :	3.20	Blowup of Q in two disjoint lines		
Picard rank:	3	Fano Degree:	38	symm.: yes
Futaki-char.:	0	Degenerations:	255692, 430424	
Cox ring: $\mathbb{C}[T_1,\dots,T_7]/\langle T_1T_2+T_3T_4+T_5T_6^2T_7\rangle$			$\deg(\underline{T}) = \begin{pmatrix} 0&1&1&0&1&0&0 \\ 1&1&1&1&0&1&0 \\ 1&0&0&1&0&0&1 \end{pmatrix}$	

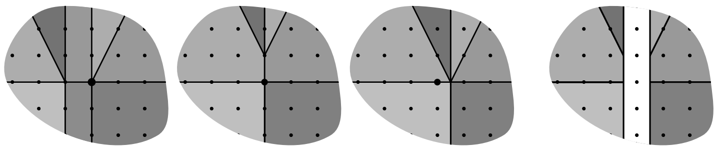


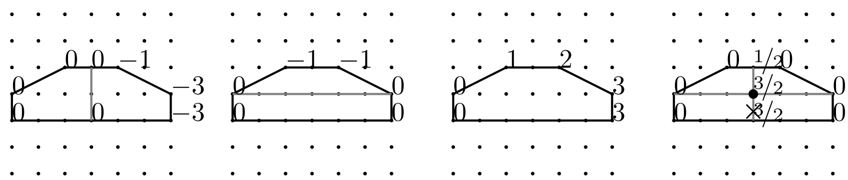
Name :	3.21	Blowup of $\mathbb{P}^1\times\mathbb{P}^2$ in a curve of degree (2,1)		
Picard rank:	3	Fano Degree:	38	symm.: no
Futaki-char.:	$\binom{7/8}{7/8}$	Degenerations:	429943	
Cox ring: $\mathbb{C}[T_1,\dots,T_7]/\langle T_1T_2^2+T_3T_4^2+T_5T_6\rangle$			$\deg(\underline{T}) = \begin{pmatrix} 1&0&1&0&1&0&0 \\ 0&1&0&1&1&1&0 \\ 0&1&0&1&2&0&1 \end{pmatrix}$	



Name :	3.22	Blowup of $\mathbb{P}^1 \times \mathbb{P}^2$ in a conic in $\{0\} \times \mathbb{P}^2$			
Picard rank:	3	Fano Degree:	40	symm.:	no
Futaki-char.:	$\binom{0}{-2/3}$	Degenerations:	430417, 519928		

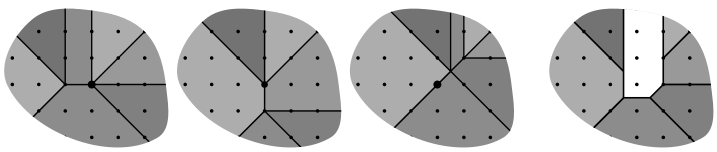
Cox ring: $\mathbb{C}[T_1, \ldots, T_7]/\langle T_1T_2 + T_3T_4 + T_5^2 \rangle$ $\quad \deg(\underline{T}) = \begin{pmatrix} 1 & 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$

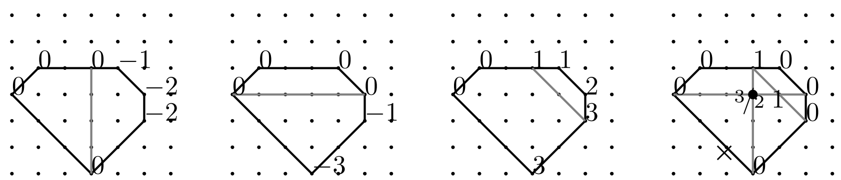




Name :	3.23	Blowup of Q in a point and the strict transform of a line passing through			
Picard rank:	3	Fano Degree:	42	symm.:	no
Futaki-char.:	$\binom{-13/12}{-53/24}$	Degenerations:	254876, 430425		

Cox ring: $\mathbb{C}[T_1, \ldots, T_7]/\langle T_1T_2 + T_3T_4 + T_5T_6^2 \rangle$ $\quad \deg(\underline{T}) = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 & 1 \end{pmatrix}$

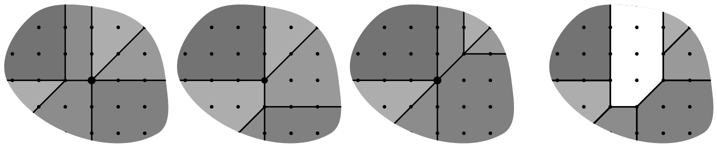


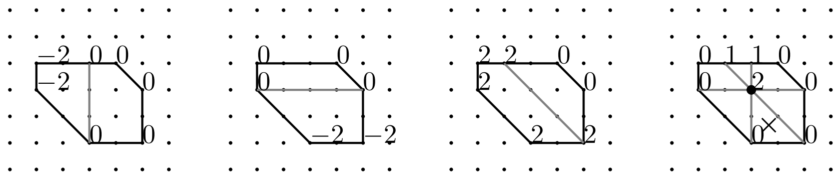


Name :	3.24	Blowup of W in $(0:0:1,*:*:0)$

Picard rank:	3	Fano Degree:	42	symm.:	no
Futaki-char.:	$\binom{2/3}{-4/3}$	Degenerations:	429952, 430423		

Cox ring: $\mathbb{C}[T_1,\ldots,T_7]/\langle T_1T_2+T_3T_4+T_5T_6\rangle$ $\quad \deg(\underline{T})=\begin{pmatrix} 0&1&0&1&1&0&0\\ 1&0&0&1&0&1&-1\\ 1&0&1&0&0&1&0 \end{pmatrix}$

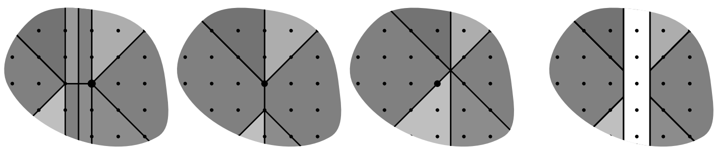


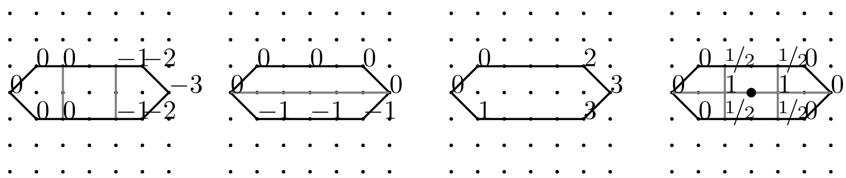


Name :	4.4	Blow up of Q in 2 non-colinear points and the strict transform of a conic passing through both of them

Picard rank:	4	Fano Degree:	32	symm.:	yes
Futaki-char.:	0	Degenerations:	254352, 430357		

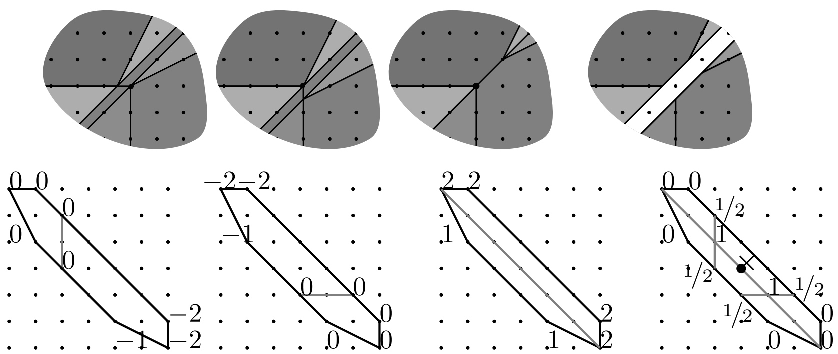
Cox ring: $\mathbb{C}[T_1,\ldots,T_8]/\langle T_1T_2^2T_3+T_4T_5+T_6^2\rangle$ $\quad \deg(\underline{T})=\begin{pmatrix} 0&1&0&2&0&1&0&1\\ 1&0&1&2&0&1&0&1\\ 1&0&1&1&1&1&1&1\\ 0&0&0&0&0&0&1&1 \end{pmatrix}$





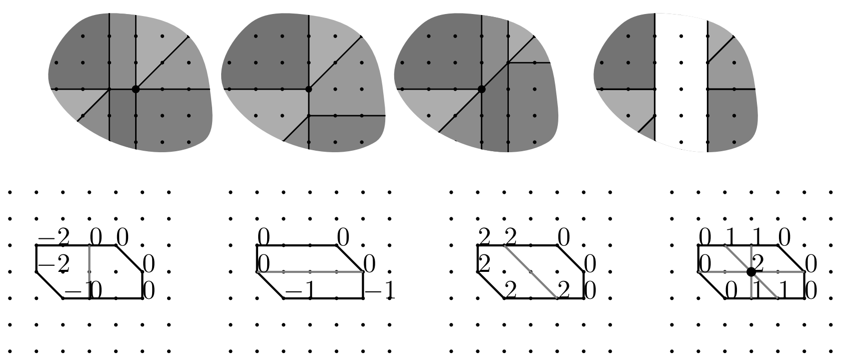
Name :	4.5	Blowup of $\mathbb{P}^1\times\mathbb{P}^2$ in curves of deg (2,1) and (1,0)			
Picard rank:	3	Fano Degree:	32	symm.:	no
Futaki-char.:	$\binom{5/24}{5/24}$	Degenerations:	255339		

Cox ring: $\mathbb{C}[T_1,\ldots,T_8]/\langle T_1T_2^2+T_3T_4^2+T_5T_6\rangle$ $\quad\deg(\underline{T})=\begin{pmatrix}1&0&1&0&1&0&0&0\\1&1&1&1&2&1&0&0\\0&1&0&1&2&0&1&0\end{pmatrix}$

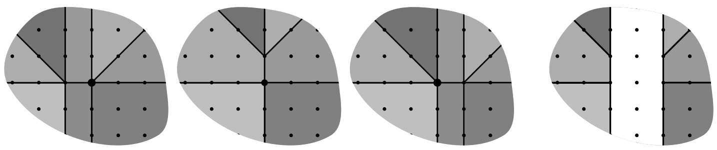


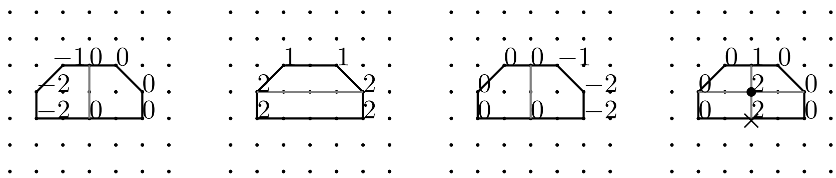
Name :	4.7	Blowup of W in $(0:0:1,*:*:0)$ and $(*:*:0,0:0:1)$			
Picard rank:	4	Fano Degree:	36	symm.:	yes
Futaki-char.:	0	Degenerations:	254603, 255837		

Cox ring: $\mathbb{C}[T_1,\ldots,T_8]/\langle T_1T_2+T_3T_4+T_5T_6\rangle$ $\quad\deg(\underline{T})=\begin{pmatrix}0&1&1&0&1&0&0&0\\1&0&1&0&0&1&0&1\\1&0&0&1&0&1&0&0\\0&0&0&0&0&0&1&1\end{pmatrix}$



NAME :	4.8	Blowup of $(\mathbb{P}^1)^3$ in a curve of degree $(0,1,1)$			
PICARD RANK:	4	FANO DEGREE:	38	SYMM.:	no
FUTAKI-CHAR.:	$\binom{0}{-13/12}$	DEGENERATIONS:	255836, 428555		
COX RING: $\mathbb{C}[T_1,\dots,T_8]/\langle T_1T_2+T_3T_4+T_5T_6\rangle$		$\deg(\underline{T}) = \begin{pmatrix} 0&1&1&0&1&0&0&0 \\ 1&0&1&0&0&1&0&0 \\ 1&0&0&1&0&1&0&1 \\ 0&0&0&0&0&0&1&1 \end{pmatrix}$			





A CALCULATING THE FUTAKI CHARACTER

The Appendix is devoted to the, somewhat technical, proof of Theorem 3.15. For an invariant ample Cartier divisor D we have the divisorial polytope $\Psi : \square \to \operatorname{Div}_{\mathbb{R}} \mathbb{P}^1$, with $\square \subset M_{\mathbb{R}}$. We cannot directly use Donaldson's result for the toric case, since the dimension of the global sections of $\mathcal{O}(kD)$ does not grow like the number of lattice points of a multiple of a polytope. But as it turns out it grows like the number of "lattice points" of a weighted sum of polytopes. More precisely, we consider the abelian group $\widetilde{\Pi}(V)$ generated by all polytopes in some $\mathbb{R}$-vector space V with the relations $[\Delta]+[\nabla] = [\Delta\cup\nabla]+[\Delta\cap\nabla]$, whenever $\Delta, \nabla, \Delta \cup \nabla$ are polytopes. A subset of V which is covered by polyhedra $\nabla = \bigcup_{i\in I} \Delta_i$ can be naturally identified with an element of this group

$$[\nabla] := \sum_{J\subset I} (-1)^{|J|+1} \cdot \left[\bigcap\nolimits_{j\in J} \Delta_j\right]. \tag{6}$$

In particular this applies for the relative boundary $d\Delta$ of a polytope. We also introduce a boundary operator ∂ for the class of a polytope in $\widetilde{\Pi}$.

$$\partial[\Delta] := \begin{cases} [d\Delta] & ,\operatorname{codim}(\Delta) = 0 \\ 2[\Delta] - [d\Delta] & ,\operatorname{codim}(\Delta) = 1, \\ 0 & ,\ \text{else.} \end{cases} \tag{7}$$

For an object $\boldsymbol{\Delta} = \sum_i a_i[\Delta_i] \subset \widetilde{\Pi}$ simply by linear continuation we have well defined notions of dilation, boundary, volume, numbers of lattice point, and barycenter:

$$
\begin{aligned}
k * \mathbf{\Delta} &:= \sum_i a_i \cdot [k\Delta_i],\\
\partial \mathbf{\Delta} &:= \sum_i a_i \cdot \partial[\Delta_i],\\
\mathrm{vol}(\mathbf{\Delta}) &:= \sum_i a_i \cdot \mathrm{vol}(\Delta_i),\\
N(\mathbf{\Delta}) &:= \sum_i a_i \cdot N(\Delta_i),\\
\mathrm{bc}(\mathbf{\Delta}) &:= \sum_i a_i \cdot \frac{\mathrm{vol}(\Delta_i)}{\mathrm{vol}(\mathbf{\Delta})} \cdot \mathrm{bc}(\Delta_i).
\end{aligned}
$$

Similarly we can extend the facet volume vol_1 and the facet barycenter bc_1 to $\widetilde{\Pi}$, where vol_1 and bc_1 are defined to be zero for polytopes of codimension $\neq 1$. More generally, we may integrate linear forms on V over elements of $\widetilde{\Pi}$.

Proposition A.1. *For an element $\mathbf{\Delta} \subset \widetilde{\Pi}(\mathbb{R}^m)$ we have the following asymptotic formula for the number of lattice points*

$$
N(k * \mathbf{\Delta}) = (\mathrm{vol}\, \mathbf{\Delta}) \cdot k^m + \frac{\mathrm{vol}_1 \partial \Delta}{2} \cdot k^{m-1} + \mathcal{O}(k^{m-2}).
$$

Proof. This is a well known fact for polytopes of maximal dimension (see also [Don02, Proposition 4.1.3] for a proof). The same fact follows for weighted sums of maximal dimensional polytopes just by linearity. Our definition (7) of a boundary $\partial[P]$ makes it work for polytopes of lower dimension, as well. $\square$

We are now going to associate to a divisorial polytope Ψ an element $[\Psi]$ of $\widetilde{\Pi}(M_\mathbb{R} \times \mathbb{R})$. We consider the natural projection $M_\mathbb{R} \times \mathbb{R} \to M_\mathbb{R}$ and inclusion $N_\mathbb{R} \hookrightarrow N_\mathbb{R} \times \mathbb{R}$. First, we need some notation. For a function $f : \mathbb{R}^n \supset U \to \mathbb{R}$ with $f \geq R$ let $\Delta^R(f)$ be the region enclosed by its graph and the R-level:

$$
\Delta^R(f) = \{(u, a) \in M_\mathbb{R} \times \mathbb{R} \mid u \in D, R \leq a \leq f(u)\}.
$$

If f is non-negative we set $\Delta(f) := \Delta^0(f)$. Now, for a concave piecewise affine function f on $\square$ we denote its graph by $\widehat{f}$ and associate an element $[f] \subset \widetilde{\Pi}$ which correspond to the area enclosed by the graph of f and the 0-level. More precisely, we choose an integer $R \leq 0$ such that $f \geq R$ and define

$$
[f] \;=\; [\Delta^R(f)] \;-\; [\square \times [R, 0]] \tag{8}
$$

Note, that $[f]$ does not depend on the particular choice of R. Now, to a divisorial polytope $\Psi : \square \to \mathrm{Div}_\mathbb{R} \mathbb{P}^1$ we associate a class $[\Psi] \in \widetilde{\Pi}(M_\mathbb{R} \times \mathbb{R})$ by

$$
[\Psi] := [\widetilde{\square}] + \sum_P [\Psi_P]. \tag{9}
$$

Here, we set $\widetilde{\square} := \square \times \{0\}$. The following proposition justifies this definition.

Proposition A.2. *For an ample divisor D on a T-variety $X(\mathcal{S})$ of dimension n and the corresponding divisorial polytope $\Psi = \Psi_D$ we have*

i. $(D)^n = \operatorname{vol}\Psi = \operatorname{vol}[\Psi]$,

ii. $\dim H^0(X, \mathcal{O}(D)) = N([\Psi])$.

iii. $\dim H^0(X, \mathcal{O}(D))_u = N_u([\Psi])$.

Here, $N_u([\Psi])$ is the number of lattice point that project to $u \in M_{\mathbb{R}}$ if we consider the projection $M_{\mathbb{R}} \times \mathbb{R} \to M_{\mathbb{R}}$ for every element in the sum (9).

Proof. By Theorem 3.8 we only have to prove $\operatorname{vol}\Psi = \operatorname{vol}[\Psi]$ to obtain (i). The only summand of (9) that contributes to the volume is $[\Psi_P]$, since all other polytopes have lower dimension. Now, by the basic rules of integration we have $\int_\square \Psi_P = \operatorname{vol}[\Psi_P]$. Hence, we obtain

$$\operatorname{vol}[\Psi] = \sum_P \operatorname{vol}[\Psi_P] = \sum_P \int_\square \Psi_P = \int_\square \deg \Psi = \operatorname{vol}\Psi.$$

We prove (iii). Remember, that the global sections of weight $u \in \square \cap M$ are given by $H^0(\mathbb{P}^1, \mathcal{O}(\Psi(u)))$. Hence, we have to check that the number N_u of lattice points of $[\Psi]$, that project to u equals $\dim H^0(\mathbb{P}^1, \mathcal{O}(\Psi(u)))$. Looking at (9) and (8) gives

$$N_u = 1 + \sum_P (\lfloor \Psi_P(u) - R_P \rfloor + 1 - (1 - R_P)) = 1 + \sum_P \lfloor \Psi_P(u) \rfloor = 1 + \deg\lfloor \Psi(u) \rfloor,$$

since the intervals $[R_P, \Psi_P(u)]$ and $[R_P, 0]$ contain $\lfloor \Psi_y(u) - R_P \rfloor + 1$ and $1 - R_P$ integers, respectively. We have $\dim H^0(X, \mathcal{O}(\Psi(u)) = 1 + \deg\lfloor \Psi(u) \rfloor$ and the claim follows. Obviously (ii) follows from (iii). ☐

Theorem A.3. *The Futaki character $F_D : N_{\mathbb{R}} \to \mathbb{R}$ of an divisor D with corresponding divisorial polytope $\Psi = \Psi_D$ is given by*

$$F_D(v) \ = \ \int_{\partial[\Psi]} v \ - \ \frac{\operatorname{vol}_1 \partial[\Psi]}{\operatorname{vol}\Psi} \int_{[\Psi]} v. \tag{10}$$

Proof. We are using exactly the same arguments as in the proof of Theorem 4.2.1 in [Don02]. For an element $v \in N$ we are interested in the total weight of $H^0(X, \mathcal{O}(D))$ with respect to this one-parameter subgroup. By Proposition A.2 it is given by

$$w(H^0(X, \mathcal{O}(D))) = \sum_{u \in \square \cap M} \langle u, v \rangle \cdot N_u([\Psi])$$

Note, that we have $[\Psi_{kD}] = k * [\Psi_D]$, by [IS10, Proposition 3.1]. We are now constructing an element $\boldsymbol{Q}$ of $\widetilde{\Pi}(M_{\mathbb{R}} \times \mathbb{R})$ fulfilling

$$N(k * \boldsymbol{Q}) - N(k * [\Psi]) = \sum_{u \in \square \cap M} \langle u, v \rangle \cdot N_u(k * [\Psi]). \tag{11}$$

For every elementary summand $[\Delta_i]$ of (9) (which is the class of a polytope) we are using the construction from [Don02, Section 4.2], i.e. we fix an integer R, such that $\langle \cdot, v\rangle \leq R$ on Δ_i and define

$$Q_i := \{(u,t) \mid u \in \Delta_i, 0 \leq t \leq R - \langle u, v\rangle\}.$$

Now, by Proposition A.2 and Proposition A.1 we have

$$l_k = N(k * [\Psi]) = (\operatorname{vol}[\Psi]) \cdot k^m + \frac{\operatorname{vol}_1 \partial[\Psi]}{2} \cdot k^{m-1} + \mathcal{O}(k^{m-2}).$$

Moreover, from the proof of [Don02, Theorem 4.2.1] we have

$$N(kQ_i) = k^{m+1} \int_{\Delta_i} (R - v) + \frac{k^m}{2} \int_{\partial \Delta_i} (R - v) + N(k\Delta_i) + \mathcal{O}(k^{m-1}).$$

By linearity and (11) we obtain

$$w_k = k^{m+1} \int_{[\Psi]} (R - v) + \frac{k^m}{2} \int_{\partial[\Psi]} (R - v) + \mathcal{O}(k^{m-1})$$

Now, plugging in the coefficient in Definition 3.13 gives the desired result. □

Corollary A.4.

$$F_D = (\operatorname{vol}_1 \partial[\Psi]) \cdot (\overline{\operatorname{bc}_1}(\partial[\Psi]) - \overline{\operatorname{bc}}([\Psi])),$$

where $\overline{\operatorname{bc}}$ *and* $\overline{\operatorname{bc}_1}$, *respectively denotes the projection of the barycenters in* $M_{\mathbb{R}} \times \mathbb{R}$ *to* $M_{\mathbb{R}}$.

Proof. If we plug into the right hand side of (10) the elements of a lattice basis $e_1, \dots, e_n$ of $N \times \mathbb{Z}$, by the definition of a barycenter we obtain the coordinates of

$$L = (\operatorname{vol}_1 \partial[\Psi])(\operatorname{bc}_1(\partial[\Psi]) - \operatorname{bc}([\Psi]))$$

with respect to the dual basis $e_1^*, \dots, e_n^*$. By definition L is an element of $(N_{\mathbb{R}} \times \mathbb{R})^* = M_{\mathbb{R}} \times \mathbb{R}$ and $F_D \in N_{\mathbb{R}}^*$ is just the restriction $L|_{N_{\mathbb{R}}}$, i.e. the projection of L to $M_{\mathbb{R}}$. □

To simplify our notation we set $\Delta := \Delta(\deg \Psi)$ and denote by Δ^∂ the part of $\Delta \to \Box$ that projects to $\partial\Box$.

Lemma A.5. *For the volume and barycenter of* $[\Psi]$ *we get*

$$\overline{\operatorname{bc}}([\Psi]) = \overline{\operatorname{bc}}(\Delta)$$
$$\operatorname{vol}([\Psi]) = \operatorname{vol}(\Delta).$$

Proof. In the proof of Proposition A.2 we have seen already that $\mathrm{vol}([\Psi]) = \int_\square \deg \Psi$ and hence $\mathrm{vol}([\Psi]) = \mathrm{vol}(\Delta)$. For the projected barycenter we use the same argument. The only summands that contribute to $\overline{\mathrm{bc}}([\Psi])$ are $[\Psi_P]$. We may interpret $\overline{\mathrm{bc}}([\Psi_P])$ as a linear form on $v \in N_\mathbb{R}$ and by definition of the barycenter and basics of integration we have

$$\begin{aligned}
\mathrm{vol}([\Psi_P]) \cdot \overline{\mathrm{bc}}([\Psi_P])(v) &= \int_{\Delta^{R_P}(\Psi_P)} v - \int_{\square\times[R_P,0]} v \\
&= \int_\square (\Psi_P - R_P)\cdot v + R_P \cdot \int_\square v \\
&= \int_\square \Psi_P \cdot v.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\mathrm{vol}([\Psi]) \cdot \overline{\mathrm{bc}}[\Psi] &= \sum_P \mathrm{vol}([\Psi_P]) \cdot \overline{\mathrm{bc}}[\Psi_P] \\
&= \sum_P \int_\square \Psi_P \cdot v \\
&= \int_\square \deg \Psi \cdot v \\
&= \mathrm{vol}(\Delta)\overline{\mathrm{bc}}(\deg \Psi) \\
&= \mathrm{vol}([\Psi])\overline{\mathrm{bc}}(\deg \Psi).
\end{aligned}$$

□

LEMMA A.6. *If we assume that Ψ_P is constant for $P \notin \{P_1, \dots, P_r\}$, then we can calculate volume and barycenter of $\partial[\Psi]$ as follows.*

$$\begin{aligned}
\overline{\mathrm{bc}_1}(\partial[\Psi]) &= \overline{\mathrm{bc}_1}\left([\Delta^\partial] + (2-r)[\widetilde{\square}] + \sum_{i=1}^r [\widehat{\Psi}_{P_i}]\right), \\
\mathrm{vol}_1(\partial[\Psi]) &= \mathrm{vol}_1\left([\Delta^\partial] + (2-r)[\widetilde{\square}] + \sum_{i=1}^r [\widehat{\Psi}_{P_i}]\right),
\end{aligned}$$

where $\widehat{\Psi}_{P_i}$ denotes the graph of Ψ_{P_i}.

Proof. When calculating vol_1 and bc_1 we have to consider only summands of $\partial[\Psi]$ of codimension one. For every P we have

$$\partial[\Psi_P] = [\widehat{\Psi}_P] - [\square \times R_P] + [\Delta(\Psi_P|_{\partial\square})] - [\partial\square \times [R_P, 0]] + \dots,$$

where ... consists of lower dimensional summands. Now, we have $\overline{\mathrm{bc}_1}[\square \times R_P] = \overline{\mathrm{bc}_1}[\square]$ and $\mathrm{vol}_1[\square \times R_P] = \mathrm{vol}_1[\square]$. Applying Lemma A.5 to every facet of $\square$

gives

$$\overline{\mathrm{bc}_1}\left(\sum_P [\Delta^{R_P}(\Psi_P|_{\partial\Box})] - [\partial\Box \times [R_P,0]]\right) = \overline{\mathrm{bc}_1}[\Delta^\partial],$$
$$\mathrm{vol}_1\left(\sum_P [\Delta^{R_P}(\Psi_P|_{\partial\Box})] - [\partial\Box \times [R_P,0]]\right) = \mathrm{vol}_1[\Delta^\partial].$$

Hence,

$$\overline{\mathrm{bc}_1}\partial \sum_{i=1}^r [\Psi_{P_i}] = \overline{\mathrm{bc}_1}\left([\Delta^\partial] - r\cdot[\Box] + \sum_{i=1}^r [\widehat{\Psi}_{P_i}]\right)$$
$$\mathrm{vol}_1\,\partial \sum_{i=1}^r [\Psi_{P_i}] = \mathrm{vol}_1\left([\Delta^\partial] - r\cdot[\Box] + \sum_{i=1}^r [\widehat{\Psi}_{P_i}]\right)$$

Looking at (9) and taking into account our definition of the boundary for lower dimensional polytopes we see, that we still have to add $\overline{\mathrm{bc}_1}(\partial\widetilde{\Box}) = 2\overline{\mathrm{bc}_1}(\widetilde{\Box})$ or $\mathrm{vol}_1(\partial\widetilde{\Box}) = 2\,\mathrm{vol}_1(\widetilde{\Box})$, respectively which gives the desired result. $\Box$

For a polytope $\nabla \subset M_{\mathbb{R}} \times \mathbb{R}$ we define ∇^+ to be the part of ∇ lying above the 0-level with respect to the last component. Similarly we define ∇^- to be the part below the 0-level. We define the *pyramid* $\mathrm{pyr}(\nabla)$ as the class $\mathrm{pyr}(\nabla) := [\mathrm{conv}(\nabla^+, 0)] - [\mathrm{conv}(\nabla^-, 0)]$ in $\widetilde{\Pi}$. The pyramid operator extends by linearity to $\widetilde{\Pi}$.

Now we have the following equivalent to Lemma A.6 for the full-dimensional volume and barycenter.

Lemma A.7. *If we assume that $\Psi_P \equiv 0$ for $P \notin \{P_1, \dots, P_\ell\}$, then we can calculate volume and barycenter of $[\Psi]$ as follows.*

$$\overline{\mathrm{bc}}[\Psi] \;=\; \overline{\mathrm{bc}}\left(\mathrm{pyr}\left([\Delta^\partial] \;+\; \sum_{i=1}^\ell [\widehat{\Psi}_{P_i}]\right)\right),$$
$$\mathrm{vol}[\Psi] \;=\; \mathrm{vol}\left(\mathrm{pyr}\left([\Delta^\partial] \;+\; \sum_{i=1}^\ell [\widehat{\Psi}_{P_i}]\right)\right),$$

Proof. For a polytope $\nabla \subset M_{\mathbb{R}} \times \mathbb{R}$ and a point $u \in M_{\mathbb{R}}$ we denote by ∇_u the length of its fiber over $u \in M_{\mathbb{R}}$. Again we may extend this notion linearly to $\widetilde{\Pi}$. For proving the claim, by Lemma A.5 it's sufficient to show that

$$\deg \Psi(u) = \mathrm{pyr}(\Delta^\partial)_u + \sum_{i=1}^r \mathrm{pyr}(\widehat{\Psi}_{P_i})_u.$$

For a point $u = w \in \partial\Box$ this equality holds, since we have $\mathrm{pyr}(\widehat{\Psi}_{P_i})_w = 0$ and $\mathrm{pyr}(\Delta^\partial)_w = \deg\Psi(w)$ by definition. Moreover, for the point $u = 0$ the equality is fulfilled, as well, since, $\mathrm{pyr}(\Delta^\partial)_0 = 0$ and $\mathrm{pyr}(\widehat{\Psi}_{P_i})_0 = \Psi_{P_i}$ holds. Now, the equality follows by linearity for all points on the line segment connecting w and 0, hence, for all points in $\Box$. $\Box$

Proof of Theorem 3.15. Consider a Fano T-variety given by an f-divisor $\mathcal{S}$ with nontrivial slices $\mathcal{S}_{P_1}, \ldots, \mathcal{S}_{P_r}$. First we choose a special representation for the canonical divisor $K_{\mathbb{P}^1} = \sum_{j=1}^{r-2} P_{r+j} - \sum_{i=1}^{r} P_i$. Where $P_1, \ldots, P_{2r-2}$ are distinct points on $\mathbb{P}^1$. Now, by Remark 3.6 the corresponding divisorial polytope fulfills

i. $\Psi_P \equiv 0$ for $P \neq P_i$, $i = 1, \ldots 2r-2$,

ii. $\Psi_P \equiv -1$ for $P = P_i$, $i = r+1, \ldots 2r-2$,

iii. The facets of the graph of Ψ_P have lattice distance 1 to the origin for $P \neq P_i, i = 1, \ldots 2r-2$,

iv. the facets F of $\Box$ with $\deg\Psi|_F \not\equiv 0$ have lattice distance 1 from the origin.

By Corollary A.4 we have

$$F(X) = \mathrm{vol}_1([\Psi])(\overline{\mathrm{bc}_1}\partial[\Psi] - \overline{\mathrm{bc}}[\Psi])$$

We set

$$C := [\Delta^\partial] + (2-r)[\Box \times \{-1\}] + \sum_{i=1}^{\ell}[\widehat{\Psi}_{P_i}].$$

Now, by Lemma A.6 we have $\mathrm{vol}_1([\Psi]) = \mathrm{vol}_1(C)$ and $\overline{\mathrm{bc}_1}([\Psi]) = \overline{\mathrm{bc}_1}(C)$. Moreover, by Lemma A.7 we obtain $\mathrm{vol}([\Psi]) = \mathrm{vol}(\mathrm{pyr}(C))$ and $\overline{\mathrm{bc}}([\Psi]) = \overline{\mathrm{bc}}(\mathrm{pyr}(C))$. Since, C is a sum over polytopes in lattice distance 1 from the origin we may apply Lemma A.8 and obtain

$$\begin{aligned} F(X) &= \mathrm{vol}_1(\partial[\Psi])(\overline{\mathrm{bc}_1}(\partial[\Psi]) - \overline{\mathrm{bc}}[\Psi]) \\ &= \mathrm{vol}_1(C) \cdot \big(\overline{\mathrm{bc}_1}(C) - \overline{\mathrm{bc}}(\mathrm{pyr}(C))\big) \\ &= \mathrm{vol}(\mathrm{pyr}(C)) \cdot \overline{\mathrm{bc}}(\mathrm{pyr}(C)) \\ &= \mathrm{vol}[\Psi] \cdot \overline{\mathrm{bc}}([\Psi]) \\ &= \mathrm{vol}\,\Delta \cdot \overline{\mathrm{bc}}(\Delta). \end{aligned}$$

$\Box$

Lemma A.8. *Assume that $\Delta_1, \ldots, \Delta_r \subset M_{\mathbb{R}} \times \mathbb{R}$ are polytopes of codimension 1 with lattice distance 1 from the origin. Then for $C = \sum a_i \cdot [\Delta_i]$ we have*

$$\mathrm{vol}_1(C) \cdot (\overline{\mathrm{bc}_1}(C) - \overline{\mathrm{bc}}(\mathrm{pyr}(C))) = \mathrm{vol}(\mathrm{pyr}(C)) \cdot \overline{\mathrm{bc}}(\mathrm{pyr}(C)).$$

Proof. We set $n = \dim M_{\mathbb{R}} + 1$. Now, by elementary geometry we have

$$\mathrm{vol}_1 \Delta_i = n \cdot \mathrm{vol}(\mathrm{pyr}(\Delta_i)), \qquad \overline{\mathrm{bc}_1}(\Delta_i) = \frac{n}{n-1} \cdot \overline{\mathrm{bc}}(\mathrm{pyr}(\Delta_i)).$$

It follows that $\mathrm{vol}_1 C = n \cdot \mathrm{vol}\,\mathrm{pyr}(C)$. Hence, we get

$$\begin{aligned}\overline{\mathrm{bc}_1}(C) = \sum a_i \frac{\mathrm{vol}_1(\Delta_i)}{\mathrm{vol}_1(C)} \overline{\mathrm{bc}_1}(\Delta_i) &= \sum a_i \frac{\mathrm{vol}(\mathrm{pyr}(\Delta_i))}{\mathrm{vol}(\mathrm{pyr}(C))} \cdot \frac{n}{n-1} \cdot \overline{\mathrm{bc}}(\mathrm{pyr}(\Delta_i)) \\ &= \frac{n}{n-1} \cdot \overline{\mathrm{bc}}(\mathrm{pyr}(C)).\end{aligned}$$

Now, plugging in $\frac{n}{n-1} \cdot \overline{\mathrm{bc}}(\mathrm{pyr}(C))$ for $\overline{\mathrm{bc}_1}(C)$ into $\mathrm{vol}_1(C) \cdot (\overline{\mathrm{bc}_1}(C) - \overline{\mathrm{bc}}(\mathrm{pyr}(C)))$ gives the desired result. ☐

References

[AH06] Klaus Altmann and Jürgen Hausen. Polyhedral divisors and algebraic torus actions. *Math. Ann.*, 334(3):557–607, 2006.

[AHHL12] I. Arzhantsev, J. Hausen, E. Herppich, and A. Liendo. The automorphism group of a variety with torus action of complexity one. *Arxiv preprint arXiv:1202.4568*, 2012.

[AHS08] Klaus Altmann, Jürgen Hausen, and Hendrik Süß. Gluing affine torus actions via divisorial fans. *Transformation Groups*, 13(2):215–242, 2008.

[AIP+11] Klaus Altmann, Nathan O. Ilten, Lars Petersen, Hendrik Süß, and Robert Vollmert. The Geometry of T-Varieties. In Piotr Pragacz, editor, *Contributions to Algebraic Geometry*, Impanga Lect Notes, 2011, arXiv:1102.5760.

[BDK+13] G. Brown, S. Davis, A. Kasprzyk, M. Kerber, O. Sisask, and S. Tawn. The graded ring database webpage. http://grdb.lboro.ac.uk, 20042013.

[BS99] Victor V. Batyrev and Elena Selivanova. Einstein-Kähler metrics on symmetric toric Fano manifolds. *J. Reine Angew. Math.*, 512:225–236, 1999.

[CDS14] Xiuxiong Chen, Simon Donaldson, and Song Sun. Khlereinstein metrics and stability. *International Mathematics Research Notices*, 2014(8):2119–2125, 2014, http://imrn.oxfordjournals.org/content/2014/8/2119.full.pdf+html.

[DH82] J.J. Duistermaat and G.J. Heckman. On the variation in the cohomology of the symplectic form of the reduced phase space. *Invent. Math.*, 69:259–268, 1982.

[Don02] Simon K. Donaldson. Scalar curvature and stability of toric varieties. *J. Differ. Geom.*, 62(2):289–349, 2002.

[Fut83] A. Futaki. An obstruction to the existence of Einstein Kähler metrics. *Invent. Math.*, 73:437–443, 1983.

[HS10] Jürgen Hausen and Hendrik Süß. The Cox ring of an algebraic variety with torus action. *Advances in Mathematics*, 225(2):977 – 1012, 2010.

[IS10] Nathan O. Ilten and Hendrik Süß. Polarized T-varieties of complexity one. *Michigan Mathmatical Journal*, 2010. to be published.

[IV12] Nathan Owen Ilten and Robert Vollmert. Deformations of rational T-varieties. *J. Algebr. Geom.*, 21(3):531–562, 2012.

[Kas10] Alexander Mieczyslaw Kasprzyk. Canonical toric fano threefolds. *Canadian Journal of Mathematics*, 62(6):1293–1309, 2010.

[KKMSD73] George Kempf, Finn Knudsen, David Mumford, and Bernard Saint-Donat. *Toroidal embeddings I.* Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin, 1973.

[KT03] Yael Karshon and Susan Tolman. Complete invariants for Hamiltonian torus actions with two dimensional quotients. *J. Symplectic Geom.*, 2(1):25–82, 2003.

[KT14] Yael Karshon and Susan Tolman. Classification of Hamiltonian torus actions with two-dimensional quotients. *Geom. Topol.*, 18(2):669–716, 2014.

[LS10] Alvaro Liendo and Hendrik Süß. Normal singularities with torus actions. *Tohoku Mathematical Journal*, May 2010. to be published.

[Mab87] Toshiki Mabuchi. Einstein-Kähler forms, Futaki invariants and convex geometry on toric Fano varieties. *Osaka J. Math.*, 24:705–737, 1987.

[MM86] Shigefumi Mori and Shigeru Mukai. Classification of Fano 3-folds with $B_2 \geq 2$. I. Nagata, M. (ed.) et al., Algebraic and topological theories. Papers from the symposium dedicated to the memory of Dr. Takehiko Miyata held in Kinosaki, October 30-November 9, 1984. Tokyo: Kinokuniya Company Ltd. 496-545 (1986)., 1986.

[PS11] Lars Petersen and Hendrik Süß. Torus invariant divisors. *Israel Journal of Mathematics*, 182:481–505, 2011.

[Süß13] Hendrik Süß. Kähler-Einstein metrics on symmetric Fano T-varieties. *Advances in Mathematics*, 246:100 – 113, 2013.

[Tia87] Gang Tian. On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$. *Invent. Math.*, 89:225–246, 1987.

[Tia12] G. Tian. K-stability and Kähler-Einstein metrics. *ArXiv e-prints*, November 2012, 1211.4669.

[Tim08] Dmitri Timashev. Torus actions of complexity one. Harada, Megumi (ed.) et al., Toric topology. International conference, Osaka, Japan, May 28–June 3, 2006. Providence, RI: American Mathematical Society (AMS). Contemporary Mathematics 460, 349-364 (2008)., 2008.

[Tit66] Jacques Tits. Normalisateurs de tores. I: Groupes de Coxeter etendus. *J. Algebra*, 4:96–116, 1966.

[WZ04] Xu-Jia Wang and Xiaohua Zhu. Kähler–Ricci solitons on toric manifolds with positive first Chern class. *Adv. Math.*, 188(1):87–103, 2004.

Hendrik Suess
School of Mathematics
Maxwell Institute
University of Edinburgh,
J.C. Maxwell Building
The King's Buildings
Mayfield Road
Edinburgh EH9 3JZ
United Kingdom