Twisted C^*-Algebras Associated to Finitely Aligned Higher-Rank Graphs

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Abstract. We introduce twisted relative Cuntz-Krieger algebras associated to finitely aligned higher-rank graphs and give a comprehensive treatment of their fundamental structural properties. We establish versions of the usual uniqueness theorems and the classification of gauge-invariant ideals. We show that all twisted relative Cuntz-Krieger algebras associated to finitely aligned higher-rank graphs are nuclear and satisfy the UCT, and that for twists that lift to real-valued cocycles, the K-theory of a twisted relative Cuntz-Krieger algebra is independent of the twist. In the final section, we identify a sufficient condition for simplicity of twisted Cuntz-Krieger algebras associated to higher-rank graphs which are not aperiodic. Our results indicate that this question is significantly more complicated than in the untwisted setting.

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1 Introduction

In [12], Kumjian and Pask introduced higher-rank graphs (or k-graphs) and their C^*-algebras. They considered only higher-rank graphs which are *row-finite* and have *no sources*, the same simplifying assumptions that were made in the first papers on graph C^*-algebras [14, 13, 1]. The theory was later expanded [23] to include the more general finitely aligned k-graphs; in this case the C^*-algebraic relations that describe the Cuntz-Krieger algebra were determined by first analysing an associated Toeplitz algebra [22]. Interpolating between the Toeplitz algebra and the Cuntz-Krieger algebra are the relative Cuntz-Krieger algebras, which were introduced in [26], and then used in [25] to determine the gauge-invariant-ideal structure of the Cuntz-Krieger algebras of finitely aligned k-graphs. Simplicity of the Cuntz-Krieger algebra of a finitely aligned higher-rank graph was completely characterised in [18].

In [16], Kumjian, Pask and Sims studied the structure theory of the twisted Cuntz-Krieger algebra $C^*(\Lambda, c)$ associated to a row-finite higher-rank graph Λ with no sources and a $\mathbb{T}$-valued categorical 2-cocycle c on Λ. They subsequently proved [17] that for cocycles c that lift to $\mathbb{R}$-valued cocycles, the K-theory of $C^*(\Lambda, c)$ is the same as that of $C^*(\Lambda)$. In this paper, we extend these results to finitely aligned k-graphs, and identify the gauge-invariant ideal structure of twisted relative Cuntz-Krieger algebras of higher-rank graphs (the Cuntz-Krieger algebra and the Toeplitz algebra are special cases). We also establish a sufficient condition for simplicity of $C^*(\Lambda, c)$ when Λ is row-finite with no sources and c is induced by the degree map from a cocycle on $\mathbb{Z}^k$. The sufficient condition for simplicity is new and requires an intricate analysis of the subalgebra generated by spanning elements in $C^*(\Lambda, c)$ whose initial and final projections coincide, and relies on a local decomposition of this subalgebra as a direct sum of noncommutative tori.

We have organised this paper as follows. Section 2 introduces the necessary background about k-graphs and their cohomology. In Section 3, we introduce the twisted Toeplitz algebra, the twisted relative Cuntz-Krieger algebras and the twisted Cuntz-Krieger algebra of a finitely aligned k-graph with respect to a $\mathbb{T}$-valued cocycle c. Following the program of [22, 25] we prove a version of Coburn's theorem for the twisted Toeplitz algebra, and versions of an Huef and Raeburn's gauge-invariant uniqueness theorem and the Cuntz-Krieger uniqueness theorem for the twisted relative Cuntz-Krieger algebras. We give a sufficient condition for the twisted Cuntz-Krieger algebra to be simple and purely infinite. In Section 4, we adapt the analysis of [25] to give a complete listing of the gauge-invariant ideals in a twisted relative Cuntz-Krieger algebra. This is new even in the untwisted setting, since the results of [25] only apply to the Cuntz-Krieger algebra. In Section 5, we modify arguments of [25, §8] to show that every twisted relative Cuntz-Krieger algebra is Morita equivalent to a crossed product of an AF algebra by $\mathbb{Z}^k$, and is therefore nuclear and satisfies the UCT. In Section 6, we combine ideas from [23] and [17] to show that if the twisting cocycle arises by exponentiation of a real-valued 2-cocycle, then the K-groups of the twisted relative Cuntz-Krieger algebra are isomorphic to those of the corresponding untwisted algebra.

Since our proofs of the results discussed in the preceding paragraph follow the lines of proofs of earlier results, our treatment is mostly quite brief, relying heavily on reference to the existing arguments, and we provide additional detail only where a nontrivial change is needed. One (perhaps surprising) example of the latter is the matter of ascertaining which diagonal projections in a twisted relative Cuntz-Krieger algebra are nonzero; it turns out that neither the arguments used for the row-finite case in [16] nor those used for the untwisted case in [26] can easily be adapted to our setting, so we use a different approach employing filters (see [9] and [3]) in a k-graph Λ and the path-space representation. This approach simplifies and streamlines even the untwisted setting [26], and substantially improves upon the argument used for twisted C^*-algebras of row-finite k-graphs with no sources in [16].

In the final section, we consider simplicity of twisted k-graph C^*-algebras. For untwisted C^*-algebras it is proved in [18] that the Cuntz-Krieger algebra of a finitely aligned k-graph Λ is simple if and only if Λ is cofinal and aperiodic (other relative Cuntz-Krieger algebras are never simple). In the twisted situation, the "if" implication in this result follows from more or less the same argument (see Corollary 3.18); and necessity of cofinality also persists, although the argument of [18] requires some modification. However, for twisted C^*-algebras, aperiodicity of Λ is not necessary for simplicity of $C^*(\Lambda, c)$: when $\mathbb{N}^2$ is regarded as a 2-graph it is certainly not aperiodic, but its twisted C^*-algebras are the rotation algebras (see [15, Example 7.7]), whose simplicity or otherwise depends on the twisting cocycle [27]. Recent work on primitive ideals in k-graph C^*-algebras [4] shows that if Λ is row-finite with no sources and is cofinal, then the primitive ideals of $C^*(\Lambda)$ are indexed by characters of a subgroup $\mathrm{Per}(\Lambda)$ of $\mathbb{Z}^k$. We consider cofinal row-finite k-graphs with no sources, and cocycles which are pulled back along the degree map from cocycles c of $\mathbb{Z}^k$. We show that if the associated skew-symmetric bicharacter cc^* restricts to a nondegenerate bicharacter of $\mathrm{Per}(\Lambda)$ (see [19, 20, 27]) — the condition which characterises simplicity of the noncommutative torus $C^*(\mathrm{Per}(\Lambda), c)$ — then the associated twisted k-graph C^*-algebra is simple.

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2 Background

Throughout this paper, $\mathbb{N}$ denotes the natural numbers including 0, and $\mathbb{N}^k$ is the monoid of k-tuples of natural numbers under coordinatewise addition. We denote the generators of $\mathbb{N}^k$ by $e_1, \dots, e_k$ and we write n_i for the i^{th} coordinate of $n \in \mathbb{N}^k$, so that $n = \sum_i n_i \cdot e_i$. For $m, n \in \mathbb{N}^k$, we write $m \vee n$ and $m \wedge n$ for their coordinatewise maximum and minimum. We regard $\mathbb{N}^k$ as a partially ordered set with $m \leq n$ if and only if $m_i \leq n_i$ for all i. When convenient we will regard $\mathbb{N}^k$ as a category with one object and composition given by addition.

A *k-graph* is a countable category Λ endowed with a functor $d : \Lambda \to \mathbb{N}^k$ that satisfies the following factorisation property: whenever $d(\lambda) = m + n$ there are unique $\mu \in d^{-1}(m)$ and $\nu \in d^{-1}(n)$ such that $\lambda = \mu\nu$. We write Λ^n for $d^{-1}(n)$. Since the identity morphisms of Λ are idempotents and d is a functor, the identity morphisms belong to Λ^0. Hence the codomain and domain maps in the category Λ determine maps $r, s : \Lambda \to \Lambda^0$. We then have $r(\lambda)\lambda = \lambda = \lambda s(\lambda)$ for all λ, and the factorisation property implies that $\Lambda^0 = \{\mathrm{id}_o : o \in \mathrm{Obj}(\Lambda)\}$.

Since we are thinking of Λ as a kind of graph, we call its morphisms *paths*; the paths Λ^0 of degree 0 are called *vertices*.
Given $\lambda \in \Lambda$ and $E \subseteq \Lambda$ we write $\lambda E := \{\lambda\mu : \mu \in E, r(\mu) = s(\lambda)\}$, and $E\lambda$ is defined similarly. In particular, if $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ then $v\Lambda^n = \{\lambda \in \Lambda^n : r(\lambda) = v\}$.
For $\mu, \nu \in \Lambda$, we define

$$\begin{aligned}\mathrm{MCE}(\mu,\nu) &:= \{\lambda \in \Lambda^{d(\mu)\vee d(\nu)} : \lambda = \mu\alpha = \nu\beta \text{ for some } \alpha,\beta\}\\ &= \mu\Lambda \cap \nu\Lambda \cap \Lambda^{d(\mu)\vee d(\nu)}.\end{aligned}$$

We say that Λ is *finitely aligned* if $\mathrm{MCE}(\mu,\nu)$ is always finite (possibly empty). Let $v \in \Lambda^0$ and $E \subseteq v\Lambda$. We say that E is *exhaustive* if for every $\lambda \in v\Lambda$ there exists $\mu \in E$ such that $\mathrm{MCE}(\lambda,\mu) \neq \emptyset$. Equivalently, E is exhaustive if $E\Lambda \cap \lambda\Lambda \neq \emptyset$ for all $\lambda \in v\Lambda$. Following [26], we write $\mathrm{FE}(\Lambda)$ for the collection of all finite exhaustive sets E in Λ such that $E \cap \Lambda^0 = \emptyset$. Each $E \in \mathrm{FE}(\Lambda)$ is a subset of $v\Lambda$ for some v. Given any $v \in \Lambda^0$ and $E \subseteq v\Lambda$, we write $r(E) := v$. If $E \in \mathrm{FE}(\Lambda)$ with $r(E) = v$, we say $E \in v\,\mathrm{FE}(\Lambda)$. If $\mathcal{E} \subseteq \mathrm{FE}(\Lambda)$ and $v \in \Lambda^0$, then $v\mathcal{E} = \mathcal{E} \cap v\,\mathrm{FE}(\Lambda)$.
Following [16], if A is an abelian group, we say that $c : \{(\mu,\nu) \in \Lambda \times \Lambda : s(\mu) = r(\nu)\} \to A$ is a *normalised cocycle* if $c(\lambda,\mu) + c(\lambda\mu,\nu) = c(\mu,\nu) + c(\lambda,\mu\nu)$ for all composable λ,μ,ν, and $c(r(\lambda),\lambda) = 0 = c(\lambda, s(\lambda))$ for all λ. (In the special case $A = \mathbb{T}$, we will write the group operation multiplicatively and the identity element as 1, and write $\overline{c}$ for the inverse $\overline{c}(\lambda,\mu) = \overline{c(\lambda,\mu)}$ of $c \in \underline{Z}^2(\Lambda,\mathbb{T})$.) We write $\underline{Z}^2(\Lambda, A)$ for the collection of all normalised cocycles, which forms a group under pointwise addition in A. In the cohomology introduced in [16], the coboundary map $\underline{\delta}^1$ carries a function $b : \Lambda \to A$ to the map $\underline{\delta}^1 b \in \underline{Z}^2(\Lambda, A)$ given by $(\underline{\delta}^1 b)(\mu,\nu) = b(\mu) + b(\nu) - b(\mu\nu)$. The elements of the range of $\underline{\delta}^1$ are called *coboundaries*. Cocycles $c, c' \in \underline{Z}^2(\Lambda, A)$ are *cohomologous* if $c' - c$ is a coboundary.

3 The core and the uniqueness theorems

In this section we prove uniqueness theorems for twisted relative k-graph C^*-algebras: a version of Coburn's theorem for the twisted Toeplitz algebra, and versions of the gauge-invariant uniqueness theorem and Cuntz-Krieger uniqueness theorem for twisted relative Cuntz-Krieger algebras. We finish with a sufficient condition for the twisted Cuntz-Krieger algebra of a k-graph to be simple and purely infinite.
The following definition parallels [22, Definition 7.1].

Definition 3.1. Let Λ be a finitely aligned k-graph and let $c \in \underline{Z}^2(\Lambda,\mathbb{T})$. A *Toeplitz-Cuntz-Krieger* (Λ, c)*-family* in a C^*-algebra A is a collection $t = \{t_\lambda : \lambda \in \Lambda\} \subseteq A$ satisfying

(TCK1) $\{t_v : v \in \Lambda^0\}$ is a set of mutually orthogonal projections;

(TCK2) $t_\mu t_\nu = c(\mu,\nu)t_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;

(TCK3) $t_\lambda^* t_\lambda = t_{s(\lambda)}$ for all $\lambda \in \Lambda$; and

(TCK4) $t_\mu t_\mu^* t_\nu t_\nu^* = \sum_{\lambda \in \mathrm{MCE}(\mu,\nu)} t_\lambda t_\lambda^*$ for all $\mu, \nu \in \Lambda$, where empty sums are interpreted as zero.

We write $C^*(t)$ for $C^*(\{t_\lambda : \lambda \in \Lambda\}) \subseteq A$.

By [21, Proposition A.4], relation (TCK3) ensures that the t_λ are partial isometries, so that $t_\lambda t_\lambda^* t_\lambda = t_\lambda$ for all λ.

Lemma 3.2. *Let Λ be a finitely aligned k-graph, $c \in \underline{Z}^2(\Lambda, \mathbb{T})$ and t a Toeplitz-Cuntz-Krieger (Λ, c)-family. The projections $\{t_\mu t_\mu^* : \mu \in \Lambda\}$ pairwise commute, and $\{t_\mu t_\mu^* : \mu \in \Lambda^n\}$ is a collection of mutually orthogonal projections for each $n \in \mathbb{N}^k$. For $\mu, \nu, \eta, \zeta \in \Lambda$ we have*

$$t_\nu^* t_\eta = \sum_{\nu\alpha = \eta\beta \in \mathrm{MCE}(\nu,\eta)} \overline{c(\nu,\alpha)} c(\eta,\beta) t_\alpha t_\beta^* \quad \text{and}$$
$$t_\mu t_\nu^* t_\eta t_\zeta^* = \sum_{\nu\alpha = \eta\beta \in \mathrm{MCE}(\nu,\eta)} c(\mu,\alpha)\overline{c(\nu,\alpha)} c(\eta,\beta)\overline{c(\zeta,\beta)} t_{\mu\alpha} t_{\zeta\beta}^*.$$

In particular $C^*(t) = \overline{\operatorname{span}}\{t_\mu t_\nu^* : s(\mu) = s(\nu)\}$.

Proof. The first assertion follows from (TCK4) as $\mathrm{MCE}(\mu,\nu) = \mathrm{MCE}(\nu,\mu)$. If $d(\mu) = d(\nu)$ and $\mu \neq \nu$, then the factorisation property ensures that $\mu\Lambda \cap \nu\Lambda = \emptyset$, and in particular $\mathrm{MCE}(\mu,\nu) = \emptyset$. Hence $t_\mu t_\mu^* t_\nu t_\nu^* = 0$ by (TCK4); so the $t_\mu t_\mu^*$ where $\mu \in \Lambda^m$ are mutually orthogonal.
For the first displayed equation, we use (TCK4), (TCK2), and then (TCK2) to calculate:

$$\begin{aligned} t_\nu^* t_\eta = t_\nu^*(t_\nu t_\nu^* t_\eta t_\eta^*) t_\eta &= \sum_{\nu\alpha = \eta\beta \in \mathrm{MCE}(\nu,\eta)} t_\nu^* t_{\nu\alpha} t_{\eta\beta}^* t_\eta \\ &= \sum_{\nu\alpha = \eta\beta \in \mathrm{MCE}(\nu,\eta)} \overline{c(\nu,\alpha)} c(\eta,\beta) t_\nu^* t_\nu t_\alpha t_\beta^* t_\eta^* t_\eta \\ &= \sum_{\nu\alpha = \eta\zeta \in \mathrm{MCE}(\nu,\eta)} \overline{c(\nu,\alpha)} c(\eta,\beta) t_\alpha t_\beta^*. \end{aligned}$$

Multiplying this expression on the left by t_μ and on the right by t_ζ and then applying (TCK2) yields the second displayed equation. It follows that $\overline{\operatorname{span}}\{t_\mu t_\nu^* : \mu, \nu \in \Lambda\}$ is closed under multiplication and hence equal to $C^*(t)$, so the final assertion follows from the observation that if $s(\mu) \neq s(\nu)$, then $t_\mu t_\nu^* = t_\mu t_{s(\mu)} t_{s(\nu)} t_\nu^* = 0$ by (TCK3) and (TCK1). $\square$

As a consequence of Lemma 3.2, products of the form $\prod_{\lambda \in E}(t_v - t_\lambda t_\lambda^*)$ for finite subsets E of Λ are well-defined, and so we can make the following definition.

Definition 3.3. For $\mathcal{E} \subseteq \mathrm{FE}(\Lambda)$, we say that a Toeplitz Cuntz-Krieger (Λ, c)-family t is a *relative Cuntz-Krieger* $(\Lambda, c; \mathcal{E})$*-family* if

(CK) for every $v \in E^0$ and $E \in v\mathcal{E}$, we have $\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) = 0$.

A relative Cuntz-Krieger $(\Lambda, c; \mathrm{FE}(\Lambda))$-family is called a *Cuntz-Krieger* (Λ, c)*-family.*

Remark 3.4. If $v \in \Lambda^0$ and $r(\lambda) = v$ then we have $t_v t_\lambda t_\lambda^* = c(v, \lambda) t_{v\lambda} t_\lambda^* = t_\lambda t_\lambda^*$. Lemma 3.2 implies in particular that $\{t_\lambda t_\lambda^* : \lambda \in v\Lambda^n\}$ is a set of mutually orthogonal projections, so $\sum_{\lambda \in F} t_\lambda t_\lambda^*$ is a projection for every finite $F \subseteq v\Lambda^n$. Since $t_v\Big(\sum_{\lambda \in F} t_\lambda t_\lambda^*\Big) = \sum_{\lambda \in F} t_\lambda t_\lambda^*$, we conclude that $t_v \geq \sum_{\lambda \in F} t_\lambda t_\lambda^*$. So relations (TCK1)–(TCK4) (for $c \equiv 1$) imply relation (4) of [22, Definition 7.1], and in particular a Toeplitz-Cuntz-Krieger $(\Lambda, 1)$-family as defined here is the same thing as a Toeplitz-Cuntz-Krieger Λ-family in the sense of [22]. Lemma 3.2 also shows that a relative Cuntz-Krieger $(\Lambda, 1; \mathcal{E})$-family is the same thing as a relative Cuntz-Krieger Λ-family in the sense of [26]

The t_λ in any Toeplitz-Cuntz-Krieger (Λ, c)-family are partial isometries, and hence have norm 0 or 1, and the same is then true for the $t_\mu t_\nu^*$. It is straightforward (following the strategy of, for example, [21, Propositions 1.20 and 1.21]) to show that there is a C^*-algebra $\mathcal{T}C^*(\Lambda, c)$ generated by a Toeplitz-Cuntz-Krieger family $s^c_{\mathcal{T}} := \{s^c_{\mathcal{T}}(\lambda) : \lambda \in \Lambda\}$ which is universal in the sense that every Toeplitz-Cuntz-Krieger (Λ, c)-family t induces a homomorphism $\pi_t : \mathcal{T}C^*(\Lambda, c) \to C^*(t)$ satisfying $\pi_t(s^c_{\mathcal{T}}(\lambda)) = t_\lambda$ for all λ.
Given a set $\mathcal{E} \subseteq \mathrm{FE}(\Lambda)$, let $J_{\mathcal{E}} \subseteq \mathcal{T}C^*(\Lambda, c)$ be the ideal generated by $\big\{\prod_{\lambda \in E}(s^c_{\mathcal{T}}(r(E)) - s^c_{\mathcal{T}}(\lambda) s^c_{\mathcal{T}}(\lambda)^*) : E \in \mathcal{E}\big\}$. Then the quotient

$$C^*(\Lambda, c; \mathcal{E}) := \mathcal{T}C^*(\Lambda, c)/J_{\mathcal{E}}$$

is, by construction, universal for relative Cuntz-Krieger $(\Lambda, c; \mathcal{E})$-families. We denote each $s^c_{\mathcal{T}}(\lambda) + J_{\mathcal{E}}$ by $s^c_{\mathcal{E}}(\lambda)$, so that $s^c_{\mathcal{E}} := \{s^c_{\mathcal{E}}(\lambda) : \lambda \in \Lambda\}$ is a universal $(\Lambda, c; \mathcal{E})$-family in $C^*(\Lambda, c; \mathcal{E})$. In the special case where $\mathcal{E} = \mathrm{FE}(\Lambda)$ we simply write $s^c(\lambda)$ for $s^c_{\mathrm{FE}(\Lambda)}(\lambda)$, and we denote $C^*(\Lambda, c; \mathrm{FE}(\Lambda))$ by $C^*(\Lambda, c)$. It follows almost immediately from the universal property that $C^*(\Lambda, c; \mathcal{E})$ depends only on the cohomology class of c:

Proposition 3.5. *Let Λ be a finitely aligned k-graph and let $\mathcal{E}$ be a subset of* $\mathrm{FE}(\Lambda)$. *Suppose that $c_1, c_2 \in \underline{Z}^2(\Lambda, \mathbb{T})$ are cohomologous; say $c_1 = \underline{\delta}^1 b c_2$. Then there is an isomorphism $\pi : C^*(\Lambda, c_1; \mathcal{E}) \to C^*(\Lambda, c_2; \mathcal{E})$ such that $\pi(s^{c_1}_{\mathcal{E}}(\lambda)) = b(\lambda) s^{c_2}_{\mathcal{E}}(\lambda)$ for each $\lambda \in \Lambda$.*

Proof. The proof is essentially that of [16, Proposition 5.6]: the formula $t_\lambda := b(\lambda) s^{c_2}_{\mathcal{E}}(\lambda)$ defines a relative Cuntz-Krieger $(\Lambda, c_1; \mathcal{E})$-family in $C^*(\Lambda, c_2; \mathcal{E})$ and therefore induces a homomorphism π carrying each $s^{c_1}_{\mathcal{E}}(\lambda)$ to $b(\lambda) s^{c_2}_{\mathcal{E}}(\lambda)$. Interchanging the roles of c_1 and c_2 and replacing b with $\overline{b}$ yields an inverse for π, and the result follows. □

The universal property of each $C^*(\Lambda, c; \mathcal{E})$ ensures that there is a homomorphism $\gamma^c : \mathbb{T}^k \to \operatorname{Aut}(C^*(\Lambda, c; \mathcal{E}))$ such that $\gamma^c_z(s^c_{\mathcal{E}}(\lambda)) = z^{d(\lambda)} s^c_{\mathcal{E}}(\lambda)$ for all λ. An $\varepsilon/3$-argument shows that γ^c is strongly continuous. Averaging over γ^c gives a faithful conditional expectation

$$\Phi^c_{\mathcal{E}} : C^*(\Lambda, c; \mathcal{E}) \to C^*(\Lambda, c; \mathcal{E})^{\gamma^c} := \{a \in C^*(\Lambda, c; \mathcal{E}) : \gamma^c_z(a) = a \text{ for all } z\}.$$

Since $\gamma^c_z(s^c_{\mathcal{E}}(\mu) s^c_{\mathcal{E}}(\nu)^*) = z^{d(\mu)-d(\nu)} s^c_{\mathcal{E}}(\mu) s^c_{\mathcal{E}}(\nu)^*$ and since the $s^c_{\mathcal{E}}(\mu) s^c_{\mathcal{E}}(\nu)^*$ span a dense subspace of $C^*(\Lambda, c; \mathcal{E})$, we see that $\Phi^c_{\mathcal{E}}$ is characterised by

$$\Phi^c_{\mathcal{E}}(s^c_{\mathcal{E}}(\mu) s^c_{\mathcal{E}}(\nu)^*) = \delta_{d(\mu),d(\nu)} s^c_{\mathcal{E}}(\mu) s^c_{\mathcal{E}}(\nu)^*,$$

and so $C^*(\Lambda, c; \mathcal{E})^{\gamma^c} = \overline{\operatorname{span}}\{s^c_{\mathcal{E}}(\mu) s^c_{\mathcal{E}}(\nu)^* : d(\mu) = d(\nu)\}$.
For $E \subseteq r(\lambda)\Lambda$, we write

$$\operatorname{Ext}(\lambda; E) := \bigcup_{\mu \in E} \{\alpha : \lambda\alpha \in \operatorname{MCE}(\lambda, \mu)\}.$$

We say that a subset $\mathcal{E}$ of $\operatorname{FE}(\Lambda)$ is *satiated* if it satisfies

(S1) if $F \in \mathcal{E}$ and $\lambda \in r(F)\Lambda \setminus \{r(F)\}$, then $F \cup \{\lambda\} \in \mathcal{E}$;

(S2) if $F \in \mathcal{E}$ and $\lambda \in r(F)\Lambda \setminus F\Lambda$, then $\operatorname{Ext}(\lambda; F) \in \mathcal{E}$;

(S3) if $F \in \mathcal{E}$ and $\lambda, \lambda\lambda' \in F$ with $\lambda' \neq s(\lambda)$, then $F \setminus \{\lambda\lambda'\} \in \mathcal{E}$;

(S4) if $F \in \mathcal{E}$, $\lambda \in F$ and $G \in \mathcal{E}$ with $r(G) = s(\lambda)$, then $F \setminus \{\lambda\} \cup \lambda G \in \mathcal{E}$.

Lemma 5.3 of [26] shows that the sets constructed in (S1)–(S4) belong to $\operatorname{FE}(\Lambda)$, and so $\operatorname{FE}(\Lambda)$ always contains $\mathcal{E}$ and satisfies (S1)–(S4). Given $\mathcal{E} \subseteq \operatorname{FE}(\Lambda)$, we write $\overline{\mathcal{E}}$ for the intersection of all satiated subsets of $\operatorname{FE}(\Lambda)$ containing $\mathcal{E}$. We call $\overline{\mathcal{E}}$ the *satiation* of $\mathcal{E}$.
Our (S1)–(S4) are different from those of [26]; but any set that can be constructed by our (S1)–(S4) can be constructed from an application of the corresponding operation from [26], and conversely any set that can be constructed by any of (S1)–(S4) from [26] can be obtained from finitely many applications of the operations discussed above. So our definition of a satiated set agrees with that of [26].

Notation 3.6. Let Λ be a finitely aligned k-graph, let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$ and let t be a Toeplitz-Cuntz-Krieger (Λ, c)-family. For $\lambda \in \Lambda$, we write $q_\lambda := t_\lambda t^*_\lambda$.
For $v \in \Lambda^0$ and a finite subset E of $v\Lambda$, we define

$$\Delta(t)^E := \prod_{\lambda \in E} (q_{r(E)} - q_\lambda).$$

In the universal algebras $C^*(\Lambda, c; \mathcal{E})$, we write $p^c_{\mathcal{E}}(\lambda) := s^c_{\mathcal{E}}(\lambda) s^c_{\mathcal{E}}(\lambda)^* \in C^*(\Lambda, c; \mathcal{E})$, and then $\Delta(s^c_{\mathcal{E}})^E = \prod_{\lambda \in E} \big(p^c_{\mathcal{E}}(r(E)) - p^c_{\mathcal{E}}(\lambda)\big)$.

Lemma 3.7. *Let Λ be a finitely aligned k-graph, let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$ and let t be a Toeplitz-Cuntz-Krieger (Λ, c)-family. If $\emptyset \neq E \subseteq v\Lambda$ is finite and $\mu \in v\Lambda$, then*

$$\Delta(t)^E t_\mu = t_\mu \Delta(t)^{\mathrm{Ext}(\mu;E)}.$$

Proof. Fix $\lambda \in E$ and use Lemma 3.2 to calculate

$$\begin{aligned}\Delta(t)^E t_\mu &= \Delta(t)^{E\setminus\{\lambda\}}(q_v - t_\lambda t_\lambda^*)t_\mu t_{s(\mu)}^* \\ &= \Delta(t)^{E\setminus\{\lambda\}} t_\mu - \sum_{\lambda\alpha=\mu\beta\in\mathrm{MCE}(\lambda,\mu)} c(\lambda,\alpha)\overline{c(\lambda,\alpha)}c(\mu,\beta)\overline{c(s(\mu),\beta)}t_{\lambda\alpha}t_\beta^*.\end{aligned}$$

For each α, β we have $t_{\lambda\alpha} = t_{\mu\beta} = \overline{c(\mu,\beta)}t_\mu t_\beta$, and we deduce that

$$\Delta(t)^E t_\mu = \Delta(t)^{E\setminus\{\lambda\}} t_\mu\Big(q_{s(\mu)} - \sum_{\beta\in\mathrm{Ext}(\mu;\{\lambda\})} q_\beta\Big).$$

The elements $\beta \in \mathrm{Ext}(\mu;\{\lambda\})$ all have degree $(d(\mu) \vee d(\lambda)) - d(\mu)$, and so the q_β are mutually orthogonal. Hence $q_{s(\mu)} - \sum_{\beta\in\mathrm{Ext}(\mu;\{\lambda\})} q_\beta = \prod_{\beta\in\mathrm{Ext}(\mu;\{\lambda\})}(q_{s(\mu)} - q_\beta)$. Now an induction on $|E|$ using that $\mathrm{Ext}(\mu;E) = \bigcup_{\lambda\in E}\mathrm{Ext}(\mu;\{\lambda\})$ proves the lemma. □

Proposition 3.12 of [10] implies that $\mathrm{Ext}(\mu\nu;E) = \mathrm{Ext}(\nu;\mathrm{Ext}(\mu;E))$ for all composable μ,ν. If $\mu \in E\Lambda$, then $\Delta(s_{\mathcal{T}}^c)^E \leq s_{\mathcal{T}}^c(r(\mu)) - p_{\mathcal{T}}^c(\mu)$, forcing $\Delta(s_{\mathcal{T}}^c)^E s_{\mathcal{T}}^c(\mu) = 0$. So Lemma 3.7 and condition (S2) imply that for a satiated subset $\mathcal{E} \subseteq \mathrm{FE}(\Lambda)$, the ideal $J_{\mathcal{E}}$ generated by $\{\Delta(s_{\mathcal{T}}^c)^E : E \in \mathcal{E}\}$ satisfies

$$J_{\mathcal{E}} = \overline{\mathrm{span}}\{s_{\mathcal{T}}^c(\mu)\Delta(s_{\mathcal{T}}^c)^E s_{\mathcal{T}}^c(\nu)^* : s(\mu) = s(\nu) \text{ and } E \in s(\mu)\mathcal{E}\}. \tag{3.1}$$

We introduce the notion of a filter on a k-graph. For details, see [3, Section 3] (taking $P = \mathbb{N}^k$). We call a subset S of Λ a *filter* if

(F1) $\lambda\Lambda \cap S \neq \emptyset$ implies $\lambda \in S$; and

(F2) $\mu, \nu \in S$ implies $\mathrm{MCE}(\mu,\nu) \cap S \neq \emptyset$.

For $\mu,\nu \in \Lambda$, we write $\mu \leq \nu$ if $\nu = \mu\nu'$. Since $\lambda \in \mathrm{MCE}(\mu,\nu)$ implies $\mu,\nu \leq \lambda$, Condition (F2) implies that each $(S, \leq)$ is a directed set. Furthermore, for $\mu,\nu \in \Lambda$, distinct elements of $\mathrm{MCE}(\mu,\nu)$ have the same degree, and so themselves have no common extensions. Hence condition (F2) implies that $|\mathrm{MCE}(\mu,\nu) \cap S| = 1$ for $\mu,\nu \in S$.
For a satiated set $\mathcal{E} \subseteq \mathrm{FE}(\Lambda)$, we say that a filter S is $\overline{\mathcal{E}}$-compatible if, whenever $\lambda \in S$ and $E \in s(\lambda)\overline{\mathcal{E}}$, we have $\lambda E \cap S \neq \emptyset$.

Remark 3.8. It is straightforward to check that the $\overline{\mathcal{E}}$-compatible boundary paths of [26] are in bijection with $\overline{\mathcal{E}}$-compatible filters via the map $x \mapsto \{x(0,n) : n \leq d(x)\}$.

Let $\{h_\lambda : \lambda \in \Lambda\}$ be the usual orthonormal basis for $\ell^2(\Lambda)$. Routine calculations show that the formula $T_\mu h_\nu := \delta_{s(\mu),r(\nu)} c(\mu,\nu) h_{\mu\nu}$ determines a Toeplitz-Cuntz-Krieger (Λ, c)-family in $\mathcal{B}(\ell^2(\Lambda))$. Let $\pi_T : \mathcal{T}C^*(\Lambda, c) \to \mathcal{B}(\ell^2(\Lambda))$ be the representation induced by the universal property of $\mathcal{T}C^*(\Lambda, c)$.

Proposition 3.9. *Let Λ be a finitely aligned k-graph and let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Let $\mathcal{E} \subseteq \mathrm{FE}(\Lambda)$. Then the $s^c_{\mathcal{E}}(v)$ are all nonzero, and for each $v \in \Lambda^0$ and finite $F \subseteq v\Lambda$, we have $\Delta(s^c_{\mathcal{E}})^F = 0$ if and only if either $v \in F$ or $F \in \overline{\mathcal{E}}$. We have $J_{\mathcal{E}} = J_{\overline{\mathcal{E}}}$ and $C^*(\Lambda, c; \mathcal{E}) = C^*(\Lambda, c; \overline{\mathcal{E}})$.*

To prove the proposition, we need to be able to tell when $a \in \mathcal{T}C^*(\Lambda, c)$ does not belong to $J_{\mathcal{E}}$. We present the requisite statement as a separate result because we will use it again in Section 7 (see Lemma 7.2).

Proposition 3.10. *Let Λ be a finitely aligned k-graph and let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Let $\mathcal{E} \subseteq \mathrm{FE}(\Lambda)$. Then*

$$J_{\mathcal{E}} = \{a \in \mathcal{T}C^*(\Lambda, c) : \lim_{\lambda \in S} \pi_T(a) h_\lambda = 0 \text{ for every } \overline{\mathcal{E}}\text{-compatible filter } S\}.$$

Here we only need the "$\subseteq$" containment — we will use it immediately to establish Proposition 3.9, which we will use in turn to prove the gauge-invariant uniqueness theorem and then our characterisation of the gauge-invariant ideals in $C^*(\Lambda, c; \mathcal{E})$. With this catalogue of gauge-invariant ideals in hand, it is then easy to establish the reverse inclusion. So we prove the "$\subseteq$" inclusion now, and defer the proof of the "$\supseteq$" inclusion until after Theorem 4.6.

Proof of $\subseteq$ in Proposition 3.10. Fix $v \in \Lambda^0$, an element $E \in \mathcal{E}$ with $r(E) = v$ and an $\overline{\mathcal{E}}$-compatible filter S. We claim that $\Delta(T)^E h_\lambda = 0$ for large $\lambda \in S$. To see this, we consider two cases. First suppose that $v \notin S$. Then (F1) ensures that $S \cap v\Lambda = \emptyset$ and so $T_v h_\lambda = 0$ for all $\lambda \in S$. Since $\Delta(T)^E \leq T_v$, it follows that $\Delta(T)^E h_\lambda = 0$ for all $\lambda \in S$. Now suppose that $v \in S$. Since S is $\overline{\mathcal{E}}$-compatible and $E \in \mathcal{E}$, there exists $\lambda \in E \cap S$. Hence $\Delta(T)^E \leq T_v - T_\lambda T^*_\lambda$. For $\mu \in S$ with $\lambda \leq \mu$, we therefore have $\Delta(T)^E h_\mu = 0$. Hence $\Delta(T)^E h_\lambda = 0$ for large $\lambda \in S$, proving the claim. It is now elementary to deduce that any finite linear combination of the form $a = \sum_{\mu,\nu \in F} a_{\mu,\nu} s^c_{\mathcal{T}}(\mu) \Delta(s^c_{\mathcal{T}})^E s^c_{\mathcal{T}}(\nu)^*$ where the E belong to $\mathcal{E}$ satisfies $\pi_T(a) h_\lambda = 0$ for large $\lambda \in E$. A continuity argument (details appear in [28, Lemma 3.4.14]) using this and (3.1) then completes the proof. ☐

Lemma 3.11. *Let Λ be a finitely aligned k-graph and let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Let $\mathcal{E} \subseteq \mathrm{FE}(\Lambda)$. For each $v \in \Lambda^0$ there exists an $\overline{\mathcal{E}}$-compatible filter S such that $\|T_v h_\lambda\| = 1$ for all $\lambda \in S$, and for each $E \in \mathrm{FE}(\Lambda) \setminus \overline{\mathcal{E}}$, there exists an $\overline{\mathcal{E}}$-compatible filter S such that $\|\Delta(T)^E h_\lambda\| = 1$ for all $\lambda \in S$.*

Proof. Fix $v \in \Lambda^0$. The argument of [26, Lemma 4.7] shows that there exists an $\overline{\mathcal{E}}$-compatible filter S that contains v. Hence $\|T_v h_\lambda\| = \|h_\lambda\| = 1$ for all $\lambda \in S$. Now fix $E \in \mathrm{FE}(\Lambda) \setminus \mathcal{E}$. Then the argument of [26, Lemma 4.7] implies that there is an $\overline{\mathcal{E}}$-compatible filter S such that $r(E) \in S$ but $E\Lambda \cap S = \emptyset$. Thus

$T_\mu T_\mu^* h_\lambda = 0$ for all $\mu \in E$ and $\lambda \in S$, and so $\|\Delta(T)^E h_\lambda\| = \|T_{r(E)} h_\lambda\| = 1$ for all $\lambda \in S$. $\square$

Proof of Proposition 3.9. For the "if" implication, observe that if $v \in F$ then $\Delta_v \leq q_v - q_v = 0$. To see that $F \in \overline{\mathcal{E}}$ implies $\Delta(s^c_{\mathcal{E}})^F = 0$, one checks that the calculations in [26] that establish the corresponding statement for the untwisted C^*-algebra $C^*(\Lambda; \mathcal{E})$ are also valid in the twisted C^*-algebra. Since we clearly have $J_{\mathcal{E}} \subseteq J_{\overline{\mathcal{E}}}$ we deduce from this that $J_{\mathcal{E}} = J_{\overline{\mathcal{E}}}$, and therefore that $C^*(\Lambda, c; \mathcal{E}) = C^*(\Lambda, c; \overline{\mathcal{E}})$.

For the "only if" implication, first observe that we have just seen that $J_{\overline{\mathcal{E}}} = J_{\mathcal{E}}$. So it suffices to show that $s^c_{\mathcal{T}}(v) \notin J_{\overline{\mathcal{E}}}$ for all $v \in \Lambda^0$ and that if $E \subset v\Lambda$ is finite, does not contain v and does not belong to $\overline{\mathcal{E}}$, then $\Delta(s^c_{\mathcal{T}})^E \notin J_{\overline{\mathcal{E}}}$.

The "$\subseteq$" containment in Proposition 3.10 and the first assertion of Lemma 3.11 combine to show that $s^c_{\mathcal{T}}(v) \notin J_{\overline{\mathcal{E}}}$ for each $v \in \Lambda^0$. Likewise, the "$\subseteq$" containment in Proposition 3.10 combined with the second assertion of Lemma 3.11 shows that $\Delta(s^c_{\mathcal{T}})^E \notin J_{\overline{\mathcal{E}}}$ for all $E \in \mathrm{FE}(\Lambda) \setminus \overline{\mathcal{E}}$. Finally, suppose that $F \subseteq v\Lambda \setminus \{v\}$ is finite and does not belong to $\mathrm{FE}(\Lambda)$. Then there exists $\lambda \in v\Lambda$ such that $\mathrm{Ext}(\lambda; F) = \emptyset$. It is not hard to check (see, for example, the argument of [26, Lemma 4.4]) that $\{\mu : \mu\mu' \in \lambda S \text{ for some } \mu'\}$ is an $\overline{\mathcal{E}}$-compatible filter which does not intersect F. So as above, we have $\|\pi_T(\Delta(s^c_{\mathcal{T}})^F) h_\lambda\| = 1$ for all $\lambda \in S$ and yet another application of the "$\subseteq$" containment in Proposition 3.10 implies that $\Delta(s^c_{\mathcal{T}})^F \notin J_{\overline{\mathcal{E}}}$. $\square$

Given a finite subset E of Λ and an element $\mu \in E$, we write $T(E; \mu) := \{\mu' \in s(\mu)\Lambda \setminus \{s(\mu)\} : \mu\mu' \in E\}$.

Recall from [23] that if E is a finite subset of a finitely aligned k-graph Λ, then there is a finite subset F of Λ which contains E and has the property that if $\mu, \nu, \sigma, \tau \in F$ with $d(\mu) = d(\nu)$ and $d(\sigma) = d(\tau)$, then $\mu\alpha$ and $\tau\beta$ belong to F for every $\nu\alpha = \sigma\beta \in \mathrm{MCE}(\nu, \sigma)$. The smallest such set is denoted ΠE and is closed under minimal common extensions (just take $\mu = \nu$ and $\tau = \sigma$). Suppose that $E = \Pi E$. The defining property of ΠE ensures that $T(E; \mu) = T(E; \nu)$ whenever $\mu, \nu \in E$ satisfy $d(\mu) = d(\nu)$. Fix $c \in \underline{Z}^2(\Lambda, \mathbb{T})$ and a Toeplitz-Cuntz-Krieger (Λ, c)-family t. For $\mu, \nu \in E$ (possibly equal) with $d(\mu) = d(\nu)$, we write

$$\Theta(t)^E_{\mu,\nu} = t_\mu \Delta(t)^{T(E;\mu)} t^*_\nu.$$

LEMMA 3.12. *Let Λ be a finitely aligned k-graph, let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$ and let t be a Toeplitz-Cuntz-Krieger (Λ, c)-family. Suppose that $E \subseteq \Lambda$ is finite and satisfies $E = \Pi E$. Then $M(t)_E := \operatorname{span}\{t_\mu t^*_\nu : \mu, \nu \in E, d(\mu) = d(\nu)\}$ is a finite dimensional C^*-subalgebra of $C^*(t)$, and $\{\Theta(t)^E_{\mu,\nu} : \mu, \nu \in E, d(\mu) = d(\nu), s(\mu) = s(\nu), \Delta(t)^{T(E;\mu)} \neq 0\}$ is a family of nonzero matrix units spanning $M(t)_E$.*

Proof. Using Lemma 3.2, it is easy to see that $\operatorname{span}\{t_\mu t^*_\nu : \mu, \nu \in E, d(\mu) = d(\nu)\}$ is closed under multiplication and hence a subalgebra of $C^*(t)$. Since $T(E; \mu) = T(E; \nu)$ whenever $d(\mu) = d(\nu)$ and $s(\mu) = s(\nu)$, we have $(\Theta(t)^E_{\mu,\nu})^* =$

$\Theta(t)^E_{\nu,\mu}$, so $M(t)_E$ is a *-subalgebra of $C^*(t)$. It is finite-dimensional by definition, and then it is automatically norm-closed and hence a C^*-subalgebra.
The argument of [23, Lemma 3.9(ii)], using Lemma 3.7 twice in place of [23, Lemma 3.10] shows that $\Theta(t)^E_{\mu,\nu}\Theta(t)^E_{\eta,\xi} = \delta_{\nu,\eta}\Theta(t)^E_{\mu,\xi}$. Since $\Delta(t)^{T(E;\mu)} = t^*_\mu\Theta(t)^E_{\mu,\nu}t_\nu$ and since $\Theta(t)^E_{\mu,\nu} = t_\mu\Delta(t)^{T(E;\mu)}t^*_\nu$, we have $\Theta(t)^E_{\mu,\nu} = 0 \iff \Delta(t)^{T(E;\mu)} = 0$. The arguments of [23, Proposition 3.5 and Corollary 3.7] only involve products of elements of the form $t_\mu t^*_\mu$, and these satisfy the same relations in a Toeplitz-Cuntz-Krieger (Λ, c)-family as in a Toeplitz-Cuntz-Krieger Λ-family. So the proof of [23, Corollary 3.7] implies that $t_\mu t^*_\mu = \sum_{\mu\mu'\in E}\Delta(t)^{T(E;\mu\mu')}$ for $\mu \in E$. Now we follow the proof of [23, Corollary 3.11] to see that

$$t_\mu t^*_\nu = \sum_{\mu\alpha\in E} c(\mu,\alpha)\overline{c(\nu,\alpha)}\Theta(t)^E_{\mu\alpha,\nu\alpha}. \tag{3.2}$$

Hence the $\Theta(t)^E_{\mu,\nu}$ span $M(t)_E$. □

THEOREM 3.13. *Let Λ be a finitely aligned k-graph, let $c \in \underline{Z}^2(\Lambda,\mathbb{T})$, and let $\mathcal{E}$ be a satiated subset of* FE(Λ). *Suppose that t is a relative Cuntz-Krieger $(\Lambda, c; \mathcal{E})$-family. The induced homomorphism $\pi^{\mathcal{E}}_t : C^*(\Lambda, c; \mathcal{E}) \to C^*(t)$ restricts to an injective homomorphism of $C^*(\Lambda, c; \mathcal{E})^{\gamma^c}$ if and only if every t_v is nonzero, and $\Delta(t)^F \neq 0$ for all $F \in$* FE$(\Lambda) \setminus \mathcal{E}$.

Proof. Proposition 3.9 implies that if $t_v = 0$ for some $v \in E^0$ or $\Delta(t)^F = 0$ for some $F \in \mathrm{FE}(\Lambda) \setminus \mathcal{E}$, then $\pi^{\mathcal{E}}_t$ is not injective on $C^*(\Lambda)^\gamma$.
Now suppose that each $t_v \neq 0$ and that $\Delta(t)^F \neq 0$ whenever $F \in \mathrm{FE}(\Lambda) \setminus \mathcal{E}$. Since every finite subset of Λ is contained in a finite subset E satisfying $E = \Pi E$, we have

$$C^*(\Lambda, c; \mathcal{E})^{\gamma^c} = \overline{\bigcup_{\Pi E = E} M(s^c_{\mathcal{E}})^E},$$

and so it suffices to show that $\pi^{\mathcal{E}}_t$ is injective (and hence isometric) on each $M(s^c_{\mathcal{E}})^E$. Fix E such that $\Pi E = E$. Since matrix algebras are simple, Lemma 3.12 implies that it suffices to show that $\Delta(t)^{T(E;\lambda)} = 0$ implies $\Delta(s^c_{\mathcal{E}})^{T(E;\lambda)} = 0$ for each $\lambda \in E$. We consider two cases. If $T(E;\lambda) \in \mathrm{FE}(\Lambda)$, then $\Delta(t)^{T(E;\lambda)} = 0$ implies $T(E;\lambda) \in \mathcal{E}$ by hypothesis, and then $\Delta(s^c_{\mathcal{E}})^{T(E;\lambda)} = 0$ as well. Now suppose that $T(E;\lambda) \notin \mathrm{FE}(\Lambda)$. It suffices to show that $\Delta(t)^{T(E;\lambda)} \neq 0$. Since $T(E;\lambda) \notin \mathrm{FE}(\Lambda)$ and $T(E;\lambda) \cap \Lambda^0 = \emptyset$, the set $T(E;\lambda)$ is not exhaustive. Fix $\mu \in s(\lambda)\Lambda$ such that $\mathrm{MCE}(\mu,\alpha) = \emptyset$ for every $\alpha \in T(E;\lambda)$. Since $t^*_\mu t_\mu = t_{s(\mu)}$ is nonzero, $q_\mu = t_\mu t^*_\mu$ is also nonzero. Lemma 3.2 implies that $q_\alpha q_\mu = 0$ for all $\alpha \in T(E;\lambda)$. Since the q_η all commute and q_μ is a projection, we deduce that

$$\Delta(t)^{T(E;\lambda)}q_\mu = \prod_{\alpha\in T(E;\lambda)}\big((t_{s(\lambda)} - q_\alpha)q_\mu\big) = q_\mu \neq 0.$$

Hence $\Delta(t)^{T(E;\lambda)} \neq 0$. □

From this flow a number of uniqueness theorems, all based on the following standard idea from [5].

Lemma 3.14. *Let A be a C^*-algebra, $\Phi : A \to A$ a faithful conditional expectation, and $\pi : A \to B$ a C^*-homomorphism. Suppose that there is a linear map $\Psi : B \to B$ such that $\Psi \circ \pi = \pi \circ \Phi$. Then π is injective if and only if $\pi|_{\Phi(A)}$ is injective.*

Proof. The "only if" implication is obvious. For the "if" implication, suppose that $\pi(a) = 0$. Then $\pi(\Phi(a^*a)) = \Psi(\pi(a^*a)) = 0$. Since π is injective on the range of Φ and Φ is faithful, we deduce that $a = 0$. □

The next result is a version of Coburn's theorem for $\mathcal{T}C^*(\Lambda, c)$.

Theorem 3.15. *Let Λ be a finitely aligned k-graph and let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Let t be a Toeplitz-Cuntz-Krieger (Λ, c)-family. Then the induced homomorphism π_t of $\mathcal{T}C^*(\Lambda, c)$ is injective if and only if every $t_v \neq 0$ and $\Delta(t)^E \neq 0$ for every $E \in \mathrm{FE}(\Lambda)$.*

Proof. Averaging over the gauge action on $\mathcal{T}C^*(\Lambda, c)$ defines a faithful conditional expectation $\Phi^{\gamma^c} : \mathcal{T}C^*(\Lambda, c) \to \mathcal{T}C^*(\Lambda, c)^{\gamma^c}$ that is characterised by $\Phi^{\gamma^c}(s^c_{\mathcal{T}}(\mu) s^c_{\mathcal{T}}(\nu)^*) = \delta_{d(\mu),d(\nu)} s^c_{\mathcal{T}}(\mu) s^c_{\mathcal{T}}(\nu)^*$ for $\mu, \nu \in \Lambda$. Lemma 3.12 shows that $\mathcal{T}C^*(\Lambda, c)^{\gamma^c}$ is AF, and $\overline{\operatorname{span}}\{s^c_{\mathcal{T}}(\mu) s^c_{\mathcal{T}}(\mu)^* : \mu \in \Lambda\}$ is its canonical diagonal subalgebra. So there is a faithful conditional expectation $\Phi^c_D : \mathcal{T}C^*(\Lambda, c)^{\gamma^c} \to \overline{\operatorname{span}}\{s^c_{\mathcal{T}}(\mu) s^c_{\mathcal{T}}(\mu)^* : \mu \in \Lambda\}$ satisfying $\Phi^c_D(s^c_{\mathcal{T}}(\mu) s^c_{\mathcal{T}}(\nu)^*) = \delta_{\mu,\nu} s^c_{\mathcal{T}}(\mu) s^c_{\mathcal{T}}(\mu)^*$. So $\Phi := \Phi^c_D \circ \Phi^{\gamma^c}$ is a faithful conditional expectation satisfying $\Phi(s^c_{\mathcal{T}}(\mu) s^c_{\mathcal{T}}(\nu)^*) = \delta_{\mu,\nu} s^c_{\mathcal{T}}(\mu) s^c_{\mathcal{T}}(\mu)^*$ for all $\mu, \nu \in \Lambda$.
The argument of [22, Proposition 8.9] (this uses Lemmas 8.5–8.8 of the same paper; all the arguments go through for twisted TCK families) gives a norm-decreasing linear map Ψ from $C^*(t)$ to $\overline{\operatorname{span}}\{t_\lambda t^*_\lambda : \lambda \in \Lambda\}$ such that $\Psi \circ \pi_t = \pi_t \circ \Phi$. Now Lemma 3.14 proves the result. □

We also obtain the following version of an Huef and Raeburn's gauge-invariant uniqueness theorem [11].

Theorem 3.16. *Let Λ be a finitely aligned k-graph, let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$, and let $\mathcal{E}$ be a satiated subset of $\mathrm{FE}(\Lambda)$. Suppose that t is a relative Cuntz-Krieger $(\Lambda, c; \mathcal{E})$-family in a C^*-algebra B and that there is a strongly continuous action β of $\mathbb{T}^k$ on B such that $\beta_z(t_\lambda) = z^{d(\lambda)} t_\lambda$ for all $\lambda \in \Lambda$. Then the induced homomorphism $\pi^{\mathcal{E}}_t : C^*(\Lambda, c; \mathcal{E}) \to C^*(t)$ is injective if and only if every t_v is nonzero, and $\Delta(t)^F \neq 0$ for all $F \in \mathrm{FE}(\Lambda) \setminus \mathcal{E}$.*

Proof. Lemma 3.14 applied to the expectation Φ obtained from averaging over γ^c and the expectation Ψ obtained by averaging over β shows that $\pi^{\mathcal{E}}_t$ is injective if and only if it restricts to an injection on $C^*(\Lambda, c; \mathcal{E})^{\gamma^c}$. So the result follows from Theorem 3.13. □

We also obtain a version of the Cuntz-Krieger uniqueness theorem (cf. [26, Theorem 6.3]). Given a filter S and a path μ with $s(\mu) \in S$, we write $\ell_\mu(S)$ for the filter $\{\nu \in \Lambda : \mu S \cap \nu\Lambda \neq \emptyset\}$. We say that a filter $S \subseteq \Lambda$ is *separating* if whenever $s(\mu) = s(\nu) \in S$ and there is a filter T such that $\ell_\mu(S) \cup \ell_\nu(S) \subseteq T$, we have $\mu = \nu$.

Theorem 3.17. *Let Λ be a finitely aligned k-graph, let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$, and let $\mathcal{E}$ be a satiated subset of $\mathrm{FE}(\Lambda)$. Suppose that for every $v \in \Lambda^0$ there is a separating $\mathcal{E}$-compatible filter S such that $r(S) = v$, and that for every $F \in \mathrm{FE}(\Lambda) \setminus \mathcal{E}$ there is a separating $\mathcal{E}$-compatible filter S such that $r(S) = r(F)$ and $S \cap F\Lambda = \emptyset$. Suppose that t is a relative Cuntz-Krieger $(\Lambda, c; \mathcal{E})$-family in a C^*-algebra B. Then the induced homomorphism $\pi_t^{\mathcal{E}} : C^*(\Lambda, c; \mathcal{E}) \to C^*(t)$ is injective if and only if every t_v is nonzero, and $\Delta(t)^F \neq 0$ for all $F \in \mathrm{FE}(\Lambda) \setminus \mathcal{E}$.*

Proof. Using Remark 3.8, we see that our hypothesis about the existence of separating $\mathcal{E}$-compatible filters is equivalent to Condition (C) of [26]. Now one follows the proof of [26, Theorem 6.3]; the only place where the cocycle c comes up is in the displayed calculation at the top of [26, page 866], where the formula for $P_2\Theta(t)^{\Pi E}_{\lambda,\mu}P_2$ picks up a factor of $c(\lambda, x(0,N))\overline{c(\mu, x(0,N))}$; but the resulting elements still form a family of matrix units, so the rest of the argument proceeds without change. $\square$

The hypothesis of Theorem 3.17 simplifies significantly in the key case where $\mathcal{E} = \mathrm{FE}(\Lambda)$. Recall from [18, Definition 3.3] that a finitely aligned k-graph Λ is *cofinal* if, for all $v, w \in \Lambda^0$ there exists a finite exhaustive subset E of $v\Lambda$ (here E may contain v) such that $w\Lambda s(\alpha) \neq \emptyset$ for every $\alpha \in E$. Recall from [18] that a k-graph Λ is *aperiodic* if whenever μ, ν are distinct paths with the same source, there exists $\tau \in s(\mu)\Lambda$ such that $\mathrm{MCE}(\mu\tau, \nu\tau) = \emptyset$.

Corollary 3.18. *Let Λ be a finitely aligned k-graph and let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Suppose Λ is aperiodic. Then a homomorphism $\pi : C^*(\Lambda, c) \to B$ is injective if and only if every $\pi(s_v^c) \neq 0$. If Λ is cofinal then $C^*(\Lambda, c)$ is simple.*

Proof. Condition (A) of [10], reinterpreted using Remark 3.8, requires that for every $v \in \Lambda^0$ there is a separating $\mathrm{FE}(\Lambda)$-compatible filter S containing v. So the implication (i) $\Longrightarrow$ (iii) of [18, Proposition 3.6] implies that the hypotheses of Theorem 3.17 are satisfied with $\mathcal{E} = \mathrm{FE}(\Lambda)$. This proves the first assertion. Now suppose that Λ is cofinal. Then the argument of (ii) $\Longrightarrow$ (iii) of [18, Theorem 5.1] carries over unchanged to the twisted setting to show that if I is an ideal of $C^*(\Lambda, \mathcal{E})$ and $s_v^c \in I$ for some v, then $s_w^c \in I$ for every $w \in \Lambda^0$ and hence $I = C^*(\Lambda, c)$. So $C^*(\Lambda, c)$ is simple by the preceding paragraph. $\square$

Recall from [8] that if Λ is a finitely aligned k-graph, then a *generalised cycle* in Λ is a pair $(\mu, \nu) \in \Lambda$ such that $r(\mu) = r(\nu)$, $s(\mu) = s(\nu)$ and $\mathrm{MCE}(\mu\tau, \nu) \neq \emptyset$ for all $\tau \in s(\mu)\Lambda$. We say that a generalised cycle (μ, ν) has an *entrance* if there exists $\tau \in s(\nu)\Lambda$ such that $\mathrm{MCE}(\mu, \nu\tau) = \emptyset$. If $c \in \underline{Z}^2(\Lambda, \mathbb{T})$ and t is a Toeplitz-Cuntz-Krieger (Λ, c)-family such that $t_{s(\tau)} \neq 0$, then $V := t_\mu t_\nu^*$

satisfies $q_\mu = VV^* \sim V^*V = q_\nu > q_\nu - q_{\nu\tau} \geq q_\mu$, where $\sim$ denotes Murray-von Neumann equivalence. Hence q_μ is infinite.

Proposition 3.19. *Let Λ be a finitely aligned k-graph with no sources, and let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Suppose that Λ is aperiodic and that for every $v \in \Lambda^0$ there is a generalised cycle (μ, ν) with an entrance such that $v\Lambda r(\mu) \neq \emptyset$. Then every hereditary subalgebra of $C^*(\Lambda, c)$ contains an infinite projection. If Λ is cofinal, then $C^*(\Lambda, c)$ is simple and purely infinite.*

Proof. Let $v \in \Lambda^0$ and choose a generalised cycle (μ, ν) with an entrance such that $v\Lambda r(\mu) \neq \emptyset$; say $\lambda \in v\Lambda r(\mu)$. Then $s^c(v) \geq s^c(\lambda)s^c(\lambda)^* \sim s^c(\lambda)^* s^c(\lambda) \geq p^c(\mu)$ is infinite. Using this in place of [25, Lemma 8.13], we can now follow the proof of [25, Proposition 8.8] (including the proof of [25, Lemma 8.12]). □

4 Gauge-invariant ideals

In this section we list the gauge-invariant ideals of each $C^*(\Lambda, c; \mathcal{E})$. There is by now a fairly standard program for this (the basic idea goes back to [6] and [11]). The standard arguments are applied to the untwisted Cuntz-Krieger algebras of finitely aligned k-graphs in [25], and we follow the broad strokes of that treatment. The twist does not affect the arguments much, but we need to make a few adjustments to pass from $C^*(\Lambda, c)$ to $C^*(\Lambda, c; \mathcal{E})$. So we give more detail here than in the preceding section.

Definition 4.1. Let Λ be a finitely aligned k-graph and let $\mathcal{E}$ be a satiated subset of $\mathrm{FE}(\Lambda)$. We say that $H \subseteq \Lambda^0$ is *hereditary* if $s(H\Lambda) \subseteq H$, and that it is *$\mathcal{E}$-saturated* if whenever $E \in \mathcal{E}$ and $s(E) \subseteq H$ we have $r(E) \in H$.

Recall from [25, Lemma 4.1] that if Λ is a finitely aligned k-graph and $H \subseteq \Lambda^0$ is hereditary, then $\Lambda \setminus \Lambda H$ is a finitely aligned k-graph under the operations and degree map inherited from Λ.

Lemma 4.2. *Let Λ be a finitely aligned k-graph and let $\mathcal{E}$ be a satiated subset of* $\mathrm{FE}(\Lambda)$. *Suppose that $H \subseteq \Lambda^0$ is hereditary and $\mathcal{E}$-saturated. Then*

$$\mathcal{E}_H := \{E \setminus EH : E \in \mathcal{E}, r(E) \notin H\}$$

is a subset of $\mathrm{FE}(\Lambda \setminus \Lambda H)$.

Proof. No element F of $\mathcal{E}_H$ contains $r(F)$ because no element E of $\mathcal{E}$ contains $r(E)$. The elements of $\mathcal{E}_H$ are nonempty because H is $\mathcal{E}$-saturated. Fix $F \in \mathcal{E}_H$, say $r(F) = v$, and fix $\lambda \in v\Lambda \setminus \Lambda H$. Choose $E \in \mathcal{E}$ such that $F = E \setminus EH$. We have to show that there exists $\mu \in F$ such that $\mathrm{MCE}(\mu, \lambda) \cap (\Lambda \setminus \Lambda H) \neq \emptyset$. We consider two cases. First suppose that $\lambda \in E\Lambda$, say $\lambda = \mu\mu'$ with $\mu \in E$. Since H is hereditary and $s(\lambda) = s(\mu') \notin H$, we have $s(\mu) = r(\mu') \notin H$, so $\mu \in F$ satisfies $\lambda \in \mathrm{MCE}(\mu, \lambda) \cap (\Lambda \setminus \Lambda H)$. Now suppose that $\lambda \notin E\Lambda$. Then $\mathrm{Ext}(\lambda; E) \in \mathcal{E}$ by (S2). Since H is $\mathcal{E}$-saturated and $s(\lambda) \notin H$, it follows that $s(\mathrm{Ext}(\lambda; E)) \not\subseteq H$. So there exists $\mu \in E$ and $\mu\alpha = \lambda\beta \in \mathrm{MCE}(\mu, \lambda)$ with

$s(\alpha) \notin H$. Since H is hereditary, it follows that $s(\mu) \notin H$, so $\mu \in F$ and $\mathrm{MCE}(\mu, \lambda) \cap (\Lambda \setminus \Lambda H) \neq \emptyset$. $\square$

Lemma 4.3. *Let Λ be a finitely aligned k-graph, let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$, and let $\mathcal{E}$ be a satiated subset of* $\mathrm{FE}(\Lambda)$. *Let I be an ideal of $C^*(\Lambda, c; \mathcal{E})$. Then*

1. $H_I := \{v \in \Lambda^0 : s^c_{\mathcal{E}}(v) \in I\}$ *is hereditary and $\mathcal{E}$-saturated;*
2. $\mathcal{B}_I := \{F \in \mathrm{FE}(\Lambda \backslash \Lambda H_I) : \Delta(s^c_{\mathcal{E}})^F \in I\}$ *is a satiated subset of* $\mathrm{FE}(\Lambda \backslash \Lambda H)$*; and*
3. $\mathcal{E}_{H_I} \subseteq \mathcal{B}_I$.

Proof. (1). If $r(\lambda) \in H_I$, then $s^c_{\mathcal{E}}(s(\lambda)) = s^c_{\mathcal{E}}(\lambda)^* s^c_{\mathcal{E}}(r(\lambda)) s^c_{\mathcal{E}}(\lambda) \in I$, and hence H_I is hereditary. To see that it is $\mathcal{E}$-saturated, suppose that $E \in \mathcal{E}$ and $s(E) \subseteq H$, and let $v = r(E)$. The calculation of [22, Proposition 8.6] implies that $s^c_{\mathcal{E}}(v) = \Delta(s^c_{\mathcal{E}})^{\vee E} + \sum_{\mu \in \vee E} s^c_{\mathcal{E}}(\mu) \Delta(s^c_{\mathcal{E}})^{T(\vee E; \mu)} s^c_{\mathcal{E}}(\mu)^*$. Since $E \in \mathcal{E}$, condition (S1) implies that $\vee E \in \mathcal{E}$, and so $\Delta(s^c_{\mathcal{E}}) = 0$. Since H is hereditary, each $s^c_{\mathcal{E}}(\mu) \Delta(s^c_{\mathcal{E}})^{T(\vee E; \mu)} s^c_{\mathcal{E}}(\mu)^* \leq s^c_{\mathcal{E}}(\mu) s^c_{\mathcal{E}}(s(\mu)) s^c_{\mathcal{E}}(\mu)^* \in I$, giving $s^c_{\mathcal{E}}(v) \in I$.
(2). Write $\Gamma := \Lambda \setminus \Lambda H$, and fix $F \in \mathcal{B}_I$, so that $\Delta(s^c_{\mathcal{E}})^F \in I$. To see that $\mathcal{B}_I$ satisfies (S1), fix $\lambda \in r(F)\Gamma \setminus \{r(F)\}$. Then $\Delta(s^c_{\mathcal{E}})^{F \cup \{\lambda\}} = \Delta(s^c_{\mathcal{E}})^F \big(p^c_{\mathcal{E}}(r(F)) - p^c_{\mathcal{E}}(\lambda)\big) \in I$. For (S2), fix $F \in \mathcal{B}_I$ and $\lambda \in r(F)\Gamma \setminus F\Gamma$. Then Lemma 3.7 gives

$$\Delta(s^c_{\mathcal{E}})^{\mathrm{Ext}(\lambda; F)} = s^c_{\mathcal{E}}(\lambda)^* \Delta(s^c_{\mathcal{E}})^F s^c_{\mathcal{E}}(\lambda) \in I.$$

For (S3), suppose $\lambda, \lambda\lambda' \in F$ with $\lambda' \neq s(\lambda)$. Then $p^c_{\mathcal{E}}(r(\lambda)) - p^c_{\mathcal{E}}(\lambda\lambda') \geq p^c_{\mathcal{E}}(r(F)) - p^c_{\mathcal{E}}(\lambda) \geq \Delta(s^c_{\mathcal{E}})^{F \setminus \lambda\lambda'}$, so $\Delta(s^c_{\mathcal{E}})^{F \setminus \lambda\lambda'} = \Delta(s^c_{\mathcal{E}})^{F \setminus \lambda\lambda'} (p^c_{\mathcal{E}}(r(F)) - p^c_{\mathcal{E}}(\lambda\lambda')) = \Delta(s^c_{\mathcal{E}})^F \in I$. For (S4), suppose that $\lambda \in F$ and $G \in s(\lambda)\mathcal{B}_I$. Let $F' := F \setminus \{\lambda\} \cup \lambda G$. For $\alpha \in G$, both $p^c_{\mathcal{E}}(r(F))$ and $p^c_{\mathcal{E}}(\lambda)$ dominate $(p^c_{\mathcal{E}}(\lambda) - p^c_{\mathcal{E}}(\lambda\alpha))$, giving $(p^c_{\mathcal{E}}(r(F)) - p^c_{\mathcal{E}}(\lambda))(p^c_{\mathcal{E}}(\lambda) - p^c_{\mathcal{E}}(\lambda\alpha)) = 0$. Hence

$$\begin{aligned}
&\prod_{\alpha \in G} (p^c_{\mathcal{E}}(r(F)) - p^c_{\mathcal{E}}(\lambda\alpha)) \\
&\qquad = \prod_{\alpha \in G} \Big(\big(p^c_{\mathcal{E}}(r(F)) - p^c_{\mathcal{E}}(\lambda)\big) + \big(p^c_{\mathcal{E}}(\lambda) - p^c_{\mathcal{E}}(\lambda\alpha)\big) \Big) \\
&\qquad = \sum_{H \subseteq G} \Big(p^c_{\mathcal{E}}(r(F)) - p^c_{\mathcal{E}}(\lambda)\Big)^{|H|} \prod_{\alpha \in G \setminus H} \Big(p^c_{\mathcal{E}}(\lambda) - p^c_{\mathcal{E}}(\lambda\alpha)\Big) \\
&\qquad = (p^c_{\mathcal{E}}(r(F)) - p^c_{\mathcal{E}}(\lambda)) + \prod_{\alpha \in G} (p^c_{\mathcal{E}}(\lambda) - p^c_{\mathcal{E}}(\lambda\alpha)).
\end{aligned}$$

Hence

$$\Delta(s^c_{\mathcal{E}})^{F'} = \Delta(s^c_{\mathcal{E}})^{F \setminus \{\lambda\}} (p^c_{\mathcal{E}}(r(F)) - p^c_{\mathcal{E}}(\lambda)) + \Delta(s^c_{\mathcal{E}})^{F \setminus \{\lambda\}} \prod_{\alpha \in G} (p^c_{\mathcal{E}}(\lambda) - p^c_{\mathcal{E}}(\lambda\alpha)).$$

We have $\Delta(s^c_{\mathcal{E}})^{F \setminus \{\lambda\}} (p^c_{\mathcal{E}}(r(F)) - p^c_{\mathcal{E}}(\lambda)) = \Delta(s^c_{\mathcal{E}})^F \in I$, and a quick computation using that range projections commute shows that $\prod_{\alpha \in G} (p^c_{\mathcal{E}}(\lambda) - p^c_{\mathcal{E}}(\lambda\alpha)) = s^c_{\mathcal{E}}(\lambda) \Delta(s^c_{\mathcal{E}})^G s^c_{\mathcal{E}}(\lambda)^* \in I$. Hence $\Delta(s^c_{\mathcal{E}})^{F'} \in I$.

(3). Fix $F \in \mathcal{E}_{H_I}$, and choose $E \in \mathcal{E}$ with $r(E) \notin H$ such that $F = E \setminus EH$. Then $\Delta(s^c_{\mathcal{E}})^E = 0$. Let q_I be the quotient map $a \mapsto a + I$. For $\mu \in EH$, we have $q_I(p^c_{\mathcal{E}}(\mu)) = 0$, and so $q_I(p^c_{\mathcal{E}}(r(E)) - p^c_{\mathcal{E}}(\mu)) = q_I(p^c_{\mathcal{E}}(r(E))$. Hence

$$\begin{aligned} 0 = q_I\big(\Delta(s^c_{\mathcal{E}})^E\big) &= \prod_{\mu \in E} \big(q_I(p^c_{\mathcal{E}}(r(E)) - p^c_{\mathcal{E}}(\mu))\big) \\ &= \prod_{\mu \in F} \big(q_I(p^c_{\mathcal{E}}(r(E)) - p^c_{\mathcal{E}}(\mu))\big) \prod_{\mu \in EH} q_I(p^c_{\mathcal{E}}(r(E)) = q_I(\Delta(s^c_{\mathcal{E}})^F). \end{aligned}$$

So $\Delta(s^c_{\mathcal{E}})^F \in I$ and hence $F \in \mathcal{B}_I$. □

Suppose that Λ is a finitely aligned k-graph, that $c \in \underline{Z}^2(\Lambda, \mathbb{T})$ and that $\mathcal{E} \subseteq \mathrm{FE}(\Lambda)$ is satiated. Suppose that $H \subseteq \Lambda^0$ is hereditary and $\mathcal{E}$-saturated and that $\mathcal{B} \subseteq \mathrm{FE}(\Lambda \setminus \Lambda H)$ is satiated and contains $\mathcal{E}_H$. We write $I^c_{H,\mathcal{B}}$ for the ideal of $C^*(\Lambda, c; \mathcal{E})$ generated by $\{s^c_{\mathcal{E}}(v) : v \in H\} \cup \{\Delta(s^c_{\mathcal{E}})^E : E \in \mathcal{B}\}$.
Observe that the cocycle c restricts to a cocycle, which we also denote c, on the subgraph $\Lambda \setminus \Lambda H$.

Theorem 4.4. *Suppose that Λ is a finitely aligned k-graph, that $c \in \underline{Z}^2(\Lambda, \mathbb{T})$ and that $\mathcal{E} \subseteq \mathrm{FE}(\Lambda)$ is satiated. Suppose that $H \subseteq \Lambda^0$ is hereditary and $\mathcal{E}$-saturated and that $\mathcal{B} \subseteq \mathrm{FE}(\Lambda \setminus \Lambda H)$ is satiated and contains $\mathcal{E}_H$. There is a homomorphism $\pi_{H,\mathcal{B}} : C^*(\Lambda, c; \mathcal{E}) \to C^*(\Lambda \setminus \Lambda H, c; \mathcal{B})$ such that*

$$\pi_{H,\mathcal{B}}(s^c_{\mathcal{E}}(\lambda)) = \begin{cases} s^c_{\mathcal{B}}(\lambda) & \text{if } s(\lambda) \notin H \\ 0 & \text{if } s(\lambda) \in H. \end{cases} \tag{4.1}$$

We have $\ker(\pi_{H,\mathcal{B}}) = I_{H,\mathcal{B}}$, and $H = H_{I_{H,\mathcal{B}}}$ and $\mathcal{B} = \mathcal{B}_{I_{H,\mathcal{B}}}$.

Proof. Routine calculations show that the formula for $\pi_{H,\mathcal{B}}$ defines a Toeplitz-Cuntz-Krieger (Λ, c)-family in $C^*(\Lambda \setminus \Lambda H, c; \mathcal{B})$. That $\mathcal{E}_H \subseteq \mathcal{B}$ ensures that this family also satisfies (CK). Hence the universal property of $C^*(\Lambda, c; \mathcal{E})$ gives a homomorphism $\pi_{H,\mathcal{B}}$ satisfying (4.1), which is surjective because its image contains the generators of $C^*(\Lambda \setminus \Lambda H, c; \mathcal{B})$.
To see that $\ker(\pi_{H,\mathcal{B}}) = I_{H,\mathcal{B}}$, observe that the generators of $I_{H,\mathcal{B}}$ belong to $\ker(\pi_{H,\mathcal{B}})$, and so $\pi_{H,\mathcal{B}}$ descends to a homomorphism $\tilde{\pi} : C^*(\Lambda, c; \mathcal{E})/I_{H,\mathcal{B}} \to C^*(\Lambda \setminus \Lambda H, c; \mathcal{B})$. It suffices to show that $\tilde{\pi}$ is injective, and we do this by constructing an inverse for $\tilde{\pi}$. Define elements $\{t_\lambda : \lambda \in \Lambda \setminus \Lambda H\}$ in $C^*(\Lambda, c; \mathcal{E})/I_{H,\mathcal{B}}$ by $t_\lambda = s^c_{\mathcal{E}}(\lambda) + I_{H,\mathcal{B}}$. Since the relations (TCK1)–(TCK3) for $\Lambda \setminus \Lambda H$ families hold in any Toeplitz-Cuntz-Krieger Λ family, t satisfies (TCK1)–(TCK3). For $\lambda, \mu \in \Lambda \setminus \Lambda H$, it is straightforward to check that $\mathrm{MCE}_{\Lambda \setminus \Lambda H}(\lambda, \mu) = \mathrm{MCE}_\Lambda(\lambda, \mu) \setminus \Lambda H$. If $\eta \in \mathrm{MCE}_\Lambda(\lambda, \mu) \cap \Lambda H$, then q_η is the zero element of $C^*(\Lambda, c; \mathcal{E})/I_{H,\mathcal{B}}$, and it follows that the t_λ satisfy (TCK4). They satisfy (CK) because $E \in \mathcal{B}$ implies $\Delta(s^c_{\mathcal{E}})^E \in I_{H,B}$ so that $\Delta(t)^E = 0$. Now the universal property of $C^*(\Lambda \setminus \Lambda H, c; \mathcal{B})$ provides a homomorphism $\pi_t : C^*(\Lambda \setminus \Lambda H, c; \mathcal{B}) \to C^*(\Lambda, c; \mathcal{E})/I_{H,\mathcal{B}}$. The map π_t is an inverse for $\tilde{\pi}$ on generators, and so $\tilde{\pi}$ is injective.

For the last assertion, observe that we have $H \subseteq H_{I_{H,\mathcal{B}}}$ and $\mathcal{B} \subseteq \mathcal{B}_{I_{H,\mathcal{B}}}$ by definition. For the reverse inclusions, observe that if $v \notin H$ then Proposition 3.9 applied to $C^*(\Lambda \setminus \Lambda H, c; \mathcal{B})$ implies that $\pi_{H,\mathcal{B}}(s^c_{\mathcal{E}}(v)) \neq 0$, so that $v \notin H_{I_{H,\mathcal{B}}}$. Similarly if $E \in \mathrm{FE}(\Lambda \setminus \Lambda H) \setminus \mathcal{B}$, then Proposition 3.9 implies that $\pi_{H,\mathcal{B}}(\Delta(s^c_{\mathcal{E}})^E) \neq 0$ and hence $E \notin \mathcal{B}_{I_{H,\mathcal{B}}}$. □

Corollary 4.5. *With the hypotheses of Theorem 4.4, the homomorphism $\pi_{H,\mathcal{B}}$ descends to an isomorphism $\phi_{H,\mathcal{B}} : C^*(\Lambda, c; \mathcal{E})/I_{H,\mathcal{B}} \to C^*(\Lambda \setminus \Lambda H, c; \mathcal{B})$ such that*

$$\phi_{H,\mathcal{B}}(s^c_{\mathcal{E}}(\lambda) + I_{H,\mathcal{B}}) = s^c_{\mathcal{B}}(\lambda) \text{ for all } \lambda \in \Lambda \setminus \Lambda H.$$

We are now ready to give a listing of the gauge-invariant ideals of $C^*(\Lambda, c; \mathcal{E})$. Let Λ be a finitely aligned k-graph and $\mathcal{E}$ a satiated subset of $\mathrm{FE}(\Lambda)$. We define $\mathrm{SH}_{\mathcal{E}} \times S$ to be the collection of all pairs $(H, \mathcal{B})$ such that $H \subseteq \Lambda^0$ is hereditary and $\mathcal{E}$-saturated, and $\mathcal{B} \subseteq \mathrm{FE}(\Lambda \setminus \Lambda H)$ is satiated and satisfies $\mathcal{E}_H \subseteq \mathcal{B}$.

Theorem 4.6. *Let Λ be a finitely aligned k-graph, $\mathcal{E}$ a satiated subset of $\mathrm{FE}(\Lambda)$, and $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. The map $(H, \mathcal{B}) \mapsto I_{H,\mathcal{B}}$ is a bijection from $\mathrm{SH}_{\mathcal{E}} \times S$ to the collection of gauge-invariant ideals of $C^*(\Lambda, c; \mathcal{E})$. We have $I_{H_1,\mathcal{B}_1} \subseteq I_{H_2,\mathcal{B}_2}$ if and only if both of the following hold: $H_1 \subseteq H_2$; and whenever $E \in \mathcal{B}_1$ and $r(E) \notin H_2$, we have $E \setminus EH_2 \in \mathcal{B}_2$.*

Proof. The final statement of Theorem 4.4 implies that $(H, \mathcal{B}) \mapsto I_{H,\mathcal{B}}$ is injective. To see that it is surjective, fix a gauge-invariant ideal I of $C^*(\Lambda, c; \mathcal{E})$. We must show that $I = I_{H_I,\mathcal{B}_I}$. We clearly have $I_{H_I,\mathcal{B}_I} \subseteq I$, and so the quotient map $q_I : C^*(\Lambda, c; \mathcal{E}) \to C^*(\Lambda, c; \mathcal{E})/I$ determines a homomorphism $\tilde{q}_I : C^*(\Lambda, c; \mathcal{E})/I_{H_I,\mathcal{B}_I} \to C^*(\Lambda, c; \mathcal{E})/I$. We must show that $\tilde{q}_I$ is injective. Let $\phi_{H_I,\mathcal{B}_I} : C^*(\Lambda, c; \mathcal{E})/I_{H_I,\mathcal{B}_I} \to C^*(\Lambda \setminus \Lambda H_I, c; \mathcal{B}_I)$ be the isomorphism of Corollary 4.5, and let $\theta := \tilde{q}_I \circ \phi_{H_I,\mathcal{B}_I}$. Since I is gauge-invariant, the gauge action on $C^*(\Lambda, c; \mathcal{E})$ descends to an action β on $C^*(\Lambda, c; \mathcal{E})/I$ such that $\beta_z \circ \theta = \theta \circ \gamma_z$ for all z. For each $v \in (\Lambda \setminus \Lambda H_I)^0$ we have $s^c_{\mathcal{E}}(v) \notin I$ by definition of H_I, and so each $\theta(s^c_{\mathcal{B}_I}(v)) \neq 0$. Similarly, if $F \in \mathrm{FE}(\Lambda \setminus \Lambda H) \setminus \mathcal{B}_I$, then $\Delta(s^c_{\mathcal{E}})^F \notin I$ by definition of $\mathcal{B}_I$, and so $\theta(\Delta(s^c_{\mathcal{E}})^F) \neq 0$. So the gauge-invariant uniqueness theorem (Theorem 3.16) implies that θ is injective. Hence $\tilde{q}_I$ is injective.
The proof of the final statement is identical to that of [25, Theorem 6.2]. □

We are now ready to prove the $\supseteq$ containment from Proposition 3.10 as promised in Section 3.

Proof of $\supseteq$ in Proposition 3.10. Let

$$J := \{a \in \mathcal{T}C^*(\Lambda, c) : \lim_{\lambda \in S} \pi_T(a) h_\lambda = 0 \text{ for every } \overline{\mathcal{E}}\text{-compatible filter } S\}.$$

This J is clearly a linear subspace of $\mathcal{T}C^*(\Lambda)$, and an $\varepsilon/3$-argument shows that it is norm-closed. If S is an $\overline{\mathcal{E}}$-compatible filter and $s(\lambda) = r(S)$, then λS is a cofinal subset of the $\overline{\mathcal{E}}$-compatible filter $\ell_\lambda(S)$. So if $a \in J$, then $\lim_{\mu \in S} \pi_T(a s_\lambda) h_\mu = \lim_{\nu \in \lambda S} \pi_T(a) h_\nu = 0$, giving $aT_\lambda \in J$. Similarly, since

$\ell^*_\lambda(S)$ is an $\mathcal{E}$-compatible filter whenever S is, for $a \in J$ and $\lambda \in \Lambda$ we have $\lim_{\mu\in S} \pi_T(as^*_\lambda)h_\mu = \lim_{\nu\in\ell^*_\lambda(S)} \pi_T(a)h_\nu = 0$, so that $as^*_\lambda \in J$. Clearly $T_\lambda a$ and $T^*_\lambda a$ belong to J. So J is an ideal of $\mathcal{T}C^*(\Lambda)$. Proposition 3.10 implies that $\Delta(s^c_{\mathcal{T}})^E \in J$ for all $E \in \mathcal{E}$. Lemma 3.11 implies that $s^c_{\mathcal{T}}(v) \notin \pi_T(J)$ for all $v \in \Lambda^0$ and that $\Delta(s^c_{\mathcal{T}})^E \notin J$ for $E \in \mathrm{FE}(\Lambda) \setminus \mathcal{E}$. So $H_J = \emptyset = H_{J_\mathcal{E}}$, and $\mathcal{B}_J = \mathcal{E} = \mathcal{B}_{J_\mathcal{E}}$. So the result will follow from Theorem 4.6 once we establish that J is gauge-invariant.

Fix $a \in J$ and $z \in \mathbb{T}^k$; we must show that $\gamma_z(a) \in J$. There is a unitary U_z in $\mathcal{B}(\ell^2(\Lambda))$ given by $U_z(h_\lambda) = z^{d(\lambda)}h_\lambda$. We have $U_z T_\lambda U^*_z = z^{d(\lambda)}T_\lambda$ for all z, λ, and so $\mathrm{Ad}\, U_z \circ \pi_T = \pi_T \circ \gamma_z$. Fix an $\overline{\mathcal{E}}$-compatible filter S. For $\lambda \in S$, we have

$$\|\pi(\gamma_z(a))h_\lambda\| = \|U_z\pi(a)U^*_z h_\lambda\| \le \|U_z\|\|\pi(a)\overline{z}^{d(\lambda)}h_\lambda\| = \|\pi(a)h_\lambda\|.$$

So $\lim_{\lambda\in S} \|\pi(\gamma_z(a))h_\lambda\| \le \lim_{\lambda\in S} \|\pi(a)h_\lambda\| = 0$, giving $\gamma_z(a) \in J$. □

5 Nuclearity and the Universal Coefficient Theorem

We show that each $C^*(\Lambda, c; \mathcal{E})$ is nuclear and satisfies the Universal Coefficient Theorem. Given a set X we write $\mathcal{K}_X$ for the C^*-algebra generated by nonzero matrix units $\{\theta_{x,y} : x, y \in X\}$.

Lemma 5.1. *Let Λ be a finitely aligned k-graph and let $\mathcal{E}$ be a satiated subset of $\mathrm{FE}(\Lambda)$. Suppose that $b : \Lambda^0 \to \mathbb{Z}^k$ satisfies $d(\lambda) = b(s(\lambda)) - b(r(\lambda))$ for all $\lambda \in \Lambda$. Let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$ and let t be a relative Cuntz-Krieger $(\Lambda, c; \mathcal{E})$-family with each $t_v \neq 0$.*

1. *For $n \in \mathbb{N}^k$ there is an isomorphism $\overline{\mathrm{span}}\{t_\mu t^*_\nu : b(s(\mu)) = b(s(\nu)) = n\} \cong \bigoplus_{b(v)=n} \mathcal{K}_{\Lambda v}$ which carries each $t_\mu t^*_\nu$ to $\theta_{\mu,\nu}$.*
2. *If $s(\mu) = s(\nu)$ and $s(\eta) = s(\zeta)$, then $t_\mu t^*_\nu t_\eta t^*_\zeta \in \overline{\mathrm{span}}\{t_\alpha t^*_\beta : b(s(\alpha)) = b(s(\beta)) = b(s(\mu)) \vee b(s(\nu))\}$.*
3. *If $N \subseteq \mathbb{N}^k$ is finite and $m, n \in N$ implies $m \vee n \in N$, then $B(t)_N := \overline{\mathrm{span}}\{t_\mu t^*_\nu : b(s(\mu)) = b(s(\nu)) \in N\}$ is an AF algebra.*

Proof. (1). We just have to check that the $t_\mu t^*_\nu$ are matrix units. So suppose that $s(\mu) = s(\nu) = v$ and $s(\eta) = s(\zeta) = w$ and $b(v) = b(w) = n$. Then

$$t_\mu t^*_\nu t_\eta t^*_\zeta = \sum_{\nu\alpha=\eta\zeta\in\mathrm{MCE}(\nu,\eta)} c(\mu,\alpha)\overline{c(\nu,\alpha)}c(\zeta,\beta)\overline{c(\eta,\beta)}t_{\mu\alpha}t^*_{\zeta\beta}.$$

If $r(\nu) \neq r(\eta)$, then $\mathrm{MCE}(\nu, \eta) = \emptyset$ and so $t_\mu t^*_\nu t_\eta t^*_\zeta = 0$. If $r(\nu) = r(\eta)$, then we have $d(\nu) = b(s(\nu)) - b(r(\nu)) = n - b(r(\nu))$ and similarly $d(\eta) = n - b(r(\eta))$ so that $d(\nu) = d(\eta)$, giving $t_\mu t^*_\nu t_\eta t^*_\zeta = \delta_{\nu,\eta} t_\mu t^*_\nu t_\nu t^*_\zeta = \delta_{\nu,\eta} t_\mu t^*_\zeta$.

(2). Suppose that $b(s(\nu)) = m$ and $b(s(\eta)) = n$, and that $\lambda \in \mathrm{MCE}(\nu, \eta)$. Since $\mathrm{MCE}(\nu, \eta) \neq \emptyset$, we have $r(\nu) = r(\eta)$, and in particular $b(r(\nu)) = b(r(\eta)) = p$, say. Thus $m - d(\nu) = p = n - d(\eta)$. We have $d(\lambda) = d(\mu) \vee d(\nu) = (m \vee n) - p$

and so $b(s(\lambda)) = b(r(\lambda)) + (d(\mu) \vee d(\nu)) = m \vee n$. So the result follows from Lemma 3.2.
(3). The proof is by induction on $|N|$. For $|N| = 1$, the result follows from part (1). Fix N with $|N| \geq 2$, and suppose as an inductive hypothesis that $B(t)_M$ is AF whenever $|M| < |N|$. Pick a minimal $n \in N$ and observe that $M := N \setminus \{n\}$ is closed under $\vee$, so that B_M is AF. Part 2 implies that B_M is an ideal of B_N and B_N/B_M is a quotient of $B_{\{n\}}$ and therefore is AF. Since extensions of AF algebras by AF algebras are AF, the result follows. □

Recall that if Λ is a finitely aligned k-graph, then $\Lambda \times_d \mathbb{Z}^k$ is the skew-product k-graph which is equal as a set to $\Lambda \times \mathbb{Z}^k$ and has structure maps $r(\lambda, n) = (r(\lambda), n)$, $s(\lambda, n) = (s(\lambda), n + d(\lambda))$, $(\lambda, n)(\mu, n + d(\lambda)) = (\lambda\mu, n)$, and $d(\lambda, n) = d(\lambda)$. For $c \in \underline{Z}^2(\Lambda, \mathbb{T})$, the map $c \times 1 : \big((\lambda, n), (\mu, n + d(\lambda))\big) \mapsto c(\lambda, \mu)$ belongs to $\underline{Z}^2(\Lambda \times_d \mathbb{Z}^k, \mathbb{T})$.

Corollary 5.2. *Let Λ be a finitely aligned k-graph, $\mathcal{E}$ a satiated subset of* $\mathrm{FE}(\Lambda)$*, and $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Then $C^*(\Lambda, c; \mathcal{E})$ is Morita equivalent to the crossed-product of an AF algebra by $\mathbb{Z}^k$. In particular, it belongs to the bootstrap class $\mathcal{N}$ of [24], and so is nuclear and satisfies the UCT.*

Proof. Let $\Gamma := \Lambda \times_d \mathbb{Z}^k$, let $c' := c \times 1$ and let $\mathcal{F} := \mathcal{E} \times \mathbb{Z}^k$. The argument of [25, Lemma 8.3] shows that $\mathcal{F} \subseteq \mathrm{FE}(\Gamma)$ and that the crossed-product of $C^*(\Lambda, c; \mathcal{E})$ by the gauge action is isomorphic to $C^*(\Gamma, c'; \mathcal{F})$. Since the subalgebras B_N of $C^*(\Gamma, c'; \mathcal{F})$ described in Lemma 5.1 are AF algebras and satisfy $C^*(\Gamma, c'; \mathcal{F}) = \overline{\bigcup_N B_N}$, and since the class of AF algebras is closed under direct limits, $C^*(\Gamma, c'; \mathcal{F})$ is AF. Now Takai duality implies that $C^*(\Lambda, c; \mathcal{E})$ is Morita equivalent to a crossed product of an AF algebra by $\mathbb{Z}^k$, and the result follows. □

6 K-theory

In this section we follow the program of [17] to show that if c has the form $c(\mu, \nu) = e^{i\omega(\mu,\nu)}$ for some $\omega \in \underline{Z}^2(\Lambda, \mathbb{R})$, then the K-theory of $C^*(\Lambda, c; \mathcal{E})$ is isomorphic to that of $C^*(\Lambda; \mathcal{E})$.

Theorem 6.1. *Let Λ be a finitely aligned k-graph and suppose that $\mathcal{E} \subseteq \mathrm{FE}(\Lambda)$ is satiated. Suppose that $\omega \in \underline{Z}^2(\Lambda, \mathbb{R})$, and define $c \in \underline{Z}^2(\Lambda, \mathbb{T})$ by $c(\mu, \nu) = e^{i\omega(\mu,\nu)}$. Then $C^*(\Lambda, c; \mathcal{E})$ is unital if and only if Λ^0 is finite. There is an isomorphism: $K_*(C^*(\Lambda, c; \mathcal{E}) \cong K_*(C^*(\Lambda; \mathcal{E}))$ taking $[s^c_{\mathcal{E}}(v)]$ to $[s_{\mathcal{E}}(v)]$ for each $v \in \Lambda^0$, and taking $[1_{C^*(\Lambda,c;\mathcal{E})}]$ to $[1_{C^*(\Lambda;\mathcal{E})}]$ if Λ^0 is finite.*

The proof of Theorem 6.1 appears at the end of the section; we have to do some preliminary work first.
Following [17, Definition 2.1], given a finitely aligned k-graph Λ, a locally compact abelian group A and a cocycle $\omega \in \underline{Z}^2(\Lambda, A)$, a *Toeplitz c-representation of* (Λ, A) on a C^*-algebra B consists of a map $\phi : \Lambda \to \mathcal{M}(B)$ and a homomorphism $\pi : C^*(A) \to \mathcal{M}(B)$ such that $\phi(\lambda)\pi(f) \in B$ for all $\lambda \in \Lambda$ and $f \in C^*(A)$, and such that

(R1) $\pi(f)\phi(\lambda) = \phi(\lambda)\pi(f)$ for all $\lambda \in \Lambda$ and $f \in C^*(A)$;

(R2) $\{\phi(v) : v \in \Lambda^0\}$ is a set of mutually orthogonal projections and $\sum_{v\in\Lambda^0} \phi(v) \to 1$ strictly in $\mathcal{M}(B)$;

(R3) $\phi(\lambda)\phi(\mu) = \pi(\omega(\lambda, \mu))\phi(\lambda\mu)$ whenever $s(\lambda) = r(\mu)$; and

(R4) the $\phi(\lambda)$ satisfy (TCK3) and (TCK4).

If the $\phi(\lambda)$ satisfy relation (CK) with respect to a satiated subset $\mathcal{E}$ of $\mathrm{FE}(\Lambda)$, then (ϕ, π) is an *$\mathcal{E}$-relative c-representation* of (Λ, A).

Calculations identical to those of Lemma 3.2 show that given a Toeplitz c-representation of (Λ, A), the C^*-algebra $C^*(\phi, \pi) := C^*\{\phi(\lambda)\pi(f) : f \in C^*(A), \lambda \in \Lambda\}$ is spanned by $\{\phi(\mu)\pi(f)\phi(\nu)^* : s(\mu) = s(\nu), f \in C^*(A)\}$, and that

$$\phi(\nu)^*\phi(\eta) = \sum_{\nu\alpha=\eta\beta\in\mathrm{MCE}(\nu,\eta)} \phi(\alpha)\pi(\omega(\eta, \beta))\pi(\omega(\nu, \alpha))^*\phi(\beta)^*,$$

where we are identifying elements of A with the corresponding multiplier unitaries of $C^*(A)$. There is a universal C^*-algebra $C^*(\Lambda, A, \omega; \mathcal{E})$ for $\mathcal{E}$-relative c-representations of (Λ, A), and we denote the universal $\mathcal{E}$-relative representation by $(\iota_\Lambda^{\mathcal{E},\omega}, \iota_A^{\mathcal{E},\omega})$. The argument of [17, Lemma 2.4] shows that $C^*(\Lambda, A, \omega; \mathcal{E})$ is canonically isomorphic to $C^*(\Lambda, A, \omega'; \mathcal{E})$ if ω and ω' are cohomologous. Let $\widehat{A}$ denote the Pontryagin dual of A. As in Proposition 2.5 of [17], the algebra $C^*(\Lambda, A, \omega; \mathcal{E})$ is a $C_0(\widehat{A})$-algebra with respect to the inclusion $i_A^{\mathcal{E},\omega} : C^*(A) \cong C_0(\widehat{A}) \hookrightarrow Z\mathcal{M}(C^*(\Lambda, A, \omega; \mathcal{E}))$, and for each $\chi \in \widehat{A}$ there is a homomorphism $\pi_\chi : C^*(\Lambda, A, \omega; \mathcal{E}) \to C^*(\Lambda, \chi \circ \omega; \mathcal{E})$ satisfying $\pi_\chi(\iota_\Lambda^{\mathcal{E},\omega}(\lambda)\iota_A^{\mathcal{E},\omega}(f)) = f(\chi)s_{\mathcal{E}}^c(\lambda)$. Moreover, π_χ descends to an isomorphism of the fibre $C^*(\Lambda, A, \omega; \mathcal{E})_\chi$ with $C^*(\Lambda, \chi \circ \omega; \mathcal{E})$.

When $\omega \equiv 1$, the universal properties of $C^*(\Lambda, A, 1, \mathcal{E})$ and $C^*(\Lambda, 1; \mathcal{E}) \otimes C^*(A)$ give an isomorphism $C^*(\Lambda, A, 1; \mathcal{E}) \cong C^*(\Lambda, 1; \mathcal{E}) \otimes C^*(A)$.

We now consider a finitely aligned k-graph Γ endowed with a map $b : \Gamma^0 \to \mathbb{Z}^k$ such that $d(\lambda) = b(s(\lambda)) - b(r(\lambda))$ for all $\lambda \in \Gamma$. Our application is when Γ is the skew-product $\Lambda \times_d \mathbb{Z}^k$ as in the preceding section, but it will keep our notation simpler to deal with the general situation.

The proof of Lemma 8.2 of [25] shows that given a finite subset E of Γ, there is a minimal finite $\tilde{E} \subseteq \Gamma$ such that $E \subseteq \tilde{E}$ and whenever $\mu, \nu, \eta, \zeta \in \tilde{E}$ with $s(\mu) = s(\nu)$, $s(\eta) = s(\zeta)$ and $\nu\alpha = \eta\beta \in \mathrm{MCE}(\nu, \eta)$, we have $\mu\alpha, \zeta\beta \in \tilde{E}$. (This set plays a similar role to the set ΠE used to examine the core in Section 3, and is constructed in a similar way.) The set $\tilde{E}$ satisfies $\tilde{E} = \vee\tilde{E}$ and has the property that $T(\tilde{E}; \mu) = T(\tilde{E}; \nu)$ whenever $\mu, \nu \in \tilde{E}$ and $s(\mu) = s(\nu)$.

Let $\omega \in \underline{Z}^2(\Gamma, A)$ and let (ϕ, π) be a Toeplitz ω-representation of (Γ, A). For $v \in \Lambda^0$ and a finite $E \subseteq v\Gamma$, let $\Delta(\phi)^E := \prod_{\lambda\in E} \phi(v) - \phi(\lambda)\phi(\lambda)^*$. Since the $\pi(\omega(\mu, \nu))$ are unitaries and each $\pi(\omega(r(\mu), \mu)) = 1_{\mathcal{M}(B)}$, the calculations of Lemma 3.7 show that

$$\Delta(\phi)^E\phi(\mu) = \phi(\mu)\Delta(\phi)^{\mathrm{Ext}(\mu;E)}. \tag{6.1}$$

The $\phi(\mu)\phi(\mu)^*$ satisfy the same commutation relations as the $t_\mu t_\mu^*$ in a Toeplitz-Cuntz-Krieger family. So if $E = \tilde{E}$, then since $\tilde{E} = \vee\tilde{E}$ the argument of [23, Corollary 3.7] gives

$$\phi(\lambda)\phi(\lambda)^* = \sum_{\lambda\lambda' \in E} \Delta(\phi)^{T(E;\lambda\lambda')}. \tag{6.2}$$

The argument of Lemma 3.12 shows that $\Theta(\phi)^E_{\mu,\nu} := \phi(\mu)\Delta(\phi)^{T(E;\mu)}\phi(\nu)^*$ defines matrix units, and that for $\mu, \nu \in E$ with $s(\mu) = s(\nu)$, we have

$$\phi(\mu)\phi(\nu)^* = \sum_{\mu\alpha \in E} \pi(\omega(\mu,\alpha))\pi(\omega(\nu,\alpha))^*\Theta(\phi)^E_{\mu\alpha,\nu\alpha}. \tag{6.3}$$

LEMMA 6.2. *Let Γ be a finitely aligned k-graph and suppose that $b : \Lambda^0 \to \mathbb{Z}^k$ satisfies $d(\lambda) = b(s(\lambda)) - b(r(\lambda))$ for all λ. Suppose that $\mathcal{F} \subseteq \mathrm{FE}(\Gamma)$ is satiated. Let A be a locally compact abelian group and consider $\omega \in \underline{Z}^2(\Gamma, A)$. If $E = \tilde{E} \subseteq \Gamma$ then*

$$\begin{aligned} M_E^{\mathcal{F},\omega} &:= \operatorname{span}\{\iota_\Gamma^{\mathcal{F},\omega}(\mu)\iota_A^{\mathcal{F},\omega}(f)\iota_\Gamma^{\mathcal{F},\omega}(\nu)^* : \mu,\nu \in E, f \in C^*(A)\} \\ &= \operatorname{span}\{\Theta(\iota_\Gamma^{\mathcal{F},\omega})^E_{\mu,\nu}\iota_A^{\mathcal{F},\omega}(f) : \mu,\nu \in E, f \in C^*(A)\}, \end{aligned}$$

and there is an isomorphism of $M_E^{\mathcal{F},\omega}$ onto $\bigoplus_{v \in s(E)} M_{Ev}(\mathbb{C}) \otimes C^*(A)$ that carries each spanning element $\Theta(\iota_\Gamma^{\mathcal{F},\omega})^E_{\mu,\nu}\iota_A^{\mathcal{F},\omega}(f)$ to $\theta_{\mu,\nu} \otimes f$.

Proof. Equation (6.3) establishes the displayed equation. We saw above that the $\Theta(\iota_\Gamma^{\mathcal{F},\omega})^E_{\mu,\nu}$ are matrix units, and they commute with the $\iota_A^{\mathcal{F},\omega}(f)$ because the range of $\iota_A^{\mathcal{F},\omega}$ is central. Now we follow the argument of Lemma 4.1 of [17]: The universal property of $\bigoplus_{v \in s(E)} M_{Ev}(\mathbb{C}) \otimes C^*(A)$ gives a surjection $\psi : \bigoplus_{v \in s(E)} M_{Ev}(\mathbb{C}) \otimes C^*(A) \to M_E^{\mathcal{F},\omega}$ such that $\psi(\theta_{\mu,\nu} \otimes f) = \Theta(\iota_\Gamma^{\mathcal{F},\omega})^E_{\mu,\nu}\iota_A^{\mathcal{F},\omega}(f)$. For each $\chi \in \widehat{A}$, and each $f \in \widehat{A}$ such that $f(\chi) = 1$, the canonical homomorphism $\pi_\chi : C^*(\Gamma, A, \omega; \mathcal{F}) \to C^*(\Gamma, \chi \circ \omega; \mathcal{E})$ carries the $\Theta(\iota_\Gamma^{\mathcal{E},\omega})^E_{\mu,\nu}\iota_A^{\mathcal{E},\omega}(f)$ to the matrix units $\theta(s_{\mathcal{E}}^{\chi\circ\omega})^E_{\mu,\nu}$, and Lemma 3.12 shows that any given $\theta(s_{\mathcal{E}}^{\chi\circ\omega})^E_{\mu,\nu}$ is nonzero if and only if $\theta(s_{\mathcal{E}})^E_{\mu,\nu}$ is nonzero; so π_χ determines an isomorphism $\big(M_E^{\mathcal{E},\omega}\big)_\chi \cong \bigoplus_{v \in s(E)} M_{Ev}(\mathbb{C})$. So ψ descends to an isomorphism of each fibre in the trivial bundle $\bigoplus_{v \in s(E)} M_{Ev}(\mathbb{C}) \otimes C^*(A)$, and so is isometric by [29, Proposition C.10(c)]. □

LEMMA 6.3. *Let Γ be a finitely aligned k-graph. Let A be a locally compact abelian group and consider $\omega \in \underline{Z}^2(\Lambda, A)$. Let (ϕ, π) be a Toeplitz ω-representation of (Γ, A). Suppose that $E \subseteq F$ are finite subsets of Γ satisfying $E = \tilde{E}$ and $F = \tilde{F}$. For each $\mu \in F$ there is a unique maximal $\iota_\mu \in E$ such that $\mu \in \iota_\mu\Gamma$, and we have*

$$\Theta(\phi)^E_{\mu,\nu} = \sum_{\mu\alpha \in F, \iota_{\mu\alpha} = \mu} \pi(\omega(\mu,\alpha))\pi(\omega(\nu,\alpha))^*\Theta(\phi)^F_{\mu\alpha,\nu\alpha}.$$

Proof. Let $N = \bigvee\{d(\lambda) : \lambda \in E, \mu \in \lambda\Gamma\}$. Then $N \le d(\mu)$ and since $E = \vee E$, factorising $\mu = \iota_\mu \mu'$ with $d(\iota_\mu) = N$ gives the desired ι_μ. We claim that for $\mu \in E$ and $\lambda \in F$ we have $\Theta(\phi)^E_{\mu,\mu}\Theta(\phi)^F_{\lambda,\lambda} = \delta_{\iota_\lambda,\mu}\Theta(\phi)^F_{\lambda,\lambda}$. First suppose that $\iota_\lambda = \mu$. We have

$$\begin{aligned}\Theta(\phi)^E_{\mu,\mu}&\Theta(\phi)^F_{\lambda,\lambda}\\ &= \phi(\mu)\phi(\mu)^* \prod_{\mu' \in T(E;\mu)} \big(\phi(\mu)\phi(\mu)^* - \phi(\mu\mu')\phi(\mu\mu')^*\big)\phi(\lambda)\phi(\lambda)^*\Theta(\phi)^F_{\lambda,\lambda}.\end{aligned}$$

For $\mu' \in T(E;\mu)$ the maximality of $\iota_\lambda = \mu$ ensures that $\lambda \notin \mu\mu'\Lambda$. Since $F = \vee F$ and contains E, we deduce that if $\mu\mu'\alpha = \lambda\beta \in \mathrm{MCE}(\mu\mu', \lambda)$, then $\phi(\lambda)\phi(\lambda)^* - \phi(\lambda\beta)\phi(\lambda\beta)^*$ is a factor in $\Theta(\phi)^F_{\lambda,\lambda}$, and so $\phi(\mu\mu')\phi(\mu\mu')^*\Theta(\phi)^F_{\lambda,\lambda} = 0$. So the preceding displayed equation collapses to

$$\Theta(\phi)^E_{\mu,\mu}\Theta(\phi)^F_{\lambda,\lambda} = \phi(\mu)\phi(\mu)^*\Theta(\phi)^F_{\lambda,\lambda} = \Theta(\phi)^F_{\lambda,\lambda}.$$

Now suppose that $\mu \neq \iota_\lambda$. Since ι_λ is the maximal initial segment of λ in E, we have two cases to consider: either $\iota_\lambda \in \mu\Lambda \setminus \{\mu\}$ or $\lambda \notin \mu\Lambda$. First suppose that $\iota_\lambda \in \mu\Lambda \setminus \{\mu\}$. Then $\phi(\mu)\phi(\mu)^* - \phi(\iota_\lambda)\phi(\iota_\lambda)^* \ge \Theta(\phi)^E_{\mu,\mu}$. Since $\big(\phi(\mu)\phi(\mu)^* - \phi(\iota_\lambda)\phi(\iota_\lambda)^*\big) \perp \phi(\lambda)\phi(\lambda)^* \ge \Theta(\phi)^F_{\lambda,\lambda}$ it follows that $\Theta(\phi)^E_{\mu,\mu}\Theta(\phi)^F_{\lambda,\lambda} = 0$. Now suppose that $\lambda \notin \mu\Lambda$. Then $\mu\alpha = \lambda\beta \in \mathrm{MCE}(\mu, \lambda)$ implies $\beta \in T(F;\lambda)$, and then the argument of the preceding paragraph gives $\phi(\mu)\phi(\mu)^*\Theta(\phi)^F_{\lambda,\lambda} = 0$. Hence $\Theta(\phi)^E_{\mu,\mu}\Theta(\phi)^F_{\lambda,\lambda} = 0$. This proves the claim.
Now fix $\mu, \nu \in E$. Equations (6.1) and (6.2) imply that

$$\Theta(\phi)^E_{\mu,\nu} = \sum_{\mu\alpha \in F} \Theta(\phi)^F_{\mu\alpha,\mu\alpha}\Theta(\phi)^E_{\mu,\mu}\phi(\mu)\phi(\nu)^*,$$

and the claim reduces this to

$$\sum_{\substack{\mu\alpha \in F\\ \iota_{\mu\alpha}=\mu}} \Theta(\phi)^F_{\mu\alpha,\mu\alpha}\phi(\mu)\phi(\nu)^* = \sum_{\substack{\mu\alpha \in F\\ \iota_{\mu\alpha}=\mu}} \Theta(\phi)^F_{\mu\alpha,\mu\alpha}\phi(\mu\alpha)\phi(\mu\alpha)^*\phi(\mu)\phi(\nu)^*.$$

A by-now familiar computation using the cocycle identity transforms this into

$$\sum_{\mu\alpha \in F, \iota_{\mu\alpha}=\mu} \pi(\omega(\mu,\alpha))\pi(\omega(\nu,\alpha))^*\Theta(\phi)^F_{\mu\alpha,\mu\alpha}\phi(\mu\alpha)\phi(\nu\alpha)^*,$$

and another application of (6.1) completes the proof. □

Now given $\Gamma, \mathcal{F}$ and ω as above and a finite subset $E = \tilde{E}$ of Γ, we define a map $\iota^E : \Gamma \to \Gamma$ as follows: if $\lambda \in E\Gamma$, then ι^E_λ is the maximal element of E such that $\lambda \in \iota^E_\lambda\Gamma$. For all other $\lambda \in \Gamma$ we set $\iota^E_\lambda = \lambda$. We define $\tau^E : \Gamma \to \Gamma$ by $\lambda = \iota_\lambda\tau_\lambda$ for all λ.

THEOREM 6.4. *Let Γ be a finitely aligned k-graph. Suppose that there is a map $b : \Gamma^0 \to \mathbb{Z}^k$ such that $d(\lambda) = b(s(\lambda)) - b(r(\lambda))$ for all λ and that $\mathcal{F} \subseteq \mathrm{FE}(\Gamma)$ is* *satiated. Let A be a locally compact abelian group and consider $\omega \in \underline{Z}^2(\Lambda, A)$. There is an isomorphism $C^*(\Lambda, A, \omega; \mathcal{F}) \cong C^*(\Lambda; \mathcal{F}) \otimes C^*(A)$ which carries $\iota_\Gamma^{\mathcal{F},\omega}(\lambda)\iota_A^{\mathcal{F},\omega}(f)\iota_\Gamma^{\mathcal{F},\omega}(\lambda)^*$ to $s_{\mathcal{F}}(\lambda)s_{\mathcal{F}}(\lambda)^* \otimes f$ for all $\lambda \in \Lambda$.*

Proof. Fix an increasing sequence $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ of subsets of Γ such that each $E_i = \tilde{E}_i$ and $\bigcup_i E_i = \Gamma$. Recursively define maps $\kappa_i : \{(\mu, \nu) \in \Gamma : s(\mu) = s(\nu)\} \to \mathcal{M}C^*(A)$ by $\kappa_0 \equiv 1_{\mathcal{M}C^*(A)}$ and

$$\kappa_{i+1}(\mu, \nu) = \kappa_i(\mu, \nu) - \omega(\iota_\mu^{E_i}, \tau_\mu^{E_i}) + \omega(\iota_\nu^{E_i}, \tau_\nu^{E_i}).$$

For each i, define a linear map $\psi_i : M_{E_i}^{\mathcal{F},\omega} \to M_{E_i}^{\mathcal{F}}$ by

$$\psi_i\big(\iota_A^{\mathcal{F},\omega}(f)\Theta(\iota_\Gamma^{\mathcal{F},\omega})_{\mu,\nu}^{E_i}\big) = \iota_A^{\mathcal{F},\omega}(\kappa_i(\mu, \nu))\iota_A^{\mathcal{F},\omega}(f)\Theta(\iota_\Gamma^{\mathcal{F}})_{\mu,\nu}^{E_i}.$$

An induction argument shows that each κ_i satisfies $\kappa_i(\nu, \mu) = -\kappa_i(\mu, \nu)$ and that $\kappa_i(\lambda, \mu) + \kappa_i(\mu, \nu) = \kappa_i(\lambda, \nu)$, and so ψ_i preserves adjoints and multiplication. Two applications of Lemma 6.2 show that ψ_i is an isomorphism. Using the definition of the κ_i and Lemma 6.3, we calculate:

$$\begin{aligned}
&\psi_i(\Theta(\iota_\Gamma^{\mathcal{F},\omega})_{\mu,\nu}) \\
&\quad = \iota_A^{\mathcal{F},\omega}(\kappa_i(\mu, \nu))\Theta(\iota_\Gamma^{\mathcal{F}})_{\mu,\nu}^{E_i} \\
&\quad = \sum_{\mu\alpha \in E_{i+1}, \iota_{\mu\alpha} = \mu} \iota_A^{\mathcal{F},\omega}(\kappa_i(\mu, \nu))\Theta(\iota_\Gamma^{\mathcal{F}})_{\mu\alpha,\nu\alpha}^{E_{i+1}} \\
&\quad = \sum_{\mu\alpha \in E_{i+1}, \iota_{\mu\alpha} = \mu} \iota_A^{\mathcal{F},\omega}(\kappa_{i+1}(\mu\alpha, \nu\alpha) + \omega(\mu, \alpha) - \omega(\nu, \alpha))\Theta(\iota_\Gamma^{\mathcal{F}})_{\mu\alpha,\nu\alpha}^{E_{i+1}} \\
&\quad = \sum_{\mu\alpha \in E_{i+1}, \iota_{\mu\alpha} = \mu} \psi_{i+1}\big(\iota_A^{\mathcal{F},\omega}(\omega(\mu, \alpha))\iota_A^{\mathcal{F},\omega}(\omega(\nu, \alpha))^*\Theta(\iota_\Gamma^{\mathcal{F},\omega})_{\mu\alpha,\nu\alpha}^{E_{i+1}}\big) \\
&\quad = \psi_{i+1}(\Theta(\iota_\Gamma^{\mathcal{F},\omega})_{\mu,\nu}^{E_i}).
\end{aligned}$$

Hence there is an isomorphism $\psi_\infty : C^*(\Gamma, A, \omega; \mathcal{F}) \to C^*(\Gamma, A; \mathcal{F})$ such that $\psi_\infty|_{M_{E_i}^{\mathcal{F},\omega}} = \psi_i$. The formula (6.2) gives $\psi_\infty\big(\iota_\Gamma^{\mathcal{F},\omega}(\lambda)\iota_A^{\mathcal{F},\omega}(f)\iota_\Gamma^{\mathcal{F},\omega}(\lambda)^*\big) = s_{\mathcal{F}}(\lambda)s_{\mathcal{F}}(\lambda)^* \otimes f$ for each λ and f. $\square$

If A is a $C_0(X)$-algebra in the sense that there is a nondegenerate homomorphism $i : C_0(X) \to \mathcal{ZM}(A)$, and if $I \subseteq X$ is closed, then we write $A|_I$ for the quotient of A by the ideal generated by $\{i(f) : f|_I = 0\}$.

COROLLARY 6.5. *Let Λ be a finitely aligned k-graph and suppose that $\mathcal{E} \subseteq \mathrm{FE}(\Lambda)$ is satiated. Suppose that $\omega \in \underline{Z}^2(\Lambda, \mathbb{R})$. There is an isomorphism $C^*(\Lambda \times_d \mathbb{Z}^k, A, \omega \times 1; \mathcal{E} \times \mathbb{Z}^k)|_{[0,1]} \cong C^*(\Lambda \times_d \mathbb{Z}^k; \mathcal{E} \times \mathbb{Z}^k) \otimes C([0, 1])$ that carries $\iota_{\Lambda \times_d \mathbb{Z}^k}^{\mathcal{E} \times \mathbb{Z}^k, \omega \times 1}(\lambda)\iota_A^{\mathcal{E} \times \mathbb{Z}^k, \omega \times 1}(f)\iota_{\Lambda \times_d \mathbb{Z}^k}^{\mathcal{E} \times \mathbb{Z}^k, \omega \times 1}(\lambda)^*$ to $s_{\mathcal{E} \times \mathbb{Z}^k}^{\chi \circ \omega}(\lambda)s_{\mathcal{E} \times \mathbb{Z}^k}^{\chi \circ \omega}(\lambda)^* \otimes f$ for all λ, f. In particular, evaluation at any character $\chi \in [0, 1] \subseteq \widehat{\mathbb{R}}$ determines an isomorphism $K_*(C^*(\Lambda \times_d \mathbb{Z}^k, A, \omega \times 1; \mathcal{E} \times \mathbb{Z}^k)|_{[0,1]}) \cong K_*(C^*(\Lambda \times_d \mathbb{Z}^k, \chi \circ \omega; \mathcal{E} \times \mathbb{Z}^k))$ which carries $\Big[\iota_{\Lambda \times_d \mathbb{Z}^k}^{\mathcal{E} \times \mathbb{Z}^k, \omega \times 1}(\lambda)\iota_{\Lambda \times_d \mathbb{Z}^k}^{\mathcal{E} \times \mathbb{Z}^k, \omega \times 1}(\lambda)^*\Big]$ to $\Big[s_{\mathcal{E} \times \mathbb{Z}^k}^{\chi \circ \omega}(\lambda)s_{\mathcal{E} \times \mathbb{Z}^k}^{\chi \circ \omega}(\lambda)^*\Big]$.*

Proof. Use Theorem 6.4 and the Künneth theorem (see [17, Corollary 4.3]). $\square$

We can now prove the main result of the section, Theorem 6.1

Proof of Theorem 6.1. We follow the proof of [17, Theorem 5.4]. As in Corollary 5.2, the argument of [25, Lemma 8.3] shows that $C^*(\Lambda \times_d \mathbb{Z}^k, \mathbb{R}, \omega \times 1; \mathcal{E} \times \mathbb{Z}^k)$ is isomorphic to the crossed product of $C^*(\Lambda, \mathbb{R}, \omega; \mathcal{E})$ by the gauge action and that the inclusion of $C^*(\Lambda, \mathbb{R}, \omega; \mathcal{E})$ into the crossed product takes the K_0 class of an $\iota_\Lambda^{\mathcal{E},\omega}(v)$ to the class of $\iota_{\Lambda\times_d\mathbb{Z}^k}^{\mathcal{E}\times\mathbb{Z}^k;\omega\times 1}((v,0))$.
The argument of [17, Lemma 5.2] goes through more or less verbatim (substitute "relative Cuntz-Krieger family" for "Cuntz-Krieger family" as necessary) to prove that translation in the $\mathbb{Z}^k$ coordinate on $\Lambda \times_d \mathbb{Z}^k$ induces the action $\hat{\gamma}$ of $\mathbb{Z}^k$ on $C^*(\Lambda \times_d \mathbb{Z}^k, \mathbb{R}, \omega; \mathcal{E})$ that is dual to the gauge-action, that the projection $P_0 = \sum_{v\in\Lambda^0} \iota_{\Lambda\times_d\mathbb{Z}^k}^{\mathcal{E}\times\mathbb{Z}^k;\omega\times 1}((v,0))$ is full in $C^*(\Lambda \times_d \mathbb{Z}^k, \mathbb{R}, \omega \times 1; \mathcal{E} \times \mathbb{Z}^k) \times_{\hat{\gamma}} \mathbb{Z}^k$ and that the corner it determines is isomorphic to $C^*(\Lambda, \mathbb{R}, \omega; \mathcal{E})$ via an isomorphism that takes the generating projection associated to $(v, 0) \in (\Lambda \times_d \mathbb{Z}^k)^0$ to the generating projection associated to $v \in \Lambda$.
Lemma 5.3 of [17] implies that the inclusion of $C^*(\mathbb{R})$ in the centre of $C^*(\Lambda \times_d \mathbb{Z}^k, \mathbb{R}, \omega \times 1; \mathcal{E} \times \mathbb{Z}^k) \times_{\hat{\gamma}} \mathbb{Z}^k$ makes it into a $C_0(\mathbb{R})$ algebra whose fibre over $\chi \in \widehat{\mathbb{R}}$ is $C^*(\Lambda \times_d \mathbb{Z}^k, \chi \circ (\omega \times 1); \mathcal{E} \times \mathbb{Z}^k) \times_{\hat{\gamma}\chi\circ\omega} \mathbb{Z}^k$. Since Corollary 6.5 implies that evaluation at each point of $[0, 1]$ induces an isomorphism in K-theory on $C^*(\Lambda \times_d \mathbb{Z}^k, \mathbb{R}, \omega \times 1; \mathcal{E} \times \mathbb{Z}^k)|_{[0,1]}$, Theorem 5.1 of [17] implies that the same is true of the crossed product; applying this at $t = 0$ and $t = 1$ gives the result. $\square$

7 Simplicity

In Corollary 3.18 we showed that if Λ is cofinal and aperiodic in the sense of [18], then each $C^*(\Lambda, c)$ is simple. In the untwisted setting, these conditions are also necessary (see [18, Theorem 3.4]), but the example of rotation algebras, discussed in the final paragraph of the introduction, shows that this is not to be expected in the twisted setting. In this section we give a sufficient condition for simplicity of twisted C^*-algebras associated to k-graphs that are not aperiodic.
A *bicharacter* of $\mathbb{Z}^k$ is a map $c : \mathbb{Z}^k \times \mathbb{Z}^k \to \mathbb{T}$ such that

$$c(m, n)c(m, n') = c(m, n + n') \qquad \text{and} \qquad c(m, n)c(m', n) = c(m + m', n).$$

This implies that $c(0, n) = c(m, 0) = 1$ and that $c(-m, n) = \overline{c(m, n)} = c(m, -n)$ for all m, n. A bicharacter is *skew-symmetric*[1] if it has the additional property that $c(n, m) = \overline{c(m, n)}$ for all m, n.
For $c \in Z^2(\mathbb{Z}^k, \mathbb{T})$, we write c^* for the cocycle $c^*(m, n) = \overline{c(n, m)}$ (note the reversal of variables; so $c^* \neq \overline{c}$). Proposition 3.2 of [19] shows that the product cc^*, given by $(cc^*)(m, n) = c(m, n)\overline{c(n, m)}$, is a skew-symmetric bicharacter, and moreover that the map $c \mapsto cc^*$ has kernel $B^2(\mathbb{Z}^k, \mathbb{T})$ and determines

[1]symplectic in the language of [19].

an isomorphism of $H^2(\mathbb{Z}^k, \mathbb{T})$ onto the group of skew-symmetric bicharacters of $\mathbb{Z}^k$. It follows immediately (or see [2]) that $H^2(\mathbb{Z}^k, \mathbb{T})$ is isomorphic to $\mathbb{T}^{k(k-1)/2}$, and that every class has a representative that is a bicharacter: given $c \in Z^2(\mathbb{Z}^k, \mathbb{T})$, form the skew-symmetric bicharacter cc^*, and let $\tilde{c}$ be the unique bicharacter of $\mathbb{Z}^k$ such that $\tilde{c}(e_i, e_j) = cc^*(e_i, e_j)$ if $i > j$ and $\tilde{c}(e_i, e_j) = 1$ if $i \leq j$. Then $\tilde{c}\tilde{c}^*(e_i, e_j) = cc^*(e_i, e_j)$ for all i, j, and since both are bicharacters, the two agree. Hence [19, Proposition 3.2] discussed above shows that c and $\tilde{c}$ represent the same class in $H^2(\mathbb{Z}^k, \mathbb{T})$.

A skew-symmetric bicharacter c is called *nondegenerate* if there is no nonzero $m \in \mathbb{Z}^k$ such that $c(m, n) = 1$ for all n. So if $c \in Z^2(\mathbb{Z}^k, \mathbb{T})$ is a bicharacter, then cc^* is nondegenerate if and only if there is no nonzero m such that $c(n, m) = c(m, n)$ for all n (equivalently, no nonzero m satisfies $c(m, e_i) = c(e_i, m)$ for all $i \in \{1, \cdots, k\}$).

Given $c \in Z^2(\mathbb{Z}^k, \mathbb{T})$, the noncommutative torus $A(c)$ is the universal algebra generated by unitaries $\{U_m : m \in \mathbb{Z}^k\}$ satisfying $U_m U_n = c(m, n) U_{m+n}$ for all $m, n \in \mathbb{Z}^k$. An argument like that of Proposition 3.5 shows that if c_1 and c_2 are cohomologous in $Z^2(\mathbb{Z}^k, \mathbb{T})$, then $A(c_1) \cong A(c_2)$. So the remarks in the preceding paragraph (see also Remark 1.2 of [20]) show that $A(c)$ is universal for unitaries $\{U_i : i \leq k\}$ satisfying $U_i U_j = cc^*(e_i, e_j) U_j U_i$. Theorem 3.7 of [27] combined with 1.8 of [7] (see [20, Theorem 1.9]) shows that the noncommutative torus $A(c)$ is simple if and only if cc^* is nondegenerate.

To state the main result of the section, observe that if Λ is row-finite and has no sources, then $v\Lambda^n \in \mathrm{FE}(\Lambda)$ for each $v \in \Lambda^0$ and $n \in \mathbb{N}^k \setminus \{0\}$. Hence a filter S is $\mathrm{FE}(\Lambda)$-compatible if and only if it is maximal in the sense that $S \cap \Lambda^n \neq \emptyset$ for all $n \in \mathbb{N}^k$. We call such filters *ultrafilters*. Recall that if S is an ultrafilter, and $s(\mu) = r(S)$, then $\ell_\mu(S)$ is the filter $\{\nu \in \lambda : \nu\Lambda \cap \mu S \neq \emptyset\}$, which is then also an ultrafilter.

Under the bijection of Remark 3.8, the ultrafilters of Λ correspond to the infinite paths used in [12], and if this bijection carries the infinite path x to the ultrafilter S, then for $\mu \in \Lambda r(x)$, it carries the infinite path μx to the ultrafilter $\ell_\mu(S)$.

As in [4], for μ, ν in a row-finite k-graph Λ with no sources, we write $\mu \sim \nu$ if $\mu x = \nu x$ for every infinite path x in $s(\mu)\Lambda^\infty$. Equivalently, $\mu \sim \nu$ if $\ell_\mu(S) = \ell_\nu(S)$ for every ultrafilter S such that $r(S) = s(\mu)$. We define $\mathrm{Per}(\Lambda)$ to be the subgroup of $\mathbb{Z}^k$ generated by $\{d(\mu) - d(\nu) : \mu \sim \nu\}$.

In this section, if $c \in Z^2(\mathbb{Z}^k, \mathbb{T})$ and Λ is a k-graph with degree functor d, then we abuse notation slightly and write $c \circ d$ for the cocycle $c \circ d(\lambda, \mu) := c(d(\lambda), d(\mu))$.

Theorem 7.1. *Let Λ be a row-finite k-graph with no sources, and take $c \in Z^2(\mathbb{Z}^k, \mathbb{T})$. Suppose that $(cc^*)|_{\mathrm{Per}(\Lambda)}$ is nondegenerate. Then $C^*(\Lambda, c \circ d)$ is simple if and only if Λ is cofinal.*

For our first couple of results, we continue to work in the generality of finitely aligned k-graphs. We begin by showing that cofinality of Λ is necessary for simplicity of $C^*(\Lambda, c)$.

Lemma 7.2. *Let Λ be a finitely aligned k-graph, and suppose that $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Suppose that Λ is not cofinal. Then $C^*(\Lambda, c)$ is not simple.*

Proof. By [18, Theorem 5.1] there exists $v \in \Lambda^0$ and an $\mathrm{FE}(\Lambda)$-compatible filter S such that $v\Lambda s(\lambda) = \emptyset$ for all $\lambda \in S$ (that is, $v\Lambda s(S) = \emptyset$). Consider

$$\mathcal{H}_S := \overline{\operatorname{span}}\{h_\mu : s(\mu) \in s(S)\} \subseteq \ell^2(\Lambda).$$

Let $\{T_\lambda : \lambda \in \Lambda\}$ be the twisted Toeplitz-Cuntz-Krieger family on $\ell^2(\Lambda)$ described just before Proposition 3.9, and let π_T be the associated representation of $\mathcal{T}C^*(\Lambda)$. For $\mu, \nu \in \Lambda$ and a basis element h_η of $\mathcal{H}_S$ we have

$$T_\mu T_\nu^* h_\eta = \begin{cases} c(\mu, \eta')\overline{c(\nu, \eta')} h_{\mu\eta'} & \text{if } \eta = \nu\eta' \\ 0 & \text{otherwise.} \end{cases}$$

Since $s(\mu\eta') = s(\eta) \in s(S)$, it follows that $\mathcal{H}_S$ is an invariant subspace of $\ell^2(\Lambda)$ for π_T. Hence restriction determines a representation $\pi_T^{\mathcal{H}_S}$ of $\mathcal{T}C^*(\Lambda, c)$ on $\mathcal{H}_S$. We have $T_v h_\eta = 0$ for all $\eta \in \Lambda s(S)$ because $v\Lambda s(S) = \emptyset$. Thus $s_{\mathcal{T}}^c(v) \in \ker(\pi_T^{\mathcal{H}_S})$.
Let $J = J_{\mathrm{FE}(\Lambda)}$ be the ideal of $\mathcal{T}C^*(\Lambda, c)$ generated by $\{\Delta(s_{\mathcal{T}}^c)^E : E \in \mathrm{FE}(\Lambda)\}$, so that $\mathcal{T}C^*(\Lambda, c)/J = C^*(\Lambda, c)$. Proposition 3.10 implies that $\lim_{\mu \in S} \pi_T(a) h_\mu = 0$ for all $a \in J$. Since $\|\pi_T(s_{\mathcal{T}}^c(r(S)))h_\mu\| = \|h_\mu\| = 1$ for all $\mu \in S$, we deduce that

$$\left\|\pi_T(s_{\mathcal{T}}^c(r(S)) - a)|_{\mathcal{H}_S}\right\| \geq 1 \text{ for all } a \in J. \tag{7.1}$$

Let $I := \pi_T^{\mathcal{H}_S}(J)$. Then I is an ideal of $\pi_T^{\mathcal{H}_S}(\mathcal{T}C^*(\Lambda, c))$. Let q_I be the quotient map from $\pi_T^{\mathcal{H}_S}(\mathcal{T}C^*(\Lambda, c))$ to $\pi_T^{\mathcal{H}_S}(\mathcal{T}C^*(\Lambda, c))/I$. Equation (7.1) implies that $\|q_I \circ \pi_T^{\mathcal{H}_S}(s_{\mathcal{T}}^c(r(S)))\| = 1$, and so $q_I \circ \pi_T^{\mathcal{H}_S}$ induces a nonzero homomorphism ρ of $C^*(\Lambda, c)$. The preceding paragraph shows that $q_I \circ \pi_T^{\mathcal{H}_S}(s_{\mathcal{T}}^c(v)) = 0$. Hence ρ is neither faithful nor trivial, and it follows that $C^*(\Lambda, c)$ is not simple. □

Proposition 7.3. *Let Λ be a finitely aligned k-graph, and let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. There is a faithful conditional expectation*

$$\Theta : C^*(\Lambda, c) \to D_\Lambda^c := \overline{\operatorname{span}}\{s^c(\lambda)s^c(\lambda)^* : \lambda \in \Lambda\}$$

given by $\Theta(s^c(\mu)s^c(\nu)^*) = \delta_{\mu,\nu} s^c(\mu)s^c(\mu)^*$.

Proof. Averaging over the gauge action γ gives a faithful conditional expectation $\Phi : C^*(\Lambda, c) \to C^*(\Lambda, c)^\gamma = \overline{\operatorname{span}}\{s^c(\mu)s^c(\nu)^* : d(\mu) = d(\nu)\}$ such that $\Phi(s^c(\mu)s^c(\nu)^*) = \delta_{d(\mu),d(\nu)} s^c(\mu)s^c(\nu)^*$. Lemma 3.12 shows that $C^*(\Lambda, c)^\gamma$ can be written as the closure of an increasing union of finite-dimensional subalgebras $M(s^c)_E$ with nested diagonal subalgebras whose union is dense in D_Λ^c. Equation (3.2) shows that if $\mu \neq \nu$ then $s^c(\mu)s^c(\nu)^*$ belongs to the span of the off-diagonal matrix units in $M(s^c)_E$ for sufficiently large E. So the canonical faithful expectation Ψ from the AF algebra $C^*(\Lambda, c)^\gamma$ onto its diagonal subalgebra satisfies $\Theta(s^c(\mu)s^c(\nu)^*) = \delta_{\mu,\nu} s^c(\mu)s^c(\mu)^*$. Now $\Theta = \Psi \circ \Phi$ is the desired expectation. □

We now restrict our attention to row-finite k-graphs with no sources. Recall that in this setting the Cuntz-Krieger relations for $C^*(\Lambda, c)$ imply that

$$\sum_{\lambda \in v\Lambda^n} s^c(\lambda)s^c(\lambda)^* = s^c(v) \text{ for every } v \in \Lambda^0 \text{ and } n \in \mathbb{N}^k. \tag{7.2}$$

It follows from this and (TCK2) that

$$s^c(\lambda)s^c(\lambda)^* = \sum_{\mu \in \lambda\Lambda^n} s^c(\mu)s^c(\mu)^* \quad \text{for all } \lambda \in \Lambda \text{ and } n \in \mathbb{N}^k. \tag{7.3}$$

Lemma 7.4. *Let Λ be a row-finite k-graph with no sources. Suppose that $\mu \sim \nu$ in Λ. Then* $\mathrm{MCE}(\mu, \nu) = \mu\Lambda \cap \Lambda^{d(\mu)\vee d(\nu)} = \nu\Lambda \cap \Lambda^{d(\mu)\vee d(\nu)}$.

Proof. For any $\mu\alpha \in \mu\Lambda \cap \Lambda^{d(\mu)\vee d(\nu)}$ and any ultrafilter S with $s(\alpha) \in S$, we have $\ell_{\mu\alpha}(S) = \ell_\mu(\ell_\alpha(S)) = \ell_\nu(\ell_\alpha(S)) = \ell_{\nu\alpha}(S)$, and so $\mu\alpha \in \nu\Lambda \cap \Lambda^{d(\mu)\vee d(\nu)}$. So $\mu\Lambda \cap \Lambda^{d(\mu)\vee d(\nu)} \subseteq \nu\Lambda \cap \Lambda^{d(\mu)\vee d(\nu)}$, and symmetry gives the reverse inclusion. Since $\mathrm{MCE}(\mu, \nu) = \mu\Lambda \cap \nu\Lambda \cap \Lambda^{d(\mu)\vee d(\nu)}$, the result follows. □

Recall from [4, Theorem 4.2] that if Λ is cofinal, then $\mathrm{Per}(\Lambda) = \{d(\mu) - d(\nu) : \mu \sim \nu\}$. Since $\mathrm{Per}(\Lambda)$ it is a subgroup of $\mathbb{Z}^k$, it is isomorphic to $\mathbb{Z}^l$ for some $l \leq k$. The collection

$$H_{\mathrm{Per}} = \{v \in \Lambda^0 : \text{ whenever } m, n \in \mathbb{N}^k, m - n \in \mathrm{Per}(\Lambda) \text{ and } \lambda \in v\Lambda^m \text{ there exists } \mu \in v\Lambda^n \text{ such that } \lambda \sim \mu\}$$

is a nonempty hereditary subset of Λ^0. For $m, n \in \mathbb{N}^k$ such that $m-n \in \mathrm{Per}(\Lambda)$, there is a bijection $\theta_{m,n} : H_{\mathrm{Per}}\Lambda^m \to H_{\mathrm{Per}}\Lambda^n$ such that $\mu \sim \theta_{m,n}(\mu)$ for all μ; this $\theta_{m,n}$ preserves the range and source maps.

Suppose that $\mu \sim \nu \in \Lambda$, and write $m = d(\mu)$ and $n = d(\nu)$. Let t be a Cuntz-Krieger (Λ, c)-family. Then (7.3) implies that $t_\mu t_\mu^* = \sum_{\eta \in \mu\Lambda^{(m\vee n)-m}} t_\eta t_\eta^*$, and then Lemma 7.4 implies that $t_\mu t_\mu^* = t_\mu t_\mu^* t_\nu t_\nu^*$; that is, $t_\mu t_\mu^* \leq t_\nu t_\nu^*$. Symmetry gives the reverse inequality, and hence

$$t_\mu t_\mu^* = t_\nu t_\nu^* \qquad \text{whenever } \mu \sim \nu. \tag{7.4}$$

For the following proposition, we remind the reader that $A(c)$ denotes the noncommutative torus, described at the beginning of this section.

Proposition 7.5. *Let Λ be a row-finite k-graph with no sources. Let $c \in Z^2(\mathbb{Z}^k, \mathbb{T})$, and suppose that $(cc^*)|_{\mathrm{Per}(\Lambda)}$ is nondegenerate. Suppose that t is a Cuntz-Krieger $(\Lambda, c \circ d)$-family such that each $t_v \neq 0$. Suppose that F is a finite subset of $H_{\mathrm{Per}}\Lambda^n$, and let $\widetilde{F} := \{\nu \in \Lambda : \nu \sim \mu$ for some $\mu \in F\}$. There is an injective homomorphism $\bigoplus_{\lambda \in F} \rho_t^\lambda : \bigoplus_{\lambda \in F} A(c|_{\mathrm{Per}(\Lambda)}) \to C^*(t)$ such that $\rho_t^\lambda(U_{d(\lambda)-d(\nu)}) = t_\lambda t_\nu^*$ whenever $\lambda \in F$ and $\nu \sim \lambda$.*

Proof. As discussed at the beginning of the section, we may assume that c is a bicharacter.

If $\mu, \nu \in \widetilde{F}$ and $\mu \not\sim \nu$, then there exist distinct $\lambda, \lambda' \in F$ such that $\mu \sim \lambda$ and $\nu \sim \lambda'$. Equation (7.4) gives $t_\mu t_\mu^* = t_\lambda t_\lambda^* \perp t_{\lambda'} t_{\lambda'}^* = t_\nu t_\nu^*$. Hence

$$C^*(\{t_\mu t_\nu^* : \mu, \nu \in \widetilde{F} \text{ and } \mu \sim \nu\}) = \bigoplus_{\lambda \in F} C^*(\{t_\mu t_\nu^* : \mu \sim \lambda \sim \nu\}).$$

Since each t_v is nonzero, so is each $t_\mu t_\nu^*$, and so each summand $C^*(\{t_\mu t_\nu^* : \mu \sim \lambda \sim \nu\})$ is nontrivial.

Fix $\lambda \in F$. It now suffices to show that there is an injective homomorphism $\rho_s : A(c|_{\mathrm{Per}(\Lambda)}) \to t_\lambda t_\lambda^* C^*(t) t_\lambda t_\lambda^*$ such that $\rho_s(U_{d(\mu)-d(\nu)}) = t_\mu t_\nu^*$ whenever $\mu \sim \lambda \sim \nu$. To see this, for each $m \in \mathrm{Per}(\Lambda)$, express $m = p(m) - q(m)$ where $p(m), q(m) \in \mathbb{N}^k$ and $p(m) = n$ whenever $m \leq n$. Since $r(\lambda) \in H_{\mathrm{Per}}$, we can define

$$V_m := \sum_{\mu \in r(\lambda)\Lambda^{p(m)}} t_\lambda t_\lambda^* t_\mu t^*_{\theta_{p(m),q(m)}(\mu)}. \tag{7.5}$$

We have

$$V_m V_m^* = \sum_{\mu,\nu \in r(\lambda)\Lambda^{p(m)}} t_\lambda t_\lambda^* t_\mu t^*_{\theta_{p(m),q(m)}(\mu)} t_{\theta_{p(m),q(m)}(\nu)} t_\nu^* t_\lambda t_\lambda^*.$$

Each $d(\theta_{p(m),q(m)}(\mu)) = q(m) = d(\theta_{p(m),q(m)}(\nu))$ and $\theta_{p(m),q(m)}$ is a bijection, so the Cuntz-Krieger relation (7.2) gives

$$V_m V_m^* = \sum_{\mu \in r(\lambda)\Lambda^p} t_\lambda t_\lambda^* t_\mu t_\mu^* t_\lambda t_\lambda^*,$$

and then (7.2) again reduces this to $t_\lambda t_\lambda^*$.

To see that $V_m^* V_m$ is also equal to $t_\lambda t_\lambda^*$, observe that

$$\begin{aligned} V_m^* V_m &= \sum_{\mu,\nu \in r(\lambda)\Lambda^{p(m)}} t_{\theta_{p(m),q(m)}(\mu)} t_\mu^* t_\lambda t_\lambda^* t_\nu t^*_{\theta_{p(m),q(m)}(\nu)} \\ &= \sum_{\substack{\mu,\nu \in r(\lambda)\Lambda^{p(m)} \\ \eta \in \lambda\Lambda^{p(m)}}} t_{\theta_{p(m),q(m)}(\mu)} t_\mu^* t_\eta t_\eta^* t_\nu t^*_{\theta_{p(m),q(m)}(\nu)}. \end{aligned}$$

For each $\eta \in \lambda\Lambda^{p(m)}$, factorise $\eta = \alpha_\eta \beta_\eta$ with $d(\alpha_\eta) = p(m)$ and $d(\beta_\eta) = n$. For $\mu \in r(\lambda)\Lambda^{p(m)}$ and $\eta \in \lambda\Lambda^{p(m)}$ we have $d(\mu) = p(m) = d(\alpha_\eta)$, so the first displayed equation in Lemma 3.2 implies that $t_\mu^* t_{\alpha_\eta} = \delta_{\mu,\alpha_\eta} t_{s(\mu)}$. Thus

$$V_m^* V_m = \sum_{\eta \in \lambda\Lambda^{p(m)}} t_{\theta_{p(m),q(m)}(\alpha_\eta)} t_{\beta_\eta} t_{\beta_\eta}^* t^*_{\theta_{p(m),q(m)}(\alpha_\eta)}.$$

If $\mu \sim \nu$ then the factorisation property gives $\mu\tau \sim \nu\tau$ for all τ, and so $\theta_{m+p,n+p}(\mu\tau) = \theta_{m,n}(\mu)\tau$ whenever $\mu \in \Lambda^m$ and $\tau \in s(\mu)\Lambda^p$. This and (7.4) give

$$\begin{aligned} V_m^* V_m &= \sum_{\eta \in \lambda\Lambda^{p(m)}} t_{\theta_{p(m)+n,q(m)+n}(\eta)} t^*_{\theta_{p(m)+n,q(m)+n}(\eta)} \\ &= \sum_{\eta \in \lambda\Lambda^{p(m)}} t_\eta t_\eta^* = t_\lambda t_\lambda^*. \end{aligned} \tag{7.6}$$

Hence the V_m are unitaries in $t_\lambda t_\lambda^* C^*(t) t_\lambda t_\lambda^*$.
Combining (7.5) and (7.6) we now have

$$\begin{aligned} \sum_{\mu \in r(\lambda)\Lambda^{p(m)}} t_\lambda t_\lambda^* t_\mu t^*_{\theta_{p(m),q(m)}(\mu)} = V_m = V_m t_\lambda t_\lambda^* \\ = \sum_{\mu \in r(\lambda)\Lambda^{p(m)}} t_\lambda t_\lambda^* t_\mu t^*_{\theta_{p(m),q(m)}(\mu)} t_\lambda t_\lambda^*, \end{aligned}$$

and then symmetry gives

$$V_m = \sum_{\mu \in r(\lambda)\Lambda^{p(m)}} t_\mu t^*_{\theta_{p(m),q(m)}(\mu)} t_\lambda t_\lambda^*.$$

Fix m, m' in $\operatorname{Per}(\Lambda)$, and write $p := p(m)$, $p' := p(m')$, $q := q(m)$ and $q' := q(m')$. For the next few calculations, put

$$C_{m,m'} := c(p,p')\overline{c(q,p')}c(p',q)\overline{c(q',q)}.$$

Then relation (7.2) gives

$$\begin{aligned} V_m V_{m'} &= \sum_{\mu \in r(\lambda)\Lambda^p} t_\lambda t_\lambda^* t_\mu t^*_{\theta_{p,q}(\mu)} \sum_{\eta \in r(\lambda)\Lambda^{p'}} t_\eta t^*_{\theta_{p',q'}(\eta)} t_\lambda t_\lambda^* \\ &= \sum_{\substack{\mu \in r(\lambda)\Lambda^p,\ \mu' \in s(\mu)\Lambda^{p'} \\ \eta \in r(\lambda)\Lambda^{p'},\ \eta' \in s(\eta)\Lambda^q}} C_{m,m'} t_\lambda t_\lambda^* t_{\mu\mu'} t^*_{\theta_{p,q}(\mu)\mu'} t_{\eta\eta'} t^*_{\theta_{p',q'}(\eta)\eta'} t_\lambda t_\lambda^* \\ &= C_{m,m'} \sum_{\substack{\mu \in r(\lambda)\Lambda^p,\ \mu' \in s(\mu)\Lambda^{p'} \\ \eta \in r(\lambda)\Lambda^{p'},\ \eta' \in s(\eta)\Lambda^q}} \delta_{\theta_{p,q}(\mu)\mu', \eta\eta'} t_\lambda t_\lambda^* t_{\mu\mu'} t^*_{\theta_{p',q'}(\eta)\eta'} t_\lambda t_\lambda^*. \end{aligned}$$

If $\theta_{p,q}(\mu)\mu' = \eta\eta'$, then $\mu\mu' \sim \theta_{p,q}(\mu)\mu' = \eta\eta' \sim \theta_{p',q'}(\eta)\eta'$; and conversely $\mu\mu' \sim \theta_{p',q'}(\eta)\eta'$ implies $\theta_{p,q}(\mu)\mu' = \eta\eta'$ by a symmetric argument. So the final line of the preceding displayed equation becomes

$$C_{m,m'} \sum_{\substack{\mu \in r(\lambda)\Lambda^p,\ \mu' \in s(\mu)\Lambda^{p'} \\ \zeta \in r(\lambda)\Lambda^{q'},\ \zeta' \in s(\eta)\Lambda^q \\ \mu\mu' \sim \zeta\zeta'}} t_\lambda t_\lambda^* t_{\mu\mu'} t^*_{\zeta\zeta'} t_\lambda t_\lambda^*. \tag{7.7}$$

Let $a := p(m+m')$ and $b := q(m+m')$. Then there are $h, l \in \mathbb{N}^k$ such that $p+p'+h = a+l$ and $q+q'+h = b+l$. For $\alpha \in \Lambda^{p+p'}$ and $\beta \in \Lambda^{q+q'}$, we have $\alpha \sim \beta$ if and only if there is an injection $\tau \mapsto (\eta(\tau), \rho(\tau), \zeta(\tau))$ from $s(\alpha)\Lambda^h$ to $r(\alpha)\Lambda^a \times \Lambda^l s(\alpha) \times r(\alpha)\Lambda^b$ such that $\alpha\tau = \eta(\tau)\rho(\tau)$ and $\beta\tau = \zeta(\tau)\rho(\tau)$ and $\eta(\tau) \sim \zeta(\tau)$ for each τ. Using this, and applying relation (7.2), we obtain

$$\begin{aligned}
&\sum_{\substack{\mu\in r(\lambda)\Lambda^p,\ \mu'\in s(\mu)\Lambda^{p'}\\ \zeta\in r(\lambda)\Lambda^{q'},\ \zeta'\in s(\eta)\Lambda^q\\ \mu\mu'\sim\zeta\zeta',}} t_{\mu\mu'}t^*_{\zeta\zeta'}\\
&\qquad= \sum_{\substack{\mu\in r(\lambda)\Lambda^p,\ \mu'\in s(\mu)\Lambda^{p'}\\ \zeta\in r(\lambda)\Lambda^{q'},\ \zeta'\in s(\eta)\Lambda^q\\ \mu\mu'\sim\zeta\zeta',\ \tau\in s(\mu')\Lambda^h}} c(p+p',h)\overline{c(q+q',h)}t_{\mu\mu'\tau}t^*_{\zeta\zeta'\tau}\\
&\qquad= \sum_{\substack{\mu\in r(\lambda)\Lambda^p,\ \mu'\in s(\mu)\Lambda^{p'}\\ \zeta\in r(\lambda)\Lambda^{q'},\ \zeta'\in s(\eta)\Lambda^q\\ \mu\mu'\sim\zeta\zeta',\ \tau\in s(\mu')\Lambda^h}} c(m+m',h)t_{\mu\mu'\tau}t^*_{\zeta\zeta'\tau}.
\end{aligned}$$

Hence the expression (7.7) for $V_m V_m^*$ gives

$$\begin{aligned}
V_m V_{m'} &= C_{m,m'}c(m+m',h)\sum_{\eta\in r(\lambda)\Lambda^a,\ \rho\in s(\eta)\Lambda^l} t_\lambda t_\lambda^* t_{\eta\rho}t^*_{\theta_{a,b}(\eta)\rho}t_\lambda t_\lambda^*\\
&= C_{m,m'}c(m+m',h)\overline{c(a,l)}c(b,l)\\
&\qquad\sum_{\eta\in r(\lambda)\Lambda^a,\ \rho\in s(\eta)\Lambda^l} t_\lambda t_\lambda^* t_{\alpha'}t_\tau t_\tau^* t^*_{\theta_{a,b}(\alpha')}t_\lambda t_\lambda^*\\
&= c(p,p')\overline{c(q,p')}c(p',q)\overline{c(q',q)}c(m+m',h-l)V_{m+m'}.
\end{aligned}$$

Rearranging this expression for $V_m V_{m'}$ and the symmetric expression

$$V_{m'}V_m = c(p',p)\overline{c(q',p)}c(p,q')\overline{c(q,q')}c(m+m',h-l)V_{m+m'},$$

and cancelling the occurrences of $c(m+m',h-l)$, we obtain

$$\begin{aligned}
V_m V_{m'} = c(p,p')&\overline{c(q,p')c(p,q')}c(q,q')\\
&\overline{c(p',p)}c(p',q)\overline{c(q',q)}c(q',p)V_{m'}V_m.
\end{aligned} \tag{7.8}$$

Using repeatedly that c is a bicharacter, we calculate:

$$\begin{aligned}
c(p,p')&\overline{c(q,p')c(p,q')}c(q,q')\overline{c(p',p)}c(p',q)\overline{c(q',q)}c(q',p)\\
&= c(p-q,p')c(-p+q,q')c(p',-p+q)c(q',p-q)\\
&= c(p-q,p')c(p-q,-q')\overline{c(p',p-q)c(-q',p-q)}\\
&= c(m,m')\overline{c(m',m)}.
\end{aligned}$$

Thus (7.8) becomes

$$V_m V_{m'} = c(m,m')\overline{c(m',m)} V_{m'} V_m.$$

Choose generators $g_1, \dots, g_l$ for $\operatorname{Per}(\Lambda)$. Then each V_m is a scalar multiple of a product of the V_{g_i}, and so $C^*(\{V_m : m \in \operatorname{Per}(\Lambda)\})$ is generated by the V_{g_i}, which satisfy $V_{g_i} V_{g_j} = (cc^*)(g_i, g_j) V_{g_j} V_{g_i}$. Since $c_{\mathrm{Per}} := c|_{\operatorname{Per}(\Lambda)}$ is nondegenerate, [27, Theorem 3.7] implies that there is an isomorphism $\rho^\lambda_t : A(c_{\mathrm{Per}}) \to C^*(\{V_m : m \in \operatorname{Per}(\Lambda)\})$ carrying each U_m to V_m.
Recall that we chose $p(m) = n$ whenever $m \leq n$. So if $\lambda \sim \nu$ then

$$V_{n-d(\nu)} = t_\lambda t^*_\lambda \sum_{\mu \in r(\lambda)\Lambda^n} t_\mu t^*_{\theta_{n,d(\nu)}(\mu)} = t_\lambda t^*_{\theta_{n,d(\nu)}(\lambda)} = t_\lambda t^*_\nu.$$

So ρ^λ_t carries $U_{d(\lambda)-d(\nu)}$ to $t_\lambda t^*_\nu$ whenever $\nu \sim \lambda$. Hence $\bigoplus_{\lambda \in F} \rho^\lambda_t : \bigoplus_{\lambda \in F} A(c_{\mathrm{Per}}) \to C^*(t)$ is the desired homomorphism. □

Lemma 7.6. *Let Λ be a row-finite k-graph with no sources. Suppose that $\mu, \nu \in \Lambda$ satisfy $s(\mu) = s(\nu)$. Then $\mu \sim \nu$ if and only if* $\operatorname{MCE}(\mu\tau, \nu\tau) \neq \emptyset$ *for all $\tau \in s(\mu)\Lambda$.*

Proof. First suppose that $\mu \sim \nu$, and fix $\tau \in s(\mu)\Lambda$. There is an ultrafilter S with $\tau \in S$, and we have $\ell_\mu(S) = \ell_\nu(S)$. Since S is an ultrafilter, so is $\ell_\mu(S)$, and so $\ell_\mu(S) \cap \Lambda^{d(\mu\tau) \vee d(\nu\tau)} \neq \emptyset$, and then the unique element of $\ell_\mu(S) \cap \Lambda^{d(\mu\tau)\vee d(\nu\tau)}$ belongs to $\operatorname{MCE}(\mu\tau, \nu\tau)$.
Now suppose that $\operatorname{MCE}(\mu\tau, \nu\tau) \neq \emptyset$ for all τ. Using the correspondence between ultrafilters and infinite paths of Remark 3.8 together with the topology on the space of infinite paths described in [12, Definition 2.4 and Lemma 2.6], we see that the sets $Z(\lambda) = \{S : S \text{ is an ultrafilter and } \lambda \in S\}$ are a basis of compact open sets for a Hausdorff topology on the set of all ultrafilters. Fix an ultrafilter S with $s(\mu) \in S$. For each $\tau \in S$, we have $\operatorname{MCE}(\mu\tau, \nu\tau) \neq \emptyset$, and hence $Z(\mu\tau) \cap Z(\nu\tau) \neq \emptyset$. Since the $Z(\mu\tau) \cap Z(\nu\tau)$ are compact and decreasing with respect to the partial ordering $\leq$ on S, we deduce that $\bigcap_{\tau \in S} Z(\mu\tau) \cap Z(\nu\tau) \neq \emptyset$; say $T \in \bigcap_{\tau \in S} Z(\mu\tau) \cap Z(\nu\tau)$. For $\tau \leq \tau' \in S$, we have $S' \cap \Lambda^{d(\mu\tau)} = \{\mu\tau\}$ for all $S' \in Z(\mu\tau')$. Hence $T \cap \Lambda^{d(\mu\tau)} \subseteq \{\mu\tau\}$ for all $\tau \in S$. Since $T \cap \Lambda^n \neq \emptyset$ for all n, we deduce that $T \cap \Lambda^{d(\mu\tau)} = \{\mu\tau\}$ for all $\tau \in S$. So $\ell_\mu(S) \subseteq T$, and since $\ell_\mu(S)$ is an ultrafilter, the two are equal. Symmetry gives $\ell_\nu(S) = T$ as well, so $\ell_\mu(S) = \ell_\nu(S)$. □

Proposition 7.7. *Let Λ be a row-finite k-graph with no sources. Let $c \in Z^2(\mathbb{Z}^k, \mathbb{T})$ and suppose that $(cc^*)|_{\operatorname{Per}(\Lambda)}$ is nondegenerate. Suppose that t is a Cuntz-Krieger $(\Lambda, c \circ d)$-family with each t_v nonzero. Suppose that $a : H_{\mathrm{Per}}\Lambda \times H_{\mathrm{Per}}\Lambda \to \mathbb{C}$ is finitely supported. Then*

$$\Big\| \sum_{\mu,\nu \in \Lambda} a_{\mu,\nu} t_\mu t^*_\nu \Big\| \geq \Big\| \sum_{\mu \sim \nu} a_{\mu,\nu} t_\mu t^*_\nu \Big\|.$$

Proof. Again as discussed at the beginning of the section, we may assume that c is a bicharacter.
Let $N = \bigvee_{a_{\mu,\nu}\neq 0} d(\mu)$. Using the Cuntz-Krieger relation, we can rewrite

$$\sum_{\mu,\nu} a_{\mu,\nu} t_\mu t_\nu^* = \sum_{\mu,\nu} \sum_{\mu\mu' \in \Lambda^N} a_{\mu,\nu} c(\mu,\mu')\overline{c(\nu,\mu')} t_{\mu\mu'} t_{\nu\mu'}^*;$$

so we may assume without loss of generality that $a_{\mu,\nu} \neq 0$ implies $\mu \in \Lambda^N$. Since the $t_\mu t_\mu^*$ for $\mu \in \Lambda^N$ are mutually orthogonal, and since $\mu \sim \nu$ implies $t_\mu t_\mu^* = t_\nu t_\nu^*$, we have $\overline{\operatorname{span}}\{t_\mu t_\nu^* : \mu \in \Lambda^N, \nu \sim \mu\} = \bigoplus_{\mu\in\Lambda^N} \overline{\operatorname{span}}\{t_\mu t_\nu^* : \mu \sim \nu\}$, and so $\big\| \sum_{\mu\sim\nu} a_{\mu,\nu} t_\mu t_\nu^* \big\| = \max_{\mu\in\Lambda^N} \big\| \sum_{\nu\sim\mu} a_{\mu,\nu} t_\mu t_\nu^* \big\|$.
Choose $\mu_{\max}$ such that

$$\Big\| \sum_{\nu\sim\mu_{\max}} a_{\mu_{\max},\nu} t_{\mu_{\max}} t_\nu^* \Big\| \geq \Big\| \sum_{\nu'\sim\mu'} a_{\mu',\nu'} t_{\mu'} t_{\nu'}^* \Big\|$$

for all $\mu' \in \Lambda^N$. Let $v_{\max} = s(\mu_{\max})$. We claim that if $\mu, \nu \in \Lambda$ satisfy $s(\mu) = s(\nu) = v_{\max}$ but $\mu \not\sim \nu$, then there exists $\tau \in v_{\max}\Lambda$ such that $\operatorname{MCE}(\mu\tau, \nu\tau) = \emptyset$. Indeed, Lemma 7.6 implies that either there exists $\tau \in v_{\max}\Lambda^{d(\mu)\vee d(\nu)-d(\mu)}$ such that $\mu\tau \notin \operatorname{MCE}(\mu,\nu)$ or there exists $\tau \in v_{\max}\Lambda^{d(\mu)\vee d(\nu)-d(\nu)}$ such that $\nu\tau \notin \operatorname{MCE}(\mu,\nu)$. We consider the first case; the second is symmetric. We have $\mu\tau \in \Lambda^{d(\mu)\vee d(\nu)}$, and so $\mu\tau = \alpha\alpha'$ for some $\alpha \in \Lambda^{d(\nu)} \setminus \{\nu\}$. In particular, factoring $\mu\tau = \eta\zeta$ with $d(\eta) = d(\nu)$, we have $\eta \neq \nu$, and so $\operatorname{MCE}(\mu\tau, \nu\tau) = \emptyset$. An induction using that Λ has no sources now shows that there exists $\tau \in v_{\max}\Lambda$ such that $d(\tau) > d(\mu)\vee d(\nu)$ whenever $a_{\mu,\nu} \neq 0$ and $\operatorname{MCE}(\mu\tau, \nu\tau) = \emptyset$ whenever $s(\mu) = s(\nu) = v_{\max}$ and $\mu \not\sim \nu$.
Let $l := d(\tau)$. For $\mu, \nu \in F$ with $a_{\mu,\nu} \neq 0$, we have

$$\begin{aligned}&\textstyle\sum_{\lambda\in Fv_{\max}\cap\Lambda^N} t_{\lambda\tau} t_{\lambda\tau}^* t_\mu t_\nu^* t_{\lambda\tau} t_{\lambda\tau}^* \\ &\quad= \begin{cases} c(N,l)\overline{c(d(\nu),l)} t_{\mu\tau} t_{\nu\tau}^* t_{\mu\tau} t_{\mu\tau}^* & \text{if } s(\mu) = v_{\max} \text{ and } \mu \sim \nu \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

If $\mu \sim \nu$ then $\mu\tau \sim \nu\tau$ giving $t_{\mu\tau} t_{\mu\tau}^* = t_{\nu\tau} t_{\nu\tau}^*$, and hence $t_{\nu\tau}^* t_{\mu\tau} t_{\mu\tau}^* = t_{\nu\tau}^*$. That is,

$$\begin{aligned}&\textstyle\sum_{\lambda\in Fv_{\max}\cap\Lambda^N} t_{\lambda\tau} t_{\lambda\tau}^* t_\mu t_\nu^* t_{\lambda\tau} t_{\lambda\tau}^* \\ &\quad= \begin{cases} c(N,l)\overline{c(d(\nu),l)} t_{\mu\tau} t_{\nu\tau}^* & \text{if } s(\mu) = v_{\max} \text{ and } \mu \sim \nu \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Since c is a bicharacter, the map $m \mapsto c(m,l)$ is a homomorphism of $\operatorname{Per}(\Lambda)$ into $\mathbb{T}$. The universal property of $A(c|_{\operatorname{Per}(\Lambda)})$ ensures that it admits an automorphism α_l such that $\alpha_l(U_m) = c(m,l)U_m$ for all $m \in \operatorname{Per}(\Lambda)$. An application of Proposition 7.5 with $F = \{\mu_{\max}\tau\}$ gives an injective homomorphism $\rho_1 : A(c|_{\operatorname{Per}(\Lambda)}) \to C^*(t)$ which carries $U_{N-d(\nu)}$ to $t_{\mu_{\max}\tau} t_{\nu\tau}^*$ whenever $\nu \sim \mu_{\max}$;

so $\rho_1 \circ \alpha_l$ carries $U_{N-d(\nu)}$ to $\sum_{\lambda \in F v_{\max} \cap \Lambda^N} t_{\lambda\tau} t^*_{\lambda\tau}$. Proposition 7.5 applied to $F = \{\mu_{\max}\}$ gives an injective homomorphism $\rho_2 : A(c|_{\mathrm{Per}(\Lambda)}) \to C^*(t)$ which carries $U_{N-d(\nu)}$ to $t_{\mu_{\max}} t^*_\nu$ whenever $\nu \sim \mu_{\max}$. Since these homomorphisms are injective, they are isometric, and so the map

$$\rho_2 \circ \alpha_l^{-1} \circ \rho_1^{-1} : \overline{\operatorname{span}}\{t_{\mu_{\max}\tau} t^*_{\nu\tau} : \nu \sim \mu_{\max}\} \to \overline{\operatorname{span}}\{t_{\mu_{\max}} t^*_\nu : \nu \sim \mu_{\max}\}$$

is isometric and carries spanning elements to spanning elements. Since the $t_{\lambda\tau} t^*_{\lambda\tau}$ are mutually orthogonal projections, the map $a \mapsto \sum_\lambda t_{\lambda\tau} t^*_{\lambda\tau} a t_{\lambda\tau} t^*_{\lambda\tau}$ is norm-decreasing, and so we have

$$\begin{aligned}
\Big\| \sum_{\mu \sim \nu} & a_{\mu,\nu} t_\mu t^*_\nu \Big\| \\
&= \Big\| \sum_{\nu \sim \mu_{\max}} a_{\mu_{\max},\nu} t_{\mu_{\max}} t^*_\nu \Big\| \\
&= \Big\| \rho_2 \circ \alpha_l^{-1} \circ \rho_1^{-1} \Big(\sum_{\lambda \in F v_{\max} \cap \Lambda^n} t_{\lambda\tau} t^*_{\lambda\tau} \Big(\sum_{\nu \sim \mu_{\max}} a_{\mu_{\max},\nu} t_{\mu_{\max}} t^*_\nu \Big) t_{\lambda\tau} t^*_{\lambda\tau} \Big) \Big\| \\
&= \Big\| \sum_{\lambda \in F v_{\max} \cap \Lambda^n} t_{\lambda\tau} t^*_{\lambda\tau} \sum_{\mu,\nu \in \Lambda} a_{\mu,\nu} t_\mu t^*_\nu t_{\lambda\tau} t^*_{\lambda\tau} \Big\| \\
&\leq \Big\| \sum_{\mu,\nu \in \Lambda} a_{\mu,\nu} t_\mu t^*_\nu \|,
\end{aligned}$$

as required. □

Proof of Theorem 7.1. Since Λ is cofinal, the saturation of H_{Per} is all of Λ^0. Hence Lemma 4.3 implies that $P_{H_{\mathrm{Per}}} C^*(\Lambda, c \circ d) P_{H_{\mathrm{Per}}}$ is a full corner of $C^*(\Lambda, c \circ d)$. So it suffices to show that $P_{H_{\mathrm{Per}}} C^*(\Lambda, c \circ d) P_{H_{\mathrm{Per}}}$ is simple. Let $\Gamma = H_{\mathrm{Per}} \Lambda$. The gauge-invariant uniqueness theorem gives a canonical isomorphism $C^*(\Gamma, c \circ d) \cong P_{H_{\mathrm{Per}}} C^*(\Lambda, c \circ d) P_{H_{\mathrm{Per}}}$. So it suffices to show that $C^*(\Gamma, c \circ d)$ is simple.

Suppose π is a homomorphism $C^*(\Gamma, c \circ d) \to B$, and let $s := \{s_\lambda : \lambda \in \Gamma\}$ be the universal Cuntz-Krieger $(\Gamma, c \circ d)$-family. First suppose that $\pi(s_v) = 0$ for some v. Then $\ker(\pi)$ contains the gauge-invariant ideal generated by s_v, which in turn, by Theorem 4.6, contains s_w for every w in the saturated hereditary set generated by v. Since Γ is cofinal it follows that $\pi(s_w) = 0$ for all w, and then $\pi(s_\mu s^*_\nu) = \pi(s_{r(\mu)} s_\mu s^*_\nu) = 0$ for all μ, ν, and hence $\pi = 0$. Now suppose that $\pi(s_v) \neq 0$ for all v; we must show that π is injective. Let $t_\mu := \pi(s_\mu)$ for all μ. Since H_{Per} is all of Γ^0, Proposition 7.7 implies that $\big\| \sum a_{\mu,\nu} t_\mu t^*_\nu \big\| \geq \big\| \sum_{\mu \sim \nu} a_{\mu,\nu} t_\mu t^*_\nu \big\|$ for all finitely supported a, and so there is a norm-decreasing linear map $\Psi : C^*(t) \to \overline{\operatorname{span}}\{t_\mu t^*_\nu : \mu \sim \nu\}$ such that $\Psi(t_\mu t^*_\nu) = t_\mu t^*_\nu$ if $\mu \sim \nu$ and is zero otherwise. The same argument gives a norm-decreasing linear map Φ from $C^*(\Gamma, c \circ d)$ to $\overline{\operatorname{span}}\{s_\mu s^*_\nu : \mu \sim \nu\}$. This is faithful on positive elements because the faithful conditional expectation Θ of Proposition 7.3 satisfies $\Theta = \Theta \circ \Psi$. Clearly $\pi \circ \Phi = \Psi \circ \pi$. Fix a finite linear combination $\sum_{\mu\sim\nu} a_{\mu,\nu} s_\mu s_\nu^*$. Using the Cuntz-Krieger relation as in the proof of Proposition 7.7, we may assume that there exists $n \in \mathbb{N}^k$ such that $a_{\mu,\nu} \neq 0$ implies that $\mu \in \Lambda^n$. Applying Proposition 7.5 to the Cuntz-Krieger $(\Gamma, c \circ d)$-families s and t and with $F = \{\mu : a_{\mu,\nu} \neq 0\}$ and composing the resulting homomorphisms, we obtain an isometric map from $C^*(\{s_\mu s_\nu^* : \lambda \in F, \mu \sim \lambda \sim \nu\})$ to $C^*(\{t_\mu t_\nu^* : \lambda \in F, \mu \sim \lambda \sim \mu\})$ which carries each $s_\mu s_\nu^*$ to $t_\mu t_\nu^*$. Hence $\big\| \sum_{\mu\sim\nu} a_{\mu,\nu} s_\mu s_\nu^* \big\| = \big\| \sum_{\mu\sim\nu} a_{\mu,\nu} t_\mu t_\nu^* \big\|$. Since a was arbitrary, we deduce that $\pi|_{\overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu\sim\nu\}}$ is injective. So Lemma 3.14 shows that π is injective. □

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