An Optimal Extension Theorem for 1-Forms and the Lipman-Zariski Conjecture

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Abstract. Let X be a normal variety. Assume that for some reduced divisor $D \subset X$, logarithmic 1-forms defined on the snc locus of (X, D) extend to a log resolution $\widetilde{X} \to X$ as logarithmic differential forms. We prove that then the Lipman-Zariski conjecture holds for X. This result applies in particular if X has log canonical singularities. Furthermore, we give an example of a 2-form defined on the smooth locus of a three-dimensional log canonical pair $(X, \emptyset)$ which acquires a logarithmic pole along an exceptional divisor of discrepancy zero, thereby improving on a similar example of Greb, Kebekus, Kovács and Peternell.

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1. Introduction

1.A. Main results. The Lipman-Zariski conjecture [Lip65] asserts the following.

Conjecture 1.1. *Let X be a complex variety such that the tangent sheaf $\mathscr{T}_X := \mathscr{H}om_{\mathscr{O}_X}(\Omega^1_X, \mathscr{O}_X)$ is locally free. Then X is smooth.*

Despite being almost 50 years old, this conjecture remains open in general. It is therefore natural to consider special cases. The conjecture is known to hold in the case that the singular locus of X has codimension at least 3 by the work of Flenner [Fle88] and for complete intersections by the work of Källström [Käl11]. Recently, Jörder [Jör13] proved the conjecture under the assumption that $\mathscr{T}_X$ locally has a basis consisting of *commuting* vector fields. Earlier work on Conjecture 1.1 includes [SS72], [Hoc75], and [SvS85].

In a slightly different direction, one may consider varieties with only singularities arising in the minimal model program. The minimal model program aims at the birational classification of varieties, and it is well-known that even if one is interested only in smooth varieties, running the mmp requires dealing with singular models. For us the most important classes will be *klt* (Kawamata log terminal) and *log canonical*. Klt singularities form the largest class of singularities where most of the mmp is known to work, while the class of log canonical singularities is the largest class where the relevant notions of the mmp make sense. It is also a more stable class than that of klt singularities; log canonical singularities and their non-normal versions play an important role in compactifications of moduli spaces of canonically polarized varieties. For the precise definitions, we refer to [KM98, Sec. 2.3]. Greb-Kebekus-Kovács-Peternell [GKKP11, Theorem 6.1] showed Conjecture 1.1 for klt spaces.

In this paper, we prove the following new special case of Conjecture 1.1.

Theorem 1.2 (Lipman-Zariski conjecture given an extension theorem). *Let X be a normal complex variety, and let $\pi\colon \widetilde{X} \to X$ be a log resolution. Assume that the sheaf*

$$\pi_*\Omega^1_{\widetilde{X}}(\log \widetilde{D})$$

is reflexive for some snc divisor $\widetilde{D} \subset \widetilde{X}$. Then the Lipman-Zariski conjecture holds for X, i.e., $\mathscr{T}_X$ being locally free implies that X is smooth.

Note that Lipman [Lip65, Thm. 3] proved that if $\mathscr{T}_X$ is locally free, then X is normal. Hence the normality assumption in our theorem is not a real restriction.

Also note that the reflexivity assumption in the theorem is equivalent to saying that logarithmic 1-forms defined on the snc locus of $(X, \pi_*\widetilde{D})$ extend to logarithmic 1-forms on $(\widetilde{X}, \widetilde{D})$ – cf. [GKKP11, Rem. 1.5.2].

By [GKKP11, Thm. 1.5], we have the following immediate corollary.

Corollary 1.3 (Lipman-Zariski conjecture for log canonical pairs). *Let (X, D) be a complex log canonical pair. Then the Lipman-Zariski conjecture holds for X.*

By the same method of proof as in Theorem 1.2, we obtain the following result.

Theorem 1.4 (Extension theorem for 1-forms on log canonical pairs). *Let (X, D) be a complex log canonical pair, and let $\pi\colon \widetilde{X} \to X$ be a log resolution of (X, D). Then the sheaf*

$$\pi_* \Omega^1_{\widetilde{X}}(\log \widetilde{D})$$

is reflexive, where $\widetilde{D}$ is any reduced divisor such that

$$\operatorname{Exc}(\pi) \wedge \pi^{-1}(\lfloor D \rfloor) \subseteq \operatorname{supp} \widetilde{D} \subseteq \pi^{-1}(\lfloor D \rfloor).$$

Here $\lfloor D \rfloor$ denotes the coefficient-wise round-down of D. In our case, $\lfloor D \rfloor$ is simply the union of all components of D that have coefficient one. The expression $\operatorname{Exc}(\pi) \wedge \pi^{-1}(\lfloor D \rfloor)$ denotes the largest divisor contained in both $\operatorname{Exc}(\pi)$ and $\pi^{-1}(\lfloor D \rfloor)$.
In a similar fashion as above, the reflexivity assertion is equivalent to saying that any logarithmic 1-form defined on the snc locus of $(X, \pi_* \widetilde{D})$ can be extended to $\widetilde{X}$, possibly acquiring logarithmic poles along $\widetilde{D}$.
Theorem 1.4 should be compared to the extension theorem [GKKP11, Thm. 1.5]. There the conclusion is similar, with $\widetilde{D}$ replaced by $\widehat{D}$, the largest reduced divisor contained in $\pi^{-1}(\text{non-klt locus})$. (The non-klt locus is the smallest closed subset $W \subset X$ such that (X, D) is klt away from W. Note that this contains $\lfloor D \rfloor$.) Theorem 1.4 says that [GKKP11, Thm. 1.5] is not optimal: we allow logarithmic poles only along a smaller divisor. For example, if $D = \emptyset$ but X is not klt, e.g. if X is a cone over an abelian variety, then $\widetilde{D} = 0$ while $\widehat{D}$ is nonzero.

1.B. Further results. We show that Theorem 1.4 in turn is optimal, both with respect to the pole divisor $\widetilde{D}$ and with respect to the degree of the forms considered. To be more precise, concerning the first point we prove the following.

Theorem 1.5 (Optimality of Theorem 1.4). *Let (X, D) and π be as in Theorem 1.4, and let $\widetilde{D} = \pi^{-1}(\lfloor D \rfloor)$. Assume that one of the following holds:*

(1.5.1) X *is $\mathbb{Q}$-factorial, or*

(1.5.2) $\dim X = 2$.

Then for any divisor B such that $\pi_*^{-1} \lfloor D \rfloor \le B \lneq \widetilde{D}$, the sheaf $\pi_* \Omega^1_{\widetilde{X}}(\log B)$ is not *reflexive.*

As to the second point, we show that for resolutions of log canonical pairs, forms of higher degree may acquire logarithmic poles along exceptional divisors of discrepancy strictly greater than -1, even if the boundary divisor of the pair is empty. This improves upon an example given in [GKKP11, Ex. 3.2]. The precise statement is as follows.

THEOREM 1.6 (Non-extension over klt places, cf. Theorem 7.4). *There exists a three-dimensional complex log canonical pair* $(X, \emptyset)$ *with empty boundary such that there is a divisor* $E_0 \subset \widetilde{X}$ *of discrepancy* 0 *in some log resolution* $\widetilde{X} \to X$*, and a* 2*-form on the smooth locus of* X *that acquires a logarithmic pole along* E_0 *when pulled back to* $\widetilde{X}$*.*

1.C. OVERVIEW OF PROOFS. The proofs of our main results are based on the two auxiliary Theorems 2.1 and 3.1. The purpose of these theorems is to shrink the divisor along which we allow pulled-back 1-forms to acquire logarithmic poles. Theorem 2.1 deals with non-exceptional components, while Theorem 3.1 handles the exceptional ones.
To prove Theorem 1.2, we first apply Theorem 2.1 and then Theorem 3.1 in order to shrink that pole divisor to zero. Then an argument going back to [SvS85, (1.6)] completes the proof. To prove Theorem 1.4, we take [GKKP11, Thm. 16.1] as our starting point and then apply Theorems 2.1 and 3.1.

1.D. RECENT WORK BY DRUEL. In [Dru13, Thm. 1.1], Druel has recently obtained Corollary 1.3 by an independent proof. He employs a cutting-down procedure to reduce to the surface case, where the main work is done. Note however that this case is essentially already contained in [SvS85]. To be more precise, if $x \in X$ is a normal surface singularity with smooth locus U and $\pi\colon \widetilde{X} \to X$ is a log resolution with exceptional divisor E, then [SvS85, Cor. 1.4] says that the map

$$H^0(U, \Omega^1_U) \Big/ H^0(\widetilde{X}, \Omega^1_{\widetilde{X}}) \longrightarrow H^0(U, \Omega^2_U) \Big/ H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(\log E))$$

induced by differentiation is injective. If $x \in X$ is log canonical, then by definition the right-hand side is zero, hence so is the left-hand side. This means that all 1-forms defined on U extend to $\widetilde{X}$ without poles. Now the argument given in [SvS85, (1.6)] shows that if $\mathscr{T}_X$ is free, then $x \in X$ is in fact smooth.

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1.F. NOTATION, DEFINITIONS, AND CONVENTIONS. Throughout this paper, we work over the field of complex numbers $\mathbb{C}$.
A *pair* (X, D) consists of a normal variety X over $\mathbb{C}$ and an effective $\mathbb{R}$-Weil divisor D on X.
Let (X, D) be a pair and $x \in X$ a point. We say that (X, D) is *snc at* x if there exists a Zariski-open neighbourhood $U \subseteq X$ of x such that U is smooth and $\operatorname{supp} D \cap U$ is either empty, or a divisor with simple normal crossings. The pair (X, D) is called an *snc pair* or simply *snc* if it is snc at every point of X.
For the definitions of *klt* and *log canonical* pairs, we refer to [KM98, Sec. 2.3].

Given a pair (X, D), let $(X, D)_{\mathrm{reg}}$ denote the maximal open subset of X where (X, D) is snc, and $(X, D)_{\mathrm{sing}}$ its complement, with the induced reduced subscheme structure.
Let (X, D) be a pair. A *log resolution* of (X, D) is a proper birational morphism $\pi\colon \widetilde{X} \to X$ such that $\widetilde{X}$ is smooth, both the pre-image $\pi^{-1}(\operatorname{supp} D)$ of $\operatorname{supp} D$ and the exceptional set $E = \operatorname{Exc}(\pi)$ are of pure codimension one in $\widetilde{X}$, and $(\widetilde{X}, \widetilde{D} + E)$ is an snc pair where $\widetilde{D} = \pi^{-1}(\operatorname{supp} D)_{\mathrm{red}}$ is the reduced divisor supported on $\pi^{-1}(\operatorname{supp} D)$.
Let D be a divisor on a normal variety, and let $D = \sum a_i D_i$ be its decomposition into irreducible components. The *round-down* $\lfloor D \rfloor$ of D is defined to be $\sum \lfloor a_i \rfloor D_i$, where $\lfloor a_i \rfloor$ is the largest integer less than or equal to a_i.
Let D_1, D_2 be divisors on a normal variety. Then $D_1 \vee D_2$ denotes the smallest divisor that contains both D_1 and D_2, while $D_1 \wedge D_2$ denotes the largest divisor that is contained in both D_1 and D_2.

2. Dropping non-exceptional divisors

In this section we prove that if the extension theorem holds for a pair $(X, \Gamma + \Delta)$ where Δ is a reduced effective divisor, then it also holds for (X, Γ). More precisely we prove the following.

Theorem 2.1 (Dropping non-exceptional divisors). *Let X be a normal variety and $\pi\colon \widetilde{X} \to X$ a log resolution of X. Assume that the sheaf*

$$\pi_* \Omega^1_{\widetilde{X}}(\log \widetilde{D})$$

is reflexive for some snc divisor $\widetilde{D}$. Let Δ be a reduced effective divisor on X such that $\operatorname{supp} \Delta \subset \pi_* \widetilde{D}$. *Then the sheaf*

$$\pi_* \Omega^1_{\widetilde{X}}(\log \widetilde{B})$$

is also reflexive, where $\widetilde{B} = \widetilde{D} - \pi_*^{-1} \Delta$.

Proof. Notice that one may assume that Δ is irreducible and conclude the general case via replacing $\widetilde{D}$ by $\widetilde{B}$ and iterating the process for all irreducible components of Δ. For simplicity let us denote $\pi_*^{-1}\Delta$ by $\widetilde{\Delta}$. Consider the following short exact sequence given by the residue map, cf. [EV92, 2.3(b)]:

$$0 \to \Omega^1_{\widetilde{X}}(\log \widetilde{B}) \to \Omega^1_{\widetilde{X}}(\log \widetilde{D}) \to \mathscr{O}_{\widetilde{\Delta}} \to 0.$$

Pushing this forward via π gives

$$0 \to \pi_* \Omega^1_{\widetilde{X}}(\log \widetilde{B}) \to \pi_* \Omega^1_{\widetilde{X}}(\log \widetilde{D}) \to \mathscr{Q} \to 0,$$

where $\mathscr{Q} \subset \pi_* \mathscr{O}_{\widetilde{\Delta}}$. In particular, $\mathscr{Q}$ is supported on Δ and it is torsion-free as an $\mathscr{O}_\Delta$-module. It follows that the only associated prime of $\mathscr{Q}$ has height 1. Then the statement follows from [Har80, Cor. 1.5]. $\square$

3. Dropping certain exceptional divisors

The aim of the present section is to show that if ω is a logarithmic 1-form on a smooth variety whose poles are contained in an exceptional divisor, then ω in fact does not have any poles.

Theorem 3.1 (Dropping exceptional divisors). *Let X be a normal variety and $\pi\colon \widetilde{X} \to X$ a log resolution. Let E be a reduced π-exceptional divisor. Then the natural inclusion map*

$$H^0\big(\widetilde{X}, \Omega^1_{\widetilde{X}}\big) \hookrightarrow H^0\big(\widetilde{X}, \Omega^1_{\widetilde{X}}(\log E)\big)$$

is an isomorphism. Equivalently, the inclusion $\pi_*\Omega^1_{\widetilde{X}} \subset \pi_*\Omega^1_{\widetilde{X}}(\log E)$ is an isomorphism of sheaves.

Theorem 3.1 is a consequence of the following two propositions.

Proposition 3.2 (Theorem 3.1 for isolated singularities). *Let X be a normal variety and $\pi\colon \widetilde{X} \to X$ a log resolution. Let E be a reduced divisor which is mapped to a single point by π. Then the natural inclusion map*

$$H^0\big(\widetilde{X}, \Omega^1_{\widetilde{X}}\big) \hookrightarrow H^0\big(\widetilde{X}, \Omega^1_{\widetilde{X}}(\log E)\big)$$

is an isomorphism.

Proposition 3.3. *Proposition 3.2 implies Theorem 3.1.*

Proposition 3.2 was first observed by Wahl in the case of surfaces, cf. [Wah85, Lemma 1.3]. Our proof of Proposition 3.3 follows the lines of [GKK10, Section 7.D].

3.A. The first Chern class.

We collect some well-known facts about the first Chern class of a line bundle.

Notation 3.4 (First Chern class). Let X be a smooth variety and $\mathscr{L} \in \operatorname{Pic}(X)$ a line bundle. The first Chern class $c_1(\mathscr{L}) \in H^1(X, \Omega^1_X)$ is the image of $\mathscr{L}$ under the map $\operatorname{Pic}(X) = H^1(X, \mathscr{O}^*_X) \to H^1(X, \Omega^1_X)$ induced by the map $\mathrm{d}\log\colon \mathscr{O}^*_X \to \Omega^1_X$ that sends $f \mapsto f^{-1}\mathrm{d}f$.

Lemma 3.5 (Connecting homomorphism of the residue sequence). *Let X be a smooth variety and $E \subset X$ an snc divisor, consisting of irreducible components $E_1, \ldots, E_k$. Consider the short exact sequence*

$$0 \to \Omega^1_X \to \Omega^1_X(\log E) \to \bigoplus_{i=1}^{k} \mathscr{O}_{E_i} \to 0 \tag{3.6}$$

given by the residue map (cf. [EV92, 2.3(a)]*). The associated connecting homomorphism*

$$\delta\colon \bigoplus_{i=1}^{k} H^0(E_i, \mathscr{O}_{E_i}) \to H^1(X, \Omega^1_X)$$

sends

$$\mathbf{1}_{E_i} \mapsto c_1(\mathscr{O}_X(E_i)), \quad 1 \le i \le k.$$

Here $\mathbf{1}_{E_i}$ *denotes the function that is constant with value* 1 *on* E_i *and vanishes on the other components.*

Proof. This is well-known, and easy to prove by a Čech cohomology computation. $\square$

For a proof of the following fact see [For81, Paragraph 17].

Fact 3.7 (Residue map on curves). *Let C be a smooth projective curve. Then there is a canonically defined linear map* $\operatorname{Res}: H^1(C, \Omega^1_C) \to \mathbb{C}$, *which is an isomorphism.* $\square$

Lemma 3.8 (Residue and degree). *If* $\mathscr{L} \in \operatorname{Pic}(C)$ *is a line bundle on a smooth projective curve, then* $\operatorname{Res}(c_1(\mathscr{L})) = \deg \mathscr{L}$.

Proof. For $P \in C$ a point and $\mathscr{L} = \mathscr{O}_C(P)$, the claim is easily seen to be true from the description of Res given in [For81, Thm. 17.3]. By linearity, this is enough. $\square$

3.B. Proof of Proposition 3.2. We may assume X to be affine of dimension ≥ 2. Let $E_1, \ldots, E_k$ be the irreducible components of E. Consider the short exact sequence (3.6),

$$0 \to \Omega^1_{\widetilde{X}} \to \Omega^1_{\widetilde{X}}(\log E) \to \bigoplus_{i=1}^k \mathscr{O}_{E_i} \to 0.$$

By the corresponding long exact sequence, it suffices to show injectivity of the induced map

$$\delta\colon \bigoplus_{i=1}^k H^0(E_i, \mathscr{O}_{E_i}) \to H^1(\widetilde{X}, \Omega^1_{\widetilde{X}}).$$

Note that we may assume π to be a projective morphism and then $\widetilde{X}$ is quasi-projective. After choosing a (locally closed) embedding of $\widetilde{X}$ into a projective space, let $H \subset \widetilde{X}$ be the intersection of general hyperplanes $H_1, \ldots, H_{\dim X - 2} \subset \widetilde{X}$. (If X is a surface, then $H = \widetilde{X}$.) We formulate the properties of H in a separate lemma.

Lemma 3.9. *Using the notation introduced above we have that* $(H, E|_H)$ *is an snc surface pair. Furthermore, for any* i, $C_i := E_i|_H$ *is irreducible (in particular, nonempty), and* $\pi|_H$ *is proper and birational onto its image.*

Proof. If $\dim X = 2$, then there is nothing to prove, so we may assume that $\dim X \geq 3$. In particular the intersection of two irreducible divisors on $\widetilde{X}$ is still positive dimensional and hence if one of them is ample, then the intersection is connected. We will use this fact below.

We proceed inductively, cutting by one hyperplane at a time. First we cut by H_1. By Bertini's theorem $E + H_1$ is snc and H_1 is connected and hence irreducible by the above discussion. Hence $(H_1, E|_{H_1})$ is an snc pair, and the $E_i|_{H_1}$ are smooth and irreducible for all i by Bertini again. It is clear that $\pi|_{H_1}$ is proper and since H_1 is general, $\pi|_{H_1}$ is birational.

Now we are in the same situation as before cutting by H_1, so we may apply the same argument again and obtain the statement for $H_1 \cap H_2$. After finitely many steps, we arrive at H. $\square$

The image $\pi(H)$ need not be normal, but we may normalize it and get a birational morphism $H \to \pi(H)^\nu$, which contracts all the C_i. Then negative definiteness ([KM98, Lemma 3.40]) asserts that the intersection matrix $A := (C_i \cdot C_j)$ is invertible.
Recall that we need to show the injectivity of δ. To this end, think of the C_i as smooth projective curves in $\widetilde{X}$, consider the restriction morphism

$$r\colon H^1(\widetilde{X}, \Omega^1_{\widetilde{X}}) \to \bigoplus_{i=1}^{k} H^1(C_i, \Omega^1_{C_i}),$$

and observe that the composition

$$r \circ \delta\colon \bigoplus_{i=1}^{k} H^0(E_i, \mathscr{O}_{E_i}) \to \bigoplus_{i=1}^{k} H^1(C_i, \Omega^1_{C_i})$$

is an isomorphism: On the left-hand side, choose the basis consisting of the functions $\mathbf{1}_{E_i}$, and on each summand of $\bigoplus_{i=1}^{k} H^1(C_i, \Omega^1_{C_i})$, choose the basis canonically determined by the residue map of Fact 3.7. By Lemmas 3.5 and 3.8 the map $r \circ \delta$ with respect to these bases is given by the matrix A. We have already noted that this matrix is invertible. $\square$

3.C. Proof of Proposition 3.3. We will make essential use of the following proposition.

Proposition 3.10 (Negativity lemma, see [GKK10, Proposition 7.5]). *Let $\varphi\colon \widetilde{Y} \to Y$ be a projective birational morphism between normal quasi-projective varieties of dimension ≥ 2, where $\widetilde{Y}$ is smooth. Let $y \in Y$ be a point whose preimage $\varphi^{-1}(y)$ has (not necessarily pure) codimension one and let $F_0, \ldots, F_k \subset \varphi^{-1}(y)$ be the reduced divisorial components. If all the F_i are smooth and $\sum e_i F_i$ is a nonzero effective divisor, then there is a $0 \leq j \leq k$ such that $e_j \neq 0$ and $h^0(F_j, \mathscr{O}_{\widetilde{Y}}(\sum e_i F_i)|_{F_j}) = 0$.* $\square$

Proof of Proposition 3.3. We may assume X to be affine of dimension ≥ 2. Let

$$\sigma \in H^0\big(\widetilde{X}, \Omega^1_{\widetilde{X}}(\log E)\big) \tag{3.11}$$

be a logarithmic 1-form. Assuming Proposition 3.2, we will show that

$$\sigma \in H^0\big(\widetilde{X}, \Omega^1_{\widetilde{X}}\big). \tag{3.12}$$

To this end, we will consider an irreducible component of $E' \subset E$ for which $\dim \pi(E')$ is maximal among the irreducible components of E, and for any such E' we will show that $\sigma \in H^0\big(\widetilde{X}, \Omega^1_{\widetilde{X}}(\log(E - E'))\big)$. Then replace E by $E - E'$ and repeat the argument until E disappears.

No.	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\dim X$	2	3	3	4	4	4	5	5	5	5	$\cdots$
$\operatorname{codim}_X \pi(E')$	2	2	3	2	3	4	2	3	4	5	$\cdots$

We proceed by induction on pairs of numbers $\big(\dim X, \operatorname{codim}_X \pi(E')\big)$ ordered lexicographically as indicated in the following table:
In order to simplify notation, we number the irreducible components E_i of E such that $E' = E_0$ and $\pi(E_i) = \pi(E_0)$ if and only if $0 \le i \le k$, for some k. Let e_i be the pole orders of σ along the E_i. These are the minimal non-negative numbers such that

$$\sigma \in H^0\big(\widetilde{X}, \Omega^1_{\widetilde{X}} \otimes \mathscr{O}_{\widetilde{X}}(\textstyle\sum e_i E_i)\big).$$

By (3.11), we already know all the e_i are either 0 or 1, and our aim is to show that $e_0 = 0$.
Start of induction. This is the case $\dim X = \operatorname{codim}_X \pi(E_0) = 2$. For surfaces, every exceptional divisor is contracted to a point, so Proposition 3.2 applies.
Inductive step. We distinguish two possibilities: the divisor E_0 may be mapped to a point by π, or it may be mapped to a positive-dimensional variety.
If $\dim \pi(E_0) = 0$, then by the choice of E_0, every exceptional divisor contained in E is contracted to a point, so Proposition 3.2 applies again.
If $\dim \pi(E_0) > 0$, choose general hyperplanes $H_1, \ldots, H_{\dim \pi(E_0)} \subset X$, let H be the intersection $H_1 \cap \cdots \cap H_{\dim \pi(E_0)}$ and $\widetilde{H}$ the preimage $\pi^{-1}(H)$. Applying [GKKP11, Lemmas 2.23 and 2.24], we obtain that H is normal and $\pi|_{\widetilde{H}}$ is a log resolution. The intersection $H \cap \pi(E_0)$ is finite, but nonempty. Shrinking X, we may assume that $H \cap \pi(E_0)$ consists of a single point, say x. Now set $F_x = (\pi|_{\widetilde{H}})^{-1}(x)$ and

$$F_{x,i} = F_x \cap E_i = (\pi|_{E_i})^{-1}(x).$$

Then F_x is the union of the $F_{x,i}$.

Claim 3.13. The subsets $F_{x,0}, \ldots, F_{x,k}$ are smooth, irreducible, and have codimension one in $\widetilde{H}$, while the other $F_{x,i}$ are empty.

Proof. If $0 \le i \le k$, then being a general fibre of $\pi|_{E_i}$, $F_{x,i}$ is smooth of dimension $\dim E_i - \dim \pi(E_0) = \dim \widetilde{H} - 1$. Since $F_{x,i} = \widetilde{H} \cap E_i$, it is an ample divisor on E_i, hence connected and by being smooth it is also irreducible. On the other hand, if $i > k$, then by the choice of $E' = E_0$, we have that $\pi(E_0) \not\subset \pi(E_i)$, and hence $x \notin \pi(E_i)$ and so $F_{x,i} = \emptyset$. $\square$

Claim 3.13 implies that it is possible to apply Proposition 3.10 to $\pi|_{\widetilde{H}} : \widetilde{H} \to H$, $x \in H$, and $F_{x,0}, \ldots, F_{x,k}$, which we will do later.
Now consider the dual of the normal bundle sequence for $\widetilde{H} \subset \widetilde{X}$,

$$0 \longrightarrow N^*_{\widetilde{H}/\widetilde{X}} \longrightarrow \Omega^1_{\widetilde{X}}|_{\widetilde{H}} \xrightarrow{\rho} \Omega^1_{\widetilde{H}} \longrightarrow 0,$$

twist it with $\mathscr{F} := \mathscr{O}_{\widetilde{H}}(\sum e_i E_i|_{\widetilde{H}})$, and restrict to $F_{x,j}$, for $0 \le j \le k$:

$$\begin{tikzcd} N^*_{\widetilde{H}/\widetilde{X}} \otimes \mathscr{F} \arrow[r, "\alpha"] \arrow[d] & \Omega^1_{\widetilde{X}}|_{\widetilde{H}} \otimes \mathscr{F} \arrow[r, "\beta"] \arrow[d, "r_j"] & \Omega^1_{\widetilde{H}} \otimes \mathscr{F} \arrow[d] \\ N^*_{\widetilde{H}/\widetilde{X}} \otimes \mathscr{F}|_{F_{x,j}} \arrow[r, "\alpha_j"] & \Omega^1_{\widetilde{X}}|_{\widetilde{H}} \otimes \mathscr{F}\big|_{F_{x,j}} \arrow[r, "\beta_j"] & \Omega^1_{\widetilde{H}} \otimes \mathscr{F}|_{F_{x,j}}. \end{tikzcd}$$

Since H has smaller dimension than X, the induction hypothesis gives us that $\beta(\sigma|_{\widetilde{H}})$ has no poles, that is

$$\beta(\sigma|_{\widetilde{H}}) \in H^0(\widetilde{H}, \Omega^1_{\widetilde{H}}) \subset H^0(\widetilde{H}, \Omega^1_{\widetilde{H}} \otimes \mathscr{F}). \tag{3.14}$$

Recall that we want to show that $e_0 = 0$. We will show more generally that $e_j = 0$ for all $0 \le j \le k$. So, assume to the contrary that there is an index j with $e_j = 1$. By the definition of the e_i, $\sigma|_{\widetilde{H}}$ as a section in $\Omega^1_{\widetilde{X}}|_{\widetilde{H}} \otimes \mathscr{F}$ does not vanish along $F_{x,j}$. But by (3.14), $\beta(\sigma|_{\widetilde{H}})$ does vanish along $F_{x,j}$. So $r_j(\sigma|_{\widetilde{H}})$ is a nonzero global section in $\ker \beta_j$, which means that $H^0(F_{x,j}, N^*_{\widetilde{H}/\widetilde{X}} \otimes \mathscr{F}|_{F_{x,j}}) \neq 0$.

Now note that $N_{\widetilde{H}/\widetilde{X}}|_{F_{x,j}}$ is trivial, because $N_{\widetilde{H}/\widetilde{X}}$ is the pullback of $N_{H/X}$. Hence from $H^0(F_{x,j}, N^*_{\widetilde{H}/\widetilde{X}} \otimes \mathscr{F}|_{F_{x,j}}) \neq 0$ it follows that $H^0(F_{x,j}, \mathscr{F}|_{F_{x,j}}) \neq 0$. Since this holds for all j with $e_j = 1$, we have a contradiction to Proposition 3.10, showing in particular that $e_0 = 0$ and thus completing the proof of Proposition 3.3. □

4. Proof of Theorem 1.2

The aim of the present section is to prove Theorem 1.2. First, for the reader's convenience we recall some facts about resolutions of singularities.

Lemma 4.1 (Reflexivity is independent of the choice of resolution). *Let X be a normal variety such that $\pi_*\Omega^1_{\widetilde{X}}$ is reflexive for some resolution $\pi\colon \widetilde{X} \to X$. Then $\varphi_*\Omega^1_{X'}$ is reflexive for any resolution $\varphi\colon X' \to X$.*

Proof. Let $\psi\colon \widehat{X} \to X$ be a resolution of X that dominates both $\widetilde{X}$ and X', i.e. we have the following commutative diagram.

$$\begin{tikzcd} & \widehat{X} \arrow[dl, "\widehat{\pi}"'] \arrow[dd, "\psi"] \arrow[dr, "\widehat{\varphi}"] & \\ \widetilde{X} \arrow[dr, "\pi"'] & & X' \arrow[dl, "\varphi"] \\ & X & \end{tikzcd}$$

Since $\widetilde{X}$ and X' are smooth, we have $\widehat{\pi}_*\Omega^1_{\widehat{X}} = \Omega^1_{\widetilde{X}}$ and $\widehat{\varphi}_*\Omega^1_{\widehat{X}} = \Omega^1_{X'}$. We obtain

$$\varphi_*\Omega^1_{X'} = \varphi_*\widehat{\varphi}_*\Omega^1_{\widehat{X}} = \psi_*\Omega^1_{\widehat{X}} = \pi_*\widehat{\pi}_*\Omega^1_{\widehat{X}} = \pi_*\Omega^1_{\widetilde{X}}.$$

Since $\pi_*\Omega^1_{\widetilde{X}}$ is reflexive by assumption, so is $\varphi_*\Omega^1_{X'}$. □

The next theorem is a special case of [GKK10, Cor. 4.7].

Theorem 4.2 (Functorial resolutions). *Let X be a normal variety. Then there exists a resolution $\varphi : X' \to X$ with the property that $\varphi_*\mathscr{T}_{X'}$ is reflexive, i.e. for any vector field ξ on some open subset $U \subset X$, there is a vector field $\widetilde{\xi}$ on $\varphi^{-1}(U)$ that agrees with ξ wherever φ is an isomorphism.*

Sketch of proof. It is a classical fact [Kau65, Satz 3] that vector fields on X are in one-to-one correspondence with *local* $\mathbb{C}$*-actions* on X. Loosely speaking, a local $\mathbb{C}$-action is a $\mathbb{C}$-action such that $t \bullet z$ is only defined for sufficiently small values of $|t|$, dependent on z. For any local $\mathbb{C}$-action, the action map $\mathbb{C} \times X \supset U \to X$ is a smooth morphism.
By [Kol07, Thm. 3.45], there exists a *resolution functor* $\mathscr{R}$ which to any variety assigns a resolution in such a way that smooth morphisms between varieties can be lifted to the resolutions. The resolutions output by $\mathscr{R}$ are called *functorial resolutions*. Let $\varphi\colon X' \to X$ be the functorial resolution of X. Applying the functor $\mathscr{R}$ to the action map associated to a vector field ξ on X, we obtain a diagram

$$\begin{tikzcd} \mathbb{C} \times X' \arrow[d, "\mathrm{id}\times\varphi"'] & U' \arrow[l, hook'] \arrow[r] \arrow[d] & X' \arrow[d, "\varphi"] \\ \mathbb{C} \times X & U \arrow[l, hook'] \arrow[r] & X. \end{tikzcd}$$

One then checks that the map $U' \to X'$ is a local $\mathbb{C}$-action, giving rise to a vector field $\widetilde{\xi}$ on X' which extends ξ as desired.
For a rigorous proof of Theorem 4.2, the reader may consult [GKK10, Sec. 4]. □

Proof of Theorem 1.2. By assumption, we have a log resolution $\pi : \widetilde{X} \to X$ such that $\pi_*\Omega^1_{\widetilde{X}}(\log \widetilde{D})$ is reflexive. We may uniquely write $\widetilde{D} = \widetilde{D}_{bir} + E$, where E is exceptional and no component of $\widetilde{D}_{bir}$ is exceptional. By Theorem 2.1, $\pi_*\Omega^1_{\widetilde{X}}(\log E)$ is reflexive. Then by Theorem 3.1, also $\pi_*\Omega^1_{\widetilde{X}}$ is reflexive.
Let $\varphi\colon X' \to X$ be the functorial resolution from Theorem 4.2, so that vector fields on X can be lifted to X'. By Lemma 4.1, $\varphi_*\Omega^1_{X'}$ is reflexive. Now the proof of [GKKP11, Theorem 6.1] applies verbatim to show that if $\mathscr{T}_X$ is locally free, then X is smooth. □

5. Proof of Theorem 1.4

By Theorem 2.1 we may assume that $\widetilde{D} = \pi^{-1}(\lfloor D \rfloor)_{\mathrm{red}}$. Let E denote the exceptional locus of π. Let $\widehat{D} = \pi_*^{-1}\lfloor D \rfloor + E$. Note that $\widehat{D}$ is obtained from $\widetilde{D}$ by adding finitely many irreducible π-exceptional divisors whose image via π is not contained in $\lfloor D \rfloor$. By [GKKP11, Thm. 16.1], we know that $\pi_*\Omega^1_{\widetilde{X}}(\log \widehat{D})$ is reflexive. Let E_1 be an irreducible component of $\widehat{D}$ that is not contained in $\widetilde{D}$. As we observed above, this means that $\pi(E_1) \not\subset \lfloor D \rfloor$, so by localizing near the general point of $\pi(E_1)$, that is, by further shrinking X, we may assume that $\lfloor D \rfloor = \emptyset$. In this case, $\widehat{D}$ is π-exceptional, hence Theorem 3.1 implies that $\pi_* \Omega^1_{\widetilde{X}}(\log \widehat{D} - E_1)$ is reflexive. We may iterate this process as long as $\widehat{D}$ is larger than $\widetilde{D}$ and so the statement follows. □

6. Optimality of Theorem 1.4

In this section, we show that extension of differential forms as in Theorem 1.4 fails in many cases if one shrinks the pole divisor $\widetilde{D}$ further, or if one considers forms of higher degree.

6.A. Shrinking $\widetilde{D}$ further.

The aim of this subsection is to prove Theorem 1.5. First we need two lemmas.

Lemma 6.1 (Strictly logarithmic poles). *Let X be a smooth variety and $f \in \mathscr{O}_X(X)$ a regular function such that its reduced zero set, $D = \{f = 0\}_{\mathrm{red}} \subset X$, is a divisor with simple normal crossings. Then $\mathrm{d}\log f \in H^0(X, \Omega^1_X(\log D))$, and $\mathrm{d}\log f \notin H^0(X, \Omega^1_X(\log B))$ for any reduced divisor $0 \le B < D$.*

Proof. This follows directly from the definition of logarithmic differentials. □

Lemma 6.2 (Non-reflexivity). *Let (X, Σ) be a pair, where Σ is a reduced divisor, $\pi : \widetilde{X} \to X$ a log resolution of (X, Σ), and $\widetilde{D}$ the largest reduced divisor contained in $\pi^{-1}(\Sigma)$. Let E_0 be an irreducible π-exceptional divisor that is mapped into an effective divisor D whose support is contained in Σ and which is $\mathbb{Q}$-Cartier at the general point of $\pi(E)$. Then the sheaf $\pi_* \Omega^1_{\widetilde{X}}(\log B)$ is not reflexive for any reduced divisor $\pi^{-1}_* \Sigma \le B \le \widetilde{D} - E_0$.*

Proof. By the assumptions, there is an open set $U \subset X$ with $\pi(E_0) \cap U \neq \emptyset$, and a function $f \in \mathscr{O}_X(U)$ cutting out some multiple of D. Set $g = \pi^* f \in \mathscr{O}_{\widetilde{X}}(\pi^{-1}(U))$. By Lemma 6.1,

$$\mathrm{d}\log g \in H^0\big(U \setminus \pi(\mathrm{Exc}(\pi)), \pi_* \Omega^1_{\widetilde{X}}(\log B)\big)$$

but

$$\mathrm{d}\log g \notin H^0\big(U, \pi_* \Omega^1_{\widetilde{X}}(\log B)\big).$$

So $\mathrm{d}\log g$ cannot be extended over the codimension ≥ 2 subset $\pi(\mathrm{Exc}(\pi))$. This implies that $\pi_* \Omega^1_{\widetilde{X}}(\log B)$ is not reflexive. □

Proof of Theorem 1.5. Let E_0 be a component of $\widetilde{D} - B$. Then E_0 is π-exceptional and $\pi^{-1}_* D \le B \le \widetilde{D} - E_0$. If we are in case (1.5.1), Lemma 6.2 applies immediately. Hence we may assume we are in case (1.5.2). Then $\pi(E_0)$ is a point $p \in X$, and we have $p \in D \subset X$, because $E_0 \subset \widetilde{D} \subset \pi^{-1}(\lfloor D \rfloor)$ set-theoretically. We will show that D is $\mathbb{Q}$-Cartier at p, then the claim follows from Lemma 6.2.

By shrinking X we may assume that (X, D) is snc away from p. For $0 < \varepsilon \le 1$, the pair $(X, (1 - \varepsilon)D)$ is numerically dlt [KM98, Ntn. 4.1, Lem. 3.41]. By [KM98, Prop. 4.11], X is $\mathbb{Q}$-factorial. In particular, D is $\mathbb{Q}$-Cartier. □

6.B. Other values of p. The analogue of Theorem 1.4 does not hold for p-forms with $p \geq 2$. Counterexamples may be obtained by taking a p-dimensional normal Gorenstein singularity $z \in Z$ which is log canonical but not klt (notice that this only exists if $p \geq 2$), and considering the product $X = Z \times \mathbb{C}^{n-p}$, for $n \geq p$ arbitrary.
Let σ be a local generator for ω_Z and replace Z with a neighbourhood of z where σ is everywhere defined. Then $\mathrm{pr}_1^*\sigma \in H^0(X_{\mathrm{reg}}, \Omega^p_{X_{\mathrm{reg}}})$ will not be extendable without logarithmic poles on any resolution of singularities of X.
This way one obtains counterexamples to the analogue of Theorem 1.4 for any $p \geq 2$ in arbitrary dimension $n \geq p$.

7. Non-Extension without poles over klt places

In this section, we consider a reduced log canonical pair (X, D) and a log resolution $\pi\colon \widetilde{X} \to X$ of (X, D). Deviating slightly from our previous notation, we let

$$\begin{aligned} E &= \pi_*^{-1}D + \mathrm{Exc}(\pi), \\ E_{\mathrm{nklt}} &= \text{sum of all divisors in } E \text{ with discrepancy } -1, \\ E_{\mathrm{nklt}} \vee \pi^{-1}(D) &= \text{sum of all divisors contained in } E_{\mathrm{nklt}} \text{ or in } \pi^{-1}(D), \\ \widetilde{D} &= \text{largest reduced divisor contained in } \pi^{-1}(\text{non-klt locus}). \end{aligned}$$

pt Then we obviously have

$$E_{\mathrm{nklt}} \leq E_{\mathrm{nklt}} \vee \pi^{-1}(D) \leq \widetilde{D} \leq E.$$

The extension theorem [GKKP11, Thm. 1.5] states that the sheaves $\pi_*\Omega^p_{\widetilde{X}}(\log \widetilde{D})$ are reflexive for all values of p. In [GKKP11, Section 3.B], it was observed that basically by the definition of discrepancy, even the sheaf $\pi_*\Omega^n_{\widetilde{X}}(\log E_{\mathrm{nklt}})$ is reflexive, where $n = \dim X$. This leads to the following natural question.

Question 7.1. *Are the sheaves $\pi_*\Omega^p_{\widetilde{X}}(\log E_{\mathrm{nklt}})$ also reflexive when $p < n$?*

The answer turns out to be "no". However, in the counterexample given in [GKKP11, Ex. 3.2], X is the quadric cone, D consists of two rulings, and the exceptional divisor where extension fails is contained in the preimage of D. This means that [GKKP11, Ex. 3.2] does not answer the following refined version of Question 7.1:

Question 7.2. *Are the sheaves $\pi_*\Omega^p_{\widetilde{X}}\big(\log\big(E_{\mathrm{nklt}} \vee \pi^{-1}(D)\big)\big)$ reflexive when $p < n$?*

In this section, we will give an example showing that even this question has to be answered negatively. First we need a lemma.

Lemma 7.3 (Cusp singularities). *There exists a log canonical Gorenstein surface singularity $0 \in S$ that has a log resolution $\widetilde{S} \to S$ containing two distinct exceptional curves C_1, C_2 which both have discrepancy -1 and whose intersection is non-empty.*

Proof. This follows from the classification of log canonical Gorenstein surface singularities [Kaw88, Sec. 9]. It is also possible to explicitly write down a hypersurface singularity with the desired property. Namely, consider the following polynomial in three variables:

$$f(x, y, z) = x^2(x + z) - y^2 z + z^4.$$

A tedious but routine calculation shows that the origin in $\mathbb{C}^3$ is an isolated singular point of $S = \{f = 0\} \subset \mathbb{C}^3$, so $0 \in S$ is a normal Gorenstein surface singularity. Furthermore, blowing up $0 \in X$ yields a resolution whose exceptional locus consists of a rational curve C_1 with a single node, and which has discrepancy -1. Blowing up that node, we obtain a log resolution containing an additional exceptional rational curve C_2, also of discrepancy -1, such that C_2 and the strict transform of C_1 meet in two points. $\square$

The next theorem tells us that the answer to Question 7.2 is "no".

Theorem 7.4 (Non-extension over klt places, cf. Theorem 1.6). *There exists a three-dimensional reduced log canonical pair (X, D) such that using the notation introduced at the beginning of this section, the sheaf $\pi_* \Omega^2_{\widetilde{X}}\big(\log\big(E_{\mathrm{nklt}} \vee \pi^{-1}(D)\big)\big)$ is not reflexive.*

Proof. Let $0 \in S$ and $p \in C_1 \cap C_2 \subset \widetilde{S}$ be as in Lemma 7.3. Take $X := S \times \mathbb{A}^1_{\mathbb{C}}$ and $D = \emptyset$. Then $X' := \widetilde{S} \times \mathbb{A}^1_{\mathbb{C}} \to X$ is a log resolution of (X, D). On X', blow up a point of the form (p, t) with $t \in \mathbb{A}^1_{\mathbb{C}}$ arbitrary to obtain $f \colon \widetilde{X} \to X'$. This gives a log resolution $\pi : \widetilde{X} \to X$ of (X, D). Denote by $E_0 \subset \widetilde{X}$ the exceptional divisor arising from blowing up the point (p, t), and note that its discrepancy $a(E_0, X, D) = 0$ by [KM98, Lemmata 2.29 and 2.30]. Hence we have the following diagram.

$$\begin{tikzcd}
X = S \times \mathbb{A}^1_{\mathbb{C}} \arrow[d] & X' = \widetilde{S} \times \mathbb{A}^1_{\mathbb{C}} \arrow[l] & \widetilde{X} \supset E_0 \arrow[l, "f"'] \arrow[ll, bend left, "\pi"] \\
S & &
\end{tikzcd}$$

In order to prove the claim, consider a local generator σ of the canonical sheaf ω_S in a neighborhood U of $0 \in S$, and denote its pullback to X' and $\widetilde{X}$ by σ' and $\widetilde{\sigma}$, respectively. Choose local coordinates (u, v, w) on X' and (x, y, z) on $\widetilde{X}$ such that $f \colon \widetilde{X} \to X'$ is given by

$$f(x, y, z) = (x, xy, xz)$$

in these coordinates. Then $E_0 = \{x = 0\}$. Furthermore, we may assume that the exceptional divisor of $X' \to X$ is given by the equation $vw = 0$. Then, up to a unit, we have

$$\sigma' = \mathrm{d} \log v \wedge \mathrm{d} \log w$$

by construction, so

$$\widetilde{\sigma} = f^* \sigma' = (\mathrm{d} \log x + \mathrm{d} \log y) \wedge (\mathrm{d} \log x + \mathrm{d} \log z).$$

This shows that

$$\widetilde{\sigma} \in H^0\big(\pi^{-1}(U \times \mathbb{A}^1_{\mathbb{C}}),\, \Omega^2_{\widetilde{X}}(\log E)\big)$$

has a pole along $E_0 = \{x = 0\}$. However, since $a(E_0, X, D) = 0$ and $D = \emptyset$, we see that E_0 is not contained in $E_{\mathrm{nklt}} \vee \pi^{-1}(D)$. An argument similar to the proof of Lemma 6.2 now yields that $\pi_* \Omega^2_{\widetilde{X}}\big(\log\big(E_{\mathrm{nklt}} \vee \pi^{-1}(D)\big)\big)$ is not reflexive. □

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