Hyperbolic Manifolds of Small Volume

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Abstract. We conjecture that for every dimension $n \neq 3$ there exists a noncompact hyperbolic n-manifold whose volume is smaller than the volume of any compact hyperbolic n-manifold. For dimensions $n \leq 4$ and $n = 6$ this conjecture follows from the known results. In this paper we show that the conjecture is true for arithmetic hyperbolic n-manifolds of dimension $n \geq 30$.

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1. Introduction

By *hyperbolic n-manifold* we mean a complete orientable manifold that is locally isometric to the hyperbolic n-space $\mathbf{H}^n$. Let M be a complete noncompact hyperbolic 3-manifold of finite volume. By Thurston's Dehn surgery theorem there exists infinitely many compact hyperbolic 3-manifolds obtained from M by Dehn filling [15, Section 5.8]. These manifolds all have their volume smaller than $\mathrm{vol}(M)$ [15, Theorem 6.5.6]. It is known that the Dehn filling procedure is specific to manifolds of dimension $n = 3$ (cf. [1, Section 2.1]). We believe that so is the minimality of volume of compact manifolds and that for hyperbolic manifolds in other dimensions the following conjecture should hold:

Conjecture 1. *Let N be a compact hyperbolic manifold of dimension $n \neq 3$. Then there exists a noncompact hyperbolic n-manifold M whose volume is smaller than the volume of N.*

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The set of volumes of hyperbolic n-manifolds is well-ordered (indeed, discrete if $n \neq 3$), thus the conjecture states that the smallest complete hyperbolic n-manifold is noncompact for $n \neq 3$. Conjecture 1 is known to be true for dimensions $n = 2$, 4 and 6: for these n there exist noncompact hyperbolic n-manifolds M with $|\chi(M)| = 1$ [14, 5], whereas it is a general fact that the Euler characteristic of a compact hyperbolic manifold is even (cf. [13, Theorem 4.4]). In this paper we prove the conjecture for *arithmetic* hyperbolic n-manifolds of sufficiently large dimension:

Theorem 2. *Conjecture 1 holds for arithmetic hyperbolic n-manifolds of dimension $n \geq 30$.*

A folklore conjecture states that the smallest volume is always attained on an arithmetic hyperbolic n-manifold (in both compact and noncompact cases). We will refer to this statement as the *minimal volume conjecture*. This conjecture is obvious for $n = 2$, it was proved only recently for $n = 3$ [6], and there is currently no any potential approach to resolving this conjecture in higher dimensions. If true, the minimal volume conjecture together with Theorem 2 would imply Conjecture 1 for dimensions $n \geq 30$. However, Conjecture 1 is weaker than the minimal volume and we think that it might be easier to attack it directly.

At this point we would like to mention the related picture for hyperbolic n-orbifolds. By [2, 3], we know the compact and noncompact minimal volume arithmetic hyperbolic n-orbifolds in all dimensions $n \geq 4$. This is complemented by the previous results for $n = 2$, 3 (see [loc.cit.]). It follows that the smallest arithmetic hyperbolic n-orbifold is compact for $n = 2$, 3, 4 and noncompact for $n \geq 5$. For dimensions $n \leq 9$ we do know the smallest volume noncompact hyperbolic n-orbifolds thanks to the result of Hild's thesis ([8]). It turns out that these orbifolds are arithmetic, supporting the orbifold version of the minimal volume conjecture.

Let us now describe the contents of the paper. The proof of the main theorem is based on arithmetic techniques. We start with the minimal volume arithmetic n-orbifold constructed in [2, 3], which is noncompact for $n \geq 5$. Then we construct a manifold cover of it and try to show that the volume of this manifold is still smaller than the smallest compact arithmetic n-orbifold, which is also known from our previous work. Constructing a manifold cover of an orbifold $\Gamma\backslash\mathbf{H}^n$ is equivalent to finding a torsion-free subgroup of Γ. Our method here is based on Lemma 3, which can be thought of as a variant of Minkowski's lemma that asserts that the kernel of $\mathrm{GL}_N(\mathbb{Z}) \to \mathrm{GL}_N(\mathbb{Z}/m)$ is torsion-free for $m > 2$. Minkowski's lemma is the classical tool to construct hyperbolic manifolds from arithmetic subgroups (see for instance [13, Section 4]). We observe that applying Lemma 3 on two small primes (see Section 2.3) gives better results. In particular we show in Section 3 how our method applies to the problem and proves Theorem 2 for dimensions $n \geq 33$. On the other hand, Minkowski's lemma (with $m = 3$) cannot be used to obtain a direct proof for $n < 50$.

The dimension bound in Theorem 2 can be further improved because the compact orbifolds of small volume considered above do have singularities and hence the volumes of the smallest compact manifolds are larger. In order to obtain better estimates for these volumes we use the arithmetic information encoded in the Euler characteristic. This approach works well for even dimensions (see Sections 5.1 and 5.2). In odd dimensions the Euler characteristic is zero, but we can still use a similar method. In order to do so we consider even dimensional totally geodesic suborbifolds of the compact odd dimensional orbifolds of small volume. The details of the argument are explained in Section 5.3 and its application to the case $n = 31$ is in Section 5.4. It is possible to use similar considerations for smaller dimensions but the number of variants and the computational difficulty increase rapidly for smaller n. The problem becomes very difficult for $n < 10$ and we expect that some new ideas or significant computational advances are required in order to extend our result to this range.

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2. A method of construction

2.1. Let $\mathbf{G}$ be a semisimple algebraic group defined over a number field k. We denote by V_{f} the set of finite places of k. For each $v \in V_{\mathrm{f}}$, we denote by k_v, $\mathcal{O}_v$ and $\mathfrak{f}_v$ the completion, local ring and residue field defined by v. We suppose that $\mathbf{G}$ is simply connected as an algebraic group. By definition, a *principal arithmetic subgroup* of $\mathbf{G}(k)$ is a subgroup $\Lambda_P = \mathbf{G}(k) \cap \prod_{v \in V_{\mathrm{f}}} P_v$, where $P = (P_v)_{v \in V_{\mathrm{f}}}$ is a coherent collection of parahoric subgroups $P_v \subset \mathbf{G}(k_v)$ (see [4]). By Bruhat-Tits theory (see [16]), for each $v \in V_{\mathrm{f}}$, there exists a smooth connected $\mathcal{O}_v$-group scheme $\mathbf{P}_v$ with generic fiber $\mathbf{G}$ and such that $\mathbf{P}_v(\mathcal{O}_v) = P_v$. We denote by $\mathbf{M}_v$ the $\mathfrak{f}_v$-group defined as the maximal reductive quotient of $\mathbf{P}_v/\mathfrak{f}_v$. The reduction map $P_v \to \mathbf{M}_v(\mathfrak{f}_v)$ is surjective. Moreover, since $\mathbf{G}$ is simply connected, we have that $\mathbf{P}_v/\mathfrak{f}_v$ (and thus $\mathbf{M}_v$ as well) is connected.

2.2. Let p be the rational prime above v, i.e., p is the characteristic of the residue field $\mathfrak{f}_v$. Let us denote by $P_v^* \subset P_v$ the pre-image under the reduction map of a p-Sylow subgroup of $\mathbf{M}_v(\mathfrak{f}_v)$.

Lemma 3. *Each torsion element in P_v^* has an order of the form p^s.*

Proof. Let K_v be the kernel of the map $P_v \to \mathbf{P}_v(\mathfrak{f}_v)$. Let $g \in P_v^*$ be an element of finite order q, and denote by $\overline{g}$ its image in $\mathbf{P}_v(\mathfrak{f}_v)$. It suffices to consider the case where q is prime. The image of P_v^* in $\mathbf{P}_v(\mathfrak{f}_v)$ is, by definition, the extension of a p-group by a unipotent $\mathfrak{f}_v$-group. It follows that the image of $\mathbf{P}_v^*$ is itself a p-group, and thus we have either $\overline{g} = 1$, or $q = p$. But in the first case g is contained in K_v, which is a pro-p group (see [11, Lemma 3.8]), and the conclusion follows as well. □

2.3. By choosing two primes v and w above two distinct rational primes, and replacing P_v (resp. P_w) by P_v^* (resp. P_w^*) in the coherent collection P, we obtain by Lemma 3 a torsion-free subgroup $\Lambda_P^{v,w} \subset \Lambda_P$. By strong approximation of $\mathbf{G}$, the index $[\Lambda_P : \Lambda_P^{v,w}]$ equals $[P_v : P_v^*] \cdot [P_w : P_w^*]$.
By construction, the index $[P_v : P_v^*]$ is equal to the order of $\mathbf{M}_v(\mathfrak{f}_v)$ divided by its highest p-factor. We will usually work with parahoric subgroups for which $\mathbf{M}_v$ is semisimple (for instance, P_v hyperspecial), and in this case $[P_v : P_v^*]$ can be easily computed from the type of $\mathbf{M}_v$ using the list of orders of simple groups over a finite field (see [10]).
For example, if $\mathbf{G}/k_v$ is a split group of type B_r and $P_v \subset \mathbf{G}(k_v)$ is hyperspecial, then $\mathbf{M}_v$ is semisimple of the same type and we obtain:

$$[P_v : P_v^*] = \prod_{j=1}^{r} (q_v^{2j} - 1), \tag{1}$$

where q_v denotes the cardinality of the finite field $\mathfrak{f}_v$. The product of two such expressions gives us the index of the torsion-free subgroup $\Lambda_P^{v,w}$ in Λ_P. If Λ_P acts on $\mathbf{H}^n$ we can then compute the volume of the quotient manifold $\Lambda_P^{v,w} \backslash \mathbf{H}^n$ as a product of this index and the covolume of Λ_P.

3. Construction of noncompact manifolds

We will construct noncompact hyperbolic manifolds starting from the arithmetic lattices of minimal covolume considered in [2, 3]. They are best understood as normalizers of principal arithmetic subgroups of $G = \mathrm{Spin}(n,1)$, the latter Lie group being a two-fold covering of $\mathrm{Isom}^+(\mathbf{H}^n)$.

3.1. We showed in [3, Section 2.1] that if a lattice $\Gamma \subset G$ contains the center Z of $\mathrm{Spin}(n,1)$, then the hyperbolic volume $\mathrm{vol}(\Gamma \backslash \mathbf{H}^n)$ is given by $2\mathrm{vol}(S^n)\mu(\Gamma \backslash G)$, where S^n is the n-sphere of constant curvature 1 and μ is the normalization of the Haar measure on G that was used by Prasad in [12], and that the volume is the half of this value otherwise. Suppose now that $\Gamma \subset G$ is torsion-free. Since Z has order 2 in all dimensions, we obtain in this case:

$$\mathrm{vol}(\Gamma \backslash \mathbf{H}^n) = \mathrm{vol}(S^n)\mu(\Gamma \backslash G) \tag{2}$$

If n is even, then $2\mu(\Gamma \backslash G) = |\chi(\Gamma)|$ (cf. [4, Section 4.2]) and equation (2) is equivalent to the Gauss-Bonnet formula (note that a torsion-free $\Gamma \subset G$ is isomorphic to – and thus has same Euler characteristic as – its image in $\mathrm{Isom}^+(\mathbf{H}^n)$):

$$\mathrm{vol}(\Gamma \backslash \mathbf{H}^n) = \frac{\mathrm{vol}(S^n)}{2} |\chi(\Gamma)|, \ n \text{ even}. \tag{3}$$

We recall that the volume of the n-sphere of curvature 1 is given by the following formula:

$$\mathrm{vol}(S^n) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}, \tag{4}$$

where $\Gamma(-)$ is the Gamma function (recall that for an integer m we have $\Gamma(m) = (m-1)!$).

3.2. Let $\Lambda_P \subset \mathrm{Spin}(n,1)$ be the (unique) nonuniform principal arithmetic subgroup whose normalizer realizes the smallest covolume. Like all nonuniform arithmetic hyperbolic lattices it is defined over $k = \mathbb{Q}$, so that the set V_{f} of finite places corresponds to the rational primes. The structure of the coherent collection P was determined in [2, 3]. We denote by $\mathbf{G}$ the algebraic $\mathbb{Q}$-group containing Λ_P, and we will use the notation introduced in Section 2.

3.3. Let us first deal with the case when $n = 2r$ is even. Then $\mathbf{G}$ is of type B_r and the coherent collection P determining Λ_P has the following structure (see [2]): P_v is hyperspecial unless $v = 2$ and $r \equiv 2,3\ (4)$, in which case the associated reductive $\mathbb{F}_2$-group $\mathbf{M}_2$ is semisimple of type ${}^2\mathrm{D}_r$. By Prasad's volume formula [12], the Euler characteristic of Λ_P is then given by

$$|\chi(\Lambda_P)| = 2\lambda_2(r) C(r) \prod_{j=1}^{r} \zeta(2j), \tag{5}$$

with $\lambda_q(r) = 1$ if $r \equiv 0,1\ (4)$ and $\lambda_q(r) = q^r - 1$ otherwise, and the constant $C(r)$ given by

$$C(r) = \prod_{j=1}^{r} \frac{(2j-1)!}{(2\pi)^{2j}}. \tag{6}$$

Using the functional equation for the Riemann zeta function, and expressing its value at negative integers through Bernoulli numbers, we can rewrite (5) as follows:

$$|\chi(\Lambda_P)| = 2\lambda_2(r) \prod_{j=1}^{r} \frac{|B_{2j}|}{4j}. \tag{7}$$

3.4. We can apply the construction presented in Section 2 to Λ_P with $v = 2$ and $w = 3$. We find that for all r, we have:

$$\lambda_2(r)[P_2 : P_2^*] = \prod_{j=1}^{r} (2^{2j} - 1), \tag{8}$$

the case of trivial $\lambda_2(r)$ being just the computation in (1). Let Γ be the (isomorphic) image of $\Lambda_P^{2;3}$ in $\mathrm{Isom}^+(\mathbf{H}^n)$ and $M^n = \Gamma\backslash\mathbf{H}^n$ the corresponding quotient manifold, which by the construction is noncompact and arithmetic. From the previous computation we have:

$$|\chi(M^{2r})| = 2\prod_{j=1}^{r} (4^j - 1)(9^j - 1)\frac{|B_{2j}|}{4j} \tag{9}$$

$$= 2C(r) \prod_{j=1}^{r} (4^j - 1)(9^j - 1)\zeta(2j). \tag{10}$$

n	$\lvert\chi(M^{2r})\rvert$
4	10
6	910
8	3171350
10	725639764850
12	16654568229539490250
14	54376724439679967985482572750
16	33998109351372684068956597092378802073750
18	5272397653068183031816584035192902513000228940543011250

TABLE 1. Euler characteristic of noncompact manifolds M^n

3.5. We now construct noncompact hyperbolic manifolds in odd dimensions $n > 3$. Let $r = (n+1)/2$. Then the $\mathbb{Q}$-group $\mathbf{G}$ containing Λ_P has type D_r. It is an inner form (type ${}^1\mathrm{D}_r$) unless r is even (i.e., $n \equiv 3\,(4)$), in which case it has type ${}^2\mathrm{D}_r$ and becomes inner over $\ell = \mathbb{Q}(\sqrt{-3})$. The details of the construction are given in [3], here we only briefly recall the relevant parts.
Let us first suppose that r is odd. Then P_v is hyperspecial unless $v = 2$ and $r \equiv 3\,(4)$. In the latter case the radical of $\mathbf{M}_2$ is a nonsplit one-dimensional torus, and the semisimple part of $\mathbf{M}_2$ has type ${}^2\mathrm{D}_{r-1}$. If we let

$$\lambda'_q(r) = \frac{(q^r-1)(q^{r-1}-1)}{q+1} \tag{11}$$

for $q = 2$ with $r \equiv 3\,(4)$ and $\lambda'_q(r) = 1$ otherwise, we find that in all cases with odd r we have

$$\lambda'_{q_v}(r)[P_v : P_v^*] = (q_v^r - 1)\prod_{j=1}^{r-1}(q_v^{2j}-1). \tag{12}$$

Moreover, Prasad's formula together with equations (2) and (4) shows that we have (with $C(-)$ defined in (6)):

$$\begin{aligned}
\mathrm{vol}(\Lambda_P) &= \mathrm{vol}(S^n)\lambda'_2(r)\frac{(r-1)!}{(2\pi)^r}\zeta(r)\prod_{j=1}^{r-1}\frac{(2j-1)!}{(2\pi)^{2j}}\zeta(2j) \\
&= \lambda'_2(r)\frac{C(r-1)}{2^{r-1}}\zeta(r)\prod_{j=1}^{r-1}\zeta(2j) && (13)\\
&= \lambda'_2(r)\zeta(r)\prod_{j=1}^{r-1}\frac{|B_j|}{8j}. && (14)
\end{aligned}$$

Thus in this case we obtain a manifold $M^n = \Lambda_P^{2,3}\backslash \mathbf{H}^n$, $n = 2r-1$, r odd, whose volume is given by

$$\mathrm{vol}(M^n) = \zeta(r)(2^r-1)(3^r-1)\prod_{j=1}^{r-1}\frac{(4^j-1)(9^j-1)}{8j}|B_{2j}|. \tag{15}$$

Let us finally consider the remaining case $n = 2r-1$ with r even. In this case all parahoric subgroups P_v in the collection P are hyperspecial with the only exception of $v = 3$ (which ramifies in $\ell = \mathbb{Q}(\sqrt{-3})$), where P_v is special. Since 2 does not split in ℓ, the group $\mathbf{G}$ is an outer form over $\mathbb{Q}_2$ and the $\mathbb{F}_2$-group $\mathbf{M}_2$ associated to the hyperspecial parahoric P_2 is semisimple of type ${}^2\mathrm{D}_r$. Thus we have:

$$[P_2 : P_2^*] = (2^r+1)\prod_{j=1}^{r-1}(2^{2j}-1). \tag{16}$$

For the place $v = 3$, $\mathbf{M}_3$ is semisimple of type B_{r-1}, from which we get

$$[P_3 : P_3^*] = \prod_{j=1}^{r-1}(3^{2j}-1). \tag{17}$$

Using Prasad's formula to compute the covolume of Λ_P, we finally deduce that the manifold $M^n = \Lambda_P^{2,3}\backslash \mathbf{H}^n$, $n = 2r-1$, r even, has volume

$$\mathrm{vol}(M^{2r-1}) = 3^{r-1/2}(2^r+1)\zeta_\ell(r)/\zeta(r)\prod_{j=1}^{r-1}\frac{(4^j-1)(9^j-1)}{8j}|B_{2j}|. \tag{18}$$

3.6. Formulas (15) and (18) allow us to evaluate the volume of the constructed manifolds M^n using Pari/GP. Moreover, for n even the volume can be deduced from formula (9) together with (3). We list the values obtained for dimensions $n \le 20$ in Table 2. It shows, in particular, that the volume grows rapidly with the dimension.

4. Proof of Theorem 2 for sufficiently large dimension

4.1. We now compare the volume of M^n from the previous section with the volume of the smallest compact arithmetic hyperbolic n-orbifold $O^n = \Gamma\backslash\mathbf{H}^n$. In the rest of the paper the Euler characteristic $\chi(O^n)$ is to be understood in the orbifold sense, that is, $\chi(O^n) = \chi(\Gamma)$.
We begin with even dimensions $n = 2r \ge 4$. The orbifold O^n is defined over the field $k = \mathbb{Q}(\sqrt{5})$ and its Euler characteristic is given by

$$|\chi(O^n)| = 4\cdot\overline{\lambda}_4(r)\cdot 5^{r^2+r/2}C(r)^2\prod_{j=1}^{r}\zeta_k(2j), \tag{19}$$

with $\overline{\lambda}_q(r) = 1$ if r is even and $\overline{\lambda}_q(r) = \frac{q^r-1}{2}$ if r is odd [2].

n	$\mathrm{vol}(M^n)$
4	131.594
5	273.467
6	15048.379
7	42504.453
8	47073267.939
9	770938537.303
10	7519493827964.029
11	305396253769850.768
12	98579836734072034892.809
13	36424053767874477954431.531
14	1.555 E29
15	2.059 E32
16	4.074 E40
17	6.691 E44
18	2.335 E54
19	1.797 E59
20	3.734 E70

Table 2. Approximate values of the volume of M^n

By (10) and (19), we have:

$$\frac{|\chi(O^n)|}{|\chi(M^n)|} = \frac{4 \cdot \overline{\lambda}_4(r) \cdot 5^{r^2+r/2} C(r)^2 \prod_{j=1}^r \zeta_k(2j)}{2C(r) \prod_{j=1}^r (4^j - 1)(9^j - 1)\zeta(2j)}. \tag{20}$$

Using the basic inequalities $\zeta_k(2j) > 1$ and $\prod_{j=1}^r \zeta(2j) < 2$ plus the fact that $\overline{\lambda}_4(r) \geq 1$ for all r, we obtain the lower bound

$$\frac{|\chi(O^n)|}{|\chi(M^n)|} > \frac{5^{r^2+r/2} C(r)}{\prod_{j=1}^r (4^j - 1)(9^j - 1)} > \frac{5^{r^2+r/2} C(r)}{36^{r^2/2+r/2}}. \tag{21}$$

By Stirling's formula,

$$C(r) = \prod_{j=1}^{r} \frac{(2j-1)!}{(2\pi)^{2j}} > \prod_{j=1}^{r} \frac{(2j-1)^{2j-1}}{2\pi(2\pi e)^{2j-1}}.$$

Applying the Euler–Maclaurin summation formula, we obtain

$$\log C(r) \gtrsim \frac{1}{4}(2r-1)^2 \log(2r-1),$$

which is $\gg cr^2$ (for any constant c). Hence for $r \gg 0$, we have

$$\frac{|\chi(O^n)|}{|\chi(M^n)|} \gg 1.$$

In fact, a computation shows that it is enough to take $r \geq 18$ for the ratio in the right hand side of (21) to be > 1. A more precise computation of the expression in (20) using Pari/GP shows that

$$\frac{|\chi(O^n)|}{|\chi(M^n)|} > 1, \text{ for } n = 2r \geq 34.$$

Since O^n has the smallest volume among compact arithmetic hyperbolic n-orbifolds, this proves Theorem 2 for these values of n.

4.2. Now consider the odd dimensional case $n = 2r - 1 \geq 5$.
The smallest compact odd dimensional arithmetic orbifold O^n is again defined over $k = \mathbb{Q}(\sqrt{5})$. Its volume is given by

$$\operatorname{vol}(O^n) = \frac{5^{r^2-r/2} \cdot 11^{r-1/2} \cdot (r-1)!}{2^{2r-1}\pi^r} \cdot L_{\ell_0|k}(r) \cdot C(r-1)^2 \prod_{j=1}^{r-1} \zeta_k(2j), \tag{22}$$

where ℓ_0 is the quartic field with a defining polynomial $x^4 - x^3 + 2x - 1$ and $L_{\ell_0|k} = \zeta_{\ell_0}/\zeta_k$ [3, Thoerem 1].
Note that for any $s \geq 3$ we have $L_{\ell_0|k}(s) \geq 1/\zeta_k(3) > 0.973$.
By (15), (18) and (22), we have

$$\frac{\operatorname{vol}(O^n)}{\operatorname{vol}(M^n)} = \frac{\frac{5^{r^2-r/2} \cdot 11^{r-1/2} \cdot (r-1)!}{2^{2r-1}\pi^r} \cdot L_{\ell_0|k}(r) \cdot C(r-1)^2 \prod_{j=1}^{r-1} \zeta_k(2j)}{A(r) \cdot \frac{C(r-1)}{2^{r-1}} \cdot \prod_{j=1}^{r-1} \zeta(2j)(4^j-1)(9^j-1)}, \tag{23}$$

where $A(r) = \zeta(r)(2^r - 1)(3^r - 1)$ if r is odd and $A(r) = 3^{r-1/2}(2^r + 1)\zeta_\ell(r)/\zeta(r)$, $\ell = \mathbb{Q}(\sqrt{-3})$, if r is even, and hence

$$A(r) < 6^r \cdot 2 \text{ for all } r. \tag{24}$$

Therefore,

$$\frac{\operatorname{vol}(O^n)}{\operatorname{vol}(M^n)} > \frac{5^{r^2-r/2} \cdot 11^{r-1/2} \cdot (r-1)! \cdot C(r-1)}{2^r \pi^r 6^r \cdot 4 \cdot \prod_{j=1}^{r-1}(4^j-1)(9^j-1)} \tag{25}$$

$$> \frac{5^{(r-1)(r-1/2)} C(r-1)}{36^{r^2/2-r/2}} \cdot \frac{11^{r-1/2} \cdot (r-1)!}{5^{1/2-r} \cdot 4 \cdot (12\pi)^r}. \tag{26}$$

Now the first factor has the same form as the ratio in (21), hence by the previous section it is > 1 for $r - 1 \geq 18$. It is an easy exercise to show that for such r, the second factor is > 1 as well.
We can improve the bound for r by evaluating (23) using Pari/GP. This gives

$$\frac{|\chi(O^n)|}{|\chi(M^n)|} > 1, \text{ for } n = 2r - 1 \geq 33.$$

Together with the even dimensional part it proves Theorem 2 for $n \geq 33$.

5. Lowering the dimension bound

5.1. Improving the bound for dimension requires more careful analysis. We will begin again with the even dimensional case.
The first dimension to consider is $n = 32$. Here we have a noncompact arithmetic manifold M^n with Euler characteristic $|\chi(M^n)| = 2.354\ldots \cdot 10^{228}$. Ironically, we call it "small", but our study shows that small manifolds in high dimensions tend to have huge volume. The smallest compact arithmetic n-orbifold O^n has $|\chi(O^n)| = 8.777\ldots \cdot 10^{217}$, which is less than $|\chi(M^n)|$. Using PARI/GP we can compute (see remark below) the precise value of $|\chi(O^n)|$, which is a rational number. Its denominator equals

$$D = 107887196930872715055177987172922818560000000000000000000,$$

which is $\sim 10^{56}$. A manifold covering of O^n has to have an (even) integer Euler characteristic, hence its degree is divisible by D. It follows that the volume of the smallest possible manifold cover of O^n is much larger than $\mathrm{vol}(M^n)$.

Remark 4. A way to compute the exact value of $|\chi(O^n)|$ is to first use the functional equation for ζ_k to transform equation (19) into a product of a (known) rational factor and values $\zeta_k(1-2j)$, which are rational. Using PARI/GP the numerical values of $\zeta_k(1-2j)$ can be approximated by rationals, leading to the number D (it is enough to work with precision `\p 80`). With this method based on approximation, the value obtained for D can only be used as a lower bound for the actual denominator of $|\chi(O^n)|$, but this is already sufficient for our purpose. However, a cleaner way to proceed is to use the library of special values computed by Alvaro Lozano-Robledo, who obtained them through a procedure under PARI/GP that computes generalized Bernoulli numbers (see [9]). This guarantees the correctness of the value computed for D.

In order to complete the discussion for $n = 32$ we need to consider other maximal arithmetic subgroups Γ whose covolume is $< \mathrm{vol}(M^n)$. Recall that any maximal arithmetic subgroup is a normalizer of a principal subgroup. Maximal arithmetic subgroups other than the one defining O^n can be either subgroups defined over $k = \mathbb{Q}(\sqrt{5})$ or arithmetic subgroups defined over other totally real fields. In the latter case, by [2], we know that the next smallest covolume subgroup has the field of definition $\mathbb{Q}(\sqrt{2})$. By the volume formula, the absolute value of its Euler characteristic is $> 9.071 \cdot 10^{271}$, which is already bigger than $|\chi(M^n)|$. Therefore, we are only left with the groups defined over k. The covolume of any such subgroup Γ would have an additional lambda factor in comparison with $\mathrm{vol}(O^n)$. By [2, Sections 2.3 and 3.2], the smallest possible lambda factor is $\lambda_v = \frac{q_v^{16}+1}{2}$ for $q_v = 4$, and it follows that the orbifold corresponding to the principal arithmetic subgroup Λ that Γ normalizes has an Euler characteristic of absolute value $1.884\ldots \cdot 10^{227}$ with denominator $2D$. Hence all manifold covers of this orbifold are much larger than M^n. Now, the index $[\Gamma : \Lambda]$ can be computed the same way as it is done in [2] and [3] and in particular, since the field $k = \mathbb{Q}(\sqrt{5})$ has class number one, this index is a power of 2 (whose exponent can be bounded). Using this the result just obtained immediately extends to the manifold covers of $\Gamma\backslash\mathbf{H}^{32}$. The same argument excludes the case of orbifolds with a lambda factor λ_v corresponding to $q_v = 5$. All other possible lambda factors give rise to orbifolds of volume larger than $\mathrm{vol}(M^n)$. This shows that Theorem 2 is true for $n = 32$.

5.2. Dimension $n = 30$ is treated in a similar way. Here we have a noncompact arithmetic manifold M^n with $|\chi(M^n)| = 1.252\ldots\cdot 10^{195}$. The smallest volume compact arithmetic orbifold O^n has $|\chi(O^n)| = 8.112\ldots\cdot 10^{187}$ with denominator $5.231\ldots\cdot 10^{48}$, hence its manifold covers are larger than M^n. The smallest covolume cocompact arithmetic subgroup defined over $k \neq \mathbb{Q}(\sqrt{5})$ has the field of definition $\mathbb{Q}(\sqrt{2})$ and $|\chi| > 3.116\cdot 10^{235}$, which is bigger than $|\chi(M^n)|$. For the other maximal arithmetic subgroups defined over $\mathbb{Q}(\sqrt{5})$ we have a different lambda factor in the volume formula. The values of λ for which $|\chi|$ of the principal subgroup is smaller than $|\chi(M^n)|$ are listed below:

λ	$\lvert\chi\rvert$	denominator of χ
$\frac{1}{2}(5^{15}-1)$	$2.305\ldots\cdot 10^{189}$	$823918598262407709060193357261824\cdot 10^{18}$
$\frac{1}{2}(9^{15}-1)$	$1.555\ldots\cdot 10^{193}$	$4115477513798240304995971 8912\cdot 10^{18}$
$\frac{1}{2}(11^{15}-1)$	$3.155\ldots\cdot 10^{194}$	$32109284800854974718802972372893696\cdot 10^{16}$

As before, considering maximal subgroups instead of principal subgroups does not change the picture and it follows that a manifold cover in each of these cases is larger than the noncompact manifold M^n.

5.3. Odd dimensional case presents us a different challenge: here the Euler characteristic is zero and we cannot take advantage of its integral properties in order to bound the degree of the smooth covers. One of the possible ways to proceed is to look at the orders of finite subgroups of $\pi_1(O^n)$. This indeed gives a bound for the degree, however, it is much smaller than bounds provided by the denominators of Euler characteristic in the neighboring even dimensions. Considerably stronger results can be obtained based on the following simple observation:

> Small volume arithmetic hyperbolic orbifolds tend to contain totally geodesic even dimensional suborbifolds whose Euler characteristic can be used to obtain good bounds on the degrees of the smooth covers.

Indeed, assume that a group Γ has an infinite index subgroup Γ' and a torsion-free finite index subgroup Γ_M with $[\Gamma : \Gamma_M] = f$. Then $\Gamma_M \cap \Gamma'$ is a torsion-free subgroup of Γ' of index $d \leq f$:

$$\begin{tikzcd} \Gamma \arrow[r, dash, "\infty"] \arrow[d, dash, "f"] & \Gamma' \arrow[d, dash, "d"] \\ \Gamma_M \arrow[r, dash, "\infty"] & \Gamma_M \cap \Gamma' \end{tikzcd} \tag{27}$$

This proposition can be easily checked by looking at the cosets Γ/Γ_M and $\Gamma'/(\Gamma_M \cap \Gamma')$. It appears to be very useful for bounding degrees of the smooth covers in odd dimensions.

5.4. We demonstrate the application of the method from 5.3 for dimension $n=31$. By [3, Section 3.5], the minimal volume compact orbifold O^{31} is defined by the quadratic form

$$f_{31}(x_0,x_1,\dots,x_{31}) = (3-2\sqrt{5})x_0^2 + x_1^2 + \dots x_{31}^2.$$

It has $\mathrm{vol}(O^{31}) = 2.415\ldots\cdot 10^{200} < \mathrm{vol}(M^{31}) = 3.113\ldots\cdot 10^{202}$, so we can not a priori exclude the possibility of some manifold cover of O^{31} being smaller than M^{31}.

A restriction f_{30} of the form f_{31} to the first 31 coordinates defines a totally geodesic suborbifold $O_1^{30} \subset O^{31}$ that has a non-zero Euler characteristic. In order to compute $\chi(O_1^{30})$ we need to control the associated integral structure. In what follows we will use the notation from [3] and [7]. Consider the quadratic space $(V, \frac{1}{2}f_{31})$. We have the discriminant $\delta(\frac{1}{2}f_{31}) = \frac{1}{2^{32}}(3-2\sqrt{5})$ and the Hasse-Witt invariant $\omega(V,\frac{1}{2}f_{31}) = 1$ over all places including the dyadic place $v_2 = (2)$ of $k = \mathbb{Q}(\sqrt{5})$. Comparing this data with the local description of the group Γ of O^{31} given in [3, Section 3], we conclude that Γ is isomorphic to the stabilizer of a maximal lattice L in $(V,\frac{1}{2}f_{31})$ (this can be also confirmed by comparing the covolume of $\mathrm{Stab}(L)$ computed using [7, Table 3] with $\mathrm{vol}(O^{31})$ and recalling the uniqueness of O^{31}). The lattice L is defined uniquely up to a conjugation under G but different (conjugate) lattices may have different restriction to the subspace $(V', \frac{1}{2}f_{30})$ of V. Geometrically this corresponds to choosing different totally geodesic 30-dimensional suborbifolds of O^{31}. In order to complete the computation we need to fix the lattice L. This can be done as follows: First take a maximal lattice L' in $(V', \frac{1}{2}f_{30})$. Now consider an integral lattice in V generated by L' and the vector $(0,\ldots,0,2)$. It is contained in some maximal lattice in V and we can choose this lattice to be our lattice L. This construction implies that the restriction of L onto the subspace $V' \subset V$ is the maximal lattice L' in $(V', \frac{1}{2}f_{30})$. Now we can use [7, Table 4] to compute the covolume of $\mathrm{Stab}(L')$, and hence the Euler characteristic of the corresponding suborbifold:

$$|\chi(O_1^{30})| = \frac{4^{15}-1}{2}\cdot\frac{11^{15}+1}{2}\cdot\frac{1}{4^{14}}\prod_{i=1}^{15}|\zeta_k(1-2i)| = 1.694\ldots\cdot 10^{202},$$

with denominator

$$D_1 = 2906238441842707968466291261440000000000000000000.$$

Hence by (27), the degree f of any smooth cover of O^{31} is $\geq D_1$, and so its volume

$$\mathrm{vol}(O^{31})\cdot f > 7.019\ldots\cdot 10^{247} > \mathrm{vol}(M^{31}). \tag{28}$$

It remains to check if there are other compact arithmetic 31-dimensional orbifolds that have volume $\leq \mathrm{vol}(M^{31})$. Similar to the previous discussion, these can be either defined over a different field k, or have a different splitting field ℓ, or have a non-trivial lambda factor in the volume formula. Note that by (23) the ratio

$$\frac{\mathrm{vol}(O^{31})}{\mathrm{vol}(M^{31})} > 0.007. \tag{29}$$

If we change the defining field then its discriminant in the volume formula for $\mathrm{vol}(O^{31})$ would contribute a factor of at least $(8/5)^{r^2-r/2}(1/11)^{r-1/2} > 3.021 \cdot 10^{34}$ (cf. [3]). This immediately brings it out of the range of consideration. Changing the field ℓ gives a factor $\geq \left(\frac{400}{275}\right)^{31/2} = 332.868\ldots$ (as the field ℓ that corresponds to the minimal volume orbifold has discriminant $\mathcal{D}_\ell = 275$ and the next possible value is $\mathcal{D} = 400$). This factor is already sufficiently large to make the volume $> \mathrm{vol}(M^{31})$. Finally, following [3, equation (5)], the lambda factor is bounded below by $\frac{2}{3}\left(\frac{3}{4}q_v\right)^{r_v}$ with $q_v \geq 4$, $r_v = 16$ if $q_v \neq 11$ and $r_v = 15$ for the remaining place v of k that ramifies in ℓ/k. Hence the smallest possible lambda factor is at least 28697814, which again makes the volume of the corresponding orbifolds too large. Similar to the previous sections, this argument based on principal arithmetic subgroups extends to maximal arithmetics in a straightforward way.

This finishes the proof of Theorem 2. $\square$

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