Special Values of Anticyclotomic Rankin-Selberg L-Functions

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Abstract. In this article, we construct a class of anticyclotomic p-adic Rankin-Selberg L-functions for Hilbert modular forms, generalizing the construction of Brakočević, Bertolini, Darmon and Prasanna in the elliptic case. Moreover, building on works of Hida, we give a necessary and sufficient criterion for the vanishing of the Iwasawa μ-invariant of this p-adic L-function vanishes and prove a result on the non-vanishing modulo p of central Rankin-Selberg L-values with anticyclotomic twists. These results have future applications to Iwasawa main conjecture for Rankin-Selberg convolution and Iwasawa theory for Heegner cycles.

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Introduction

The purpose of this article is to (i) construct a class of anticyclotomic Rankin-Selberg p-adic L-functions for Hilbert modular forms and study the vanishing/non-vanishing of the associated Iwasawa μ-invariant, (ii) prove a result on the non-vanishing modulo p of central Rakin-Selberg L-values with anticyclotomic twists. Let $\mathcal{F}$ be a totally real algebraic number field of degree d over $\mathbf{Q}$ and $\mathcal{K}$ be a totally imaginary quadratic extension of $\mathcal{F}$. Denote by $z \mapsto \overline{z}$ the non-trivial element in $\mathrm{Gal}(\mathcal{K}/\mathcal{F})$. Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_{\mathcal{F}})$ with unitary central character ω. Let $\pi_{\mathcal{K}}$ be a lifting of π to $\mathcal{K}$ constructed in [Jac72, Thm. 20.6]. Then $\pi_{\mathcal{K}}$ is an irreducible automorphic representation of $\mathrm{GL}_2(\mathbf{A}_{\mathcal{K}})$, which is cuspidal if π is not

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obtained from the automorphic induction from $\mathcal{K}/\mathcal{F}$. Let $\lambda : \mathbf{A}_{\mathcal{K}}^{\times}/\mathcal{K}^{\times} \to \mathbf{C}^{\times}$ be a unitary Hecke character of $\mathcal{K}^{\times}$ such that

$$\lambda|_{\mathbf{A}_{\mathcal{F}}^{\times}} = \omega^{-1}. \tag{0.1}$$

The automorphic representation $\pi_{\mathcal{K}} \otimes \lambda$ is therefore conjugate self-dual. For each place v of $\mathcal{F}$, we can associate a local L-function $L(s, \pi_{\mathcal{K}_v} \otimes \lambda_v)$ and a local epsilon factor $\varepsilon(s, \pi_{\mathcal{K}_v} \otimes \lambda_v, \psi_v)$ (which depends on a choice of non-trivial character $\psi_v : \mathcal{F}_v \to \mathbf{C}^{\times}$) to the local constituent $\pi_{\mathcal{K}_v} \otimes \lambda_v$ of $\pi_{\mathcal{K}} \otimes \lambda$ ([JL70, Thm. 2.18 (iv)]). Denote by $L(s, \pi_{\mathcal{K}} \otimes \lambda)$ the global L-function obtained by the meromorphic continuation of the Euler product of local L-functions at all *finite* places. In this paper, we study the p-adic variation of the central value $L(\frac{1}{2}, \pi_{\mathcal{K}} \otimes \lambda)$ with anticyclotomic twists under certain hypotheses.

To introduce our hypotheses precisely, we need some notation. Fix a CM-type Σ of $\mathcal{K}$. Namely, Σ is a subset of $\mathrm{Hom}(\mathcal{K}, \mathbf{C})$ such that

$$\Sigma \sqcup \overline{\Sigma} = \mathrm{Hom}(\mathcal{K}, \mathbf{C})\,;\ \Sigma \cap \overline{\Sigma} = \emptyset.$$

Then Σ induces an identification $\mathcal{K} \otimes_{\mathbf{Q}} \mathbf{R} \simeq \mathbf{C}^{\Sigma}$. We shall identify Σ with the set of archimedean places of $\mathcal{F}$ via the restriction. For each $k = \sum_{\sigma \in \Sigma} k_\sigma \sigma \in \mathbf{Z}[\Sigma]$, we write $\Gamma_\Sigma(k) = \prod_{\sigma \in \Sigma} \Gamma(k_\sigma)$ (Γ is the usual Gamma function), and if $x = (x_\sigma) \in (A^{\times})^{\Sigma}$ for an algebra A, we let $x^k := \prod_{\sigma \in \Sigma} x_\sigma^{k_\sigma}$. For a Hecke character $\chi : \mathbf{A}_{\mathcal{K}}^{\times}/\mathcal{K}^{\times} \to \mathbf{C}^{\times}$, we denote by $\chi_\infty : (\mathcal{K} \otimes_{\mathbf{Q}} \mathbf{R})^{\times} \to \mathbf{C}^{\times}$ its archimedean component, and we say χ is of infinity type (k_1, k_2) for $k_1, k_2 \in 2^{-1}\mathbf{Z}[\Sigma]$ such that $k_1 - k_2 \in \mathbf{Z}[\Sigma]$ if

$$\chi_\infty(z) = z^{k_1 - k_2}(z\overline{z})^{k_2} \text{ for all } z \in (\mathcal{K} \otimes_{\mathbf{Q}} \mathbf{R})^{\times} \simeq (\mathbf{C}^{\times})^{\Sigma}.$$

For each ideal $\mathfrak{a}$ of $\mathcal{F}$ (resp. ideal $\mathfrak{A}$ of $\mathcal{K}$), we have a unique factorization $\mathfrak{a} = \mathfrak{a}^+\mathfrak{a}^-$ (resp. $\mathfrak{A} = \mathfrak{A}^+\mathfrak{A}^-$), where $\mathfrak{a}^+$ (resp. $\mathfrak{A}^+$) is only divisible by primes split in $\mathcal{K}$ and $\mathfrak{a}^-$ (resp. $\mathfrak{A}^-$) is divisible by primes inert or ramified in $\mathcal{K}$. Let $\mathfrak{n} = \mathfrak{n}^+\mathfrak{n}^-$ be the conductor of π. We define the normalized local root number attached to $\pi_{\mathcal{K}_v}$ and λ_v for each place v by

$$\varepsilon^*(\pi_{\mathcal{K}_v}, \lambda_v) := \varepsilon(\frac{1}{2}, \pi_{\mathcal{K}_v} \otimes \lambda_v, \psi_v) \cdot \omega_v(-1).$$

We remark that the value $\varepsilon(\frac{1}{2}, \pi_{\mathcal{K}_v} \otimes \lambda_v, \psi_v)$ does not depend on the choice of ψ_v.

We assume that π has infinity type $k = \sum_\sigma k_\sigma \sigma \in \mathbf{Z}_{>0}[\Sigma]$ and λ has infinity type $(\frac{k}{2}, -\frac{k}{2})$. In other words, π_σ is a discrete series or limit of discrete series of weight k_σ with unitary central character at every archimedean place σ. In particular, this implies that $\{k_\sigma\}_{\sigma \in \Sigma}$ have the same parity and the local root number $\varepsilon^*(\pi_{\mathcal{K}_v}, \lambda_v) = +1$ at every archimedean place. We further assume the following local root number hypothesis for (π, λ):

Hypothesis A. *The local root number $\varepsilon^*(\pi_{\mathcal{K}_v}, \lambda_v) = +1$ for each $v \mid \mathfrak{n}^-$.*

In particular, the above hypothesis holds if $\mathfrak{n}^- = (1)$.

We prepare some notation in Iwasawa theory. Let p be an odd rational prime. Fix an embedding $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and isomorphism $\iota : \mathbf{C} \stackrel{\sim}{\to} \mathbf{C}_p$ once and for all. Throughout this article, we make the following assumption

$$\text{(ord)} \qquad \Sigma \text{ is } p\text{-ordinary.}$$

Let Σ_p be the set of p-adic places of $\mathcal{K}$ induced by Σ. Then the ordinary assumption (ord) means that Σ_p and its complex conjugation $\overline{\Sigma}_p$ gives a full partition of the set of p-adic places of $\mathcal{K}$. If L is a number field, we write $G_L = \mathrm{Gal}(\overline{\mathbf{Q}}/L)$ for the absolute Galois group. Denote by $\mathrm{rec}_\mathcal{K} : \mathbf{A}_\mathcal{K}^\times \to G_\mathcal{K}^{ab}$ the geometrically normalized reciprocity law. Recall that we say a continuous character $\widehat{\phi} : G_\mathcal{K}^{ab} \to \mathbf{C}_p^\times$ is locally algebraic of weight (k_1, k_2) with $k_1, k_2 \in \mathbf{Z}[\Sigma]$ if $\chi(\mathrm{rec}_\mathcal{K}(a)) = a^{k_1}\overline{a}^{k_2}$ for every $a \in (\mathcal{K} \otimes \mathbf{Q}_p)^\times$ close to 1 (See also [Ser68, Chapter III, §2]). Let $\mathcal{K}_{p^\infty}^-$ be the maximal anticyclotomic $\mathbf{Z}_p^{[\mathcal{F}:\mathbf{Q}]}$-extension of $\mathcal{K}$. Let $\Gamma^- = \mathrm{Gal}(\mathcal{K}_{p^\infty}^-/\mathcal{K})$ and let $\Lambda = \overline{\mathbf{Z}}_p[\![\Gamma^-]\!]$ be the Iwasawa algebra of $[\mathcal{F} : \mathbf{Q}]$-variables. To each locally algebraic p-adic character $\widehat{\phi} : \Gamma^- \to \mathbf{C}_p^\times$ of weight $(m, -m)$, we can associate a Hecke character $\phi : \mathbf{A}_\mathcal{K}^\times/\mathcal{K}^\times \to \mathbf{C}^\times$ of infinity type $(m, -m)$ defined dy

$$\phi(a) := \iota^{-1}(\widehat{\phi}(\mathrm{rec}_\mathcal{K}(a))a_p^{-m}\overline{a}_p^m)a_\infty^m\overline{a}_\infty^{-m},$$

where $a_p \in (\mathcal{K} \otimes_\mathbf{Q} \mathbf{Q}_p)^\times$ and $a_\infty \in (\mathcal{K} \otimes_\mathbf{Q} \mathbf{R})^\times$ are the p-component and the archimedean component of a respectively. We say $\widehat{\phi}$ is the *p-adic avatar* of ϕ. We shall call $\mathfrak{X}_p^{\mathrm{crit}}$ the set of *critical specializations*, consisting of locally algebraic p-adic characters on Γ^- of weight $(m, -m)$ with $m \in \mathbf{Z}_{\geq 0}[\Sigma]$ (See §5.4).

Our first result is the construction of the anticyclotomic p-adic L-function attached to $\pi_\mathcal{K} \otimes \lambda$. We need more notation. Let $\mathcal{D}_\mathcal{F}$ be the different of $\mathcal{F}$ and $\mathcal{D}_{\mathcal{K}/\mathcal{F}}$ be the relative different of $\mathcal{K}/\mathcal{F}$. Let $\mathfrak{N}$ be the prime-to-p conductor of $\pi_\mathcal{K} \otimes \lambda$. We have a unique decomposition $\mathfrak{N} = \mathfrak{N}^+\mathfrak{N}^-$ and fix a decomposition $\mathfrak{N}^+ = \mathfrak{F}\overline{\mathfrak{F}}$ with $(\mathfrak{F}, \overline{\mathfrak{F}}) = 1$ such that $\mathfrak{N}^-$ is only divisible by prime inert or ramified in $\mathcal{K}/\mathcal{F}$ and $\mathfrak{F}$ is only divisible by primes split in $\mathcal{K}/\mathcal{F}$. We choose $\delta \in \mathcal{K}$ such that

- $\overline{\delta} = -\delta$,
- $\mathrm{Im}\,\sigma(\delta) > 0$ for all $\sigma \in \Sigma$,
- The polarization ideal $\mathfrak{c}(\mathcal{O}_\mathcal{K}) := \mathcal{D}_\mathcal{F}^{-1}(2\delta\mathcal{D}_{\mathcal{K}/\mathcal{F}})$ is prime to $p\mathfrak{N}\overline{\mathfrak{N}}$.

Let $(\Omega_\infty, \Omega_p)$ be the complex and p-adic periods attached to $(\mathcal{K}, \Sigma)$ defined in [HT93, (4.4a), (4.4b)]. For each Hecke character χ of $\mathcal{K}^\times$, we define the p-adic multiplier $e_{\Sigma_p}(\pi, \chi)$ by

$$\text{(0.2)} \quad e_{\Sigma_p}(\pi, \chi) := \prod_{\mathfrak{P} \in \Sigma_p,\, \mathfrak{p} = \mathfrak{P}\overline{\mathfrak{P}}} \varepsilon(\frac{1}{2}, \pi_\mathfrak{p} \otimes \chi_{\overline{\mathfrak{P}}}, \psi_\mathfrak{p})L(\frac{1}{2}, \pi_\mathfrak{p} \otimes \chi_{\overline{\mathfrak{P}}})^{-2}\chi_{\overline{\mathfrak{P}}}^{-2}(-2\delta).$$

The shape of this modified p-Euler factor $e_{\Sigma_p}(\pi, \chi)$ has been suggested by J. Coates [Coa89].

Theorem A. *In addition to* (ord) *and Hypothesis A, we further assume that*

(sf) $\mathfrak{n}^-$ *is square-free.*

Then there exists an element $\mathscr{L}_{\Sigma_p}(\pi,\lambda) \in \Lambda$ *such that for every* $\widehat{\phi} \in \mathfrak{X}_p^{\mathrm{crit}}$ *of weight* $(m,-m)$, *we have*

$$\frac{\widehat{\phi}(\mathscr{L}_{\Sigma_p}(\pi,\lambda)^2)}{\Omega_p^{2(k+2m)}} = \frac{\Gamma_\Sigma(k+m)\Gamma_\Sigma(m+1)}{(\operatorname{Im}\delta)^{k+2m}(4\pi)^{k+2m+1\cdot\Sigma}} \cdot e_{\Sigma_p}(\pi,\lambda\phi) \cdot \frac{L(\frac{1}{2},\pi_{\mathcal{K}}\otimes\lambda\phi)}{\Omega_{\mathcal{K}}^{2(k+2m)}} \times [\mathcal{O}_{\mathcal{K}}^\times : \mathcal{O}_{\mathcal{F}}^\times]^2 \cdot \phi(\mathfrak{F}^{-1}),$$

where $\Omega_{\mathcal{K}} = (2\pi i)^{-1}\Omega_\infty$, $(\operatorname{Im}\delta) = (\operatorname{Im}\sigma(\delta))_{\sigma\in\Sigma}$ *and* (4π) *means the diagonal element* $(4\pi)_{\sigma\in\Sigma}$ *in* $(\mathbf{C}^\times)^\Sigma$.

If π is attached to a Hilbert new form f and χ is a Hecke character of $\mathbf{A}_{\mathcal{K}}^\times$, then the L-function $L(s,\pi_{\mathcal{K}}\otimes\chi)$ can be identified with the Rankin-Selberg L-function $L(f,\chi,s)$ of f and the theta series associated to χ by

$$L(s,\pi_{\mathcal{K}}\otimes\chi) = L(f,\chi,s+\frac{k_{mx}-1}{2}) \quad (k_{mx} := \max_{\sigma\in\Sigma} k_\sigma).$$

Therefore, $\mathscr{L}_{\Sigma_p}(\pi,\lambda)$ is the p-adic L-function that interpolates the square root of Rankin-Selberg central L-values. We shall call $L_{\Sigma_p}(\pi,\lambda) := \mathscr{L}_{\Sigma_p}(\pi,\lambda)^2$ the anticyclotomic p-adic L-function for $\pi_{\mathcal{K}}\otimes\lambda$ with respect to the p-ordinary CM type Σ. If π is obtained from the automorphic induction of a Hecke character of $\mathcal{K}^\times$, one can see, by comparing the interpolation formulas on both sides, that $L_{\Sigma_p}(\pi,\lambda)$ is a product of two p-adic Hecke L-functions for CM fields constructed by Katz [Kat78] and Hida-Tilouine [HT93] up to an explicit unit in Λ.

Remark. When $\mathcal{F} = \mathbf{Q}$, π arises from an elliptic new form f of weight k and level $\mathfrak{n}$, the construction of $L_{\Sigma_p}(\pi,\lambda)$ can be recovered from [BDP13, Theorem 5.4] under some extra assumptions $p \nmid \mathfrak{n}$ and $\mathfrak{n}^-$ is only divisible by ramified primes. In their notation, $L(f,\chi^{-1},0) = L(\frac{1-k}{2},\pi_{\mathcal{K}}\otimes\chi^{-1})$ for $\chi \in \Sigma_{cc}^{(2)}(\mathfrak{N})$ defined in the page 1094 *loc.cit.*,and our set $\mathfrak{X}_p^{\mathrm{crit}}$ corresponds to $\Sigma_{cc}^{(2)}(\mathfrak{N})$ by $\widehat{\phi} \to \lambda^{-1}\phi^{-1}|\cdot|_{\mathbf{A}_{\mathcal{K}}}^{-\frac{k}{2}}$, where $|\cdot|_{\mathbf{A}_{\mathcal{K}}}$ is the adelic absolute value of $\mathbf{A}_{\mathcal{K}}^\times$ [2]. Note that Hypothesis A on local root numbers is also imposed in the bottom of page 1903 *loc.cit.*. This kind of p-adic L-function with some extra Euler factors removed is also considered in [Bra11a] under different hypotheses.

Our second theorem is to prove the vanishing of the Iwasawa μ-invariant $\mu^-_{\pi,\lambda,\Sigma}$ of $\mathscr{L}_{\Sigma_p}(\pi,\lambda)$ under certain hypothesis. Recall that the μ-invariant $\mu^-_{\pi,\lambda,\Sigma}$ is defined by

$$\mu^-_{\pi,\lambda,\Sigma} = \inf\left\{r \in \mathbf{Q}_{\geq 0} \mid p^{-r}\mathscr{L}_{\Sigma_p}(\pi,\lambda) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{m}_p\Lambda)\right\},$$

where $\mathfrak{m}_p$ is the maximal ideal of $\overline{\mathbf{Z}}_p$. To explain our hypothesis, we recall that thanks to the works of Deligne, Carayol, Blasius-Rogawski and Taylor

[2] Our conventions for the infinity type are opposite to each other

et.al ([BR93], [Tay89], [Jar97]), there exists a finite extension $L/\mathbf{Q}_p$ and the p-adic Galois representation $\rho_p(\pi) : G_{\mathcal{F}} \to \mathrm{GL}_2(\mathcal{O}_L)$ such that $\rho_p(\pi)$ is unramified outside $p\mathfrak{n}$; for each finite place $v \nmid p\mathfrak{n}$,

$$\iota^{-1}(L(s, \rho_p(\pi)|_{W_{\mathcal{F}_v}})) = L(s + \frac{1 - k_{mx}}{2}, \pi_v^{\vee}),$$

where $W_{\mathcal{F}_v}$ is the Weil group of $\mathcal{F}_v$. Let $\mathfrak{c}_\lambda$ be the conductor of λ. For each $v \mid \mathfrak{c}_\lambda^-$, let $\Delta_{\lambda,v}$ be the finite group $\lambda(\mathcal{O}_{\mathcal{K}_v}^\times)$.

Theorem B. *With the assumptions in Theorem A, suppose further that*

(1) p is unramified in $\mathcal{F}$,
(2) the residual Galois representation

$$\overline{\rho}_p(\pi_{\mathcal{K}}) := \rho_p(\pi)|_{G_{\mathcal{K}}} \,(mod\, \mathfrak{m}_p) \text{ is absolutely irreducible,}$$

(3) $p \nmid \prod_{v(\mathfrak{c}_\lambda^-)=1} \sharp(\Delta_{\lambda,v})$.

Then $\mu^-_{\pi,\lambda,\Sigma} = 0$.

Let $\ell \neq p$ be a rational prime. We next consider the problem of the non-vanishing modulo p of L-values twisted by anticyclotomic characters of ℓ-power conductor. This problem has been studied in the literature in various settings (*cf.* [Vat03], [Hid04a], [Fin06], [Hsi12]). To state our result along this direction, we introduce some notation. Let $\mathfrak{l}$ be a prime of $\mathcal{F}$ above ℓ and let $\mathcal{K}^-_{\mathfrak{l}^\infty}$ be the anticyclotomic pro-ℓ extension in the ray class field over $\mathcal{K}$ of conductor $\mathfrak{l}^\infty$. Let $\Gamma_{\mathfrak{l}}^- := \mathrm{Gal}(\mathcal{K}^-_{\mathfrak{l}^\infty}/\mathcal{K})$ and let $\mathfrak{X}_{\mathfrak{l}}^0$ be the set consisting of finite order characters $\phi : \Gamma_{\mathfrak{l}}^- \to \mu_{\ell^\infty}$. Let χ be a Hecke character of infinity type $(\frac{k}{2} + m, -\frac{k}{2} - m)$ and of conductor $\mathfrak{c}_\chi$ with $m \in \mathbf{Z}_{\geq 0}[\Sigma]$ as before.

Theorem C. *In addition to* (ord), (sf) *and Hypothesis A, we further assume that*

(1) $(p\mathfrak{l}, \mathfrak{n}\mathfrak{c}_\chi \mathcal{D}_{\mathcal{K}/\mathcal{F}}) = 1$,
(2) the residual Galois representation $\overline{\rho}_p(\pi_{\mathcal{K}})$ is absolutely irreducible,
(3) $p \nmid \prod_{v(\mathfrak{c}_\lambda^-)=1} \sharp(\Delta_{\chi,v})$.

Then for almost all characters $\phi \in \mathfrak{X}_{\mathfrak{l}}^0$, we have

$$[\mathcal{O}_{\mathcal{K}}^\times : \mathcal{O}_{\mathcal{F}}^\times]^2 \cdot \frac{\Gamma_\Sigma(k+m)\Gamma_\Sigma(m+1)}{(\mathrm{Im}\,\delta)^{k+2m}(4\pi)^{k+2m+1\cdot\Sigma}} \cdot \frac{L(\frac{1}{2}, \pi_{\mathcal{K}} \otimes \chi\phi)}{\Omega_{\mathcal{K}}^{2(k+2m)}} \not\equiv 0 \,(mod\, \mathfrak{m}_p).$$

Here almost all means "except for finitely many $\phi \in \mathfrak{X}_{\mathfrak{l}}^-$" if $\dim_{\mathbf{Q}_\ell} F_{\mathfrak{l}} = 1$ and "for ϕ in a Zariski dense subset of $\mathfrak{X}_{\mathfrak{l}}^0$" if $\dim_{\mathbf{Q}_\ell} F_{\mathfrak{l}} > 1$ ([Hid04a, p.737]).

Note that Theorem C in particular implies a simultaneous non-vanishing of central L-values with anticyclotomic twists. This has application to the non-vanishing of Bessel models of theta lifts of GSp(4) in view of [PTB11, Thm. 3] and the existence of some explicit theta lifts [Nar13]. In addition, we would like to mention several future applications of these results in Iwasawa theory.

I. Iwasawa main conjecture for Rankin-Selberg convolution. The congruences between Eisenstein series and cusp forms on unitary groups provide a general strategy to construct elements in Selmer groups in terms of L-values and has been used to prove one-sided divisibility in Greenberg-Iwasawa main conjectures for GL_2 and CM fields ([SU14] [Hsi14a]). Usually the most difficult part in the method of Eisenstein congruence is to prove the non-vanishing modulo p of certain Eisenstein series, where the non-vanishing modulo p of L-values always play an important role. For example, Skinner and Urban use results of Finis and Vatsal to show the non-vanishing modulo p of certain Klingen-Eisenstein series on $U(2,2)$. In a recent work [Wan13b], Xin Wan applies the method of Eisenstein congruence on the unitary group $U(3,1)$ to obtain a one-sided divisibility result towards Greenberg-Iwasawa main conjecture for certain Rankin-Selberg convolution, and Theorem C is used to prove the non-vanishing modulo p of Fourier-Jacobi coefficients of certain Siegel-Eisenstein series on $U(3,1)$. His results along this direction lead to C. Skinner's work on the converse of Gross-Zagier and Kolyvagin. Moreover, combining our Theorem B, he is able to deduce Perrin-Riou's main conjecture for Heegner points [Wan13a] in some cases.

II. Iwasawa theory for Heegner cycles. An immediate consequence of Theorem C is the non-vanishing of the p-adic L-function $\mathscr{L}_{\Sigma_p}(\pi,\lambda)$. Combined with the work [BDP13] relating the p-adic logarithm of Heegner points and the values of $\mathscr{L}_{\Sigma_p}(\pi,\lambda)$ outside the range of interpolation, this gives a new proof of Cornut-Vatsal theorem on Mazur conjecture for higher Heegner points when p is split in the imaginary quadratic field. In our forthcoming work [CH14] on the Perrin-Riou's explicit reciprocity law for Heegner cycle Euler system with connection to $\mathscr{L}_{\Sigma_p}(\pi,\lambda)$ when π is associated with an elliptic new form $f\in S_k(\Gamma_0(N))$ with Deligne's p-adic Galois representation V_f, we also use this result to obtain the rank-zero case of Bloch-Kato conjecture for the Galois representation $V_f(\frac{k}{2})\otimes\widehat{\lambda}$ as well as the analogue of Mazur conjecture for the image of higher Heegner cycles under the p-adic Abel-Jacobi map (the ℓ-adic case with $\ell\neq p$ is proved by Howard [How06]).

The key ingredients in this article are the use of *normalized toric cusp forms* and the explicit calculations of their period integral formula. In representation theory, toric cusp forms are *Gross-Prasad test vectors* [GP91] in the space of cuspidal automorphic forms on $\mathrm{GL}_2(\mathbf{A}_{\mathcal{F}})$. It seems they often serve the optimal choice in the application of toric period integrals to arithmetic. For example, Afalo and Nekovář [AN10] used Gross-Prasad test vectors in the setting of definite quaternion algebras to give an extension of the work of Cornut-Vatsal on Mazur conjecture. For the reader's convenience, we recall the definition here. Let χ be a Hecke character of $\mathcal{K}^\times$ such that $\chi|_{\mathbf{A}^\times}=\omega^{-1}$ and let $\mathcal{T}\subset\mathbf{A}_{\mathcal{K}}^\times$ be the subgroup consisting of ideles $z=(z_v)\in\prod_v\mathcal{K}_v^\times$ with $z_v/\overline{z_v}\in\mathcal{O}_{\mathcal{K}_v}^\times$ for all primes v split in $\mathcal{K}$. Fixing an embedding $\iota:\mathcal{K}^\times\hookrightarrow\mathrm{GL}_2(\mathcal{F})$, we say an automorphic form $\varphi : \mathrm{GL}_2(\mathcal{F})\backslash \mathrm{GL}_2(\mathbf{A}_{\mathcal{F}}) \to \mathbf{C}$ is a toric form of character χ if

$$\varphi(g\iota(t)) = \chi^{-1}(t)\varphi(g) \text{ for all } t \in \mathcal{T}.$$

In addition, to obtain the optimal p-integrality, we will need to normalize toric forms so that their Fourier coefficients are not all congruent to zero modulo p. This is equivalent to choosing a normalized Gross-Prasad test vector in the local Whittaker model of π_v at each place v. The reader will find later that the normalization of toric forms is the most subtle and important part of this paper.

We give a rough sketch of the proofs. We begin with an outline the construction of $\mathscr{L}_{\Sigma_p}(\pi, \lambda)$ as follows.

(1) Construct a toric Hilbert modular form $\varphi_{\lambda\phi}$ of character $\lambda\phi$ for each $\widehat{\phi} \in \mathfrak{X}_p^{\mathrm{crit}}$ as above by a careful choice of toric local Whittaker functions in local Whittaker models of π (See Definition 3.1, Lemma 3.13).
(2) Make an explicit calculation of the Fourier expansion of $\varphi_{\lambda\phi}$.
(3) Via the p-adic interpolation of the Fourier expansion, construct a p-adic distribution $\mathcal{F}_\lambda$ on Γ^- valued in the space of p-adic modular forms, which interpolates toric forms $\varphi_{\lambda\phi}$ for $\widehat{\phi} \in \mathfrak{X}_p^{\mathrm{crit}}$. The p-adic L-function $\mathscr{L}_{\Sigma_p}(\pi, \lambda)$ is obtained by a weighted sum of the evaluation of this distribution $\mathcal{F}_\lambda$ at a finite set of CM points.

The evaluation of φ_χ with $\chi = \lambda\phi$ at CM points in the step (3) is essentially the toric period integral $P_\chi(\varphi_\chi)$ given by

$$P_\chi(\varphi_\chi) = \int_{\mathbf{A}^\times \mathcal{K}^\times \backslash \mathbf{A}_{\mathcal{K}}^\times} \varphi_\chi(\iota(t))\chi(t)\mathrm{d}t.$$

To prove the formula in Theorem A, we have to express the square $P_\chi(\varphi_\chi)^2$ in terms of the central L-value $L(\frac{1}{2}, \pi_{\mathcal{K}} \otimes \chi)$. This is usually referred to as an explicit Waldspurger formula. Such a formula has been exploited widely in the literature based on either explicit theta lifts ([Mur10], [Mur08], [Xue07], [Hid10a] and [BDP13]) or the technique of relative trace formula ([MW09]). In this paper we adopt a different approach, making use of a formula of Waldspurger which is indeed proved but not stated explicitly in [Wal85]. This formula decomposes the square $P_\chi(\varphi)$ of the global period toric integral into a product of local period integrals involving local Whittaker functions of φ. Explicit computation of these local integrals shows that $P_\chi(\varphi)^2$ is essentially equal to the central value of the L-function $L(s, \pi_{\mathcal{K}} \otimes \lambda)$. We emphasize that this explicit formula does not depend on the choices of special Bruhat-Schwartz functions in the classical approach of theta lifting but on choices of local Whittaker functions which reflect the arithmetic of modular forms directly via the Fourier expansion. Now with the above construction of toric forms and explicit period integral formulas, the proofs of Theorem B and Theorem C when combined with fundamental works of Hida ([Hid10b] and [Hid04a]) are reduced to a study the vanishing/non-vanishing modulo p properties of the Fourier expansions of the toric cusp form φ_λ at cusps $(\mathfrak{a}, \mathfrak{b})$ such that $\mathfrak{a}\mathfrak{b}^{-1}$ is the polarization of an abelian variety with

CM by $\mathcal{O}_{\mathcal{K}}$. We give an explicit computation of Fourier coefficients of toric forms φ_λ, with which we can study the non-vanishing modulo p property of these Fourier coefficients. These calculations are elementary but quite tedious and lengthy. Finally, the connection between the Fourier coefficients of Hilbert modular forms and the trace of Frobenius of the associated Galois representations enables us to relate the non-vanishing modulo p of Fourier coefficients of φ_λ at these cusps and the irreducibility of the residual Galois representation $\overline{\rho}_p(\pi)|_{G_{\mathcal{K}}}$.

We end this introduction by making a few remarks on our assumptions. The restriction (sf) is merely due to the computational difficulty on the local period integrals and the local Fourier coefficients, and it is expected to be unnecessary. The global assumption on the irreducibility of residual Galois representation assures that the new form associated to π is not congruent to theta series arising from $\mathcal{K}$. This assumption prevents the vanishing modulo p of $\mathscr{L}_{\Sigma_p}(\pi,\lambda)$ from the possibility that $\mathscr{L}_{\Sigma_p}(\pi,\lambda)$ is congruent to a product of two anticyclotomic Katz p-adic L-functions attached to self-dual Heck characters of the root number -1. The local assumption (3) is equivalent to saying that the local residual character $\lambda_v \pmod{\mathfrak{m}_p}$ is ramified for all $v|\mathfrak{c}_\lambda^-$. This hypothesis is used to avoid the vanishing of L-values due to *sign change* phenomenon. For example, if $\lambda_v \equiv 1 \pmod{\mathfrak{m}_p}$ is unramified at some prime $\mathfrak{q}|\mathfrak{c}_\lambda^-$ with $\pi_{\mathfrak{q}}$ special, then one can construct $\lambda' \equiv \lambda \pmod{\mathfrak{m}_p}$ such that λ' has the conductor $\mathfrak{q}^{-1}\mathfrak{c}_\lambda^-$ and the same infinity type with λ. This implies that $\pi_{\mathcal{K}} \otimes \lambda'$ has global root number -1, and hence the algebraic part of $L(\frac{1}{2}, \pi_{\mathcal{K}} \otimes \lambda)$ is congruent to $L(\frac{1}{2}, \pi_{\mathcal{K}} \otimes \lambda') = 0$. The above assumptions might be weaken with exceptional effort on the compuation of Fourier coefficients of toric forms. However, Hypothesis A is fundamental, the failure of which makes the period integral $P_\chi(\varphi_\chi)$ vanish for all χ (and hence make the results null) by a well-known theorem of Saito-Tunnell ([Sai93], [Tun83]).

This paper is organized as follows. After fixing notation and definitions in §1, we derive a key formula of Waldspurger on the decomposition of global toric period integrals into local toric period integrals (Proposition 2.1) in §2. The bulk of this article is §3, where we give the choices of local toric Whittaker functions $W_{\chi,v}$ and calculate explicitly these local period integrals attached to $W_{\chi,v}$. The explicit Waldspurger formula is proved in Theorem 3.14, and a non-vanishing modulo p of these toric Whittaker functions is proved in Proposition 3.19. After reviewing briefly theory of complex and geometric Hilbert modular forms in §4, we prove Theorem A in §5. The key ingredient is Proposition 5.5, the construction of a p-adic measure $\mathcal{F}_{\lambda,\mathfrak{c}}$ on Γ^- with values in the space of p-adic modular forms, and the p-adic L-function $\mathscr{L}_{\Sigma_p}(\pi,\lambda)$ is thus obtained by evaluating $\mathcal{F}_{\lambda,\mathfrak{c}}$ at suitable CM points. The precise evaluation formula of $\mathscr{L}_{\Sigma_p}(\pi,\lambda)^2$ is established in Theorem 5.7. In §6, we study the μ-invariant of $\mathscr{L}_{\Sigma_p}(\pi,\lambda)$ and prove Theorem B in Theorem 6.2. Finally, the non-vanishing of central L-values modulo p is considered in §7 and Theorem C is proved in Theorem 7.1.

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1. NOTATION AND DEFINITIONS

1.1. MEASURES ON LOCAL FIELDS. We fix some general notation and conventions on local fields through this article. Let $\psi_{\mathbf{Q}}: \mathbf{A}_{\mathbf{Q}}/\mathbf{Q} \to \mathbf{C}^\times$ be the additive character such that $\psi_{\mathbf{Q}}(x_\infty) = \exp(2\pi i x_\infty)$ with $x_\infty \in \mathbf{R}$. Let q be a place of $\mathbf{Q}$ and let F be a finite extension of $\mathbf{Q}_q$. Let ψ_q be the local component of ψ at q and let $\psi_F := \psi_q \circ \mathrm{T}_{F/\mathbf{Q}_q}$, where $\mathrm{T}_{F/\mathbf{Q}_q}$ is the trace from $\mathcal{F}$ to $\mathbf{Q}_q$. Let $\mathrm{d}x$ be the Haar measure on F self-dual with respect to the pairing $(x, x') \mapsto \psi_F(xx')$. Let $|\cdot|_F$ be the absolute value of $\mathcal{F}$ normalized by $\mathrm{d}(ax) = |a|_F \,\mathrm{d}x$ for $a \in F^\times$. We often simply write $|\cdot| = |\cdot|_F$ if it is clear from the context without possible confusion. We recall the definition of the local zeta function $\zeta_F(s)$. If $\mathcal{F}$ is non-archimedean, let ϖ_F be a uniformizer of F and let

$$\zeta_F(s) = \frac{1}{1 - |\varpi_F|_F^s}.$$

If F is archimedean, then

$$\zeta_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2);\ \zeta_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

The Haar measures $\mathrm{d}^\times x$ on $F^\times$ is normalized by

$$\mathrm{d}^\times x = \zeta_F(1)\,|x|_F^{-1}\,\mathrm{d}x.$$

In particular, if $F = \mathbf{R}$, then $\mathrm{d}x$ is the Lebesgue measure and $\mathrm{d}^\times x = |x|_{\mathbf{R}}^{-1}\,\mathrm{d}x$, and if $F = \mathbf{C}$, then $\mathrm{d}x$ is twice the Lebesgue measure on $\mathbf{C}$ and $\mathrm{d}^\times x = 2\pi^{-1}r^{-1}\mathrm{d}r\mathrm{d}\theta$ ($x = re^{i\theta}$).

Suppose that F is non-archimedean. Let $\mathcal{O}_F$ be the ring of integers of F and let $\mathcal{D}_F$ be the absolute different of F. Then $\mathcal{D}_F^{-1}$ is the Pontryagin dual of $\mathcal{O}_F$ with respect to ψ_F, and $\mathrm{vol}(\mathcal{O}_F, \mathrm{d}x) = |\mathcal{D}_F|_F^{\frac{1}{2}}$. If $\mu: F^\times \to \mathbf{C}^\times$ is a character of $F^\times$, define the local conductor $a(\mu)$ by

$$a(\mu) = \inf\left\{n \in \mathbf{Z}_{\geq 0} \mid \mu(x) = 1 \text{ for all } x \in (1 + \varpi_F^n\mathcal{O}_F) \cap \mathcal{O}_F^\times\right\}.$$

1.2. If L is a number field, the ring of integers of L is denoted by $\mathcal{O}_L$, $\mathbf{A}_L$ is the adele of L and $\mathbf{A}_{L,f}$ is the finite part of $\mathbf{A}_L$. For $a \in \mathbf{A}_L^\times$, we put

$$\mathfrak{il}_L(a) := a(\mathcal{O}_L \otimes \widehat{\mathbf{Z}}) \cap L.$$

Denote by G_L the absolute Galois group and by $\mathrm{rec}_L : \mathbf{A}_L^\times \to G_L^{ab}$ the geometrically normalized reciprocity law. We define $\psi_L : \mathbf{A}_L^\times/L \to \mathbf{C}^\times$ by $\psi_L(x) = \psi_{\mathbf{Q}} \circ \mathrm{Tr}_{L/\mathbf{Q}}(x)$.

Let v_p be the p-adic valuation on $\mathbf{C}_p$ normalized so that $v_p(p) = 1$. We regard L as a subfield in $\mathbf{C}$ (resp. $\mathbf{C}_p$) via $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ (resp. $\iota_p = \iota^{-1} \circ \iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$) and $\mathrm{Hom}(L, \overline{\mathbf{Q}}) = \mathrm{Hom}(L, \mathbf{C}_p)$.

Let $\overline{\mathbf{Z}}$ be the ring of algebraic integers of $\overline{\mathbf{Q}}$ and let $\overline{\mathbf{Z}}_p$ be the p-adic completion of $\overline{\mathbf{Z}}$ in $\mathbf{C}_p$. Let $\overline{\mathbf{Z}}$ be the ring of algebraic integers of $\overline{\mathbf{Q}}$ and let $\overline{\mathbf{Z}}_p$ be the p-adic completion of $\overline{\mathbf{Z}}$ in $\mathbf{C}_p$ with the maximal ideal $\mathfrak{m}_p$.

1.3. Local L-functions. Let F be a non-archimedean local filed. Let $\mu, \nu : F^\times \to \mathbf{C}^\times$ be two characters of $F^\times$. Denote by $I(\mu,\nu)$ the space consisting of smooth and $\mathrm{GL}_2(\mathcal{O}_F)$-finite functions $f : \mathrm{GL}_2(F) \to \mathbf{C}$ such that

$$f(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g) = \mu(a)\nu(d) \left|\frac{a}{d}\right|^{\frac{1}{2}} f(g).$$

Then $I(\mu,\nu)$ is an admissible representation of $\mathrm{GL}_2(F)$. Denote by $\pi(\mu,\nu)$ the unique infinite dimensional subquotient of $I(\mu,\nu)$. We call $\pi(\mu,\nu)$ a principal series if $\mu\nu^{-1} \neq |\cdot|^{\pm}$ and a special representation if $\mu\nu^{-1} = |\cdot|^{\pm}$.
Let E be a quadratic extension of F and let $\chi : E^\times \to \mathbf{C}^\times$ be a character. We recall the definition of local L-functions $L(s, \pi_{\mathcal{K}} \otimes \chi)$ ([Jac72, §20]) when $\pi = \pi(\mu,\nu)$ is a subrepresentation of $I(\mu,\nu)$. If $E = F \oplus F$, then we write $\chi = (\chi_1, \chi_2) : F^\times \oplus F^\times \to \mathbf{C}^\times$ and put

$$L(s, \pi_{\mathcal{K}} \otimes \chi) = \begin{cases} L(s, \pi \otimes \chi_1) L(s, \pi \otimes \chi_2) & \text{if } \mu\nu^{-1} \neq |\cdot|^{\pm}, \\ L(s, \mu\chi_1) L(s, \mu\chi_2) & \text{if } \mu\nu^{-1} = |\cdot|. \end{cases}$$

If E is a field, then

$$L(s, \pi_E \otimes \chi) = \begin{cases} L(s, \mu'\chi) L(s, \nu'\chi) & \text{if } \mu\nu^{-1} \neq |\cdot|^{\pm}, \\ L(s, \mu'\chi) & \text{if } \mu\nu^{-1} = |\cdot|. \end{cases}$$

Here $\mu' = \mu \circ \mathrm{N}_{E/F}$, $\nu' = \nu \circ \mathrm{N}_{E/F}$ are characters of $E^\times$.

1.4. Whittaker and Kirillov models. Let F be a local field. Let π be an irreducible admissible representation of $\mathrm{GL}_2(F)$ and let $\psi : F \to \mathbf{C}^\times$ be a non-trivial additive character. We let $\mathcal{W}(\pi,\psi)$ be the Whittaker model of π. Recall that $\mathcal{W}(\pi,\psi)$ is a subspace of smooth functions $W : \mathrm{GL}_2(F) \to \mathbf{C}$ such that

(1) $W(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) = \psi(x) W(g)$ for all $x \in F$.
(2) If v is archimedean, $W(\begin{pmatrix} a & \\ & 1 \end{pmatrix}) = O(|a|^N)$ for some positive number N.

(*cf.* [JL70, Thm. 6.3]). Let $\mathcal{K}(\pi,\psi)$ be the Kirillov model of π. If F is non-archimedean, then $\mathcal{K}(\pi,\psi)$ is a subspace of smooth $\mathbf{C}$-valued functions on $F^\times$, containing all Bruhat-Schwartz functions on $F^\times$. A function in $\mathcal{K}(\pi,\psi)$ shall be called a *local Fourier coefficient* of π. In addition, it is well known that we have the following $\mathrm{GL}_2(F)$-equivariant isomorphism

$$\begin{aligned}\mathcal{W}(\pi,\psi) &\xrightarrow{\sim} \mathcal{K}(\pi,\psi)\\ W &\mapsto \xi_W(a) := W(\begin{pmatrix} a & \\ & 1\end{pmatrix}).\end{aligned} \tag{1.1}$$

2. Waldspurger formula

Let $\mathcal{F}$ be a number field and $\mathcal{K}$ be a quadratic field extension of $\mathcal{F}$. Let $\mathbf{A} = \mathbf{A}_{\mathcal{F}}$. Let $G = \mathrm{GL}_{2\,/\mathcal{F}}$. Let π be an irreducible cuspidal automorphic representation of $G(\mathbf{A})$ with unitary central character ω. Denote by $\mathcal{A}(\pi)$ the realization of π in the space $\mathcal{A}_0(G)$ of cusp forms on $G(\mathbf{A})$. Let χ be a unitary Hecke character of $\mathcal{K}^\times$ such that $\chi|_{\mathbf{A}^\times} = \omega^{-1}$. Let $\pi_{\mathcal{K}}$ be the quadratic base change of π to the quadratic extension $\mathcal{K}/\mathcal{F}$. The existence of $\pi_{\mathcal{K}}$ is established in [Jac72]. The goal of this section is to deduce from results in [Wal85] a formula (Proposition 2.1) which expresses the central value $L(\frac{1}{2}, \pi_{\mathcal{K}} \otimes \chi)$ in terms of a product of local toric period integrals of Whittaker functions.
Let $\psi := \psi_{\mathcal{F}} : \mathbf{A}/\mathcal{F} \to \mathbf{C}^\times$ be the standard non-trivial additive character. For a place v of $\mathcal{F}$, we let $G_v = G(\mathcal{F}_v)$ and let $\chi_v : \mathcal{K}_v^\times \to \mathbf{C}^\times$ (resp. $\psi_v : \mathcal{F}_v \to \mathbf{C}^\times$) denote the local constituent of χ (resp. ψ).

2.1. For $x \in \mathcal{K}$, let $\mathrm{T}(x) := x + \overline{x}$ and $\mathrm{N}(x) = x\overline{x}$. Let $\{1, \vartheta\}$ be a basis of $\mathcal{K}$ over $\mathcal{F}$. We let $\iota : \mathcal{K} \to M_2(\mathcal{F})$ be the embedding attached to ϑ given by

$$\iota(a\vartheta + b) = \begin{pmatrix} a\mathrm{T}(\vartheta) + b & -a\mathrm{N}(\vartheta) \\ a & b \end{pmatrix} \quad (a, b \in \mathcal{F}). \tag{2.1}$$

Put

$$\boldsymbol{J} := \begin{pmatrix} -1 & \mathrm{T}(\vartheta) \\ 0 & 1 \end{pmatrix}.$$

Then $M_2(\mathcal{F}) = \iota(\mathcal{K}) \oplus \iota(\mathcal{K})\boldsymbol{J}$. It is clear that $\boldsymbol{J}^2 = 1$ and $\iota(t)\boldsymbol{J} = \boldsymbol{J}\iota(\overline{t})$ for all $t \in \mathcal{K}$.

2.2. The local bilinear form and toric integral. For each place v of $\mathcal{F}$, denote by π_v (resp. ψ_v) the local constituent of π (resp. ψ) at v. Define a $\mathbf{C}$-bilinear form $\mathbf{b}_v : \mathcal{W}(\pi_v, \psi_v) \times \mathcal{W}(\pi_v, \psi_v) \to \mathbf{C}$ by

$$\begin{aligned}\mathbf{b}_v(W_1, W_2) &:= \sum_{n=-\infty}^{\infty} \int_{\varpi^n \mathcal{O}_F^\times} W_1(\begin{pmatrix} a & \\ & 1\end{pmatrix}) W_2(\begin{pmatrix} -a & \\ & 1\end{pmatrix}) \omega^{-1}(a) \mathrm{d}^\times a \\ &= \int_{\mathcal{F}_v^\times} W_1(\begin{pmatrix} a & \\ & 1\end{pmatrix}) W_2(\begin{pmatrix} -a & \\ & 1\end{pmatrix}) \omega^{-1}(a) \mathrm{d}^\times a.\end{aligned}$$

It is known that this series converges absolutely as π_v is a local constituent of a unitary cuspidal automorphic representation. Moreover, the pairing $\mathbf{b}_v$ enjoys the property:

$$\mathbf{b}_v(\pi(g)W_1, \pi(g)W_2) = \omega(\det g)\mathbf{b}_v(W_1, W_2). \tag{2.2}$$

The pairing $\mathbf{b}_v$ thus gives rise to an isomorphism between the contragredient representation $\pi^\vee$ and $\pi\otimes\omega^{-1}$.
The local toric period integral for $W_1, W_2 \in \mathcal{W}(\pi_v,\psi_v)$ is given by

$$P(W_1,W_2,\chi_v) := \int_{\mathcal{K}_v^\times/\mathcal{F}_v^\times} \mathbf{b}_v(\pi(\iota(t))W_1,\pi(\boldsymbol{J})W_2)\chi_v(t)\mathrm{d}t\cdot\frac{L(1,\tau_{\mathcal{K}_v/\mathcal{F}_v})}{\zeta_{\mathcal{F}_v}(1)}.$$

The above integral converges as χ_v is unitary ([Wal85, LEMME 7]).

2.3. A formula of Waldspurger. Let $\Lambda(s,\pi_\mathcal{K}\otimes\chi)$ be the completed L-function of $\pi_\mathcal{K}\otimes\chi$ given by

$$\Lambda(s,\pi_\mathcal{K}\otimes\chi) := \prod_v L(s,\pi_{\mathcal{K}_v}\otimes\chi_v) = L(s,\pi_\mathcal{K}\otimes\chi)\cdot\prod_{v|\infty}L(s,\pi_{\mathcal{K}_v}\otimes\chi_v).$$

It is well known that $\Lambda(s,\pi_\mathcal{K}\otimes\chi)$ converges absolutely for $\operatorname{Re}s\gg 0$ and has meromorphic continuation to all $s\in\mathbf{C}$. Moreover, it satisfies the functional equation

$$\Lambda(s,\pi_\mathcal{K}\otimes\chi) = \varepsilon(s,\pi_\mathcal{K}\otimes\chi)\Lambda(1-s,\pi_\mathcal{K}^\vee\otimes\chi^{-1}).$$

The global toric period integral for $\varphi\in\mathcal{A}(\pi)$ is defined by

$$P_\chi(\varphi) := \int_{\mathcal{K}^\times\mathbf{A}^\times\backslash\mathbf{A}_\mathcal{K}^\times}\varphi(\iota(t))\chi(t)\mathrm{d}t.$$

The following proposition connects the global toric periods and central L-values of $\pi_\mathcal{K}\otimes\chi$.

Proposition 2.1 (Waldspurger). *Let $\varphi_1,\varphi_2\in\mathcal{A}(\pi)$ and let $W_{\varphi_1}, W_{\varphi_2}$ be the associated global Whittaker functions. We suppose that $W_{\varphi_i}=\prod_v W_{i,v}$, where $W_{i,v}\in\mathcal{W}(\pi_v,\psi_v)$ such that $W_{i,v}(1)=1$ for almost v $(i=1,2)$. Then there exists a finite set S_0 of places of $\mathcal{F}$ including all archimedean places such that for every finite set $S\supset S_0$, we have*

$$P_\chi(\varphi_1)P_\chi(\varphi_2) = \Lambda(\frac{1}{2},\pi_\mathcal{K}\otimes\chi)\cdot\prod_{v\in S}\frac{1}{L(\frac{1}{2},\pi_{\mathcal{K}_v}\otimes\chi_v)}\cdot P(W_{1,v},W_{2,v},\chi_v).$$

Proof. The proof is the combination of various formulae established in [Wal85]. We first recall some important local integrals. Let $D=G\times G$. For each place v of $\mathcal{F}$, let $\mathcal{S}_v=\mathcal{S}(M_2(\mathcal{F}_v))\otimes\mathcal{S}(\mathcal{F}_v^\times)$ and let $D_v=G_v\times G_v$. Let $r=r'\times r'': G_v\times D_v\to\operatorname{End}\mathcal{S}_v$ be the Weil representation of $G_v\times D_v$ defined in [Wal85, §I.3 p.178]
Let $\varphi\in\mathcal{A}(\pi)$ be an automorphic form in the automorphic realization of π. Recall that the global Whittaker function of φ is defined by

$$W_\varphi(g) = \int_{\mathcal{F}\backslash\mathbf{A}_F}\varphi(\begin{pmatrix}1&x\\0&1\end{pmatrix}g)\psi(-x)\mathrm{d}x.$$

Write $\mathcal{W}_v=\mathcal{W}(\pi_v,\psi_v)$. We further assume that W_φ has the factorization $W_\varphi=\prod_v W_{\varphi,v}\in\otimes'_v\mathcal{W}_v$ such that $W_{\varphi,v}(1)=1$ for almost v. For each v, let $U:\mathcal{S}_v\to\mathcal{W}_v\otimes\mathcal{W}_v$, $f_v\to U_{f_v}$ be the $G_v\times G_v$-equivaraint surjective morphism associated to W_v introduced in [Wal85, COROLLAIRE, p.187]. Define the following local integrals:

$$C(f_v) := \int_{\mathcal{F}_v^\times} U_{f_v}(\begin{pmatrix} a & \\ & 1 \end{pmatrix}, \begin{pmatrix} -a & \\ & 1 \end{pmatrix}))\omega^{-1}(a)\mathrm{d}^\times a,$$

$$B(f_v, 1) := \int_{Z_v\backslash G_v N_v} \int_{\mathcal{F}_v^\times} W_{\varphi,v}(\sigma) r'(\sigma) f_v(x, x^{-2})\mathrm{d}x\mathrm{d}\sigma,$$

$$P(f_v, \chi_v, \frac{1}{2}) := \int_{\mathcal{F}_v^\times\backslash\mathcal{K}_v^\times} B(r''(\iota(t), 1)f_v, 1)\chi_v(t)\mathrm{d}t.$$

The convergence and analytic properties of these local integrals are studied in [Wal85, LEMME 2, LEMME 3, LEMME 5]. Moreover, we have

$$B(f_v, 1) = C(f_v) \cdot \frac{1}{\zeta_{\mathcal{F}_v}(1)}.$$

For each v, we take a special test function $f_v \in \mathcal{S}_v$ such that

$$U_{f_v} = W_{1,v} \otimes \pi(\boldsymbol{J})W_{2,v}. \tag{2.3}$$

Note that f_v can be chosen to be the spherical test function $f_v^0 := \mathbb{I}_{M_2(\mathcal{O}_{\mathcal{F}_v})} \otimes \mathbb{I}_{\mathcal{O}_{\mathcal{F}_v}^\times}$ for all but finitely many v. With this particular choice of f_v, we have

$$\begin{aligned} P(f_v, \chi_v, \frac{1}{2}) &= \int_{\mathcal{F}_v^\times\backslash\mathcal{K}_v^\times} C(r''(\iota(t), 1)f_v)\mathrm{d}t \cdot \frac{1}{\zeta_{\mathcal{F}_v}(1)} \\ &= \int_{\mathcal{F}_v^\times\backslash\mathcal{K}_v^\times} \mathbf{b}_v(\pi(\iota(t))W_1, \pi(\boldsymbol{J})W_2)\chi_v(t)\mathrm{d}t \cdot \frac{1}{\zeta_{\mathcal{F}_v}(1)} \\ &= P(W_{1,v}, W_{2,v}, \chi_v) \cdot \frac{1}{L(1, \tau_{\mathcal{K}_v/\mathcal{F}_v})}. \end{aligned} \tag{2.4}$$

Let $\mathcal{S} = \otimes\mathcal{S}_v$ be the restricted product with respect to spherical test functions $\left\{f_v^0\right\}_v$. Define the theta kernel for $f := \otimes f_v \in \mathcal{S}$ by

$$\theta_f(\sigma, g) := \sum_{(x,u)\in M_2(\mathcal{F})\times\mathcal{F}^\times} r(\sigma, g)f(x, u)\ (\sigma \in G(\mathbf{A}),\ g \in G(\mathbf{A})\times G(\mathbf{A})),$$

and define the automorphic form $\theta(f, \varphi, g)$ on $G(\mathbf{A})\times G(\mathbf{A})$ by

$$\theta(f, \varphi, g) = \int_{G(\mathcal{F})\backslash G(\mathbf{A})} \varphi(\sigma)\theta_f(\sigma, g)\mathrm{d}\sigma.$$

Note that according to (2.3), we have

$$\theta(f, \varphi, g_1, g_2) = \varphi_1(g_1)\varphi_2(g_2\boldsymbol{J}).$$

We define the toric period integral $P(f, \chi)$ by

$$P(f, \chi) := \int_{[\mathcal{K}^\times\mathbf{A}^\times\backslash\mathbf{A}_{\mathcal{K}}^\times]^2} \theta(f, \varphi, \iota(t_1), \iota(t_2))\chi(t_1)\chi(\bar{t}_2)\mathrm{d}t_1\mathrm{d}t_2.$$

By the relation $\boldsymbol{J}\iota(t_2)\boldsymbol{J} = \iota(\bar{t}_2)$ and the automorphy of φ_2, we find that

$$P(f, \chi) = P_\chi(\varphi_1)P_\chi(\varphi_2).$$

Let S_0 be a finite set of places of $\mathcal{F}$ such that $W_{\varphi,v}$, $W_{i,v}$ and f_v are spherical for all $v \notin S_0$. From [Wal85, Prop. 4, p.196 and LEMME 7, p.219], we deduce the following formula for every finite set $S \supset S_0$:

$$P_\chi(\varphi_1)P_\chi(\varphi_2) = \Lambda(\frac{1}{2}, \pi_\mathcal{K} \otimes \chi) \cdot \prod_{v \in S} P(f_v, \chi_v, \frac{1}{2}) \cdot \frac{L(1, \tau_{\mathcal{K}_v/\mathcal{F}_v})}{L(\frac{1}{2}, \pi_{\mathcal{K}_v} \otimes \chi)}.$$

We thus establish the desired formula in virtue of (2.4). □

3. Toric period integrals

3.1. Notation. Throughout we suppose that $\mathcal{F}$ is a totally real number field and $\mathcal{K}$ is a totally imaginary quadratic extension of $\mathcal{F}$. We retain the notation in the introduction and §2.1. Let Σ be a fixed CM type of $\mathcal{K}$. Let π be an irreducible automorphic cuspidal representation of $\mathrm{GL}_2(\mathbf{A})$. Let $\mathfrak{n}$ be the conductor of π. Suppose that π has infinity type $k = \sum_{\sigma \in \Sigma} k_\sigma \sigma \in \mathbf{Z}_{\geq 1}[\Sigma]$. Let $m = \sum_\sigma m_\sigma \sigma \in \mathbf{Z}_{\geq 0}[\Sigma]$ and let χ be a Hecke character of infinity type $(k/2 + m, -k/2 - m)$ such that $\chi|_{\mathbf{A}^\times} = \omega^{-1}$. Let $\mathbf{h}$ be the set of finite places of $\mathcal{F}$. Recall that the set of infinite places of $\mathcal{F}$ is identified with the CM-type Σ.

In this section, we will choose a special local Whittaker function at each place v of $\mathcal{F}$ in §3.6 and calculate their associated local toric period integrals in §3.7 and §3.8. Finally, we prove in §3.10 a non-vanishing modulo p result of these local Whittaker functions. This result plays an important role in the later application to the calculation of the μ-invariant.

Let $\mathfrak{C}_\chi$ (resp. $\mathfrak{c}_\omega$) be the conductor of χ (resp. ω). Let $\mathfrak{c}_\chi = \mathfrak{C}_\chi \cap \mathcal{F}$. We further decompose $\mathfrak{n}^- = \mathfrak{n}_s^- \mathfrak{n}_r^-$, where $\mathfrak{n}_s^-$ is prime to $\mathfrak{c}_\omega$ and $\mathfrak{n}_r^-$ is only divisible by prime factors of $\mathfrak{c}_\omega$. Put

$$\begin{aligned} c_v(\chi) =& \inf \left\{ n \in \mathbf{Z}_{\geq 0} \mid \chi = 1 \text{ on } (1 + \varpi^n \mathcal{O}_E)^\times \right\}, \\ m_v(\chi, \pi) =& c_v(\chi) - v(\mathfrak{n}^-). \end{aligned} \tag{3.1}$$

It is clear that $c_v(\chi) = v(\mathfrak{c}_\chi)$. We put

$$A(\chi) = \{ v \in \mathbf{h} \mid \mathcal{K}_v \text{ is a field, } \pi_v \text{ is special and } c_v(\chi) = 0 \}.$$

Let $p > 2$ be a rational prime satisfying (ord). The assumption (ord) in particular implies that every prime factor of p in $\mathcal{F}$ splits in $\mathcal{K}$. Let Σ_p be the p-adic places induced by Σ via ι_p. Thus Σ_p and its complex conjugation $\overline{\Sigma}_p$ give a partition of the places of $\mathcal{K}$ above p. Let $\mathfrak{N}$ be the prime-to-p conductor of $\pi_\mathcal{K} \otimes \chi$. We fix a decomposition $\mathfrak{N}^+ = \mathfrak{F}\overline{\mathfrak{F}}$ such that $(\mathfrak{F}, \overline{\mathfrak{F}}) = 1$.

3.2. Galois representation attached to π. Let $\rho_p(\pi) : G_F \to \mathrm{GL}_2(\mathcal{O}_{L_\pi})$ be the p-adic Galois representation associated to π as in the introduction. Let $v \nmid p$ and let $W_{\mathcal{F}_v}$ be the local Weil group at v. Suppose that $\pi_v = \pi(\mu_v, \nu_v)$ is a subquotient of the induced representations. By the local-global compatibility

([Car86], [Tay89] and [Jar97]), we have

$$\rho_p(\pi)|_{W_{\mathcal{F}_v}} \simeq \begin{pmatrix} \mu_v^{-1}|\cdot|^{\frac{1-k_{mx}}{2}} & * \\ 0 & \nu_v^{-1}|\cdot|^{\frac{1-k_{mx}}{2}} \end{pmatrix} \quad (k_{mx} = \max_\sigma k_\sigma). \tag{3.2}$$

In particular, this implies that $\mu_v(\varpi_{\mathcal{F}_v})$ and $\nu_v(\varpi_{\mathcal{F}_v})$ are p-adic units in $\mathcal{O}_{L_\pi}^\times$.

3.3. OPEN COMPACT SUBGROUPS. For each finite place v, we put

$$K_v^0 = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_v \mid a, d \in \mathcal{O}_{\mathcal{F}_v},\, b \in \mathcal{D}_{F_v}^{-1},\, c \in \mathcal{D}_{\mathcal{F}_v},\, \det g \in \mathcal{O}_{\mathcal{F}_v}^\times \right\},$$

and for an integral ideal $\mathfrak{a}$ of $\mathcal{F}$, we put

$$K_v^0(\mathfrak{a}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_v^0 \mid c \in \mathfrak{a}\mathcal{D}_{\mathcal{F}_v},\, a-1 \in \mathfrak{a} \right\},$$
$$U_v(\mathfrak{a}) = \{ g \in \mathrm{GL}_2(\mathcal{O}_{\mathcal{F}_v}) \mid g \equiv 1 \,(\mathrm{mod}\ \mathfrak{a}) \}.$$

Let $K^0 = \prod_{v \in \mathbf{h}} K_v^0$ and $U(\mathfrak{a}) = \prod_{v \in \mathbf{h}} U_v(\mathfrak{a})$ be open compact subgroups of $\mathrm{GL}_2(\mathbf{A}_f)$.

3.4. THE CHOICES OF ϑ AND ς_v. We fix a finite idele $d_{\mathcal{F}} = (d_{\mathcal{F}_v}) \in \mathbf{A}_{\mathcal{F},f}^\times$ such that $d_{\mathcal{F}_v}$ is a generator of the absolute different $\mathcal{D}_{\mathcal{F}_v}$ at each finite place v and $d_{\mathcal{F}_v} = 1$ for $v \nmid \mathcal{D}_{\mathcal{F}}$. Fixing an integral ideal $\mathfrak{r} \subset \mathfrak{c}_\chi \mathfrak{n} \mathcal{D}_{\mathcal{K}}^2$ of $\mathcal{F}$, we choose $\vartheta \in \mathcal{K}$ such that

(d1) $\mathrm{Im}\, \sigma(\vartheta) > 0$ for all $\sigma \in \Sigma$,
(d2) $\{1, d_{\mathcal{F}_v}^{-1}\vartheta\}$ is an $\mathcal{O}_{\mathcal{F}_v}$-basis of $\mathcal{O}_{\mathcal{K}_v}$ for all $v \mid p\mathfrak{r}$,
(d3) $d_{\mathcal{F}_v}^{-1}\vartheta$ is a uniformizer of $\mathcal{K}_v$ for every v ramified in $\mathcal{K}$.

The existence of such ϑ is guaranteed by strong approximation theorem. Then ϑ is a generator of $\mathcal{K}$ over $\mathcal{F}$ and determines an embedding $\mathcal{K} \hookrightarrow M_2(\mathcal{F})$ in (2.1). Let

$$\delta = 2^{-1}(\vartheta - \overline{\vartheta}) \in \mathcal{K}^\times.$$

The condition (d2) allows us to choose $d_{\mathcal{F}_v} = 2\delta$ at split $v \mid p\mathfrak{r}$. For each finite place v, we also fix an $\mathcal{O}_{\mathcal{F}_v}$-basis $\{1, \boldsymbol{\theta}_v\}$ of $\mathcal{O}_{\mathcal{K}_v}$ such that $\boldsymbol{\theta}_v = \vartheta$ except for finitely many v and

$$\boldsymbol{\theta}_v = d_{\mathcal{F}_v}^{-1}\vartheta \text{ for } v|p\mathfrak{r}.$$

Write $\boldsymbol{\theta}_v = a_v\vartheta + b_v$ with $a_v, b_v \in \mathcal{F}_v$.

For every v split in $\mathcal{K}$, we shall fix a place w of $\mathcal{K}$ above v throughout, and decompose $\mathcal{K}_v := \mathcal{K} \otimes_{\mathcal{F}} \mathcal{F}_v = \mathcal{F}_v e_{\overline{w}} \oplus \mathcal{F}_v e_w$, where e_w and $e_{\overline{w}}$ are the idempotents attached to w and $\overline{w}$ respectively. If $v|p\mathfrak{N}^+$, we further require that $w|\mathfrak{F}\Sigma_p$, i.e. $w|\mathfrak{F}$ or $w \in \Sigma_p$. We identify $\delta \in \mathcal{K}_w = \mathcal{F}_v$ and write $\vartheta_v = \vartheta_{\overline{w}} e_{\overline{w}} + \vartheta_w e_w$ for split v.

For each place v, we define $\varsigma_v \in \mathrm{GL}_2(\mathcal{F}_v)$ as follows:

$$\begin{aligned} \varsigma_v &= \begin{pmatrix} \operatorname{Im}\sigma(\vartheta) & \operatorname{Re}\sigma(\vartheta) \\ 0 & 1 \end{pmatrix} \text{ for } v = \sigma \in \Sigma, \\ \varsigma_v &= (\vartheta_{\overline{w}} - \vartheta_w)^{-1} \begin{pmatrix} d_{\mathcal{F}_v}\vartheta_{\overline{w}} & \vartheta_w \\ d_{\mathcal{F}_v} & 1 \end{pmatrix} \text{ for split } v = w\overline{w}, \\ \varsigma_v &= \begin{pmatrix} a_v & 0 \\ -b_v & d_{\mathcal{F}_v} \end{pmatrix} \text{ for non-split finite } v. \end{aligned} \tag{3.3}$$

For $t \in \mathcal{K}_v$, we put

$$\iota_{\varsigma_v}(t) := \varsigma_v^{-1}\iota(t)\varsigma_v.$$

It is straightforward to verify that if $v = \sigma \in \Sigma$ is archimedean and $t = x+iy \in \mathbf{C}^\times$, then

$$\iota_{\varsigma_\sigma}(t) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}, \tag{3.4}$$

and if $v = w\overline{w}$ is split and $t = t_1 e_{\overline{w}} + t_2 e_w$, then

$$\iota_{\varsigma_v}(t) = \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}. \tag{3.5}$$

Moreover, for all finite places v

$$\iota_{\varsigma_v}(\mathcal{O}_{\mathcal{K}_v}^\times) = \iota_{\varsigma_v}(\mathcal{K}_v^\times) \cap K_v^0.$$

3.5. Running assumptions. In this section, we will assume Hypothesis A for (π, χ) and

$$\mathfrak{n}^- \text{ is square-free.} \tag{sf}$$

The assumption (sf) implies that π_v is an unramified special representation if $v|\mathfrak{n}_s^-$ and π_v is a ramified principal series if $v|\mathfrak{n}_r^-$. In particular, for every place v inert or ramified in $\mathcal{K}$, π_v is a sub-quotient of induced representations and the local L-function $L(s,\pi_v) \neq 1$. We shall write $\pi_v = \pi(\mu_v, \nu_v)$ such that $L(s,\pi_v) = L(s,\mu_v)$ for $v|\mathfrak{n}^-$. By the local root number formulas [JL70, Prop. 3.5,Thm. 2.18], under the assumption (sf) Hypothesis A on the sign of local root numbers is equivalent to the following condition:

$$\begin{aligned} &\text{Each } v \in A(\chi) \text{ is ramified in } \mathcal{K} \text{ and } \mu_v'\chi_v(\varpi_{\mathcal{K}_v}) = -\left|\varpi\right|^{\frac{1}{2}} \\ &\qquad (\mu_v' := \mu_v \circ \mathrm{N}_{\mathcal{K}_v/\mathcal{F}_v}). \end{aligned} \tag{R1}$$

In what follows, we fix a place v of $\mathcal{F}$. Let $F = \mathcal{F}_v$ and $E = \mathcal{K}_v$. If v is finite, let $\mathcal{O} = \mathcal{O}_F$ and let $\varpi = \varpi_{\mathcal{F}_v}$ and ϖ_E be uniformizers of $\mathcal{O}$ and $\mathcal{O}_E$ respectively. We shall suppress the subscript v and write $\pi = \pi_v$, $\chi = \chi_v$, $\varsigma = \varsigma_v$ and $\psi = \psi_v$. For $a \in F^\times$, we put

$$\mathbf{d}(a) = \begin{pmatrix} a & \\ & 1 \end{pmatrix} \in \mathrm{GL}_2(F).$$

3.6. The choice of local toric Whittaker functions. If v is finite, we let W_v^0 denote the new Whittaker function in $\mathcal{W}(\pi,\psi)$. In other words, W_v^0 is the unique Whittaker function which is invariant by $K_v^0(\mathfrak{n})$ and $W_v^0(1)=1$. The existence and uniqueness of W_v^0 are a consequence of the theory of new vectors [Cas73]. Now we introduce some special local Whittaker functions.

3.6.1. *The archimedean case.* Suppose that $v=\sigma\in\Sigma$ is an archimedean place and $F=\mathbf{R}$. Then $\pi_\sigma=\pi(|\cdot|^{\frac{k_\sigma-1}{2}},|\cdot|^{\frac{1-k_\sigma}{2}}\operatorname{sgn}^{k_\sigma})$ is the discrete series of minimal $\mathrm{SO}(2,\mathbf{R})$-type k_σ. Let $W_{k_\sigma}\in\mathcal{W}(\pi_v,\psi_v)$ be the Whittaker function given by

$$W_{k_\sigma}(z\mathbf{d}(a)\kappa_\theta)=a^{\frac{k_\sigma}{2}}e^{-2\pi a}\mathbb{I}_{\mathbf{R}_+}(a)\cdot e^{ik_\sigma\theta}\operatorname{sgn}(z)^{k_\sigma}, \tag{3.6}$$

where $z\in\mathbf{R}^\times$ and $\kappa_\theta=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$. Let V_+ and V_- be the weight raising and lowering differential operators in [JL70, p.165] given by

$$V_\pm=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\otimes 1\pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\otimes i\in\operatorname{Lie}(\mathrm{GL}_2(\mathbf{R}))\otimes_{\mathbf{R}}\mathbf{C}.$$

Define the normalized weight raising differential operator $\widetilde{V}_+$ by

$$\widetilde{V}_+=\frac{1}{(-8\pi)}\cdot V_+. \tag{3.7}$$

Then we have

$$\widetilde{V}_+^{m_\sigma}W_{k_\sigma}(g\kappa_\theta)=\widetilde{V}_+^{m_\sigma}W_{k_\sigma}(g)e^{i(k_\sigma+2m_\sigma)\theta}. \tag{3.8}$$

3.6.2. *The split case.* Suppose that $v=w\overline{w}$ is split with $w|\Sigma_p\mathfrak{F}$ if $v|p\mathfrak{N}^+$. We introduce some smooth functions $\mathbf{a}_{\chi,v}$ on $F^\times$ in the Kirillov model $\mathcal{K}(\pi,\psi)$. Write $\chi=(\chi_w,\chi_{\overline{w}}):F^\times\oplus F^\times\to\mathbf{C}^\times$. If the local L-function $L(s,\pi\otimes\chi_w)=1$, we simply put

$$\mathbf{a}_{\chi,v}(a)=\mathbb{I}_{\mathcal{O}^\times}(a)\chi_{\overline{w}}(a^{-1}).$$

Suppose that $L(s,\pi\otimes\chi_w)\neq 1$. Then $\pi=\pi(\mu,\nu)$ is a principal series or $\pi=\pi(\mu,\nu)$ is special with $\mu\nu^{-1}=|\cdot|$ and $\mu\chi_{\overline{w}}$ is unramified. If $\pi\otimes\chi_w$ is unramified, we set

$$\mathbf{a}_{\chi,v}(a)=\mathbb{I}_{\mathcal{O}}(a)\cdot\chi_{\overline{w}}^{-1}|\cdot|^{\frac{1}{2}}(a)\sum_{i+j=v(a),\,i,j\geq 0}\mu\chi_{\overline{w}}(\varpi^i)\nu\chi_{\overline{w}}(\varpi^j).$$

If $\mu_i\chi_w$ is unramified and $\mu_j\chi_w$ is ramified for $\{\mu_1,\mu_2\}=\{\mu,\nu\}$, we set

$$\mathbf{a}_{\chi,v}(a)=\mu_i|\cdot|^{\frac{1}{2}}(a)\mathbb{I}_{\mathcal{O}}(a).$$

If π is special, we set

$$\mathbf{a}_{\chi,v}(a)=\mu|\cdot|^{\frac{1}{2}}(a)\mathbb{I}_{\mathcal{O}}(a).$$

These functions $\mathbf{a}_{\chi,v}$ indeed belong to the Kirillov model $\mathcal{K}(\pi,\psi)$ in virtue of the description of the Kirillov models [Jac72, Lemma 14.3]. For each $\xi\in\mathcal{K}(\pi,\psi)$, by the isomorphism (1.1) we denote by $W_\xi \in \mathcal{W}(\pi,\psi)$ the unique Whittaker function such that $W_\xi(\mathbf{d}(a)) = \xi(a)$. We put

$$W_{\chi,v} := W_{\mathbf{a}_{\chi,v}}.$$

It follows from the choice of $W_{\chi,v}$ that

$$W_{\chi\phi,v} = W_{\chi,v} \text{ if } \phi : E^\times \to \mathbf{C}^\times \text{ is unramified.}$$

Recall that the zeta integral $\Psi(s, W, \chi_{\overline{w}})$ for $W \in \mathcal{W}(\pi,\psi)$ is defined by

$$\Psi(s, W, \chi_{\overline{w}}) := \int_{F^\times} W(\mathbf{d}(a))\chi_{\overline{w}}(a)\,|a|^{s-\frac{1}{2}}\,\mathrm{d}^\times a.$$

Then the zeta integral for $W_{\chi,v}$ satisfies the following equation:

$$\Psi(s, W_{\chi,v}, \chi_{\overline{w}}) = L(s, \pi\otimes\chi_{\overline{w}})\,|\mathcal{D}_F|^{\frac{1}{2}} \quad (\mathrm{vol}(\mathcal{O}_F^\times, \mathrm{d}^\times a) = |\mathcal{D}_F|^{\frac{1}{2}}). \tag{3.9}$$

Suppose that $v = w\overline{w}$ with $w \in \Sigma_p$. We define some p-modified Whittaker functions as follows. For each $u \in \mathcal{O}_F^\times$, we put

$$\mathbf{a}_{u,v}(a) := \mathbb{I}_{u(1+\varpi\mathcal{O})}(a)\chi_{\overline{w}}(a^{-1}) \text{ and } W_{\chi,u,v} = W_{\mathbf{a}_{u,v}}.$$

Let $\mathbf{a}^\flat_{\chi,v}(a) := \mathbb{I}_{\mathcal{O}^\times}(a)\chi_w(a^{-1})$ and let $W^\flat_{\chi,v}$ be the p-modified Whittaker function given by

$$W^\flat_{\chi,v} := W_{\mathbf{a}^\flat_{\chi,v}} = \sum_{u\in\mathcal{U}_v} W_{\chi,u,v}, \tag{3.10}$$

where $\mathcal{U}_v$ is the torsion subgroup of $\mathcal{O}^\times$. It is easy to verify that

$$\Psi(s, W^\flat_{\chi,v}, \chi_w) = 1\,;\ \pi(\begin{pmatrix} a & b \\ 0 & d\end{pmatrix})W^\flat_{\chi,v} = \chi_{\overline{w}}^{-1}(a)\chi_w^{-1}(d)W^\flat_{\chi,v} \tag{3.11}$$

for $a, d \in \mathcal{O}^\times$, $b \in \mathcal{D}_F^{-1}$.

3.6.3. *The inert and ramified case.* Suppose that v is an inert or ramified finite place. Then E is a non-archimedean local field. Define the operators $\mathcal{R}_v$ and $\mathcal{P}_{\chi,\varsigma}$ on $W \in \mathcal{W}(\pi,\psi)$ by

$$\begin{aligned}\mathcal{R}_v W(g) :=& W(g\begin{pmatrix} 1 & \\ & \varpi\end{pmatrix}),\\ \mathcal{P}_{\chi,\varsigma}W(g) :=& \mathbf{v}_E^{-1}\cdot\int_{E^\times/F^\times}\pi(\iota_\varsigma(t))W(g)\chi(t)\mathrm{d}t.\\ =& \mathbf{v}_E^{-1}\int_{E^\times/F^\times} W(g\varsigma^{-1}\iota(t)\varsigma)\chi(t)\mathrm{d}t.\end{aligned}$$

Note that

$$\mathbf{v}_E = \mathrm{vol}(E^\times/F^\times, \mathrm{d}t) = e_v\cdot|\mathcal{D}_E|_E^{\frac{1}{2}}\,|\mathcal{D}_F|^{-\frac{1}{2}}, \quad e_v = \begin{cases} 1 & \text{if } v \text{ is inert,}\\ 2 & \text{if } v \text{ is ramified.}\end{cases}$$

We define the Whittaker function $W_{\chi,v}$ by

$$W_{\chi,v} := \mathcal{P}_{\chi,\varsigma}\mathcal{R}_v^{m_v(\chi,\pi)}W_v^0. \tag{3.12}$$

3.6.4. Define the subgroup $\mathcal{T}_v$ of $E^\times$ by

$$\mathcal{T}_v = \begin{cases} \mathcal{O}_E^\times F^\times & \text{if } v \text{ is split,} \\ E^\times & \text{if } v \text{ is non-split.} \end{cases}$$

Then $\mathcal{T}_v = \left\{x \in E \mid x/\overline{x} \in \mathcal{O}_E^\times\right\}$ if v is finite.

Definition 3.1 (Toric Whittaker functions). We say that $W \in \mathcal{W}(\pi, \psi)$ is a toric Whittaker function of character χ if

$$\pi(\iota_\varsigma(t))W = \chi^{-1}(t) \cdot W \text{ for all } t \in \mathcal{T}_v.$$

Lemma 3.2. *The Whittaker functions $W_{\chi,v}$ chosen as above are toric. To be precise, we have*

(1) $\widetilde{V}_+^{m_\sigma} W_{k_\sigma}$ is a toric Whittaker function of the character $\chi_\sigma : \mathbf{C}^\times \to \mathbf{C}^\times$, $z \mapsto z^{k_\sigma+m_\sigma}\overline{z}^{-m_\sigma}\left|z\overline{z}\right|^{-k_\sigma/2}$.
(2) If v is finite, then $W_{\chi,v}$ are toric Whittaker functions of character χ_v.
(3) If $v|p$, then $W_{\chi,v}^\flat$ is toric, and for $u \in \mathcal{O}_F^\times$

$$\pi(\iota_\varsigma(t))W_{\chi,u,v} = \chi^{-1}(t)W_{\chi,u.t^{1-c},v},$$

where $u.t^{1-c} := ut_w t_{\overline{w}}^{-1}$, $t = t_{\overline{w}}e_{\overline{w}} + t_w e_w \in \mathcal{O}_E^\times$ with $w \in \Sigma_p$,

Proof. It follows immediately from the definitions of these Whittaker functions together with (3.8), (3.4) and (3.5). □

3.7. Local toric period integrals (I).

3.7.1. Define the local toric period integral for $W \in \mathcal{W}(\pi, \psi)$ by

$$\begin{aligned}\boldsymbol{P}(W,\chi) :=& P(W,W,\chi) \\ =& \int_{E^\times/F^\times} \mathbf{b}_v(\pi(\iota(t))W, \pi(\boldsymbol{J})W)\chi(t)\mathrm{d}t \cdot \frac{L(1,\tau_{E/F})}{\zeta_F(1)}.\end{aligned}$$

The main task of this section is to evaluate $\boldsymbol{P}(\pi(\varsigma)W_{\chi,v}, \chi)$. We first treat the archimedean and split cases.

3.7.2. *The archimedean case.* Suppose $v = \sigma \in \Sigma \xrightarrow{\sim} \mathrm{Hom}(\mathcal{F}, \mathbf{R})$ is an archimedean place.

Proposition 3.3. *We have*

$$\boldsymbol{P}(\pi(\varsigma)\widetilde{V}_+^{m_\sigma} W_{k_\sigma}, \chi) = 2^3 \cdot \frac{\Gamma(m_\sigma+1)\Gamma(k_\sigma+m_\sigma)}{(4\pi)^{k_\sigma+1+2m_\sigma}}.$$

Proof. Introduce the Hermitian inner product on $\mathcal{W}(\pi, \psi)$ defined by

$$\langle W_1, W_2\rangle := \mathbf{b}_v(W_1, c(W_2)), \text{ where } c(W_2)(g) := \overline{W(\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} g)}\omega(\det g).$$

Write $k = k_\sigma$ and $m = m_\sigma$. It is clear that

$$\langle W_k, W_k\rangle = (4\pi)^{-k}\Gamma(k).$$

Since $c(V_+^m W_k)$ and $\pi(\begin{pmatrix} -1 & \\ & 1 \end{pmatrix})V_+^m W_k$ are both nonzero Whittaker functions of weight $-k-2m$, there exists some constant γ such that

$$\pi(\begin{pmatrix} -1 & \\ & 1 \end{pmatrix})V_+^m W_k = \gamma \cdot c(V_+^m W_k) \iff V_+^m W_k(\mathbf{d}(a)) = \gamma \cdot \overline{V_+^m W_k(\mathbf{d}(a))}$$

for all $a \in \mathbf{R}_+$. Let $h_m(x) := V_+^m W_k(\mathbf{d}(x))$. Then $h_0(x) = W_k(\mathbf{d}(x))$ is a real-valued function in view of the definition (3.6). A simple calculation shows that

$$h_{m+1} = 2x\frac{dh_m}{dx} + (k+2m-4\pi x)h_m,$$

so by induction $h_m(x)$ takes value in $\mathbf{R}$ (*cf.* [JL70, p.189]). This implies that $\gamma = 1$. We thus have

$$\mathbf{b}_v(\pi(\varsigma)V_+^m W_k, \pi(\boldsymbol{J}\varsigma)V_+^m W_k) = \langle V_+^m W_k, V_+^m W_k\rangle \quad (\varsigma^{-1}\boldsymbol{J}\varsigma = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}).$$

To evaluate $\langle V_+^m W_k, V_+^m W_k\rangle$, note that by [JL70, p.166] we have

$$V_-^m V_+^m W_k = (-4)^m \frac{\Gamma(k+m)\Gamma(m+1)}{\Gamma(k)} \cdot W_k, \tag{3.13}$$

and hence we find that

$$\begin{aligned}\langle V_+^m W_k, V_+^m W_k\rangle =&(-1)^m\langle W_k, V_-^m V_+^m W_k\rangle\\ =&4^m\frac{\Gamma(k+m)}{\Gamma(k)}\Gamma(m+1)\langle W_k, W_k\rangle\\ =&4^m(4\pi)^{-k}\cdot\Gamma(k+m)\Gamma(m+1).\end{aligned}$$

Recall that $\mathrm{d}t = 2\pi^{-1}d\theta$ with$t = e^{i\theta}$, so $\mathrm{vol}(\mathbf{C}^\times/\mathbf{R}^\times, \mathrm{d}t) = 2\pi^{-1}\cdot\pi = 2$. Combining these with Lemma 3.2 (1), we find that

$$\begin{aligned}\boldsymbol{P}(\pi(\varsigma)\widetilde{V}_+^m W_k, \chi) &= 2\cdot(-8\pi)^{-2m}\cdot\mathbf{b}_v(\pi(\varsigma)V_+^m W_k, \pi(\boldsymbol{J}\varsigma)V_+^m W_k)\cdot\frac{\zeta_{\mathbf{R}}(2)}{\zeta_{\mathbf{R}}(1)}\\ &= 2^3(4\pi)^{-2m-1}4^{-m}\cdot\langle V_+^m W_k, V_+^m W_k\rangle\\ &= 2^3\cdot(4\pi)^{-k-2m-1}\Gamma(k+m)\Gamma(m+1). \qquad \square\end{aligned}$$

3.7.3. *The split case.* Suppose that $v = w\overline{w}$ is a finite place split in E. Recall that we have assumed $w|\Sigma_p\mathfrak{F}$ if $v|p\mathfrak{N}^+$.

Lemma 3.4. *We have*

$$\begin{aligned}&\boldsymbol{P}(\pi(\varsigma)W, \chi)\\ =&\Psi(\frac{1}{2}, W, \chi_{\overline{w}})^2\cdot\frac{L(\frac{1}{2}, \pi\otimes\chi_w)}{L(\frac{1}{2}, \pi\otimes\chi_{\overline{w}})}\cdot\varepsilon(\frac{1}{2}, \pi\otimes\chi_{\overline{w}}, \psi)\cdot\omega^{-1}\chi_{\overline{w}}^{-2}(-d_F)\omega(\det\varsigma).\end{aligned}$$

Proof. Let $\widehat{W}(g) := W(g\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})\omega^{-1}(\det g)$. By [JL70, Thm. 2.18 (iv)], we have the local functional equation:

$$\frac{\Psi(1-s,\widehat{W},\chi_{\overline{w}}^{-1})}{L(1-s,\pi^\vee\otimes\chi_{\overline{w}}^{-1})}=\varepsilon(s,\pi\otimes\chi_{\overline{w}},\psi)\cdot\frac{\Psi(s,W,\chi_{\overline{w}})}{L(s,\pi\otimes\chi_{\overline{w}})}.$$

We note that

$$\varsigma_v^{-1}\boldsymbol{J}\varsigma_v=\begin{pmatrix} 0 & d_F^{-1} \\ d_F & 0 \end{pmatrix}.$$

A straightforward computation shows that

$$\begin{aligned}
&\boldsymbol{P}(\pi(\varsigma)W,\chi)\\
=&\omega(\det\varsigma)\int_{F^\times}\int_{F^\times}W(\mathbf{d}(at_1))W(\mathbf{d}(-a)\begin{pmatrix} 0 & d_F^{-1} \\ d_F & 0 \end{pmatrix})\chi_{\overline{w}}(t_1)\omega^{-1}(a)\mathrm{d}^\times a\mathrm{d}t_1\\
=&\omega(-\det\varsigma)\omega^{-1}\chi_{\overline{w}}^{-2}(d_F)\cdot\Psi(\frac{1}{2},W,\chi_{\overline{w}})\Psi(\frac{1}{2},\widehat{W},\chi_{\overline{w}}^{-1})\\
=&\omega(\det\varsigma)\omega^{-1}\chi_{\overline{w}}^{-2}(-d_F)\Psi(\frac{1}{2},W,\chi_{\overline{w}})^2\cdot\varepsilon(\frac{1}{2},\pi\otimes\chi_{\overline{w}},\psi)\cdot\frac{L(\frac{1}{2},\pi^\vee\otimes\chi_{\overline{w}}^{-1})}{L(\frac{1}{2},\pi\otimes\chi_{\overline{w}})}.
\end{aligned}$$

The lemma thus follows. □

Proposition 3.5. *We have*

$$\begin{aligned}
&\frac{1}{L(\frac{1}{2},\pi_E\otimes\chi)}\cdot\boldsymbol{P}(\pi(\varsigma)W_{\chi,v},\chi)\\
&=|\mathcal{D}_F|\cdot\begin{cases}\varepsilon(\frac{1}{2},\pi\otimes\chi_{\overline{w}},\psi)\chi_{\overline{w}}^{-2}(-d_F) & \text{if } v\mid\mathfrak{N}^+,\\ \omega(\det\varsigma) & \text{if } v\nmid\mathfrak{N}^+.\end{cases}
\end{aligned}$$

If $v=w\overline{w}$ with $w\in\Sigma_p$, then

$$\frac{1}{L(\frac{1}{2},\pi_E\otimes\chi)}\cdot\boldsymbol{P}(\pi(\varsigma)W^\flat_{\chi,v},\chi)=\frac{\varepsilon(\frac{1}{2},\pi\otimes\chi_{\overline{w}},\psi)}{L(\frac{1}{2},\pi\otimes\chi_{\overline{w}})^2}\cdot\chi_{\overline{w}}^{-2}(d_F)\,|\mathcal{D}_F|\,.$$

Proof. The proposition follows immediately from Lemma 3.4. (3.9) and (3.11) combined with the equations $\det\varsigma=d_F$ if $v|p\mathfrak{N}^+$ and

$$\varepsilon(\frac{1}{2},\pi\otimes\chi_{\overline{w}},\psi)\cdot\omega^{-1}\chi_{\overline{w}}^{-2}(-d_F)=1 \text{ if } v\nmid\mathfrak{N}^+. \qquad \square$$

3.8. Local toric period integrals (II). In this subsection, we treat the case v is inert or ramified. A large part of the computation in this subsection is inspired by [Mur08]. Put

$$\begin{aligned}
\mathbf{w}&=\begin{pmatrix} 0 & -d_F^{-1} \\ d_F & 0 \end{pmatrix};\\
K^0(\varpi)&:=\left\{g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in K_v^0 \mid a-1\in\varpi\mathcal{O},\, c\in\varpi\mathcal{D}_F\right\}.
\end{aligned}$$

Let $\boldsymbol{\theta} = \boldsymbol{\theta}_v \in \mathcal{O}_E$ be the element chosen in §3.4 and write W^0 for the new local Whittaker function W_v^0 at v. Recall that $\{1, \boldsymbol{\theta}\}$ is an $\mathcal{O}$-basis of $\mathcal{O}_E$ and $\boldsymbol{\theta}$ is a uniformizer if E/F is ramified.

3.8.1. We prepare some elementary lemmas.

Lemma 3.6. *Suppose that $v|\mathfrak{r}$. Let m be a non-negative integer and let*

$$B^1(\mathcal{O}) = \left\{ \begin{pmatrix} 1 & x \\ 0 & d \end{pmatrix} \mid x \in \mathcal{D}_F^{-1},\, d \in \mathcal{O}^\times \right\},$$
$$N(\mathcal{D}_F^{-1}) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathcal{D}_F^{-1} \right\}.$$

If $y \in \varpi^{m+1}\mathcal{O}$, then we have

$$\mathbf{d}(\varpi^m)\iota_\varsigma(1+y\boldsymbol{\theta})\mathbf{d}(\varpi^{-m}) \in K^0(\varpi) \quad (\mathbf{d}(a) = \begin{pmatrix} a & \\ & 1 \end{pmatrix}).$$

If $y \in \varpi^r\mathcal{O}^\times$ and $0 \le r \le m$, then

$$\mathbf{d}(\varpi^m)\iota_\varsigma(1+y\boldsymbol{\theta})\mathbf{d}(\varpi^{-m}) \in N(\mathcal{D}_F^{-1}) \begin{pmatrix} \varpi^{m-r} & \\ & y\varpi^{-m} \end{pmatrix} \mathbf{w} B^1(\mathcal{O}).$$

If $y \in \varpi\mathcal{O}$, then

$$\mathbf{d}(\varpi^m)\iota_\varsigma(y+\boldsymbol{\theta})\mathbf{d}(\varpi^{-m}) \in N(\mathcal{D}_F^{-1}) \begin{pmatrix} \varpi^{m+e_v-1} & \\ & \varpi^{-m} \end{pmatrix} \mathbf{w} B^1(\mathcal{O}).$$

Proof. Recall that if $v|\mathfrak{r}$, then $\boldsymbol{\theta} = d_F^{-1}\vartheta$, $\varsigma = \begin{pmatrix} d_F & \\ & d_F^{-1} \end{pmatrix}$, and hence

$$\iota_\varsigma(x+y\boldsymbol{\theta}) = \begin{pmatrix} x + y\mathrm{T}(\boldsymbol{\theta}) & yd_F^{-1}\mathrm{N}(\boldsymbol{\theta}) \\ yd_F & x \end{pmatrix} \quad (x, y \in F).$$

Then the proof is a straightforward calculation, so we omit the details. □

Lemma 3.7. *Suppose that $\chi|_{F^\times}$ is trivial on $1+\varpi\mathcal{O}$. For each non-negative integer r, we set*

$$X_r := \int_{\varpi^r\mathcal{O}} \chi(1+y\boldsymbol{\theta})\mathrm{d}'y,$$

where $\mathrm{d}'y$ is the Haar measure on $\mathcal{O}$ such that $\mathrm{vol}(\mathcal{O}, \mathrm{d}'y) = L(1, \tau_{E/F})\,|\mathcal{D}_E|_E^{\frac{1}{2}}\,|\mathcal{D}_F|^{-\frac{1}{2}}$. Then $X_r = 0$ if $c_v(\chi) > 1$ and $0 < r < c_v(\chi)$ and $X_r = |\varpi^r| \cdot L(1, \tau_{E/F})\,|\mathcal{D}_E|_E^{\frac{1}{2}}\,|\mathcal{D}_F|^{-\frac{1}{2}}$ if $r \ge c_v(\chi)$.

Proof. Let $Q_r := 1+\varpi^r\mathcal{O}_E/1+\varpi^r\mathcal{O}$. If $0 < r < c_v(\chi)$, then χ is a non-trivial character on the group Q_r. Note that we have a bijection $\varpi^r\mathcal{O} \xrightarrow{\sim} Q_r$, $y \mapsto 1+y\boldsymbol{\theta}$ and the pull-back of the quotient measure $\mathrm{d}t$ on Q_r is $\mathrm{d}'y$. Therefore, we have

$$X_r = \int_{Q_r} \chi(t)\mathrm{d}t = \begin{cases} 0 & \text{if } 0 < r < c_v(\chi) \\ \mathrm{vol}(\varpi^r\mathcal{O}, \mathrm{d}'y) & \text{if } r \ge c_v(\chi). \end{cases}$$

This finishes the proof. $\square$

Define the matrix coefficient $\mathbf{m}^0 : \mathrm{GL}_2(F) \to \mathbf{C}$ by

$$\begin{aligned}\mathbf{m}^0(g) :=&\mathbf{b}_v(\pi(g)W^0, \pi(\begin{pmatrix} -1 & \\ & 1\end{pmatrix})W^0)\\ =&\int_{F^\times} W^0(\mathbf{d}(a)g)W^0(\mathbf{d}(a))\omega^{-1}(a)\mathrm{d}^\times a.\end{aligned}$$

Since W^0 is invariant by $K^0(\varpi)$, $\mathbf{m}^0(g)$ only depends on the double coset $K^0(\varpi)gK^0(\varpi)$ by (2.2). Put

$$m = m_v(\chi, \pi) = c_v(\chi) - v(\mathfrak{n}^-) \geq -1.$$

We set

(3.14)

$$\begin{aligned}\boldsymbol{P}^*(\pi(\varsigma)\mathcal{R}_v^m W^0, \chi) :=& \boldsymbol{P}(\pi(\varsigma)\mathcal{R}_v^m W^0, \chi) \cdot \frac{\zeta_F(1)}{L(1, \tau_{E/F})}\omega(\varpi^{-m}\det\varsigma^{-1})\\ =&\int_{E^\times/F^\times} \mathbf{b}_v(\mathcal{R}_v^{-m}\pi(\iota_\varsigma(t))\mathcal{R}_v^m W^0, \pi(\begin{pmatrix} -1 & \\ & 1\end{pmatrix})W^0)\chi(t)\mathrm{d}t\\ =&\int_{E^\times/F^\times} \mathbf{m}^0(\mathbf{d}(\varpi^m)\iota_\varsigma(t)\mathbf{d}(\varpi^{-m}))\chi(t)\mathrm{d}t.\end{aligned}$$

Here we have used the fact that $m + v(\mathrm{T}(\boldsymbol{\theta})) \geq 0$ in the second equality. It follows immediately from the definition of the projector $\mathcal{P}_{\chi,\varsigma}$ that

$$\begin{aligned}\boldsymbol{P}(\pi(\varsigma)W_{\chi,v}, \chi) =&\boldsymbol{P}(\pi(\varsigma)\mathcal{P}_{\chi,\varsigma}\mathcal{R}_v^m W^0, \chi)\\ =&\boldsymbol{P}^*(\pi(\varsigma)\mathcal{R}_v^m W^0, \chi) \cdot \frac{\omega(\varpi^m \det\varsigma)L(1, \tau_{E/F})}{\zeta_F(1)}.\end{aligned} \tag{3.15}$$

Using the decomposition

$$E^\times = F^\times(1 + \mathcal{O}\boldsymbol{\theta}) \sqcup F^\times(\varpi\mathcal{O} + \boldsymbol{\theta})$$

and Lemma 3.6, we find that

(3.16)

$$\begin{aligned}&\boldsymbol{P}^*(\pi(\varsigma)\mathcal{R}_v^m W^0, \chi)\\ =&\int_{\mathcal{O}} \chi(1 + y\boldsymbol{\theta})\mathbf{m}^0(\mathbf{d}(\varpi^m)\iota_\varsigma(1 + y\boldsymbol{\theta})\mathbf{d}(\varpi^{-m}))\mathrm{d}'y\\ &+ \int_{\varpi\mathcal{O}} \chi(x + \boldsymbol{\theta})\mathbf{m}^0(\mathbf{d}(\varpi^m)\iota_\varsigma(y + \boldsymbol{\theta})\mathbf{d}(\varpi^{-m}))\left|y + \boldsymbol{\theta}\right|_E^{-1}\mathrm{d}'y\\ =&X_{m+1} \cdot \mathbf{m}^0(1) + \sum_{r=0}^{m}\int_{\varpi^r\mathcal{O}^\times} \chi(1 + y\boldsymbol{\theta})\omega(\varpi^{-m}y)\mathrm{d}'y \cdot \mathbf{m}^0(\mathbf{d}(\varpi^{2(m-r)})\mathbf{w})\\ &+ Y_0 \cdot \omega(\varpi^{-m})\mathbf{m}^0(\mathbf{d}(\varpi^{2m+e_v-1})\mathbf{w}),\end{aligned}$$

where

$$Y_0 := \int_{\varpi\mathcal{O}} \chi(y + \boldsymbol{\theta})\mathrm{d}'y \cdot |\varpi|^{1-e_v}.$$

In what follows, we use Lemma 3.7 and (3.16) to calculate $\boldsymbol{P}(\pi(\varsigma)W_{\chi,v},\chi)$.

3.8.2. *The case $v \nmid \mathfrak{n}_r^-$.* Suppose that $v \nmid \mathfrak{n}_r^-$, i.e. the central character ω is unramified. Then (sf) implies that π is either an unramified principal series or an unramified special representation.

Proposition 3.8. *Suppose that π is an unramified principal series. Then*

$$\begin{aligned}&\frac{1}{L(\frac{1}{2},\pi_E\otimes\chi)}\cdot\boldsymbol{P}(\pi(\varsigma)W_{\chi,v},\chi)\\ =&\omega(\varpi^m)\left|\varpi^{c_v(\chi)}\right||\mathcal{D}_E|_E^{\frac{1}{2}}\cdot\begin{cases}\omega(\det\varsigma) & \text{if } c_v(\chi)=0,\\ L(1,\tau_{E/F})^2 & \text{if } c_v(\chi)>0.\end{cases}\end{aligned}$$

Proof. Since π is unramified, ω is unramified and $m=c_v(\chi)$. Write $\pi=\pi(\mu,\nu)$ and let $\alpha=\mu(\varpi)$ and $\beta=\nu(\varpi)$. The matrix coefficient $\mathbf{m}^0$ is a spherical function on $\mathrm{GL}_2(F)$ in the sense of [Car79, Definition 4.1, p.150], and $\mathbf{m}^0(g)$ only depends on the double coset $K_v^0gK_v^0$. By a standard computation (*cf.* [Wal85, LEMME 14, p.226)]), we obtain

(3.17)
$$\mathbf{m}^0(1)=\frac{\zeta_F(1)L(1,\mathrm{Ad}\,\pi)}{\zeta_F(2)}\cdot|\mathcal{D}_F|^{\frac{1}{2}}=\frac{(1+|\varpi|)\zeta_F(1)}{(1-\alpha\beta^{-1}|\varpi|)(1-\alpha^{-1}\beta|\varpi|)}\cdot|\mathcal{D}_{\mathcal{F}}|^{\frac{1}{2}};$$

$$\mathbf{m}^0(\mathbf{d}(\varpi))=\frac{|\varpi|^{\frac{1}{2}}}{1+|\varpi|}\cdot(\alpha+\beta)\cdot\mathbf{m}^0(1);\tag{3.18}$$

$$\mathbf{m}^0(\mathbf{d}(\varpi^2))=\frac{|\varpi|}{1+|\varpi|}\cdot(\alpha^2+\beta^2+(1-|\varpi|)\alpha\beta)\cdot\mathbf{m}^0(1).\tag{3.19}$$

If v is inert and $m=0$, then

$$\begin{aligned}\omega(\det\varsigma^{-1})\boldsymbol{P}(\pi(\varsigma)W^0,\chi)=&\mathbf{m}^0(1)\cdot\frac{L(1,\tau_{E/F})}{\zeta_F(1)}\cdot|\mathcal{D}_E|_E^{\frac{1}{2}}|\mathcal{D}_F|^{-\frac{1}{2}}\\ =&\frac{1}{(1-\alpha\beta^{-1}|\varpi|)(1-\alpha^{-1}\beta|\varpi|)}\cdot|\mathcal{D}_E|_E^{\frac{1}{2}}\\ =&L(\frac{1}{2},\pi_E\otimes\chi)\cdot|\mathcal{D}_E|_E^{\frac{1}{2}}.\end{aligned}$$

Suppose that either v is ramified or $m>0$ (so $v|\mathfrak{r}$ and $\det\varsigma=1$). Then we deduce from (3.16) that

$$\begin{aligned}&\boldsymbol{P}^*(\pi(\varsigma)\mathcal{R}_v^mW^0,\chi)\\ =&X_m\cdot\mathbf{m}^0(1)+\sum_{r=0}^{m-1}(X_r-X_{r+1})\omega(\varpi^{r-m})\cdot\mathbf{m}^0(\mathbf{d}(\varpi^{2(m-r)}))\\ &+Y_0\cdot\omega(\varpi^{-m})\mathbf{m}^0(\mathbf{d}(\varpi^{2m+e_v-1})).\end{aligned}\tag{3.20}$$

If v is ramified and $m = 0$, then $X_0 = |\mathcal{D}_E|_E^{\frac{1}{2}} |\mathcal{D}_F|^{-\frac{1}{2}}$ and $Y_0 = \chi(\varpi_E) |\mathcal{D}_E|_E^{\frac{1}{2}} |\mathcal{D}_F|^{-\frac{1}{2}}$. By (3.20), we find that

$$\begin{aligned}
&\boldsymbol{P}(\pi(\varsigma)W^0, \chi)\\
=&(\mathbf{m}^0(1) + \chi(\varpi_E)\mathbf{m}^0(\mathbf{d}(\varpi)))\frac{L(1, \tau_{E/F})}{\zeta_F(1)} |\mathcal{D}_E|_E^{\frac{1}{2}} |\mathcal{D}_F|^{-\frac{1}{2}}\\
=&(1 + \frac{\alpha+\beta}{1+|\varpi|} \cdot |\varpi|^{\frac{1}{2}} \chi(\varpi_E)) \cdot \mathbf{m}^0(1) \cdot \frac{|\mathcal{D}_E|_E^{\frac{1}{2}} |\mathcal{D}_F|^{-\frac{1}{2}}}{\zeta_F(1)} \quad \text{(by (3.18))}\\
=&\frac{(1 + \chi(\varpi_E)\alpha |\varpi|^{\frac{1}{2}})(1 + \chi(\varpi_E)\beta |\varpi|^{\frac{1}{2}})}{1+|\varpi|} \cdot \mathbf{m}^0(1) \cdot \frac{|\mathcal{D}_E|_E^{\frac{1}{2}} |\mathcal{D}_F|^{-\frac{1}{2}}}{\zeta_F(1)}\\
=& |\mathcal{D}_E|_E^{\frac{1}{2}} \cdot L(\frac{1}{2}, \pi_E \otimes \chi).
\end{aligned}$$

Suppose that $m > 0$. Note that since $\chi|_{\mathcal{O}^\times} = 1$, $Y_0 = -X_0$ if v is inert and $Y_0 = X_0 = 0$ if v is ramified. Combining with Lemma 3.7, (3.19) and (3.20), we find that

$$\begin{aligned}
\boldsymbol{P}(\pi(\varsigma)\mathcal{R}_v^m W^0, \chi) =& X_m \cdot (\mathbf{m}^0(1) - \omega(\varpi^{-1})\mathbf{m}^0(\mathbf{d}(\varpi^2)) \cdot \omega(\varpi^{-m})\frac{L(1, \tau_{E/F})}{\zeta_F(1)}\\
=&\omega(\varpi^m) |\varpi^m| \cdot \frac{(1 - \alpha\beta^{-1} |\varpi|)(1 - \alpha^{-1}\beta |\varpi|)}{1+|\varpi|} \cdot \mathbf{m}^0(1)\\
&\times \frac{L(1, \tau_{E/F})^2}{\zeta_F(1)} |\mathcal{D}_E|_E^{\frac{1}{2}} |\mathcal{D}_F|^{-\frac{1}{2}}\\
=&\omega(\varpi^m) |\varpi^m| |\mathcal{D}_E|_E^{\frac{1}{2}} \cdot L(1, \tau_{E/F})^2.
\end{aligned}$$

The proposition follows immediately. $\square$

Proposition 3.9. *Suppose that π is an unramified special representation. Then*

$$\begin{aligned}
&\frac{1}{L(\frac{1}{2}, \pi_E \otimes \chi)} \cdot \boldsymbol{P}(\pi(\varsigma)W_{\chi,v}, \chi)\\
=&\omega(\varpi^m) \left|\varpi^{c_v(\chi)}\right| |\mathcal{D}_E|_E^{\frac{1}{2}} \cdot \begin{cases} L(1, \tau_{E/F})^2 & \textit{if } c_v(\chi) > 0,\\ 2 & \textit{if } v \textit{ is ramified and } c_v(\chi) = 0. \end{cases}
\end{aligned}$$

Proof. Suppose that $v | \mathfrak{n}_s^-$. Then $m = m_v(\chi, \pi) = c_v(\chi) - 1$. Recall that $\pi = \pi(\mu, \nu)$ is a special representation with a unramified character μ and $\mu\nu^{-1} = |\cdot|$. We have

$$\begin{aligned}
W^0(\mathbf{d}(a)) =& \mu(a) |a|^{\frac{1}{2}} \mathbb{I}_{\mathcal{O}}(a),\\
W^0(\mathbf{d}(a)\mathbf{w}) =& - \mu(a) |a|^{\frac{1}{2}} |\varpi| \mathbb{I}_{\varpi^{-1}\mathcal{O}}(a)
\end{aligned}$$

(*cf.* [Sch02, Eq.(54)]). With the above formulas, we obtain by a direct computation that

$$\mathbf{m}^0(1) = \frac{|\mathcal{D}_{\mathcal{F}}|^{\frac{1}{2}}}{1-|\varpi|^2}\,;\ \mathbf{m}^0(\mathbf{w}) = (-|\varpi|)\cdot\mathbf{m}^0(1)\,;$$

$$\mathbf{m}^0(\begin{pmatrix}1 & \\ & \varpi\end{pmatrix}\mathbf{w}) = (-\mu(\varpi)\,|\varpi|^{-\frac{1}{2}})\cdot\mathbf{m}^0(1).$$

If $c_v(\chi)>0$, then it follows from (3.16) and Lemma 3.7 that

$$\begin{aligned}\boldsymbol{P}(\pi(\varsigma)\mathcal{R}_v^m W^0,\chi) =& X_{m+1}\cdot(\mathbf{m}^0(1)-\mathbf{m}^0(\mathbf{w}))\cdot\omega(\varpi^m)\frac{L(1,\tau_{E/F})}{\zeta_F(1)}\\ =&\omega(\varpi^m)\left|\varpi^{m+1}\right|\cdot|\mathcal{D}_E|_E^{\frac{1}{2}}\,L(1,\tau_{E/F})^2.\end{aligned}$$

If $c_v(\chi)=0$ $(m=-1)$, then v is ramified, $X_0=|\mathcal{D}_E|_E^{\frac{1}{2}}\,|\mathcal{D}_F|^{-\frac{1}{2}}$,, $Y_0=\chi(\varpi_E)\,|\mathcal{D}_E|_E^{\frac{1}{2}}\,|\mathcal{D}_F|^{-\frac{1}{2}}$, and

$$\begin{aligned}&\boldsymbol{P}(\pi(\varsigma)\mathcal{R}_v^m W^0,\chi)\\ =&\left(X_0\cdot\mathbf{m}^0(1)+Y_0\cdot\mathbf{m}^0(\begin{pmatrix}1 & \\ & \varpi\end{pmatrix}\mathbf{w})\right)\omega(\varpi^{-1})\cdot\frac{L(1,\tau_{E/F})}{\zeta_F(1)}\\ =&(1-\mu(\varpi)\chi(\varpi_E)\,|\varpi|^{-\frac{1}{2}})\,|\mathcal{D}_E|_E^{\frac{1}{2}}\,|\mathcal{D}_F|^{-\frac{1}{2}}\cdot\mathbf{m}^0(1)(1-|\varpi|)\omega(\varpi^{-1})\\ =&\frac{2\,|\mathcal{D}_E|_E^{\frac{1}{2}}\,\omega(\varpi^{-1})}{1+|\varpi|}\quad\text{(by (R1))}\\ =&2\,|\mathcal{D}_E|_E^{\frac{1}{2}}\,\omega(\varpi^{-1})\cdot L(\frac{1}{2},\pi_E\otimes\chi).\end{aligned}$$ □

3.8.3. *The case* $v|\mathfrak{n}_r^-$. We consider the case π is a ramified principal series. Recall that (sf) suggests that $\pi=\pi(\mu,\nu)$, where μ is unramified and ν is ramified, and the conductor $a(\nu)=a(\omega)=1$. Since $\chi|_{F^\times}=\omega^{-1}$, we must have $m=c_v(\chi)-1\geq 0$. Let $\boldsymbol{\delta}_v:=\boldsymbol{\theta}-\overline{\boldsymbol{\theta}}$. Let $D_{E/F}$ be the discriminant of E/F. We begin with a lemma.

Lemma 3.10. *Suppose that* $\chi|_{\mathcal{O}^\times}\neq 1$ *and* $\chi|_{1+\varpi\mathcal{O}}=1$. *Then*

$$\int_{\varpi^{-m}\mathcal{O}}\chi(y+\boldsymbol{\theta})\mathrm{d}'y=\chi(\boldsymbol{\delta}_v)\,|\boldsymbol{\delta}_v|_E^{\frac{1}{2}}\cdot\frac{\varepsilon(0,\chi^{-1},\psi_E)}{\varepsilon(-1,\omega,\psi)}\cdot L(1,\tau_{E/F})\left|D_{E/F}\right|^{\frac{1}{2}}.$$

Proof. By [HKS96, Prop. 8.2], we have

$$\begin{aligned}\int_F\chi(y+2^{-1}\boldsymbol{\delta}_v)\mathrm{d}y:=&\int_F\chi(y+2^{-1}\boldsymbol{\delta}_v)\left|y+2^{-1}\boldsymbol{\delta}_v\right|_E^{-s-\frac{1}{2}}\mathrm{d}y|_{s=-\frac{1}{2}}\\ =&\chi(\boldsymbol{\delta}_v)\,|\boldsymbol{\delta}_v|_E^{\frac{1}{2}}\cdot\frac{\varepsilon(0,\chi^{-1},\psi_E)}{\varepsilon(-1,\omega,\psi)}.\end{aligned}$$

By the assumption, for all $r\geq m+1$ we have

$$\int_{\varpi^{-r}\mathcal{O}^\times}\chi(y+\boldsymbol{\theta})\mathrm{d}y=\chi(\varpi^{-r})\cdot\int_{\mathcal{O}^\times}\chi(y)\mathrm{d}y=0.$$

Thus

$$\int_{\varpi^{-m}\mathcal{O}} \chi(y+\boldsymbol{\theta})\mathrm{d}y = \varinjlim_{r} \int_{\varpi^{-r}\mathcal{O}} \chi(y+\boldsymbol{\theta})\mathrm{d}y = \int_F \chi(y+2^{-1}\boldsymbol{\delta}_v)\mathrm{d}y.$$

The lemma follows from the fact that

$$\mathrm{d}'y = L(1,\tau_{E/F}) \left|D_{E/F}\right|^{\frac{1}{2}} \cdot \mathrm{d}y. \qquad \square$$

PROPOSITION 3.11. *We have*

$$\frac{1}{L(\frac{1}{2},\pi_E\otimes\chi)} \cdot \boldsymbol{P}(\pi(\varsigma)W_{\chi,v},\chi) = \left|\varpi^{c_v(\chi)}\right| |\mathcal{D}_E|_E^{\frac{1}{2}} \chi(\boldsymbol{\delta}_v d_F^{-1}) |\boldsymbol{\delta}_v|_E^{\frac{1}{2}} \cdot \varepsilon(0,\chi,\psi_E)$$
$$\times L(1,\tau_{E/F})^2 \cdot n_v^2,$$

where n_v is given by

$$n_v := \frac{\mu(\varpi)\,|\varpi|^{m/2}\,|\mathcal{D}_F|^{\frac{1}{4}}}{\varepsilon(0,\omega,\psi)} \in \overline{\mathbf{Z}}_{(p)}^{\times}. \tag{3.21}$$

PROOF. We first recall that if $\xi : F^\times \to \mathbf{C}^\times$ is a character of conductor $a(\xi)$, then

$$\varepsilon(s,\xi,\psi) = \xi(c)\,|c|^{s-\frac{1}{2}} \cdot \left(\sum_{a\in\mathcal{O}^\times/(1+\varpi^{a(\xi)}\mathcal{O})} \xi^{-1}(u)\psi(u/c) \right) \quad (c = d_F\varpi^{a(\xi)}).$$

By the equation $\varepsilon(s,\xi,\psi)\varepsilon(1-s,\xi^{-1},\psi) = \xi(-1)$ (*cf.* [Sch02, Eq.(7)]), we see that $\varepsilon(0,\xi,\psi)$ belongs to $\overline{\mathbf{Z}}_{(p)}^{\times}$ whenever v does not divide p and ξ takes values in $\overline{\mathbf{Z}}_{(p)}$. This shows that n_v is a p-adic unit by the discussion in §3.2.
We proceed to prove the toric integral. We have

$$\begin{aligned} W^0(\mathbf{d}(a)) =& \nu|\cdot|^{\frac{1}{2}}(a)\mathbb{I}_{\mathcal{O}}(a), \\ W^0(\mathbf{d}(a)\mathbf{w}) =& \mu|\cdot|^{\frac{1}{2}}(a)\mathbb{I}_{\varpi^{-1}\mathcal{O}}(a) \cdot \frac{\omega(d_F)\mu(\varpi^2)}{\varepsilon(0,\omega,\psi)} \end{aligned}$$

(*cf.* [Sch02, Eq.(50) and (51)]). A simple calculation shows that

$$\mathbf{m}^0(1) = 0,\ \mathbf{m}^0(\mathbf{w}) = \frac{\omega(d_F)\mu(\varpi^2)}{\varepsilon(0,\omega,\psi)\cdot(1-|\varpi|)} \cdot |\mathcal{D}_{\mathcal{F}}|^{\frac{1}{2}}.$$

It is not difficult to show that if v is ramified, then

$$Y_0 = \int_{\varpi\mathcal{O}} \chi(y+\boldsymbol{\theta})\mathrm{d}'y = 0,$$

and that if $m = c_v(\chi) - 1 > 0$, then

$$\int_{\varpi^r\mathcal{O}^\times} \chi(y^{-1}+\boldsymbol{\theta})\mathrm{d}'y = 0 \text{ for } 0 < r < m \text{ and}$$
$$\int_{\mathcal{O}} \chi(y+\boldsymbol{\theta})\mathrm{d}'y = 0.$$

From the above equations, we find that

$$\begin{aligned}
&\boldsymbol{P}^*(\pi(\varsigma)\mathcal{R}_v^m W^0,\chi)\\
=&X_{m+1}\cdot\mathbf{m}^0(1)+\sum_{r=0}^{m}\int_{\varpi^r\mathcal{O}^\times}\chi(y^{-1}+\boldsymbol{\theta})\mathrm{d}'y\cdot\omega(\varpi^{-m})\mathbf{m}^0(\mathbf{d}(\varpi^{2m-2r})\mathbf{w})\\
&+Y_0\cdot\omega(\varpi^{-m})\mathbf{m}^0(\mathbf{d}(\varpi^{2m+e_v-1})\mathbf{w})\\
=&\int_{\varpi^{-m}\mathcal{O}}\chi(y+\boldsymbol{\theta})\mathrm{d}'y\cdot\omega(\varpi^{-m})\left|\varpi^{2m}\right|\mathbf{m}^0(\mathbf{w}).
\end{aligned}$$

By Lemma 3.10, we obtain

$$\begin{aligned}
&\boldsymbol{P}(\pi(\varsigma)\mathcal{R}_v^m W^0,\chi)\\
=&\boldsymbol{P}^*(\pi(\varsigma)\mathcal{R}_v^m W^0,\chi)\cdot\omega(\varpi^m)\frac{L(1,\tau_{E/F})}{\zeta_F(1)}\\
=&\left|\varpi^{2m}\right|\chi(\boldsymbol{\delta}_v)\left|\boldsymbol{\delta}_v\right|_E^{\frac12}\left|\mathcal{D}_E\right|_E^{\frac12}\left|\mathcal{D}_F\right|^{-\frac12}\frac{\varepsilon(0,\chi^{-1},\psi_E)}{\varepsilon(-1,\omega,\psi)}\cdot\frac{\mu(\varpi^2)\omega(d_F)}{\varepsilon(0,\omega,\psi)(1-\left|\varpi\right|)}\\
&\times\frac{L(1,\tau_{E/F})^2}{\zeta_F(1)}\\
=&\frac{L(1,\tau_{E/F})^2\mu(\varpi^2)\left|\mathcal{D}_F\right|^{\frac12}\left|\varpi^{2m+1}\right|}{\varepsilon(0,\omega,\psi)^2}\cdot\left|\mathcal{D}_E\right|_E^{\frac12}\chi(\boldsymbol{\delta}_v d_F^{-1})\left|\boldsymbol{\delta}_v\right|_E^{\frac12}\varepsilon(0,\chi^{-1},\psi_E).
\end{aligned}$$

The last equality follows from

$$\varepsilon(-1,\omega,\psi)=\left|\varpi\mathcal{D}_F\right|^{-1}\varepsilon(0,\omega,\psi).$$

From the above computation and that $L(s,\pi_E\otimes\chi)=1$, the proposition follows. □

3.9. The global toric period integral. We return to the global situation. Let $W^{(p)}_{\chi,f}$ be the prime-to-p Whittaker function given by

$$W^{(p)}_{\chi,f}=\prod_{v\in\mathbf{h},\,v\nmid p}W_{\chi,v}\in\bigotimes_{v\in\mathbf{h},v\nmid p}\mathcal{W}(\pi_v,\psi_v).$$

Definition 3.12. Let $W_{\chi,\infty}:=\prod_{\sigma\in\Sigma}W_{k_\sigma}$. Define the p-modified toric Whittaker function W_χ by

$$W_\chi=W_{\chi,\infty}\cdot W^{(p)}_{\chi,f}\cdot\prod_{v|p}W^\flat_{\chi,v}\in\mathcal{W}(\pi,\psi).\tag{3.22}$$

Let $u=(u_v)\in(\mathcal{O}_\mathcal{F}\otimes_\mathbf{Z}\mathbf{Z}_p)^\times=\prod_v\mathcal{O}^\times_{\mathcal{F}_v}$. The *u-component* $W_{\chi,u}$ of W_χ is defined by

$$W_{\chi,u}=W_{\chi,\infty}\cdot W^{(p)}_{\chi,f}\cdot\prod_{v|p}W_{\chi,u_v,v}.\tag{3.23}$$

Recall that the automorphic form $\varphi_W \in \mathcal{A}(\pi)$ associated to $W \in \mathcal{W}(\pi,\psi)$ is defined by

$$\varphi_W(g) := \sum_{\beta\in\mathcal{F}} W(\begin{pmatrix}\beta & \\ & 1\end{pmatrix} g). \tag{3.24}$$

Let φ_χ (resp. $\varphi_{\chi,u}$) be the automorphic form associated to W_χ (resp. $W_{\chi,u}$). Let $\mathcal{U}_p = \prod_{v|p}\mathcal{U}_v$ be the torsion subgroup of $(\mathcal{O}_\mathcal{F}\otimes_\mathbf{Z}\mathbf{Z}_p)^\times$. It follows immediately from the definition (3.10) that

$$\varphi_\chi = \sum_{u\in\mathcal{U}_p}\varphi_{\chi,u}. \tag{3.25}$$

Choose a sufficiently small prime-to-p integral ideal $\mathfrak{n}_1$ such that $W_{\chi,v}$ is invariant by $U_v(\mathfrak{n}_1)$ for all $v\nmid p$. Let $K=\prod_v K_v \subset \mathrm{GL}_2(\mathbf{A}_f)$ be an open compact subgroup such that

$$K_v = K_v^0 \text{ if } v \mid p\,;\, K_v \subset U_v(\mathfrak{n}_1) \text{ if } v\nmid p. \tag{3.26}$$

For each positive integer n, put

$$K_1^n := \left\{ g\in K \mid g_v \equiv \begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix} \pmod{p^n} \text{ for all } v|p\right\}.$$

One can verify that W_χ and $W_{\chi,u}$ (and hence φ_χ and $\varphi_{\chi,u}$) are invariant by K_1^n for sufficiently large n. The following lemma immediately follows from Lemma 3.2,

Lemma 3.13. *Let $\mathcal{T}=\prod'_v\mathcal{T}_v\subset\mathbf{A}_\mathcal{K}^\times$. Then φ_χ is a toric automorphic form in the sense that for all $t\in\mathcal{T}$, we have*

$$\pi(\iota_\varsigma(t))\widetilde{V}_+^m\varphi_\chi = \chi^{-1}(t)\widetilde{V}_+^m\varphi_\chi.$$

In addition, for all $t\in\mathcal{T}_f=\prod'_{v\in\mathbf{h}}\mathcal{T}_v$, we have

$$\pi(\iota_\varsigma(t))\varphi_{\chi,u} = \chi^{-1}(t)\varphi_{\chi,u\cdot t^{1-c}},$$

where $u\cdot t^{1-c} := ut_{\Sigma_p}t^{-1}_{\overline{\Sigma}_p}\in(\mathcal{O}_\mathcal{F}\otimes_\mathbf{Z}\mathbf{Z}_p)^\times$.

Decompose $\mathfrak{c}_\chi^- = \mathfrak{c}_{\chi,1}^-\mathfrak{c}_{\chi,2}^-$ such that $(\mathfrak{c}_{\chi,1}^-,\mathfrak{n}_r^-)=1$ and $\mathfrak{c}_{\chi,2}^-$ has the same support with $\mathfrak{n}_r^-$. Define a constant $C'_\pi(\chi)$ by

$$\begin{aligned}C'_\pi(\chi) :=& 2^{\sharp(A(\chi))+3[\mathcal{F}:\mathbf{Q}]}\cdot \mathrm{N}_{\mathcal{F}/\mathbf{Q}}(\mathfrak{c}_\chi^-)^{-1}\omega(\mathfrak{c}_{\chi,1}^-)\omega(\mathfrak{n}_s^-)^{-1}\\ &\times\prod_{v\nmid p\mathfrak{r}}\omega(\det\varsigma_v)\cdot\prod_{w|\mathfrak{F},\, v=w\overline{w}}\varepsilon(\frac{1}{2},\pi_v\otimes\chi_{\overline{w}},\psi_v)\chi_{\overline{w}}^{-2}(-d_{\mathcal{F}_v})\\ &\times\prod_{v|\mathfrak{n}_r^-}\chi_v(-\boldsymbol{\delta}_v d_{\mathcal{F}_v}^{-1})\,|\boldsymbol{\delta}_v|_{\mathcal{K}_v}^{\frac{1}{2}}\,\varepsilon(0,\chi_v^{-1},\psi_{\mathcal{K}_v})\end{aligned} \tag{3.27}$$

Note that $C'_\pi(\chi)$ is actually a p-adic unit as $p > 2$ and $(p, \mathfrak{F}\mathfrak{n}^-) = 1$. We introduce the normalization factor $N(\pi,\chi)$ given by

$$N(\pi,\chi) := \prod_{v \in B(\chi)} L(1, \tau_{\mathcal{K}_v/\mathcal{F}_v}) n_v. \tag{3.28}$$

We have the following central value formula of the toric integral $P_\chi(\pi(\varsigma)\widetilde{V}_+^m\varphi_\chi)$.

Theorem 3.14. *We have*

$$\begin{aligned} P_\chi(\pi(\varsigma)\widetilde{V}_+^m\varphi_\chi)^2 =& \frac{\Gamma_\Sigma(k+m)\Gamma_\Sigma(m+1)}{(4\pi)^{k+2m+1\cdot\Sigma}} \cdot e_{\Sigma_p}(\pi,\chi) \cdot L(\frac{1}{2}, \pi_{\mathcal{K}} \otimes \chi) \\ & \times |D_{\mathcal{K}}|_{\mathbf{R}}^{-\frac{1}{2}} \cdot C'_\pi(\chi) N(\pi,\chi)^2, \end{aligned}$$

where $e_{\Sigma_p}(\pi,\chi)$ is the p-adic multiplier given by

$$e_{\Sigma_p}(\pi,\chi) = \prod_{w \in \Sigma_p, v = w\overline{w}} \varepsilon(\frac{1}{2}, \pi_v \otimes \chi_{\overline{w}}, \psi_v) L(\frac{1}{2}, \pi_v \otimes \chi_{\overline{w}})^{-2} \chi_{\overline{w}}^{-2}(d_{\mathcal{F}_v}).$$

Proof. Note that $\widetilde{V}_+^m\varphi_\chi$ is the automorphic form associated to the Whittaker function

$$\widetilde{V}_+^m W_\chi = \widetilde{V}_+^m W_{\chi,\infty} \cdot W_{\chi,f}^{(p)} \cdot \prod_{v|p} W_{\chi,v}^\flat.$$

Hence, by Proposition 2.1 we find that

$$\begin{aligned} & P_\chi(\pi(\varsigma)\widetilde{V}_+^m\varphi_\chi)^2 \\ =& \prod_{\sigma\in\Sigma} \boldsymbol{P}(\pi(\varsigma_\sigma)\widetilde{V}_+^{m_\sigma} W_{k_\sigma}, \chi_\sigma) \prod_{v|p} \frac{1}{L(\frac{1}{2}, \pi_{\mathcal{K}_v} \otimes \chi_v)} \cdot \boldsymbol{P}(\pi(\varsigma_v) W_{\chi,v}^\flat, \chi_v) \\ & \times \prod_{v \in \mathbf{h}, v\nmid p} \frac{1}{L(\frac{1}{2}, \pi_{\mathcal{K}_v} \otimes \chi_v)} \cdot \boldsymbol{P}(\pi(\varsigma_v) W_{\chi,v}, \chi_v) \cdot L(\frac{1}{2}, \pi_{\mathcal{K}} \otimes \chi). \end{aligned}$$

Combining the local calculations of toric integrals of our Whittaker functions (Proposition 3.3, Proposition 3.5, Proposition 3.8, Proposition 3.9 and Proposition 3.11) yields the central value formula. □

Remark 3.15. Let φ_χ^0 be the automorphic form associated to the toric Whittaker function $W_\chi^0 := W_{\chi,\infty} \cdot \prod_{v\in\mathbf{h}} W_{\chi,v}$. Then we obtain the following central value formula:

$$P_\chi(\pi(\varsigma)\widetilde{V}_+^m\varphi_\chi^0)^2 = |D_{\mathcal{K}}|_{\mathbf{R}}^{-\frac{1}{2}} \frac{\Gamma_\Sigma(k+m)\Gamma_\Sigma(m+1)}{(4\pi)^{k+2m+1\cdot\Sigma}} \cdot L(\frac{1}{2}, \pi_{\mathcal{K}} \otimes \chi) \cdot C'_\pi(\chi) N(\pi,\chi)^2.$$

3.10. Non-vanishing of the local Fourier coefficients. In order to prove the non-vanishing of our toric form φ_χ later, we calculate its local Fourier coefficients in this subsection. Define $\mathbf{a}_{\chi,v} : F^\times \to \mathbf{C}$ the *local Fourier coefficient* associated to $W_{\chi,v}$ by

$$\mathbf{a}_{\chi,v}(a) = W_{\chi,v}(\mathbf{d}(a)).$$

To obtain the optimal p-integrality of φ_χ, we need the following normalization of the $\mathbf{a}_{\chi,v}$.

Definition 3.16 (Normalized local Fourier coefficients). Let

$$B(\chi)=\{v\in\mathbf{h}\mid v\text{ is non-split with }c_v(\chi)>0\}\,.$$

For $v\in B(\chi)$, let n_v be defined as in (3.21) if $v|\mathfrak{n}_r^-$ and $n_v=1$ if $v\nmid\mathfrak{n}_r^-$. Define the normalized local Fourier coefficient $\mathbf{a}^*_{\chi,v}$ by

$$\mathbf{a}^*_{\chi,v}:=\mathbf{a}_{\chi,v}\cdot\begin{cases}n_v^{-1}L(1,\tau_{\mathcal{K}_v/\mathcal{F}_v})^{-1} & \text{if } v\in B(\chi),\\ 1 & \text{if } v\in A(\chi),\\ e_v & \text{otherwise.}\end{cases}$$

Recall that $e_v=1$ if v is unramified and $e_v=2$ if v is ramified.

Let $v\nmid p$ be a finite place. We shall show the normalized local Fourier coefficients $\mathbf{a}^*_{\chi,v}$ indeed take value in a finite extension of $\mathbf{Z}_p$ and is not identically zero modulo the maximal $\mathfrak{m}_p$ of $\overline{\mathbf{Z}}$ induced by $\iota_p:\overline{\mathbf{Q}}\hookrightarrow\mathbf{C}_p$. This is clear if v is split in view of the definition of $\mathbf{a}^*_{\chi,v}=\mathbf{a}_{\chi,v}$ in §3.6. The most difficult case is when v is inert and π is ramified at v. We begin with some formulae of $\mathbf{a}_{\chi,v}$.

Lemma 3.17. *Suppose that $c_v(\chi)=0$. Then*

$$\mathbf{a}^*_{\chi,v}(a)=\begin{cases}W_v^0(\mathbf{d}(a)) & \textit{if } v\nmid\mathfrak{n}\textit{ is unramified,}\\ W_v^0(\mathbf{d}(a))+W_v^0(\mathbf{d}(a\varpi))\chi(\varpi_{E_v}) & \textit{if } v\nmid\mathfrak{n}\textit{ is ramified.}\end{cases}$$

If $v\mid\mathfrak{n}$, then v is ramified and

$$\mathbf{a}^*_{\chi,v}(a)=\mu|\cdot|^{\frac{1}{2}}(a)\mathbb{I}_{\varpi^{-1}\mathcal{O}}(a).$$

Proof. It is well-known that if $\pi=\pi(\mu,\nu)$ is a unramified principal series, then

$$W_v^0(\mathbf{d}(a))=\mathbb{I}_{\mathcal{O}}(a)\,|a|^{\frac{1}{2}}\cdot\sum_{i+j=v(a),\,i,j\geq 0}\mu(\varpi^i)\nu(\varpi^j)$$

(*cf.* [Bum97, Thm. 4.6.5]). It follows from the definition of $W_{\chi,v}$ that $W_{\chi,v}=W_v^0$ if $v\nmid\mathfrak{n}$ is unramified and

$$W_{\chi,v}(g)=\frac{1}{2}\cdot W_v^0(g)+\frac{1}{2}\cdot W_v^0(g\mathbf{d}(\varpi))\chi(\varpi_E)\text{ if } v\nmid\mathfrak{n}\text{ is ramified }.$$

If $v|\mathfrak{n}$, then $v\in A(\chi)$. By (R1) v is ramified, and we find that

$$W_{\chi,v}(g)=\frac{1}{2}\cdot W_v^0(g\mathbf{d}(\varpi))+\frac{1}{2}\cdot W_v^0(g\mathbf{w})\omega(\varpi)\chi(\varpi_E).$$

The assertion follows from the formulas of W_v^0 in Proposition 3.9. □

To treat the case v is non-split with $c_v(\chi)>0$, i.e. $v\in B(\chi)$, we need to introduce certain partial Gauss sums. For a non-split place v, write $\pi=\pi(\mu,\nu)$ with unramified μ and $\mu\nu^{-1}(\varpi)\neq|\varpi|^{-1}$ if π is unramified or special. Define a character $\Psi_{\pi,\chi,v}:E^\times\to\mathbf{C}^\times$ by

$$\Psi_{\pi,\chi,v}(t):=\mu(\mathrm{N}(t))\cdot\chi|\cdot|_{\mathcal{K}}^{\frac{1}{2}}(t).\tag{3.29}$$

Recall that the partial Gauss sum $\widetilde{A}_\beta(\varPsi_{\pi,\chi,v})$ in [Hsi12, (4.17)] is defined by

$$\widetilde{A}_\beta(\varPsi_{\pi,\chi,v}) := \lim_{n\to\infty}\int_{\varpi^{-n}\mathcal{O}} \varPsi^{-1}_{\pi,\chi,v}(x+\boldsymbol{\theta})\psi(-d_F^{-1}\beta x)\mathrm{d}x\cdot|\mathcal{D}_F|^{-\frac{1}{2}} \quad (\beta\in F^\times).$$

Lemma 3.18. *Let $v\in B(\chi)$ be a non-split place with $c_v(\chi)>0$. Then we have*

$$\begin{aligned}\frac{e_v}{L(1,\tau_{E/F})}\cdot\mathbf{a}_{\chi,v}(a) =&\widetilde{A}_a(\varPsi_{\pi,\chi,v})\cdot\nu|\cdot|^{\frac{1}{2}}(a)\nu|\cdot|^{\frac{1}{2}}(\varpi^m)\\ &\times\begin{cases}1 & \text{if } v\nmid\mathfrak{n}^-,\\ -1 & \text{if } v\mid\mathfrak{n}^-_s,\\ |\varpi^m|\,\chi(\boldsymbol{\delta}_v)\,|\boldsymbol{\delta}_v|_E^{\frac{1}{2}}\,\frac{\varepsilon(0,\chi^{-1},\psi_E)}{\varepsilon(-1,\omega,\psi)}\,|\mathcal{D}_F|^{-\frac{1}{2}} & \text{if } v\mid\mathfrak{n}^-_r.\end{cases}\end{aligned}$$

Proof. It seems very difficult to deduce the above formula of $\mathbf{a}_{\chi,v}(a)$ by a straightforward computation, so we shall prove the formula by identifying the toric Whittaker function $W_{\chi,v}$ with the image of an explicit element in the induced representation corresponding to π_v via the Whittaker linear functional. Recall the Whittaker linear functional $\Lambda: I(\mu,\nu)\to\mathbf{C}$ ([Bum97, (6.9), p.498]) is defined by

$$\Lambda(f)=\int_F f(\begin{pmatrix}0&-1\\1&x\end{pmatrix})\psi(-x)\mathrm{d}x := \lim_{n\to\infty}\int_{\varpi^{-n}\mathcal{O}} f(\begin{pmatrix}0&-1\\1&x\end{pmatrix})\psi(-x)\mathrm{d}x.$$

Let $\varsigma=\varsigma_v=\begin{pmatrix}d_F&\\&d_F^{-1}\end{pmatrix}$ and $m=m_v(\chi,\pi)$. Define $\mathcal{P}_{\chi,\varsigma}\mathcal{R}_v^m\in\mathrm{End}_{\mathbf{C}}\,I(\mu,\nu)$ by

$$\mathcal{P}_{\chi,\varsigma}\mathcal{R}_v^m f(g)=\mathrm{vol}(E^\times/F^\times,\mathrm{d}t)^{-1}\int_{E^\times/F^\times} f(g\begin{pmatrix}1&\\&\varpi^m\end{pmatrix}\iota_\varsigma(t))\mathrm{d}t.$$

By [Sch02, Lemma 2.2.1], there exists a local new vector section $f^0\in I(\mu,\nu)^{K^0(\varpi)}$ such that

$$W_v^0(g)=\Lambda(\pi(g)f^0).$$

Put $f^0_\chi:=\mathcal{P}_{\chi,\varsigma}\mathcal{R}_v^m f^0$. Then

$$\mathbf{a}_{\chi,v}(a)=W_{\chi,v}(\mathbf{d}(a))=\Lambda(\pi(\mathbf{d}(a))f^0_\chi).$$

We thus have

$$\begin{aligned}\mathbf{a}_{\chi,v}(a)=&\nu|\cdot|^{\frac{1}{2}}(a)\int_F f^0_\chi(\begin{pmatrix}0&-1\\1&x\end{pmatrix})\psi(-ax)\mathrm{d}x\\ =&f^0_\chi(\begin{pmatrix}d_F&\\&d_F^{-1}\end{pmatrix})\,|\mathcal{D}_F|^{-1}\cdot\nu|\cdot|^{\frac{1}{2}}(a)\int_F\varPsi^{-1}_{\pi,\chi,v}(x+\boldsymbol{\theta})\psi(-d_F^{-1}ax)\mathrm{d}x\\ =&f^0_\chi(\varsigma)^*\cdot\nu|\cdot|^{\frac{1}{2}}(a)\widetilde{A}_a(\varPsi_{\pi,\chi,v})\cdot e_v^{-1}\,|\mathcal{D}_E|_E^{-\frac{1}{2}}\,\nu|\cdot|^{-\frac{1}{2}}(\varpi^m),\end{aligned}$$

where $f^0_\chi(\varsigma)^*$ is the normalized value

$$f^0_\chi(\varsigma)^*:=\nu^{-1}|\cdot|^{\frac{1}{2}}(\varpi^m)\mathbf{v}_E\cdot f^0_\chi(\varsigma)\quad(\mathbf{v}_E=e_v\,|\mathcal{D}_E|_E^{\frac{1}{2}}\,|\mathcal{D}_F|^{-\frac{1}{2}}).$$

To evaluate the value $f^0_\chi(\varsigma)^*$, we use the computation in (3.16) and obtain

$$\begin{aligned} f^0_\chi(\varsigma)^* =& \int_{\mathcal{O}} \chi(1+y\boldsymbol{\theta}) f^0(\varsigma\cdot\mathcal{R}_v^{-m}\iota_\varsigma(1+y\boldsymbol{\theta})\mathcal{R}_v^m)\mathrm{d}'y \\ &+ \int_{\varpi\mathcal{O}} \chi(y+\boldsymbol{\theta}) f^0(\varsigma\cdot\mathcal{R}_v^{-m}\iota_\varsigma(y+\boldsymbol{\theta})\mathcal{R}_v^m)\,|\mathcal{D}_E|_E^{-1}\,\mathrm{d}'y \\ =& X_{m+1}\cdot f^0(\varsigma) + \sum_{r=0}^{m}\int_{\varpi^r\mathcal{O}^\times}\chi(1+y\boldsymbol{\theta})\omega(\varpi^{-m}y)\mathrm{d}'y\cdot f^0(\varsigma\mathbf{d}(\varpi^{2(m-r)})) \\ &+ Y_0\cdot\omega(\varpi^{-m})f^0(\varsigma\mathbf{d}(\varpi^{2m+e_v-1})). \end{aligned}$$

To proceed, we need to use explicit formulas for f^0 ([Sch02, Prop. 2.1.2]). Suppose that π is a unramified principal series ($v\nmid\mathfrak{n}^-$) or special representation ($v\mid\mathfrak{n}^-_s$). Then

$$\begin{aligned} f^0_\chi(\varsigma)^* =& X_m\cdot f^0(\varsigma) + \sum_{r=0}^{m-1}(X_r - X_{r+1})\cdot\omega(\varpi^{r-m})\cdot f^0(\varsigma\mathbf{d}(\varpi^{2(m-r)})) \\ &+ Y_0\cdot\omega(\varpi^{-m})f^0(\varsigma\mathbf{d}(\varpi^{2m+e_v-1})). \end{aligned}$$

Let f^{sph} be the unique K^0_v-invariant function in $I(\mu,\nu)$ with $f^{sph}(\varsigma) = L(1,\mu\nu^{-1})\,|\mathcal{D}_{\mathcal{F}}|^{\frac{1}{2}}$. If π is an unramified principal series, then we can take $f^0 = f^{sph}$ ([Bum97, Prop. 4.6.8]), and following the computation of the case $c_v(\chi)>0$ in Proposition 3.8 we find that

$$\begin{aligned} f^0_\chi(\varsigma)^* =& X_m\cdot(f^0(\varsigma)-\omega(\varpi^{-1})f^0(\mathbf{d}(\varpi^2)\varsigma)) = X_m\cdot(1-\mu\nu^{-1}|\cdot|(\varpi))\cdot f^0(\varsigma) \\ =& |\varpi^m|\,|\mathcal{D}_E|_E^{\frac{1}{2}}\,L(1,\tau_{E/F}). \end{aligned}$$

If π is special, then

$$f^0 = f^{sph} - \mu^{-1}|\cdot|^{\frac{1}{2}}(\varpi)^{-1}\pi(\begin{pmatrix}1 & \\ & \varpi\end{pmatrix})f^{sph},$$

and following the computation of the case $c_v(\chi)>0$ in Proposition 3.9 we find that

$$\begin{aligned} f^0_\chi(\varsigma)^* =& X_{m+1}\cdot(f^0(\varsigma)-f^0(\varsigma\cdot\mathbf{w})) \\ =& X_{m+1}\cdot(-|\varpi|^{-1}+|\varpi|)f^{sph}(\varsigma) \\ =& (-1)\cdot|\varpi^m|\,|\mathcal{D}_E|_E^{\frac{1}{2}}\cdot L(1,\tau_{E/F}). \end{aligned}$$

Finally, suppose that π is a ramified principal series with ramified ν ($v\mid\mathfrak{n}^-_r$). Let $B(F)$ be the group of upper triangular matrices in $\mathrm{GL}_2(F)$. Let $f^0\in I(\mu,\nu)$ be the function supported in $B(F)\mathbf{w}N(\mathcal{D}_F^{-1})$ such that

$$f^0(\varsigma\mathbf{w}n) = |\mathcal{D}_F|^{\frac{1}{2}} \text{ for every } n\in N(\mathcal{D}_F^{-1}).$$

Then one checks easily that f^0 does the job. Following the computation in Proposition 3.11, we find that

$$\begin{aligned}&f^0_\chi(\varsigma)^*\\ =&\omega(\varpi^{-m})\left|\varpi^{2m}\right|\cdot\chi(\boldsymbol{\delta}_v)\left|\boldsymbol{\delta}_v\right|_E^{\frac{1}{2}}\frac{\varepsilon(0,\chi^{-1},\psi_E)}{\varepsilon(-1,\omega,\psi)}L(1,\tau_{E/F})\left|\mathcal{D}_E\right|_E^{\frac{1}{2}}\left|\mathcal{D}_F\right|^{-1}\cdot f^0(\varsigma\cdot\mathbf{w})\\ =&\left|\varpi^m\right|\cdot L(1,\tau_{E/F})\cdot\omega^{-1}|\cdot|(\varpi^m)\cdot\chi(\boldsymbol{\delta}_v)\frac{\varepsilon(0,\chi^{-1},\psi_E)}{\varepsilon(-1,\omega,\psi)}\left|\mathcal{D}_E\right|_E^{\frac{1}{2}}\left|\mathcal{D}_F\right|^{-\frac{1}{2}}.\end{aligned}$$

This completes the proof in all cases. $\square$

To investigate the p-integrality of $\mathbf{a}^*_{\chi,v}$, we define the local invariant $\mu_p(\Psi_{\pi,\chi,v})$ by

$$\mu_p(\Psi_{\pi,\chi,v}) := \inf_{x\in\mathcal{K}_v^\times} v_p(\Psi_{\pi,\chi,v}(x)-1). \tag{3.30}$$

By [Hsi12, (4.17)], $\widetilde{A}_\beta(\Psi_{\pi,\chi,v})$ is indeed an algebraic integer. Moreover, it is proved in [Hsi12, Lemma 6.4] that

$$\mu_p(\Psi_{\pi,\chi,v}) > 0 \iff \widetilde{A}_\beta(\Psi_{\pi,\chi,v}) \equiv 0 \,(\mathrm{mod}\ \mathfrak{m}_p) \text{ for all } \beta\in F^\times.$$

Therefore, it follows from Lemma 3.18 that if $v\in B(\chi)$, then $\mathbf{a}^*_{\chi,v}$ takes values in $\overline{\mathbf{Z}}_p$ and

$$\mathbf{a}^*_{\chi,v} \equiv 0 \,(\mathrm{mod}\ \mathfrak{m}_p) \iff \mu_p(\Psi_{\pi,\chi,v}) > 0.$$

We summarize our discussion in the following proposition.

Proposition 3.19. *Let $\mathcal{O}$ be the finite extension of $\mathcal{O}_{L_\pi}$ generated by $\left\{\mathbf{a}^*_{\chi,v}(1)\right\}_{v\in B(\chi)}$ and the values of $\widehat{\chi}$. Then we have*

1. *the normalized local Fourier coefficient $\mathbf{a}^*_{\chi,v}$ takes values in $\mathcal{O}$ for every finite place $v\nmid p$,*
2. *if either $v\notin B(\chi)$ is unramified or $v\in A(\chi)$, then $\mathbf{a}^*_{\chi,v}(1)=1$,*
3. *if $v\nmid\mathfrak{n}$ is ramified with $c_v(\chi)=0$, then $\mathbf{a}^*_{\chi,v}(\varpi^{-1})=1$,*
4. *if $v\in B(\chi)$, then $\mu_p(\Psi_{\pi,\chi,v})=0$ if and only if there exists $\eta_v\in F^\times$ such that*
$$\mathbf{a}^*_{\chi,v}(\eta_v)\not\equiv 0\,(mod\ \mathfrak{m}_p).$$

4. Review of Hilbert modular forms

In this section, we review some standard facts about Hilbert modular Shimura varieties and Hilbert modular forms.

4.1. Let $V=\mathcal{F}e_1\oplus\mathcal{F}e_2$ be a two dimensional $\mathcal{F}$-vector space and $\langle\,,\rangle : V\times V\to\mathcal{F}$ be the $\mathcal{F}$-bilinear alternating pairing defined by $\langle e_1,e_2\rangle=1$. Let $\mathcal{L}=\mathcal{O}_\mathcal{F}e_1\oplus\mathcal{D}_\mathcal{F}^{-1}e_2$ be the standard $\mathcal{O}_\mathcal{F}$-lattice in V, which is self-dual with respect to $\langle\,,\rangle$. For $g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{M}_2(\mathcal{F})$, we define an involution $g\mapsto g':=\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$. We identify vectors in V with row vectors according to the basis

e_1, e_2, so $G(\mathcal{F}) = \mathrm{GL}_2(\mathcal{F})$ has a natural right action on V. If $g \in G(\mathcal{F})$, then $g' = g^{-1} \det g$. Define a left action of G on V by $g * x := x \cdot g'$, $x \in V$.
Hereafter, we let K be an open compact subgroup of $G(\mathbf{A}_f)$ satisfying (3.26) and the following conditions:
(neat)

K is contained in $U(N')$ for some $N' \geq 3$ and $\det(K) \cap \mathcal{O}_{\mathcal{F},+}^{\times} \subset (K \cap \mathcal{O}_{\mathcal{F}}^{\times})^2$.

We also fix a prime-to-p positive integer N such that $U(N) \subset K$.

4.2. Kottwitz models. We recall Kottwitz models of Hilbert modular Shimura varieties following the exposition in [Hid04b].

Definition 4.1 (S-quadruples). Let $\Box$ be a finite set of rational primes not dividing N and let U be an open compact subgroup of K^0 such that $U(N) \subset U$. Let $\mathcal{W}_U = \mathbf{Z}_{(\Box)}[\zeta_N]$ with $\zeta_N = \exp(\frac{2\pi i}{N})$. Define the fibered category $\mathcal{A}_U^{(\Box)}$ over the category $SCH_{/\mathcal{W}_U}$ of schemes over $\mathcal{W}_U$ as follows. Let S be a locally noetherian connected $\mathcal{W}_U$-scheme and let $\overline{s}$ be a geometric point of S. The objects are abelian varieties with real multiplication (AVRM) over S of level U, i.e. a S-*quadruple* $(A, \bar{\lambda}, \iota, \overline{\eta}^{(\Box)})_S$ consisting of the following data:

(1) A is an abelian scheme of dimension d over S.
(2) $\iota : \mathcal{O}_{\mathcal{F}} \hookrightarrow \mathrm{End}_S\, A \otimes_{\mathbf{Z}} \mathbf{Z}_{(\Box)}$.
(3) λ is a prime-to-$\Box$ polarization of A over S and $\bar{\lambda}$ is the $\mathcal{O}_{\mathcal{F},(\Box),+}$-orbit of λ. Namely

$$\bar{\lambda} = \mathcal{O}_{\mathcal{F},(\Box),+}\lambda := \left\{\lambda' \in \mathrm{Hom}(A, A^t) \otimes_{\mathbf{Z}} \mathbf{Z}_{(\Box)} \mid \lambda' = \lambda \circ a,\, a \in \mathcal{O}_{\mathcal{F},(\Box),+}\right\}.$$

(4) $\overline{\eta}^{(\Box)} = \eta^{(\Box)} U^{(\Box)}$ is a $\pi_1(S, \overline{s})$-invariant $U^{(\Box)}$-orbit of the isomorphisms of $\mathcal{O}_{\mathcal{F}}$-modules $\eta^{(\Box)} : \mathcal{L} \otimes_{\mathbf{Z}} \mathbf{A}_f^{(\Box)} \xrightarrow{\sim} V^{(\Box)}(A_{\overline{s}}) := H_1(A_{\overline{s}}, \widehat{\mathbf{Z}}^{(\Box)}) \otimes_{\mathbf{Z}} \mathbf{A}_f^{(\Box)}$. Here we define $\eta^{(\Box)} g$ for $g \in G(\mathbf{A}_f^{(\Box)})$ by $\eta^{(\Box)} g(x) = \eta^{(\Box)}(g * x)$.

Furthermore, $(A, \bar{\lambda}, \iota, \overline{\eta}^{(\Box)})_S$ satisfies the following conditions:

- Let ^{t} denote the Rosati involution induced by λ on $\mathrm{End}_S\, A \otimes \mathbf{Z}_{(\Box)}$. Then $\iota(b)^t = \iota(b)$, $\forall\, b \in \mathcal{O}_{\mathcal{F}}$.
- Let e^{λ} be the Weil pairing induced by λ. Lifting the isomorphism $\mathbf{Z}/N\mathbf{Z} \simeq \mathbf{Z}/N\mathbf{Z}(1)$ induced by ζ_N to an isomorphism $\zeta : \widehat{\mathbf{Z}} \simeq \widehat{\mathbf{Z}}(1)$, we can regard e^{λ} as an $\mathcal{F}$-alternating form $e^{\lambda} : V^{(\Box)}(A) \times V^{(\Box)}(A) \to \mathcal{D}_{\mathcal{F}}^{-1} \otimes_{\mathbf{Z}} \mathbf{A}_f^{(\Box)}$. Let e^{η} denote the $\mathcal{F}$-alternating form on $V^{(\Box)}(A)$ induced by $e^{\eta}(x, x') = \langle x\eta, x'\eta \rangle$. Then
$$e^{\lambda} = u \cdot e^{\eta} \text{ for some } u \in \mathbf{A}_f^{(\Box)}.$$
- As $\mathcal{O}_{\mathcal{F}} \otimes_{\mathbf{Z}} \mathcal{O}_S$-modules, we have an isomorphism $\mathrm{Lie}\, A \simeq \mathcal{O}_{\mathcal{F}} \otimes_{\mathbf{Z}} \mathcal{O}_S$ locally under the Zariski topology of S.

For two S-quadruples $\underline{A} = (A, \bar{\lambda}, \iota, \overline{\eta}^{(\Box)})_S$ and $\underline{A'} = (A', \overline{\lambda'}, \iota', \overline{\eta'}^{(\Box)})_S$, we define morphisms by

$$\mathrm{Hom}_{\mathcal{A}_K^{(\Box)}}(\underline{A}, \underline{A'}) = \left\{\phi \in \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(A, A') \mid \phi^* \overline{\lambda'} = \bar{\lambda},\, \phi \circ \overline{\eta'}^{(\Box)} = \overline{\eta}^{(\Box)}\right\}.$$

We say $\underline{A} \sim \underline{A}'$ (resp. $\underline{A} \simeq \underline{A}'$) if there exists a prime-to-$\square$ isogeny (resp. isomorphism) in $\mathrm{Hom}_{\mathcal{A}_K^{(\square)}}(\underline{A}, \underline{A}')$.

We consider the cases when $\square = \emptyset$ and $\{p\}$. When $\square = \emptyset$ is the empty set and U is an open compact subgroup in $G(\mathbf{A}_f^{(\square)}) = G(\mathbf{A}_f)$, we define the functor $\mathcal{E}_U : SCH_{/\mathcal{W}_U} \to SETS$ by

$$\mathcal{E}_U(S) = \left\{(A, \bar{\lambda}, \iota, \overline{\eta})_S \in \mathcal{A}_K(S)\right\} / \sim .$$

By the theory of Shimura-Deligne, $\mathcal{E}_U$ is represented by Sh_U which is a quasi-projective scheme over $\mathcal{W}_U$. We define the functor $\mathfrak{E}_U : SCH_{/\mathcal{W}_U} \to SETS$ by

$$\mathfrak{E}_U(S) = \left\{(A, \bar{\lambda}, \iota, \overline{\eta}) \in \mathcal{A}_U^{(\square)}(S) \mid \eta^{(\square)}(\mathcal{L} \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}) = H_1(A_{\overline{s}}, \widehat{\mathbf{Z}})\right\} / \simeq .$$

By the discussion in [Hid04b, p.136], we have $\mathfrak{E}_K \overset{\sim}{\to} \mathcal{E}_K$ under the hypothesis (neat).
When $\square = \{p\}$ and $U = K$, we let $\mathcal{W} = \mathcal{W}_K = \mathbf{Z}_{(p)}[\zeta_N]$ and define functor $\mathcal{E}_K^{(p)} : SCH_{/\mathcal{W}} \to SETS$ by

$$\mathcal{E}_K^{(p)}(S) = \left\{(A, \bar{\lambda}, \iota, \overline{\eta}^{(p)})_S \in \mathcal{A}_{K^{(p)}}^{(p)}(S)\right\} / \sim .$$

In [Kot92], Kottwitz shows $\mathcal{E}_K^{(p)}$ is representable by a quasi-projective scheme $Sh_K^{(p)}$ over $\mathcal{W}$ if K is neat. Similarly we define the functor $\mathfrak{E}_K^{(p)} : SCH_{/\mathcal{W}} \to SETS$ by

$$\mathfrak{E}_K^{(p)}(S) = \left\{(A, \bar{\lambda}, \iota, \overline{\eta}^{(p)}) \in \mathcal{A}_K^{(p)}(S) \mid \eta^{(p)}(\mathcal{L} \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}^{(p)}) = H_1(A_{\overline{s}}, \widehat{\mathbf{Z}}^{(p)})\right\} / \simeq .$$

It is shown in [Hid04b, §4.2.1] that $\mathfrak{E}_K^{(p)} \overset{\sim}{\to} \mathcal{E}_K^{(p)}$.
Let $\mathfrak{c}$ be a prime-to-pN ideal of $\mathcal{O}_{\mathcal{F}}$ and let $\mathbf{c} \in (\mathbf{A}_f^{(pN)})^\times$ such that $\mathfrak{c} = \mathfrak{il}_{\mathcal{F}}(\mathbf{c})$. We say $(A, \lambda, \iota, \overline{\eta}^{(p)})$ is $\mathfrak{c}$-polarized if $\lambda \in \bar{\lambda}$ such that $e^\lambda = ue^\eta$, $u \in \mathbf{c}\det(K)$. The isomorphism class $[(A, \lambda, \iota, \overline{\eta}^{(p)})]$ is independent of a choice of λ in $\overline{\lambda}$ under the assumption (neat) (*cf.* [Hid04b, p.136]). We consider the functor

$$\mathfrak{E}_{\mathfrak{c},K}^{(p)}(S) = \left\{\mathfrak{c}\text{-polarized } S\text{-quadruple } [(A, \lambda, \iota, \overline{\eta}^{(p)})_S] \in \mathfrak{E}_K^{(p)}(S)\right\}.$$

Then $\mathfrak{E}_{\mathfrak{c},K}^{(p)}$ is represented by a geometrically irreducible scheme $Sh_K^{(p)}(\mathfrak{c})_{/\mathcal{W}}$, and we have

$$Sh_K^{(p)}{}_{/\mathcal{W}} = \bigsqcup_{[\mathfrak{c}] \in \mathrm{Cl}_{\mathcal{F}}^+(K)} Sh_K^{(p)}(\mathfrak{c})_{/\mathcal{W}}, \tag{4.1}$$

where $\mathrm{Cl}_{\mathcal{F}}^+(K)$ is the narrow ray class group of $\mathcal{F}$ with level $\det(K)$.

4.3. Igusa schemes. Let n be a positive integer. Define the functor $\mathcal{I}^{(p)}_{K,n} : SCH_{/\mathcal{W}} \to SETS$ by

$$S \mapsto \mathcal{I}^{(p)}_{K,n}(S) = \left\{(A, \bar{\lambda}, \iota, \eta^{(p)}, j)_S\right\} / \sim,$$

where $(A, \bar{\lambda}, \iota, \eta^{(p)})_S$ is a S-quadruple, j is a level p^n-structure, i.e. an $\mathcal{O}_{\mathcal{F}}$-group scheme morphism:

$$j : \mathcal{D}^{-1}_{\mathcal{F}} \otimes_{\mathbf{Z}} \boldsymbol{\mu}_{p^n} \hookrightarrow A[p^n],$$

and $\sim$ means modulo prime-to-p isogeny. It is known that $\mathcal{I}^{(p)}_{K,n}$ is relatively representable over $\mathcal{E}^{(p)}_K$ (*cf.* [HLS06, Lemma (2.1.6.4)]) and thus is represented by a scheme $I_{K,n}$.
Now we consider S-quintuples $(A, \lambda, \iota, \eta^{(p)}, j)_S$ such that $[(A, \lambda, \iota, \eta^{(p)})] \in \mathfrak{E}^{(p)}_{\mathfrak{c},K}(S)$. Define the functor $\mathcal{I}^{(p)}_{K,n}(\mathfrak{c}) : SCH_{/\mathcal{W}} \to SETS$ by

$$S \mapsto \mathcal{I}^{(p)}_{K,n}(\mathfrak{c})(S) = \left\{(A, \lambda, \iota, \eta^{(p)}, j)_S \text{ as above}\right\} / \simeq .$$

Then $\mathcal{I}^{(p)}_{K,n}(\mathfrak{c})$ is represented by a scheme $I_{K,n}(\mathfrak{c})$ over $Sh^{(p)}_K(\mathfrak{c})$, and $I_{K,n}(\mathfrak{c})$ can be identified with a geometrically irreducible subscheme of $I_{K,n}$ ([DR80, Thm. (4.5)]). For $n \geq n' > 0$, the natural morphism $\pi_{n,n'} : I_{K,n}(\mathfrak{c}) \to I_{K,n'}(\mathfrak{c})$ induced by the inclusion $\mathcal{D}^{-1}_{\mathcal{F}} \otimes \boldsymbol{\mu}_{p^{n'}} \hookrightarrow \mathcal{D}^{-1}_{\mathcal{F}} \otimes \boldsymbol{\mu}_{p^n}$ is finite étale. The forgetful morphism $\pi : I_{K,n}(\mathfrak{c}) \to Sh^{(p)}_K(\mathfrak{c})$ defined by $\pi : (\underline{A}, j) \mapsto \underline{A}$ is étale for all $n > 0$. Hence $I_{K,n}(\mathfrak{c})$ is smooth over $\operatorname{Spec} \mathcal{W}$. We write $I_K(\mathfrak{c})$ for $\varprojlim_n I_{K,n}(\mathfrak{c})$.

4.4. Complex uniformization. We describe the complex points $Sh_U(\mathbf{C})$ for $U \subset G(\mathbf{A}_f)$. Put

$$X^+ = \left\{\tau = (\tau_\sigma)_{\sigma \in \Sigma} \in \mathbf{C}^\Sigma \mid \operatorname{Im} \tau_\sigma > 0 \text{ for all } \sigma \in \Sigma\right\}.$$

The action of $g = (g_\sigma)_{\sigma \in \Sigma} \in G(\mathcal{F} \otimes_{\mathbf{Q}} \mathbf{R})$ with $g_\sigma = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$ and $\det g_\sigma > 0$ on X^+ is given by $\tau = (\tau_\sigma) \mapsto g\tau = \left(\frac{a_\sigma \tau_\sigma + b_\sigma}{c_\sigma \tau_\sigma + d_\sigma}\right)$. Let $\mathcal{F}_+$ be the set of totally positive elements in $\mathcal{F}$ and let $G(\mathcal{F})^+ = \{g \in G(\mathcal{F}) \mid \det g \in \mathcal{F}_+\}$. Define the complex Hilbert modular Shimura variety by

$$M(X^+, U) := G(\mathcal{F})^+ \backslash X^+ \times G(\mathbf{A}_f)/U.$$

It is well known that $M(X^+, K) \xrightarrow{\sim} Sh_U(\mathbf{C})$ by the theory of abelian varieties over $\mathbf{C}$ (*cf.* [Hid04b, § 4.2]). Now we define this isomorphism explicitly.
For $\tau = (\tau_\sigma)_{\sigma \in \Sigma} \in X^+$, we let p_τ be the isomorphism $V \otimes_{\mathbf{Q}} \mathbf{R} \xrightarrow{\sim} \mathbf{C}^\Sigma$ defined by $\mathrm{p}_\tau(ae_1 + be_2) = a\tau + b$ with $a, b \in \mathcal{F} \otimes_{\mathbf{Q}} \mathbf{R} = \mathbf{R}^\Sigma$. We can associate a AVRM to $(\tau, g) \in X^+ \times G(\mathbf{A}_f)$ as follows.

- The complex abelian variety $\mathcal{A}_g(\tau) = \mathbf{C}^\Sigma / \mathrm{p}_\tau(g * \mathcal{L})$.
- The $\mathcal{F}_+$-orbit of polarization $\overline{\langle\, , \,\rangle_{can}}$ on $\mathcal{A}_g(\tau)$ is given by the Riemann form $\langle\, , \,\rangle_{can} := \langle\, , \,\rangle \circ \mathrm{p}_\tau^{-1}$.

- The $\iota_{\mathbf{C}} : O \hookrightarrow \operatorname{End} \mathcal{A}_g(\tau) \otimes_{\mathbf{Z}} \mathbf{Q}$ is induced from the pull back of the natural $\mathcal{F}$-action on V via p_τ.
- The level structure $\eta_g : \mathcal{L} \otimes_{\mathbf{Z}} \mathbf{A}_f \xrightarrow{\sim} (g * \mathcal{L}) \otimes_{\mathbf{Z}} \mathbf{A}_f = H_1(\mathcal{A}_g(\tau), \mathbf{A}_f)$ is defined by $\eta_g(v) = g * v$.

Let $\underline{\mathcal{A}_g(\tau)}$ denote the $\mathbf{C}$-quadruple $(\mathcal{A}_g(\tau), \overline{\langle\, ,\, \rangle}_{can}, \iota_{\mathbf{C}}, K\eta_g)$. Then the map $[(\tau, g)] \mapsto [\underline{\mathcal{A}_g(\tau)}]$ gives rise to an isomorphism $M(X^+, U) \xrightarrow{\sim} Sh_U(\mathbf{C})$.
For a positive integer n, the exponential map gives the isomorphism $\exp(2\pi i -) : p^{-n}\mathbf{Z}/\mathbf{Z} \simeq \boldsymbol{\mu}_{p^n}$ and thus induces a level p^n-structure $j(g_p)$:

$$j(g_p)\colon \mathcal{D}_{\mathcal{F}}^{-1} \otimes_{\mathbf{Z}} \boldsymbol{\mu}_{p^n} \xrightarrow{\sim} \mathcal{D}_{\mathcal{F}}^{-1} e_2 \otimes_{\mathbf{Z}} p^{-n}\mathbf{Z}/\mathbf{Z} \hookrightarrow \mathcal{L} \otimes_{\mathbf{Z}} p^{-n}\mathbf{Z}/\mathbf{Z} \overset{g*}{\xrightarrow{\sim}} \mathcal{A}_g(\tau)[p^n].$$

Put

$$K_1^n := \left\{ g \in K \mid g_p \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{p^n} \right\}.$$

We have a non-canonical isomorphism:

$$\begin{aligned} M(X^+, K_1^n) &\xrightarrow{\sim} I_{K,n}(\mathbf{C}) \\ [(\tau, g)] &\mapsto [(\mathcal{A}_g(\tau), \overline{\langle\, ,\, \rangle}_{can}, \iota_{\mathbf{C}}, \overline{\eta}_g^{(p)}, j(g_p))]. \end{aligned}$$

Let $\underline{z} = \{z_\sigma\}_{\sigma \in \Sigma}$ be the standard complex coordinates of $\mathbf{C}^\Sigma$ and $d\underline{z} = \{dz_\sigma\}_{\sigma \in \Sigma}$. Then $\mathcal{O}_{\mathcal{F}}$-action on $d\underline{z}$ is given by $\iota_{\mathbf{C}}(\alpha)^* dz_\sigma = \sigma(\alpha) dz_\sigma$, $\sigma \in \Sigma \simeq \operatorname{Hom}(\mathcal{F}, \mathbf{C})$. Let $z = z_{id}$ be the coordinate corresponding to $\iota_\infty : \mathcal{F} \hookrightarrow \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$. Then

$$(\mathcal{O}_{\mathcal{F}} \otimes_{\mathbf{Z}} \mathbf{C}) dz = H^0(\mathcal{A}_g(\tau), \Omega_{\mathcal{A}_g(\tau)/\mathbf{C}}). \tag{4.2}$$

4.5. Hilbert modular forms. Let $k = \sum_\sigma k_\sigma \sigma \in \mathbf{Z}_{\geq 1}[\Sigma]$ such that

$$k_{\sigma_1} \equiv k_{\sigma_2} \equiv \cdots \equiv k_{\sigma_d} \pmod 2 \text{ for all } \sigma_1, \ldots, \sigma_d \in \Sigma.$$

For $\tau = (\tau_\sigma)_{\sigma \in \Sigma} \in X^+$ and $g = (\begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix})_{\sigma \in \Sigma} \in G(\mathcal{F} \otimes_{\mathbf{Q}} \mathbf{R})$, we put

$$\underline{J}(g, \tau)^k = \prod_{\sigma \in \Sigma} (c_\sigma \tau_\sigma + d_\sigma)^{k_\sigma}.$$

Definition 4.2. Let $k_{mx} = \max_{\sigma \in \Sigma} k_\sigma$. Denote by $\mathbf{M}_k(K_1^n, \mathbf{C})$ the space of holomorphic Hilbert modular forms of weight k and level K_1^n. Each $\mathbf{f} \in \mathbf{M}_k(K_1^n, \mathbf{C})$ is a $\mathbf{C}$-valued function $\mathbf{f} : X^+ \times G(\mathbf{A}_f) \to \mathbf{C}$ such that the function $\mathbf{f}(-, g_f) : X^+ \to \mathbf{C}$ is holomorphic for each $g_f \in G(\mathbf{A}_f)$, and for $u \in K_1^n$ and $\alpha \in G(\mathcal{F})^+$,

$$\mathbf{f}(\alpha(\tau, g_f)u) = (\det \alpha)^{-\frac{k_{mx}\Sigma + k}{2}} \underline{J}(\alpha, \tau)^k \cdot \mathbf{f}(\tau, g_f).$$

Here $\det \alpha$ is considered to be the element $(\sigma(\det \alpha))_{\sigma \in \Sigma}$ in $(\mathbf{C}^\times)^\Sigma$.

For every $\mathbf{f} \in \mathbf{M}_k(K_1^n, \mathbf{C})$,we have the Fourier expansion

$$\mathbf{f}(\tau, g_f) = \sum_{\beta \in \mathcal{F}_+ \cup \{0\}} W_\beta(\mathbf{f}, g_f) e^{2\pi i \operatorname{Tr}_{\mathcal{F}/\mathbf{Q}}(\beta\tau)}.$$

For a semi-group L in $\mathcal{F}$, let $L_+ = \mathcal{F}_+ \cap L$ and $L_{\geq 0} = L_+ \cup \{0\}$. If B is a ring, we denote by $B[\![L]\!]$ the set of all formal series

$$\sum_{\beta \in L} a_\beta q^\beta,\ a_\beta \in B.$$

Let $a, b \in (\mathbf{A}_f^{(pN)})^\times$ and let $\mathfrak{a} = \mathfrak{il}_{\mathcal{F}}(a)$ and $\mathfrak{b} = \mathfrak{il}_{\mathcal{F}}(b)$. The q-expansion of $\mathbf{f}$ at the cusp $(\mathfrak{a}, \mathfrak{b})$ is given by

$$\mathbf{f}|_{(\mathfrak{a},\mathfrak{b})}(q) = \sum_{\beta \in (N^{-1}\mathfrak{a}\mathfrak{b})_{\geq 0}} W_\beta(\mathbf{f}, \begin{pmatrix} a^{-1} & 0 \\ 0 & b \end{pmatrix}) q^\beta \in \mathbf{C}[\![(N^{-1}\mathfrak{a}\mathfrak{b})_{\geq 0}]\!]. \tag{4.3}$$

If B is a $\mathcal{W}$-algebra in $\mathbf{C}$, let $\mathbf{M}_k(\mathfrak{c}, K_1^n, B)$ be the space consisting of functions $\mathbf{f} \in \mathbf{M}_k(K_1^n, \mathbf{C})$ such that

$$\mathbf{f}|_{(\mathfrak{a},\mathfrak{b})}(q) \in B[\![(N^{-1}\mathfrak{a}\mathfrak{b})_{\geq 0}]\!] \text{ for all } (\mathfrak{a}, \mathfrak{b}) \text{ such that } \mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}.$$

4.5.1. *Tate objects.* Let $\mathscr{S}$ be a set of d linearly $\mathbf{Q}$-independent elements in $\mathrm{Hom}(\mathcal{F}, \mathbf{Q})$ such that $l(\mathcal{F}_+) > 0$ for $l \in \mathscr{S}$. If L is a lattice in $\mathcal{F}$ and n a positive integer, let $L_{\mathscr{S},n} = \{x \in L \mid l(x) > -n \text{ for all } l \in \mathscr{S}\}$ and put $B((L; \mathscr{S})) = \lim\limits_{n\to\infty} B[\![L_{\mathscr{S},n}]\!]$. To a pair $(\mathfrak{a}, \mathfrak{b})$ of two prime-to-pN fractional ideals, we can attach the Tate AVRM $Tate_{\mathfrak{a},\mathfrak{b}}(q) = \mathfrak{a}^* \otimes_{\mathbf{Z}} \mathbb{G}_m / q^{\mathfrak{b}}$ over $\mathbf{Z}((\mathfrak{a}\mathfrak{b}; \mathscr{S}))$ with O-action ι_{can} acting on $\mathfrak{a}^*$, where $\mathfrak{a}^* := \mathfrak{a}^{-1}\mathcal{D}_{\mathcal{F}}^{-1}$. As described in [Kat78], $Tate_{\mathfrak{a},\mathfrak{b}}(q)$ has a canonical $\mathfrak{a}\mathfrak{b}^{-1}$-polarization λ_{can} and also carries a canonical $\mathcal{O}_{\mathcal{F}} \otimes \mathbf{Z}((\mathfrak{a}\mathfrak{b}; \mathscr{S}))$-generator $\boldsymbol{\omega}_{can}$ of $\Omega_{Tate_{\mathfrak{a},\mathfrak{b}}}$ induced by the isomorphism $\mathrm{Lie}(Tate_{\mathfrak{a},\mathfrak{b}}(q)_{/\mathbf{Z}((\mathfrak{a}\mathfrak{b};\mathscr{S}))}) = \mathfrak{a}^* \otimes_{\mathbf{Z}} \mathrm{Lie}(\mathbb{G}_m) \simeq \mathfrak{a}^* \otimes \mathbf{Z}((\mathfrak{a}\mathfrak{b}; \mathscr{S}))$. Since $\mathfrak{a}$ is prime to p, the natural inclusion $\mathfrak{a}^* \otimes_{\mathbf{Z}} \boldsymbol{\mu}_{p^n} \hookrightarrow \mathfrak{a}^* \otimes_{\mathbf{Z}} \mathbb{G}_m$ induces a canonical level p^n-structure $\eta_{p,can} \colon \mathcal{D}_{\mathcal{F}}^{-1} \otimes_{\mathbf{Z}} \boldsymbol{\mu}_{p^n} = \mathfrak{a}^* \otimes_{\mathbf{Z}} \boldsymbol{\mu}_{p^n} \hookrightarrow Tate_{\mathfrak{a},\mathfrak{b}}(q)$. Let $\mathcal{L}_{\mathfrak{a},\mathfrak{b}} = \mathcal{L} \cdot \begin{pmatrix} \mathfrak{b} & \\ & \mathfrak{a}^{-1} \end{pmatrix} = \mathfrak{b}e_1 \oplus \mathfrak{a}^* e_2$. Then we have a level N-structure $\eta_{can}^{(p)} : N^{-1}\mathcal{L}_{\mathfrak{a},\mathfrak{b}}/\mathcal{L}_{\mathfrak{a},\mathfrak{b}} \xrightarrow{\sim} Tate_{\mathfrak{a},\mathfrak{b}}(q)[N]$ over $\mathbf{Z}[\zeta_N]((N^{-1}\mathfrak{a}\mathfrak{b}; \mathscr{S}))$ induced by the fixed primitive N-th root of unity ζ_N. We write $\underline{Tate}_{\mathfrak{a},\mathfrak{b}}$ for the Tate $\mathbf{Z}((\mathfrak{a}\mathfrak{b}; \mathscr{S}))$-quadruple $(Tate_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{can}, \iota_{can}, \overline{\eta}_{can}^{(p)}, \eta_{p,can})$ at $(\mathfrak{a}, \mathfrak{b})$.

4.5.2. *Geometric modular forms.* We collect here definitions and basic facts of geometric modular forms. The whole theory can be found in [Kat78] and [Hid04b]. Let T be the algebraic torus over $\mathcal{W}$ defined by $T(R) = (\mathcal{O}_{\mathcal{F}} \otimes_{\mathbf{Z}} R)^\times$ for every $\mathcal{W}$-algebra R. Let $k \in \mathrm{Hom}(T, \mathbb{G}_{m/\mathcal{W}})$. Let B be a $\mathcal{W}$-algebra. For a B-algebra C, we consider a triple $(\underline{A}, j, \boldsymbol{\omega})$ over C, consisting of $[(\underline{A}, j)] = [(A, \lambda, \iota, \eta^{(p)}, j)] \in I_{K,n}(\mathfrak{c})(C)$ (resp. $[(\underline{A}, j)] = [(A, \bar{\lambda}, \iota, \eta^{(p)}, j)] \in I_{K,n}(C)$) AVRM with level structures and an 1-form $\boldsymbol{\omega}$ generating $H^0(A, \Omega_{A/C})$ over $\mathcal{O}_{\mathcal{F}} \otimes_{\mathbf{Z}} C$. A *geometric modular form* f of weight k on $I_{K,n}(\mathfrak{c})$ (resp. $I_{K,n}$) over B is a rule of assigning to every triple $(\underline{A}, j, \boldsymbol{\omega})$ over C a value $f(\underline{A}, j, \boldsymbol{\omega}) \in C$ satisfying the following axioms.

(G1) $f(\underline{A}, j, \boldsymbol{\omega}) = f(\underline{A}', j', \boldsymbol{\omega}') \in C$ if $(\underline{A}, j, \boldsymbol{\omega}) \simeq (\underline{A}', j', \boldsymbol{\omega}')$ over C,

(G2) For a B-algebra homomorphism $\varphi : C \to C'$, we have
$$f((\underline{A}, j, \boldsymbol{\omega}) \otimes_C C') = \varphi(f(\underline{A}, j, \boldsymbol{\omega})),$$
(G3) $f((\underline{A}, j, a\boldsymbol{\omega}) = k(a^{-1})f(\underline{A}, j, \boldsymbol{\omega})$ for all $a \in T(C) = (\mathcal{O}_{\mathcal{F}} \otimes_{\mathbf{Z}} C)^{\times}$,
(G4) $f(\underline{Tate}_{\mathfrak{a},\mathfrak{b}}, \boldsymbol{\omega}_{can}) \in B[\![(N^{-1}\mathfrak{a}\mathfrak{b})_{\geq 0}]\!]$ at all cusps $(\mathfrak{a}, \mathfrak{b})$ in $I_{K,n}(\mathfrak{c})$ (resp. $I_{K,n}$).

For each $k \in \mathbf{Z}[\Sigma]$, we regard $k \in \mathrm{Hom}(T, \mathbb{G}_{m/\mathcal{W}})$ as the character $x \mapsto x^k$, $x \in (\mathcal{O}_{\mathcal{F}} \otimes_{\mathbf{Z}} \mathcal{W})^{\times}$. We denote by $\mathcal{M}_k(\mathfrak{c}, K_1^n, B)$ (resp. $\mathcal{M}_k(K_1^n, B)$) the space of geometric modular forms over B of weight k on $I_{K,n}(\mathfrak{c})$ (resp. $I_{K,n}$). For $f \in \mathcal{M}_k(K_1^n, B)$, we write $f|_{\mathfrak{c}} \in \mathcal{M}_k(\mathfrak{c}, K_1^n, B)$ for the restriction $f|_{I_{K,n}(\mathfrak{c})}$.
For each $f \in \mathcal{M}_k(K_1^n, \mathbf{C})$, we regard f as a holomorphic Hilbert modular form of weight k and level K_1^n by
$$f(\tau, g_f) = f(\mathcal{A}_g(\tau), \overline{\langle\, , \rangle}_{can}, \iota_{\mathbf{C}}, \overline{\eta}_g, 2\pi i dz),$$
where dz is the differential form in (4.2). By GAGA this gives rise to an isomorphism $\mathcal{M}_k(K_1^n, \mathbf{C}) \stackrel{\sim}{\to} \mathbf{M}_k(K_1^n, \mathbf{C})$ and $\mathcal{M}_k(\mathfrak{c}, K_1^n, \mathbf{C}) \stackrel{\sim}{\to} \mathbf{M}_k(\mathfrak{c}, K_1^n, \mathbf{C})$. Moreover, as discussed in [Kat78, §1.7], we have the following important identity which bridges holomorphic modular forms and geometric modular forms
$$\mathbf{f}|_{(\mathfrak{a},\mathfrak{b})}(q) = \mathbf{f}(\underline{Tate}_{(\mathfrak{a},\mathfrak{b})}, \boldsymbol{\omega}_{can}) \in \mathbf{C}[\![(N^{-1}\mathfrak{a}\mathfrak{b})_{\geq 0}]\!].$$
By the q-expansion principle, if B is $\mathcal{W}$-algebra in $\mathbf{C}$ and $\mathbf{f} \in \mathbf{M}_k(\mathfrak{c}, K_1^n, B) = \mathcal{M}_k(\mathfrak{c}, K_1^n, \mathbf{C})$, then $\mathbf{f}|_{\mathfrak{c}} \in \mathcal{M}_k(\mathfrak{c}, K_1^n, B)$.

4.5.3. *p-adic modular forms.* Let B be a p-adic $\mathcal{W}$-algebra in $\mathbf{C}_p$. Let $V(\mathfrak{c}, K, B)$ be the space of Katz p-adic modular forms over B defined by
$$V(\mathfrak{c}, K, B) := \varprojlim_m \varinjlim_n H^0(I_{K,n}(\mathfrak{c})_{/B/p^mB}, \mathcal{O}_{I_{K,n}}).$$
In other words, Katz p-adic modular forms consist of formal functions on the Igusa tower.
Let C be a B/p^mB-algebra. For each C-point $[(\underline{A}, j)] = [(A, \lambda, \iota, \eta^{(p)}, j] \in I_K(\mathfrak{c})(C) = \varprojlim_n I_{K,n}(\mathfrak{c})(C)$, the p^∞-level structure j induces an isomorphism $j_* : \mathcal{D}_{\mathcal{F}}^{-1} \otimes_{\mathbf{Z}} C \simeq \mathrm{Lie}\, A$ which in turn gives rise to a generator $\boldsymbol{\omega}(j)$ of $H^0(A, \Omega_A)$ as a $\mathcal{O}_{\mathcal{F}} \otimes_{\mathbf{Z}} C$-module. Then we have a natural injection
$$\begin{aligned} \mathcal{M}_k(\mathfrak{c}, K_1^n, B) &\hookrightarrow V(\mathfrak{c}, K, B) \\ f &\mapsto \widehat{f}(\underline{A}, j) := f(\underline{A}, j, \boldsymbol{\omega}(j)) \end{aligned} \tag{4.4}$$
which preserves the q-expansions in the sense that $\widehat{f}|_{(\mathfrak{a},\mathfrak{b})}(q) := \widehat{f}(\underline{Tate}_{\mathfrak{a},\mathfrak{b}}) = f|_{(\mathfrak{a},\mathfrak{b})}(q)$. We call $\widehat{f}$ the p-adic avatar of f.

4.6. CM points. Recall that we have fixed $\vartheta \in \mathcal{K}$ in §3.4 satisfying (d1-3) and the associated embedding $\iota : \mathcal{K} \hookrightarrow M_2(\mathcal{F})$ in (2.1). The map $\mathrm{p}_\vartheta : V \otimes_{\mathbf{Q}} \mathbf{R} \simeq \mathbf{C}^{\Sigma}$, $ae_1 + be_2 \mapsto a\vartheta + b$ yields an isomorphism $\mathrm{p}_\vartheta : V \simeq \mathcal{K}$ satisfying
$$\mathrm{p}_\vartheta(x)\alpha = \mathrm{p}_\vartheta(x\iota(\alpha)) \text{ for } x \in V, \alpha \in \mathcal{K}.$$

Let $\varsigma = \prod_v \varsigma_v \in G(\mathbf{A})$, where $\varsigma_v \in G_v$ for each place v is defined in (3.3). Let $\varsigma_f \in G(\mathbf{A}_f)$ be the finite part of ς. According to our choices of ς_v, we have

$$\varsigma_f * (\mathcal{L} \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}) = (\mathcal{L} \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}) \cdot \varsigma_f' = \mathrm{p}_\vartheta^{-1}(\mathcal{O}_\mathcal{K} \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}).$$

Define $x : \mathbf{A}_\mathcal{K}^\times \to X^+ \times G(\mathbf{A}_f)$ by

$$a = (a_\infty, a_f) \mapsto x(a) := (\vartheta_\Sigma, \iota(a_f)\varsigma_f).$$

Let $a \in (\mathbf{A}_{\mathcal{K},f}^{(pN)})^\times$ and let

$$(\underline{A}(a), j(a))_{/\mathbf{C}} = (\mathcal{A}_{\iota(a)\varsigma_f}(\vartheta_\Sigma), \overline{\langle\, , \rangle}_{can}, \iota_{can}, \eta^{(p)}(a), j(a))$$

be the $\mathbf{C}$-quintuple associated to $x(a)$ as in §4.4. The alternating pairing $\langle\, , \rangle : \mathcal{K} \times \mathcal{K} :\to \mathcal{F}$ defined by $\langle x, y \rangle = (\overline{x}y - x\overline{y})/(\vartheta - \overline{\vartheta})$ induces an isomorphism $\mathcal{O}_\mathcal{K} \wedge_{\mathcal{O}_\mathcal{F}} \mathcal{O}_\mathcal{K} = \mathfrak{c}(\mathcal{O}_\mathcal{K})^{-1}\mathcal{D}_\mathcal{F}^{-1}$ for the fractional ideal $\mathfrak{c}(\mathcal{O}_\mathcal{K}) = \mathcal{D}_\mathcal{F}^{-1}((\vartheta - \overline{\vartheta})\mathcal{D}_{\mathcal{K}/\mathcal{F}}^{-1})$. The hypothesis (d2) on ϑ implies that

$$\mathfrak{c}(\mathcal{O}_\mathcal{K}) \text{ is prime to } p\mathfrak{c}_\chi \mathfrak{n} D_{\mathcal{K}/\mathcal{F}}.$$

Note that $\mathfrak{c}(\mathcal{O}_\mathcal{K})$ descends to a fractional ideal of $\mathcal{O}_\mathcal{F}$ and that $\mathfrak{c}(\mathcal{O}_\mathcal{K})$ is the polarization of $x(1) = (\underline{A}(1), j(1))$. In addition, $x(a) = (\underline{A}(a), j(a))_{/\mathbf{C}}$ is an abelian variety with CM by $\mathcal{O}_\mathcal{K}$ with the polarization ideal of $x(a)$ given by

$$\mathfrak{c}(a) := \mathfrak{c}(\mathcal{O}_\mathcal{K})\mathrm{N}(\mathfrak{a})^{-1} \quad (\mathfrak{a} = \mathfrak{il}_\mathcal{K}(a)).$$

It thus gives rise to a complex point $[x(a)]$ in $I_K(\mathfrak{c}(a))(\mathbf{C})$. Let $\mathcal{W}_p$ be the p-adic completion of the maximal unramified extension of $\mathbf{Z}_p$ in $\mathbf{C}_p$. The general theory of CM abelian varieties ([Shi98]) combined with the criterion of Serre and Tate ([ST68]) imply that $[x(a)]$ indeed descends to a point in $I_K(\mathfrak{c}(a))(\mathcal{W}_p) \hookrightarrow I_K(\mathcal{W}_p)$, which is still denoted by $x(a)$. The collection $\{[x(a)]\}_{a \in (\mathbf{A}_{\mathcal{K},f}^{(pN)})^\times} \subset I_K(\mathcal{W}_p)$ are called *CM points* in the Hilbert modular Shimura varieties.

5. Anticyclotomic Rankin-Selberg p-adic L-functions

5.1. Toric forms.

Definition 5.1 (Toric forms). We define the complex Hilbert modular form $\mathbf{f}_\chi : X^+ \times G(\mathbf{A}_f) \to \mathbf{C}$ associated to φ_χ by

$$\begin{aligned} \mathbf{f}_\chi(\tau, g_f) =& \varphi_\chi(g) \cdot \underline{J}(g_\infty, \mathbf{i})^k (\det g_\infty)^{-\frac{k_{mx}\Sigma + k}{2}} \left|\det g\right|_\mathbf{A}^{k_{mx}/2}, \\ & (\mathbf{i} = (\sqrt{-1})_{\sigma\in\Sigma},\ g = (g_\infty, g_f),\ g_\infty \mathbf{i} = \tau,\ \det g_\infty > 0). \end{aligned} \tag{5.1}$$

Here $\det g_\infty = (\det g_\sigma)_{\sigma \in \Sigma} \in (\mathbf{R}^\times)^\Sigma$ and $\det g_\infty > 0$ means $\det g_\sigma > 0$ for all $\sigma \in \Sigma$.

Let $\mathbf{f}_\chi^*$ be the normalization of $\mathbf{f}_\chi$ given by

$$\mathbf{f}_\chi^* = N(\pi, \chi)^{-1} \left|\det \varsigma_f\right|_{\mathbf{A}_f}^{-k_{mx}/2} \cdot \mathbf{f}_\chi.$$

Let δ_k^m be the Maass-Shimura differential operator (*cf.* [HT93, (1.21)]). Then the normalized differential operator $\widetilde{V}_+^m$ defined in (3.7) is the representation theoretic avatar of δ_k^m in the following sense:

$$\delta_k^m \mathbf{f}_\chi(\tau, g_f) = (\widetilde{V}_+^m \varphi_\chi)(g_\infty, g_f)\underline{J}(g_\infty, i)^{k+2m}(\det g_\infty)^{-\frac{k_{mx}\Sigma+k+2m}{2}} |\det g|_{\mathbf{A}}^{k_{mx}/2}$$

(*cf.* [Hsi12, §4.5]). We call $\delta_k^m \mathbf{f}_\chi^*$ the normalized toric form of character χ associated with the Hilbert modular form f.

Similarly, for each $u \in (\mathcal{O}_\mathcal{F} \otimes_\mathbf{Z} \mathbf{Z}_p)^\times$, we let $\mathbf{f}_{\chi,u}^*$ be the normalized modular form associated to the u-component $\varphi_{\chi,u}$ (*cf.* $W_{\chi,u}$ in (3.23)). It is clear from (3.25) that

$$\mathbf{f}_\chi^* = \sum_{u \in \mathcal{U}_p} \mathbf{f}_{\chi,u}^*. \tag{5.2}$$

Let K_1^n be the open compact subgroup defined in (3.26). Then $\mathbf{f}_\chi^*$ and $\left\{\mathbf{f}_{\chi,u}^*\right\}_{u \in \mathcal{U}_p}$ belong to $\mathbf{M}_k(K_1^n, \mathbf{C})$ for sufficiently large n.

For $a \in (\mathbf{A}_{\mathcal{K},f}^{(p)})^\times \times (\mathcal{O}_\mathcal{K} \otimes \mathbf{Z}_p)^\times$, we consider the Hecke action $|[a]$ given by

$$|[a] : \mathbf{M}_k(\mathfrak{c}(a), K_1^n, \mathbf{C}) \to \mathbf{M}_k(\mathfrak{c}, {}_aK_1^n, \mathbf{C}) \quad ({}_aK_1^n := \iota_\varsigma(a) K_1^n \iota_\varsigma(a^{-1})),$$
$$\mathbf{f} \mapsto \mathbf{f}|[a](\tau, g_f) := \mathbf{f}(\tau, g_f \iota_\varsigma(a)).$$

The Hecke action $|[a]$ can be extended to the spaces of p-integral modular forms (*cf.* [Hsi14b, §2.6]). It follows from Lemma 3.13 immediately that

$$\mathbf{f}_{\chi,u}^*|[a] = \chi^{-1}|\cdot|_{\mathbf{A}_\mathcal{K}}^{k_{mx}/2}(a) \cdot \mathbf{f}_{\chi,u.a^{1-c}}^* \text{ for all } a \in \mathcal{T}_f \quad (u.a^{1-c} := u a_{\Sigma_p} a_{\overline{\Sigma}_p}^{-1}). \tag{5.3}$$

5.2. The toric period integral. Next we consider the toric period integral of $\mathbf{f}_\chi^*$. Let $U_\mathcal{K} = (\mathcal{K} \otimes_\mathbf{Q} \mathbf{R})^\times \times (\mathcal{O}_\mathcal{K} \otimes_\mathbf{Z} \widehat{\mathbf{Z}})^\times$ be a subgroup of $\mathbf{A}_\mathcal{K}^\times$ and let $Cl_- = \mathcal{K}^\times \mathbf{A}^\times \backslash \mathbf{A}_\mathcal{K}^\times / U_\mathcal{K}$. Let $\mathcal{R}$ be the subgroup of $\mathbf{A}_\mathcal{K}^\times$ generated by $\mathcal{K}_v^\times$ for all ramified places v and let Cl_-^{alg} be the subgroup of Cl_- generated by the image of $\mathcal{R}$ By Lemma 3.13 and the fact that $\mathcal{T} = \mathbf{A}^\times U_\mathcal{K} \mathcal{R}$, we have

$$P_\chi(\pi(\varsigma)\widetilde{V}_+^m \varphi_\chi) = \mathrm{vol}(U_\mathcal{K}, dt)\sharp(Cl_-^{\mathrm{alg}}) \cdot \sum_{[t] \in Cl_-/Cl_-^{\mathrm{alg}}} \widetilde{V}_+^m \varphi_\chi(\iota(t)\varsigma)\chi(t). \tag{5.4}$$

Let $\mathcal{D}_1$ be a set of representatives of Cl_-/Cl_-^{alg} in $(\mathbf{A}_{\mathcal{K},f}^{(pN)})^\times$. We define the χ-isotypic toric period by

$$P_\chi(\delta_k^m \mathbf{f}_\chi^*) := \sum_{a \in \mathcal{D}_1} \delta_k^m \mathbf{f}_\chi^*(x(a))\chi|\cdot|_{\mathbf{A}_\mathcal{K}}^{-k_{mx}/2}(a).$$

Proposition 5.2. *Let $D_{\mathcal{K}/\mathcal{F}}$ be the discriminant of $\mathcal{K}/\mathcal{F}$. We have*

$$P_\chi(\delta_k^m \mathbf{f}_\chi^*)^2 = \frac{\Gamma_\Sigma(k+m)\Gamma_\Sigma(m+1)}{(\mathrm{Im}\,\vartheta)^{k+2m}(4\pi)^{2m+k+1}} \cdot L(\frac{1}{2}, \pi_\mathcal{K} \otimes \chi) \cdot e_{\Sigma_p}(\pi, \chi)$$
$$\times [\mathcal{O}_\mathcal{K}^\times : \mathcal{O}_\mathcal{F}^\times]^2 \cdot C_\pi(\chi),$$

where

$$C_\pi(\chi) = C'_\pi(\chi) \cdot 4^{-[F:\mathbf{Q}]} \left|\mathrm{N}_{\mathcal{F}/\mathbf{Q}}(D_{\mathcal{K}/\mathcal{F}})\right|_{\mathbf{R}}^{\frac{1}{2}} \left(\frac{\sharp(Cl_-)h_{\mathcal{F}}}{\sharp(Cl_-^{\mathrm{alg}})h_{\mathcal{K}}}\right)^2 \in \overline{\mathbf{Z}}_{(p)}^\times$$

and $C'_\pi(\chi)$ *is defined in* (3.27).

PROOF. We first note that the ratio $\frac{\sharp(Cl_-)h_{\mathcal{F}}}{\sharp(Cl_-^{\mathrm{alg}})h_{\mathcal{K}}}$ is a power of 2, so the constant $C_\pi(\chi)$ is a p-adic unit. By definition, we have

$$\mathbf{f}_\chi(x(a)) = \varphi_\chi(\iota(a_f)\varsigma)(\operatorname{Im}\vartheta)^{-k/2} \cdot |\mathrm{N}(a)\det\varsigma_f|_{\mathbf{A}}^{k_{mx}/2}.$$

By (5.4), we find that

$$\mathrm{vol}(U_E, \mathrm{d}^\times t)\sharp(Cl_-^{\mathrm{alg}}) \cdot P_\chi(\delta_k^m \mathbf{f}_\chi^*) = \frac{1}{N(\pi,\chi)\cdot(\operatorname{Im}\vartheta)^{k/2+m}} \cdot P_\chi(\pi(\varsigma)\widetilde{V}_+^m\varphi_\chi).$$

From the well-known formula

$$2L(1,\tau_{\mathcal{K}/\mathcal{F}}) = (2\pi)^{[\mathcal{F}:\mathbf{Q}]} \cdot \frac{h_{\mathcal{K}}/h_{\mathcal{F}}}{|D_{\mathcal{K}}|_{\mathbf{R}}^{\frac{1}{2}}|D_{\mathcal{F}}|_{\mathbf{R}}^{-\frac{1}{2}} \cdot [\mathcal{O}_{\mathcal{K}}^\times : \mathcal{O}_{\mathcal{F}}^\times]},$$

we see that

$$\begin{aligned}\mathrm{vol}(U_{\mathcal{K}}, dt) =& \mathrm{vol}(\mathcal{K}^\times\mathbf{A}^\times\backslash\mathbf{A}_{\mathcal{K}}^\times, dt)\cdot\sharp(Cl_-)^{-1}\\ =& 2\pi^{-[\mathcal{F}:\mathbf{Q}]}L(1,\tau_{\mathcal{K}/\mathcal{F}})\cdot\sharp(Cl_-)^{-1} = \frac{2^{[\mathcal{F}:\mathbf{Q}]}|D_{\mathcal{F}}|_{\mathbf{R}}^{\frac{1}{2}}}{|D_{\mathcal{K}}|_{\mathbf{R}}^{\frac{1}{2}}[\mathcal{O}_{\mathcal{K}}^\times : \mathcal{O}_{\mathcal{F}}^\times]}\cdot\frac{h_{\mathcal{K}}}{h_{\mathcal{F}}\sharp(Cl_-)}.\end{aligned}$$

The proposition follows form Theorem 3.14 immediately. □

5.3. THE FOURIER EXPANSION OF $\mathbf{f}_{\chi,u}^*$. Let $u = (u_v) \in \mathcal{U}_p$. We give an expression of the Fourier expansion of $\mathbf{f}_{\chi,u}^*$. Let $W_{\chi,u,f}$ be the finite part of $W_{\chi,u}$. By the definition of $\mathbf{f}_{\chi,u}$, we have

(5.5)

$$\begin{aligned}\mathbf{f}_{\chi,u}(\tau,g_f) &= \sum_{\beta\in\mathcal{F}} W_{\chi,u,f}(\begin{pmatrix}\beta & \\ & 1\end{pmatrix}g_f)W_{\chi,\infty}(\begin{pmatrix}\beta & \\ & 1\end{pmatrix}\begin{pmatrix}y_\infty & x_\infty\\ 0 & 1\end{pmatrix})\cdot y_\infty^{-k/2}\\ &= \sum_{\beta\in\mathcal{F}_+} W_{\chi,u,f}(\begin{pmatrix}\beta & \\ & 1\end{pmatrix}g_f)\beta^{k/2}e^{2\pi i\operatorname{Tr}_{\mathcal{F}/\mathbf{Q}}(\beta\tau)}.\\ &\quad(\tau = x_\infty + iy_\infty = (x_\sigma+iy_\sigma)_{\sigma\in\Sigma}\in X^+)\end{aligned}$$

The second equality follows from the choice of Whittaker functions at the archimedean places (3.6).

We define the global prime-to-p Fourier coefficient $\mathbf{a}_\chi^{(p)} : (\mathbf{A}_f^{(p)})^\times \to \mathbf{C}$ by
(5.6)
$$\begin{aligned}\mathbf{a}_\chi^{(p)}(a) :=& N(\pi,\chi)^{-1}\cdot W^{(p)}_{\chi,f}(\begin{pmatrix} a & \\ & 1\end{pmatrix}) \quad (a=(a_v)\in \mathbf{A}_f^\times)\\ =& \prod_{v\in B(\chi)} \frac{1}{n_v L(1,\tau_{\mathcal{K}_v/\mathcal{F}_v})} W_{\chi,v}(\begin{pmatrix} a_v & \\ & 1\end{pmatrix}) \prod_{v\notin B(\chi), v\nmid p} W_{\chi,v}(\begin{pmatrix} a_v & \\ & 1\end{pmatrix})\\ =& \prod_{v\in \mathbf{h}, v\nmid p} \mathbf{a}^*_{\chi,v}(a_v).\end{aligned}$$

Here $\mathbf{a}^*_{\chi,v}$ are the normalized local Fourier coefficients defined in Definition 3.16.

Proposition 5.3. *Let $\mathfrak{c}$ be a prime-to-p ideal of $\mathcal{F}$ and let $\mathbf{c}\in(\mathbf{A}^{(p)}_{\mathcal{K},f})^\times$ such that $\mathrm{il}_{\mathcal{F}}(\mathbf{c})=\mathfrak{c}$. Then the Fourier expansion of $\mathbf{f}^*_{\chi,u}$ at the cusp $(\mathcal{O}_\mathcal{F},\mathfrak{c})$ is given by*
$$\mathbf{f}^*_{\chi,u}|_{(\mathcal{O}_\mathcal{F},\mathfrak{c})}(q) = \sum_{\beta\in(N^{-1}\mathfrak{c})_+} \mathbf{a}_\beta(\mathbf{f}^*_{\chi,u},\mathfrak{c})q^\beta,$$
where
$$\mathbf{a}_\beta(\mathbf{f}^*_{\chi,u},\mathfrak{c}) = \beta^{k/2}\mathbf{a}_\chi^{(p)}(\beta\mathbf{c}^{-1}) \prod_{w\in\Sigma_p, v|w} \chi_{\overline{w}}(\beta^{-1})\mathbb{I}_{u_v(1+\varpi_v\mathcal{O}_{\mathcal{F}_v})}(\beta).$$
In particular, $\mathbf{f}^*_{\chi,u}\in \mathbf{M}_k(K_1^n,\mathcal{O})$ by Proposition 3.19, and the Fourier expansion of $\mathbf{f}^*_\chi$ at the cusp $(\mathcal{O}_\mathcal{F},\mathfrak{c})$ is given by
$$\mathbf{f}^*_{\chi}|_{(\mathcal{O}_\mathcal{F},\mathfrak{c})}(q) = \sum_{\beta\in(N^{-1}\mathfrak{c})_+} \mathbf{a}_\beta(\mathbf{f}^*_{\chi},\mathfrak{c})q^\beta,$$
where
$$\mathbf{a}_\beta(\mathbf{f}^*_{\chi},\mathfrak{c}) = \beta^{k/2}\mathbf{a}_\chi^{(p)}(\beta\mathbf{c}^{-1}) \prod_{w\in\Sigma_p} \chi_{\overline{w}}(\beta^{-1})\cdot\mathbb{I}_{\mathcal{O}^\times_{\mathcal{F},(p)}}(\beta).$$

Proof. It follows from the definition of $W_{\chi,u}$ that
$$\begin{aligned}W_{\chi,u,f}(\begin{pmatrix} \beta\mathbf{c}^{-1} & \\ & 1\end{pmatrix}) =& W^{(p)}_{\chi,f}(\begin{pmatrix} \beta\mathbf{c}^{-1} & \\ & 1\end{pmatrix})\cdot \prod_{v|p} W_{\chi,u_v,v}(\begin{pmatrix} \beta & \\ & 1\end{pmatrix})\\ =& W^{(p)}_{\chi,f}(\begin{pmatrix} \beta\mathbf{c}^{-1} & \\ & 1\end{pmatrix})\cdot \prod_{w\in\Sigma_p, v|w} \chi^{-1}_{\overline{w}}(a_v)\mathbb{I}_{u_v(1+\varpi_v\mathcal{O}_{\mathcal{F}_v})}(\beta).\end{aligned}$$
The proposition follows from (5.5) immediately. The Fourier expansion of $\mathbf{f}^*_\chi$ follows from (5.2). □

5.4. p-adic L-functions. Now we resume the setting in the introduction. Let $\mathcal{K}^-_{p^\infty}$ be the maximal anticyclotomic $\mathbf{Z}_p^{[\mathcal{F}:\mathbf{Q}]}$-extension of $\mathcal{K}$ and let $\Gamma^- = \mathrm{Gal}(\mathcal{K}^-_{p^\infty}/\mathcal{K})$. Let $\mathcal{C}(\Gamma^-)=\mathcal{C}(\Gamma^-,\overline{\mathbf{Z}}_p)$ be the space of continuous functions $\varphi:\Gamma^-\to\overline{\mathbf{Z}}_p$. The reciprocity law $\mathrm{rec}_\mathcal{K}$ at Σ_p induces a morphism
$$\mathrm{rec}_{\Sigma_p}: (\mathcal{F}\otimes_\mathbf{Q}\mathbf{Q}_p)^\times \simeq \prod_{w\in\Sigma_p}\mathcal{K}_w^\times \xrightarrow{\mathrm{rec}_\mathcal{K}} \Gamma^-.$$

Let $\mathfrak{X}_p^{\mathrm{crit}}$ be the set of critical specializations, consisting of p-adic characters $\widehat{\phi}:\Gamma^- \to \mathbf{C}_p^\times$ such that for some $m \in \mathbf{Z}_{\geq 0}[\Sigma]$,

$$\widehat{\phi}(\mathrm{rec}_{\Sigma_p}(x)) = x^m \text{ for all } x \in (\mathcal{O}_{\mathcal{F}} \otimes \mathbf{Z}_p)^\times \text{ sufficiently close to } 1.$$

Let ϕ be an anticyclotomic Hecke character of p-power conductor and of infinity type $(m, -m)$ with $m \in \mathbf{Z}_{\geq 0}[\Sigma]$. Then ϕ is unramified outside p and $\phi|_{\mathbf{A}^\times} = 1$. The p-adic avatar $\widehat{\phi}$ of ϕ belongs to $\mathfrak{X}_p^{\mathrm{crit}}$. To be precise, let $\phi_{\Sigma_p} := \prod_{w \in \Sigma_p} \phi_w$. Then we have

$$\widehat{\phi}(\mathrm{rec}_{\Sigma_p}(x)) = \phi_{\Sigma_p}(x)x^m \text{ for every } x \in (\mathcal{F} \otimes_{\mathbf{Q}} \mathbf{Q}_p)^\times. \tag{5.7}$$

Hereafter, we let λ be a Hecke character of $\mathcal{K}^\times$ and assume that Hypothesis A and (sf) hold for (π, λ). Note that Hypothesis A and (sf) also hold for $(\pi, \lambda\phi)$. We will apply our calculations in §3 to the pair $(\pi, \chi) = (\pi, \lambda\phi)$.

LEMMA 5.4. *Let ϕ be as above. Then*

(1) $\mathbf{a}_{\lambda\phi}^{(p)} = \mathbf{a}_\lambda^{(p)}$.
(2) $C'_\pi(\lambda\phi) = C'_\pi(\lambda)\phi(\mathfrak{F})$.

PROOF. If $v \nmid p$ is split, we have remarked that $W_{\chi_v\phi_v^{-1},v} = W_{\chi_v,v}$. If v is inert or ramified, then $\phi_v = 1$ as ϕ_v is unramified and $p > 2$. Therefore, we have $W_{\chi\phi,f}^{(p)} = W_{\chi,f}^{(p)}$. Part (1) follows from the definition of $\mathbf{a}_{\lambda\phi}^{(p)}$ (5.6) immediately. Next, recall that we have defined $C'_\pi(\chi)$ for a Hecke character χ in (3.27). Since ϕ is anticyclotomic and unramified outside p, part (2) follows from the fact [Sch02, Eq.(11)]

$$\varepsilon(\frac{1}{2}, \pi_v \otimes \lambda_{\overline{w}}\phi_{\overline{w}}, \psi_v) = \varepsilon(\frac{1}{2}, \pi_v \otimes \lambda_{\overline{w}}, \psi)\phi_{\overline{w}}(\mathcal{D}_{\mathcal{F}}^2\mathfrak{F}) \quad (v = w\overline{w},\, w|\mathfrak{F}). \qquad \square$$

Let $O_p := \mathcal{O}_{\mathcal{F}} \otimes_{\mathbf{Z}} \mathbf{Z}_p$ and let $\Gamma' := \mathrm{rec}_{\Sigma_p}(1 + pO_p)$ be an open subgroup of Γ^-. Let $\{\theta(\sigma)\}_{\sigma \in \Sigma}$ be the Dwork-Katz p-adic differential operators ([Kat78, Cor. (2.6.25)]) and let $\theta^m := \prod_{\sigma \in \Sigma} \theta(\sigma)^{m_\sigma}$.

PROPOSITION 5.5. *There exists a unique p-adic distribution $\mathcal{F}_{\lambda,\mathfrak{c}} : \mathcal{C}(\Gamma^-) \to V(\mathfrak{c}, K, \overline{\mathbf{Z}}_p)$ such that*

(i) *$\mathcal{F}_{\lambda,\mathfrak{c}}$ is supported in Γ',*
(ii) *for every $\widehat{\phi} \in \mathfrak{X}_p^{\mathrm{crit}}$ of weight $(m, -m)$, we have*

$$\mathcal{F}_{\lambda,\mathfrak{c}}(\widehat{\phi}) = \theta^m \widehat{\mathbf{f}}^*_{\lambda\phi,\mathfrak{c}}.$$

PROOF. We denote by $\mathcal{F}_{\lambda,\mathfrak{c}}(q)$ the p-adic measure with values in the space of formal q-expansions such that for every $\varphi \in \mathcal{C}(\Gamma^-)$,

$$\mathcal{F}_{\lambda,\mathfrak{c}}(q)(\varphi) = \sum_{\beta \in (N^{-1}\mathfrak{c})_+} \mathbf{a}_\beta(\mathbf{f}^*_\lambda, \mathfrak{c})\varphi(\mathrm{rec}_{\Sigma_p}(\beta))q^\beta.$$

Note that $\mathbf{a}_\beta(\mathbf{f}^*_\lambda, \mathfrak{c}) = 0$ unless $\beta \in \mathcal{O}_{\mathcal{F},(p)}^\times$, and thus $\mathcal{F}_{\lambda,\mathfrak{c}}$ has support in Γ' by definition.

Let $\widehat{\phi}$ be the p-adic avatar of a Hecke character ϕ of infinity type $(m,-m)$. By [Kat78, (2.6.27)] (*cf.* [HT93, §1.7 p.205]), the q-expansion of $\theta^m\widehat{\mathbf{f}}^*_{\lambda\phi}$ is given by

$$\theta^m\widehat{\mathbf{f}}^*_{\lambda\phi}|_{(\mathcal{O}_{\mathcal{F}},\mathfrak{c})}(q)=\sum_{\beta\in(N^{-1}\mathfrak{c})_+}\mathbf{a}_\beta(\mathbf{f}^*_{\lambda\phi},\mathfrak{c})\phi_{\Sigma_p}(\beta)\beta^m q^\beta.$$

Therefore, by Lemma 5.4 and (5.7) we find that

$$\mathcal{F}_{\lambda,\mathfrak{c}}(q)(\widehat{\phi})=\theta^m\widehat{\mathbf{f}}^*_{\lambda\phi,\mathfrak{c}}(q). \tag{5.8}$$

By the q-expansion principle, this measure descends to the p-adic measure $\mathcal{F}_{\lambda,\mathfrak{c}}$ with values in the space of p-adic modular forms $V(\mathfrak{c},K,\overline{\mathbf{Z}}_p)$. □

We are ready to define the p-adic L-function.

Definition 5.6. For $a\in\mathbf{A}^\times_{\mathcal{K}}$, define $|[a]\in\mathrm{End}(\mathcal{C}(\Gamma^-))$ by $\varphi\mapsto\varphi|[a](\sigma):=\varphi(\sigma\,\mathrm{rec}_{\mathcal{K}}(a)|_{\Gamma^-})$. Fix a square root $\sqrt{C_\pi(\lambda)}\in\overline{\mathbf{Z}}_p^\times$ of the constant $C_\pi(\lambda)$ and let $\widetilde{\lambda}=\lambda\cdot|\cdot|^{-k_{mx}/2}_{\mathbf{A}_{\mathcal{K}}}$. We define the p-adic integral distribution $\mathscr{L}_{\Sigma_p}(\pi,\lambda):\mathcal{C}(\Gamma^-)\to\overline{\mathbf{Z}}_p$ by

$$\mathscr{L}_{\Sigma_p}(\pi,\lambda)(\varphi)=\frac{1}{\sqrt{C_\pi(\lambda)}}\cdot\sum_{a\in\mathcal{D}_1}\widetilde{\lambda}(a)\cdot\left(\mathcal{F}_{\lambda,\mathfrak{c}(a)}(\varphi|[a])\right)(x(a)). \tag{5.9}$$

We will still denote by $\mathscr{L}_{\Sigma_p}(\pi,\lambda)\in\overline{\mathbf{Z}}_p[\![\Gamma^-]\!]$ the corresponding power series.

We give the evaluation formula of $\mathscr{L}_{\Sigma_p}(\pi,\lambda)$ at critical specializations. Let $(\Omega_\infty,\Omega_p)\in(\mathbf{C}^\times)^\Sigma\times(\overline{\mathbf{Z}}_p^\times)^\Sigma$ be the complex and p-adic CM periods of $(\mathcal{K},\Sigma)$ introduced in [HT93, (4.4 a,b) p.211] (*cf.* (Ω,c) in [Kat78, (5.1.46), (5.1.48)]) and let $\Omega_{\mathcal{K}}=(2\pi i)^{-1}\Omega_\infty$. For each Hecke character χ of infinity type $(m,-m)$, we define the algebraic L-value by

$$L^{\mathrm{alg}}(\frac{1}{2},\pi_{\mathcal{K}}\otimes\chi):=\frac{\Gamma_\Sigma(m)\Gamma_\Sigma(k+m)}{(\mathrm{Im}\,\vartheta)^{k+2m}(4\pi)^{k+2m+1\cdot\Sigma}}\cdot\frac{L(\frac{1}{2},\pi_{\mathcal{K}}\otimes\chi)}{\Omega_{\mathcal{K}}^{2(k+2m)}}\in\overline{\mathbf{Q}}. \tag{5.10}$$

The algebraicity of this L-value is due to Shimura [Shi78].

Theorem 5.7. *Suppose that Hypothesis A and* (sf) *hold. Then for each p-adic character $\widehat{\phi}\in\mathfrak{X}_p^{\mathrm{crit}}$ of weight $(m,-m)$, we have the evaluation formula*

$$\left(\frac{\mathscr{L}_{\Sigma_p}(\pi,\lambda)(\widehat{\phi})}{\Omega_p^{k+2m}}\right)^2=[\mathcal{O}^\times_{\mathcal{K}}:\mathcal{O}^\times_{\mathcal{F}}]^2\cdot e_{\Sigma_p}(\pi,\lambda\phi)L^{\mathrm{alg}}(\frac{1}{2},\pi_{\mathcal{K}}\otimes\lambda\phi)\cdot\phi^{-1}(\mathfrak{F}).$$

Proof. It follows from [Kat78, (2.4.6), (2.6.8), (2.6.33)] that

$$\frac{1}{\Omega_p^{k+2m}}\theta^m\widehat{\mathbf{f}}^*_{\lambda\phi}(x(a))=\frac{1}{\Omega_{\mathcal{K}}^{k+2m}}\delta_k^m\mathbf{f}^*_{\lambda\phi}(x(a)).$$

We thus have

$$\begin{aligned}\frac{1}{\Omega_p^{k+2m}}\cdot\mathscr{L}_{\Sigma_p}(\pi,\lambda)(\widehat{\phi}) =&\sqrt{C_\pi(\lambda)}^{-1}\sum_{a\in\mathcal{D}_1}\widetilde{\lambda}\phi(a)\cdot\frac{1}{\Omega_p^{k+2m}}\theta^m\widehat{\mathbf{f}}^*_{\lambda\phi}(x(a))\\ =&\frac{1}{\Omega_{\mathcal{K}}^{k+2m}\cdot\sqrt{C_\pi(\lambda)}}\cdot P_{\lambda\phi}(\delta_k^m\mathbf{f}^*_{\lambda\phi}).\end{aligned}$$

Combined with Proposition 5.2 and Lemma 5.4 (2), the above equation yields the proposition. □

6. The μ-invariant of p-adic L-functions

In this section, we use the explicit computation of Fourier coefficients of $\left\{\mathbf{f}^*_{\lambda,u}\right\}_{u\in\mathcal{U}_p}$ to study the μ-invariant of the p-adic L-function $\mathscr{L}_{\Sigma_p}(\pi,\lambda)$ by the approach of Hida [Hid10b].

6.1. The t-expansion of p-adic modular forms. We begin with a brief review of the t-expansion of p-adic modular forms. A functorial point in $I_K(\mathfrak{c})$ can be written as $[(\underline{A},j)] = [(A,\lambda,\iota,\overline{\eta}^{(p)},j)]$. Enlarging $\mathcal{W}_p$ if necessary, we let $\mathcal{W}_p$ be the p-adic ring generated by the values of λ on finite ideles over the Witt ring $W(\bar{\mathbb{F}}_p)$. Let $\mathfrak{m}_{\mathcal{W}_p}$ be the maximal ideal of $\mathcal{W}_p$ and fix an isomorphism $\mathcal{W}_p/\mathfrak{m}_{\mathcal{W}_p}\stackrel{\sim}{\to}\bar{\mathbb{F}}_p$. Let $T:=\mathcal{O}^*_{\mathcal{F}}\otimes_{\mathbf{Z}}\boldsymbol{\mu}_{p^\infty}$ and let $\widehat{T}=\mathcal{O}^*_{\mathcal{F}}\otimes_{\mathbf{Z}}\widehat{\mathbb{G}}_{m/\mathcal{W}_p}$ be the formal completion. Let $\{\xi_1,\dots,\xi_d\}$ be a basis of $\mathcal{O}_{\mathcal{F}}$ over $\mathbf{Z}$ and let t be the character $1\in\mathcal{O}_{\mathcal{F}}=X^*(\mathcal{O}^*_{\mathcal{F}}\otimes_{\mathbf{Z}}\mathbb{G}_m)=\mathrm{Hom}(\mathcal{O}^*_{\mathcal{F}}\otimes_{\mathbf{Z}}\mathbb{G}_m,\mathbb{G}_m)$. Then

$$\mathcal{O}_{\widehat{T}}\stackrel{\sim}{\to}\mathcal{W}_p[\![S_1,\dots,S_d]\!]\quad(S_i=t^{\xi_i}-1).$$

Let $\mathbf{x}:=x(1)_{/\mathcal{W}_p}\in I_K(\mathfrak{c})(\mathcal{W}_p)$ be the CM point introduced in §4.6and let $x_0=\mathbf{x}\otimes_{\mathcal{W}_p}\bar{\mathbb{F}}_p=(\underline{A}_0,j_0)\in I_K(\mathfrak{c})(\bar{\mathbb{F}}_p)$ be the reduction. The theory of Serre-Tate coordinates ([Kat81]) tells us the deformation space $\widehat{S}_{x_0}$ of x_0 is isomorphic to the formal torus $\widehat{T}$, and the p^∞-level structure j_0 of A_0 induces a canonical isomorphism $\varphi_{x_0}\colon\widehat{T}_{/\mathcal{W}}\stackrel{\sim}{\to}\widehat{S}_{x_0}=\mathrm{Spf}\,\widehat{\mathcal{O}}_{I_K(\mathfrak{c}),x_0}$ (*cf.* [Hid10b, (3.15)]). We will regard the character t on $\widehat{T}$ as a function on $\widehat{S}_{x_0}$ via φ_{x_0}. Then $\mathbf{x}$ is the canonical lifting of x_0, i.e. $t(\mathbf{x})=1$. For $f\in V(\mathfrak{c},K,\mathcal{W}_p)$, we define

$$f(t):=\varphi^*_{x_0}(f)\in\mathcal{O}_{\widehat{T}}=\mathcal{W}_p[\![S_1,\dots,S_d]\!].$$

The formal power series $f(t)$ is called the *t-expansion* around x_0 of f.

6.2. The vanishing of the μ-invariant. Let $\pi_-:(\mathbf{A}^{(pN)}_{\mathcal{K},f})^\times\to\Gamma^-$ be the natural map induced by the reciprocity law. Let $Z'=\pi_-^{-1}(\Gamma')$ be a subgroup of $(\mathbf{A}^{(pN)}_{\mathcal{K},f})^\times$ and let $Cl'_-\supset Cl_-^{\mathrm{alg}}$ be the image of Z' in Cl_-. Let $\mathcal{D}'_1$ (resp. $\mathcal{D}''_1$) be a set of representative of $Cl'_-/Cl_-^{\mathrm{alg}}$ (resp. Cl_-/Cl'_-) in $(\mathbf{A}^{(pN)}_{\mathcal{K},f})^\times$. Let $\mathcal{D}_1:=\mathcal{D}'_1\mathcal{D}''_1$ be a set of representative of Cl_-/Cl_-^{alg}. Recall that $\mathcal{U}_p$ is the torsion subgroup of $O_p^\times$. Let $\mathcal{U}$ be the torsion subgroup of $\mathcal{K}^\times$ and let $\mathcal{U}^{\mathrm{alg}}=(\mathcal{K}^\times)^{1-c}\cap\mathcal{O}^\times_{\mathcal{K}}$ be a subgroup of $\mathcal{U}$. We regard $\mathcal{U}^{\mathrm{alg}}$ as a subgroup of O_p

by the imbedding induced by Σ_p. Let $\mathcal{D}_0$ be a set of representatives of $\mathcal{U}_p/\mathcal{U}^{\mathrm{alg}}$ in $\mathcal{U}_p$.

Let $\mathfrak{c} := \mathfrak{c}(\mathcal{O}_\mathcal{K})$ be the polarization ideal of the chosen CM point $x(1)$. The following theorem reduces the calculation of the μ-invariant $\mu^-_{\pi,\lambda,\Sigma}$ to the determination of p-adic valuation of the q-expansion of $\mathbf{f}^*_{\lambda,u}$.

Theorem 6.1. *Suppose that p is unramified in $\mathcal{F}$. Then*

$$\mu^-_{\pi,\lambda,\Sigma} = \inf_{\substack{(a,u)\in\mathcal{D}_1\times\mathcal{D}_0\\ \beta\in\mathcal{F}_+}} v_p(\mathbf{a}_\beta(\mathbf{f}^*_{\lambda,u},\mathfrak{c}(a))).$$

Proof. For every pair $(u,a)\in\mathcal{U}_p\times\mathcal{D}_1$, we let $\mathbf{f}^*_{u,a} := \mathbf{f}^*_{\lambda,u}|_{\mathfrak{c}(a)} \in \mathcal{M}_k(\mathfrak{c}(a),K,\mathcal{O})$. Let $\mathcal{F}_{u,a}$ be the p-adic avatar of $\mathbf{f}^*_{u,a}$. Fix a sufficient large finite extension L over $\mathbf{Q}_p$ so that χ and $\mathbf{f}^*_{u,a}|[a]$ are defined over $\mathcal{O}_L$ for all (u,a), and hence $\mathcal{F}_{u,a}|[a]\in V(\mathfrak{c},\mathcal{O}_L)$. For each $z\in Z'$, let $\langle z\rangle$ be the unique element in $1+pO_p$ such that $\mathrm{rec}_{\Sigma_p}(\langle z\rangle)=\pi_-(z)\in\Gamma^-$. For $(a,b)\in\mathcal{D}_1\times\mathcal{D}''_1$, we define

$$\widetilde{\mathcal{F}}_a(t) = \sum_{u\in\mathcal{U}_p}\mathcal{F}_{u,a}(t^{u^{-1}}),$$
$$\mathcal{F}^b(t) = \sum_{a\in b\mathcal{D}_1}\widetilde{\lambda}(ab^{-1})\widetilde{\mathcal{F}}_a|[a](t^{\langle ab^{-1}\rangle}).$$

Let $\mathscr{L}^b_\Sigma(\pi,\lambda)$ be the p-adic measure on $1+pO_p\simeq\Gamma'$ obtained by the restriction of $\mathscr{L}_{\Sigma_p}(\pi,\lambda)$ to $\pi_-(b)\Gamma'$. In other words, for each continuous function $\varphi:\Gamma'\to\overline{\mathbf{Z}}_p$, we have

$$\begin{aligned}\mathscr{L}^b_\Sigma(\pi,\lambda)(\varphi) :=&\mathscr{L}_{\Sigma_p}(\pi,\lambda)(\mathbb{I}_{b.\Gamma'}\cdot\varphi|[b^{-1}])\\ =&\sum_{a\in b\mathcal{D}'_1}\widetilde{\lambda}(a)\mathcal{F}_{\lambda,\mathfrak{c}(a)}(\varphi|[ab^{-1}])(x(a)).\end{aligned}$$

Here the second equality follows from the fact that $\mathcal{F}_{\lambda,\mathfrak{c}(a)}$ has support in Γ' (Proposition 5.5 (i)). The argument of [Hsi14b, Prop. 5.2] shows that $\mathcal{F}^b(t)$ is the power series expansion of the measure $\mathscr{L}^b_\Sigma(\pi,\lambda)$ regarded as a p-adic measure on O_p and that

$$\begin{aligned}&\mu^-_{\pi,\lambda,\Sigma} = \inf_{b\in\mathcal{D}''_1}\mu(\mathcal{F}^b),\ \text{where}\\ &\qquad\mu(\mathcal{F}^b) := \inf\left\{r\in\mathbf{Q}_{\geq 0}\mid p^{-r}\mathcal{F}^b\not\equiv 0\ (\mathrm{mod}\ \mathfrak{m}_p)\right\}.\end{aligned}$$

By (5.3) we find that

$$\widetilde{\mathcal{F}}_a(t) = \sharp(\mathcal{U}^{\mathrm{alg}})\cdot\sum_{u\in\mathcal{D}_0}\mathcal{F}_{u,a}(t^{u^{-1}}),$$

and hence

$$\mathcal{F}^b(t) = \sharp(\mathcal{U}^{\mathrm{alg}})\cdot\sum_{(u,a)\in\mathcal{D}_0\times b\mathcal{D}'_1}\widetilde{\lambda}(ab^{-1})\mathcal{F}_{u,a}|[a](t^{\langle ab^{-1}\rangle u^{-1}}).$$

Proceeding along the same lines in [Hsi14b, Thm. 5.5], we deduce the theorem from the above equation by the linear independence of p-adic modular forms modulo p acted by the automorphisms in $\mathcal{D}_0 \times \mathcal{D}_1'$ ([Hid10b, Thm. 3.20, Cor. 3.21]) and the q-expansion principle for p-adic modular forms. $\square$

Theorem 6.2. *In addition to Hypothesis A and* (sf), *we suppose that p is unramified in $\mathcal{F}$ and*

(aiK) *the residual Galois representation $\overline{\rho}_p(\pi_{\mathcal{K}})$ is absolutely irreducible.*

Then $\mu^-_{\pi,\lambda,\Sigma} = 0$ if and only if

$$\sum_{v|\mathfrak{c}^-_\lambda} \mu_p(\Psi_{\pi,\lambda,v}) = 0,$$

where $\mu_p(\Psi_{\pi,\lambda,v})$ are the local invariants defined as in (3.30).

Proof. It is not difficult to deduce from the formula of $\mathbf{a}_\beta(\mathbf{f}^*_{\lambda,u}, \mathfrak{c}(a))$ in Proposition 5.3 and Proposition 3.19 that

$$\mu_p(\Psi_{\pi,\lambda,v}) > 0 \text{ for some } v|\mathfrak{c}^-_\lambda \Rightarrow \mathbf{a}_\beta(\mathbf{f}^*_{\lambda,u}, \mathfrak{c}(a)) \equiv 0 \,(\mathrm{mod}\ \mathfrak{m}_p) \text{ for all } a \in \mathbf{A}^\times_f,$$

and hence $\mu^-_{\pi,\lambda,\Sigma} > 0$ by Theorem 6.1.

Conversely, we suppose that $\mu_p(\Psi_{\pi,\lambda,v}) = 0$ for all $v|\mathfrak{c}^-_\lambda$. We are going to show $\mu^-_{\pi,\lambda,\Sigma} = 0$ by contradiction. Assume that $\mu^-_{\pi,\lambda,\Sigma} > 0$. By Proposition 5.3 Theorem 6.1, for each $a \in \mathbf{A}^{(pN)}_{\mathcal{K},f}$ we find that

$$\begin{aligned} &\mathbf{a}_\beta(\mathbf{f}^*_{\lambda,u}, \mathfrak{c}(a)) \equiv 0 \,(\mathrm{mod}\ \mathfrak{m}_p) \text{ for all } u \in \mathcal{U}_p \text{ and } \beta \in \mathcal{F}_+ \\ \iff &\mathbf{a}^{(p)}_\lambda(\beta \mathbf{c}^{-1} \mathrm{N}(a^{-1})) \equiv 0 \,(\mathrm{mod}\ \mathfrak{m}_p) \text{ for all } \beta \in \mathcal{O}^\times_{\mathcal{F},(p)}. \end{aligned}$$

Therefore, as a function on $(\mathbf{A}^{(p)}_f)^\times$, we have

$$\text{(6.1)} \quad \begin{aligned} &\mathbf{a}^{(p)}_\lambda(a) \equiv 0 \,(\mathrm{mod}\ \mathfrak{m}_p) \text{ for all} \\ &\quad a \in \mathcal{O}^\times_{\mathcal{F},(p)} \mathbf{c}^{-1} \det(U(N)) \mathrm{N}((\mathbf{A}^{(pN)}_{\mathcal{K},f})^\times) = \mathcal{F}^\times \mathbf{c}^{-1} \mathrm{N}((\mathbf{A}^{(p)}_{\mathcal{K},f})^\times). \end{aligned}$$

By Proposition 3.19, there exists $\eta = (\eta_v) \in \prod_{v|\mathfrak{c}^-_\lambda} \mathcal{F}^\times_v$ such that $\mathbf{a}^*_{\lambda,v}(\eta_v) \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{m}_p)$ for each $v|\mathfrak{c}^-_\lambda$. We extend η to be the idele in $\mathbf{A}^\times_f$ such that $\eta_v = 1$ at $v \nmid \mathfrak{c}_\chi$. Therefore, (6.1) together with the factorization formula of $\mathbf{a}^{(p)}_\lambda$ (5.6) imply that for each uniformizer ϖ_v at $v \nmid p\mathfrak{r}$, we have

(6.2)

$$\begin{aligned} &\mathbf{a}^{(p)}_\lambda(\eta\varpi_v) \equiv 0 \,(\mathrm{mod}\ \mathfrak{m}_p) \iff W^0_v(\begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix}) \equiv 0 \,(\mathrm{mod}\ \mathfrak{m}_p) \text{ whenever} \\ &\qquad \varpi_v \in [\eta^{-1}\mathbf{c}^{-1}] := \mathcal{F}^\times \eta^{-1} \mathbf{c}^{-1} \mathrm{N}((\mathbf{A}^{(p)}_{\mathcal{K},f})^\times). \end{aligned}$$

On the other hand, by (3.2), we find that

$$\mathrm{Tr}\,\rho_p(\pi)(\mathrm{Frob}_v) = \omega(\varpi_v)^{-1} \left|\varpi_v\right|^{-k_{mx}/2} W^0_v(\begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix}) \text{ for all } v \nmid p\mathfrak{n}.$$

Let $\mathrm{rec}_{\mathcal{K}/\mathcal{F}} : \mathbf{A}_{\mathcal{K}}^{\times} \to \mathrm{Gal}(\mathcal{K}/\mathcal{F})$ be the surjection induced by the reciprocity law. Combined with (6.2), the above equation yields that

$$\begin{aligned}&\mathrm{Tr}\,\rho_p(\pi)(\mathrm{Frob}_v) \equiv 0 \,(\mathrm{mod}\ \mathfrak{m}_p) \text{ whenever}\\ &\qquad\qquad \mathrm{Frob}_v|_{\mathcal{K}} = \mathrm{rec}_{\mathcal{K}/\mathcal{F}}(\varpi_v) = \mathrm{rec}_{\mathcal{K}/\mathcal{F}}(\eta^{-1}\mathbf{c}^{-1}).\end{aligned}$$

This in particular implies that $\mathrm{rec}_{\mathcal{K}/\mathcal{F}}(\eta^{-1}\mathbf{c}^{-1})$ must be the complex conjugation c, and hence we arrive at a contradiction to (aiK) by the following Lemma 6.3. □

Lemma 6.3. *Let $p > 2$ be a prime. Let G be a finite group and $H \subset G$ be a index two subgroup. Let $\rho : G \hookrightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ be a faithful irreducible representation of G. Let $\mathrm{T} = \mathrm{Tr}\,\rho : G \to \bar{\mathbb{F}}_p$ be the trace function. Assume that*

(1) There exists an order two element $c \in G - H$,
(2) $\mathrm{T}(hc) = 0$ for all $h \in H$.

Then $\rho|_H$ is reducible.

Proof. The assumption (2) implies that $\mathrm{T}(c) = 0$, and hence $\det \rho(c) = -1$. We may assume that $\rho(c) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Suppose that $p \nmid \sharp(G)$. By the usual representation theory of finite groups, we have

$$1 = \langle \mathrm{T}, \mathrm{T}\rangle := \frac{1}{\sharp(G)} \sum_{g\in G} \mathrm{T}(g)\mathrm{T}(g^{-1}) = \frac{1}{2\sharp(H)} \sum_{h\in H} \mathrm{T}(h)\mathrm{T}(h^{-1}) = \frac{1}{2}\cdot\langle \mathrm{T}|_H, \mathrm{T}|_H\rangle.$$

Since $\langle \mathrm{T}|_H, \mathrm{T}|_H\rangle = 2$, we conclude that $\rho|_H$ is not irreducible.
Now we assume that $p \mid \sharp(H)$. For each $b \in M_2(\bar{\mathbb{F}}_p)$ with $b^2 = 0$, define the p-subgroup P_b of $\rho(H)$ by

$$P_b = \left\{h \in \rho(H) \mid h = 1 + xb \text{ for some } x \in \bar{\mathbb{F}}_p\right\}.$$

Let $h \in H$ be an element of p-power order. It is well known that $(\rho(h)-1)^2 = 0$, and hence $\mathrm{T}(h) = 2$ and $\det \rho(h) = 1$. Combined with $\mathrm{T}(hc) = 0$, these equations imply that

$$\rho(h) \in P_{b_1} \text{ or } P_{b_2} \text{ with } b_1 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, b_2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Note that either P_{b_1} or P_{b_2} is trivial. Indeed, if $h_1 \neq 1 \in P_{b_1}$ and $h_2 \neq 1 \in P_{b_2}$. Then $h_1h_2 \in H$ and $\mathrm{Tr}(h_1h_2c) \neq 0$, which is a contradiction. In particular, we conclude that elements of p-power order in H are commutative with each other and that there is only one p-Sylow subgroup of H, which we denote by P. It is clear that H normalizes P. Since $P \neq \{1\}$, there is a unique line fixed by $\rho(P)$, which is an invariant subspace of $\rho(H)$. We find that $\rho|_H$ is reducible if $p \mid \sharp(H)$. □

Remark 6.4. The assumption (3) in Theorem B in the introduction implies the vanishing of $\mu_p(\Psi_{\pi,\lambda,v})$ for all $v|\mathfrak{c}_\lambda^-$.

7. Non-vanishing of central L-values with anticyclotomic twists

In this section, we consider the problem of non-vanishing of L-values modulo p with anticyclotomic twists. Let $\ell \neq p$ be a rational prime and let $\mathfrak{l}$ be a prime of $\mathcal{F}$ above ℓ. Let $\Gamma_{\mathfrak{l}}^- := \mathrm{Gal}(\mathcal{K}_{\mathfrak{l}^\infty}^-/\mathcal{K})$ be the Galois group of the maximal anticyclotomic pro-ℓ extension $\mathcal{K}_{\mathfrak{l}^\infty}^-$ in the ray class field of $\mathcal{K}$ of conductor $\mathfrak{l}^\infty$. Let $\mathfrak{X}_{\mathfrak{l}}^0$ be the set consisting of finite order characters $\phi : \Gamma_{\mathfrak{l}}^- \to \mu_{\ell^\infty}$. Fix a Hecke character χ of infinity type $(k/2+m, -k/2-m)$. We assume

$$(p\mathfrak{l}, \mathfrak{n}D_{\mathcal{K}/\mathcal{F}}) = 1.$$

When $p \nmid D_{\mathcal{K}}$, we know the algebraic L-value $L^{\mathrm{alg}}(\frac{1}{2}, \pi_{\mathcal{K}} \otimes \chi\phi) \in \overline{\mathbf{Z}}_p$ in (5.10) in view of Theorem 5.7. Recall that $\mathfrak{m}_p$ is the maximal ideal of $\overline{\mathbf{Z}}_p$. This section is devoted to proving the following result:

Theorem 7.1. *With the same assumptions in Theorem 6.2, we further assume that*

(1) $(p\mathfrak{l}, \mathfrak{n}\mathfrak{c}_\chi D_{\mathcal{K}/\mathcal{F}}) = 1$.
(2) $\mu_p(\Psi_{\pi,\chi,v}) = 0$ *for all* $v | \mathfrak{c}_\chi^-$.

Then for almost all $\phi \in \mathfrak{X}_{\mathfrak{l}}^0$ *we have*

$$L^{\mathrm{alg}}(\frac{1}{2}, \pi_{\mathcal{K}} \otimes \chi\phi) \not\equiv 0 \,(mod\; \mathfrak{m}_p).$$

Here almost all means "except for finitely many $\phi \in \mathfrak{X}_{\mathfrak{l}}^0$*" if* $\dim_{\mathbf{Q}_\ell} F_{\mathfrak{l}} = 1$ *and "for* ϕ *in a Zariski dense subset of* $\mathfrak{X}_{\mathfrak{l}}^0$*" if* $\dim_{\mathbf{Q}_\ell} F_{\mathfrak{l}} > 1$ *(See* [Hid04a, p.737]*).*

When $\mathcal{F} = \mathbf{Q}$, an imprimitive version of the above result under different assumptions is treated in [Bra11b].

7.1. After introducing some notation, we outline the approach of Hida [Hid04a] to study this problem. We shall take $\mathfrak{r} = \mathfrak{c}_\chi \mathfrak{n} D_{\mathcal{K}/\mathcal{F}} \mathfrak{l}$ to be the fixed ideal in §3.4. For every $n \in \mathbf{Z}_{\geq 0}$, let $R_{\mathfrak{l}^n} := \mathcal{O}_{\mathcal{F}} + \mathfrak{l}^n \mathcal{O}_{\mathcal{K}}$ be the order in $\mathcal{K}$ of conductor $\mathfrak{l}^n$. Let $U_{\mathfrak{l}^n} = (\mathcal{K} \otimes_{\mathbf{Q}} \mathbf{R})^\times (R_n \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}})^\times$ and let $Cl_{\mathfrak{l}^n}^- := \mathcal{K}^\times \mathbf{A}^\times \backslash \mathbf{A}_{\mathcal{K}}^\times / U_{\mathfrak{l}^n}$ be the anticyclotomic ideal class group of conductor $\mathfrak{l}^n$. Denote by $[\cdot]_n : \mathbf{A}_{\mathcal{K}}^\times \to Cl_{\mathfrak{l}^n}^-$ the quotient map. Let $Cl_{\mathfrak{l}^\infty}^- = \varprojlim_n Cl_{\mathfrak{l}^n}^-$. Let $I_{\mathfrak{l}}$ be the $\mathfrak{l}$-adic Iwahori subgroup of $K_{\mathfrak{l}}^0$ given by

$$I_{\mathfrak{l}} = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{\mathfrak{l}}^0 \mid c \in \varpi_{\mathfrak{l}} \mathcal{D}_{\mathcal{F}_{\mathfrak{l}}} \right\}.$$

Let $K_0(\mathfrak{l}) := K^{\mathfrak{l}} I_{\mathfrak{l}} = \{g \in K \mid g_{\mathfrak{l}} \in I_{\mathfrak{l}}\}$ be an open compact subgroup of $\mathrm{GL}_2(\mathbf{A}_f)$. Recall that the $U_{\mathfrak{l}}$-operator on $\mathbf{M}_k(K_0(\mathfrak{l}), \mathbf{C})$ is given by

$$F \mid U_{\mathfrak{l}}(\tau, g_f) = \sum_{u \in \mathcal{O}_{\mathcal{F}}/\mathfrak{l}\mathcal{O}_{\mathcal{F}}} F(\tau, g_f \begin{pmatrix} \varpi_{\mathfrak{l}} & u d_{\mathcal{F}_{\mathfrak{l}}}^{-1} \\ 0 & 1 \end{pmatrix}).$$

We briefly outline the approach of Hida to prove Theorem 7.1 as follows:

(1) Construct a suitable p-adic modular form $\widehat{\mathbf{f}}_\chi^\dagger$ which is an eigenfunction of $U_{\mathfrak{l}}$-operator with p-adic unit eigenvalue.

(2) Consider Hida's measure $\varphi_\chi^\dagger$ on $Cl_{\mathfrak{l}^\infty}^-$ attached to $\widehat{\mathbf{f}}_\chi^\dagger$ (7.1) and show the evaluation formula of this measure is related to central values $L^{\mathrm{alg}}(\frac{1}{2},\pi_{\mathcal{K}}\otimes\chi\phi)$ (Proposition 7.3).

(3) The Zariski density of CM points in Hilbert modular varieties modulo p reduces the proof of Theorem 7.1 to the non-vanishing of certain Fourier coefficients of $\widehat{\mathbf{f}}_\chi^\dagger$ at some cusp ([Hid04a, Thm. 3.2 and Thm. 3.3]).

We remind, as the reader will note, that the proof is very close to Theorem 5.7 and Theorem 6.2. The essential new inputs in this section are the choice of $U_{\mathfrak{l}}$-eigenforms and the computation of local period integral at $\mathfrak{l}$.

7.2. CM POINTS OF CONDUCTOR $\mathfrak{l}^n$. Let $n\in\mathbf{Z}_{\geq 0}$. We choose $\varsigma_{\mathfrak{l}}^{(n)}\in G_{\mathfrak{l}}$ as follows. If $\mathfrak{l}=\mathfrak{L}\overline{\mathfrak{L}}$ splits in $\mathcal{K}$, writing $\vartheta=\vartheta_{\overline{\mathfrak{L}}}e_{\overline{\mathfrak{L}}}+\vartheta_{\mathfrak{L}}e_{\mathfrak{L}}$ (so $d_{\mathcal{F}_{\mathfrak{l}}}=\vartheta_{\mathfrak{L}}-\vartheta_{\overline{\mathfrak{L}}}$ is a generator of $\mathcal{D}_{\mathcal{F}_{\mathfrak{l}}}$), we put

$$\varsigma_{\mathfrak{l}}^{(n)}=\begin{pmatrix}\vartheta_{\mathfrak{L}} & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\varpi_{\mathfrak{l}}^n & \\ & 1\end{pmatrix}.$$

If $\mathfrak{l}$ is inert, then we put

$$\varsigma_{\mathfrak{l}}^{(n)}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\varpi_{\mathfrak{l}}^n & \\ & 1\end{pmatrix}.$$

Let $\varsigma^{(n)}:=\varsigma_{\mathfrak{l}}^{(n)}\prod_{v\neq\mathfrak{l}}\varsigma_v$. According to this choice of $\varsigma_{\mathfrak{l}}^{(n)}$, we have

$$\varsigma_f^{(n)}*(\mathscr{L}\otimes_{\mathbf{Z}}\widehat{\mathbf{Z}})=q_\vartheta(R_{\mathfrak{l}^n}).$$

Define $x_n:\mathbf{A}_{\mathcal{K}}^\times\to X^+\times G(\mathbf{A}_f)$ by

$$x_n(a):=(\vartheta_\Sigma,a_f\varsigma_f^{(n)}).$$

This collection $\{x_n(a)\}_{a\in\mathbf{A}_{\mathcal{K}}^\times}$ of points is called CM points of conductor $\mathfrak{l}^n$. As discussed in §4.6, $\{x_n(a))\}_{a\in(\mathbf{A}_{\mathcal{K},f}^{(p)})^\times}$ descend to CM points in $I_K(\mathcal{W}_p)$.

7.3. THE MEASURES ASSOCIATED TO $U_{\mathfrak{l}}$-EIGENFORMS. We construct the $U_{\mathfrak{l}}$-eigenform $\mathbf{f}_\chi^\dagger$ as follows. Write $\pi_{\mathfrak{l}}=\pi(\mu_{\mathfrak{l}},\nu_{\mathfrak{l}})$. Define the local Whittaker function $W_{\mathfrak{l}}^\dagger\in\mathcal{W}(\pi_{\mathfrak{l}},\psi_{\mathfrak{l}})$ by

$$W_{\mathfrak{l}}^\dagger(g)=W_{\mathfrak{l}}^0(g)-\mu_{\mathfrak{l}}|\cdot|^{\frac{1}{2}}(\mathfrak{l})W_{\mathfrak{l}}^0(g\begin{pmatrix}\varpi_{\mathfrak{l}}^{-1} & \\ & 1\end{pmatrix}).$$

It is not difficult to verify that

- $W_{\mathfrak{l}}^\dagger$ is invariant by $I_{\mathfrak{l}}$,
- $W_{\mathfrak{l}}^\dagger(\begin{pmatrix}a & \\ & 1\end{pmatrix})=\nu_{\mathfrak{l}}|\cdot|^{\frac{1}{2}}(a)\mathbb{I}_{\mathcal{O}_{F_{\mathfrak{l}}}}(a)$,
- $W_{\mathfrak{l}}^\dagger$ is an $U_{\mathfrak{l}}$-eigenfunction with the eigenvalue $\nu_{\mathfrak{l}}(\varpi_{\mathfrak{l}})\,|\varpi_{\mathfrak{l}}|^{-\frac{1}{2}}$.

Define the normalized global Whittaker function $W^{\dagger}_{\chi}$ by

$$W^{\dagger}_{\chi}:=N(\pi,\chi)^{-1}\left|\det\varsigma_f\right|_{\mathbf{A}_f}^{-k_{mx}/2}\cdot\prod_{\sigma\in\Sigma}W_{k_\sigma}\cdot\prod_{v\in\mathbf{h},v\neq\mathfrak{l}}W_{\chi,v}\cdot W^{\dagger}_{\mathfrak{l}},$$

where $N(\pi,\chi)$ is the normalization factor in (3.28). Let $\varphi^{\dagger}_{\chi}$ be the automorphic form associated to $W^{\dagger}_{\chi}$ as in (3.24) and let $\mathbf{f}^{\dagger}_{\chi}$ be the associated Hilbert modular form as in Definition 5.1.

The following lemma follows from the choice of our Whittaker function $W^{\dagger}_{\chi}$ and the construction of $\mathbf{f}^{\dagger}_{\chi}$.

Lemma 7.2. *Recall that $\mathcal{R}$ is the group generated by $\mathcal{K}_v^{\times}$ for all ramified places v in $\mathbf{A}_{\mathcal{K}}^{\times}$. We have*

(1) $\mathbf{f}^{\dagger}_{\chi}$ is toric of character χ outside $\mathfrak{l}$, and

$$\mathbf{f}^{\dagger}_{\chi}(x_n(ta))=\mathbf{f}^{\dagger}_{\chi}(x_n(t))\chi^{-1}|\cdot|_{\mathbf{A}_{\mathcal{K}}}^{k_{mx}/2}(a)\ \textit{for all } a\in\mathcal{R}\cdot(R_{\mathfrak{l}^n}\otimes_{\mathbf{Z}}\widehat{\mathbf{Z}})^{\times}.$$

(2) $\mathbf{f}^{\dagger}_{\chi\phi}=\mathbf{f}^{\dagger}_{\chi}$ for every $\phi\in\mathfrak{X}^0_{\mathfrak{l}}$.

Proof. Part (1) follows immediately from the fact that $W^{\dagger}_{\chi}$ is a toric Whittaker function outside $\mathfrak{l}$ in view of Lemma 3.2. In addition, for every $\phi\in\mathfrak{X}^0_{\mathfrak{l}}$, ϕ is anticyclotomic and unramified outside $\mathfrak{l}$. We thus have $W^{\dagger}_{\chi}=W^{\dagger}_{\chi\phi}$, which verifies part (2) (*cf.* Lemma 5.4). □

Following [Hid04a, (3.9)], we define a p-adic $\overline{\mathbf{Z}}_p$-valued measure $\varphi^{\dagger}_{\chi}$ on $Cl^-_{\mathfrak{l}^\infty}$ attached to the p-adic avatar $\widehat{\mathbf{f}}^{\dagger}_{\chi}$ of $\mathbf{f}^{\dagger}_{\chi}$ as follows. For a locally constant function $\phi: Cl^-_{\mathfrak{l}^\infty}\to\overline{\mathbf{Z}}_p$ factoring through $Cl^-_{\mathfrak{l}^n}$, we define

$$\int_{Cl^-_{\mathfrak{l}^\infty}}\phi\,\mathrm{d}\varphi^{\dagger}_{\chi}=\alpha_{\mathfrak{l}}^{-n}\sum_{[a]_n\in Cl^-_{\mathfrak{l}^n}}\theta^m\widehat{\mathbf{f}}^{\dagger}_{\chi}(x_n(a))\widehat{\chi}(a)\phi([a]_n),\tag{7.1}$$

where $\alpha_{\mathfrak{l}}=\nu_{\mathfrak{l}}(\varpi_{\mathfrak{l}})\left|\varpi_{\mathfrak{l}}\right|^{\frac{-k_{mx}}{2}}$ and $\widehat{\chi}$ is the p-adic avatar of $\chi|\cdot|_{\mathbf{A}_{\mathcal{K}}}^{-k_{mx}/2}$. One checks that the right hand side does not depend on the choice of n since $\mathbf{f}^{\dagger}_{\chi}$ is an $U_{\mathfrak{l}}$-eigenform with the eigenvalue $\alpha_{\mathfrak{l}}$.

Let $\phi\in\mathfrak{X}^0_{\mathfrak{l}}$ be a character of conductor $\mathfrak{l}^n$. We view ϕ as a character on $Cl^-_{\mathfrak{l}^n}$ by the reciprocity law. Following the arguments in Proposition 5.2 and Theorem 5.7, we can write the measure as a toric period integral of $\widetilde{V}^m_+\varphi^{\dagger}_{\chi\phi}$:

$$\begin{aligned}&\frac{\Omega_{\mathcal{K}}^{k+2m}}{\Omega_p^{k+2m}}\cdot\int_{Cl^-_{\mathfrak{l}^\infty}}\phi\,\mathrm{d}\varphi^{\dagger}_{\chi}\\ &=\alpha_{\mathfrak{l}}^{-n}\operatorname{vol}(U_{\mathfrak{l}^n},dt)^{-1}\frac{1}{(\operatorname{Im}\vartheta)^{k/2+m}}\cdot P_{\chi\phi}(\pi(\varsigma^{(n)})\widetilde{V}^m_+\varphi^{\dagger}_{\chi})\\ &=\operatorname{vol}(U_{\mathcal{K}},dt)^{-1}\frac{1}{(\operatorname{Im}\vartheta)^{k/2+m}}\cdot\frac{\alpha_{\mathfrak{l}}^{-n}}{L(1,\tau_{\mathcal{K}_{\mathfrak{l}}/\mathcal{F}_{\mathfrak{l}}})\left|\varpi_{\mathfrak{l}}\right|^n}\cdot P_{\chi\phi}(\pi(\varsigma^{(n)})\widetilde{V}^m_+\varphi^{\dagger}_{\chi\phi}).\end{aligned}\tag{7.2}$$

Here we used the fact that

$$\operatorname{vol}(U_{\mathfrak{l}^n},dt)=\operatorname{vol}(U_{\mathcal{K}},dt)\cdot L(1,\tau_{\mathcal{K}_{\mathfrak{l}}/\mathcal{F}_{\mathfrak{l}}})\left|\varpi_{\mathfrak{l}}\right|^n.$$

We have the following evaluation formula.

Proposition 7.3. *Suppose that* $(\mathfrak{l}, \mathfrak{c}_\chi \mathfrak{n} D_{\mathcal{K}/\mathcal{F}}) = 1$. *For* $\phi \in \mathfrak{X}_{\mathfrak{l}}^0$ *of conductor* $\mathfrak{l}^n$ *with* $n \geq 1$, *we have*

$$\left(\frac{1}{\Omega_p^{k+2m}} \cdot \int_{Cl_{\mathfrak{l}^\infty}^-} \phi \mathrm{d}\varphi_\chi^\dagger\right)^2 = \frac{|\varpi_{\mathfrak{l}}|^{-n}}{\alpha_{\mathfrak{l}}^{2n}} \cdot [\mathcal{O}_{\mathcal{K}}^\times : \mathcal{O}_{\mathcal{F}}^\times]^2 \cdot L^{\mathrm{alg}}(\frac{1}{2}, \pi_{\mathcal{K}} \otimes \chi\phi) \cdot C_\pi(\chi)\phi(\mathfrak{F}).$$

Proof. In view of (7.2), it remains to compute $P_{\chi\phi}(\pi(\varsigma^{(n)})\widetilde{V}_+^m \varphi_{\chi\phi}^\dagger)^2$, which can be written as a product of local toric period integrals as in the proof of Theorem 3.14. We have computed these local period integrals in §3.7 and §3.8 except for the local integral at $\mathfrak{l}$, which will be carried out in the following Lemma 7.4. The desired formula is obtained by combining these calculations. □

Lemma 7.4. *Suppose that* $\chi_{\mathfrak{l}}$ *is unramified and* $\phi \in \mathfrak{X}_{\mathfrak{l}}^0$ *has conductor* $\mathfrak{l}^n$, $n \geq 1$. *Then*

$$\boldsymbol{P}(\pi(\varsigma_{\mathfrak{l}}^{(n)})W_{\mathfrak{l}}^\dagger, \chi\phi) = |\mathcal{D}_{\mathcal{K}_{\mathfrak{l}}}|_{\mathcal{K}_{\mathfrak{l}}}^{\frac{1}{2}} \cdot \omega_{\mathfrak{l}}(\varpi_{\mathfrak{l}}^n) \, |\varpi_{\mathfrak{l}}^n| \, L(1, \tau_{\mathcal{K}_{\mathfrak{l}}/\mathcal{F}_{\mathfrak{l}}})^2.$$

Proof. Write $F = \mathcal{F}_{\mathfrak{l}}$ (resp. $E = \mathcal{K}_{\mathfrak{l}}$) and $\varpi = \varpi_{\mathfrak{l}}$. For $t \in E$, we put

$$\iota_\varsigma^{(n)}(t) := (\varsigma_{\mathfrak{l}}^{(n)})^{-1}\iota(t)\varsigma_{\mathfrak{l}}^{(n)}.$$

First we suppose $\mathfrak{l}$ is split. A direct computation shows that

$$\begin{aligned}\iota_\varsigma^{(n)}(t) &= \begin{pmatrix} 1 & \varpi^{-n}d_F^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t_{\mathfrak{L}} & \\ 0 & t_{\overline{\mathfrak{L}}} \end{pmatrix} \begin{pmatrix} 1 & -\varpi^{-n}d_F^{-1} \\ 0 & 1 \end{pmatrix};\\ (\varsigma_{\mathfrak{l}}^{(n)})^{-1}\boldsymbol{J}\varsigma_{\mathfrak{l}}^{(n)} &= \begin{pmatrix} 1 & 0 \\ \varpi^n d_F & -1 \end{pmatrix}.\end{aligned}$$

By the definition of $W_{\mathfrak{l}}^\dagger$, we find that

$$\begin{aligned}&\omega^{-1}(\det \varsigma_{\mathfrak{l}}^{(n)})\boldsymbol{P}(\pi(\varsigma_{\mathfrak{l}}^{(n)})W_{\mathfrak{l}}^\dagger, \chi\phi)\\ =&\int_{\mathcal{O}_F^\times} \psi(-d_F^{-1}\varpi^{-n}x)\phi_{\mathfrak{L}}(x)\mathrm{d}^\times x \cdot \int_{\mathcal{O}_F} \psi(d_F^{-1}\varpi^{-n}a)\phi_{\mathfrak{L}}^{-1}(a)\mathrm{d}^\times a\\ =&\varepsilon(1, \phi_{\mathfrak{L}}^{-1}, \psi)\phi_{\mathfrak{L}}(-1)\varepsilon(1, \phi_{\mathfrak{L}}, \psi) \cdot \zeta_F(1)^2\\ =&|\varpi^n \mathcal{D}_F| \, L(1, \tau_{E/F})^2.\end{aligned}$$

This proves the formula in the split case. Now we suppose that $\mathfrak{l}$ is inert. We shall retain the notation in §3.8. Define $\mathbf{m}^\dagger : G_{\mathfrak{l}} \to \mathbf{C}$ by

$$\mathbf{m}^\dagger(g) := \mathbf{b}_{\mathfrak{l}}(\pi(g)W_{\mathfrak{l}}^\dagger, W_{\mathfrak{l}}^\dagger).$$

Then $\mathbf{m}^\dagger(g)$ only depends on the double coset $I_{\mathfrak{l}} g I_{\mathfrak{l}}$. Put

$$\boldsymbol{P}^* := \int_{E^\times/F^\times} \mathbf{m}^\dagger(\iota_\varsigma^{(n)}(t))\chi\phi(t)dt.$$

It is clear that

$$\boldsymbol{P}(\pi(\varsigma_{\mathfrak{l}}^{(n)})W_{\mathfrak{l}}^{\dagger},\chi\phi)=\boldsymbol{P}^{*}\cdot\frac{L(1,\tau_{E/F})\omega(\det\varsigma_{\mathfrak{l}}^{(n)})}{\zeta_{F}(1)}. \tag{7.3}$$

For $y\in\varpi^{r}\mathcal{O}_{F}^{\times}$, it is easy to verify that $\iota_{\varsigma}^{(n)}(1+y\boldsymbol{\theta})\in I_{\mathfrak{l}}$ if $r\geq n$ and

$$\iota_{\varsigma}^{(n)}(1+y\boldsymbol{\theta})\in I_{\mathfrak{l}}\mathbf{w}\begin{pmatrix}\varpi^{n-r}&\\&\varpi^{r-n}\end{pmatrix}I_{\mathfrak{l}}\text{ if }0\leq r<n\quad(\mathbf{w}=\begin{pmatrix}0&-d_{F}^{-1}\\d_{F}&0\end{pmatrix}).$$

If $x\in\varpi\mathcal{O}_{F}$, then

$$\iota_{\varsigma}^{(n)}(x+\boldsymbol{\theta})\in I_{\mathfrak{l}}\mathbf{w}\begin{pmatrix}\varpi^{n}&\\&\varpi^{-n}\end{pmatrix}I_{\mathfrak{l}}.$$

Note that $n=c_{\mathfrak{l}}(\phi)=c_{\mathfrak{l}}(\chi\phi)$ as in (3.1). Combined with the above observations and Lemma 3.7, a direct computation shows that

$$\begin{aligned}\boldsymbol{P}^{*}&=X_{n}\cdot\mathbf{m}^{\dagger}(1)+(-X_{n})\cdot\mathbf{m}^{\dagger}(\mathbf{w}\begin{pmatrix}\varpi&\\&\varpi^{-1}\end{pmatrix})\\&=\mathbf{b}_{\mathfrak{l}}(W_{\mathfrak{l}}^{0}-\pi(\begin{pmatrix}\varpi^{-1}&\\&\varpi\end{pmatrix})W_{\mathfrak{l}}^{0},W_{\mathfrak{l}}^{\dagger})\cdot X_{n}\\&=(\frac{\mu_{\mathfrak{l}}(\varpi)}{1-|\varpi|}-\frac{\nu_{\mathfrak{l}}(\varpi)}{1-\mu_{\mathfrak{l}}^{-1}\nu_{\mathfrak{l}}|\cdot|(\varpi)})\cdot\frac{1-\mu_{\mathfrak{l}}^{-1}\nu_{\mathfrak{l}}|\cdot|(\varpi)}{\mu_{\mathfrak{l}}(\varpi)-\nu_{\mathfrak{l}}(\varpi)}\cdot|\mathcal{D}_{F}|^{\frac{1}{2}}X_{n}\\&=\frac{1}{1-|\varpi|}\cdot L(1,\tau_{E/F})\,|\varpi^{n}|\,|\mathcal{D}_{E}|_{E}^{\frac{1}{2}}.\end{aligned}$$

The formula in the inert case follows from (7.3) immediately. $\square$

7.4. Proof of Theorem 7.1. We prove Theorem 7.1 in this subsection. By the evaluation formula Proposition 7.3, it boils down to proving that

$$\int_{Cl_{\infty}^{-}}\phi\,\mathrm{d}\varphi_{\chi}^{\dagger}\not\equiv 0\,(\mathrm{mod}\ \mathfrak{m}_{p})\text{ for almost all }\phi\in\mathfrak{X}_{\mathfrak{l}}^{0}. \tag{7.4}$$

By [Hid04a, Thm. 3.2, 3.3] together with the toric property of $\mathbf{f}_{\chi}^{\dagger}$ Lemma 7.2 (*cf.* [Hsi12, Lemma 6.1 and Remark 6.2]), the validity of (7.4) is reduced to verifying the following condition:

(H′) For every $u\in\mathcal{O}_{\mathcal{F}_{\mathfrak{l}}}$ and a positive integer r, there exist $\beta\in\mathcal{O}_{\mathcal{F},(p)}^{\times}$ and $a\in(\mathbf{A}_{\mathcal{K},f}^{(pN)})^{\times}$ such that $\beta\equiv u\,(\mathrm{mod}\ \mathfrak{l}^{r})$ and

$$\mathbf{a}_{\beta}(\mathbf{f}_{\chi}^{\dagger},\mathfrak{c}(a))\not\equiv 0\,(\mathrm{mod}\ \mathfrak{m}_{p}).$$

The verification of (H′) under the assumptions (aiK) and $\mu_{p}(\Psi_{\pi,\chi,v})=0$ for all $v|\mathfrak{c}_{\chi}^{-}$ follows from the same argument in Theorem 6.2. Note that for a polarization ideal $\mathfrak{c}(a)$ ($\mathfrak{c}=\mathfrak{c}(\mathcal{O}_{\mathcal{K}})$, $a\in(\mathbf{A}_{\mathcal{K},f}^{(p\mathfrak{l})})^{\times}$) and a totally positive $\beta\in\mathcal{O}_{\mathcal{F},(p)}^{\times}\cap\mathcal{O}_{\mathcal{F}_{\mathfrak{l}}}$, we have $(\mathfrak{c}(a),p\mathfrak{l})=1$ and

$$\mathbf{a}_{\beta}(\mathbf{f}_{\chi}^{\dagger},\mathfrak{c}(a))=\beta^{k/2}\prod_{v\nmid p\mathfrak{l}}\mathbf{a}_{\chi,v}^{*}(\beta\mathbf{c}_{v}^{-1}\mathrm{N}(a_{v}^{-1}))\cdot\nu_{\mathfrak{l}}|\cdot|_{\mathcal{F}_{\mathfrak{l}}}^{\frac{1}{2}}(\beta)\quad(\mathfrak{il}_{\mathcal{F}}(\mathbf{c})=\mathfrak{c}).$$

Let $u \in \mathcal{O}_{\mathcal{F}_\mathfrak{l}}$ and let $\eta^u = (\eta^u_v)$ be the idele in $\mathbf{A}^\times$ such that $\mathbf{a}^*_{\lambda,v}(\eta^u_v) \not\equiv 0 \pmod{\mathfrak{m}_p}$ for all $v|\mathfrak{c}^-_\chi$, $\eta^u_\mathfrak{l} = u$ and $\eta^u_v = 1$ for all $v \nmid \mathfrak{l}\mathfrak{c}^-_\chi$. To verify (H′), we simply proceed the Galois argument in Theorem 6.2, replacing η in by η^u therein. This completes the proof of Theorem 7.1.

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