A Conductor Formula for Completed Group Algebras

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Received: November 25, 2013

Communicated by Otmar Venjakob

Abstract. Let $\mathfrak{o}$ be the ring of integers in a finite extension of $\mathbb{Q}_p$. If G is a finite group and Γ is a maximal $\mathfrak{o}$-order containing the group ring $\mathfrak{o}[G]$, Jacobinski's conductor formula gives a complete description of the central conductor of Γ into $\mathfrak{o}[G]$ in terms of characters of G. We prove a similar result for completed group algebras $\mathfrak{o}[[G]]$, where G is a p-adic Lie group of dimension 1. We will also discuss several consequences of this result.

2010 Mathematics Subject Classification: 16H10, 16H20, 11R23
Keywords and Phrases: central conductor, completed group algebras, extensions of lattices, Fitting invariants

Introduction

Let $\mathfrak{o}$ be the ring of integers in a number field K (or in a finite extension K of $\mathbb{Q}_p$) and consider the group ring $\mathfrak{o}[G]$ of a finite group G over $\mathfrak{o}$. The central conductor $\mathcal{F}(\mathfrak{o}[G])$ consists of all elements x in the center of $\mathfrak{o}[G]$ such that $x\Gamma \subseteq \mathfrak{o}[G]$, where $\Gamma \subseteq K[G]$ is a chosen maximal $\mathfrak{o}$-order containing $\mathfrak{o}[G]$, i.e.

$$\mathcal{F}(\mathfrak{o}[G]) = \{x \in \zeta(\mathfrak{o}[G]) \mid x\Gamma \subseteq \mathfrak{o}[G]\}.$$

Here, we write $\zeta(\Lambda)$ for the center of a ring Λ. A result of Jacobinski [Ja66] (see also [CR81, Theorem 27.13]) gives a complete description of the central conductor in terms of the irreducible characters of G. More precisely, we have

$$\mathcal{F}(\mathfrak{o}[G]) = \bigoplus_{\chi} \frac{|G|}{\chi(1)} \mathfrak{D}^{-1}(\mathfrak{o}[\chi]/\mathfrak{o}), \tag{1}$$

[1]The author acknowledges financial support provided by the DFG

where $\mathfrak{D}^{-1}(\mathfrak{o}[\chi]/\mathfrak{o})$ denotes the inverse different of $\mathfrak{o}[\chi]$, the ring of integers in $K(\chi) := K(\chi(g) \mid g \in G)$, with respect to $\mathfrak{o}$, and the sum runs through all absolutely irreducible characters of G modulo the following Galois action: If χ is an absolutely irreducible character of G and σ belongs to $\mathrm{Gal}(K(\chi)/K)$, then σ acts on χ as ${}^{\sigma}\chi(g) = \sigma(\chi(g))$ for every $g \in G$. Jacobinski's main interest was in determining annihilators of Ext; in fact, he showed that

$$\mathcal{F}(\mathfrak{o}[G]) \cdot \mathrm{Ext}^1_{\mathfrak{o}[G]}(M, N) = 0$$

for all $\mathfrak{o}[G]$-lattices M and $\mathfrak{o}[G]$-modules N. For instance, it can be deduced from this result that $|G|/\chi(1)$ annihilates $\mathrm{Ext}^1_{\mathfrak{o}[G]}(M_\chi, N)$ if M_χ is an $\mathfrak{o}[G]$-lattice such that $K \otimes_{\mathfrak{o}} M_\chi$ is absolutely simple with character χ. Later, Roggenkamp [Ro71] showed that the annihilators obtained in this way are in fact the best possible in a certain precise sense.

In this article we consider completed group algebras $\mathfrak{o}[[G]]$, where $\mathfrak{o}$ denotes the ring of integers in a finite extension K of $\mathbb{Q}_p$ and G is a p-adic Lie group of dimension 1. Hence G may be written as a semi-direct product $H \rtimes \Gamma$ with finite H and a cyclic pro-p-group Γ, isomorphic to $\mathbb{Z}_p$. We will exclude the special case $p = 2$, as we will make heavily use of results of Ritter and Weiss [RW04] (where the underlying prime is assumed to be odd) on the total ring of fractions $\mathcal{Q}^K(G)$ of $\mathfrak{o}[[G]]$. However, it turns out that the results provided by Ritter and Weiss are not sufficient for our purposes and so we shall have to determine the structure of $\mathcal{Q}^K(G)$ in more detail, thereby generalizing and extending results of Lau [La12] (where $K = \mathbb{Q}_p$ and G is pro-p). We will do this in the first section. In section 2 we provide the necessary preparations for our main theorem including results on reduced traces and conductors. The main theorem will then be stated and proved in section 3. More precisely, if we define the central conductor in complete analogy to the group ring case, then we have an inclusion

$$\bigoplus_{\chi/\sim} \frac{|H| w_\chi}{\chi(1)} \cdot \mathfrak{D}^{-1}(\mathfrak{o}_\chi/\mathfrak{o})\mathfrak{o}_\chi[[\Gamma'_\chi]] \subseteq \mathcal{F}(\mathfrak{o}[[G]]), \tag{2}$$

where the sum runs through all absolutely irreducible characters of G with open kernel up to a certain explicit equivalence relation. Moreover, w_χ is the index of a certain subgroup (depending on χ) in G and $\mathfrak{o}_\chi$ denotes the ring of integers in $K_\chi := K(\chi(h) \mid h \in H)$. Finally, Γ'_χ is a cyclic pro-p-group which has an explicitly determined topological generator. We will give precise definitions later in the text. The inclusion (2) is not far from being an equality. In fact, we show that its 'χ-part' becomes an equality after localization at any height 1 prime ideal of $\mathfrak{o}_\chi[[\Gamma'_\chi]]$ that does not contain p. It is an equality if $G = H \times \Gamma$ is a direct product or if no skewfields occur in the Wedderburn decomposition of $\mathcal{Q}^K(G)$ (in fact it suffices to suppose that the Schur indices are not divisible by p). Moreover, we will also explicitly determine the central conductor whenever G is a pro-p-group; we use this to give an example where

(2) is a proper inclusion.

The proof of Jakobinski's central conductor formula does not carry over unchanged to the present situation for two reasons. First, the completed group algebra is an order over the power series ring $\mathfrak{o}[[T]]$, but there is no canonical choice of embedding of $\mathfrak{o}[[T]]$ into $\zeta(\mathfrak{o}[[G]])$. Secondly and more seriously, the ring $\mathfrak{o}[[T]]$ is a regular local ring, but it is not a Dedekind domain. Furthermore, even if we localize at a height one prime ideal, the residue field will not be finite. Hence we do not have the well elaborated theory of maximal orders over discrete valuation rings with finite residue field at our disposal. In the aforementioned two cases, when $G = H \times \Gamma$ is a direct product or when no skewfields occur in the Wedderburn decomposition, we will overcome this problem by replacing our chosen maximal $\mathfrak{o}[[T]]$-order by a suitable maximal $\mathfrak{o}$-order. When G is a pro-p-group, it is the explicit description of the occurring skewfields due to Lau [La12] which allows us to determine the central conductor.

Finally, we derive some consequences in section 4. In particular, we obtain results for the corresponding Ext-groups in analogy to the group ring case. We also apply our main result to the theory of non-commutative Fitting invariants introduced by the author [Ni10] and further developed in [JN13]. This theory may be applied to $\mathfrak{o}[[G]]$-modules even if G is non-abelian, but in contrast to the commutative case, the Fitting invariant of a finitely presented $\mathfrak{o}[[G]]$-module M might not be contained in the annihilator of M. To obtain annihilators one has to multiply by a certain ideal $\mathcal{H}(\mathfrak{o}[[G]])$ of $\zeta(\mathfrak{o}[[G]])$ which is hard to determine in general. However, it is easily seen that $\mathcal{H}(\mathfrak{o}[[G]])$ always contains the central conductor so that our main theorem provides a method to compute explicit annihilators of a finitely presented $\mathfrak{o}[[G]]$-module, at least, if we are able to compute its Fitting invariant.

Acknowledgement. The author is indebted to Henri Johnston for his many suggestions and helpful remarks.

1 The total ring of fractions of a completed group algebra

Let p be an odd prime and let G be a profinite group containing a finite normal subgroup H such that $G/H \simeq \Gamma$ for a pro-p-group Γ, isomorphic to $\mathbb{Z}_p$; thus G can be written as a semi-direct product $H \rtimes \Gamma$ and is a p-adic Lie group of dimension 1. We denote the completed group algebra $\mathbb{Z}_p[[G]]$ by $\Lambda(G)$ that is

$$\Lambda(G) := \mathbb{Z}_p[[G]] = \varprojlim \mathbb{Z}_p[G/N],$$

where the inverse limit is taken over all open normal subgroups N of G. If K is a finite field extension of $\mathbb{Q}_p$ with ring of integers $\mathfrak{o}$, we put $\Lambda^{\mathfrak{o}}(G) := \mathfrak{o} \otimes_{\mathbb{Z}_p} \Lambda(G) = \mathfrak{o}[[G]]$. We fix a topological generator γ of Γ and choose a natural number n such that γ^{p^n} is central in G. Since we also have that $\Gamma^{p^n} \simeq \mathbb{Z}_p$, there is an isomorphism $\mathfrak{o}[[\Gamma^{p^n}]] \simeq \mathfrak{o}[[T]]$ induced by $\gamma^{p^n} \mapsto 1+T$. Here, $R := \mathfrak{o}[[T]]$ denotes the power series ring in one variable over $\mathfrak{o}$. If we view $\Lambda^{\mathfrak{o}}(G)$ as an R-module (or more generally as a left $R[H]$-module), there is a decomposition

$$\Lambda^{\mathfrak{o}}(G) = \bigoplus_{i=0}^{p^n-1} R[H]\gamma^i.$$

Hence $\Lambda^{\mathfrak{o}}(G)$ is finitely generated and free as an R-module and is an R-order in the separable $L := Quot(R)$-algebra $\mathcal{Q}^K(G) := L \otimes_R \Lambda^{\mathfrak{o}}(G)$. Note that $\mathcal{Q}^K(G)$ is obtained from $\Lambda^{\mathfrak{o}}(G)$ by inverting all regular elements. By [RW04, Lemma 1] we have $\mathcal{Q}^K(G) = K \otimes_{\mathbb{Q}_p} \mathcal{Q}(G)$, where $\mathcal{Q}(G) := \mathcal{Q}^{\mathbb{Q}_p}(G)$ denotes the total ring of fractions of $\Lambda(G)$.
Let $\mathbb{Q}_p^c$ be an algebraic closure of $\mathbb{Q}_p$ and denote by $\mathrm{Irr}\,(G)$ the set of absolutely irreducible $\mathbb{Q}_p^c$-valued characters of G with open kernel. Fix $\chi \in \mathrm{Irr}\,(G)$. Let η be an irreducible constituent of $\mathrm{res}\,_H^G\chi$. Then G acts on η as $\eta^g(h) = \eta(g^{-1}hg)$ for $g \in G$, $h \in H$, and we set

$$St(\eta) := \{g \in G : \eta^g = \eta\}, \quad e(\eta) := \frac{\eta(1)}{|H|}\sum_{h\in H}\eta(h^{-1})h, \quad e_\chi := \sum_{\eta \mid \mathrm{res}\,_H^G\chi} e(\eta).$$

Choose a finite Galois extension E of K such that the character χ has a realization V_χ over E. By [RW04, Corollary to Proposition 6] e_χ is a primitive central idempotent of $\mathcal{Q}^E(G)$. In fact, it is shown that every primitive central idempotent of $\mathcal{Q}^c(G) := \mathbb{Q}_p^c \otimes_{\mathbb{Q}_p} \mathcal{Q}(G)$ is an e_χ, and $e_\chi = e_{\chi'}$ if and only if $\chi = \chi' \otimes \rho$ for some character ρ of G of type W (i.e. $\mathrm{res}\,_H^G\rho = 1$).
By Clifford theory [CR81, Proposition 11.4] the irreducible constituents of $\mathrm{res}\,_H^G\chi$ are precisely the conjugates of η under the action of G, each occurring with the same multiplicity z_χ. By [RW04, Lemma 4] we have $z_\chi = 1$ and thus equalities

$$\mathrm{res}\,_H^G\chi = \sum_{i=0}^{w_\chi-1}\eta^{\gamma^i}, \quad e_\chi = \sum_{i=0}^{w_\chi-1} e(\eta^{\gamma^i}) = \frac{\chi(1)}{|H|w_\chi}\sum_{h\in H}\chi(h^{-1})h, \tag{3}$$

where $w_\chi := [G : St(\eta)]$. Note that w_χ is a power of p as H is a subgroup of $St(\eta)$. We now put

$$K_\chi := K(\chi(h) \mid h \in H) \subseteq K(\eta) := K(\eta(h) \mid h \in H)$$

and note that $K_\chi = K_{\chi\otimes\rho}$ whenever ρ is of type W. As $g^{-1}Hg = H$, we have $K(\eta^g) = K(\eta)$ for every $g \in G$ and thus $K(\eta)$ does not depend on the particular choice of irreducible constituent of $\mathrm{res}\,_H^G\chi$.
We let $\sigma \in \mathrm{Gal}(\mathbb{Q}_p^c/K)$ act on χ as ${}^\sigma\chi(g) = \sigma(\chi(g))$ for all $g \in G$ and similarly on characters of H. Note that the actions on $\mathrm{res}\,_H^G\chi$ and η factor through $\mathrm{Gal}(K_\chi/K)$ and $\mathrm{Gal}(K(\eta)/K)$, respectively.

Lemma 1.1. *The degree $[K(\eta) : K_\chi]$ divides w_χ; in particular, it is a power of p.*

Proof. Fix σ in $\mathrm{Gal}(K(\eta)/K_\chi)$. As σ acts trivially on $\mathrm{res}\,^G_H\chi$, the character ${}^\sigma\eta$ also is an irreducible constituent of $\mathrm{res}\,^G_H\chi$. Hence there are irreducible constituents $\eta_1, \dots, \eta_s$ of $\mathrm{res}\,^G_H\chi$ such that $\mathrm{res}\,^G_H\chi$ may be written as

$$\mathrm{res}\,^G_H\chi = \sum_{i=1}^{s} \sum_{\sigma\in\mathrm{Gal}(K(\eta_i)/K_\chi)} {}^\sigma\eta_i.$$

Then $w_\chi\eta(1) = \chi(1) = \sum_{i=1}^{s}[K(\eta_i) : K_\chi]\eta_i(1) = s[K(\eta) : K_\chi]\eta(1)$ as desired. ∎

By [RW04, Proposition 5] there is a unique element $\gamma_\chi \in \zeta(\mathcal{Q}^c(G)e_\chi)$ such that γ_χ acts trivially on V_χ and $\gamma_\chi = \gamma^{w_\chi} \cdot c = c \cdot \gamma^{w_\chi}$, where $c \in (\mathbb{Q}_p^c[H]e_\chi)^\times$. An analysis of the proof in fact shows that $c \in (E[H]e_\chi)^\times$ if both χ and η have realizations over E. We can and do assume this in the following. Again by [RW04, Proposition 5] the element γ_χ generates a procyclic p-subgroup Γ_χ of $(\mathcal{Q}^E(G)e_\chi)^\times$. Moreover, γ_χ induces an isomorphism $\mathcal{Q}^E(\Gamma_\chi) \xrightarrow{\simeq} \zeta(\mathcal{Q}^E(G)e_\chi)$ by [RW04, Proposition 6].

We let $\sigma \in \mathrm{Gal}(E/K)$ act on $\mathcal{Q}^E(G) = E \otimes_K \mathcal{Q}^K(G)$ as $\sigma(x \otimes y) = \sigma(x) \otimes y$ for all $x \in E$ and $y \in \mathcal{Q}^K(G)$. We then have

$$\sigma(e_\chi) = e_{{}^\sigma\chi}, \quad \sigma(\gamma_\chi) = \gamma_{{}^\sigma\chi} \tag{4}$$

for every $\sigma \in \mathrm{Gal}(E/K)$; here, the latter equality follows from the uniqueness of $\gamma_{{}^\sigma\chi}$. By (3) we have $\sigma(e_\chi) = e_\chi$ whenever $\sigma \in \mathrm{Gal}(E/K_\chi)$; in particular, we have an action of $\mathrm{Gal}(E/K_\chi)$ on $\zeta(\mathcal{Q}^E(G)e_\chi)$.

For a finite extension F of $\mathbb{Q}_p$ let U_F^1 denote the group of principal units in F.

Lemma 1.2. *Fix $\chi \in \mathrm{Irr}\,(G)$. Then there is a principal unit $x = x_\chi \in U_E^1$ with the following properties.*

(i) The element $\gamma'_\chi := x\gamma_\chi$ is invariant under the action of $\mathrm{Gal}(E/K_\chi)$ on $\zeta(\mathcal{Q}^E(G)e_\chi)$.

(ii) Let $m \geq 0$ be an integer. If γ^{p^m} acts trivially on V_χ, then x^{p^m} belongs to $U^1_{K_\chi}$.

Moreover, if Γ'_χ denotes the procyclic p-subgroup generated by γ'_χ, then

$$\mathcal{Q}^E(\Gamma_\chi)^{\mathrm{Gal}(E/K_\chi)} = \mathcal{Q}^{K_\chi}(\Gamma'_\chi).$$

Proof. We first observe that all claims do not depend on the choice of E. To see this let F be a second finite Galois extension of K such that the character χ can be realized over F. Replacing F by EF we may assume $E \subseteq F$. If there is an $x \in U_E^1$ with (i) and (ii), then we may use the same x for F

as $U_E^1 \subseteq U_F^1$. Conversely, suppose there is an $x \in U_F^1$ which fulfills (i) and (ii) with E replaced by F. Let $\sigma \in \mathrm{Gal}(F/E)$ be arbitrary. Then σ acts trivially on γ_χ as $\gamma_\chi \in \zeta(\mathcal{Q}^E(G)e_\chi)$, and it acts trivially on γ'_χ by (i). Hence $\sigma(x) = \sigma(\gamma'_\chi)\sigma(\gamma_\chi)^{-1} = \gamma'_\chi\gamma_\chi^{-1} = x$ and thus $x \in U_F^1 \cap E = U_E^1$ as desired. Finally, as $\mathrm{Gal}(F/E)$ acts trivially on γ_χ, we have $\mathcal{Q}^F(\Gamma_\chi)^{\mathrm{Gal}(F/E)} = \mathcal{Q}^E(\Gamma_\chi)$, and thus the last statement of the Lemma does not depend on E either.
We now may assume that E is obtained from K_χ by adjoining roots of unity. More precisely, the group $G/\ker(\chi)$ is finite as χ has open kernel, and by [CR81, Theorem 15.16] we may take $E = K_\chi(\zeta)$, where ζ is a root of unity of order $|G/\ker(\chi)|$. Let us denote by $\mu_p(E)$ the group of all p-power roots of unity in E. Let $\sigma \in \mathrm{Gal}(E/K_\chi)$ be arbitrary. Then $\mathrm{res}_H^G \chi = \mathrm{res}_H^G {}^\sigma\chi$ and thus ${}^\sigma\chi = \chi \otimes \rho_\sigma$ for some character ρ_σ of type W. Hence $\sigma(\gamma_\chi) = \gamma_{{}^\sigma\chi} = \zeta_\sigma\gamma_\chi$ with $\zeta_\sigma := \rho_\sigma(\gamma)^{-w_\chi} \in \mu_p(E)$ by [RW04, Proposition 5]. The assignment

$$\mathrm{Gal}(E/K_\chi) \to \mu_p(E), \quad \sigma \mapsto \zeta_\sigma = \gamma_\chi^{\sigma-1}$$

is a 1-cocycle. Let E_0 be the maximal unramified extension of K_χ in E. Then $E = E_0(\mu_p(E))$ and thus $H^1(\mathrm{Gal}(E/E_0), \mu_p(E)) = 1$ by [NSW08, Proposition 9.1.6]. By the inflation-restriction sequence this yields an isomorphism

$$H^1(\mathrm{Gal}(E_0/K_\chi), \mu_p(E_0)) \simeq H^1(\mathrm{Gal}(E/K_\chi), \mu_p(E)). \tag{5}$$

The natural map $H^1(\mathrm{Gal}(E_0/K_\chi), \mu_p(E_0)) \to H^1(\mathrm{Gal}(E_0/K_\chi), U_{E_0}^1)$ is trivial, as the latter group vanishes by [NSW08, Proposition 7.1.2]. This and (5) imply that there is an $a \in \mu_p(E) \cdot U_{E_0}^1 \subseteq U_E^1$ such that $\zeta_\sigma = a^{\sigma-1}$ for all $\sigma \in \mathrm{Gal}(E/K_\chi)$. We put $x := a^{-1}$; then $\gamma'_\chi := x\gamma_\chi$ is easily seen to be invariant under the action of $\mathrm{Gal}(E/K_\chi)$.
Now let $m \geq 0$ be an integer such that γ^{p^m} acts trivially on V_χ. Then also $c^{p^m} = (\gamma_\chi\gamma^{-w_\chi})^{p^m}$ acts trivially on V_χ and belongs to $(E[H]e_\chi)^\times$. So we must have $c^{p^m} = e_\chi$. We deduce that $\gamma_\chi^{p^m} = \gamma^{p^m w_\chi}e_\chi$ and thus $\sigma(\gamma_\chi^{p^m}) = \gamma_\chi^{p^m}$ for every $\sigma \in \mathrm{Gal}(E/K_\chi)$. We obtain

$$\begin{aligned} \sigma(x^{p^m}) &= \sigma((\gamma'_\chi)^{p^m})\sigma(\gamma_\chi^{p^m})^{-1} \\ &= (\gamma'_\chi)^{p^m}(\gamma_\chi^{p^m})^{-1} \\ &= x^{p^m}; \end{aligned}$$

hence x^{p^m} belongs to $U_E^1 \cap K_\chi = U_{K_\chi}^1$.
For the last assertion, we observe that U_E^1 is a $\mathbb{Z}_p$-module and therefore

$$\Lambda^{\mathfrak{o}_E}(\Gamma_\chi) = \Lambda^{\mathfrak{o}_E}(\Gamma'_\chi) \tag{6}$$

as $(\gamma'_\chi)^z = x^z\gamma_\chi^z$ for every $z \in \mathbb{Z}_p$ (see also [Hi93, Lemma 1, p.199]). Hence also $\mathcal{Q}^E(\Gamma_\chi) = \mathcal{Q}^E(\Gamma'_\chi)$ and thus

$$\mathcal{Q}^E(\Gamma_\chi)^{\mathrm{Gal}(E/K_\chi)} = \mathcal{Q}^E(\Gamma'_\chi)^{\mathrm{Gal}(E/K_\chi)} = \mathcal{Q}^{K_\chi}(\Gamma'_\chi)$$

as desired. $\square$

REMARK 1.3. *The principal unit x and the element γ'_χ are unique up to a principal unit in K_χ. To see this let $y \in U_E^1$ be a second principal unit which fulfills (i) and (ii) of Lemma 1.2 with γ'_χ replaced by $\gamma''_\chi := y\gamma_\chi$. Then $yx^{-1} = \gamma''_\chi(\gamma'_\chi)^{-1}$ is a principal unit which is invariant under $\mathrm{Gal}(E/K_\chi)$ and thus belongs to $U^1_{K_\chi}$.*

EXAMPLE. Assume that $G = H \times \Gamma$ is a direct product. Then $w_\chi = 1$ and $\mathrm{res}^G_H \chi = \eta$ is irreducible. If $\chi(\gamma) = \chi(1)$, then $\gamma_\chi = \gamma e_\chi$ and we may choose $\gamma'_\chi = \gamma_\chi$. If $\chi(\gamma) \neq \chi(1)$, we may write $\chi = \chi' \otimes \rho$ with $\chi'(\gamma) = \chi'(1) = \chi(1)$ and some character ρ of type W. By [RW04, Proposition 5] we have $\gamma_\chi = \gamma_{\chi'}\rho(\gamma)^{-1} = \rho(\gamma)^{-1}\gamma e_\chi$. Then $x = \rho(\gamma)$ fulfills (i) and (ii) of Lemma 1.2 and we find that $\gamma'_\chi = \gamma_{\chi'} = \gamma e_\chi = \gamma'_{\chi'}$.

DEFINITION 1.4. *Let $\chi, \psi \in \mathrm{Irr}\,(G)$. We say that χ and ψ are* equivalent over K *(and write $\chi \sim_K \psi$) if there is $\sigma \in \mathrm{Gal}(K_\chi/K)$ such that ${}^\sigma(\mathrm{res}^G_H\chi) = \mathrm{res}^G_H\psi$.*

Note that if $\chi \sim_K \psi$, then we have $K_\chi = K_\psi$ and via (4) an isomorphism $\mathcal{Q}^{K_\chi}(\Gamma'_\chi) \simeq \mathcal{Q}^{K_\psi}(\Gamma'_\psi)$. Moreover, we have $\chi \sim_K \chi \otimes \rho$ whenever ρ is a character of type W.

PROPOSITION 1.5. *Let G be a p-adic Lie group of dimension 1 and let K be a finite extension of $\mathbb{Q}_p$. Then there is an isomorphism*

$$\zeta(\mathcal{Q}^K(G)) \simeq \bigoplus_{\chi \in \mathrm{Irr}\,(G)/\sim_K} \mathcal{Q}^{K_\chi}(\Gamma'_\chi),$$

where the sum runs through all $\chi \in \mathrm{Irr}\,(G)$ up to the above equivalence relation.

Proof. Since there are only finitely many central primitive idempotents e_χ of $\mathcal{Q}^c(G)$, we may choose a finite Galois extension E of K such that $E[H]$ contains each e_χ. We will use the fact that the center of $\mathcal{Q}^K(G)$ coincides with the $\mathrm{Gal}(E/K)$-invariants of $\zeta(\mathcal{Q}^E(G))$. For this let σ be an element of $\mathrm{Gal}(E/K)$. Fix $\chi, \psi \in \mathrm{Irr}\,(G)$. Then $\sigma(e_\chi) = e_{^\sigma\chi}$, and $\sigma(e_\chi) = e_\psi$ if and only if $\mathrm{res}^G_H\psi = {}^\sigma(\mathrm{res}^G_H\chi)$. In particular, the action of $\mathrm{Gal}(E/K)$ on e_χ factors through $\mathrm{Gal}(K_\chi/K)$, and we may write

$$\zeta(\mathcal{Q}^K(G)) = \bigoplus_{\chi \in \mathrm{Irr}\,(G)/\sim_K} \zeta(\mathcal{Q}^K(G)\varepsilon_\chi), \quad \varepsilon_\chi := \sum_{\sigma \in \mathrm{Gal}(K_\chi/K)} \sigma(e_\chi) \in \zeta(\mathcal{Q}^K(G)). \tag{7}$$

Now let $\beta \in \zeta(\mathcal{Q}^K(G)\varepsilon_\chi)$. We view β as an element in $\zeta(\mathcal{Q}^E(G)\varepsilon_\chi)$ which is invariant under $\mathrm{Gal}(E/K)$. We may therefore write

$$\beta = (\beta_\sigma)_\sigma \in \bigoplus_{\sigma \in \mathrm{Gal}(K_\chi/K)} \zeta(\mathcal{Q}^E(G)\sigma(e_\chi)) \simeq \bigoplus_{\sigma \in \mathrm{Gal}(K_\chi/K)} \mathcal{Q}^E(\Gamma_{^\sigma\chi})$$

where, by abuse of notation, σ denotes also a chosen lift of σ in $\mathrm{Gal}(E/K)$. As we have already mentioned before, the uniqueness of γ_χ implies that $\gamma_{^\sigma\chi} =$

$\sigma(\gamma_\chi)$ for every $\sigma \in \mathrm{Gal}(E/K)$; thus β is determined by β_1, and β_1 lies in $\mathcal{Q}^E(\Gamma_\chi)^{\mathrm{Gal}(E/K_\chi)} = \mathcal{Q}^{K_\chi}(\Gamma'_\chi)$, where the equality is Lemma 1.2. Now $\beta \mapsto \beta_1$ induces an isomorphism $\zeta(\mathcal{Q}^K(G)\varepsilon_\chi) \simeq \mathcal{Q}^{K_\chi}(\Gamma'_\chi)$. $\square$

Remark 1.6. *In the special case, where $K = \mathbb{Q}_p$ and G is a pro-p-group, a similar result has been established by Lau [La12] using a different method. The same is true for Corollary 1.9 below.*

Corollary 1.7. *Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathfrak{o}$. Choose a maximal R-order $\tilde{\Lambda}^{\mathfrak{o}}(G)$ containing $\Lambda^{\mathfrak{o}}(G)$. Then*

$$\zeta(\tilde{\Lambda}^{\mathfrak{o}}(G)) \simeq \bigoplus_{\chi \in \mathrm{Irr}\,(G)/\sim_K} \Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi),$$

where $\mathfrak{o}_\chi$ denotes the ring of integers in K_χ.

Proof. This follows from Proposition 1.5 as $\Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)$ is the integral closure of R in $\mathcal{Q}^{K_\chi}(\Gamma'_\chi)$. $\square$

Remark 1.8. *Here, $\tilde{\Lambda}^{\mathfrak{o}}(G)$ is an order over $R = \mathfrak{o}[[T]]$, where we have identified $1+T$ with γ^{p^n} for a chosen large n. Let us fix a $\chi \in \mathrm{Irr}\,(G)$ in each equivalence class over K. If $m = m(\chi)$ is sufficiently large, then γ^{p^m} acts trivially on V_χ and hence $\gamma_\chi^{p^m} = (\gamma^{p^m})^{w_\chi} e_\chi$. Enlarging n and m if necessary (for the finitely many χ), we may assume that $p^n = p^m \cdot w_\chi$. Hence we may also assume that the inclusion*

$$R = \mathfrak{o}[[T]] \rightarrowtail \Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)$$

is induced by $1+T \mapsto \gamma_\chi^{p^n/w_\chi} = x^{-p^m}(\gamma'_\chi)^{p^m} \in \Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)$, where x^{-p^m} belongs to $U^1_{K_\chi}$ by Lemma 1.2. We will fix such an n for the rest of the paper.

Corollary 1.9. *The algebra $\mathcal{Q}^K(G)$ has Wedderburn decomposition*

$$\mathcal{Q}^K(G) \simeq \bigoplus_{\chi \in \mathrm{Irr}\,(G)/\sim_K} (D_\chi)_{n_\chi \times n_\chi},$$

where $n_\chi \in \mathbb{N}$ and D_χ is a skewfield with center $\mathcal{Q}^{K_\chi}(\Gamma'_\chi)$. If s_χ denotes the Schur index of D_χ, then we have an equality $\chi(1) = n_\chi s_\chi$.

Proof. All assertions are immediate from Proposition 1.5 apart from the last equality. Let us denote the simple component $(D_\chi)_{n_\chi \times n_\chi}$ by A_χ. With E

sufficiently large as in Proposition 1.5 we compute

$$\begin{aligned}
(n_\chi s_\chi)^2 &= \dim_{\mathcal{Q}^{K_\chi}(\Gamma'_\chi)}(A_\chi) \\
&= [K_\chi : K]^{-1} \cdot \dim_{\mathcal{Q}^K(\Gamma'_\chi)}(A_\chi) \\
&\overset{(i)}{=} [K_\chi : K]^{-1} \cdot \dim_{\mathcal{Q}^E(\Gamma_\chi)}(E \otimes_K A_\chi) \\
&\overset{(ii)}{=} [K_\chi : K]^{-1} \cdot \sum_{\sigma \in \mathrm{Gal}(K_\chi/K)} \dim_{\mathcal{Q}^E(\Gamma_\chi)}(\mathcal{Q}^E(G) e_{\sigma\chi}) \\
&\overset{(iii)}{=} [K_\chi : K]^{-1} \cdot \sum_{\sigma \in \mathrm{Gal}(K_\chi/K)} {}^\sigma\chi(1)^2 \\
&= \chi(1)^2
\end{aligned}$$

Here, (i) follows from the equality $\mathcal{Q}^E(\Gamma'_\chi) = \mathcal{Q}^E(\Gamma_\chi)$ which was established in the proof of Lemma 1.2 (confer equation (6)). The isomorphism $E \otimes_K A_\chi \simeq \bigoplus_{\sigma \in \mathrm{Gal}(K_\chi/K)} \mathcal{Q}^E(G) e_{\sigma\chi}$ implies (ii), and (iii) is shown in the proof of [RW04, Proposition 6]. ☐

Remark 1.10. *In the case $K = \mathbb{Q}_p$ and G a pro-p-group one can determine the occurring skewfields explicitly (see [La12, Theorem 1] and also (15) below).*

Theorem 1.11. *Fix $\chi \in \mathrm{Irr}\,(G)$ and let $A_\chi = (D_\chi)_{n_\chi \times n_\chi}$ be the corresponding simple component of $\mathcal{Q}^K(G)$. Then $E \otimes_K A_\chi$ splits whenever one (and hence every) irreducible constituent η of $\mathrm{res}\,^G_H \chi$ can be realized over E. More precisely, we then have an isomorphism*

$$E \otimes_K A_\chi \simeq \bigoplus_{\sigma \in \mathrm{Gal}(K_\chi/K)} \left(\mathcal{Q}^E(\Gamma'_\chi)\right)_{\chi(1) \times \chi(1)}.$$

Proof. Recall that $L = \mathcal{Q}^K(\Gamma^{p^n})$ for our fixed sufficiently large n. Let E be a finite extension of K and let $L' := E \otimes_K L = \mathcal{Q}^E(\Gamma^{p^n})$. As L'-vector space and more generally as left $L'[H]$-module, we have a decomposition

$$\mathcal{Q}^E(G) = \bigoplus_{i=0}^{p^n - 1} L'[H]\gamma^i.$$

Now fix $\chi \in \mathrm{Irr}\,(G)$ and let η be an irreducible constituent of $\mathrm{res}\,^G_H \chi$. Suppose that η has a realization over E. Recall the definition (7) of ε_χ. As $K_\chi \subseteq K(\eta) \subseteq E$, we have a decomposition

$$E \otimes_K A_\chi = \mathcal{Q}^E(G)\varepsilon_\chi = \bigoplus_{\sigma \in \mathrm{Gal}(K_\chi/K)} \mathcal{Q}^E(G)\sigma(e_\chi).$$

As the centers of $\mathcal{Q}^E(G)\sigma(e_\chi)$, $\sigma \in \mathrm{Gal}(K_\chi/K)$ are isomorphic fields via (4) (compare also the proof of Proposition 1.5), it suffices to show that $\mathcal{Q}^E(G)e_\chi$ splits.

Since E is a subfield of L' we have $L'[H]e(\eta) \simeq L'_{\eta(1)\times\eta(1)}$ and similarly for every other irreducible constituent of $\mathrm{res}^G_H \chi$. We obtain

$$\begin{aligned}\mathcal{Q}^E(G)e_\chi &= \bigoplus_{i=0}^{p^n-1} L'[H]e_\chi \gamma^i \\ &= \bigoplus_{i=0}^{p^n-1}\bigoplus_{j=0}^{w_\chi-1} L'[H]e(\eta^{\gamma^j})\gamma^i \\ &\simeq \bigoplus_{i=0}^{p^n-1}\bigoplus_{j=0}^{w_\chi-1} L'_{\eta(1)\times\eta(1)}\gamma^i,\end{aligned}$$

where we have used equation (3) for the second equality. We now choose an indecomposable idempotent $f_\eta = f_\eta e(\eta)$ of $L'[H]e(\eta) \simeq L'_{\eta(1)\times\eta(1)}$. Observe that for a second indecomposable idempotent f'_η of $L'[H]e(\eta)$ we have an isomorphism $f_\eta L'[H]f'_\eta \simeq L'$. As $\mathcal{Q}^E(G)e_\chi$ is a simple algebra over its center $\mathcal{Q}^E(\Gamma'_\chi)$ by Corollary 1.9, and f_η is also an idempotent in $\mathcal{Q}^E(G)e_\chi$, it suffices to show that $f_\eta\mathcal{Q}^E(G)e_\chi f_\eta$ is a field, namely $\mathcal{Q}^E(\Gamma'_\chi)$. For this, we first observe that $f_{\eta,i} := \gamma^i f_\eta \gamma^{-i}$ is also an indecomposable idempotent for every $0 \le i < p^n$, and belongs to $L'[H]e(\eta^{\gamma^i})$ as $\gamma^i e(\eta)\gamma^{-i} = e(\eta^{\gamma^i})$ and H is normal in G. However, $e(\eta) = e(\eta^{\gamma^i})$ if and only if w_χ divides i, and thus

$$f_\eta L'[H] f_{\eta,i} \simeq \begin{cases} L' & \text{if } w_\chi \mid i \\ 0 & \text{otherwise.}\end{cases}$$

Since $e(\eta^{\gamma^j})\gamma^i f_\eta = f_{\eta,i}e(\eta^{\gamma^j})\gamma^i$ we therefore have

$$f_\eta \mathcal{Q}^E(G)e_\chi f_\eta \simeq \bigoplus_{\substack{i=0\\ w_\chi \mid i}}^{p^n-1} L'\gamma^i = \bigoplus_{i=0}^{w_\chi^{-1}p^n-1} L'\gamma^{w_\chi i}.$$

We conclude that $\dim_{L'}(f_\eta\mathcal{Q}^E(G)e_\chi f_\eta) = w_\chi^{-1}p^n$. However, recall that $L' = \mathcal{Q}^E(\Gamma^{p^n})$ and γ^{p^n} identifies with $\gamma_\chi^{p^n/w_\chi} = x^{-p^n/w_\chi}(\gamma'_\chi)^{p^n/w_\chi}$ by Remark 1.8, where $x^{-p^n/w_\chi} \in U^1_{K_\chi} \subseteq (L')^\times$. As L'-vector space we therefore have a decomposition

$$\mathcal{Q}^E(\Gamma'_\chi) = \bigoplus_{i=0}^{w_\chi^{-1}p^n-1} L'(\gamma'_\chi)^i.$$

Thus we also have $\dim_{L'}(\mathcal{Q}^E(\Gamma'_\chi)) = w_\chi^{-1}p^n$. Since $\mathcal{Q}^E(\Gamma'_\chi) = \zeta(\mathcal{Q}^E(G)e_\chi)$ is contained in $f_\eta\mathcal{Q}^E(G)e_\chi f_\eta$, we are done. ∎

Corollary 1.12. *There is a finite Galois extension E of K such that*

$$\mathcal{Q}^E(G) \simeq \bigoplus_{\chi \in \mathrm{Irr}\,(G)/W} (\mathcal{Q}^E(\Gamma_\chi))_{\chi(1)\times\chi(1)},$$

where the sum runs through all $\chi \in \mathrm{Irr}\,(G)$ modulo twists with characters of type W. In particular, no skewfields occur in the Wedderburn decomposition of the semi-simple algebra $\mathcal{Q}^E(G)$.

Proof. This is an immediate consequence of Theorem 1.11 and Corollary 1.9 once we observe that $\mathcal{Q}^E(\Gamma_\chi) = \mathcal{Q}^E(\Gamma'_\chi)$ by (6) if E is sufficiently large. $\square$

Corollary 1.13. *Write $\eta(1) = s_\eta n_\eta$, where s_η denotes the Schur index of η. Then s_χ divides $s_\eta[K(\eta) : K_\chi]$ and n_η divides n_χ.*

Proof. By definition of the Schur index there is a field E of minimal degree s_η over $K(\eta)$ such that η can be realized over E. However,

$$\begin{aligned} E \otimes_K A_\chi &\simeq E \otimes_K \mathcal{Q}^{K_\chi}(\Gamma'_\chi) \otimes_{\mathcal{Q}^{K_\chi}(\Gamma'_\chi)} A_\chi \\ &\simeq \bigoplus_{\sigma \in \mathrm{Gal}(K_\chi/K)} E \otimes_{K_\chi} \mathcal{Q}^{K_\chi}(\Gamma'_\chi) \otimes_{\mathcal{Q}^{K_\chi}(\Gamma'_\chi)} A_\chi \\ &\simeq \bigoplus_{\sigma \in \mathrm{Gal}(K_\chi/K)} \mathcal{Q}^{E}(\Gamma'_\chi) \otimes_{\mathcal{Q}^{K_\chi}(\Gamma'_\chi)} A_\chi \end{aligned}$$

splits by Theorem 1.11. Now [Re03, Theorem 28.5] implies that s_χ divides

$$[\mathcal{Q}^E(\Gamma'_\chi) : \mathcal{Q}^{K_\chi}(\Gamma'_\chi)] = [E : K_\chi] = s_\eta[K(\eta) : K_\chi].$$

Moreover, we have

$$w_\chi s_\eta n_\eta = w_\chi \eta(1) = \chi(1) = n_\chi s_\chi \mid n_\chi s_\eta [K(\eta) : K_\chi] \mid n_\chi s_\eta w_\chi,$$

where the last divisibility is due to Lemma 1.1. Thus n_η divides n_χ as claimed. $\square$

Example. Assume that H is a p-group. Then $s_\eta = 1$ for all irreducible characters η of H by a result of Roquette [Ro58]. However, Lau [La12] gives examples, where $s_\chi = [K(\eta) : K_\chi]$ is non-trivial.

Let us denote the global dimension of a ring Λ by gl.dim (Λ). The following lemma is only needed to justify Remark 3.6 below and can be skipped on a first reading.

Lemma 1.14. *Let A be a separable L-algebra of finite dimension over L and let Λ be a maximal R-order in A. If A is split, then there is a maximal $\mathfrak{o}$-order Δ in Λ such that $R \otimes_{\mathfrak{o}} \Delta = \Lambda$ (and thus $L \otimes_{\mathfrak{o}} \Delta = A$).*

Proof. There are natural numbers $k > 0$ and n_i, $1 \leq i \leq k$ such that $A = \bigoplus_{i=1}^k A_i$ and $A_i \simeq L_{n_i \times n_i}$. We put $\tilde{\Lambda}_i := R_{n_i \times n_i}$ and $\tilde{\Lambda} := \bigoplus_{i=1}^k \tilde{\Lambda}_i$. Observe that this is a maximal R-order in A by [Re03, Theorem 8.7]. Then $\tilde{\Delta} := \bigoplus_{i=1}^k \mathfrak{o}_{n_i \times n_i}$ is a maximal $\mathfrak{o}$-order in $\bigoplus_{i=1}^k K_{n_i \times n_i}$ and has $R \otimes_{\mathfrak{o}} \tilde{\Delta} = \tilde{\Lambda}$ and $L \otimes_{\mathfrak{o}} \tilde{\Delta} = A$. Since global dimension is invariant under Morita equivalence (cf. [Ra69, Corollary, p. 476]), we have

$$\text{gl.dim}\ (\tilde{\Lambda}_i) = \text{gl.dim}\ (R) = 2.$$

Moreover, $\tilde{\Lambda}_i/\text{rad}(\tilde{\Lambda}_i)$ is a matrix ring over the residue field of R and thus a simple artinian ring. As likewise Λ decomposes into $\Lambda = \bigoplus_{i=1}^k \Lambda_i$, where each Λ_i is a maximal R-order in A_i, we may apply [Ra69, Theorem 5.4] in each simple component: there is a unit $a_i \in A_i$ such that $\Lambda_i = a_i^{-1}\tilde{\Lambda}_i a_i$. We put $a := \sum_{i=1}^k a_i$ so that $\Lambda = a^{-1}\tilde{\Lambda}a$. We further put $\Delta := a^{-1}\tilde{\Delta}a$. Then Δ is a maximal $\mathfrak{o}$-order in Λ, and we have

$$R \otimes_{\mathfrak{o}} \Delta = a^{-1}(R \otimes_{\mathfrak{o}} \tilde{\Delta})a = a^{-1}\tilde{\Lambda}a = \Lambda$$

as desired. □

2 Traces and conductors

Let R be a noetherian integrally closed domain with quotient field L. If A is a simple L-algebra, we denote by $\text{tr}_{A/L}$ the reduced trace from A to L. If A is separable, then by [Re03, Theorem 9.26] the reduced trace gives rise to a symmetric associative nondegenerate bilinear form $A \times A \to L$ which sends (a, b) to $\text{tr}_{A/L}(ab)$. Note that if $A = L'$ is a field, then $\text{tr}_{L'/L} = \text{Tr}_{L'/L}$ is the ordinary trace of fields. Now let Λ be an R-order in A. By a Λ-lattice we mean a finitely generated Λ-module which is torsionfree as R-module. We consider duality with respect to the trace form: For each full left Λ-lattice M in A (that is $A \otimes_\Lambda M = A$) we associate the dual right Λ-lattice

$$D(M/R) := \left\{x \in A \mid \text{tr}_{A/L}(xM) \subseteq R\right\}.$$

In particular, applying this construction to the two-sided Λ-lattice Λ we obtain a two-sided Λ-lattice $D(\Lambda/R)$ which (by abuse of language) we will call the *inverse different* of Λ with respect to R. If $D(\Lambda/R)$ happens to be invertible, we call $\mathfrak{D}(\Lambda/R) := D(\Lambda/R)^{-1}$ the *different* of Λ with respect to R.

Lemma 2.1. *Let R be a noetherian integrally closed domain with quotient field L and let A be a simple separable L-algebra. Let $L \subseteq L' \subseteq \zeta(A)$ be fields and denote the integral closure of R in L' by R'. Then for every maximal R-order Λ in A, we have*

$$D(\Lambda/R) \supseteq D(\Lambda/R')D(R'/R)$$

with equality if $D(R'/R)$ is invertible.

Proof. As Λ is also a maximal R'-order by [Re03, Theorem 10.5], we have $\Lambda = R'\Lambda$ and $D(\Lambda/R')$ is defined. Let $a \in D(\Lambda/R')$ and $b \in D(R'/R)$ be arbitrary. Then

$$\mathrm{tr}_{A/L}(ab\Lambda) = \mathrm{tr}_{L'/L}(\mathrm{tr}_{A/L'}(ab\Lambda)) = \mathrm{tr}_{L'/L}(b\mathrm{tr}_{A/L'}(a\Lambda)) \subseteq \mathrm{tr}_{L'/L}(bR') \subseteq R$$

as desired. Now assume that $D(R'/R) = \mathfrak{D}(R'/R)^{-1}$ is invertible. We conclude

$$\begin{aligned} x \in D(\Lambda/R) &\iff \mathrm{tr}_{L'/L}(\mathrm{tr}_{A/L'}(x\Lambda)) \subseteq R \\ &\iff \mathrm{tr}_{L'/L}(\mathrm{tr}_{A/L'}(xR'\Lambda)) \subseteq R \\ &\iff \mathrm{tr}_{L'/L}(R'\mathrm{tr}_{A/L'}(x\Lambda)) \subseteq R \\ &\iff \mathrm{tr}_{A/L'}(x\Lambda) \subseteq D(R'/R) \\ &\iff \mathrm{tr}_{A/L'}(x\mathfrak{D}(R'/R)\Lambda) \subseteq R' \\ &\iff x\mathfrak{D}(R'/R) \subseteq D(\Lambda/R') \\ &\iff x \in D(\Lambda/R')D(R'/R). \end{aligned}$$

□

Now let A be a semisimple L-algebra and let Tr be the ordinary trace map from A to L. For every $x \in A$ we associate the homomorphism $\mathrm{Tr}_x \in \mathrm{Hom}_L(A, L)$ defined by $\mathrm{Tr}_x(a) := \mathrm{Tr}(xa)$ for all $a \in A$. Let Λ be an R-order in A. For a full left Λ-lattice M in A we may also consider duality with respect to the ordinary trace form:

$$D_{\mathrm{ord}}(M/R) := \{x \in A \mid \mathrm{Tr}(xM) \subseteq R\} = \{x \in A \mid \mathrm{Tr}_x(M) \subseteq R\}$$

is a right Λ-lattice in A. We have a canonical homomorphism of right Λ-modules

$$\delta_M : D_{\mathrm{ord}}(M/R) \to M^+ := \mathrm{Hom}_R(M, R), \quad x \mapsto \mathrm{Tr}_x|_M, \tag{8}$$

where $\lambda \in \Lambda$ acts on $f \in M^+$ as $f^\lambda(m) := f(\lambda m)$ for all $m \in M$. Similar observations hold for full right Λ-lattices in A.
Recall that an R-module M is called *reflexive* if the canonical map $M \to M^{++}$, $m \mapsto [f \mapsto f(m)]$ is an isomorphism.

Proposition 2.2. *Let R be a noetherian integrally closed domain with quotient field L. Let A be a semisimple L-algebra and let Λ be an R-order in A. If the ordinary trace Tr gives rise to a nondegenerate bilinear form $A \times A \to L$, $(a, b) \mapsto \mathrm{Tr}(ab)$, then the homomorphism δ_M in (8) is an isomorphism for every full Λ-lattice M in A. In particular, $D_{\mathrm{ord}}(D_{\mathrm{ord}}(M/R)/R) = M$ if and only if M is reflexive.*

Proof. As the bilinear form $A \times A \to L$, $(a, b) \mapsto \mathrm{Tr}(ab)$ is nondegenerate, we have $\mathrm{Tr}_x = \mathrm{Tr}_y$ if and only if $x = y$. It follows that the map $\delta_A : A \to \mathrm{Hom}_L(A, L)$, $x \mapsto \mathrm{Tr}_x$ is injective and thus an isomorphism of L-vector spaces as both sides have the same dimension over L. It is also a homomorphism of right A-modules. Now let M be a full left Λ-lattice in A. Then δ_M is just the restriction of δ_A to $D_{\mathrm{ord}}(M/R)$ and thus injective. However, by [NSW08, Remark p.268] restriction to M yields an isomorphism

$$\begin{array}{rcl} M^+ & \simeq & \{\phi \in \mathrm{Hom}_L(A, L) \mid \phi(M) \subseteq R\} \\ & = & \{\mathrm{Tr}_x \mid x \in A,\ \mathrm{Tr}_x(M) \subseteq R\} \\ & = & \{\mathrm{Tr}_x \mid x \in D_{\mathrm{ord}}(M/R)\}. \end{array}$$

Hence the image of δ_M is M^+ as claimed. $\square$

We now return to the case $L = Quot(R)$, where $R = \mathfrak{o}[[T]]$ is the power series ring in one variable over the ring of integers $\mathfrak{o}$ in K. Recall that as an L-vector space we have $\mathcal{Q}^K(G) = \bigoplus_{i=0}^{p^n-1} L[H]\gamma^i = \bigoplus_{i=0}^{p^n-1} L\gamma^i[H]$. We denote by Tr the ordinary trace map from $\mathcal{Q}^K(G)$ to L.

LEMMA 2.3. *The elements $\gamma^i h$, $0 \leq i < p^n$, $h \in H$ form an L-basis of $\mathcal{Q}^K(G)$ and*

$$\mathrm{Tr}\,(\gamma^i h) = \begin{cases} p^n|H| & \text{if } \gamma^i = h = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Its dual basis with respect to Tr *is given by $(p^n|H|)^{-1}h^{-1}\gamma^{-i}$, $0 \leq i < p^n$, $h \in H$.*

Proof. It is clear that these elements form an L-basis of $\mathcal{Q}^K(G)$. Now let $0 \leq i < p^n$ and $h \in H$. We use the same basis to compute $\mathrm{Tr}\,(\gamma^i h)$. For every $0 \leq j < p^n$ and $h' \in H$, we have to write $\gamma^i h \gamma^j h'$ as an L-linear combination of these basis elements. However, $\gamma^i h \gamma^j h' = \gamma^{i+j} h_j h'$ with $h_j := \gamma^{-j} h \gamma^j \in H$. This actually belongs to this basis if $i + j < p^n$; if $i + j \geq p^n$, then $\gamma^{i+j} h_j h' = \gamma^{p^n} \gamma^{i+j-p^n} h_j h'$ with $\gamma^{p^n} \in L = \mathcal{Q}^K(\Gamma^{p^n})$. Since only the diagonal entries of the corresponding $p^n|H| \times p^n|H|$ matrix contribute to $\mathrm{Tr}\,(\gamma^i h)$, we now suppose that $\gamma^j h' = \gamma^{i+j} h_j h'$ if $i + j < p^n$, and that $\gamma^j h' = \gamma^{i+j-p^n} h_j h'$ if $i + j \geq p^n$. In both cases we must have $h_j = 1$ and thus $h = 1$. In the first case, we also have $j = i + j$ which forces $i = 0$, whereas in the second case $j = i + j - p^n$ which gives $i = p^n$. This contradicts $i < p^n$ and so the latter case does not occur. We see that this matrix has all diagonal entries equal to zero if $\gamma^i h \neq 1$; hence $\mathrm{Tr}\,(\gamma^i h) = 0$ in this case. That $\mathrm{Tr}\,(1) = p^n|H|$ is clear. It is now easily checked that $(p^n|H|)^{-1}h^{-1}\gamma^{-i}$, $0 \leq i < p^n$, $h \in H$ is the dual basis. $\square$

LEMMA 2.4. *Let K' be a finite field extension of K with ring of integers $\mathfrak{o}'$ and let a be a principal unit in some finite extension of K'. Let $n \in \mathbb{N}_0$ and suppose that a^{p^n} belongs to $U^1_{K'}$. Consider $\Lambda^{\mathfrak{o}'}(\Gamma)$ as R-order via the embedding*

$$\iota_{a,n} : R \rightarrowtail \Lambda^{\mathfrak{o}'}(\Gamma), \quad 1 + T \mapsto (a\gamma)^{p^n}.$$

Then the inverse different

$$D(\Lambda^{\mathfrak{o}'}(\Gamma)/R) = \left\{ x \in \mathcal{Q}^{K'}(\Gamma) \mid \mathrm{Tr}_{\mathcal{Q}^{K'}(\Gamma)/L}(x\Lambda^{\mathfrak{o}'}(\Gamma)) \subseteq R \right\}$$

is given by

$$D(\Lambda^{\mathfrak{o}'}(\Gamma)/R) = p^{-n}\mathfrak{D}^{-1}(\mathfrak{o}'/\mathfrak{o})\Lambda^{\mathfrak{o}'}(\Gamma),$$

where $\mathfrak{D}^{-1}(\mathfrak{o}'/\mathfrak{o})$ *denotes the usual inverse different of* $\mathfrak{o}'$ *with respect to* $\mathfrak{o}$*. In particular,* $D(\Lambda^{\mathfrak{o}'}(\Gamma)/R)$ *is invertible.*

Proof. First, we consider the case $a = 1$ and $n = 0$. If $x_1, \ldots, x_k$ form an $\mathfrak{o}$-basis of $\mathfrak{o}'$, then $x_1, \ldots, x_k$ are also an $\mathfrak{o}[[T]]$-basis of $\mathfrak{o}'[[T]]$ which is isomorphic to $\Lambda^{\mathfrak{o}'}(\Gamma)$ via $1 + T \mapsto \gamma$. Hence its dual basis with respect to the ordinary trace $\mathrm{Tr}_{K'/K}$ of fields, is also a dual basis with respect to $\mathrm{Tr}_{\mathcal{Q}^{K'}(\Gamma)/L}$. Hence $D(\Lambda^{\mathfrak{o}'}(\Gamma)/R)$ equals $\mathfrak{D}^{-1}(\mathfrak{o}'/\mathfrak{o})\Lambda^{\mathfrak{o}'}(\Gamma)$ and is invertible in this case. For the general case let $\iota'_{a,n} : \Lambda^{\mathfrak{o}'}(\Gamma) \rightarrowtail \Lambda^{\mathfrak{o}'}(\Gamma)$ be induced from $\gamma \mapsto (a\gamma)^{p^n}$. Then $\iota_{a,n} = \iota'_{a,n} \circ \iota_{1,0}$. We may therefore assume $\mathfrak{o} = \mathfrak{o}'$ by the first part of the proof and an application of Lemma 2.1. The inverse different only depends upon the image of R in $\Lambda^{\mathfrak{o}}(\Gamma)$. However, $\iota_{a,n}(R) = \iota_{1,n}(R)$ as $a^{p^n} \in U^1_{K'}$. Thus we may also assume $a = 1$. But in this case the result follows from Lemma 2.3 with $G = \Gamma$. □

By Corollary 1.9 we may write $\mathcal{Q}^K(G) = \bigoplus_{\chi \in \mathrm{Irr}(G)/\sim_K} A_\chi$, where $A_\chi = (D_\chi)_{n_\chi \times n_\chi}$, $n_\chi \in \mathbb{N}$ and D_χ is a skewfield with Schur index s_χ and center $\mathcal{Q}^{K_\chi}(\Gamma'_\chi)$. Again by Corollary 1.9, we have $\chi(1) = s_\chi n_\chi$. Hence the ordinary trace may be written as

$$\mathrm{Tr} = \sum_{\chi \in \mathrm{Irr}(G)/\sim_K} \chi(1)\mathrm{tr}_\chi, \tag{9}$$

where tr_χ denotes the reduced trace from A_χ to L. Moreover, we have

$$\mathrm{tr}_\chi = \mathrm{Tr}_{\mathcal{Q}^{K_\chi}(\Gamma'_\chi)/L} \circ \mathrm{tr}_{A_\chi/\mathcal{Q}^{K_\chi}(\Gamma'_\chi)}, \tag{10}$$

where $\mathrm{Tr}_{\mathcal{Q}^{K_\chi}(\Gamma'_\chi)/L}$ denotes the ordinary trace of fields and $\mathrm{tr}_{A_\chi/\mathcal{Q}^{K_\chi}(\Gamma'_\chi)}$ denotes the reduced trace from A_χ into its center. Recall from Remark 1.8 that we have fixed a sufficiently large integer $n \geq 0$ such that $R = \mathfrak{o}[[T]]$ embeds into $\Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)$ via $1 + T \mapsto (x^{-1}\gamma'_\chi)^{p^n/w_\chi}$. Now Lemma 2.1 and Lemma 2.4 (with the embedding $\iota_{a,m}$, where $a = x^{-1}$ and $p^m = p^n/w_\chi$) imply that the following definition does not depend on the choice of the embedding.

Definition 2.5. *Choose a maximal* R*-order* $\tilde{\Lambda}^{\mathfrak{o}}(G)$ *containing* $\Lambda^{\mathfrak{o}}(G)$*. We have a decomposition* $\tilde{\Lambda}^{\mathfrak{o}}(G) = \bigoplus_{\chi \in \mathrm{Irr}(G)/\sim_K} \tilde{\Lambda}^{\mathfrak{o}}_\chi(G)$*, where each* $\tilde{\Lambda}^{\mathfrak{o}}_\chi(G)$ *is a maximal* R*-order in* A_χ*. For sufficiently large* n *we call the two-sided* $\tilde{\Lambda}^{\mathfrak{o}}_\chi(G)$*-lattice*

$$D_\chi(\tilde{\Lambda}^{\mathfrak{o}}(G)) = D_{\mathrm{norm}}(\tilde{\Lambda}^{\mathfrak{o}}_\chi(G)/R) := p^n D(\tilde{\Lambda}^{\mathfrak{o}}_\chi(G)/R) = \\ = p^n \cdot \left\{ x \in A_\chi \mid \mathrm{tr}_\chi(x\tilde{\Lambda}^{\mathfrak{o}}_\chi(G)) \subseteq R \right\}$$

the normalized inverse different.

Definition 2.6. *Let $\Lambda \subseteq \tilde{\Lambda}$ be a pair of rings. Then*

$$(\tilde{\Lambda} : \Lambda)_l := \left\{ x \in \tilde{\Lambda} \mid x\tilde{\Lambda} \subseteq \Lambda \right\}$$

is called the left conductor *of $\tilde{\Lambda}$ into Λ. Similarly,*

$$(\tilde{\Lambda} : \Lambda)_r := \left\{ x \in \tilde{\Lambda} \mid \tilde{\Lambda}x \subseteq \Lambda \right\}$$

is called the right conductor *of $\tilde{\Lambda}$ into Λ.*

Proposition 2.7. *Let $\tilde{\Lambda}^o(G)$ be a maximal R-order containing $\Lambda^o(G)$. Then $\Lambda^o(G)$, the maximal order $\tilde{\Lambda}^o(G)$ as well as the left conductor $(\tilde{\Lambda}^o(G) : \Lambda^o(G))_l$ and the right conductor $(\tilde{\Lambda}^o(G) : \Lambda^o(G))_r$ are reflexive (and thus free) R-modules.*

Proof. We first observe that R is a 2-dimensional regular local ring and thus every reflexive R-module is free (and vice versa) by [NSW08, Proposition 5.1.9]. As $\Lambda^o(G)$ is free over R, it is also reflexive. Moreover, every maximal R-order is reflexive by [Re03, Theorem 11.4]. Finally, let $M := (\tilde{\Lambda}^o(G) : \Lambda^o(G))_r$ (the argument for $(\tilde{\Lambda}^o(G) : \Lambda^o(G))_l$ is similar). We denote by $P(R)$ the set of prime ideals of R of height 1. As M is R-torsionfree, the proof of [NSW08, Proposition 5.1.8] shows that we have an injection $M \rightarrowtail M^{++} = \bigcap_{\mathfrak{p} \in P(R)} M_{\mathfrak{p}}$, where the equality is [NSW08, Lemma 5.1.2]. Now let $x \in \bigcap_{\mathfrak{p} \in P(R)} M_{\mathfrak{p}}$ be arbitrary; we have to show that $x \in M$. For every $\mathfrak{p} \in P(R)$ we have $x \in \tilde{\Lambda}^o(G)_{\mathfrak{p}}$ and $\tilde{\Lambda}^o(G)_{\mathfrak{p}} x \subseteq \Lambda^o(G)_{\mathfrak{p}}$. Thus $x \in \bigcap_{\mathfrak{p} \in P(R)} \tilde{\Lambda}^o(G)_{\mathfrak{p}} = \tilde{\Lambda}^o(G)$ and

$$\tilde{\Lambda}^o(G)x = \bigcap_{\mathfrak{p} \in P(R)} \tilde{\Lambda}^o(G)_{\mathfrak{p}} x \subseteq \bigcap_{\mathfrak{p} \in P(R)} \Lambda^o(G)_{\mathfrak{p}} = \Lambda^o(G),$$

that is $x \in M$. ∎

We now establish the following analogue of Jacobinski's conductor formula [CR81, Theorem 27.8].

Theorem 2.8. *Let $\tilde{\Lambda}^o(G)$ be a maximal R-order containing $\Lambda^o(G)$. Then*

$$(\tilde{\Lambda}^o(G) : \Lambda^o(G))_l = (\tilde{\Lambda}^o(G) : \Lambda^o(G))_r = \bigoplus_{\chi \in \mathrm{Irr}\,(G)/\sim_K} \frac{|H|}{\chi(1)} D_\chi(\tilde{\Lambda}^o(G)).$$

Proof. Put $M := (\tilde{\Lambda}^o(G) : \Lambda^o(G))_r$; then M is the largest left $\tilde{\Lambda}^o(G)$-lattice in $\Lambda^o(G)$. So its ordinary dual $D_{\mathrm{ord}}\,(M/R)$ is the smallest right $\tilde{\Lambda}^o(G)$-lattice containing $D_{\mathrm{ord}}\,(\Lambda^o(G)/R)$. Since $\Lambda^o(G)$ is free over R with basis $\gamma^i h$, $0 \leq i < p^n$, $h \in H$, we therefore have

$$D_{\mathrm{ord}}\,(M/R) = D_{\mathrm{ord}}\,(\Lambda^o(G)/R)\tilde{\Lambda}^o(G) = (p^n|H|)^{-1}\tilde{\Lambda}^o(G),$$

where we have used Lemma 2.3 for the second equality. However, M is reflexive by Proposition 2.7 and so Proposition 2.2 yields

$$M = D_{\mathrm{ord}}\,(D_{\mathrm{ord}}\,(M/R)/R) = \\ = D_{\mathrm{ord}}\,((p^n|H|)^{-1}\tilde{\Lambda}^{\mathfrak{o}}(G)/R) = p^n|H|D_{\mathrm{ord}}\,(\tilde{\Lambda}^{\mathfrak{o}}(G)/R).$$

The theorem now follows from the equalities

$$\begin{aligned} D_{\mathrm{ord}}\,(\tilde{\Lambda}^{\mathfrak{o}}(G)/R) &= \bigoplus_{\chi\in \mathrm{Irr}\,(G)/\sim_K} D_{\mathrm{ord}}\,(\tilde{\Lambda}^{\mathfrak{o}}_{\chi}(G)/R) \\ D_{\mathrm{ord}}\,(\tilde{\Lambda}^{\mathfrak{o}}_{\chi}(G)/R) &= \chi(1)^{-1}D(\tilde{\Lambda}^{\mathfrak{o}}_{\chi}(G)/R) \\ &= (\chi(1)p^n)^{-1}D_{\chi}(\tilde{\Lambda}^{\mathfrak{o}}(G)), \end{aligned}$$

as a similar argument works for $(\tilde{\Lambda}^{\mathfrak{o}}(G) : \Lambda^{\mathfrak{o}}(G))_l$. ◻

3 A formula for the central conductor

Definition 3.1. *Let $\tilde{\Lambda}^{\mathfrak{o}}(G)$ be a maximal R-order containing $\Lambda^{\mathfrak{o}}(G)$. Then the* central conductor *of $\tilde{\Lambda}^{\mathfrak{o}}(G)$ into $\Lambda^{\mathfrak{o}}(G)$ is defined to be*

$$\begin{aligned} \mathcal{F}(\Lambda^{\mathfrak{o}}(G)) = \mathcal{F}(\tilde{\Lambda}^{\mathfrak{o}}(G)/\Lambda^{\mathfrak{o}}(G)) &:= \zeta(\Lambda^{\mathfrak{o}}(G)) \cap (\tilde{\Lambda}^{\mathfrak{o}}(G) : \Lambda^{\mathfrak{o}}(G))_l \\ &= \zeta(\Lambda^{\mathfrak{o}}(G)) \cap (\tilde{\Lambda}^{\mathfrak{o}}(G) : \Lambda^{\mathfrak{o}}(G))_r \\ &= \left\{ x \in \zeta(\Lambda^{\mathfrak{o}}(G)) \mid x\tilde{\Lambda}^{\mathfrak{o}}(G) \subseteq \Lambda^{\mathfrak{o}}(G) \right\}. \end{aligned}$$

Remark 3.2. *We will see below that the central conductor only depends on $\Lambda^{\mathfrak{o}}(G)$ and not on the choice of maximal order containing $\Lambda^{\mathfrak{o}}(G)$. Hence the notation $\mathcal{F}(\Lambda^{\mathfrak{o}}(G))$ is justified.*

By Corollary 1.9 we may write $\mathcal{Q}^K(G) = \bigoplus_{\chi\in \mathrm{Irr}\,(G)/\sim_K} A_{\chi}$, where each $A_{\chi} \simeq (D_{\chi})_{n_{\chi}\times n_{\chi}}$ is simple. Similarly, $\tilde{\Lambda}^{\mathfrak{o}}(G)$ decomposes into $\bigoplus_{\chi\in \mathrm{Irr}\,(G)/\sim_K} \tilde{\Lambda}^{\mathfrak{o}}_{\chi}(G)$, where each $\tilde{\Lambda}^{\mathfrak{o}}_{\chi}(G)$ is a maximal R-order in A_{χ} with center $\Lambda^{\mathfrak{o}_{\chi}}(\Gamma'_{\chi})$. The central conductor is an ideal in $\zeta(\tilde{\Lambda}^{\mathfrak{o}}(G)) = \bigoplus_{\chi\in \mathrm{Irr}\,(G)/\sim_K} \Lambda^{\mathfrak{o}_{\chi}}(\Gamma'_{\chi})$ and so we may write

$$\mathcal{F}(\tilde{\Lambda}^{\mathfrak{o}}(G)/\Lambda^{\mathfrak{o}}(G)) = \bigoplus_{\chi\in \mathrm{Irr}\,(G)/\sim_K} \mathcal{F}_{\chi}(\tilde{\Lambda}^{\mathfrak{o}}(G)/\Lambda^{\mathfrak{o}}(G)),$$

where each $\mathcal{F}_{\chi}(\tilde{\Lambda}^{\mathfrak{o}}(G)/\Lambda^{\mathfrak{o}}(G))$ is an ideal in $\Lambda^{\mathfrak{o}_{\chi}}(\Gamma'_{\chi})$. We define

$$d_{\chi} := \mathcal{Q}^{K_{\chi}}(\Gamma'_{\chi}) \cap D(\tilde{\Lambda}^{\mathfrak{o}}_{\chi}(G)/\Lambda^{\mathfrak{o}_{\chi}}(\Gamma'_{\chi})).$$

We denote by P_{χ} the set of prime ideals of $\Lambda^{\mathfrak{o}_{\chi}}(\Gamma'_{\chi})$ of height 1.

Lemma 3.3. *The fractional $\Lambda^{\mathfrak{o}_{\chi}}(\Gamma'_{\chi})$-ideal d_{χ} is a reflexive $\Lambda^{\mathfrak{o}_{\chi}}(\Gamma'_{\chi})$-module which does not depend on the choice of maximal order $\tilde{\Lambda}^{\mathfrak{o}}(G)$.*

Proof. We first show that d_χ is reflexive. As it is torsionfree, we have an injection $d_\chi \rightarrowtail d_\chi^{++} = \bigcap_{\mathfrak{p}\in P_\chi}(d_\chi)_\mathfrak{p}$. Now let $x \in \bigcap_{\mathfrak{p}\in P_\chi}(d_\chi)_\mathfrak{p}$. Then clearly $x \in \mathcal{Q}^{K_\chi}(\Gamma'_\chi)$ and

$$\mathrm{tr}_{A_\chi/\mathcal{Q}^{K_\chi}(\Gamma'_\chi)}(x\tilde{\Lambda}^\mathfrak{o}_\chi(G)_\mathfrak{p}) \subseteq \Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)_\mathfrak{p}$$

for all $\mathfrak{p} \in P_\chi$. This implies

$$\begin{aligned} \mathrm{tr}_{A_\chi/\mathcal{Q}^{K_\chi}(\Gamma'_\chi)}(x\tilde{\Lambda}^\mathfrak{o}_\chi(G)) &\subseteq \bigcap_{\mathfrak{p}\in P_\chi} \mathrm{tr}_{A_\chi/\mathcal{Q}^{K_\chi}(\Gamma'_\chi)}(x\tilde{\Lambda}^\mathfrak{o}_\chi(G)_\mathfrak{p}) \\ &\subseteq \bigcap_{\mathfrak{p}\in P_\chi} \Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)_\mathfrak{p} \\ &= \Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi). \end{aligned}$$

Thus x belongs to $\mathcal{Q}^{K_\chi}(\Gamma'_\chi) \cap D(\tilde{\Lambda}^\mathfrak{o}_\chi(G)/\Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)) = d_\chi$ as claimed.
Now let $\breve{\Lambda}^\mathfrak{o}(G)$ be a second maximal R-order containing $\Lambda^\mathfrak{o}(G)$. Then likewise $\breve{\Lambda}^\mathfrak{o}(G)$ decomposes into $\bigoplus_{\chi\in \mathrm{Irr}\,(G)/\sim_K} \breve{\Lambda}^\mathfrak{o}_\chi(G)$, where each $\breve{\Lambda}^\mathfrak{o}_\chi(G)$ is a maximal R-order in A_χ. We put

$$\breve{d}_\chi := \mathcal{Q}^{K_\chi}(\Gamma'_\chi) \cap D(\breve{\Lambda}^\mathfrak{o}_\chi(G)/\Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)).$$

Fix a height 1 prime ideal $\mathfrak{p} \in P_\chi$. By [AG60, Proposition 3.5] there is a unit $a_\chi \in A_\chi$ such that $\breve{\Lambda}^\mathfrak{o}_\chi(G)_\mathfrak{p} = a_\chi\tilde{\Lambda}^\mathfrak{o}_\chi(G)_\mathfrak{p}a_\chi^{-1}$. Now let $x \in \mathcal{Q}^{K_\chi}(\Gamma'_\chi)$. Then

$$\begin{aligned} \mathrm{tr}_{A_\chi/\mathcal{Q}^{K_\chi}(\Gamma'_\chi)}(x\breve{\Lambda}^\mathfrak{o}_\chi(G)_\mathfrak{p}) &= \mathrm{tr}_{A_\chi/\mathcal{Q}^{K_\chi}(\Gamma'_\chi)}(xa_\chi\tilde{\Lambda}^\mathfrak{o}_\chi(G)_\mathfrak{p}a_\chi^{-1}) \\ &= \mathrm{tr}_{A_\chi/\mathcal{Q}^{K_\chi}(\Gamma'_\chi)}(a_\chi x\tilde{\Lambda}^\mathfrak{o}_\chi(G)_\mathfrak{p}a_\chi^{-1}) \\ &= \mathrm{tr}_{A_\chi/\mathcal{Q}^{K_\chi}(\Gamma'_\chi)}(x\tilde{\Lambda}^\mathfrak{o}_\chi(G)_\mathfrak{p}), \end{aligned}$$

where the second equality holds, since x is central in A_χ. Hence

$$\mathrm{tr}_{A_\chi/\mathcal{Q}^{K_\chi}(\Gamma'_\chi)}(x\breve{\Lambda}^\mathfrak{o}_\chi(G)_\mathfrak{p}) \subseteq \Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)_\mathfrak{p} \Longleftrightarrow \mathrm{tr}_{A_\chi/\mathcal{Q}^{K_\chi}(\Gamma'_\chi)}(x\tilde{\Lambda}^\mathfrak{o}_\chi(G)_\mathfrak{p}) \subseteq \Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)_\mathfrak{p},$$

that is $(d_\chi)_\mathfrak{p} = (\breve{d}_\chi)_\mathfrak{p}$ for all $\mathfrak{p} \in P_\chi$. As both d_χ and $\breve{d}_\chi$ are reflexive, this implies $d_\chi = \breve{d}_\chi$ as desired. □

Let π_χ be a prime element in $\mathfrak{o}_\chi$ and put $\mathfrak{p}_\chi := \pi_\chi\Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi) \in P_\chi$.

Lemma 3.4. *There is an integer $r_\chi \le 0$ such that $d_\chi = \mathfrak{p}_\chi^{r_\chi}$. If the Schur index s_χ is not divisible by p, then $r_\chi = 0$ and thus $d_\chi = \Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)$.*

Proof. Let Δ_χ be a maximal order in D_χ. Then $(\Delta_\chi)_{n_\chi\times n_\chi}$ is a maximal order in A_χ by [Re03, Theorem 8.7]. Let $\alpha \in (\Delta_\chi)_{n_\chi\times n_\chi}$ be the matrix with a 1 in the upper left corner and zeros everywhere else. Let $x \in d_\chi$. Then in particular

$$\mathrm{tr}_{A_\chi/\mathcal{Q}^{K_\chi}(\Gamma'_\chi)}(x\alpha) = x \cdot \mathrm{tr}_{A_\chi/\mathcal{Q}^{K_\chi}(\Gamma'_\chi)}(\alpha) = x \cdot s_\chi \in \Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi). \tag{11}$$

Here, the first equality follows from linearity of the reduced trace as x belongs to $\mathcal{Q}^{K_\chi}(\Gamma'_\chi)$. For the second equality we compute

$$\mathrm{tr}_{A_\chi/\mathcal{Q}^{K_\chi}(\Gamma'_\chi)}(\alpha) = \mathrm{tr}_{D_\chi/\mathcal{Q}^{K_\chi}(\Gamma'_\chi)}(1) = s_\chi.$$

As s_χ is an integer, it follows from (11) that $x \in \Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)_\mathfrak{p}$ for every $\mathfrak{p} \in P_\chi$, $\mathfrak{p} \neq \mathfrak{p}_\chi$. However, d_χ is reflexive by Lemma 3.3 and thus we must have $d_\chi = \mathfrak{p}_\chi^{r_\chi}$ for some integer r_χ. Since obviously $\Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi) \subseteq d_\chi$, we have $r_\chi \leq 0$. Finally, if p does not divide s_χ, then by (11) we have $x \in \Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)_\mathfrak{p}$ for every $\mathfrak{p} \in P_\chi$ including $\mathfrak{p}_\chi$ and thus $r_\chi = 0$ in this case. $\square$

THEOREM 3.5. *Let $\tilde{\Lambda}^\mathfrak{o}(G)$ be a maximal order containing $\Lambda^\mathfrak{o}(G)$. Then $\mathcal{F}(\tilde{\Lambda}^\mathfrak{o}(G)/\Lambda^\mathfrak{o}(G))$ does not depend on the choice of maximal order $\tilde{\Lambda}^\mathfrak{o}(G)$ and we have an equality*

$$\mathcal{F}(\Lambda^\mathfrak{o}(G)) = \mathcal{F}(\tilde{\Lambda}^\mathfrak{o}(G)/\Lambda^\mathfrak{o}(G)) = \bigoplus_{\chi \in \mathrm{Irr}\,(G)/\sim_K} \frac{|H| w_\chi}{\chi(1)} \cdot \mathfrak{D}^{-1}(\mathfrak{o}_\chi/\mathfrak{o}) d_\chi.$$

Here, $d_\chi = \mathfrak{p}_\chi^{r_\chi}$ for some integer $r_\chi \leq 0$. In particular, we have an inclusion

$$\frac{|H| w_\chi}{\chi(1)} \cdot \mathfrak{D}^{-1}(\mathfrak{o}_\chi/\mathfrak{o}) \Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi) \subseteq \mathcal{F}_\chi(\tilde{\Lambda}^\mathfrak{o}(G)/\Lambda^\mathfrak{o}(G)), \tag{12}$$

which becomes an equality after localization at every $\mathfrak{p} \in P_\chi$ different from $\mathfrak{p}_\chi$. Moreover, the inclusion (12) is an equality in each of the following cases:

(i) $G = H \times \Gamma$ is a direct product.

(ii) The Schur index s_χ is not divisible by p.

(iii) There is a maximal $\mathfrak{o}_\chi$-order Δ_χ contained in $\tilde{\Lambda}^\mathfrak{o}_\chi(G)$ such that $\Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi) \otimes_{\mathfrak{o}_\chi} \Delta_\chi = \tilde{\Lambda}^\mathfrak{o}_\chi(G)$.

REMARK 3.6. *Fix a p-adic Lie-group G of dimension 1. Then (ii) and (iii) hold for all χ whenever K is sufficiently large. This follows from Theorem 1.11 and Lemma 1.14.*

REMARK 3.7. *If p does not divide the order of the commutator subgroup of G, then $\Lambda^\mathfrak{o}(G)$ is a direct sum of matrix rings over commutative rings by [JN13, Proposition 4.5]. In particular, no skewfields occur in the Wedderburn decomposition of $\mathcal{Q}^K(G)$ and thus (ii) holds for every $\chi \in \mathrm{Irr}\,(G)$.*

Proof of Theorem 3.5. Let us define

$$\delta_\chi^{-1} := p^n \mathfrak{D}^{-1}(\Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)/R) = w_\chi \cdot \mathfrak{D}^{-1}(\mathfrak{o}_\chi/\mathfrak{o}) \Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi),$$

where the second equality follows from Lemma 2.4 and Remark 1.8.

LEMMA 3.8. *We have an equality $D_\chi(\tilde{\Lambda}^\mathfrak{o}(G)) = D(\tilde{\Lambda}^\mathfrak{o}_\chi(G)/\Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)) \cdot \delta_\chi^{-1}$.*

Proof. This is a special case of Lemma 2.1. $\square$

By Theorem 2.8 and the definition of the central conductor we obtain

$$\mathcal{F}_\chi(\tilde{\Lambda}^{\mathfrak{o}}(G)/\Lambda^{\mathfrak{o}}(G)) = \Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi) \cap \left(\frac{|H|}{\chi(1)} \cdot D_\chi(\tilde{\Lambda}^{\mathfrak{o}}(G))\right)$$

for each $\chi \in \mathrm{Irr}\,(G)$. Hence it must be shown that

$$\Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi) \cap \left(\frac{|H|}{\chi(1)} \cdot D_\chi(\tilde{\Lambda}^{\mathfrak{o}}(G))\right) = \frac{|H|}{\chi(1)}\delta_\chi^{-1} d_\chi \tag{13}$$

for each character χ. We note that

$$\frac{|H|}{\chi(1)}\delta_\chi^{-1} d_\chi \subseteq \frac{|H|}{\chi(1)} D_\chi(\tilde{\Lambda}^{\mathfrak{o}}(G)) \subseteq \tilde{\Lambda}^{\mathfrak{o}}(G);$$

so each element of $\frac{|H|}{\chi(1)}\delta_\chi^{-1} d_\chi$ is integral over R, and thus lies in $\Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)$. This gives one inclusion in (13). Now let $y \in \Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)$. Then by Lemma 3.8 we have

$$y \in \frac{|H|}{\chi(1)} \cdot D_\chi(\tilde{\Lambda}^{\mathfrak{o}}(G)) \iff y\delta_\chi \subseteq \frac{|H|}{\chi(1)} D(\tilde{\Lambda}^{\mathfrak{o}}_\chi(G)/\Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)) \iff y\delta_\chi \subseteq \frac{|H|}{\chi(1)} d_\chi$$

giving the reverse inclusion in (13). Lemma 3.3 and Lemma 3.4 imply the remaining assertions apart from the claim that $r_\chi = 0$ in case (i) and (iii).
Let us first assume that $G = H \times \Gamma$ is a direct product. Choose a maximal $\mathfrak{o}$-order $\mathfrak{M}(H)$ containing $\mathfrak{o}[H]$. Let $\chi \in \mathrm{Irr}\,(G)$. Then $\mathrm{res}^G_H \chi = \eta$ is an irreducible character of H and there is a character ρ of type W such that $\chi \otimes \rho$ is trivial on Γ. As $\chi \sim_K \chi \otimes \rho$, we may henceforth assume that $\rho = 1$. We then have $w_\chi = 1$, $K_\chi = K(\eta)$, $e_\chi = e(\eta)$ and $\gamma_\chi = \gamma e_\chi = \gamma'_\chi$. Then $\mathfrak{M}(H)\varepsilon_\chi$ is a maximal $\mathfrak{o}_\chi$-order such that $\tilde{\Lambda}^{\mathfrak{o}}_\chi(G) := \Lambda^{\mathfrak{o}_\chi}(\Gamma) \otimes_{\mathfrak{o}_\chi} \mathfrak{M}(H)\varepsilon_\chi$ is a maximal R-order in A_χ. It follows that (i) is a special case of (iii).
Now assume that (iii) holds and let $y \in (d_\chi)_{\mathfrak{p}_\chi}$ be arbitrary. We have to show that $y \in \Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)_{\mathfrak{p}_\chi}$. As the reduced trace is $\mathcal{Q}^{K_\chi}(\Gamma'_\chi)$-linear, we may alter y by a unit in $\Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)_{\mathfrak{p}_\chi}$ so that we may assume that $y = \pi_\chi^k$ for an appropriate integer k; in particular, we then have $y \in K_\chi$. By hypothesis there is a maximal $\mathfrak{o}_\chi$-order Δ_χ contained in $\Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)$ such that $\mathcal{Q}^{K_\chi}(\Gamma'_\chi) \otimes_{\mathfrak{o}_\chi} \Delta_\chi = A_\chi$. Hence an $\mathfrak{o}_\chi$-basis of Δ_χ is also a $\mathcal{Q}^{K_\chi}(\Gamma'_\chi)$-basis of A_χ which we may use to compute the reduced trace. Hence $y \in K_\chi$ has

$$\mathrm{tr}_{K_\chi \otimes_{\mathfrak{o}_\chi} \Delta_\chi / K_\chi}(y\Delta_\chi) \subseteq \Lambda^{\mathfrak{o}_\chi}(\Gamma'_\chi)_{\mathfrak{p}_\chi} \cap K_\chi = \mathfrak{o}_\chi$$

and we have to show that $y \in \mathfrak{o}_\chi$; in other words we shall show that $K_\chi \cap \mathfrak{D}^{-1}(\Delta_\chi/\mathfrak{o}_\chi) = \mathfrak{o}_\chi$. Suppose that this is not true. Then $\mathfrak{o}_\chi$ is properly contained in $K_\chi \cap \mathfrak{D}^{-1}(\Delta_\chi/\mathfrak{o}_\chi)$. Then $\mathfrak{p}_\chi^{-1} \subseteq \mathfrak{D}^{-1}(\Delta_\chi/\mathfrak{o}_\chi)$ and thus $\mathfrak{p}_\chi\Delta_\chi$ contains $\mathfrak{D}(\Delta_\chi/\mathfrak{o}_\chi)$. However, if $\mathfrak{P}_\chi$ denotes the radical of Δ_χ, then $\mathfrak{p}_\chi\Delta_\chi = \mathfrak{P}_\chi^e$ for some positive integer e and $\mathfrak{D}(\Delta_\chi/\mathfrak{o}_\chi) = \mathfrak{P}_\chi^{e-1}$ by [Re03, Theorems 20.3 and 14.3], a contradiction. $\square$

Example. Assume that $G = H \times \Gamma$ is a direct product and thus

$$\Lambda^{\mathfrak{o}}(G) = \Lambda^{\mathfrak{o}}(\Gamma)[H] = \Lambda^{\mathfrak{o}}(\Gamma) \otimes_{\mathfrak{o}} \mathfrak{o}[H]. \tag{14}$$

Let $\chi \in \mathrm{Irr}\,(G)$. As we have seen in the proof of Theorem 3.5 we then have $w_\chi = 1$, $K_\chi = K(\eta)$ and $\gamma'_\chi = \gamma e_\chi$. Hence

$$\mathcal{F}(\Lambda^{\mathfrak{o}}(G)) = \bigoplus_{\chi \in \mathrm{Irr}\,(G)/\sim_K} \frac{|H|}{\chi(1)} \cdot \mathfrak{D}^{-1}(\mathfrak{o}_\chi/\mathfrak{o})\Lambda^{\mathfrak{o}_\chi}(\Gamma) = \Lambda^{\mathfrak{o}}(\Gamma) \otimes_{\mathfrak{o}} \mathcal{F}(\mathfrak{o}[H])$$

which can be shown more directly using (14) and Jacobinski's central conductor formula (1). So the main obstacle to derive Theorem 3.5 directly from Jacobinski's result is the fact that in general $G = H \rtimes \Gamma$ is only semi-direct.

We finally determine the integers r_χ whenever G is a pro-p-group. For simplicity we will also assume that $K = \mathbb{Q}_p$. Then $K_\chi = \mathbb{Q}_{p,\chi}$ and $\mathfrak{o}_\chi =: \mathbb{Z}_{p,\chi}$ denotes the ring of integers in $\mathbb{Q}_{p,\chi}$. Let $\mathfrak{Q}_\eta$ and $\mathfrak{Q}_\chi$ be the maximal ideals in $\mathbb{Z}_p[\eta]$ and $\mathbb{Z}_{p,\chi}$, respectively. If x is a rational number, we let $\lfloor x \rfloor \in \mathbb{Z}$ denote the largest integer such that $\lfloor x \rfloor \leq x$.

Theorem 3.9. *Assume that G is a pro-p-group and a p-adic Lie group of dimension 1. Fix $\chi \in \mathrm{Irr}\,(G)$ and let $A_\chi \simeq (D_\chi)_{n_\chi \times n_\chi}$ be the corresponding simple component of $\mathcal{Q}(G)$. Write $\mathfrak{D}(\mathbb{Z}_p[\eta]/\mathbb{Z}_{p,\chi}) = \mathfrak{Q}_\eta^{k_\chi}$ for some integer $k_\chi \geq 0$. Then D_χ is a cyclic skewfield with Schur index $s_\chi = [\mathbb{Q}_p(\eta) : \mathbb{Q}_{p,\chi}]$ and $r_\chi = -\lfloor \frac{k_\chi}{s_\chi} \rfloor$.*

Proof. By [La12, Theorem 1] we know that D_χ is a cyclic skewfield with Schur index $s_\chi = [\mathbb{Q}_p(\eta) : \mathbb{Q}_{p,\chi}]$. More precisely, the extension $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ is cyclic and totally ramified, as G is pro-p and thus $\mathbb{Q}_p(\eta)$ is a subfield of $\mathbb{Q}_p(\zeta_{p^m})$ for some positive integer m. Let σ be a generator of the Galois group $\mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi})$. Then by [La12, Theorem 1] the skewfield D_χ is given by

$$D_\chi \simeq \bigoplus_{i=0}^{s_\chi - 1} \left(\mathcal{Q}^{\mathbb{Q}_p(\eta)}(\Gamma^{s_\chi}) \right) \gamma^i, \tag{15}$$

where γ^{s_χ} is a topological generator of $\Gamma^{s_\chi} \simeq \mathbb{Z}_p$ and for $x \in \mathcal{Q}^{\mathbb{Q}_p(\eta)}(\Gamma^{s_\chi})$ we have $\gamma x = \sigma(x)\gamma$ (note that this is not the same Γ as in $G = H \rtimes \Gamma$; it is just a topological group isomorphic to $\mathbb{Z}_p$ which we again denote by Γ). Then D_χ has center $\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma^{s_\chi})$ which is isomorphic to $\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma'_\chi)$ under the isomorphism (15). Now let $d = \sum_{i=0}^{s_\chi - 1} x_i \gamma^i \in D_\chi$ be arbitrary, $x_i \in \mathcal{Q}^{\mathbb{Q}_p(\eta)}(\Gamma^{s_\chi})$, $0 \leq i < s_\chi$. We claim that

$$\mathrm{tr}_{\,D_\chi/\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma^{s_\chi})}(d) = \mathrm{Tr}_{\,\mathcal{Q}^{\mathbb{Q}_p(\eta)}(\Gamma^{s_\chi})/\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma^{s_\chi})}(x_0). \tag{16}$$

For this one only has to check that we have an isomorphism

$$\begin{array}{rcl} \mathbb{Q}_p(\eta)\otimes_{\mathbb{Q}_{p,\chi}} D_\chi & \simeq & (\mathcal{Q}^{\mathbb{Q}_p(\eta)}(\Gamma))_{s_\chi\times s_\chi} \\ x_0 & \mapsto & \begin{pmatrix} x_0 & & & \\ & \sigma(x_0) & & \\ & & \ddots & \\ & & & \sigma^{s_\chi-1}(x_0) \end{pmatrix} \\ \gamma & \mapsto & \begin{pmatrix} 0 & & & \gamma \\ \gamma & \ddots & & 0 \\ & \ddots & 0 & \vdots \\ & & \gamma & 0 \end{pmatrix} \end{array}$$

From this one can easily compute the reduced trace in (16). Now put

$$\Sigma_\chi := \bigoplus_{i=0}^{s_\chi-1}\left(\Lambda^{\mathbb{Z}_p[\eta]}(\Gamma^{s_\chi})\right)\gamma^i \subseteq D_\chi.$$

Then Σ_χ is a $\Lambda^{\mathbb{Z}_{p,\chi}}(\Gamma^{s_\chi})$-order in D_χ. Moreover, $\mathfrak{Q}_\eta\Sigma_\chi$ is a two-sided ideal of Σ_χ, since $\gamma\mathfrak{Q}_\eta = \sigma(\mathfrak{Q}_\eta)\gamma = \mathfrak{Q}_\eta\gamma$. We consider the $\Lambda^{\mathbb{Z}_{p,\chi}}(\Gamma^{s_\chi})_{\mathfrak{p}_\chi}$-order $\Sigma_{\mathfrak{p}_\chi} := (\Sigma_\chi)_{\mathfrak{p}_\chi}$ in D_χ. Then $\mathfrak{Q}_\eta\Sigma_{\mathfrak{p}_\chi}$ is again two-sided and we have

$$(\mathfrak{Q}_\eta\Sigma_{\mathfrak{p}_\chi})^{s_\chi} = \mathfrak{Q}_\chi\Sigma_{\mathfrak{p}_\chi} = \mathfrak{p}_\chi\Sigma_{\mathfrak{p}_\chi} \subseteq \mathrm{rad}(\Sigma_{\mathfrak{p}_\chi}).$$

Hence also $\mathfrak{Q}_\eta\Sigma_{\mathfrak{p}_\chi} \subseteq \mathrm{rad}(\Sigma_{\mathfrak{p}_\chi})$ by [Re03, §6, Exercise 3]. We now observe that $\gamma \mapsto 1+T$ induces an isomorphism

$$\Sigma_\chi/\mathfrak{Q}_\eta\Sigma_\chi \simeq \mathbb{F}_p[[T]].$$

Moreover, this isomorphism induces

$$\Sigma_{\mathfrak{p}_\chi}/\mathfrak{Q}_\eta\Sigma_{\mathfrak{p}_\chi} \simeq \mathbb{F}_p((T))$$

and thus the inclusion $\mathfrak{Q}_\eta\Sigma_{\mathfrak{p}_\chi} \subseteq \mathrm{rad}(\Sigma_{\mathfrak{p}_\chi})$ is an equality. Moreover, the order $\Sigma_{\mathfrak{p}_\chi}$ is quasi-local. As $\mathfrak{Q}_\eta$ is a principal ideal, there is a short exact sequence

$$0 \to \Sigma_{\mathfrak{p}_\chi} \to \Sigma_{\mathfrak{p}_\chi} \to \mathbb{F}_p((T)) \to 0$$

and thus the projective dimension of $\mathbb{F}_p((T))$ considered as left $\Sigma_{\mathfrak{p}_\chi}$-module equals 1. Now [Ra69, Corollary 1.3] gives that also gl.dim $(\Sigma_{\mathfrak{p}_\chi}) = 1$. It then follows from [AG60, Theorem 2.3] that $\Sigma_{\mathfrak{p}_\chi}$ is a maximal $\Lambda^{\mathbb{Z}_{p,\chi}}(\Gamma^{s_\chi})_{\mathfrak{p}_\chi}$-order. By (16) we find that

$$\begin{array}{rcl} \mathrm{tr}_{\,D_\chi/\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma^{s_\chi})}(\Sigma_{\mathfrak{p}_\chi}) & = & \mathrm{Tr}_{\,\mathcal{Q}^{\mathbb{Q}_p(\eta)}(\Gamma^{s_\chi})/\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma^{s_\chi})}(\Lambda^{\mathbb{Z}_p[\eta]}(\Gamma^{s_\chi})_{\mathfrak{p}_\chi}) \\ & = & \mathrm{Tr}_{\,\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}}(\mathbb{Z}_p[\eta])\Lambda^{\mathbb{Z}_{p,\chi}}(\Gamma^{s_\chi})_{\mathfrak{p}_\chi} \\ & = & \mathfrak{Q}_\chi^{r'_\chi}\Lambda^{\mathbb{Z}_{p,\chi}}(\Gamma^{s_\chi})_{\mathfrak{p}_\chi}, \end{array}$$

where $r'_\chi \in \mathbb{Z}$ is maximal such that

$$\mathbb{Z}_p[\eta] \subseteq \mathfrak{Q}_\chi^{r'_\chi} \mathfrak{D}(\mathbb{Z}_p[\eta]/\mathbb{Z}_{p,\chi})^{-1} = \mathfrak{Q}_\eta^{s_\chi r'_\chi - k_\chi}$$

and thus $r'_\chi = \lfloor \frac{k_\chi}{s_\chi} \rfloor$. Now let $x \in \mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma^{s_\chi})$. Then x belongs to $(d_\chi)_{\mathfrak{p}_\chi}$ if and only if

$$\mathrm{tr}_{\,D_\chi/\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma^{s_\chi})}(x \cdot \Sigma_{\mathfrak{p}_\chi}) = x \cdot \mathrm{tr}_{\,D_\chi/\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma^{s_\chi})}(\Sigma_{\mathfrak{p}_\chi}) \subseteq \Lambda^{\mathbb{Z}_{p,\chi}}(\Gamma^{s_\chi})_{\mathfrak{p}_\chi}$$

and thus if and only if $x \in \mathfrak{p}_\chi^{-r'_\chi}$. Therefore we have $r_\chi = -r'_\chi = -\lfloor \frac{k_\chi}{s_\chi} \rfloor$ as desired. □

Example. We continue the example in [La12, p. 1233]. For this let $p = 3$ and let H be the cyclic group of order 9. Choose a generator h of H. We put $G = H \rtimes \Gamma$, where the action of Γ on H is determined by $\gamma h \gamma^{-1} = h^4$. Let η be the irreducible character of H with $\eta(h) = \zeta_9$. Then $St(\eta) = H \times \langle \gamma^3 \rangle$ and $\chi = \mathrm{ind}_{St(\eta)}^G \chi'$ with $\chi'(h) = \eta(h)$ and $\chi'(\gamma^3) = 1$ is an irreducible character of G with open kernel. We have $\chi(1) = w_\chi = 3$ and $\mathbb{Q}_p(\eta) = \mathbb{Q}_p(\zeta_9)$, $\mathbb{Q}_{p,\chi} = \mathbb{Q}_p(\zeta_3)$. We see that $s_\chi = [\mathbb{Q}_p(\zeta_9) : \mathbb{Q}_p(\zeta_3)] = 3$. Moreover, the different $\mathfrak{D}(\mathbb{Z}_p[\zeta_9]/\mathbb{Z}_p[\zeta_3])$ is easily computed and we find that $k_\chi = 3$. Thus $r_\chi = -\lfloor \frac{k_\chi}{s_\chi} \rfloor = -1$ is non-trivial and (12) is a proper inclusion in this case.

4 Consequences of the central conductor formula

In this section we derive several consequences of our main Theorem 3.5.

4.1 Extensions of lattices

Corollary 4.1. *Let M be a $\Lambda^{\mathfrak{o}}(G)$-lattice and let N be a finitely generated $\Lambda^{\mathfrak{o}}(G)$-module. Then*

$$\left(\bigoplus_{\chi \in \mathrm{Irr}\,(G)/\sim_K} \frac{|H| w_\chi}{\chi(1)} \cdot \mathfrak{D}^{-1}(\mathfrak{o}_\chi/\mathfrak{o}) d_\chi \right) \cdot \mathrm{Ext}^i_{\Lambda^{\mathfrak{o}}(G)}(M, N)$$

is finite for all integers $i \geq 1$.

Proof. An R-module M is finite if and only if it is pseudo-null, i.e. $M_{\mathfrak{p}} = 0$ for every height one prime ideal $\mathfrak{p}$ of R. We do induction on the integer i. As $R_{\mathfrak{p}}$ is a Dedekind domain for every height one prime ideal $\mathfrak{p}$, the case $i = 1$ is now immediate from Theorem 3.5 and [CR81, Theorem 29.4]. For k sufficiently large, there is an exact sequence

$$M' \rightarrowtail \Lambda^{\mathfrak{o}}(G)^k \twoheadrightarrow M.$$

Then M' also is a $\Lambda^{\mathfrak{o}}(G)$-lattice. Applying $\mathrm{Hom}_{\Lambda^{\mathfrak{o}}(G)}(-,N)$ to the above exact sequence gives isomorphisms

$$\mathrm{Ext}^{j}_{\Lambda^{\mathfrak{o}}(G)}(M',N) \simeq \mathrm{Ext}^{j+1}_{\Lambda^{\mathfrak{o}}(G)}(M,N)$$

for all integers $j \geq 1$. The case $j = i-1$ gives the induction step. $\square$

EXAMPLE. The stronger statement that the central conductor annihilates $\mathrm{Ext}^{i}_{\Lambda^{\mathfrak{o}}(G)}(M,N)$ is not true. Assume that $G = \Gamma$ and $K = \mathbb{Q}_p$. Then Corollary 4.1 simply asserts that $\mathrm{Ext}^{i}_{\Lambda(\Gamma)}(M,N)$ is finite for every finitely generated torsionfree $\Lambda(\Gamma)$-module M and every finitely generated $\Lambda(\Gamma)$-module N. Suppose that $\mathrm{Ext}^{i}_{\Lambda(\Gamma)}(M,N) = 0$ for every finitely generated $\Lambda(\Gamma)$-module N; then M is projective and thus free as $\Lambda(\Gamma)$-module. However, there are many examples of torsionfree $\Lambda(\Gamma)$-modules which are not free (take for example the maximal ideal of $\Lambda(\Gamma)$).

For $\chi \in \mathrm{Irr}\,(G)$ we put $\mathfrak{a}_{\chi}^{-1} := (\mathfrak{D}^{-1}(\mathfrak{o}_{\chi}/\mathfrak{o}) \cap K) \cdot R \supseteq R$.

COROLLARY 4.2. *We have*

$$\bigcap_{\chi \in \mathrm{Irr}\,(G)/\sim_K} \frac{|H| w_{\chi}}{\chi(1)} \cdot \mathfrak{a}_{\chi}^{-1} \subseteq R \cap \mathcal{F}(\Lambda^{\mathfrak{o}}(G)).$$

In fact, if $r_{\chi} = 0$ for all $\chi \in \mathrm{Irr}\,(G)$, then a proof similar to that of [CR81, Theorem 27.13(ii)] shows that the inclusion in Corollary 4.2 is an equality.
For every $\Lambda^{\mathfrak{o}}(G)$-module M let $\Upsilon(M) := \left\{ e_{\chi} \mid e_{\chi} \cdot \mathcal{Q}^K(G) \otimes_{\Lambda^{\mathfrak{o}}(G)} M = 0 \right\}$.

COROLLARY 4.3. *Let M and N be $\Lambda^{\mathfrak{o}}(G)$-lattices and let $i \geq 1$ be an integer. Then*

$$\left(\bigcap_{e_{\chi} \notin \Upsilon(M)} \frac{|H| w_{\chi}}{\chi(1)} \cdot \mathfrak{a}_{\chi}^{-1} \right) \mathrm{Ext}^{i}_{\Lambda^{\mathfrak{o}}(G)}(M,N)$$

is finite. In particular, $\left(\bigcap_{\chi \in \mathrm{Irr}\,(G)/\sim_K} \frac{|H| w_{\chi}}{\chi(1)} \cdot \mathfrak{a}_{\chi}^{-1} \right) \cdot \mathrm{Ext}^{i}_{\Lambda^{\mathfrak{o}}(G)}(M,N)$ *is finite for every $\Lambda^{\mathfrak{o}}(G)$-lattices M and N and every integer $i \geq 1$.*

Proof. The last assertion is an immediate consequence of Corollary 4.1 and Corollary 4.2. The first assertion is also easy and is shown exactly in the same way as [CR81, Theorem 29.9]. $\square$

COROLLARY 4.4. *Let M and N be $\Lambda^{\mathfrak{o}}(G)$-lattices and assume that $\mathcal{Q}^K(G) \otimes_{\Lambda^{\mathfrak{o}}(G)} M$ is absolutely simple. Then there is a unique idempotent $e_{\chi} \notin \Upsilon(M)$ and for every integer $i \geq 1$ we have that $\frac{|H| w_{\chi}}{\chi(1)} \cdot \mathrm{Ext}^{i}_{\Lambda^{\mathfrak{o}}(G)}(M,N)$ is finite.*

Example. Assume that p divides $|H|$ and that $K = \mathbb{Q}_p$. We consider $\Lambda(\Gamma)$ as $\Lambda(G)$-module in the natural way; then $\mathcal{Q}(G) \otimes_{\Lambda(G)} \Lambda(\Gamma)$ is absolutely simple and e_{χ_0} is the unique idempotent not contained in $\Upsilon(\Lambda(\Gamma))$, where χ_0 denotes the trivial character. Moreover, we have $\chi_0(1) = 1$ and $w_{\chi_0} = 1$. Let $\Delta(G, \Gamma)$ be the kernel of the augmentation map aug : $\Lambda(G) \twoheadrightarrow \Lambda(\Gamma)$ which sends each $h \in H$ to 1. We then have an exact sequence

$$\mathrm{Hom}_{\Lambda(G)}(\Lambda(\Gamma), \Delta(G, \Gamma)) \rightarrowtail \mathrm{Hom}_{\Lambda(G)}(\Lambda(\Gamma), \Lambda(G)) \xrightarrow{\mathrm{aug}_*} \mathrm{Hom}_{\Lambda(G)}(\Lambda(\Gamma), \Lambda(\Gamma)) \twoheadrightarrow \mathrm{Ext}^1_{\Lambda(G)}(\Lambda(\Gamma), \Delta(G, \Gamma)).$$

We have a canonical isomorphism $\mathrm{Hom}_{\Lambda(G)}(\Lambda(\Gamma), \Lambda(\Gamma)) \simeq \Lambda(\Gamma)$, $f \mapsto f(1)$ and the image of aug_* in $\Lambda(\Gamma)$ under this identification is given by $\mathrm{aug}\left((\sum_{h \in H} h)\Lambda(G)\right) = |H|\Lambda(\Gamma)$. We therefore have a canonical isomorphism

$$\mathrm{Ext}^1_{\Lambda(G)}(\Lambda(\Gamma), \Delta(G, \Gamma)) \simeq \Lambda(\Gamma)/|H|\Lambda(\Gamma);$$

in particular, this module is not finite. However, $|H|\mathrm{Ext}^1_{\Lambda(G)}(\Lambda(\Gamma), \Delta(G, \Gamma)) = 0$ is finite.

4.2 Non-commutative Fitting invariants

For the following we refer the reader to [Ni10]. Let A be a separable L-algebra and Λ be an R-order in A, finitely generated as R-module, where R is an integrally closed complete commutative noetherian local domain with field of quotients L. Let N and M be two $\zeta(\Lambda)$-submodules of an R-torsionfree $\zeta(\Lambda)$-module. Then N and M are called $\mathrm{nr}(\Lambda)$-*equivalent* if there exists an integer n and a matrix $U \in \mathrm{Gl}_n(\Lambda)$ such that $N = \mathrm{nr}(U) \cdot M$, where $\mathrm{nr} : A \to \zeta(A)$ denotes the reduced norm map which extends to matrix rings over A in the obvious way. We denote the corresponding equivalence class by $[N]_{\mathrm{nr}(\Lambda)}$. We say that N is $\mathrm{nr}(\Lambda)$-contained in M (and write $[N]_{\mathrm{nr}(\Lambda)} \subseteq [M]_{\mathrm{nr}(\Lambda)}$) if for all $N' \in [N]_{\mathrm{nr}(\Lambda)}$ there exists $M' \in [M]_{\mathrm{nr}(\Lambda)}$ such that $N' \subseteq M'$. We will say that x is contained in $[N]_{\mathrm{nr}(\Lambda)}$ (and write $x \in [N]_{\mathrm{nr}(\Lambda)}$) if there is $N_0 \in [N]_{\mathrm{nr}(\Lambda)}$ such that $x \in N_0$. Now let M be a finitely presented (left) Λ-module and let

$$\Lambda^a \xrightarrow{h} \Lambda^b \twoheadrightarrow M \tag{17}$$

be a finite presentation of M. We identify the homomorphism h with the corresponding matrix in $M_{a \times b}(\Lambda)$ and define $S(h) = S_b(h)$ to be the set of all $b \times b$ submatrices of h if $a \geq b$. The Fitting invariant of h over Λ is defined to be

$$\mathrm{Fitt}_\Lambda(h) = \begin{cases} [0]_{\mathrm{nr}(\Lambda)} & \text{if} \quad a < b \\ \left[\langle \mathrm{nr}(H) \mid H \in S(h) \rangle_{\zeta(\Lambda)}\right]_{\mathrm{nr}(\Lambda)} & \text{if} \quad a \geq b. \end{cases}$$

We call $\mathrm{Fitt}_\Lambda(h)$ a *Fitting invariant* of M over Λ. One defines $\mathrm{Fitt}^{\max}_\Lambda(M)$ to be the unique Fitting invariant of M over Λ which is maximal among all Fitting invariants of M with respect to the partial order "$\subseteq$".

We now specialize to the situation in this article, where Λ is $\Lambda^{\mathfrak{o}}(G)$. Then Theorem 3.5 and [JN13, Corollary 6.2] imply the following result.

Corollary 4.5. *Let M be a finitely presented $\Lambda^{\mathfrak{o}}(G)$-module. Then*

$$\left(\bigoplus_{\chi\in\mathrm{Irr}\,(G)/\sim_K}\frac{|H|w_\chi}{\chi(1)}\cdot\mathfrak{D}^{-1}(\mathfrak{o}_\chi/\mathfrak{o})d_\chi\right)\cdot\mathrm{Fitt}^{\max}_{\Lambda^{\mathfrak{o}}(G)}(M)\subseteq\mathrm{Ann}_{\Lambda^{\mathfrak{o}}(G)}(M).$$

Together with Corollary 4.2 and [Ni10, Lemma 3.4] this yields the following corollary.

Corollary 4.6. *Let M be a finitely presented $\Lambda^{\mathfrak{o}}(G)$-module. Then*

$$\left(\bigcap_{e_\chi\notin\Upsilon(M)}\frac{|H|w_\chi}{\chi(1)}\cdot\mathfrak{a}_\chi^{-1}\right)\cdot\mathrm{Fitt}^{\max}_{\Lambda^{\mathfrak{o}}(G)}(M)\subseteq\mathrm{Ann}_{\Lambda^{\mathfrak{o}}(G)}(M).$$

Remark 4.7. *Note that in fact $\mathcal{H}(\Lambda^{\mathfrak{o}}(G))\cdot\mathrm{Fitt}^{\max}_{\Lambda^{\mathfrak{o}}(G)}(M)\subseteq\mathrm{Ann}_{\Lambda^{\mathfrak{o}}(G)}(M)$, where $\mathcal{H}(\Lambda^{\mathfrak{o}}(G))$ is a certain ideal of $\zeta(\Lambda^{\mathfrak{o}}(G))$ which always contains the central conductor. In general, however, this containment is not an equality. Though the ideal $\mathcal{H}(\Lambda^{\mathfrak{o}}(G))$ is hard to determine in general, considerable progress is made in [JN13]; in particular, by [JN13, Proposition 4.5] one knows that $\mathcal{H}(\Lambda^{\mathfrak{o}}(G))$ equals $\zeta(\Lambda^{\mathfrak{o}}(G))$ (to wit: is best possible) if and only if p does not divide the order of the commutator subgroup of G (which is finite).*

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