Trace Class Operators, Regulators, and Assembly Maps in K-Theory

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Abstract. Let G be a group and let KH be homotopy algebraic K-theory. We prove that if G satisfies the rational KH isomorphism conjecture for the group algebra $\mathcal{L}^1[G]$ with coefficients in the algebra of trace-class operators in Hilbert space, then it also satisfies the K-theoretic Novikov conjecture for the group algebra over the integers, and the rational injectivity part of the Farrell-Jones conjecture with coefficients in any number field.

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1 Introduction

Let G be a group, $\mathcal{F}in$ the family of its finite subgroups, and $\mathcal{E}(G, \mathcal{F}in)$ the classifying space. Let $\mathcal{L}^1$ be the algebra of trace-class operators in an infinite dimensional, separable Hilbert space over the complex numbers. Consider the rational assembly map in homotopy algebraic K-theory

$$H_p^G(\mathcal{E}(G, \mathcal{F}in), KH(\mathcal{L}^1)) \otimes \mathbb{Q} \to KH_p(\mathcal{L}^1[G]) \otimes \mathbb{Q}. \quad (1)$$

The rational KH-isomorphism conjecture ([1, Conjecture 7.3]) predicts that (1) is an isomorphism; it follows from a theorem of Yu ([14], [4]) that it is always injective. In the current article we prove the following.

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Theorem 2. *Assume that* (1) *is surjective. Let* $n \equiv p+1 \mod 2$. *Then:*

i) The rational assembly map for the trivial family

$$H_n^G(\mathcal{E}(G,\{1\}),K(\mathbb{Z}))\otimes\mathbb{Q}\to K_n(\mathbb{Z}[G])\otimes\mathbb{Q} \tag{3}$$

is injective.

ii) For every number field F*, the rational assembly map*

$$H_n^G(\mathcal{E}(G,\mathcal{F}in),K(F))\otimes\mathbb{Q}\to K_n(F[G])\otimes\mathbb{Q} \tag{4}$$

is injective.

We remark that the K-theory Novikov conjecture asserts that part i) of the theorem above holds for all G, and that part ii) is equivalent to the rational injectivity part of the K-theory Farrell-Jones conjecture for number fields ([11, Conjectures 51 and 58 and Proposition 70]).

The idea of the proof of Theorem 2 is to use an algebraic, equivariant version of Karoubi's multiplicative K-theory. The latter theory assigns groups $MK_n(\mathfrak{A})$ $(n\ge 1)$ to any unital Banach algebra $\mathfrak{A}$, which fit into a long exact sequence

$$HC_{n-1}^{\mathrm{top}}(\mathfrak{A})\to MK_n(\mathfrak{A})\to K_n^{\mathrm{top}}(\mathfrak{A})\overset{Sch_n^{\mathrm{top}}}{\to} HC_{n-2}^{\mathrm{top}}(\mathfrak{A}).$$

Here HC^{top} is the cyclic homology of the completed cyclic module $C_n^{\mathrm{top}}(\mathfrak{A}) = \mathfrak{A}\hat{\otimes}\dots\hat{\otimes}\mathfrak{A}$ ($n+1$ factors), ch^{top} is the Connes-Karoubi Chern character with values in its periodic cyclic homology $HP_*^{\mathrm{top}}(\mathfrak{A})$, and S is the periodicity operator. Karoubi introduced a multiplicative Chern character

$$\mu_n : K_n(\mathfrak{A})\to MK_n(\mathfrak{A}). \tag{5}$$

In particular if $\mathcal{O}$ is the ring of integers in a number field F one can consider the composite

$$K_n(\mathcal{O})\to K_n(\mathbb{C})^{\hom(F,\mathbb{C})}\to MK_n(\mathbb{C})^{\hom(F,\mathbb{C})}. \tag{6}$$

By comparing this map with the Borel regulator, Karoubi showed in [7] that (6) is rationally injective. It follows that

$$K_n(\mathbb{Z})\to MK_n(\mathbb{C}) \tag{7}$$

is rationally injective. In the current paper we assign, to every unital $\mathbb{C}$-algebra A, groups $\kappa_n(A)$ $(n\in\mathbb{Z})$ which fit into a long exact sequence

$$HC_{n-1}(A/\mathbb{C})\to\kappa_n(A)\to KH_n(\mathcal{L}^1\otimes_{\mathbb{C}}A)\overset{\mathrm{Tr}Sch_n}{\to} HC_{n-2}(A/\mathbb{C}).$$

Here $HC(/\mathbb{C})$ is algebraic cyclic homology of $\mathbb{C}$-algebras, ch is the algebraic Connes-Karoubi Chern character and Tr is induced by the operator trace. We also introduce a character

$$\tau_n : K_n(A)\to\kappa_n(A)\ \ (n\in\mathbb{Z}). \tag{8}$$

If $\mathfrak{A}$ is a finite dimensional Banach algebra and $n \geq 1$ then $\kappa_n(\mathfrak{A}) = MK_n(\mathfrak{A})$ and (5) identifies with (8) (Proposition 20). Both κ and τ have equivariant versions, so that if X is a G-space and A is a $\mathbb{C}$-algebra, we have an assembly map

$$H_n^G(X, \kappa(A)) \to \kappa_n(A[G]).$$

Let $\mathcal{F}cyc$ be the family of finite cyclic subgroups. We show in Proposition 30 that the map

$$H_n^G(\mathcal{E}(G, \mathcal{F}cyc), \kappa(\mathbb{C})) \to H_n^G(\mathcal{E}(G, \mathcal{F}in), \kappa(\mathbb{C}))$$

is an isomorphism, and compute $H_n^G(\mathcal{E}(G, \mathcal{F}cyc), \kappa(\mathbb{C})) \otimes \mathbb{Q}$ in terms of the finite cyclic subgroups of G. We use this and the rational injectivity of (7) to show, in Proposition 44, that the map

$$H_n^G(\mathcal{E}(G, \{1\}), K(\mathbb{Z})) \to H_n^G(\mathcal{E}(G, \mathcal{F}in), K(\mathbb{C})) \xrightarrow{\tau} H_n^G(\mathcal{E}(G, \mathcal{F}in), \kappa(\mathbb{C})) \quad (9)$$

is rationally injective. It is well-known [11, Proposition 76] that the map

$$H_n^G(\mathcal{E}(G, \mathcal{F}cyc), K(R)) \otimes \mathbb{Q} \to H_n^G(\mathcal{E}(G, \mathcal{F}in), K(R)) \otimes \mathbb{Q}$$

is an isomorphism for every unital ring R. In particular, we may substitute $\mathcal{F}cyc$ for $\mathcal{F}in$ in (4). We use this together with Proposition 30 and the rational injectivity of

$$K_n(F) \to K_n(\mathbb{C})^{\hom(F,\mathbb{C})} \to MK_n(\mathbb{C})^{\hom(F,\mathbb{C})}$$

(see Remark 24), to show in Proposition 48 that if $m \geq 1$, $\mathcal{C}yc_m$ is the family of cyclic subgroups whose order divides m, and ζ_m is a primitive m-root of 1, then the composite

$$\begin{tikzcd} H_n^G(\mathcal{E}(G, \mathcal{C}yc_m), K(F)) \otimes \mathbb{Q} \arrow[r] \arrow[dr, dotted] & H_n^G(\mathcal{E}(G, \mathcal{C}yc_m), K(\mathbb{C}))^{\hom(F(\zeta_m),\mathbb{C})} \otimes \mathbb{Q} \arrow[d, "\tau"] \\ & H_n^G(\mathcal{E}(G, \mathcal{F}cyc), \kappa(\mathbb{C}))^{\hom(F(\zeta_m),\mathbb{C})} \otimes \mathbb{Q} \end{tikzcd} \quad (10)$$

is injective. Since the map $\operatorname{colim}_m \mathcal{E}(G, \mathcal{C}yc_m) \to \mathcal{E}(G, \mathcal{F}cyc)$ is an equivalence, it follows that if the rational assembly map

$$H_n^G(\mathcal{E}(G, \mathcal{F}cyc), \kappa(\mathbb{C})) \otimes \mathbb{Q} \to \kappa_n(\mathbb{C}[G]) \otimes \mathbb{Q} \quad (11)$$

is injective then so are both (3) and (4). We show in Corollary 41 that if (1) is surjective, then (11) is injective for $n \equiv p+1 \mod 2$. This proves Theorem 2. The rest of this paper is organized as follows. In Section 2 we define $\kappa_n(A)$ and the map $\tau_n : K_n(A) \to \kappa_n(A)$. By definition, if $n \leq 0$, then $\kappa_n(A) = KH_n(A \otimes_{\mathbb{C}} \mathcal{L}^1)$ and τ_n is the identity map (16). We show in Proposition 20 that if $n \geq 1$ and $\mathfrak{A}$ is a finite dimensional Banach algebra, then $\kappa_n(\mathfrak{A}) = MK_n(\mathfrak{A})$

and $\tau_n = \mu_n$. Karoubi's regulators and his injectivity results are recalled in Theorem 23. We use Karoubi's theorem to prove, in Lemma 26, that if F is a number field, C a cyclic group of order m, n a multiple of m, and ζ_n a primitive n-root of 1, then the composite

$$K_*(F[C]) \to K_*(F(\zeta_n)[C]) \to \\ K_*(\mathbb{C}[C])^{\hom(F(\zeta_n),\mathbb{C})} \xrightarrow{\mu} MK_*(\mathbb{C}[C])^{\hom(F(\zeta_n),\mathbb{C})}$$

is rationally injective. The main result of Section 3 is Proposition 30, which computes $H_n^G(\mathcal{E}(G,\mathcal{F}cyc),\kappa(\mathbb{C}))\otimes\mathbb{Q}$ in terms of group homology and of the groups $\kappa_*(\mathbb{C}[C])$ for $C\in\mathcal{F}cyc$. The resulting formula is similar to existing formulas for equivariant K and cyclic homology, which are used in its proof ([3], [9], [10], [11],[12]). In Section 4 we show that the rational $\kappa(\mathbb{C})$-assembly map is injective whenever the rational $KH(\mathcal{L}^1)$-assembly map is surjective (Corollary 41). For this we use the fact that for every $m\ge 1$, the assembly map

$$H_*^G(\mathcal{E}(G,\mathcal{C}yc_m),HC(\mathbb{C}/\mathbb{C}))\to HC_*(\mathbb{C}[G])$$

has a natural left inverse π_m, which makes the following diagram commute

$$\begin{array}{ccc} H_*^G(\mathcal{E}(G,\mathcal{C}yc_m),KH(\mathcal{L}^1))\otimes\mathbb{Q} & \longrightarrow & KH_*(\mathcal{L}^1[G])\otimes\mathbb{Q} \\ \downarrow{\scriptstyle \mathrm{Tr}Sch} & & \downarrow{\scriptstyle \mathrm{Tr}Sch} \\ H_{*-2}^G(\mathcal{E}(G,\mathcal{C}yc_m),HC(\mathbb{C}/\mathbb{C})) & \xleftarrow{\pi_m} & HC_{*-2}(\mathbb{C}[G]). \end{array}$$

Hence for every n we have an inclusion

$$\mathrm{Tr}Sch(H_{n+1}^G(\mathcal{E}(G,\mathcal{C}yc_m),KH(\mathcal{L}^1))\otimes\mathbb{Q})\subset\pi_m\mathrm{Tr}Sch(KH_{n+1}(\mathcal{L}^1[G])\otimes\mathbb{Q}). \tag{12}$$

We show in Proposition 39 that the rational assembly map

$$H_n^G(\mathcal{E}(G,\mathcal{C}yc_m),\kappa(\mathbb{C}))\otimes\mathbb{Q}\to\kappa_n(\mathbb{C}[G])\otimes\mathbb{Q}$$

is injective if and only if the inclusion (12) is an equality. Corollary 41 is immediate from this. Section 5 is concerned with proving that (9) and (10) are injective (Propositions 44 and 48). Finally in Section 6 we show that if the identity holds in (12) for $m=1$ then (3) is injective (Theorem 51) and that if it holds for m, then

$$H_n^G(\mathcal{E}(G,\mathcal{C}yc_m),K(F))\otimes\mathbb{Q}\to K_n(F[G])\otimes\mathbb{Q}$$

is injective for every number field F (Theorem 52).

2 The character $\tau : K(A)\to\kappa(A)$

2.1 Definition of τ

Let A be a $\mathbb{C}$-algebra and $k\subset\mathbb{C}$ a subfield. Write $C(A/k)$ for Connes' cyclic module and $HH(A/k)$, $HC(A/k)$, $HN(A/k)$ and $HP(A/k)$ for the associated

Hochschild, cyclic, negative cyclic and periodic cyclic chain complexes. When $k = \mathbb{Q}$ we omit it from our notation; thus for example, $HH(A) = HH(A/\mathbb{Q})$. As usual, we write S, B and I for the maps appearing in Connes' SBI sequence. To simplify notation we shall make no distinction between a chain complex and the spectrum the Dold-Kan corresponce associates to it. We write KH for Weibel's homotopy algebraic K-theory and K^{nil} for the fiber of the comparison map $K \to KH$. We have a map of fibration sequences [2, §11.3]

$$\begin{tikzcd} K^{\mathrm{nil}}(A) \arrow[r] \arrow[d, "\nu"'] & K(A) \arrow[r] \arrow[d] & KH(A) \arrow[d, "ch"] \\ HC(A)[-1] \arrow[r, "B"'] & HN(A) \arrow[r, "I"'] & HP(A). \end{tikzcd}$$

Here ch is the Connes-Karoubi character. Write $\mathcal{B}$ for the algebra of bounded operators in an infinite dimensional, separable Hilbert space, and $\mathcal{L}^1 \lhd \mathcal{B}$ for the ideal of trace class operators. Recall from [6] that HP satisfies excision; in particular, the canonical map $HP(\mathcal{L}^1 \otimes_{\mathbb{C}} A) \to HP(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A)$ is a quasi-isomorphism. We shall abuse notation and write Sch for the map that makes the following diagram commute

$$\begin{tikzcd} KH(\mathcal{L}^1 \otimes_{\mathbb{C}} A)[+1] \arrow[r, "ch"] \arrow[d, "Sch"'] & HP(\mathcal{L}^1 \otimes_{\mathbb{C}} A)[+1] \arrow[d, "\wr"] \\ HC(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A)[-1] & HP(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A)[+1]. \arrow[l, "S"] \end{tikzcd} \tag{13}$$

By [5, Theorems 6.5.3 and 7.1.1], the map $\nu : K^{nil}(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A) \to HC(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A)[-1]$ is an equivalence, and thus the map Sch fits into a fibration sequence

$$KH(\mathcal{L}^1 \otimes_{\mathbb{C}} A)[+1] \xrightarrow{Sch} HC(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A)[-1] \to K(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A).$$

On the other hand the operator trace $\mathrm{Tr} : \mathcal{L}^1 \to \mathbb{C}$ induces a map of cyclic modules

$$\begin{gathered} \mathrm{Tr} : C(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A) \to C(A/\mathbb{C}) \\ \mathrm{Tr}(b_0 \otimes a_0 \otimes \cdots \otimes b_n \otimes a_n) = \mathrm{Tr}(b_0 \cdots b_n) a_0 \otimes \cdots \otimes a_n. \end{gathered} \tag{14}$$

Note that Tr is defined on $b_0 \cdots b_n$ since at least one of the b_i is in $\mathcal{L}^1$. In particular Tr induces a chain map

$$\mathrm{Tr} : HC(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A) \to HC(A/\mathbb{C}). \tag{15}$$

We define $\kappa(A)$ as the homotopy cofiber of the composite of (15) and the map Sch of (13)

$$\kappa(A) := \mathrm{hocofi}(KH(\mathcal{L}^1 \otimes_{\mathbb{C}} A)[+1] \xrightarrow{\mathrm{Tr} Sch} HC(A/\mathbb{C})[-1]).$$

Thus because, by definition, cyclic homology vanishes in negative degrees, we have

$$\kappa_n(A) = KH_n(\mathcal{L}^1 \otimes_{\mathbb{C}} A) \quad (n \le 0). \tag{16}$$

By construction, there is an induced map $K(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A) \to \kappa(A)$ which fits into a commutative diagram

$$\begin{tikzcd}
KH(\mathcal{L}^1 \otimes_{\mathbb{C}} A)[+1] \arrow[r, equal] \arrow[d, "Sch"] & KH(\mathcal{L}^1 \otimes_{\mathbb{C}} A)[+1] \arrow[d] \\
HC(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A)[-1] \arrow[r] \arrow[d] & HC(A/\mathbb{C})[-1] \arrow[d] \\
K(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A) \arrow[r] & \kappa(A).
\end{tikzcd} \tag{17}$$

A choice of a rank one projection p gives a map $A \to \mathcal{L}^1 \otimes_{\mathbb{C}} A$, $a \mapsto p \otimes a$, and therefore a map $K(A) \to K(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A)$. We shall be interested in the composite

$$\tau : K(A) \to K(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A) \to \kappa(A). \tag{18}$$

2.2 Comparison with Karoubi's multiplicative Chern character

Suppose now that $\mathfrak{A}$ is a unital Banach algebra. Let $\Delta^{\mathrm{diff}}_{\bullet}\mathfrak{A} = C^\infty(\Delta_\bullet, \mathfrak{A})$ be the simplicial algebra of $\mathfrak{A}$-valued C^∞-functions on the standard simplices. Write $KV^{\mathrm{diff}}(\mathfrak{A})$ for the diagonal of the bisimplicial space $[n] \mapsto BGL(\Delta^{\mathrm{diff}}_n \mathfrak{A})$. We have

$$K^{\mathrm{top}}_n(\mathfrak{A}) = \pi_n KV^{\mathrm{diff}}(\mathfrak{A}) \quad (n \ge 1).$$

Consider the fiber $\mathcal{F}(\mathfrak{A}) = \mathrm{hofiber}(BGL^+(\mathfrak{A}) \to KV^{\mathrm{diff}}(\mathfrak{A}))$. We have a homotopy fibration

$$\Omega BGL^+(\mathfrak{A}) \to \Omega KV^{\mathrm{diff}}(\mathfrak{A}) \to \mathcal{F}(\mathfrak{A}). \tag{19}$$

Let $\hat{\otimes}$ be the projective tensor product of Banach spaces and let $C^{\mathrm{top}}(\mathfrak{A})$ be the cyclic module with $C^{\mathrm{top}}(\mathfrak{A})_n = \mathfrak{A}\hat{\otimes}\dots\hat{\otimes}\mathfrak{A}$ ($n+1$ factors). Write $HC^{\mathrm{top}}(\mathfrak{A})$ and $HP^{\mathrm{top}}(\mathfrak{A})$ for the cyclic and periodic cyclic complexes of $C^{\mathrm{top}}(\mathfrak{A})$. In [8] (see also [7, §7]), Max Karoubi constructs a map $ch^{\mathrm{rel}} : \mathcal{F}(\mathfrak{A}) \to HC^{\mathrm{top}}(\mathfrak{A})[-1]$ and defines his multiplicative K-groups as the homotopy groups

$$MK_n(\mathfrak{A}) = \pi_n(\mathrm{hofiber}(KV^{\mathrm{diff}}(\mathfrak{A}) \to HC^{\mathrm{top}}(\mathfrak{A})[-2])) \quad (n \ge 1).$$

He further defines the multiplicative Chern character as the induced map $\mu_n : K_n(\mathfrak{A}) \to MK_n(\mathfrak{A})$ $(n \ge 1)$.

Proposition 20. *Let $\mathfrak{A}$ be a unital Banach algebra, and let $n \ge 1$. Then there is a natural map $\kappa_n(\mathfrak{A}) \to MK_n(\mathfrak{A})$ which makes the following diagram* *commute*

$$\begin{tikzcd} K_n(\mathfrak{A}) \arrow[r, "\tau_n"] \arrow[dr, "\mu_n"'] & \kappa_n(\mathfrak{A}) \arrow[d] \\ & MK_n(\mathfrak{A}). \end{tikzcd}$$

If furthermore $\mathfrak{A}$ is finite dimensional, then $\kappa_n(\mathfrak{A}) \to MK_n(\mathfrak{A})$ is an isomorphism.

Proof. Consider the simplicial ring

$$\Delta_\bullet\mathfrak{A} : [n] \mapsto \Delta_n\mathfrak{A} = \mathfrak{A}[t_0, \dots, t_n]/\langle 1 - \sum_{i=0}^{n} t_i\rangle.$$

Let $KV(\mathfrak{A})$ be the diagonal of the bisimplicial set $BGL(\Delta_\bullet\mathfrak{A})$. We have a homotopy commutative diagram

$$\begin{tikzcd} KV(\mathcal{L}^1 \otimes_{\mathbb{C}} \mathfrak{A}) \arrow[r] \arrow[d] & HC(\mathcal{B} \otimes_{\mathbb{C}} \mathfrak{A} : \mathcal{L}^1 \otimes_{\mathbb{C}} \mathfrak{A})[-2] \arrow[r, "\mathrm{Tr}"] \arrow[d] & HC(\mathfrak{A}/\mathbb{C})[-2] \arrow[d] \\ KV^{\mathrm{diff}}(\mathcal{L}^1 \hat{\otimes} \mathfrak{A}) \arrow[r] & HC^{\mathrm{top}}(\mathcal{B} \hat{\otimes} \mathfrak{A} : \mathcal{L}^1 \hat{\otimes} \mathfrak{A})[-2] \arrow[r, "\mathrm{Tr}"] & HC^{\mathrm{top}}(\mathfrak{A})[-2] \\ KV^{\mathrm{diff}}(\mathfrak{A}) \arrow[r] \arrow[u] & HC^{\mathrm{top}}(\mathfrak{A})[-2] \arrow[u] \arrow[ur, equal] & \end{tikzcd} \tag{21}$$

By [5, Lemma 3.2.1 and Theorem 6.5.3(ii)] and [13, Proposition 1.5] (or [2, Proposition 5.2.3]), the natural map

$$KV_n(\mathcal{L}^1 \otimes_{\mathbb{C}} \mathfrak{A}) \to KH_n(\mathcal{L}^1 \otimes_{\mathbb{C}} \mathfrak{A})$$

is an isomorphism for $n \geq 1$. It follows from this that for $n \geq 1$, the group $\kappa_n(\mathfrak{A})$ is isomorphic to π_n of the fiber of the composite of the first row of diagram (21). On the other hand, by Karoubi's density theorem, the map $KV^{\mathrm{diff}}(\mathfrak{A}) \to KV^{\mathrm{diff}}(\mathcal{L}^1 \hat{\otimes} \mathfrak{A})$ is an equivalence; inverting it and taking fibers and homotopy groups, we get a natural map $\kappa_n(\mathfrak{A}) \to MK_n(\mathfrak{A})$ $(n \geq 1)$. The commutativity of the diagram of the proposition is clear. If now $\mathfrak{A}$ is finite dimensional, then $\mathfrak{A} \hat{\otimes} V = \mathfrak{A} \otimes_{\mathbb{C}} V$ for any locally convex vector space V. Hence the map $HC(\mathfrak{A}/\mathbb{C}) \to HC^{\mathrm{top}}(\mathfrak{A})$ is the identity map. Furthermore, by [5, Theorem 3.2.1], the map $KV(\mathcal{L}^1 \otimes_{\mathbb{C}} \mathfrak{A}) \to KV^{\mathrm{diff}}(\mathcal{L}^1 \hat{\otimes} \mathfrak{A})$ is an equivalence. It follows that $\kappa_n(\mathfrak{A}) \to MK_n(\mathfrak{A})$ is an isomorphism for all $n \geq 1$, finishing the proof. □

Example 22. We have

$$\kappa_n(\mathbb{C}) = \begin{cases} \mathbb{C}^* & n \geq 1, \text{ odd.} \\ \mathbb{Z} & n \leq 0, \text{ even.} \\ 0 & \text{otherwise.} \end{cases}$$

2.3 REGULATORS

In view of Proposition 20 above we may substitute τ for μ in the theorem below.

THEOREM 23. [7, Théorème 7.20] *Let $\mathcal{O}$ be the ring of integers in a number field F. Write $F \otimes \mathbb{R} \cong \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$; put $r = r_1 + r_2$. Then the inclusion $\mathcal{O} \subset \mathbb{C}^r$ followed by the the map $\mu_n : K_n(\mathbb{C})^r \to MK_n(\mathbb{C})^r$ induces a monomorphism $K_n(\mathcal{O}) \otimes \mathbb{Q} \to MK_n(\mathbb{C})^r \otimes \mathbb{Q}$ $(n \geq 1)$.*

Remark 24. It follows from classical results of Quillen that the map $K_n(\mathcal{O}) \to K_n(F)$ is a rational isomorphism for $n \geq 2$. Thus $K_n(F) \to MK_n(\mathbb{C})^r$ is rationally injective for $n \geq 2$. Moreover, $K_1(F) \to MK_1(\mathbb{C})^r$ is injective too, since the map $\tau_1 : K_1(\mathbb{C}) \to MK_1(\mathbb{C})$ is the identity of $\mathbb{C}^*$. Observe that the isomorphism $F \otimes \mathbb{R} \cong \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ of Theorem 23 is not canonical; it implies choosing r_2 nonreal embeddings $F \to \mathbb{C}$ out of the total $2r_2$, so that no two of them differ by complex conjugation. On the other hand the map

$$\iota : F \to \mathbb{C}^{\hom(F,\mathbb{C})}, \quad \iota(x)_\sigma = \sigma(x)$$

is canonical. Moreover, the composite

$$\mathrm{reg}_n(F) : K_n(F) \to K_n(\mathbb{C})^{\hom(F,\mathbb{C})} \to MK_n(\mathbb{C})^{\hom(F,\mathbb{C})} \tag{25}$$

is still a rational monomorphism. Indeed the map of the theorem is obtained by composing (25) with a projection $MK_n(\mathbb{C})^{\hom(F,\mathbb{C})} \to MK_n(\mathbb{C})^r$.

LEMMA 26. *Let F be a number field, C a cyclic group of order m, n a multiple of m and ζ_n a primitive n-th root of 1. Then the composite map*

$$K_*(F[C]) \to K_*(F(\zeta_n)[C]) \xrightarrow{\iota} K_*(\mathbb{C}[C])^{\hom(F(\zeta_n),\mathbb{C})} \xrightarrow{\mu} MK_*(\mathbb{C}[C])^{\hom(F(\zeta_n),\mathbb{C})}$$

is rationally injective.

Proof. Let $G_m = \mathrm{Gal}(F(\zeta_m)/F)$; if M is a G_m-module, write M^{G_m} for the fixed points. By [9, Lemma 8.4], the map $F[C] \to F(\zeta_m)[C]$ induces an isomorphism $K_*(F[C]) \otimes \mathbb{Q} \to K_*(F(\zeta_m)[C])^{G_m} \otimes \mathbb{Q}$. In particular, $K_*(F[C]) \to K_*(F(\zeta_m)[C])$ is rationally injective. Now if $\sigma : F(\zeta_m) \to E$ is a field homomorphism, then $K_*(E[C]) = K_*(E)^m$, and the map $K_*(F(\zeta_m)[C]) \to K_*(E[C])$ decomposes into a direct sum of m copies of the map $K_*(F(\zeta_m)) \to K_*(E)$. In particular this applies when $E \in \{F(\zeta_n), \mathbb{C}\}$. In view of Theorem 23 and Remark 24, it follows that both $K_*(F(\zeta_m)[C]) \to MK_*(\mathbb{C}[C])^{\hom(F(\zeta_m),\mathbb{C})}$ and $K_*(F(\zeta_n)[C]) \to MK_*(\mathbb{C}[C])^{\hom(F(\zeta_n),\mathbb{C})}$ are rationally injective. Summing up, we have a commutative diagram

$$\begin{tikzcd}
K_*(F[C]) \arrow[d] & \\
K_*(F(\zeta_m)[C]) \arrow[r] \arrow[d] & MK_*(\mathbb{C}[C])^{\hom(F(\zeta_m),\mathbb{C})} \arrow[d] \\
K_*(F(\zeta_n)[C]) \arrow[r] & MK_*(\mathbb{C}[C])^{\hom(F(\zeta_n),\mathbb{C})}
\end{tikzcd}$$

We have shown that the first vertical map on the left and the two horizontal maps are rationally injective. Since the vertical map on the right is injective, we conclude that the composite of the left column followed by the bottom horizontal arrow is a rational monomorphism, finishing the proof. □

3 Rational computation of equivariant κ-homology

Let G be a group and let $\mathrm{Or}G$ be its orbit category. For $G/H \in \mathrm{Or}G$, let $\mathcal{G}(G/H) = \mathcal{G}^G(G/H)$ be the transport groupoid. It follows from [4, §3] that the diagram (17) can be promoted to a commutative diagram of $\mathrm{Or}G$-spectra whose columns are homotopy fibrations

$$\begin{tikzcd}
KH(\mathcal{L}^1 \otimes_{\mathbb{C}} A[\mathcal{G}(G/H)])[+1] \arrow[d, "Sch"] \arrow[dr, "\mathrm{Tr}Sch"] & \\
HC(\mathcal{B} \otimes_{\mathbb{C}} A[\mathcal{G}(G/H)] : \mathcal{L}^1 \otimes_{\mathbb{C}} A[\mathcal{G}(G/H)])[-1] \arrow[r] \arrow[d] & HC(A[\mathcal{G}(G/H)]/\mathbb{C})[-1] \arrow[d] \\
K(\mathcal{B} \otimes_{\mathbb{C}} A[\mathcal{G}(G/H)] : \mathcal{L}^1 \otimes_{\mathbb{C}} A[\mathcal{G}(G/H)]) \arrow[r] & \kappa(A[\mathcal{G}(G/H)]).
\end{tikzcd}$$

If now X is any G-simplicial set, then taking G-equivariant homology yields a diagram whose columns are again homotopy fibrations

$$\begin{tikzcd}
H^G(X, KH(\mathcal{L}^1 \otimes_{\mathbb{C}} A))[+1] \arrow[r, equal] \arrow[d, "Sch"] & H^G(X, KH(\mathcal{L}^1 \otimes_{\mathbb{C}} A))[+1] \arrow[d] \\
H^G(X, HC(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A))[-1] \arrow[r, "\mathrm{Tr}"] \arrow[d] & H^G(X, HC(A/\mathbb{C}))[-1] \arrow[d] \\
H^G(X, K(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A)) \arrow[r] & H^G(X, \kappa(A)).
\end{tikzcd} \tag{27}$$

Hence

$$H^G(X, \kappa(A)) = \mathrm{hocofi}(H^G(X, KH(\mathcal{L}^1 \otimes_{\mathbb{C}} A))[+1] \to H^G(X, HC(A/\mathbb{C}))[-1]). \tag{28}$$

Similarly, a choice of rank one projection induces a map of OrG-spectra

$$K(A[\mathcal{G}(G/H)]) \to K(\mathcal{B} \otimes_{\mathbb{C}} A[\mathcal{G}(G/H)] : \mathcal{L}^1 \otimes_{\mathbb{C}} A[\mathcal{G}(G/H)]).$$

Taking equivariant homology we obtain a map

$$H^G(X, K(A)) \to H^G(X, K(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A)).$$

Composing this map with the bottom arrow in diagram (27) we obtain an equivariant character

$$\tau : H^G(X, K(A)) \to H^G(X, \kappa(A)). \tag{29}$$

In what follows we shall be interested in several families of finite subgroups of a given group. We write $\mathcal{F}in$ and $\mathcal{F}cyc$ for the family of finite subgroups and the subfamily of those finite subgroups that are cyclic. If $m \geq 1$ we write $\mathcal{C}yc_m$ for the family of those cyclic subgroups whose order divides m. If G is a group and $\mathcal{F}$ a family of subgroups, we write $\mathcal{E}(G, \mathcal{F})$ for the corresponding classifying space. If $H \subset G$ is a subgroup in the family $\mathcal{F}$, we write (H) for the conjugacy class of H and

$$(\mathcal{F}) = \{(H) : H \in \mathcal{F}\}$$

for the set of all conjugacy classes of subgroups of G in the family $\mathcal{F}$. If G is a group and $C \subset G$ is a cyclic subgroup, we write $N_G C$ for its normalizer, $Z_G C$ for its centralizer, and put

$$W_G C = N_G C / Z_G C.$$

If $C \subset G$ is a finite cyclic group, and $A(C)$ is its Burnside ring, then there is a canonical isomorphism $A(C) \otimes \mathbb{Q} \cong \mathbb{Q}^{\mathcal{F}in C}$. We write $\theta_C \in A(C) \otimes \mathbb{Q}$ for the element corresponding to the characteristic function $\chi_C \in \mathbb{Q}^{\mathcal{F}in C}$.

Proposition 30. *Let G be a group. Then the map $H^G_*(\mathcal{E}(G, \mathcal{F}cyc), \kappa(\mathbb{C})) \to H^G_*(\mathcal{E}(G, \mathcal{F}in), \kappa(\mathbb{C}))$ is an isomorphism and*

$$H^G_n(\mathcal{E}(G, \mathcal{F}cyc), \kappa(\mathbb{C})) \otimes \mathbb{Q} = \bigoplus_{p+q=n} \bigoplus_{(C) \in (\mathcal{F}cyc)} H_p(Z_G C, \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \theta_C \cdot \kappa_q(\mathbb{C}[C]) \otimes \mathbb{Q} \tag{31}$$

Proof. If H is a finite subgroup, then the equivalence $KH(\mathcal{L}^1) \xrightarrow{\sim} K^{\mathrm{top}}(\mathcal{L}^1) \overset{\sim}{\leftarrow} K^{\mathrm{top}}(\mathbb{C})$ induces an equivalence $KH(\mathcal{L}^1[\mathcal{G}(G/H)]) \overset{\sim}{\to} K^{\mathrm{top}}(C^*(\mathcal{G}(G/H)))$. Hence if X is a $(G, \mathcal{F}in)$-complex, we have an equivalence

$$H^G(X, KH(\mathcal{L}^1)) \xrightarrow{\sim} H^G(X, K^{\mathrm{top}}(\mathbb{C})). \tag{32}$$

Thus the map

$$H^G(\mathcal{E}(G, \mathcal{F}cyc), KH(\mathcal{L}^1)) \to H^G(\mathcal{E}(G, \mathcal{F}in), KH(\mathcal{L}^1))$$

is a weak equivalence because

$$H^G(\mathcal{E}(G,\mathcal{F}cyc),K^{\mathrm{top}}(\mathbb{C}))\to H^G(\mathcal{E}(G,\mathcal{F}in),K^{\mathrm{top}}(\mathbb{C}))$$

is ([11, Proposition 69]). Similarly,

$$H^G(\mathcal{E}(G,\mathcal{F}cyc),HC(\mathbb{C}/\mathbb{C}))\to H^G(\mathcal{E}(G,\mathcal{F}in),HC(\mathbb{C}/\mathbb{C}))$$

is an equivalence (see [12, §9] or [3, §7]). From (28) and what we have just proved, it follows that $H^G(\mathcal{E}(G,\mathcal{F}cyc),\kappa(\mathbb{C}))\to H^G(\mathcal{E}(G,\mathcal{F}in),\kappa(\mathbb{C}))$ is an equivalence. This shows the first assertion of the proposition. From (32), [10, Theorem 0.7] and [11, Theorem 172], we get

$$\begin{aligned}H_n^G(\mathcal{E}(G,\mathcal{F}cyc),KH(\mathcal{L}^1))\otimes\mathbb{Q}=&\\ \bigoplus_{p+q=n}\bigoplus_{(C)\in(\mathcal{F}cyc)}&H_p(Z_GC,\mathbb{Q})\otimes_{\mathbb{Q}[W_GC]}\theta_C\cdot K_q^{\mathrm{top}}(\mathbb{C}[C])\otimes\mathbb{Q}.\end{aligned}$$

Next write $\mathrm{con}_f(G)$ for the conjugacy classes of elements of G of finite order, and $Gen(C)$ for the set of all generators of $C\in\mathcal{F}cyc$. By using [12, Lemma 7.4] and the argument of the proof of [4, Proposition 2.2.1] we obtain

$$\begin{aligned}&H_n^G(\mathcal{E}(G,\mathcal{F}cyc),HC(\mathbb{C}/\mathbb{C}))=\\&=\bigoplus_{p+q=n}\bigoplus_{(g)\in\mathrm{con}_f(G)}H_p(Z_G(\langle g\rangle),\mathbb{Q})\otimes HC_q(\mathbb{C}/\mathbb{C})\\&=\bigoplus_{p+q=n}\bigoplus_{(C)\in(\mathcal{F}cyc)}H_p(Z_GC,\mathbb{Q})\otimes_{\mathbb{Q}[W_G(C)]}\mathrm{map}(Gen(C),HC_q(\mathbb{C}/\mathbb{C}))\\&=\bigoplus_{p+q=n}\bigoplus_{(C)\in(\mathcal{F}cyc)}H_p(Z_GC,\mathbb{Q})\otimes_{\mathbb{Q}[W_G(C)]}\theta_C\cdot HC_q(\mathbb{C}[C]/\mathbb{C})\end{aligned}$$

It follows from the proof of Proposition 20 that under the isomorphism (32) and the identity $H_*^G(-,HC(\mathbb{C}/\mathbb{C}))=H_*^G(-,HC^{\mathrm{top}}(\mathbb{C}))$ the map $\mathrm{Tr}Sch$ identifies with $\mathrm{Tr}Sch^{\mathrm{top}}$. Hence, by naturality, the map

$$(\mathrm{Tr}Sch)_n:H_{n+1}^G(\mathcal{E}(G,\mathcal{F}cyc),KH(\mathcal{L}^1))\to H_{n-1}^G(\mathcal{E}(G,\mathcal{F}cyc),HC(\mathbb{C}/\mathbb{C}))$$

is induced by the maps

$$\mathrm{Tr}Sch_q^{\mathrm{top}}:K_{q+1}^{\mathrm{top}}(\mathbb{C}[C])\to HC_{q-1}^{\mathrm{top}}(\mathbb{C}[C])=HC_{q-1}(\mathbb{C}[C]/\mathbb{C}).$$

The computation of $H_n^G(\mathcal{E}(G,\mathcal{F}in),\kappa(\mathbb{C}))\otimes\mathbb{Q}$ is now immediate from this. □

Remark 33. We have an equivalence of $(G,\mathcal{F}cyc)$-spaces

$$\operatorname*{colim}_m\mathcal{E}(G,\mathcal{C}yc_m)\xrightarrow{\cong}\mathcal{E}(G,\mathcal{F}cyc)$$

where the colimit is taken with respect to the partial order of divisibility. Hence for every OrG-spectrum E,

$$H_*^G(\mathcal{E}(G,\mathcal{F}cyc),E) = \operatorname*{colim}_m H_*^G(\mathcal{E}(G,\mathcal{C}yc_m),E).$$

Moreover it is clear from the proof of Proposition 30 that for every m the map

$$H_*^G(\mathcal{E}(G,\mathcal{C}yc_m),\kappa(\mathbb{C}))\otimes\mathbb{Q} \to H_*^G(\mathcal{E}(G,\mathcal{F}cyc),\kappa(\mathbb{C}))\otimes\mathbb{Q}$$

is the inclusion

$$\bigoplus_{p+q=n}\bigoplus_{(C)\in(\mathcal{C}yc_m)} H_p(Z_GC,\mathbb{Q})\otimes_{\mathbb{Q}[W_GC]}\theta_C\cdot\kappa_q(\mathbb{C}[C])\otimes\mathbb{Q}\hookrightarrow$$
$$\bigoplus_{p+q=n}\bigoplus_{(C)\in(\mathcal{F}cyc)} H_p(Z_GC,\mathbb{Q})\otimes_{\mathbb{Q}[W_GC]}\theta_C\cdot\kappa_q(\mathbb{C}[C])\otimes\mathbb{Q}.$$

4 Conditions equivalent to the rational injectivity of the κ assembly map

Let G be a group. As shown in the proof of Proposition (30), we have a direct sum decomposition

$$H_n^G(\mathcal{E}(G,\mathcal{F}cyc),HC(\mathbb{C}/\mathbb{C})) =$$
$$\bigoplus_{p+q=n}\bigoplus_{(C)\in(\mathcal{F}cyc)} H_p(Z_GC,\mathbb{Q})\otimes_{\mathbb{Q}[W_G(C)]}\theta_C\cdot HC_q(\mathbb{C}[C]/\mathbb{C}).$$

By the same proof, for each p,q we have

$$\bigoplus_{(C)\in(\mathcal{F}cyc)} H_p(Z_GC,\mathbb{Q})\otimes_{\mathbb{Q}[W_G(C)]}\theta_C\cdot HC_q(\mathbb{C}[C]/\mathbb{C}) =$$
$$\bigoplus_{(g)\in\mathrm{con}_f(G)} H_p(Z_G\langle g\rangle,\mathbb{Q})\otimes HC_q(\mathbb{C}/\mathbb{C}). \quad (34)$$

On the other hand we also have a decomposition

$$HC_n(\mathbb{C}[G]/\mathbb{C}) = \bigoplus_{(g)\in\mathrm{con}(G)} HC_n^{(g)}(\mathbb{C}[G]/\mathbb{C}).$$

The assembly map identifies

$$H_n^G(\mathcal{E}(G,\mathcal{F}cyc),HC(\mathbb{C}/\mathbb{C})) = \bigoplus_{(g)\in\mathrm{con}_f(G)} HC_n^{(g)}(\mathbb{C}[G]/\mathbb{C}).$$

Thus there is a projection

$$\pi_n^{\mathcal{F}cyc}: HC_n(\mathbb{C}[G]/\mathbb{C})\to H_n^G(\mathcal{E}(G,\mathcal{F}cyc),HC(\mathbb{C}/\mathbb{C}))$$

which is left inverse to the asembly map. By composing the map

$$\mathrm{Tr}Sch : KH_{n+1}(\mathcal{L}^1[G]) \to HC_{n-1}(\mathbb{C}[G]/\mathbb{C})$$

with the projection above, we obtain a map

$$\pi_{n-1}^{\mathcal{F}cyc}\mathrm{Tr}Sch : KH_{n+1}(\mathcal{L}^1[G]) \otimes \mathbb{Q} \to H_{n-1}^G(\mathcal{E}(G,\mathcal{F}cyc), HC(\mathbb{C}/\mathbb{C})). \tag{35}$$

Next, if $m \geq 1$ then

$$H_n^G(\mathcal{E}(G,\mathcal{C}yc_m), HC(\mathbb{C}/\mathbb{C})) = \bigoplus_{(g)\in \mathrm{con}_f(G), g^m=1} HC_n^{(g)}(\mathbb{C}[G]/\mathbb{C}).$$

Thus we also have a map

$$\pi_{n-1}^{\mathcal{C}yc_m}\mathrm{Tr}Sch : KH_{n+1}(\mathcal{L}^1[G]) \otimes \mathbb{Q} \to H_{n-1}^G(\mathcal{E}(G,\mathcal{C}yc_m), HC(\mathbb{C}/\mathbb{C})). \tag{36}$$

In the following proposition we use the following notation. We write

$$\begin{aligned} &H_n^G(\mathcal{E}(G,\mathcal{C}yc_m), \kappa(\mathbb{C}) \otimes \mathbb{Q})^+ := \\ &\qquad \bigoplus_{p+q=n, q\geq 1} \bigoplus_{(C)\in(\mathcal{C}yc_m)} H_p(Z_GC, \mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \theta_C \cdot \kappa_q(\mathbb{C}[C]) \otimes \mathbb{Q} \end{aligned} \tag{37}$$

and

$$\begin{aligned} &H_n^G(\mathcal{E}(G,\mathcal{C}yc_m), \kappa(\mathbb{C}) \otimes \mathbb{Q})^- := \\ &\qquad \bigoplus_{p+q=n, q\leq 0} \bigoplus_{(C)\in(\mathcal{C}yc_m)} H_p(Z_GC, \mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \theta_C \cdot \kappa_q(\mathbb{C}[C]) \otimes \mathbb{Q}. \end{aligned} \tag{38}$$

Note that, by Proposition 30, $H_n^G(\mathcal{E}(G,\mathcal{C}yc_m), \kappa(\mathbb{C})) \otimes \mathbb{Q}$ is the direct sum of (37) and (38).

Proposition 39. *Let G be a group, $n \in \mathbb{Z}$ and $m \geq 1$. The following are equivalent.*

i) The rational assembly map

$$H_n^G(\mathcal{E}(G,\mathcal{C}yc_m), \kappa(\mathbb{C})) \otimes \mathbb{Q} \to \kappa_n(\mathbb{C}[G]) \otimes \mathbb{Q} \tag{40}$$

is injective.

ii) The restriction of the rational assembly map to the summand (37) is injective.

iii) The image of the map (36) coincides with the image of

$$\mathrm{Tr}Sch : H_{n+1}^G(\mathcal{E}(G,\mathcal{C}yc_m), KH(\mathcal{L}^1)) \otimes \mathbb{Q} \to H_{n-1}^G(\mathcal{E}(G,\mathcal{C}yc_m), HC(\mathbb{C}/\mathbb{C})).$$

Proof. It is clear that i) implies ii). Assume that ii) holds and consider the following commutative diagram with exact columns:

$$\begin{tikzcd}
H^G_{n+1}(\mathcal{E}(G,\mathcal{C}yc_m),KH(\mathcal{L}^1))\otimes\mathbb{Q} \arrow[r] \arrow[d, "\mathrm{Tr}Sch"] & KH_{n+1}(\mathcal{L}^1[G])\otimes\mathbb{Q} \arrow[d] \arrow[dl, "(36)"] \\
H^G_{n-1}(\mathcal{E}(G,\mathcal{C}yc_m),HC(\mathbb{C}/\mathbb{C})) \arrow[r] \arrow[d] & HC_{n-1}(\mathbb{C}[G]) \arrow[d] \\
H^G_{n}(\mathcal{E}(G,\mathcal{C}yc_m),\kappa(\mathbb{C}))\otimes\mathbb{Q} \arrow[r] \arrow[d] & \kappa_n(\mathbb{C}[G])\otimes\mathbb{Q} \arrow[d] \\
H^G_{n}(\mathcal{E}(G,\mathcal{C}yc_m),KH(\mathcal{L}^1))\otimes\mathbb{Q} \arrow[r] & KH_{n}(\mathcal{L}^1[G])\otimes\mathbb{Q}.
\end{tikzcd}$$

Let x be an element of the kernel of the map of part i), that is of the first map above bottom in the diagram above. Write $x = x_+ + x_-$, with x_+ in (37) and x_- in (38). The image of x under the vertical map must be zero, since by Yu's theorem ([14], see also [4]), the bottom horizontal map is injective. By (16) and the proof of Proposition 30, this implies that $x_- = 0$, proving that ii) implies i). Next assume y is an element in the image of (36) which is not in the image of the vertical map TrSch in the diagram above. Then the image of y under the vertical map is a nonzero element of the kernel of the next horizontal map. Thus i) implies iii). The converse is also clear, using Yu's theorem again. □

Corollary 41. *Let G be a group and let $n,p\in\mathbb{Z}$ with $n\equiv p+1 \mod 2$. Assume that the map*

$$H^G_p(\mathcal{E}(G,\mathcal{F}cyc),KH(\mathcal{L}^1))\otimes\mathbb{Q}\to KH_p(\mathcal{L}^1[G])\otimes\mathbb{Q} \tag{42}$$

is surjective. Then the map (40) is injective for every $m\ge 1$.

Proof. By Yu's theorem ([14],[4]) the map (42) is always injective; under our current assumptions, it is an isomorphism. Moreover, by [5, Theorem 6.5.3], the groups $KH_p(\mathcal{L}^1[G])$ depend only on the parity of p. It follows that condition iii) of Proposition 39 holds for every m and every $n\equiv p+1 \mod 2$. This concludes the proof. □

5 Rational injectivity of the equivariant regulators

Let G be a group. By composing the equivariant character (29) with the map induced by the inclusion $\mathbb{Z}\subset\mathbb{C}$ we obtain a map

$$H^G_*(\mathcal{E}(G,\{1\}),K(\mathbb{Z}))\to H^G_*(\mathcal{E}(G,\mathcal{F}in),K(\mathbb{C}))\xrightarrow{\tau} H^G_*(\mathcal{E}(G,\mathcal{F}in),\kappa(\mathbb{C})). \tag{43}$$

Proposition 44. *The map (43) is rationally injective.*

Proof. We have

$$H_n^G(\mathcal{E}(G,\{1\}),K(\mathbb{Z}))\otimes\mathbb{Q}=\bigoplus_{p+q=n}H_p(G,\mathbb{Q})\otimes K_q(\mathbb{Z})\otimes\mathbb{Q}. \tag{45}$$

By Theorem 23, the regulators $K_q(\mathbb{Z})\to K_q(\mathbb{C})\to\kappa_q(\mathbb{C})=MK_q(\mathbb{C})$ induce a monomorphism from (45) to

$$\bigoplus_{p+q=n}H_p(G,\mathbb{Q})\otimes\kappa_q(\mathbb{C})\otimes\mathbb{Q}. \tag{46}$$

The map (43) tensored with $\mathbb{Q}$ is the composite of the above monomorphism with the inclusion of (46) as a direct summand in (31). $\square$

Let F be a number field, G a group and $m\geq 1$. Let ζ_m be a primitive m^{th} root of 1. The map $\mathcal{E}(G,\mathcal{C}yc_m)\to\mathcal{E}(G,\mathcal{F}cyc)$, together with the inclusion

$$F\subset F(\zeta_m)\xrightarrow{\iota}\mathbb{C}^{\hom(F(\zeta_m),\mathbb{C})}$$

and the character $\tau: K(\mathbb{C})\to\kappa(\mathbb{C})$, induce a homomorphism

$$H_*^G(\mathcal{E}(G,\mathcal{C}yc_m),K(F))\to H_*^G(\mathcal{E}(G,\mathcal{F}cyc),\kappa(\mathbb{C}))^{\hom(F(\zeta_m),\mathbb{C})}. \tag{47}$$

Proposition 48. *The map (47) is rationally injective.*

Proof. By [9, Theorem 0.3], we have

$$\begin{aligned}&H_n^G(\mathcal{E}(G,\mathcal{C}yc_m),K(F))=\\&\qquad\bigoplus_{p+q=n}\ \bigoplus_{(C)\in(\mathcal{C}yc_m)}H_p(Z_GC,\mathbb{Q})\otimes_{\mathbb{Q}[Z_GC]}\theta_C\cdot K_q(F[C])\otimes\mathbb{Q}.\end{aligned} \tag{49}$$

By Lemma 26 the maps $K_q(F[C])\to\kappa_q(\mathbb{C}[C])^{\hom(F(\zeta_m),\mathbb{C})}$ with $C\in\mathcal{C}yc_m$ induce a rational monomorphism from (49) to

$$\bigoplus_{p+q=n}\ \bigoplus_{(C)\in(\mathcal{C}yc_m)}H_p(Z_GC,\mathbb{Q})\otimes_{\mathbb{Q}[Z_GC]}\theta_C\cdot\kappa_q(\mathbb{C}[C])^{\hom(F(\zeta_m),\mathbb{C})}. \tag{50}$$

The map (47) tensored with $\mathbb{Q}$ is the composite of the above monomorphism with the inclusion of (50) as a summand in (31). $\square$

6 Comparing conjectures and assembly maps

Theorem 51. *Let G be a group. Assume that the equivalent conditions of Proposition 39 hold for G with $m=1$. Then the assembly map*

$$H_n^G(\mathcal{E}(G,\{1\}),K(\mathbb{Z}))\to K_n(\mathbb{Z}[G])$$

is rationally injective. In particular this is the case whenever G satisfies the rational KH-isomorphism conjecture with $\mathcal{L}^1$-coefficients.

Proof. Immediate from Proposition 44 and Corollary 41. □

THEOREM 52. *Let G be a group and $m \geq 1$. Assume that the equivalent conditions of Proposition 39 hold for G and m. Then for every number field F, the assembly map*

$$H_n^G(\mathcal{E}(G, \mathcal{C}yc_m), K(F)) \to K_n(F[G])$$

is rationally injective. If moreover the condition holds for all m –as is the case, for example, if G satisfies the rational KH-isomorphism conjecture with $\mathcal{L}^1$-coefficients- then G satisfies the rational injectivity part of the K-theory isomorphism conjecture with coefficients in any number field.

Proof. Immediate from Proposition 48 and Corollary 41. □

REFERENCES

[1] Arthur Bartels and Wolfgang Lück, *Isomorphism conjecture for homotopy K-theory and groups acting on trees*, J. Pure Appl. Algebra 205 (2006), no. 3, 660–696. ↑439

[2] Guillermo Cortiñas, *Algebraic v. topological K-theory: a friendly match*, Topics in algebraic and topological K-theory, Lecture Notes in Math., vol. 2008, Springer, Berlin, 2011, pp. 103–165. ↑443, 445

[3] Guillermo Cortiñas and Eugenia Ellis, *Isomorphism conjectures with proper coefficients*, J. Pure Appl. Algebra 218 (2014), 1224–1263, DOI 10.1016/j.jpaa.2013.11.016. ↑442, 449

[4] Guillermo Cortiñas and Gisela Tartaglia, *Operator ideals and assembly maps in K-theory.*, Proc. Amer. Math. Soc. 142 (2014), 1089–1099. ↑439, 447, 449, 452

[5] Guillermo Cortiñas and Andreas Thom, *Comparison between algebraic and topological K-theory of locally convex algebras*, Adv. Math. 218 (2008), no. 1, 266–307. ↑443, 445, 452

[6] Joachim Cuntz and Daniel Quillen, *Excision in bivariant periodic cyclic cohomology*, Invent. Math. 127 (1997), no. 1, 67–98. ↑443

[7] Max Karoubi, *Homologie cyclique et K-théorie*, Astérisque 149 (1987), 147 (French, with English summary). ↑440, 444, 446

[8] ———, *Sur la K-théorie multiplicative*, Cyclic cohomology and noncommutative geometry (Waterloo, ON, 1995), Fields Inst. Commun., vol. 17, Amer. Math. Soc., Providence, RI, 1997, pp. 59–77 (French, with English summary). ↑444

[9] Wolfgang Lück, *Chern characters for proper equivariant homology theories and applications to K- and L-theory*, J. Reine Angew. Math. 543 (2002), 193–234. ↑442, 446, 453

[10] ———, *The relation between the Baum-Connes conjecture and the trace conjecture*, Invent. Math. 149 (2002), no. 1, 123–152. ↑442, 449

[11] Wolfgang Lück and Holger Reich, *The Baum-Connes and the Farrell-Jones conjectures in K- and L-theory*, Handbook of K-theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 703–842. ↑440, 441, 442, 449

[12] ———, *Detecting K-theory by cyclic homology*, Proc. London Math. Soc. (3) 93 (2006), no. 3, 593–634. ↑442, 449

[13] Charles A. Weibel, *Homotopy algebraic K-theory*, Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987), Contemp. Math., vol. 83, Amer. Math. Soc., Providence, RI, 1989, pp. 461–488. ↑445

[14] Guoliang Yu, *The algebraic K-theory Novikov conjecture for group algebras over the ring of Schatten class operators*, available at arXiv:1106.3796. ↑439, 452

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