On Quadratic Diophantine Equations in Four Variables and Orders Associated with Lattices

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Abstract. This paper treats certain lattices in ternary quadratic spaces, which are obtained from the data of a non-zero element and a maximal lattice in a quaternary space. Each class in the genus of such a lattice with respect to the special orthogonal group corresponds to an isomorphism class in the genus of an order associated with the lattice in a quaternion algebra. Using this result together with the principle of Shimura, we show that the number of classes of the primitive solutions of a quadratic Diophantine equation in four variables coincides with the type number of the order under suitable conditions on the given element. We illustrate this result by explicit examples.

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Introduction

Let (V, φ) be a nondegenerate quadratic space of dimension 4 over a number field F, that is, V is a four-dimensional vector space over F and φ is a nondegenerate symmetric F-bilinear form on V. For an element h of V such that $\varphi[h](=\varphi(h, h)) \neq 0$, we put

$$W = (Fh)^{\perp} = \{x \in V \mid \varphi(x, h) = 0\}$$

and consider a quadratic space (W, ψ) of dimension 3 over F with the restriction ψ of φ to W. In this paper the special orthogonal group SO^{ψ} of ψ is regarded as the subgroup $\{\gamma \in SO^{\varphi} \mid h\gamma = h\}$ of the orthogonal group SO^{φ} of φ. The orthogonal group O^{ψ} of ψ is also identified with O^{φ} in a similar manner. For a maximal lattice L in (V, φ) we put

$$L[q,\ \mathfrak{b}] = \{x \in V \mid \varphi[x] = q,\ \varphi(x, L) = \mathfrak{b}\}, \quad D(L) = \{\alpha \in O^{\varphi}_{\mathbf{A}} \mid L\alpha = L\},$$

where $q = \varphi[h]$, $\mathfrak{b} = \varphi(h, L)$, and the subscript $\mathbf{A}$ means adelization. Since $L[q, \mathfrak{b}]$ is stable under $\Gamma^{\cdot}(L) = O^{\varphi} \cap D(L)$, the set $L[q, \mathfrak{b}]/\Gamma^{\cdot}(L)$ is meaningful in an obvious way.

In the sense of Shimura [9, Introduction I], $L[q, \mathfrak{b}]$ is the set of *primitive solutions* of the equation $\varphi[x] = q$. Our interest in this paper is basically the set

$$L[q, \mathfrak{b}]/\Gamma^{\cdot}(L)$$

consisting of the classes of such solutions. By the principle of [9, Theorem 11.6], *each class of solutions of $L_i[q, \mathfrak{b}]$ modulo $\Gamma^{\cdot}(L_i)$ for $i \in J$ corresponds to a class of O^{ψ} relative to an open subgroup $O^{\psi}_{\mathbf{A}} \cap D(L)$ in $O^{\psi}_{\mathbf{A}}$*. Here $\{L_i\}_{i \in J}$ is a set of representatives for the O^{φ}-classes in the O^{φ}-genus of L for which $L_i[q, \mathfrak{b}] \neq \emptyset$ (see also (4.1) below).

Now we consider the lattice $L \cap W$ in (W, ψ) and the even Clifford algebra $A^+(W)$ of ψ, which is a quaternion algebra over F since the dimension of W is 3. Let $A(N)$ be the order generated by $\mathfrak{g}$ and N in the Clifford algebra of ψ, and put $A^+(N) = A^+(W) \cap A(N)$ for an integral lattice N in (W, ψ). Here $\mathfrak{g}$ is the ring of all integers of F and the terms *integral* and *maximal* are given in §1.1. To $L \cap W$ we can associate an order $\mathfrak{O}$ in $A^+(W)$, containing $A^+(L \cap W)$, whose discriminant is given by (3.22) below. The purpose of this paper is to define such an order $\mathfrak{O}$, to give a bijection of the SO^{ψ}-classes in the SO^{ψ}-genus of $L \cap W$ onto the isomorphism classes in the genus of $\mathfrak{O}$, and to prove

$$\sum_{i \in J} \# \{L_i[q, \mathfrak{b}]/\Gamma^{\cdot}(L_i)\} = t(\mathfrak{O}) \tag{0.1}$$

through the above principle under suitable conditions on $h \in L[q, \mathfrak{b}]$, where the genus of $\mathfrak{O}$ is defined by the set $\{\alpha^{-1}\mathfrak{O}\alpha \mid \alpha \in A^+(W)^{\times}_{\mathbf{A}}\}$ and $t(\mathfrak{O})$ is the type number of $\mathfrak{O}$.

To obtain the order $\mathfrak{O}$, we proceed in a similar manner to [10, §4.6] under mild conditions on $h \in L[q, \mathfrak{b}]$ (Proposition 3.3 (3)). As for the bijection, given a lattice N in the genus of $L \cap W$, put $N = (L \cap W)\tau(\alpha)$ with $\alpha \in A^+(W)^{\times}_{\mathbf{A}}$. Here τ is a surjective homomorphism of $G^+(W)_{\mathbf{A}}$ onto $SO^{\psi}_{\mathbf{A}}$ and $G^+(W)$ is the even Clifford group of ψ, which is given by $A^+(W)^{\times}$. Then our bijection is defined by $N \longmapsto \alpha^{-1}\mathfrak{O}\alpha$ (Theorem 3.4 (2)). This result mainly relies on a description of $L \cap W$ in $A^+(W)$ by means of $A^+(L \cap W)$ (cf. [4, Theorem 2.1]). Now in Proposition 4.3, we shall prove that $O^{\psi}\varepsilon D(L \cap W) = O^{\psi}\varepsilon(O^{\psi}_{\mathbf{A}} \cap D(L))$ for every $\varepsilon \in O^{\psi}_{\mathbf{A}}$ under several assumptions on $h \in L[q, \mathfrak{b}]$. Because W is odd-dimensional, the class number of the genus of $L \cap W$ with respect to O^{ψ} equals that with respect to SO^{ψ}. Hence by virtue of the principle mentioned above, the number $\sum_{i \in J} \# \{L_i[q, \mathfrak{b}]/\Gamma^{\cdot}(L_i)\}$ is equal to the number of SO^{ψ}-classes in the SO^{ψ}-genus of $L \cap W$. Our main result (0.1) follows from this fact and the above bijection.

It should be noted that the genus of $\mathfrak{O}$ is determined by the discriminant if $\mathfrak{O}$ has squarefree discriminant, for instance, if $\mathfrak{O}$ is maximal. When the quaternion algebra is totally definite, there are formulas for computing the type number of such an order; see Peters [5, Satz 14] or Pizer [6, Theorem A and Theorem B], etc.. In Section 5 we will take up totally-positive definite forms φ and employ their numerical tables [5, Tabelle 2] and [6, Table 1] for type numbers. It seems that there are few numerical examples for the type number of the genus of an order whose discriminant is not squarefree and for the class number of the genus of a lattice which is neither maximal nor unimodular with respect to a definite form. Here we assume that h satisfies the conditions in Proposition 4.3. Then (0.1) contains the case of non-maximal (and often non-unimodular) $L \cap W$, more clearly, the case that $\mathfrak{O}$ has at most one higher-power prime $\mathfrak{p}^e$ $(e > 1)$ in its discriminant, where $\mathfrak{p}$ is a prime ideal of F (see also (4.10) below).

To see the existence of such an element h, as an application of Proposition 4.3, let $B_{r,\infty}$ be a definite quaternion algebra over $\mathbf{Q}$ ramified exactly at a prime number r and take a prime number d prime to r so that $d \equiv 1 \pmod 4$. In Theorem 5.1 we shall show:
For every odd prime number p prime to dr and $0 \leq n \in \mathbf{Z}$, except when n is odd and p remains prime in $\mathbf{Q}(\sqrt{d})$, there exists a maximal lattice L in (V, φ) over $\mathbf{Q}$ of invariants $\{4, \mathbf{Q}(\sqrt{d}), B_{r,\infty}, 4\}$ (see (1.5) for the definitions) such that $L[dp^n, 2^{-1}d\mathbf{Z}] \neq \emptyset$. And moreover, formula (0.1) is valid for $h \in L[dp^n, 2^{-1}d\mathbf{Z}]$ with an order $\mathfrak{O}$ in $B_{r,\infty}$ of discriminant $rp^n\mathbf{Z}$.
For example, let us take (V, φ) of invariants $\{4, \mathbf{Q}(\sqrt{29}), B_{2,\infty}, 4\}$. By [9, Theorem 12.14 (vi)] the number of O^φ-classes in the O^φ-genus of maximal lattices in (V, φ) equals the number $\#\{\Lambda[29, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)\}$, where Λ is a maximal lattice in a five-dimensional quadratic space over $\mathbf{Q}$ with respect to the quadratic form defined by the sum of five squares. In [9, §12.15], $\#\{\Lambda[29, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)\}$ was determined, and it is 3. Hence the genus of maximal lattices in (V, φ) consists of three O^φ-classes. For details, see the last part of Section 4.1 in the text. We can also see the representatives $\{L_1, L_2, L_3\}$ of such classes by means of a bijection in Lemma 4.1 applied to the set $\Lambda[29, \mathbf{Z}]$. Then we have the following numerical table:

p	$N_1(29p)$	$N_2(29p)$	$N_3(29p)$	$t(2, p)$	$\mathfrak{c}(29p)$
1	2	0	0	1	1
5	0	2	0	1	1
7	0	0	2	1	1
13	0	2	2	2	2
23	0	0	6	1	1
53	0	10	6	3	3
59	24	8	6	3	3

Here we put $N_i(29p) = \#L_i[29p, 2^{-1} \cdot 29\mathbf{Z}]$, $t(2, p) = t(\mathfrak{O})$ with an order $\mathfrak{O}$

in $B_{2,\,\infty}$ of discriminant $2p\mathbf{Z}$, and $\mathfrak{c}(29p) = \sum_{i=1}^{3} \#\{L_i[29p,\, 2^{-1}\cdot 29\mathbf{Z}]/\Gamma^{\cdot}(L_i)\}$; we quoted the type number $t(2,\, p)$ in [6, Table 1]. Therefore we have

$$\#\left\{L_i[29\cdot 59,\; 2^{-1}\cdot 29\mathbf{Z}]/\Gamma^{\cdot}(L_i)\right\} = 1 \quad \text{for } i = 1,\; 2,\; 3,$$

for instance. It is noted that $\#\Gamma(L_1) = 48$, $\#\Gamma(L_2) = 8$, and $\#\Gamma(L_3) = 6$, where $\Gamma(L_i) = SO^{\varphi} \cap D(L_i)$. In Section 5.3 we shall give a few numerical tables for $r = 2$ and $d = 5,\, 13,\, 17,\, 29$ including the above cases. As a special case of Theorem 5.1 we have Corollary 5.2, which states that for *any* d as in Theorem 5.1 *only one* O^{φ}-class in the genus satisfies $L[d,\, 2^{-1}d\mathbf{Z}] \neq \emptyset$ with a lattice L in the class and then (0.1) precisely gives $\#\{L[d,\, 2^{-1}d\mathbf{Z}]/\Gamma^{\cdot}(L)\} = 1$, provided the type number of $B_{r,\,\infty}$ is 1.
The existence of a maximal lattice in Theorem 5.1 can be shown in a similar way to [3, Proposition 4.3] by means of two explicit formulas concerning $\#L[dp^n,\, 2^{-1}d\mathbf{Z}]$ and $\#L[dp^n,\, 2^{-1}\mathbf{Z}]$. Those formulas will be given in (5.2) and (5.3) without detailed proofs because of the length of the paper. The author hopes to prove it in a subsequent paper.

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Notation. We denote by $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$ the ring of rational integers, the field of rational numbers, and the field of real numbers, respectively.
If R is an associative ring with identity element and if M is an R-module, then we write $R^{\times}$ for the group of all invertible elements of R and M_n^m the R-module of $m \times n$-matrices with entries in M. We set $R^{\times 2} = \{a^2 \mid a \in R^{\times}\}$. For a finite set X, we denote by $\#X$ the number of elements in X. We set $[a] = \mathrm{Max}\{n \in \mathbf{Z} \mid n \leq a\}$.
Let V be a vector space over a field F of characteristic 0, and $GL(V)$ the group of all F-linear automorphisms of V. We let $GL(V)$ act on V on the *right*.
Let F be an algebraic number field and $\mathfrak{g}$ the ring of all algebraic integers in F. For a fractional ideal of F we often call it a $\mathfrak{g}$-ideal. Let $\mathbf{a}$, $\mathbf{h}$, and $\mathbf{r}$ be the sets of archimedean primes, nonarchimedean primes, and real archimedean primes of F, respectively. We denote by F_v the completion of F at $v \in \mathbf{a} \cup \mathbf{h}$, and by $F_{\mathbf{A}}$ and $F_{\mathbf{A}}^{\times}$ the adele ring and the idele group of F, respectively. We often identify v with the prime ideal of F corresponding to $v \in \mathbf{h}$, and write x_v for the image of x under the embedding of F into $\mathbf{R}$ over $\mathbf{Q}$ at $v \in \mathbf{r}$. For $v \in \mathbf{h}$ we denote by $\mathfrak{g}_v$, $\mathfrak{p}_v$, and π_v the maximal order, the prime ideal, and a prime element of F_v, respectively. If K is a quadratic extension of F, we denote by $D_{K/F}$ the relative discriminant of K over F, and put $K_v = K \otimes_F F_v$ for $v \in \mathbf{h}$.

By a $\mathfrak{g}$-*lattice*, or simply a *lattice* L in a vector space V over a number field or nonarchimedean local field F, we mean a finitely generated $\mathfrak{g}$-submodule in V containing a basis of V. By an *order* in a quaternion algebra B over F we mean a subring of B containing $\mathfrak{g}$ that is a $\mathfrak{g}$-lattice in B. For a symmetric F-bilinear form φ on V and two subspaces X and Y of V, we denote by $X \oplus Y$ the direct sum of X and Y if $\varphi(x, y) = 0$ for every $x \in X$ and $y \in Y$; then we also denote by $\varphi|_X$ the restriction of φ to X. When X is an object defined over a number field F, we often denote by X_v the localization at a prime v if it is meaningful. For given local objects X_v in the text with $v \in \mathbf{a} \cup \mathbf{h}$, we put $X_{\mathbf{a}} = \prod_{v \in \mathbf{a}} X_v$ and $X_{\mathbf{h}} = \prod_{v \in \mathbf{h}} X_v$.

1 Preliminaries for quadratic forms

1.1 Quadratic spaces and Clifford algebras

Let F be an algebraic number field or its completion at a prime. Throughout the paper we often call the former a global field and the latter a local field when it is nonarchimedean. Let (V, φ) be a quadratic space over F, that is, V is a vector space over F and φ is a symmetric F-bilinear form on V. In this paper we consider only a nondegenerate form φ. We put $\varphi[x] = \varphi(x, x)$ for $x \in V$. We define the orthogonal group and the special orthogonal group of φ by

$$O^{\varphi}(V) = O^{\varphi} = \{\gamma \in GL(V) \mid \varphi(x\gamma, y\gamma) = \varphi(x, y)\},$$
$$SO^{\varphi}(V) = SO^{\varphi} = \{\gamma \in O^{\varphi}(V) \mid \det(\gamma) = 1\}.$$

We denote by $A(\varphi) = A(V)$ the Clifford algebra of φ and by $A^+(\varphi) = A^+(V)$ the even Clifford algebra of φ. For $x \in A(V)$ we mean x^* the image of x under the canonical involution of $A(V)$. We define the even Clifford group $G^+(V)$ of (V, φ) by

$$G^+(V) = \{\alpha \in A^+(V)^{\times} \mid \alpha^{-1} V \alpha = V\}. \tag{1.1}$$

We denote by τ a homomorphism defined as follows:

$$\tau : G^+(V) \longrightarrow SO^{\varphi}(V) \quad \text{via} \quad x\tau(\alpha) = \alpha^{-1} x \alpha \quad \text{for } x \in V. \tag{1.2}$$

This is surjective and the kernel is $F^{\times}$; see [9, Theorem 3.6], for example.

For a $\mathfrak{g}$-lattice L in V we put

$$\widetilde{L} = L^{\sim} = \{x \in V \mid 2\varphi(x, L) \subset \mathfrak{g}\}. \tag{1.3}$$

We call L *integral* with respect to φ if $\varphi[L] \subset \mathfrak{g}$. We note that $L \subset \widetilde{L}$ if L is integral. By a $\mathfrak{g}$-*maximal*, or simply a *maximal*, lattice L with respect to φ, we understand a $\mathfrak{g}$-lattice L in V which is maximal among $\mathfrak{g}$-lattices on which the values $\varphi[x]$ are contained in $\mathfrak{g}$. For an integral lattice L in V with respect to φ, we denote by $A(L)$ the subring of $A(V)$ generated by $\mathfrak{g}$ and L. We also put

$$A^+(L) = A^+(V) \cap A(L). \tag{1.4}$$

Then $A(L)$ (resp. $A^+(L)$) is an order in $A(V)$ (resp. $A^+(V)$) (cf. [9, §8.2]).

For a global field F and $v \in \mathbf{a} \cup \mathbf{h}$, we put $V_v = V \otimes_F F_v$ and denote by φ_v the F_v-bilinear extension of φ to V_v; we then put $(V, \varphi)_v = (V_v, \varphi_v)$. For $v \in \mathbf{h}$ and a $\mathfrak{g}_v$-maximal lattice L_v in V_v, $(V, \varphi)_v$ has a *Witt decomposition* as follows (cf. [9, Lemma 6.5]): $V_v = Z_v \oplus \sum_{i=1}^{r_v}(F_v e_i + F_v f_i)$ and $L_v = N_v + \sum_{i=1}^{r_v}(\mathfrak{g}_v e_i + \mathfrak{g}_v f_i)$ with some elements e_i and f_i $(i = 1, \cdots, r_v)$ such that $\varphi_v(e_i, e_j) = \varphi_v(f_i, f_j) = 0$ and $2\varphi_v(e_i, f_j) = 1$ or 0 according as $i = j$ or $i \neq j$. Here $Z_v = \{z \in V_v \mid \varphi_v(z, e_i) = \varphi_v(z, f_i) = 0 \text{ for every } i\}$, on which φ_v is anisotropic; $N_v = \{x \in Z_v \mid \varphi_v[x] \in \mathfrak{g}_v\}$, which is a unique $\mathfrak{g}_v$-maximal lattice in Z_v with respect to φ_v. The dimension t_v of Z_v is uniquely determined by φ_v and $0 \leq t_v \leq 4$ for $v \in \mathbf{h}$ (cf. [9, Theorem 7.6 (ii)]). We call Z_v a *core subspace* of $(V, \varphi)_v$ and t_v the *core dimension* of (V, φ) at v. For convenience, we also call a subspace U_v of V_v anisotropic if φ_v is so on U_v.

For $\mathfrak{g}$-lattices L and M in V over a global or local field F, we denote by $[L/M]$ a $\mathfrak{g}$-ideal of F generated over $\mathfrak{g}$ by $\det(\alpha)$ of all F-linear automorphisms α of V such that $L\alpha \subset M$. If F is a global field, then $[L/M] = \prod_{v\in\mathbf{h}}[L_v/M_v]$ with the localization $[L/M]_v = [L_v/M_v]$ at each v. Following [11, §6.1], in both global and local F, we call $[\widetilde{L}/L]$ the *discriminant ideal* of (V, φ) if L is a $\mathfrak{g}$-maximal lattice in V with respect to φ. This is independent of the choice of L. If F is a local field, the discriminant ideal of φ coincides with that of a core subspace of φ.

By the *invariants* of (V, φ) over a number field F, we understand a set of data

$$\{n,\ F(\sqrt{\delta}),\ Q(\varphi),\ \{s_v(\varphi)\}_{v\in\mathbf{r}}\}, \tag{1.5}$$

where n is the dimension of V, $F(\sqrt{\delta})$ is the *discriminant field* of φ with $\delta = (-1)^{n(n-1)/2}\det(\varphi)$, $Q(\varphi)$ is the *characteristic quaternion algebra* of φ, and $s_v(\varphi)$ is the *index* of φ at $v \in \mathbf{r}$. For these definitions, the reader is referred to [11, §1.1, 3.1, and 4.1] (cf. also [4, (1.6)]). By virtue of [11, Theorem 4.2], the isomorphism class of (V, φ) is determined by $\{n, F(\sqrt{\delta}), Q(\varphi), \{s_v(\varphi)\}_{v\in\mathbf{r}}\}$ and vice versa.

The characteristic algebra $Q(\varphi_v)$ is also defined for φ_v at $v \in \mathbf{a} \cup \mathbf{h}$ (cf. [11, §3.1]). By [11, Lemma 3.3] the isomorphism class of $(V, \varphi)_v$ is determined by $\{n, F_v(\sqrt{\delta}), Q(\varphi_v)\}$ if $v \in \mathbf{h}$. As for $v \in \mathbf{a}$, it is determined by $\{n, s_v(\varphi)\}$ if $v \in \mathbf{r}$, and by the dimension n if $v \notin \mathbf{r}$. If $v \in \mathbf{r}$, then $Q(\varphi_v)$ is given by [11, (4.2a) and (4.2b)], for example. If $v \notin \mathbf{r}$, then $Q(\varphi_v) = M_2(\mathbf{C})$, where $\mathbf{C}$ is the field of complex numbers.

Let $SO^\varphi(V)_\mathbf{A}$ (resp. $O^\varphi(V)_\mathbf{A}$) be the adelization of $SO^\varphi(V)$ (resp. $O^\varphi(V)$) in the usual sense (cf. [9, §9.6]). For $\alpha \in SO^\varphi(V)_\mathbf{A}$ and a $\mathfrak{g}$-lattice L in V, we denote by $L\alpha$ the $\mathfrak{g}$-lattice in V whose localization at each $v \in \mathbf{h}$ is given by

$L_v\alpha_v$. We put

$$C(L) = \{\alpha \in SO^\varphi(V)_{\mathbf{A}} \mid L\alpha = L\}, \quad C(L_v) = SO^{\varphi_v}(V_v) \cap C(L), \quad (v \in \mathbf{h})$$
$$\Gamma(L) = SO^\varphi(V) \cap C(L).$$

Then the map $\alpha \longmapsto L\alpha^{-1}$ gives a bijection of $SO^\varphi \backslash SO^\varphi_{\mathbf{A}}/C(L)$ onto $\{L\alpha \mid \alpha \in SO^\varphi_{\mathbf{A}}\}/SO^\varphi$. We call $\{L\alpha \mid \alpha \in SO^\varphi_{\mathbf{A}}\}$ the SO^φ-*genus* of L, $\{L\gamma \mid \gamma \in SO^\varphi\}$ the SO^φ-*class* of L, and $\#\{SO^\varphi \backslash SO^\varphi_{\mathbf{A}}/C(L)\}$ the *class number* of SO^φ relative to $C(L)$ or the class number of the genus of L with respect to SO^φ. It is known that all $\mathfrak{g}$-maximal lattices in V with respect to φ form a single SO^φ-genus. Let $A^+(V)^\times_{\mathbf{A}}$ (resp. $G^+(V)_{\mathbf{A}}$) be the adelization of $A^+(V)^\times$ (resp. $G^+(V)$). We can extend τ of (1.2) to a homomorphism of $G^+(V)_{\mathbf{A}}$ onto $SO^\varphi_{\mathbf{A}}$. We denote it by the same symbol τ (cf. [9, §9.10]).

For a $\mathfrak{g}$-lattice L in V, $q \in F$, and a $\mathfrak{g}$-ideal $\mathfrak{b}$ of F, we put

$$L[q] = \{x \in L \mid \varphi[x] = q\}, \quad L[q,\ \mathfrak{b}] = \{x \in V \mid \varphi[x] = q,\ \varphi(x,\ L) = \mathfrak{b}\}.$$

Here $\varphi(x,\ L) = \{\varphi(x,\ y) \mid y \in L\}$, which becomes a $\mathfrak{g}$-ideal of F. Suppose F is a nonarchimedean local field. Let V have dimension $n > 2$ and L be a $\mathfrak{g}$-maximal lattice in V with respect to φ. Then [9, Theorem 10.5] due to Shimura shows that

$$L[q,\ \mathfrak{b}] = hC(L), \tag{1.6}$$

provided $h \in L[q,\ \mathfrak{b}]$ (cf. also [12, Theorem 1.3]).

For a quaternion algebra B over F, we put $2\beta(x,\ y) = xy^\iota + yx^\iota$ for $x, y \in B$ with the main involution ι of B. For an order $\mathfrak{o}$ in B it is known that $[\widetilde{\mathfrak{o}}/\mathfrak{o}] = d(\mathfrak{o})^2$ with an integral ideal $d(\mathfrak{o})$ of F. Here $\widetilde{\mathfrak{o}}$ is defined by (1.3) with β. The ideal $d(\mathfrak{o})$ is called the *discriminant* of $\mathfrak{o}$. If F is a number field and $\mathfrak{o}$ is a maximal order, then $d(\mathfrak{o})$ is the product of all prime ideals ramified in B, which is called the discriminant of B and denoted by D_B. We set

$$T(\mathfrak{o}) = \{\alpha \in B^\times_{\mathbf{A}} \mid \alpha\mathfrak{o} = \mathfrak{o}\alpha\}, \quad T(\mathfrak{o}_v) = B_v^\times \cap T(\mathfrak{o}) \quad (v \in \mathbf{a} \cup \mathbf{h}),$$
$$\Gamma^*(\mathfrak{o}) = B^\times \cap T(\mathfrak{o}),$$

where $B^\times_{\mathbf{A}}$ is the adelization of $B^\times$, $B_v = B \otimes_F F_v$, and $\mathfrak{o}_v = \mathfrak{o} \otimes_{\mathfrak{g}} \mathfrak{g}_v$. The number $\#\{T(\mathfrak{o})\backslash B^\times_{\mathbf{A}}/B^\times\}$ is called the *type number* of $\mathfrak{o}$. Let $U = B^\times_{\mathbf{a}} \prod_{v \in \mathbf{h}} \mathfrak{o}_v^\times$ in $B^\times_{\mathbf{A}}$. Then the number of $U \setminus B^\times_{\mathbf{A}}/B^\times$ is called the *class number* of $\mathfrak{o}$.

Here we introduce two symbols below, which will be used throughout the paper. Let F be a nonarchimedean local field and $\mathfrak{p}$ the prime ideal of F. For $b \in F^\times$ we set

$$(F(\sqrt{b})/\mathfrak{p}) = \begin{cases} 1 & \text{if } F(\sqrt{b}) = F, \\ -1 & \text{if } F(\sqrt{b}) \text{ is an unramified quadratic extension of } F, \\ 0 & \text{if } F(\sqrt{b}) \text{ is a ramified quadratic extension of } F. \end{cases}$$

For a quaternion algebra B over F we set

$$\chi(B) = \begin{cases} 1 & \text{if } B \cong M_2(F), \\ -1 & \text{if } B \text{ is a division algebra.} \end{cases}$$

1.2 Ternary quadratic spaces

We recall some basic facts on 3-dimensional quadratic spaces (W, ψ) over a number field or its completion F. The characteristic algebra $Q(\psi)$ is given by $A^+(W)$ by definition. The core dimension s_v of (W, ψ) at $v \in \mathbf{h}$ is determined by

$$s_v = \begin{cases} 1 & \text{if } Q(\psi_v) = M_2(F_v), \\ 3 & \text{if } Q(\psi_v) \text{ is a division algebra.} \end{cases} \tag{1.7}$$

This can be seen from [11, §3.2] and the proof of [11, Lemma 3.3].

There are isomorphisms of (W, ψ) onto $(A^+(W)^\circ, d\nu^\circ)$ with $d \in F^\times$. Here $A^+(W)^\circ = \{x \in A^+(W) \mid x^* = -x\}$, $\nu[x] = xx^*$ for $x \in A^+(W)$, and ν° is the restriction of ν to $A^+(W)^\circ$. Let us explain such isomorphisms, following [9, §7.3].

Take an orthogonal basis $\{k_1, k_2, k_3\}$ of W with respect to ψ, namely, an F-basis $\{k_i\}$ of W such that $\psi(k_i, k_j) = 0$ for $i \neq j$. Under the identification of W with the corresponding subspace in the Clifford algebra $A(W)$, put $\xi = k_1k_2k_3 \in A(W)^\times$; then $F + F\xi$ is the center of $A(W)$. We see that $A^+(W) = F + Fk_1k_2 + Fk_1k_3 + Fk_2k_3$ and $W\xi = Fk_1k_2 + Fk_1k_3 + Fk_2k_3$. By [9, Theorem 2.8 (ii)], $A^+(W)$ is a quaternion algebra over F; the main involution coincides with the canonical involution $*$ restricted to $A^+(W)$. Then the mapping $x \longmapsto x\xi$ gives an F-linear isomorphism of W onto $A^+(W)^\circ$ such that $(x\xi)(x\xi)^* = \xi\xi^*\psi[x]$ for $x \in W$. Putting $\nu[y] = yy^*$ for $y \in A^+(W)$, we have an isomorphism

$$(W, \psi) \cong (A^+(W)^\circ, (\xi\xi^*)^{-1}\nu^\circ) \quad \text{via } x \longmapsto x\xi. \tag{1.8}$$

We note that $\xi\xi^* \in \det(\psi)F^{\times 2}$, since $\xi\xi^* = \psi[k_1]\psi[k_2]\psi[k_3] \in F^\times$.

Let $G^+(W)$ be the even Clifford group of (W, ψ) as in (1.1) and τ the homomorphism defined in (1.2). By the definition of $A^+(W)^\circ$, $\alpha^{-1}A^+(W)^\circ\alpha = A^+(W)^\circ$ for $\alpha \in A^+(W)^\times$. Hence we have $G^+(W) = A^+(W)^\times$. Moreover, under the isomorphism (1.8) we can understand that

$$x\tau(\alpha) = \alpha^{-1}x\xi\alpha\xi^{-1}$$

for $x \in W$ and $\alpha \in A^+(W)^\times$.

Now, the pair $(A^+(W), \nu)$ can be viewed as a quaternary quadratic space over F. We note that $\nu(x, y) = 2^{-1}Tr_{A^+(W)/F}(xy^*)$ for $x, y \in A^+(W)$. For an integral lattice N in W with respect to ψ, we consider the order $A^+(N)$ in $A^+(W)$ defined by (1.4). Its discriminant $d(A^+(N))$ is given by $[A^+(N)\tilde{}/A^+(N)] = d(A^+(N))^2$, where $A^+(N)\tilde{}$ is defined by (1.3) with ν. By

[4, Lemma 1.1], $d(A^+(N)) = 2^{-1}[\widetilde{N}/N]$. It is noted that *if the order $A^+(N)$ is maximal in $A^+(W)$ for an integral lattice N in $(W,\, \psi)$, then N is $\mathfrak{g}$-maximal with respect to ψ.* The converse is not true; namely, in general, $A^+(N)$ is not maximal even if N is a maximal lattice.

2 Orthogonal complements in quaternary spaces

2.1 Invariants and discriminant ideals

Let $(V,\, \varphi)$ be a 4-dimensional quadratic space over a number field F. The characteristic algebra $Q(\varphi)$ is determined by $A(\varphi) \cong M_2(Q(\varphi))$ by definition. Set $B = Q(\varphi)$ and $K = F(\sqrt{\delta})$ with $\delta = \det(\varphi)$. The core dimension t_v of $(V,\, \varphi)$ at $v \in \mathbf{h}$ is determined by

$$t_v = \begin{cases} 0 & \text{if } F_v(\sqrt{\delta}) = F_v \text{ and } Q(\varphi_v) = M_2(F_v), \\ 4 & \text{if } F_v(\sqrt{\delta}) = F_v \text{ and } Q(\varphi_v) \text{ is a division algebra,} \\ 2 & \text{if } F_v(\sqrt{\delta}) \neq F_v. \end{cases} \tag{2.1}$$

This can be seen from [11, §3.2] and the proof of [11, Lemma 3.3].
For $h \in V$ such that $\varphi[h] = q \neq 0$ we put

$$W = (Fh)^{\perp} = \{x \in V \mid \varphi(x,\, h) = 0\}. \tag{2.2}$$

Then $(W,\, \psi)$ is a nondegenerate ternary quadratic space over F with the restriction ψ of φ to W and $(V,\, \varphi) = (W,\, \psi) \oplus (Fh,\, \varphi|_{Fh})$. The invariants of $(W,\, \psi)$ are given by $\{3,\, F(\sqrt{-\delta q}),\, Q(\psi),\, \{s_v(\psi)\}_{v \in \mathbf{r}}\}$, which are independent of the choice of h so that $\varphi[h] = q$. The characteristic algebra $Q(\psi) = A^+(W)$ is determined by the local algebras $Q(\psi_v)$ for all primes v of F. Then by [2, Theorem 1.1 (1)], $Q(\psi_v) = M_2(F_v)$ holds exactly in the following cases:

$$\begin{aligned} &\delta \in F_v^{\times 2} \text{ and } v \nmid D_B, \\ &\delta \notin F_v^{\times 2}, v \nmid D_B, \text{ and } q \in \kappa_v[K_v^{\times}], \\ &\delta \notin F_v^{\times 2}, v \mid D_B, \text{ and } q \notin \kappa_v[K_v^{\times}], \\ &v \in \mathbf{r}, q_v > 0, \text{ and } s_v(\varphi) = 0,\ 2, \\ &v \in \mathbf{r}, q_v < 0, \text{ and } s_v(\varphi) = 0,\ -2, \\ &v \in \mathbf{a} \text{ such that } v \notin \mathbf{r}, \end{aligned}$$

where κ_v is the norm form of K_v. It should be noted that

$$M_2(Q(\varphi)) \cong Q(\psi) \otimes_F \{K,\ q\}, \tag{2.3}$$

where F is a number field or its completion and $\{K,\, q\}$ is the quaternion algebra over F defined in [4, (1.12)] if $K \neq F$; we set $\{K,\, q\} = M_2(F)$ if $K = F$ (see also [9, §1.10]). This (2.3) can be seen from [11, Theorem 7.4 (i)]. The index at $v \in \mathbf{r}$ is given by $s_v(\psi) = s_v(\varphi) - 1$ if $q_v > 0$ and $s_v(\psi) = s_v(\varphi) + 1$ if $q_v < 0$. The core dimension of $(W,\, \psi)$ at $v \in \mathbf{h}$ is determined by (1.7).

The Clifford algebra $A(W)$ can be viewed as a subalgebra of $A(V)$ with the restriction ψ. Then $A^+(W) = \{x \in A^+(V) \mid xh = hx\}$ and $G^+(W) = \{\alpha \in G^+(V) \mid \alpha h = h\alpha\}$ by [9, Lemma 3.16]. The canonical involution of $A(W)$ coincides with $*$ of $A(V)$ restricted to $A(W)$. In particular, such an involution $*$ gives the main involution of the quaternion algebra $A^+(W)$.

Let L and M be $\mathfrak{g}$-maximal lattices in V and W with respect to φ and ψ, respectively. The discriminant ideals of φ and ψ are given as follows:

$$[\widetilde{L}/L] = D_{K/F}\mathfrak{e}^2, \tag{2.4}$$

$$[\widetilde{M}/M] = 2\mathfrak{a}^{-1}D^2_{Q(\psi)} \cap 2\mathfrak{a}, \tag{2.5}$$

where $\mathfrak{e}$ is the product of all the prime ideals which are ramified in B and which do not ramify in K; we understand $D_{K/F} = \mathfrak{g}$ if $K = F$; we put $\delta q\mathfrak{g} = \mathfrak{a}\mathfrak{b}^2$ with a squarefree integral ideal $\mathfrak{a}$ and a $\mathfrak{g}$-ideal $\mathfrak{b}$ of F. These (2.4) and (2.5) can be obtained by applying [11, Theorem 6.2] to (V, φ) and the complement (W, ψ).
The intersection $L \cap W$ is an integral $\mathfrak{g}$-lattice in W with respect to ψ. It can be seen that $[(L\cap W)\tilde{}/L \cap W] = [M/L \cap W]^2[\widetilde{M}/M]$ and $[M/L \cap W]$ is an integral ideal, which is independent of the choice of M; see [2, Lemma 2.2 (6)]. Moreover there is a $\mathfrak{g}$-ideal $\mathfrak{b}(q)$ of F such that

$$[M/L \cap W] = \mathfrak{b}(q)(2\varphi(h, L))^{-1} \tag{2.6}$$

by [2, Theorem 4.2]. We note that $2\varphi(h, L)$ must contain $\mathfrak{b}(q)$ and that $2\varphi(h, L) \subset \mathfrak{g}$ if $h \in L$. The ideal $\mathfrak{b}(q)$ is determined by

$$2q[\widetilde{L}/L] = \mathfrak{b}(q)^2[\widetilde{M}/M] \tag{2.7}$$

(cf. [2, (4.1)]). Combining these, we obtain $[(L \cap W)\tilde{}/L \cap W] = 2q[\widetilde{L}/L](2\varphi(h, L))^{-2}$. Now to $L\cap W$ we associate the order $A^+(L\cap W)$ defined by (1.4). Its discriminant is given by

$$d(A^+(L \cap W)) = 2^{-1}[(L \cap W)\tilde{}/L \cap W] = q[\widetilde{L}/L](2\varphi(h, L))^{-2}. \tag{2.8}$$

It is noted that the discriminant of $A^+(W)$ divides $q[\widetilde{L}/L](2\varphi(h, L))^{-2}$. We also note that if $d(A^+(L \cap W))$ is squarefree, then $2\varphi(h, L)$ must be $\mathfrak{b}(q)$ in (2.6), that is, $L \cap W$ is maximal in W.

For our later use, let us state a *weak* Witt decomposition of the local space $(V, \varphi)_v$ whose core dimension t_v is 0 or 2. We fix a nonarchimedean prime v of F and drop the subscript v. Let L be a $\mathfrak{g}$-maximal lattice in V with respect to φ. We first note that φ is isotropic as t is 0 or 2. Let K be the discriminant algebra of φ defined by $K = F \times F$ if $t = 0$ and by $K = F(\sqrt{\det(\varphi)})$ if $t = 2$; also let κ be the norm form defined by $2\kappa(x, y) = \kappa[x + y] - \kappa[x] - \kappa[y]$ and

$\kappa[(a, b)] = ab$ for $x, y, (a, b) \in K$ if $t = 0$, and by $2\kappa(x, y) = xy^\rho + x^\rho y$ for $x, y \in K$ with a nontrivial automorphism ρ of K over F if $t = 2$. Because K is embeddable in $A^+(V)$, we identify K with the image of it. Then there is a weak Witt decomposition as follows (cf. [4, (1.19) and (1.20)]):

$$V = Kg \oplus (Fe + Ff), \qquad L = \mathfrak{r}g + (\mathfrak{g}e + \mathfrak{g}f),$$
$$(Kg,\ \varphi) \cong (K,\ c\kappa) \quad \text{via } xg \longmapsto x \tag{2.9}$$

with some elements e and f of V such that $\varphi[e] = \varphi[f] = 0$ and $2\varphi(e, f) = 1$, and $g \in V$ such that $g^2 = c \in F^\times$. Here $\mathfrak{r} = \mathfrak{g} \times \mathfrak{g}$ if $t = 0$ and $\mathfrak{r}$ is the maximal order of K if $t = 2$. We may assume that $c = 1$ if $t = 0$, $c \in \mathfrak{g}^\times$ if $(K/\mathfrak{p}) = -1$ and $\chi(Q(\varphi)) = +1$, $c \in \pi\mathfrak{g}^\times$ if $(K/\mathfrak{p}) = -1$ and $\chi(Q(\varphi)) = -1$, and $c \in \mathfrak{g}^\times$ if $(K/\mathfrak{p}) = 0$. We also note that $A^+(Kg) = K$ and $xg = gx^*$ for $x \in K$, where $(a, b)^* = (b, a)$ for $(a, b) \in K$ if $t = 0$ and the involution $*$ gives a nontrivial automorphism of K over F if $t = 2$.

2.2 The genus of $L \cap W$

Let (V, φ) be a quaternary quadratic space over a number field F and (W, ψ) as in §2.1 with a fixed element h of V such that $\varphi[h] \neq 0$.

Lemma 2.1. *Let L be a $\mathfrak{g}$-maximal lattice in V with respect to φ. Then $A^+(L \cap W) = A^+(L) \cap A^+(W)$ for every $h \in V$ such that $\varphi[h] \neq 0$. The discriminant of $A^+(L \cap W)$ is given by* (2.8).

This follows from the similar result [4, Lemma 3.2] on local orders $A^+(L_v \cap W_v)$ by localization. We next restate [4, Corollary 2.2] which is a conclusion from the main result of [4]:

Theorem 2.2. *Let (V, φ) be a quaternary quadratic space over a number field F and L a $\mathfrak{g}$-maximal lattice in V with respect to φ. For $h \in V$ such that $\varphi[h] \neq 0$ put $W = (Fh)^\perp$ and let ψ be the restriction of φ to W. Put $\mathfrak{o} = A^+(L \cap W)$. Then $C(L \cap W) = \tau(T(\mathfrak{o}))$ and $\Gamma(L \cap W) = \tau(\Gamma^*(\mathfrak{o}))$ hold. Consequently, the map $N \longmapsto A^+(N)$ gives a bijection of the $SO^\psi(W)$-classes in the $SO^\psi(W)$-genus of $L \cap W$ onto the conjugacy classes in the genus of $\mathfrak{o}$ which is the set $\{\alpha^{-1}\mathfrak{o}\alpha \mid \alpha \in A^+(W)^\times_{\mathbf{A}}\}$.*

3 An order associated with $L \cap W$

3.1 The local case

We first recall some general notation and results, following [9, §8 Part I]. For a quadratic space (V, φ) over a local field F, take a $\mathfrak{g}$-maximal lattice L in V with respect to φ. We define a subgroup J_V of $G^+(V)$ by

$$J_V = \{\alpha \in G^+(V) \mid \tau(\alpha) \in C(L),\ \alpha\alpha^* \in \mathfrak{g}^\times\}. \tag{3.1}$$

Put $E_V = G^1(V) \cap J_V$, where $G^1(V) = \{\alpha \in G^+(V) \mid \alpha\alpha^* = 1\}$ is the spin group of φ. If the dimension of V is even more than 2, then by virtue of [9, Theorem 8.9] specialized to this case,

$$[C(L) : \tau(J_V)] = \begin{cases} 1 & \text{if } t = 0, \text{ or } t = 2,\ (K/\mathfrak{p}) = -1, \text{ and } Q(\varphi) = M_2(F), \\ 2 & \text{otherwise,} \end{cases} \tag{3.2}$$

where t is the core dimension of (V, φ) and $K = F(\sqrt{\delta})$ is the discriminant field of φ. If the dimension of V is odd more than 1, then by [9, Theorem 8.9] and [12, Theorem 1.8 (ii)],

$$[C(L) : \tau(J_V)] = \begin{cases} 1 & \text{if } t = 1 \text{ and } \delta \in \mathfrak{g}^\times F^{\times 2}, \\ 2 & \text{otherwise.} \end{cases} \tag{3.3}$$

Let (V, φ) be a quaternary quadratic space over F. For $h \in V$ such that $\varphi[h] = q \neq 0$, put $W = (Fh)^\perp$ and let ψ be the restriction of φ to W. Let $K = F(\sqrt{\delta})$ be the discriminant field of φ. Also let L and M be $\mathfrak{g}$-maximal lattices in V and W with respect to φ and ψ, respectively. We define J_V in $G^+(V)$ by (3.1) with L and J_W in $G^+(W)$ with M. Let S_V^+ (resp. S_W^+) be the order in $A^+(V)$ (resp. $A^+(W)$) generated by E_V and $A^+(L)$ (resp. by E_W and $A^+(M)$) except the case where $t = 2$, $(K/\mathfrak{p}) = -1$, and $\chi(Q(\varphi)) = -1$ (resp. where $t = 0$ and $q \in \pi\mathfrak{g}^\times F^{\times 2}$, or $t = 2$, $\delta q \in \pi\mathfrak{g}^\times F^{\times 2}$, and $Q(\varphi) = M_2(F)$); in which cases we put

$$S_V^+ = A^+(V) \cap S_V \quad \text{if } t = 2,\ (K/\mathfrak{p}) = -1, \text{ and } \chi(Q(\varphi)) = -1, \tag{3.4}$$

$$S_W^+ = A^+(W) \cap S_W \quad \text{if } \begin{cases} t = 0 \text{ and } q \in \pi\mathfrak{g}^\times F^{\times 2}, \\ t = 2,\ \delta q \in \pi\mathfrak{g}^\times F^{\times 2}, \text{ and } Q(\psi) = M_2(F), \end{cases} \tag{3.5}$$

where S_V (resp. S_W) is a unique maximal order in $A(V)$ (resp. $A(W)$) containing E_V and $A(L)$ (resp. E_W and $A(M)$) given by [9, Theorem 8.6 (i)]. By [9, Theorem 8.6 (ii)] these S_V^+ and S_W^+ are maximal orders except in cases (3.4) and (3.5). It should be noted that we can prove this fact in a similar way to the proof of [9, Theorem 8.6 (ii)] even for the case which does not satisfy the assumption [9, (8.1)]. For the same reason we also see that $S_W = A(M)$ in case (3.5). In all cases,

$$J_V = G^+(V) \cap (S_V^+)^\times, \tag{3.6}$$

$$J_W = G^+(W) \cap (S_W^+)^\times = (S_W^+)^\times. \tag{3.7}$$

In fact, [9, Proposition 8.8 (ii)] together with $G^+(W) = A^+(W)^\times$ implies (3.7) except in case (3.5). As for (3.5), there is an order in $A(W)$ containing J_W and M by [9, Lemma 8.4 (ii)]. In view of the uniqueness of S_W and $E_W \subset J_W$, S_W

contains J_W and M, and hence [9, Proposition 8.8 (i)] is applicable to the case (3.5). This proves (3.7). Similarly we have (3.6).
Now, $A^+(W) = \{x \in A^+(V) \mid xh = hx\}$ and $G^+(W) = \{\alpha \in G^+(V) \mid \alpha h = h\alpha\}$ as mentioned in §2.1. It can be seen that

$$G^+(W) \cap J_V = (A^+(W) \cap S_V^+)^\times. \tag{3.8}$$

Thus $G^+(W) \cap J_V$ is the unit group of an order $A^+(W) \cap S_V^+$ in $A^+(W)$.

Lemma 3.1. *In the above setting the following assertions hold:*

(1) $[S_W^+/A^+(M)]$ *is given by*

$$\begin{cases} \mathfrak{p} & \text{if } t = 4 \text{ and } q \in \mathfrak{g}^\times F^{\times 2}, \\ & \text{or } t = 2,\ \delta q \in \mathfrak{g}^\times F^{\times 2}, \text{ and } \chi(Q(\psi)) = -1, \\ \mathfrak{g} & \text{otherwise.} \end{cases} \tag{3.9}$$

Here S_W^+ may or may not be maximal when $t = 0$ and $q \in \pi\mathfrak{g}^\times F^{\times 2}$ or when $t = 2$, $\delta q \in \pi\mathfrak{g}^\times F^{\times 2}$, and $Q(\psi) = M_2(F)$.

(2) *Assume that $q \in \mathfrak{g}^\times F^{\times 2}$ if $t = 2$, K/F is unramified, and $Q(\varphi)$ is a division algebra. Then $[A^+(W) \cap S_V^+/A^+(L \cap W)]$ is given by*

$$\begin{cases} \mathfrak{p} & \text{if } t = 4 \text{ and } q \in \mathfrak{g}^\times F^{\times 2}, \\ & \text{or } t = 2,\ (K/\mathfrak{p}) = -1, \text{ and } \chi(Q(\varphi)) = -1, \\ \mathfrak{g} & \text{otherwise.} \end{cases} \tag{3.10}$$

Proof. Let s be the core dimension of (W, ψ). In view of (1.7), (2.3), and (2.1), we observe that

$$s = 1 \text{ and } \delta q \in \pi\mathfrak{g}^\times F^{\times 2} \iff \begin{cases} t = 0 \text{ and } q \in \pi\mathfrak{g}^\times F^{\times 2}, \\ t = 2,\ \delta q \in \pi\mathfrak{g}^\times F^{\times 2}, \text{ and } Q(\psi) = M_2(F), \end{cases} \tag{3.11}$$

$$s = 3 \text{ and } \delta q \in \mathfrak{g}^\times F^{\times 2} \iff \begin{cases} t = 4 \text{ and } q \in \mathfrak{g}^\times F^{\times 2}, \\ t = 2,\ \delta q \in \mathfrak{g}^\times F^{\times 2}, \text{ and } \chi(Q(\psi)) = -1. \end{cases} \tag{3.12}$$

Then we can verify that

$$S_V^+ = A^+(L) \text{ if } \begin{cases} t = 0, \\ t = 2 \quad \text{except the case } (K/\mathfrak{p}) = -1 \text{ and } \chi(Q(\varphi)) = -1, \end{cases} \tag{3.13}$$

$$S_W^+ = A^+(M) \iff \begin{cases} s = 1, \\ s = 3 \text{ and } \delta q \in \pi\mathfrak{g}^\times F^{\times 2}. \end{cases} \tag{3.14}$$

In fact, if $s = 1$ and $\delta q \in \pi\mathfrak{g}^\times F^{\times 2}$, then the 'if'-part of (3.14) follows from (3.11), (3.5), and $S_W = A(M)$. If $s = 3$ and $\delta q \in \pi\mathfrak{g}^\times F^{\times 2}$, then $S_W^+ = A^+(M)$ because the discriminant of $A^+(M)$ is $\mathfrak{p}$. If $s = 3$ and $\delta q \in \mathfrak{g}^\times F^{\times 2}$, then since S_W^+ is a maximal order in the division algebra $A^+(W)$, it has discriminant $\mathfrak{p}$. Note that the discriminant of $A^+(M)$ is $\mathfrak{p}^2$. Hence $S_W^+ \neq A^+(M)$. Further, observing $A^+(M) \subset S_W^+ \subset (S_W^+)\tilde{} \subset A^+(M)\tilde{}$ and applying [2, Lemma 2.2 (3)] with the norm form ν of $A^+(W)$, we have $[S_W^+/A^+(M)] = \mathfrak{p}$. The remaining parts follow from [9, Theorem 8.6 (vi)].

From (3.14) and (3.12) we see that

$$S_W^+ \neq A^+(M) \Longleftrightarrow \begin{cases} t = 4 \text{ and } q \in \mathfrak{g}^\times F^{\times 2}, \\ t = 2,\ \delta q \in \mathfrak{g}^\times F^{\times 2}, \text{ and } \chi(Q(\psi)) = -1. \end{cases}$$

In this case $[S_W^+/A^+(M)] = \mathfrak{p}$, which proves (1).

To prove (2), it is sufficient to observe the two cases that $t = 4$ or that $t = 2$, $(K/\mathfrak{p}) = -1$, and $\chi(Q(\varphi)) = -1$ by (3.13) and Lemma 2.1.

If $t = 4$ and $q \in \pi\mathfrak{g}^\times F^{\times 2}$, then $A^+(L \cap W) \subset A^+(W) \cap S_V^+ \subset S_W^+ = A^+(M)$. Thus $A^+(W) \cap S_V^+ = A^+(L \cap W)$ because $L \cap W$ is maximal.

Suppose that $t = 4$ and $q \in \mathfrak{g}^\times F^{\times 2}$. Then $A^+(L \cap W)$ has discriminant $\mathfrak{p}^2$ and by Lemma 2.1, $A^+(L \cap W) \subset A^+(W) \cap S_V^+ \subset S_W^+$ in the division algebra $A^+(W)$. We employ the setting and notation in the case where $q_0 \in \mathfrak{g}^\times$ and $(K_1/\mathfrak{p}) = 0$ in [4, §4.4]. In [4, (3.31)] observing $(g_2g_3)(g_2g_3)^* \in \pi^2\mathfrak{g}^\times$, we set

$$\mathfrak{O} = \mathfrak{g} + \mathfrak{g}g_1g_2 + \mathfrak{g}g_1g_3 + \mathfrak{g}\pi^{-1}g_2g_3. \tag{3.15}$$

This is an order in $A^+(W)$ which contains but does not coincide with $A^+(L \cap W)$. Hence $\mathfrak{O}$ is a unique maximal order S_W^+ in $A^+(W)$. Now in the present setting, $(V, \varphi) = (B, \beta)$ and L is a unique maximal order $\mathfrak{o}$ in $B = Q(\varphi)$ with the norm form β. To see the order S_V^+ in $A^+(V)$, we here recall an F-linear mapping p defined in [9, §7.4 (B)]:

$$p : V \longrightarrow M_2(B) \quad \text{via} \quad p(x) = \begin{bmatrix} 0 & x \\ x^\iota & 0 \end{bmatrix},$$

where ι is the main involution of B. Then $A^+(V)$ and S_V^+ are given by

$$A^+(V) = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \mid x,\ y \in B \right\}, \quad S_V^+ = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \mid x,\ y \in \mathfrak{o} \right\}.$$

Under the identification of V with $p(V)$ and of W with $p(W)$, $A^+(L \cap W)$ and S_W^+ are given by [4, (3.31)] and (3.15), respectively. Then we see that

$$\pi^{-1}g_2g_3 = \frac{1}{\pi}\begin{bmatrix} 0 & g_2 \\ g_2^\iota & 0 \end{bmatrix}\begin{bmatrix} 0 & g_3 \\ g_3^\iota & 0 \end{bmatrix} = \begin{bmatrix} \pi^{-1}g_2g_3^\iota & 0 \\ 0 & \pi^{-1}g_2^\iota g_3 \end{bmatrix},$$
$$\beta[\pi^{-1}g_2g_3^\iota] = \pi^{-2} \cdot \pi ac \cdot \pi^{2-2(\kappa-k)}(1-c) \in \mathfrak{g}^\times.$$

Thus both $\pi^{-1}g_2g_3^\iota$ and $\pi^{-1}g_2^\iota g_3$ belong to $\mathfrak{o}$, so that $\pi^{-1}g_2g_3 \in S_V^+$. Therefore S_V^+ contains S_W^+, which implies that $A^+(W)\cap S_V^+$ is the maximal order S_W^+. For the other cases S_W^+ can be observed in a similar manner; we have then

$$S_W^+ = \begin{cases} \mathfrak{g} + \mathfrak{g}1_B\omega + \mathfrak{g}1_B(v\omega) + \mathfrak{g}\pi^{-1}\omega(v\omega) & \text{if } (K_1/\mathfrak{p}) = -1 \text{ and } \mathfrak{p}\nmid 2, \\ \mathfrak{g} + \mathfrak{g}1_B\omega + \mathfrak{g}1_B(u\omega) + \mathfrak{g}\pi^{-1}\omega(u\omega) & \text{if } (K_1/\mathfrak{p}) = -1 \text{ and } \mathfrak{p}\mid 2, \\ \mathfrak{g} + \mathfrak{g}\sqrt{s}\omega + \mathfrak{g}\sqrt{s}(v\omega) + \mathfrak{g}\pi^{-1}\omega(v\omega) & \text{if } (K_1/\mathfrak{p}) = 1. \end{cases}$$

Here the notation is the same as in each case of [4, §3.4]. Consequently $A^+(W)\cap S_V^+ = S_W^+$ in each case. This settles the case where $t=4$ and $q\in\mathfrak{g}^\times F^{\times 2}$.
Suppose that $t=2$, $(K/\mathfrak{p})=-1$, and $\chi(Q(\varphi))=-1$. In this case S_V^+ is defined by (3.4) with the maximal order S_V in $A(V)$. Let $q\in\pi^{2\ell}\mathfrak{g}^\times$ with $\ell\in\mathbf{Z}$. Then $\mathfrak{b}(q)=\mathfrak{p}^\ell$ as was seen in the case of $q\notin\varphi[Kg]$ in [4, §3.2] with $g^2\in\pi\mathfrak{g}^\times$ in (2.9). Since $A^+(W)$ is a division algebra, S_W^+ is a unique maximal order in $A^+(W)$ of discriminant $\mathfrak{p}$. We have by (2.8),

$$\begin{gathered} A^+(L\cap W)\subset A^+(W)\cap S_V\subset S_W^+, \\ \mathfrak{p}^{2\ell+2}(2\varphi(h,\,L))^{-2}\subset d(A^+(W)\cap S_V)\subset\mathfrak{p}. \end{gathered} \tag{3.16}$$

Now put $2\varphi(h,\,L)=\mathfrak{p}^m$, which satisfies $m\le\ell$. We observe that $q\pi^{-m}e+\pi^m f\in L[q,\,2^{-1}\mathfrak{p}^m]=hC(L)$ by (1.6) with the same notation as in the proof of [4, Lemma 3.1]. Then identifying W with that in [4, (3.1)] and employing the isomorphism Ψ of $A(V)$ in the proof of [4, Lemma 3.1], we can find the structure of $A^+(W)\cap S_V$ as follows:

$$\Psi(A^+(W)\cap S_V)=\mathfrak{r}+\pi^{-m}\mathfrak{r}\eta, \tag{3.17}$$

where $\mathfrak{r}=\mathfrak{g}[\xi]$ is the maximal order of K and η is given by [4, (3.3)]. From this together with [4, (3.4)] we have $[A^+(W)\cap S_V/A^+(L\cap W)]=[\mathfrak{r}/\mathfrak{f}]=\mathfrak{p}$, where $\mathfrak{f}=\mathfrak{g}+g^2\mathfrak{g}\xi$. To see (3.17), we recall by [9, Theorem 8.6 (iii)] that $\Psi(S_V)=M_2(Q)$, where $Q=\mathfrak{r}+\mathfrak{r}g$ is a maximal order in the division algebra $Q(\varphi)=A(Kg)=K+Kg$. Then (3.17) can be seen from this and [4, (3.2)]. This completes the proof. □

Lemma 3.2. *Let the notation be the same as in Lemma 3.1 with h and L. Then the following assertions hold:*

(1) *Define an order $\mathfrak{O}$ in $A^+(W)$ by*

$$\mathfrak{O}=\begin{cases} S_W^+ & \text{if } t=4, \\ A^+(W)\cap S_V & \text{if } t=2,\ (K/\mathfrak{p})=-1,\ \text{and } \chi(Q(\varphi))=-1, \\ A^+(L\cap W) & \text{otherwise.} \end{cases} \tag{3.18}$$

Then $G^+(W)\cap J_V=\mathfrak{O}^\times$.

(2) *Assume that* $q = \varphi[h] \in \mathfrak{g}^\times F^{\times 2}$ *and* $2\varphi(h, L) = \mathfrak{b}(q)$ *if* $t = 2$*,* K/F *is unramified, and* $Q(\varphi)$ *is a division algebra. Let* $\mathfrak{O}$ *be the order defined by* (3.18)*. Then* $\mathfrak{O}$ *is a unique order in* $A^+(W)$*, containing* $A^+(L \cap W)$*, of discriminant*

$$\begin{cases} \mathfrak{p} & \text{if } t = 4, \\ & \text{or } t = 2,\ (K/\mathfrak{p}) = -1, \text{ and } \chi(Q(\varphi)) = -1, \\ q[\widetilde{L}/L](2\varphi(h,\, L))^{-2} & \text{otherwise.} \end{cases} \tag{3.19}$$

In particular, $\mathfrak{O}$ *is a unique maximal order in the division algebra* $A^+(W)$ *when* $t = 2$*,* $(K/\mathfrak{p}) = -1$*, and* $\chi(Q(\varphi)) = -1$*. Moreover, if* $L \cap W \subset M$*, then* $\mathfrak{O} \subset S_W^+$*.*

Proof. To prove (1), let $\mathfrak{O}$ be the order given by (3.18). From Lemma 3.1 it can be seen that $\mathfrak{O} = A^+(W) \cap S_V^+$. Thus we have $G^+(W) \cap J_V = \mathfrak{O}^\times$ by (3.8), which proves (1).
To prove (2), by Lemma 3.1 (2) we see that $A^+(W) \cap S_V^+ \neq A^+(L \cap W)$ if and only if $t = 4$ and $q \in \mathfrak{g}^\times F^{\times 2}$ or if $t = 2$, $(K/\mathfrak{p}) = -1$, and $\chi(Q(\varphi)) = -1$. If $t = 4$ and $q \in \mathfrak{g}^\times F^{\times 2}$, then $A^+(W) \cap S_V^+ = S_W^+$ as seen in the proof of Lemma 3.1 (2). If $t = 2$, $(K/\mathfrak{p}) = -1$, and $\chi(Q(\varphi)) = -1$, then, by our assumption, $q \in \pi^{2\ell}\mathfrak{g}^\times$ and $2\varphi(h, L) = \mathfrak{b}(q) = \mathfrak{p}^\ell$ with $\ell \in \mathbf{Z}$. Thus applying (3.16) to $m = \ell$, we have $\mathfrak{p}^2 \subset d(A^+(W) \cap S_V) \subset \mathfrak{p}$. Because $A^+(W) \cap S_V \neq A^+(L \cap W)$, $A^+(W) \cap S_V$ must be maximal in $A^+(W)$. Consequently, if $\mathfrak{O} \neq A^+(L \cap W)$, it is a maximal order which is uniquely determined by discriminant $\mathfrak{p}$. As for the case of $\mathfrak{O} = A^+(L \cap W)$, the discriminant is given by (2.8). Summing up these, we have the uniqueness of $\mathfrak{O}$. To prove the last assertion, suppose that $L \cap W \subset M$. Then $A^+(L \cap W) \subset A^+(M) \subset S_W^+$, which shows $\mathfrak{O} \subset S_W^+$ when $\mathfrak{O} = A^+(L \cap W)$. If $\mathfrak{O} \neq A^+(L \cap W)$, then $\mathfrak{O}$ is maximal in $A^+(W)$. Since S_W^+ is also maximal, we have $\mathfrak{O} = S_W^+$. Hence $\mathfrak{O} \subset S_W^+$ holds if $L \cap W \subset M$. This proves (2). ☐

3.2 The global case

Let (V, φ) and (W, ψ) be the quadratic spaces over a number field F in the setting of §2.2 with an element h of V such that $\varphi[h] = q \in F^\times$. Let L and M be $\mathfrak{g}$-maximal lattices in V and W with respect to φ and ψ, respectively. Put

$$J_V = G^+(V)_{\mathbf{a}} \prod_{v \in \mathbf{h}} J_{V_v}, \qquad J_W = G^+(W)_{\mathbf{a}} \prod_{v \in \mathbf{h}} J_{W_v}, \tag{3.20}$$

where J_{V_v} and J_{W_v} are given in §3.1. We have an order S_W^+ in $A^+(W)$ determined by $S_{W_v}^+$ for all $v \in \mathbf{h}$, where $S_{W_v}^+$ is the order in $A^+(W_v)$ given in §3.1; notice that $S_{W_v}^+ = A^+(M_v)$ for almost all v.
Let us here insert a remark on the order in $A^+(W)$ given in [12, Lemma 5.3 (ii)]. By applying that lemma to M, we have an order $\mathfrak{O}_0$ containing $A^+(M)$.

Then $[(\mathfrak{O}_0)_v/A^+(M_v)]$ is the same ideal as in (3.9) for each $v \in \mathbf{h}$. This can be seen in the proof of [12, Lemma 5.3 (ii)]. Hence the order $\mathfrak{O}_0$ coincides with S_W^+ in the present situation.

Proposition 3.3. *Let the notation be the same as above with $h \in V$ and L. Also let $K = F(\sqrt{\delta})$ be the discriminant field of φ. Then the following assertions hold:*

(1) *Let $\mathfrak{O}$ be the order in $A^+(W)$ whose localization at $v \in \mathbf{h}$ is the local order defined by* (3.18). *Then*

$$G^+(W)_{\mathbf{A}} \cap J_V = A^+(W)_{\mathbf{a}}^\times \mathfrak{O}_{\mathbf{h}}^\times. \tag{3.21}$$

(2) *$T(A^+(M)) = T(S_W^+)$ and $J_W = A^+(W)_{\mathbf{a}}^\times (S_W^+)_{\mathbf{h}}^\times$. Moreover $J_W \subset T(A^+(M))$ and $G^+(W)_{\mathbf{A}} \cap J_V \subset T(A^+(L \cap W))$.*

(3) *Assume that $q = \varphi[h] \in \mathfrak{g}_v^\times F_v^{\times 2}$ and $2\varphi(h, L)_v = \mathfrak{b}(q)_v$ for every $v \in \mathbf{h}$ such that $t_v = 2$, K_v/F_v is unramified, and $Q(\varphi)_v$ is a division algebra. Let $\mathfrak{O}$ be the order given in (1). Then $\mathfrak{O}$ is a unique order in $A^+(W)$, containing $A^+(L \cap W)$, of discriminant*

$$q[\widetilde{L}/L](2\varphi(h, L))^{-2}\mathfrak{f}^{-1}. \tag{3.22}$$

Here $\mathfrak{f}$ is the product of all the prime ideals $\mathfrak{p}$ of F such that $t_{\mathfrak{p}} = 4$ and $q \in \mathfrak{g}_{\mathfrak{p}}^\times F_{\mathfrak{p}}^{\times 2}$, or that $t_{\mathfrak{p}} = 2$, $K_{\mathfrak{p}}/F_{\mathfrak{p}}$ is unramified, and $Q(\varphi)_{\mathfrak{p}}$ is a division algebra.

(4) *Under the assumptions of (3) suppose $L \cap W \subset M$. Then $\mathfrak{O} \subset S_W^+$ and $G^+(W)_{\mathbf{A}} \cap J_V \subset J_W$.*

Proof. To prove (1), we see that

$$\begin{aligned} &G^+(W)_{\mathbf{A}} \cap J_V = \\ &\quad = G^+(W)_{\mathbf{A}} \cap (G^+(V)_{\mathbf{a}} \prod_{v \in \mathbf{h}} (J_V)_v) = G^+(W)_{\mathbf{a}} \prod_{v \in \mathbf{h}} (G^+(W)_v \cap (J_V)_v). \end{aligned}$$

Since $G^+(W)_v \cap (J_V)_v = \mathfrak{O}_v^\times$ by Lemma 3.2 (1), we have (3.21).
From (3.14), $(S_W^+)_v$ is generated by $G^1(W)_v$ and $A^+(M)_v$ if $\delta \in \mathfrak{g}^\times F_v^{\times 2}$ and $\chi(Q(\psi)_v) = -1$, and $(S_W^+)_v = A^+(M)_v$ otherwise $v \in \mathbf{h}$. Since

$$\alpha^{-1} G^1(W)_v \alpha = G^1(W)_v \quad \text{for every } \alpha \in A^+(W)_v^\times,$$

we have $T(A^+(M)) \subset T(S_W^+)$. Conversely, for $\alpha \in A^+(W)_{\mathbf{A}}^\times$

$$\alpha^{-1} S_W^+ \alpha = S_W^+ \Longrightarrow M\tau(\alpha) = M \Longrightarrow \alpha^{-1} A^+(M) \alpha = A^+(M).$$

This is because $C(M) = \tau(T(S_W^+))$ by [12, Lemma 5.4]. Thus $T(A^+(M)) = T(S_W^+)$. Let x be an element of J_W. Since $\tau(x) \in \tau(J_W) \subset C(M)$, together

with $C(M) = \tau(T(A^+(M)))$, there is an element y of $T(A^+(M))$ such that $\tau(x) = \tau(y)$. Hence $x = ay$ with some $a \in F_{\mathbf{A}}^\times$. As $F_{\mathbf{A}}^\times \subset T(A^+(M))$, we have $J_W \subset T(A^+(M))$. Similarly let $x \in G^+(W)_{\mathbf{A}} \cap J_V$. Since $\tau(x) \in SO_{\mathbf{A}}^\psi \cap \tau(J_V) \subset C(L \cap W)$, together with $C(L\cap W) = \tau(T(A^+(L\cap W)))$ by Theorem 2.2, there is an element y of $T(A^+(L\cap W))$ such that $\tau(x) = \tau(y)$. From this, noticing $F_{\mathbf{A}}^\times \subset T(A^+(L\cap W))$, we have $G^+(W)_{\mathbf{A}} \cap J_V \subset T(A^+(L\cap W))$. This proves (2).
To prove (3), we take the order $\mathfrak{O}$ of (1). Since Lemma 3.2 (2) is applicable to $\mathfrak{O}_v$ for each $v \in \mathbf{h}$ under the assumption of (3), $\mathfrak{O}_v$ contains $A^+(L\cap W)_v$ and has the discriminant given by (3.19). Also when $\mathfrak{O}_v \neq A^+(L\cap W)_v$, $[\mathfrak{O}_v/A^+(L\cap W)_v] = \mathfrak{p}_v$ by Lemma 3.1 (2). Thus by applying [2, Lemma 2.2 (3)] to $\mathfrak{O}$ and $A^+(L\cap W)$ with the norm form ν of $A^+(W)$, we have

$$\begin{aligned}[\widetilde{\mathfrak{O}}/\mathfrak{O}] &= [A^+(L\cap W)^{\sim}/A^+(L\cap W)][\mathfrak{O}/A^+(L\cap W)]^{-2} \\ &= (q[\widetilde{L}/L](2\varphi(h,\, L))^{-2})^2 \prod_{\mathfrak{p}|\mathfrak{f}} \mathfrak{p}^{-2},\end{aligned}$$

where $\mathfrak{f}$ is the ideal in the statement of (3). This gives (3.22). Now, let $\mathfrak{O}'$ be an order in $A^+(W)$, containing $A^+(L\cap W)$, whose discriminant is given by (3.22). Then the localization $\mathfrak{O}'_v$ at $v \in \mathbf{h}$ contains $A^+(L\cap W)_v$ and has the discriminant of (3.19). By Lemma 3.2 (2), $\mathfrak{O}'_v = \mathfrak{O}_v$ for every v. Hence we have $\mathfrak{O}' = \mathfrak{O}$, which shows the uniqueness of $\mathfrak{O}$.
Keeping the assumptions of (3), let $L \cap W \subset M$. Then applying Lemma 3.2 (2) with localization, we have $\mathfrak{O} \subset S_W^+$. Thus $G^+(W)_{\mathbf{A}} \cap J_V \subset J_W$ by (3.8) and (3.7). This proves (4). ☐

THEOREM 3.4. *Let the notation and assumption be the same as in Proposition 3.3 (3) and $\mathfrak{O}$ the order in $A^+(W)$ given in that proposition. Then the following assertions hold:*

(1) $C(L\cap W) = \tau(T(\mathfrak{O}))$ *and* $\Gamma(L\cap W) = \tau(\Gamma^*(\mathfrak{O}))$.

(2) *The map $(L\cap W)\tau(\alpha) \longmapsto \alpha^{-1}\mathfrak{O}\alpha$ gives a bijection of the $SO^\psi(W)$-classes in the $SO^\psi(W)$-genus of $L\cap W$ onto the conjugacy classes in the genus of $\mathfrak{O}$ which is the set $\{\alpha^{-1}\mathfrak{O}\alpha \mid \alpha \in A^+(W)_{\mathbf{A}}^\times\}$.*

(3) *The type number of $\mathfrak{O}$ equals the type number of $A^+(L\cap W)$ and consequently is equal to the class number of the genus of $L\cap W$ with respect to $SO^\psi(W)$.*

Proof. In view of Lemma 3.1, $\mathfrak{O}_v \neq A^+(L\cap W)_v$ if and only if $t_v = 4$ and $q \in \mathfrak{g}_v^\times F_v^{\times 2}$ or if $t_v = 2$, $(K/v) = -1$, and $\chi(Q(\varphi)_v) = -1$ for $v \in \mathbf{h}$. Since Lemma 3.2 (2) is applicable in our assumption, $\mathfrak{O}_v$ is a unique maximal order in the division algebra $A^+(W)_v$ in both cases. Furthermore, $(L\cap W)_v$ is a unique maximal lattice in the anisotropic space $(W,\, \psi)_v$ becuase $2\varphi(h,\, L)_v = \mathfrak{b}(q)_v$. Thus it can be found that

$$\alpha^{-1}\mathfrak{O}\alpha = \mathfrak{O} \Longleftrightarrow \alpha^{-1}A^+(L\cap W)\alpha = A^+(L\cap W)$$

for $\alpha \in A^+(W)^\times_{\mathbf{A}}$. This combined with Theorem 2.2 proves (1).
To prove (2), let N be an arbitrary $\mathfrak{g}$-lattice in the genus of $L \cap W$. Since N is integral we have the order $A^+(N)$ in $A^+(W)$. Taking $\alpha \in A^+(W)^\times_{\mathbf{A}}$ so that $N = (L \cap W)\tau(\alpha)$, we can put $\mathfrak{O}(N) = \alpha^{-1}\mathfrak{O}\alpha$ in $A^+(W)$. In fact, if $N = (L \cap W)\tau(\alpha')$ with some $\alpha' \in A^+(W)^\times_{\mathbf{A}}$, then $(L \cap W)\tau(\alpha(\alpha')^{-1}) = L \cap W$, whence $\alpha(\alpha')^{-1}$ belongs to $T(A^+(L \cap W)) = T(\mathfrak{O})$ by (1). This shows $(\alpha')^{-1}\mathfrak{O}\alpha' = \alpha^{-1}\mathfrak{O}\alpha$, namely, $\mathfrak{O}(N)$ is independent of the choice of α. Moreover this is a unique order of discriminant $q[\widetilde{L}/L](2\varphi(h, L))^{-2}\mathfrak{f}^{-1}$ containing $A^+(N)$, where $\mathfrak{f}$ is the ideal in (3.22). Indeed, since $\mathfrak{O}$ contains $A^+(L \cap W)$ and has the discriminant given by (3.22), the order $\mathfrak{O}(N)$ contains $A^+(N)$ and has the same discriminant. The uniqueness of $\mathfrak{O}(N)$ can be reduced to that of $\mathfrak{O}$. Our assertion (2) can be verified by using this fact and (1). Assertion (3) is a consequence from (2). This completes the proof. $\square$

4 Quadratic Diophantine equations in four variables

4.1 Quadratic Diophantine equations

Let (V, φ) be a quadratic space of dimension n over a number field F and L a $\mathfrak{g}$-lattice in V. We recall that

$$L[q,\ \mathfrak{b}] = \{x \in V \mid \varphi[x] = q,\ \varphi(x,\, L) = \mathfrak{b}\},$$

and this set is stable under $\Gamma(L)$.
For $h \in V$ such that $\varphi[h] = q \neq 0$ we set (W, ψ) as in (2.2). Assume that L is $\mathfrak{g}$-maximal with respect to φ and $n > 2$. Then

$$\sum_{i \in I} \#\{L_i[q,\ \mathfrak{b}]/\Gamma(L_i)\} = \#\left\{SO^\psi \setminus SO^\psi_{\mathbf{A}}/(SO^\psi_{\mathbf{A}} \cap C(L))\right\}, \tag{4.1}$$

where $\mathfrak{b} = \varphi(h, L)$, $\{L_i\}_{i \in I}$ is a set of representatives for the SO^φ-classes in the SO^φ-genus of L for which $L_i[q, \mathfrak{b}] \neq \emptyset$, and SO^ψ is regarded as the subgroup $\{\gamma \in SO^\varphi \mid h\gamma = h\}$ of SO^φ. This is a consequence from the main theorem of quadratic Diophantine equations due to Shimura [9, Theorem 11.6] (cf. also [12, Theorem 2.2 and (2.7)]). For a $\mathfrak{g}$-lattice N in V we put

$$D(N) = \{\alpha \in O^\varphi(V)_{\mathbf{A}} \mid N\alpha = N\}, \qquad \Gamma^{\cdot}(N) = O^\varphi(V) \cap D(N)$$

as denoted in the Introduction. Then formula (4.1) is valid for $(O^\varphi, O^\psi, D(L), \Gamma^{\cdot}(L_i), J)$ in place of $(SO^\varphi, SO^\psi, C(L), \Gamma(L_i), I)$ by [9, Theorem 11.6 (iii) and (v)], where $\{L_i\}_{i \in J}$ is a set of representatives for the O^φ-classes in the O^φ-genus of L for which $L_i[q, \mathfrak{b}] \neq \emptyset$ and O^ψ is regarded as the subgroup $\{\gamma \in O^\varphi \mid h\gamma = h\}$ of O^φ. We note that the O^φ-genus of L coincides with the SO^φ-genus of L and that the class number of O^φ relative to $D(L)$ equals the class number of SO^φ relative to $C(L)$ when n is odd; see [9, Lemma 9.23 (i)], for example.

Now we pay attention to the following; if the number of the right-hand side of (4.1) coincides with $\#\{SO^{\psi} \setminus SO^{\psi}_{\mathbf{A}}/C(L\cap W)\}$, then the left-hand side of (4.1) is given by the class number of the genus of $L\cap W$. Concerning this, there is a result [9, Proposition 11.13] for odd-dimensional spaces and also its analogue [2, Proposition 4.4] for even-dimensional spaces whose discriminant fields are the base fields. In Proposition 4.3 below we shall prove another analogue of [9, Proposition 11.13] to quaternary case.

As for the representatives of classes in the genus of $L\cap W$, by virtue of the principle in [9, Theorem 11.6 (i)], we have the following:

Lemma 4.1. *Let the notation be as above. Fix an element h of $L[q, \mathfrak{b}]$ $(q \neq 0)$ and set (W, ψ) as in (2.2). Then the map*

$$k\Gamma(L_i) \longmapsto (L_i \cap (Fk)^{\perp})\gamma^{-1}SO^{\psi}$$

defines a well-defined surjection of the union of the sets $L_i[q, \mathfrak{b}]/\Gamma(L_i)$ for $i \in I$ onto the SO^{ψ}-classes in the SO^{ψ}-genus of $L\cap W$ with $\gamma \in SO^{\varphi}$ such that $k = h\gamma$ for $k \in L_i[q, \mathfrak{b}]$ and $i \in I$. In particular, if $SO^{\psi}\varepsilon(SO^{\psi}_{\mathbf{A}} \cap C(L)) = SO^{\psi}\varepsilon C(L\cap W)$ for every $\varepsilon \in SO^{\psi}_{\mathbf{A}}$, then the map is bijective. Moreover the assertions are true for $(O^{\psi}, J, \Gamma^{\cdot}(L_i), D(L), D(L\cap W))$ in place of $(SO^{\psi}, I, \Gamma(L_i), C(L), C(L\cap W))$.

Proof. For $k \in L_i[q, \mathfrak{b}]$ with $i \in I$ there is $\gamma \in SO^{\varphi}$ such that $k = h\gamma$ as $\varphi[k] = \varphi[h]$ by [9, Lemma 1.5 (ii)]. We set $L = L_i\alpha_i$ with $\alpha_i \in SO^{\varphi}_{\mathbf{A}}$. We may assume that $(\alpha_i)_v = 1$ for $v \in \mathbf{a}$. Since $h, k(\alpha_i)_v \in L_v[q, \mathfrak{b}_v]$ for $v \in \mathbf{h}$, by (1.6), $h = k(\alpha_i)_v\alpha_v$ with some $\alpha_v \in C(L_v)$ for each v. Putting $\alpha_v = \gamma_v^{-1}$ for $v \in \mathbf{a}$, we have $\alpha \in C(L)$ whose component is α_v for every prime v. Then by [9, Theorem 11.6 (i)] the map $k \longmapsto \gamma\alpha_i\alpha$ induces a well-defined bijection of $\bigcup_{i\in I} L_i[q, \mathfrak{b}]/\Gamma(L_i)$ onto $SO^{\psi} \setminus SO^{\psi}_{\mathbf{A}}/(SO^{\psi}_{\mathbf{A}} \cap C(L))$. Obviously $\gamma\alpha_i\alpha \longmapsto (L\cap W)(\gamma\alpha_i\alpha)^{-1}$ gives a surjection of $SO^{\psi} \setminus SO^{\psi}_{\mathbf{A}}/(SO^{\psi}_{\mathbf{A}} \cap C(L))$ onto the SO^{ψ}-classes in the genus of $L\cap W$. On the other hand, we can consider a $\mathfrak{g}$-lattice $L_i \cap (Fk)^{\perp}$ in the complement $(Fk)^{\perp}$, which is isomorphic to $(L_i \cap (Fk)^{\perp})\gamma^{-1}$ in W under γ^{-1}. Then by localization $(L\cap W)_v(\gamma\alpha_i\alpha)_v^{-1} = \{L_v(\alpha_i)_v^{-1} \cap (F_vh)^{\perp}(\alpha_i\alpha)_v^{-1}\}\gamma^{-1} = (L_i \cap (Fk)^{\perp})_v\gamma^{-1}$ for every $v \in \mathbf{h}$. This determines $(L\cap W)(\gamma\alpha_i\alpha)^{-1} = (L_i \cap (Fk)^{\perp})\gamma^{-1}$. We have thus the desired surjection. Clearly this map is bijective under the assumption in the statement. The assertions for O^{φ} can be handled in a similar way. □

Here we apply Lemma 4.1 to the quadratic form defined by the sum of five squares; the result will be used in Section 5.3.

Let $X = \mathbf{Q}^1_5$ and define Φ by $\Phi[x] = x \cdot {}^t x$ for $x \in X$. The pair (X, Φ) defines a quadratic space over $\mathbf{Q}$ whose invariants are $\{5, \mathbf{Q}, B_{2,\infty}, 5\}$. These invariants can be determined by [11, (Q.5)] because of $(X, \Phi) \cong (B_{2,\infty}, \beta) \oplus (\mathbf{Q}e, \Phi|_{\mathbf{Q}e})$ with some $e \in X$ so that $\Phi[e] = 1$, where β is the norm form of $B_{2,\infty}$. Let Λ be a $\mathbf{Z}$-maximal lattice in (X, Φ). It is known that $\#\{O^{\Phi} \setminus O^{\Phi}_{\mathbf{A}}/D(\Lambda)\} = 1$; see [9, §12.12], for example. By [9, Lemma 12.13 (i)], $\Lambda[d, \mathbf{Z}] \neq \emptyset$ for every squarefree positive integer d. Fixing $k_0 \in \Lambda[d, \mathbf{Z}]$, we put $V = (\mathbf{Q}k_0)^\perp$ and $L = \Lambda \cap V$. Then by [9, Theorem 12.14 (ii)], L is a $\mathbf{Z}$-maximal lattice in V with respect to the restriction φ of Φ to V. By virtue of (4.1) for O^Φ, $\#\{\Lambda[d, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)\} = \#\{O^\varphi \backslash O^\varphi_{\mathbf{A}}/(O^\varphi_{\mathbf{A}} \cap D(\Lambda))\}$ holds. Suppose that d is an odd prime number. Then [9, Proposition 11.13 (iii)] is applicable to $k_0 \in \Lambda[d, \mathbf{Z}]$. We have thus

$$O^\varphi \varepsilon (O^\varphi_{\mathbf{A}} \cap D(\Lambda)) = O^\varphi \varepsilon D(L) \tag{4.2}$$

for every $\varepsilon \in O^\varphi_{\mathbf{A}}$. Therefore $\#\{\Lambda[d, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)\}$ equals the number of O^φ-classes in the O^φ-genus of $\mathbf{Z}$-maximal lattices in (V, φ). This result can be found in [9, Theorem 12.14 (vi)]; the class number of SO^φ relative to $C(L)$ equals $\#\{\Lambda[d, \mathbf{Z}]/\Gamma(\Lambda)\}$ by the same theorem. Moreover (V, φ) has invariants $\{4, \mathbf{Q}(\sqrt{d}), B_{2,\infty}, 4\}$, which can be seen by applying [2, Theorem 1.1 (2)] to (X, Φ) and d.

In view of (4.2), by Lemma 4.1 we have a bijection

$$k\Gamma^{\cdot}(\Lambda) \longmapsto (\Lambda \cap (\mathbf{Q}k)^\perp)\gamma^{-1}O^\varphi \tag{4.3}$$

of $\Lambda[d, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)$ onto the O^φ-classes in the genus of L with some $\gamma \in O^\Phi$ so that $k = k_0\gamma$ for every odd prime number d. A method of determining the set $\Lambda[d, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)$ is explained in [9, §12.15]. In that explanation the case of $d = 29$ is treated and the result $\#\{\Lambda[29, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)\} = 3$ is obtained with explicit representatives for $\Lambda[29, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)$. Hence the class number of O^φ relative to $D(L)$ is equal to 3, as mentioned in the Introduction. In Section 5.3 we shall list the representatives for $\Lambda[d, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)$ and the corresponding lattices under the map (4.3) for $d = 5,\ 13,\ 17$, and 29.

4.2 Results for quaternary spaces

To apply our results in the previous section to quadratic Diophantine equations, let us assume $n = 4$ in the setting of §4.1 and take an element h of $L[q, \mathfrak{b}]$. Under suitable conditions on q and $\mathfrak{b}$, we have an order $\mathfrak{O}$ defined in Proposition 3.3 (3). The order satisfies inequalities

$$t(\mathfrak{O}) \leq \#\{SO^\psi \setminus SO^\psi_{\mathbf{A}}/(SO^\psi_{\mathbf{A}} \cap C(L))\} \leq c(\mathfrak{O}). \tag{4.4}$$

Here $t(\mathfrak{O})$ (resp. $c(\mathfrak{O})$) is the type number (resp. the class number) of $\mathfrak{O}$. To show (4.4), we observe that

$$SO^\psi_{\mathbf{A}} \cap \tau(J_V) \subset SO^\psi_{\mathbf{A}} \cap C(L) \subset C(L \cap W).$$

Since the kernel of τ is $F^\times_{\mathbf{A}}$, the class number of $\mathfrak{O}$ is more than $\#\{SO^\psi \setminus SO^\psi_{\mathbf{A}}/(SO^\psi_{\mathbf{A}} \cap \tau(J_V))\}$ by (3.21). Further by Theorem 3.4 the type number of $\mathfrak{O}$ equals $\#\{SO^\psi \setminus SO^\psi_{\mathbf{A}}/C(L \cap W)\}$. This proves (4.4).

Corollary 4.2. *Let the notation and assumption be as in Proposition 3.3 (3) and $\mathfrak{O}$ the order in $A^+(W)$ defined in that proposition with an element h of $L[q, \mathfrak{b}]$. Also let I (resp. J) be a set of representatives α for $SO^{\varphi} \setminus SO^{\varphi}_{\mathbf{A}}/C(L)$ (resp. $O^{\varphi}\setminus O^{\varphi}_{\mathbf{A}}/D(L)$) for which $L\alpha^{-1}[q, \mathfrak{b}] \neq \emptyset$. Then the following inequalities hold:*

$$t(\mathfrak{O}) \leq \sum_{\alpha\in J} \#\left\{L\alpha^{-1}[q,\ \mathfrak{b}]/\Gamma^{\cdot}(L\alpha^{-1})\right\} \leq$$

$$\leq \sum_{\alpha\in I} \#\left\{L\alpha^{-1}[q,\ \mathfrak{b}]/\Gamma(L\alpha^{-1})\right\} \leq c(\mathfrak{O}).$$

Moreover, assume that $2\varphi(h, L)_v$ contains $\mathfrak{p}_v^{[\nu/2]}$ with $q\mathfrak{g}_v = \mathfrak{p}_v^{\nu}$ for every $v \in \mathbf{h}$ such that $t_v = 2$ and K_v/F_v is ramified. Then the formula in [10, (1.9)] is applicable to h and it can be given as follows:

$$\sum_{y} \#\left\{L\tau(y)^{-1}[q,\ \mathfrak{b}]/\tau(G^+(V) \cap yJ_Vy^{-1})\right\} = c(\mathfrak{O}). \tag{4.5}$$

Here y runs over a set of all representatives for $G^+(V)\setminus G^+(V)_{\mathbf{A}}/J_V$ such that $G^+(W)_{\mathbf{A}} \cap G^+(V)yJ_V \neq \emptyset$.

Proof. To prove the first assertion, we recall that $\#\{SO^{\psi} \setminus SO^{\psi}_{\mathbf{A}}/C(L\cap W)\} = \#\{O^{\psi} \setminus O^{\psi}_{\mathbf{A}}/D(L \cap W)\}$, because W is odd-dimensional. Since $O^{\psi}_{\mathbf{A}} \cap D(L)$ is contained in $D(L \cap W)$, by formula (4.1) for O^{φ}, we have the first inequality. Here we may assume that $\{L\alpha^{-1}\}_{\alpha\in J} \subset \{L\alpha^{-1}\}_{\alpha\in I}$. Clearly $\#\left\{L\alpha^{-1}[q, \mathfrak{b}]/\Gamma^{\cdot}(L\alpha^{-1})\right\} \leq \#\left\{L\alpha^{-1}[q, \mathfrak{b}]/\Gamma(L\alpha^{-1})\right\}$ for every $\alpha \in J$. Then the desired inequalities follow from these and (4.4) combined with (4.1).
To prove (4.5), put $q = q_0\pi_v^{2\ell}$ and $2\varphi(h, L)_v = \mathfrak{p}_v^m$ with $q_0 \in \mathfrak{g}_v^{\times} \cup \pi_v\mathfrak{g}_v^{\times}$ and $\ell,\ m \in \mathbf{Z}$ for $v \in \mathbf{h}$. In order to apply [10, (1.9)], we have to verify that $hC(L) = h\tau(J_V)$ in $V_{\mathbf{A}} = V \otimes_F F_{\mathbf{A}}$. In view of (3.2) it is sufficient to observe the local cases where (i) $t_v = 4$, (ii) $t_v = 2$ and $(K/v) = 0$, (iii) $t_v = 2$, $(K/v) = -1$, and $\chi(Q(\varphi)_v) = -1$. Our argument is basically the same as in [10, §4.3], and so we give only an outline of the proof to avoid a repetition of the same argument. Put $C_v = C(L_v)$ and $J_v = J_{V_v}$.
(i) Through an isomomorphism of $Q(\varphi)_v$ onto $A^+(W)_v$ we have $hC_v = h\tau(J_v)$ in the same way as in §4.3 (i) of [10]. We note that $C_v = SO_v^{\varphi}$ and $C(L_v\cap W_v) = SO_v^{\psi}$.
(ii) Assume that $2\varphi(h, L)_v \supset \mathfrak{p}_v^{\ell}$. In a Witt decomposition of φ in (2.9) with $g^2 \in \mathfrak{g}_v^{\times}$, take the same element $\omega_v \in K_v^{\times} = G^+(K_vg)$ as in §4.3 (ii) of [10]. We take $k_v = q\pi_v^{-m}e + \pi_v^m f$; then $k_v\tau(\omega_v) = k_v$. In a similar manner to [10, §4.3 (ii)] we have $\tau(\omega_v) \in C_v$ and $\omega_v \notin J_v$, from which it follows that $k_vC_v = k_v\tau(J_v)$. Since, by our assumption, $k_v \in L_v[q,\ 2^{-1}\mathfrak{p}_v^m]$, we have $k_v \in hC_v$ by (1.6). Thus the criterion [10, (1.10)] is applicable to k_v; we have $hC_v = h\tau(J_v)$.
(iii) In a Witt decomposition of φ in (2.9) with $g^2 \in \pi_v\mathfrak{g}_v^{\times}$, we take $\omega_v =$

$g(q\pi_v^{-m}e - \pi_v^m f) \in A^+(V_v)^\times$ and $k_v = q\pi_v^{-m}e + \pi_v^m f$. Then it can be seen that

$$\nu(\omega_v) = \omega_v\omega_v^* = -qg^2 \in \pi_v q\mathfrak{g}_v^\times, \qquad k_v\tau(\omega_v) = k_v,$$
$$L_v\tau(\omega_v) = \mathfrak{r}_v g + \mathfrak{g}_v q^{-1}\pi_v^{2m} f + \mathfrak{g}_v q\pi_v^{-2m} e.$$

Because $m = \ell$ by the assumption on (iii), we have $\nu(\omega_v) \in \pi_v^{2\ell+1}\mathfrak{g}_v^\times$ and $L_v\tau(\omega_v) = L_v$. Hence $k_v C_v = k_v\tau(J_v)$ by the same way as in §4.3 (iii) of [10]. Since $k_v \in L_v[q,\, 2^{-1}\mathfrak{p}_v^\ell] = hC_v$, by [10, (1.10)], we have $hC_v = h\tau(J_v)$. Accordingly $hC(L) = h\tau(J_V)$ holds. Therefore [10, (1.9)] is applicable and the formula is given by

$$\begin{aligned}&\sum_y \#\left\{L\tau(y)^{-1}[q,\ \mathfrak{b}]/\tau(G^+(V)\cap yJ_Vy^{-1})\right\}\\ =\ &\#\left\{G^+(W)\setminus G^+(W)_{\mathbf{A}}/(G^+(W)_{\mathbf{A}}\cap J_V)\right\},\end{aligned} \tag{4.6}$$

where y runs over all representatives for $G^+(V) \setminus G^+(V)_{\mathbf{A}}/J_V$ for which $G^+(W)_{\mathbf{A}} \cap G^+(V)yJ_V \neq \emptyset$. Since $G^+(W)_{\mathbf{A}} \cap J_V = A^+(W)_{\mathbf{a}}^\times \mathfrak{O}_{\mathbf{h}}^\times$ by Proposition 3.3 (3), (4.6) equals the class number of $\mathfrak{O}$. Thus we obtain (4.5). ☐

Let v be a prime of F in case (i), (ii), or (iii) of the proof of Corollary 4.2. As can be seen in the proof, there is an element ω_v of $G^+(V_v)$ such that $h\tau(\omega_v) = h$, $L_v\tau(\omega_v) = L_v$, and $\omega_v \notin J_{V_v}$. This together with (3.2) shows that

$$[SO_v^\psi \cap C(L_v) : SO_v^\psi \cap \tau(J_{V_v})] = [C(L_v) : \tau(J_{V_v})] \tag{4.7}$$

for every $v \in \mathbf{h}$ under the two assumptions that $2\varphi(h,\, L)_v \supset \mathfrak{p}_v^{[\nu/2]}$ if $(K/v) = 0$ and that $\nu \in 2\mathbf{Z}$ and $2\varphi(h,\, L)_v = \mathfrak{b}(q)_v$ if $(K/v) = -1$ and $\chi(Q(\varphi)_v) = -1$. Here K is the discriminant field of φ and $q\mathfrak{g}_v = \mathfrak{p}_v^\nu$. In the same assumptions we also see that

$$SO_v^\psi \cap \tau(J_{V_v}) = \tau(\mathfrak{O}_v^\times). \tag{4.8}$$

These facts (4.7) and (4.8) are often useful in the application to quadratic Diophantine equations with four variables.

As for formula (4.1) for O^φ, we can state the following proposition:

Proposition 4.3. *Let $(V,\, \varphi)$ be a quadratic space of dimension 4 over a number field F and $K = F(\sqrt{\delta})$ the discriminant field of φ. For an element h of V such that $\varphi[h] = q \neq 0$ put $W = (Fh)^\perp$ and let ψ be the restriction of φ to W. Identify $O^\psi(W)$ with $\{\gamma \in O^\varphi(V) \mid h\gamma = h\}$. Let L be a $\mathfrak{g}$-maximal lattice in V with respect to φ. Also let $\mathfrak{f}_1$ be the product of all primes $v \in \mathbf{h}$ such that $2\varphi(h,\, L)_v \neq \mathfrak{b}(q)_v$. Suppose that for $v \in \mathbf{h}$,*

(1) *$v \nmid 2$ and $\varphi(h,\, L)_v^2 = q\mathfrak{g}_v$ if $(K/v) = 0$ and $\chi(Q(\psi)_v) = -1$.*

(2) *$q\mathfrak{g}_v$ is a square ideal of F_v if $(K/v) = -1$ and $\chi(Q(\varphi)_v) = -1$.*

(3) $\mathfrak{f}_1$ *consists of the primes* v *such that* $t_v = 0$ *or that* $v \nmid 2$, $(K/v) = -1$, $\chi(Q(\varphi)_v) = +1$, *and* $q\mathfrak{g}_v$ *is a square ideal of* F_v.

Here t_v *is the core dimension of* φ *at* v. *Let* λ *be the number of prime factors of* $\mathfrak{f}_1\mathfrak{f}_2$, *where* $\mathfrak{f}_2$ *is the product of all primes* $v \in \mathbf{h}$ *such that* $v \nmid \mathfrak{f}_1$, $t_v \neq 4$, $(K/v) \neq 0$, *and* $q\mathfrak{g}_v$ *is not a square ideal of* F_v. *Then* $[D(L\cap W) : O^{\psi}_{\mathbf{A}} \cap D(L)] = [C(L \cap W) : SO^{\psi}_{\mathbf{A}} \cap C(L)] = 2^{\lambda}$. *Moreover, if* $\lambda \leq 1$, *then* $O^{\psi}\varepsilon D(L \cap W) = O^{\psi}\varepsilon(O^{\psi}_{\mathbf{A}} \cap D(L))$ *for every* $\varepsilon \in O^{\psi}_{\mathbf{A}}$.

Before stating the proof, we note a simple fact. Let $G(V)$ be the Clifford group of φ. Then the homomorphism τ of (1.2) gives a surjection of $G(V)$ onto $O^{\varphi}(V)$, because V is even-dimensional.

Proof. In view of assumptions (2) and (3), we can take the order $\mathfrak{O}$ in Proposition 3.3 (3). Put $q\mathfrak{g}_v = \mathfrak{p}_v^{\nu_v}$ with $\nu_v \in \mathbf{Z}$ for $v \in \mathbf{h}$. We note that $v \mid \mathfrak{f}_2$ if and only if $t_v = 0$, $v \nmid \mathfrak{f}_1$, and ν_v is odd, or if $(K/v) = -1$, $v \nmid \mathfrak{f}_1$, and ν_v is odd.
Suppose $v \nmid \mathfrak{f}_1\mathfrak{f}_2$. Then $(L \cap W)_v$ is maximal in $(W, \psi)_v$. If $t_v = 0$, then ψ_v is isotropic and $\delta q\mathfrak{g}_v$ is square, which is because ν_v must be even by $v \nmid \mathfrak{f}_2$. Since $C(L_v) = \tau(J_{V_v})$ by (3.2), we have $SO^{\psi}_v \cap C(L_v) = \tau(\mathfrak{O}_v^{\times})$ by (4.8). Clearly $\mathfrak{O}_v = A^+(L \cap W)_v$ by calculating the discriminant. Note that $C(L_v \cap W_v) = \tau(A^+(L_v \cap W_v)^{\times})$ by [12, Lemma 5.4]. Hence we have $C(L_v \cap W_v) = SO^{\psi}_v \cap C(L_v)$. If $t_v = 4$, then $D(L_v) = O^{\varphi}_v$ and $C(L_v) = SO^{\varphi}_v$. Also $D(L_v \cap W_v) = O^{\psi}_v$ and $C(L_v \cap W_v) = SO^{\psi}_v$ as ψ_v is anisotropic. Hence we have $D(L_v \cap W_v) = O^{\psi}_v \cap D(L_v)$ and $C(L_v \cap W_v) = SO^{\psi}_v \cap C(L_v)$. Assume $t_v = 2$ and $(K/v) = -1$. Then $\delta q\mathfrak{g}_v$ must be square. If $Q(\varphi)_v = M_2(F_v)$, then ψ_v is isotropic. In the same way as in the case $t_v = 0$ we see that $C(L_v \cap W_v) = SO^{\psi}_v \cap C(L_v)$. If $Q(\varphi)_v$ is a division algebra, then ψ_v is anisotropic and $A^+(W)_v$ is a division algebra. Notice that $C(L_v \cap W_v) = \tau(A^+(W)_v^{\times})$ as $L_v \cap W_v$ is maximal. Our order $\mathfrak{O}_v$ has discriminant $\mathfrak{p}_v$ by (3.22), whence it is maximal in $A^+(W)_v$. Observe that $A^+(W)_v^{\times} = F_v^{\times}(\mathfrak{O}_v^{\times} \cup \mathfrak{O}_v^{\times}\omega)$ with some $\omega \in A^+(W)_v^{\times}$ so that ω^2 is a prime element of F_v. Since $SO^{\psi}_v \cap \tau(J_{V_v}) = \tau(\mathfrak{O}_v^{\times})$ by (4.8), we have $[C(L_v \cap W_v) : SO^{\psi}_v \cap \tau(J_{V_v})] = 2$. In view of (4.7) together with $[C(L_v) : \tau(J_{V_v})] = 2$ by (3.2), $SO^{\psi}_v \cap C(L_v)$ must coincide with $C(L_v \cap W_v)$. Assume $(K/v) = 0$. If $Q(\psi)_v = M_2(F_v)$, we take a Witt decomposition of φ_v in (2.9) with $g \in V_v$ so that $g^2 \in \mathfrak{g}_v^{\times}$. Then $\mathfrak{r}_v g$ is a maximal lattice in the core subspace $(K_v g, \varphi_v)$. Our assumption $Q(\psi)_v = M_2(F_v)$ implies that there is an element k of $K_v g$ such that $\varphi_v[k] = q = \varphi[h]$. Since the lattice $\mathfrak{r}_v g \cap (F_v k)^{\perp}$ is maximal in the complement $(F_v k)^{\perp}$ in $K_v g$ as $(K_v g, \varphi_v)$ is anisotropic, we have $2\varphi_v(k, \mathfrak{r}_v g) = \mathfrak{b}(q)_v = 2\varphi(h, L)_v$. Thus [9, Proposition 11.12 (iv) and (v)] are applicable to h. We have $C(L_v \cap W_v) = SO^{\psi}_v \cap C(L_v)$ and $D(L_v \cap W_v) = O^{\psi}_v \cap D(L_v)$. Similarly for the case where $Q(\psi)_v$ is a division algebra, under the assumption (1), we have $C(L_v \cap W_v) = SO^{\psi}_v \cap C(L_v)$ and $D(L_v \cap W_v) = O^{\psi}_v \cap D(L_v)$.
Suppose $v \mid \mathfrak{f}_1$. By assumption (3) such a prime satisfies either (i) $t_v = 0$ or (ii) $v \nmid 2$, $(K/v) = -1$, $Q(\varphi)_v = M_2(F_v)$, and ν_v is even. In both cases (i) and (ii),

$\mathfrak{O}_v = A^+(L\cap W)_v$ and $SO_v^\psi \cap C(L_v) = \tau(\mathfrak{O}_v^\times)$. Moreover, we can prove that

$$T(\mathfrak{O}_v) = F_v^\times(\mathfrak{O}_v^\times \cup \mathfrak{O}_v^\times \eta) \tag{4.9}$$

with some element η of $A^+(W)_v^\times$ such that $\eta\mathfrak{O}_v^\times = \mathfrak{O}_v^\times\eta$ and $\eta\eta^*\mathfrak{g}_v = \mathfrak{g}_v$ or $\eta\eta^*\mathfrak{g}_v = \mathfrak{p}_v$ according as ν_v is even or odd. This can be handled in a similar way to the proof of [3, Theorem 3.1] for Case (i) and to [3, §3.4] for Case (ii). (We will determine the index $[A^+(M_v)^\times : A^+(L_v\cap W_v)^\times]$ in a subsequent paper, which may be used in the proof of (4.9).) We have therefore $[C(L_v\cap W_v) : \tau(\mathfrak{O}_v^\times)] = 2$.

Suppose $v \mid \mathfrak{f}_2$. Then $(L\cap W)_v$ is maximal and $\mathfrak{O}_v = A^+(L\cap W)_v$. If $t_v = 0$, we have $SO_v^\psi \cap C(L_v) = \tau(\mathfrak{O}_v^\times)$. Since $q\mathfrak{g}_v$ is not square, by (3.5) and (3.7), $J_{W_v} = A^+(L_v\cap W_v)^\times$. Hence $[C(L_v\cap W_v) : \tau(\mathfrak{O}_v^\times)] = 2$ by (3.3). If $(K/v) = -1$, then $Q(\varphi)_v$ must be $M_2(F_v)$ under the assumption (2) as ν_v is odd. Applying (3.2) and (4.8), we have $SO_v^\psi \cap C(L_v) = \tau(\mathfrak{O}_v^\times)$. Hence by the same way as in the case $t_v = 0$, $[C(L_v\cap W_v) : \tau(\mathfrak{O}_v^\times)] = 2$.

To prove $[D(L_v\cap W_v) : O_v^\psi \cap D(L_v)] = [C(L_v\cap W_v) : SO_v^\psi \cap C(L_v)]$, we shall show that $[O_v^\psi \cap D(L_v) : SO_v^\psi \cap C(L_v)] = 2$ because $[D(L_v\cap W_v) : C(L_v\cap W_v)] = 2$ by [9, Lemma 6.8]. It is sufficient to investigate the following cases; (a) $t_v = 0$, (b) $(K/v) = -1$ and $\chi(Q(\varphi)_v) = +1$, (c) $(K/v) = -1$ and $\chi(Q(\varphi)_v) = -1$. In cases (a) and (b) we can verify the desired fact by the same technique as in the proof of [9, Proposition 11.12 (v)]; see the case $\widetilde{L} = L$ and $t \neq 1$ in that proof. As for case (c), we first note $\nu_v \in 2\mathbf{Z}$ by our assumption (2); put $\ell = \nu_v/2$. Since $(L\cap W)_v$ must be maximal under assumption (3), $2\varphi(h, L)_v = \mathfrak{b}(q)_v = \mathfrak{p}_v^\ell$ by (2.6) and (2.7). Now we take our setting and notation to be those in Case (iii) of the proof of Corollary 4.2. By (1.6), $h\alpha = k_v$ with some $\alpha \in C(L_v)$. Under such an α we may identify h, $(W, \psi)_v$, and $(L\cap W)_v$ with k_v, $K_vg \oplus F_v(q\pi_v^{-\ell}e - \pi_v^\ell f)$, and $\mathfrak{r}_vg + \mathfrak{p}_v^{-\ell}(q\pi_v^{-\ell}e - \pi_v^\ell f)$, respectively. Looking at the lattice $\mathfrak{r}_vg$ in the subspace (K_vg, φ_v) of $(V, \varphi)_v$, we can find $\gamma_0 \in O(K_vg)$ such that $\det(\gamma_0) = -1$ and $(\mathfrak{r}_vg)\gamma_0 = \mathfrak{r}_vg$ by [9, Lemma 6.8]. Extend γ_0 to an element γ of $GL(V_v)$ by setting γ to be the identity map on $(K_vg)^\perp$. Then $\gamma \in O_v^\varphi$, $h\gamma = h$, $\det(\gamma) = -1$, and $L_v\gamma = L_v$. This shows $[O_v^\psi \cap D(L_v) : SO_v^\psi \cap C(L_v)] = 2$. Summing up all these results, we obtain the first assertion.

To prove the second assertion, we borrow the idea of the proof of [9, Proposition 11.13 (ii)]. When there is no prime v dividing $\mathfrak{f}_1\mathfrak{f}_2$, we have $D(L\cap W) = O_{\mathbf{A}}^\psi \cap D(L)$, and so our assertion is obvious. Hereafter we assume $\lambda = 1$. For $\varepsilon \in O_{\mathbf{A}}^\psi$ put $\Lambda = L\varepsilon^{-1}$, which is a $\mathfrak{g}$-maximal lattice in (V, φ). We consider $\tau(h)$ of O^ψ. Put $\mathfrak{a} = \varphi[h](2\varphi(h, L))^{-2}$. Let $c \in F_{\mathbf{A}}^\times$ so that $2c\varphi(h, L) = \mathfrak{g}$; then $2\varphi_v(c_vh, \Lambda_v) = \mathfrak{g}_v$ and $\varphi_v[c_vh]\mathfrak{g}_v = \varphi_v[c_vh]\varphi_v(2c_vh, \Lambda_v)^{-2} = \mathfrak{a}_v$ for every $v \in \mathbf{h}$.

Suppose $\mathfrak{a}_v = \mathfrak{g}_v$. Then $\varphi_v[c_vh] \in \mathfrak{g}_v^\times$ and $2\varphi_v(c_vh, \Lambda_v) = \mathfrak{g}_v$. Hence c_vh belongs to Λ_v and also it is invertible in the order $A(\Lambda)_v$. Since this order contains Λ_v by definition, $A(\Lambda)_v \cap V_v = \Lambda_v$ by [9, Lemma 8.4 (iii)]. Thus we have $\Lambda_v\tau(h) = h^{-1}A(\Lambda)_vh \cap V_v = \Lambda_v$.

Let $v \nmid \mathfrak{f}_1\mathfrak{f}_2$. If $t_v = 0$ or $(K/v) = -1$, then $q\mathfrak{g}_v = \mathfrak{b}(q)_v^2$ as ν_v is even. We have $\mathfrak{a}_v = \mathfrak{g}_v$, whence $\Lambda_v\tau(h) = \Lambda_v$. If $t_v = 4$, then Λ_v is a unique maximal lattice in the anisotropic space $(V, \varphi)_v$. Hence $\Lambda_v\tau(h) = \Lambda_v$. If $(K/v) = 0$ and $Q(\psi)_v = M_2(F_v)$, we take a Witt decomposition of φ_v as in the same case of the proof of the first assertion with Λ_v in place of L_v. Because $2\varphi_v(h, \Lambda_v) = 2\varphi(h, L)_v = \mathfrak{b}(q)_v$, by the same manner as in that proof, we can find $\alpha \in C(\Lambda_v)$ so that $h\alpha = k$ with some $k \in K_v g$. Then $\tau(h) = \alpha\tau(k)\alpha^{-1}$ by [9, Lemma 3.8 (ii)]. We see that

$$\Lambda_v\tau(h) = \{(\mathfrak{r}_v g)\tau(k) + \mathfrak{g}_v e\tau(k) + \mathfrak{g}_v f\tau(k)\}\alpha^{-1} = \Lambda_v.$$

If $(K/v) = 0$ and $Q(\psi)_v$ is a division algebra, then $\mathfrak{a}_v = \mathfrak{g}_v$ by assumption (1), which leads $\Lambda_v\tau(h) = \Lambda_v$.

Let $v \mid \mathfrak{f}_1\mathfrak{f}_2$. We take a weak Witt decomposition of φ_v as in (2.9) with Λ_v in place of L_v. Put $q_v = \varphi_v[c_v h]$ and $k = q_v e + f$. We see that $\mathfrak{a}_v = [M/L \cap W]_v^2$ if $v \mid \mathfrak{f}_1$ and $\nu_v \in 2\mathbf{Z}$, $\mathfrak{a}_v = [M/L \cap W]_v^2\mathfrak{p}_v$ if $t_v = 0$ and $\nu_v \notin 2\mathbf{Z}$, and $\mathfrak{a}_v = \mathfrak{p}_v$ if $v \mid \mathfrak{f}_2$ and $(K/v) = -1$, where M is a maximal lattice in (W, ψ). Since $q_v \in \mathfrak{a}_v$, it belongs to $\mathfrak{g}_v$. Hence we have $k \in \Lambda_v[q_v, 2^{-1}\mathfrak{g}_v]$. By (1.6) there is $\alpha \in C(\Lambda_v)$ so that $(c_v h)\alpha = k$. Moreover $\tau(k) = \alpha^{-1}\tau(h)\alpha$ by [9, Lemma 3.8 (ii)]. Then α gives an isomorphism of W_v onto $W' = (F_v k)^\perp$ such that $(\Lambda_v \cap W_v)\alpha = \Lambda_v \cap W'$. Observe that $\Lambda_v \cap W' = \mathfrak{r}_v g + \mathfrak{g}_v(q_v e - f)$. Employing [9, Lemma 3.10], we can find that

$$(\Lambda_v \cap W')\tau(k) = \{-x - a(q_v e - f) \mid x \in \mathfrak{r}_v g,\ a \in \mathfrak{g}_v\} = \Lambda_v \cap W',$$
$$\Lambda_v\tau(k) = \{-x + q_v ae + q_v^{-1}bf \mid x \in \mathfrak{r}_v g,\ a, b \in \mathfrak{g}_v\} \neq \Lambda_v,$$

because $q_v \in \mathfrak{a}_v \subset \mathfrak{p}_v$ as seen above. Thus we have $(\Lambda_v \cap W_v)\tau(h) = \Lambda_v \cap W_v$ but $\Lambda_v\tau(h) \neq \Lambda_v$. To sum up, $\tau(h)$ is an element of O^ψ such that $(\Lambda \cap W)\tau(h) = \Lambda \cap W$ and $\Lambda\tau(h) \neq \Lambda$.

Now, observe $D(\Lambda \cap W) = \varepsilon D(L \cap W)\varepsilon^{-1}$ and $O^\psi_{\mathbf{A}} \cap D(\Lambda) = \varepsilon(O^\psi_{\mathbf{A}} \cap D(L))\varepsilon^{-1}$. Since $[D(L \cap W) : O^\psi_{\mathbf{A}} \cap D(L)] = 2$ by $\lambda = 1$, we have $[D(\Lambda \cap W) : O^\psi_{\mathbf{A}} \cap D(\Lambda)] = 2$. By our result on $\tau(h)$ we obtain

$$D(\Lambda \cap W) = (O^\psi_{\mathbf{A}} \cap D(\Lambda)) \cup \tau(h)(O^\psi_{\mathbf{A}} \cap D(\Lambda)).$$

Then our assertion follows from this and $\tau(h) \in O^\psi$. □

As a consequence, assuming that $h \in L[q, \mathfrak{b}]$ satisfies all the assumptions with $\lambda \leq 1$ in Proposition 4.3, by formula (4.1) for O^φ together with Proposition 4.3 and Theorem 3.4 (3), we obtain

$$\sum_{\alpha \in J} \#\left\{L\alpha^{-1}[q,\ \mathfrak{b}]/\Gamma^{\cdot}(L\alpha^{-1})\right\} = t(\mathfrak{O}), \tag{4.10}$$

where J is a set of representatives α for $O^\varphi \backslash O^\varphi_{\mathbf{A}}/D(L)$ for which $L\alpha^{-1}[q, \mathfrak{b}] \neq \emptyset$ and $\mathfrak{O}$ is the order in $A^+(W)$ defined in Proposition 3.3 (3) with h. It should be remarked that the discriminant of $\mathfrak{O}$ has at most one higher-power prime $\mathfrak{p}^e$ $(e > 1)$ if h satisfies $\lambda \leq 1$. Note that formula (4.5) permits several such primes in the discriminant of $\mathfrak{O}$ if h satisfies the assumptions of Corollary 4.2. For example, the reader is referred to our notes after the proof of [3, Proposition 4.3], in which $\mathfrak{O}$ has discriminant $2 \cdot 5^2 g^2 \mathbf{Z}$ for a squarefree odd positive integer g prime to 5.

5 Applications and numerical examples

5.1 Applications to $\{4,\ \mathbf{Q}(\sqrt{d}),\ B_{r,\infty},\ 4\}$

Theorem 5.1. *Let $B_{r,\infty}$ be a definite quaternion algebra over $\mathbf{Q}$ ramified only at a prime number r. Take a quadratic space $(V,\,\varphi)$ over $\mathbf{Q}$ whose invariants are $\{4,\ \mathbf{Q}(\sqrt{d}),\ B_{r,\infty},\ 4\}$ with a prime number d prime to r such that $d \equiv 1 \pmod 4$. Then for every odd prime number p prime to dr and $0 \leq n \in \mathbf{Z}$ there exist $\mathbf{Z}$-maximal lattices L and L' in $(V,\,\varphi)$ such that*

$$L[dp^n,\ 2^{-1}d\mathbf{Z}] \neq \emptyset, \qquad L'[dp^n,\ 2^{-1}\mathbf{Z}] \neq \emptyset, \tag{5.1}$$

except when $n \notin 2\mathbf{Z}$ and $\left(\frac{d}{p}\right) = -1$. Moreover the following formulas are valid:

$$\sum_{\alpha \in I} \frac{\#L\alpha^{-1}[dp^n,\ 2^{-1}d\mathbf{Z}]}{[\Gamma(L\alpha^{-1}) : 1]} = \frac{r-1}{24} \cdot \begin{cases} 1 & \text{if } n = 0, \\ p^{n-1}\left(p + \left(\frac{d}{p}\right)\right) & \text{if } n \geq 1, \end{cases} \tag{5.2}$$

$$\sum_{\alpha \in I} \frac{\#L'\alpha^{-1}[dp^n,\ 2^{-1}\mathbf{Z}]}{[\Gamma(L'\alpha^{-1}) : 1]} = \frac{(r-1)(d^2-1)}{48} \cdot \begin{cases} 1 & \text{if } n = 0, \\ p^{n-1}\left(p + \left(\frac{d}{p}\right)\right) & \text{if } n \geq 1, \end{cases} \tag{5.3}$$

$$\sum_{\alpha \in J} \#\left\{L\alpha^{-1}[dp^n,\ 2^{-1}d\mathbf{Z}]/\Gamma^{\cdot}(L\alpha^{-1})\right\} = t(\mathfrak{O}). \tag{5.4}$$

Here $\left(\frac{d}{p}\right)$ is the quadratic residue symbol, I (resp. J) is a complete set of representatives for $SO^{\varphi} \setminus SO^{\varphi}_{\mathbf{A}}/C(L)$ (resp. $O^{\varphi} \setminus O^{\varphi}_{\mathbf{A}}/D(L)$), $\mathfrak{O}$ is an order in the algebra $A^+(W)$, which is isomorphic to $B_{r,\infty}$, of discriminant $rp^n\mathbf{Z}$ containing $A^+(L \cap W)$, and $W = (\mathbf{Q}h)^{\perp}$ with $h \in L[dp^n,\ 2^{-1}d\mathbf{Z}]$.

It is noted that $L\alpha^{-1}[dp^n,\ 2^{-1}d\mathbf{Z}]$ or $L'\alpha^{-1}[dp^n,\ 2^{-1}\mathbf{Z}]$ may be empty for some $\alpha \in I$ or some $\alpha \in J$.

Proof. First of all, under the assumption that $L[dp^n,\ 2^{-1}d\mathbf{Z}] \neq \emptyset$ and $L'[dp^n,\ 2^{-1}\mathbf{Z}] \neq \emptyset$ with some maximal L and L' in V, we can derive formulas (5.2) and (5.3). We should mention that the proof will be given in a subsequent paper and that these formulas will be used in the present proof to show (5.1).

By [8, Proposition 1.8], for any positive integer q there is a $\mathbf{Z}$-maximal lattice L in $(V,\,\varphi)$ such that $L[q] \neq \emptyset$. Let $h \in L[dp^n]$ with $0 \leq n \in \mathbf{Z}$ and take the complement (W, ψ) as in (2.2). Since $d \equiv 1 \pmod 4$ and $\left(\frac{d}{p}\right) = 1$ if n is odd, the quaternion algebra $\{\mathbf{Q}(\sqrt{d}),\, dp^n\}$ is $M_2(\mathbf{Q})$. Hence $Q(\psi) = B_{r,\infty}$ by (2.3) and so the invariants of ψ are $\{3,\, \mathbf{Q}(\sqrt{-p^n}),\, B_{r,\infty},\, 3\}$. We have then $\mathfrak{b}(dp^n) = dp^\ell$ with $\ell = [n/2]$. Noticing $\mathfrak{b}(dp^n) \subset 2\varphi(h,\, L) \subset \mathbf{Z}$ as noted in §2.1, we see that

$$L[dp^n] = \bigcup_{i=0}^{\ell} \{L[dp^n,\ 2^{-1}dp^i\mathbf{Z}] \cup L[dp^n,\ 2^{-1}p^i\mathbf{Z}]\}. \tag{5.5}$$

Applying the explicit formula of [8, Theorem 1.5 (II)] to $L[dp^n]$, we can derive that

$$\sum_{\alpha \in I} \frac{\# L\alpha^{-1}[dp^n]}{[\Gamma(L\alpha^{-1}) : 1]} = \frac{(r-1)\left(d^2 + \left(\frac{d}{p}\right)^n\right)}{48} \sum_{i=0}^{2\ell} \left(\frac{d}{p}\right)^{n+i} p^i. \tag{5.6}$$

We here recall our assumption that $\left(\frac{d}{p}\right) = 1$ if n is odd. We put

$$R[q] = \sum_{\alpha \in I} \frac{\# L\alpha^{-1}[q]}{\#\Gamma(L\alpha^{-1})}, \qquad R[q,\, \mathfrak{b}] = \sum_{\alpha \in I} \frac{\# L\alpha^{-1}[q,\ \mathfrak{b}]}{\#\Gamma(L\alpha^{-1})} \tag{5.7}$$

for $q \in \mathbf{Z}$ and a $\mathbf{Z}$-ideal $\mathfrak{b}$ of $\mathbf{Q}$.

Suppose $n = 2\ell$ with $0 \le \ell \in \mathbf{Z}$. We shall prove (5.1) by induction on ℓ. If $\ell = 0$, then $\mathfrak{b}(d) = d\mathbf{Z}$ and $L[d] = L[d,\, 2^{-1}d\mathbf{Z}] \cup L[d,\, 2^{-1}\mathbf{Z}]$ by (5.5). Because $L[d] \neq \emptyset$, either $L[d,\, 2^{-1}d\mathbf{Z}]$ or $L[d,\, 2^{-1}\mathbf{Z}]$ must be nonempty. If $L[d,\, 2^{-1}d\mathbf{Z}] \neq \emptyset$, then formula (5.2) is valid as mentioned above. Combining this with (5.6), we have $R[d,\, 2^{-1}\mathbf{Z}] = R[d] - R[d,\, 2^{-1}d\mathbf{Z}] = 48^{-1}(r-1)(d^2-1)$. This implies that there is some $\alpha \in SO^{\varphi}_{\mathbf{A}}$ so that $L\alpha^{-1}[d,\, 2^{-1}\mathbf{Z}] \neq \emptyset$. Conversely, if $L[d,\, 2^{-1}\mathbf{Z}] \neq \emptyset$, we have $R[d,\, 2^{-1}d\mathbf{Z}] = 24^{-1}(r-1)$ in the same way, whence $L\alpha^{-1}[d,\, 2^{-1}d\mathbf{Z}] \neq \emptyset$ with some $\alpha \in SO^{\varphi}_{\mathbf{A}}$. As a consequence we can find maximal lattices L and L' in $(V,\, \varphi)$ such that $L[d,\, 2^{-1}d\mathbf{Z}] \neq \emptyset$ and $L'[d,\, 2^{-1}\mathbf{Z}] \neq \emptyset$. This settles the case $\ell = 0$. Suppose $\ell > 0$. In view of (5.5) we have

$$R[dp^n] = \sum_{i=0}^{\ell} \left\{R[dp^n,\ 2^{-1}dp^i\mathbf{Z}] + R[dp^n,\ 2^{-1}p^i\mathbf{Z}]\right\}. \tag{5.8}$$

Observe that the mapping $x \longmapsto xp^i$ gives a bijection of $L\alpha^{-1}[dp^{2(\ell-i)},\, 2^{-1}d\mathbf{Z}]$ onto $L\alpha^{-1}[dp^n,\, 2^{-1}dp^i\mathbf{Z}]$ for $i \neq 0$ and $\alpha \in SO^{\varphi}_{\mathbf{A}}$ for which $L\alpha^{-1}[dp^n,\, 2^{-1}dp^i\mathbf{Z}] \neq \emptyset$. Similarly $L\alpha^{-1}[dp^{2(\ell-i)},\, 2^{-1}\mathbf{Z}]$ is mapped onto $L\alpha^{-1}[dp^n,\, 2^{-1}p^i\mathbf{Z}]$ under the above bijection if $i \neq 0$ and $L\alpha^{-1}[dp^n,\, 2^{-1}p^i\mathbf{Z}] \neq \emptyset$. By our induction, (5.2) and (5.3) for $2(\ell - i)$ in place of n are valid for

$i \neq 0$. Thus we see that

$$\begin{aligned}
& R[dp^n,\ 2^{-1}d\mathbf{Z}] + R[dp^n,\ 2^{-1}\mathbf{Z}] \\
= \quad & R[dp^n] - \sum_{i=0}^{\ell} \left\{ R[dp^{2(\ell-i)},\ 2^{-1}d\mathbf{Z}] + R[dp^{2(\ell-i)},\ 2^{-1}\mathbf{Z}] \right\} \\
= \quad & \frac{(r-1)(d^2+1)}{48} \cdot p^{2\ell-1} \left(p + \left(\frac{d}{p} \right) \right). \qquad (5.9)
\end{aligned}$$

This shows that either $L_1[dp^n, 2^{-1}d\mathbf{Z}]$ or $L_1[dp^n, 2^{-1}\mathbf{Z}]$ is not empty with some maximal lattice L_1 in V. Now, if $L_1[dp^n, 2^{-1}d\mathbf{Z}] \neq \emptyset$, then formula (5.2) is valid. Combining these results with (5.9), we have $R[dp^n, 2^{-1}\mathbf{Z}] \neq 0$, which implies that $L_1\alpha^{-1}[dp^n, 2^{-1}\mathbf{Z}] \neq \emptyset$ with some $\alpha \in SO^{\varphi}_{\mathbf{A}}$. Conversely, if $L_1[dp^n, 2^{-1}\mathbf{Z}] \neq \emptyset$, we have $L_1\alpha^{-1}[dp^n, 2^{-1}d\mathbf{Z}] \neq \emptyset$ with $\alpha \in SO^{\varphi}_{\mathbf{A}}$ by the same way. Consequently we have maximal lattices L_1 and L_1' in (V, φ) such that $L_1[dp^n, 2^{-1}d\mathbf{Z}] \neq \emptyset$ and $L_1'[dp^n, 2^{-1}\mathbf{Z}] \neq \emptyset$. This completes our induction on $\ell = n/2$.

The case of odd n can be proved similarly, which together with the case of even n shows (5.1) for every integer $n \geq 0$. At the same time we obtain formulas (5.2) and (5.3).

As for (5.4), observe first that the conditions of (1) and (2) in Proposition 4.3 are satisfied for $h \in L[dp^n, 2^{-1}d\mathbf{Z}]$ because r, d, and p are distinct prime numbers. Further $(L \cap W)_v$ is not maximal if and only if $v = p$ as $\mathfrak{b}(dp^n) = dp^{\ell}\mathbf{Z}$, except when $\ell = [n/2] = 0$, that is, when $n = 0$ or 1. Then we easily see that condition (3) of that proposition is satisfied; for instance, if p remains prime in $\mathbf{Q}(\sqrt{d})$, then n must be even by our assumption, and so p satisfies (3). The ideal $\mathfrak{f}_2$ of Proposition 4.3 in the present situation is $\mathbf{Z}$, except when $n = 1$. If $n = 0$ or 1, then $L \cap W$ is maximal. Also $\mathfrak{f}_2 = \mathbf{Z}$ or $p\mathbf{Z}$ according as $n = 0$ or 1. To sum up, Proposition 4.3 is applicable to $h \in L[dp^n, 2^{-1}d\mathbf{Z}]$ for every $0 \leq n \in \mathbf{Z}$. Hence (5.4) follows from (4.10). $\square$

We note that when $n \notin 2\mathbf{Z}$ and $\left(\frac{d}{p}\right) = -1$ in Theorem 5.1, $L[dp^n, 2^{-1}d\mathbf{Z}] = \emptyset$ for any maximal lattice L and $L'[dp^n, 2^{-1}\mathbf{Z}] \neq \emptyset$ with some maximal lattice L' in (V, φ).

Formulas (5.2) and (5.3) can be derived by means of the mass formula due to Shimura [9, (13.18)], combined with a result in a subsequent paper as mentioned in the proof of Theorem 5.1.

It should be remarked about (5.4) that the type number of $\mathfrak{O}$ is not determined by discriminant, but by the genus of $\mathfrak{O}$. (In other words, by Theorem 3.4 (2), the ideal $[(L \cap W)\tilde{}/L \cap W]$ does *not* determine the genus of $L \cap W$.) However, if $L \cap W$ is maximal, that is, if $\mathfrak{O}$ has squarefree discriminant, $t(\mathfrak{O})$ is determined by the discriminant. In fact, $\mathfrak{O}$ is maximal or an order of squarefree discriminant $rp\mathbf{Z}$ according as $n = 0$ or 1. By a result due to Eichler [1, Satz 3], any order $\mathfrak{O}'$ of discriminant $rp\mathbf{Z}$ belongs to the genus of $\mathfrak{O}$ in the sense that $\mathfrak{O}' = y^{-1}\mathfrak{O}y$ with some $y \in A^+(W)^{\times}_{\mathbf{A}}$. The similar fact is true for maximal orders, which have discriminant $r\mathbf{Z}$. Accordingly in either case $n = 0$ or 1 the discriminant $rp^n\mathbf{Z}$ certainly determines the genus of $\mathfrak{O}$. Moreover the discriminant does not depend on d. In view of these together with (5.4), we can conclude

COROLLARY 5.2. *Let the notation be as in Theorem 5.1. Then for $n = 0$ or 1 the number of the left-hand side of* (5.4) *is independent of the choice of d. Especially, if the type number of orders in $B_{r,\infty}$ of discriminant $rp^n\mathbf{Z}$ is* 1, *for any prime number d prime to rp^n such that $d \equiv 1 \pmod 4$ and $\left(\frac{d}{p}\right)^n = 1$ there exists only one O^φ-class in the genus of maximal lattices in (V, φ) of $\{4, \mathbf{Q}(\sqrt{d}), B_{r,\infty}, 4\}$ such that $L_1[dp^n, 2^{-1}d\mathbf{Z}] \neq \emptyset$ and*

$$L_1[dp^n,\ 2^{-1}d\mathbf{Z}] = h\Gamma^{\cdot}(L_1)$$

with a lattice L_1 in the class and $h \in L_1[dp^n, 2^{-1}d\mathbf{Z}]$.

In Table 1 of Section 5.3 below we shall see a few numerical examples for $r = 2$ and $n = 0$ supporting this fact.

5.2 EXAMPLES FOR REAL QUADRATIC FIELDS

Let V be a totally definite quaternion algebra over F of discriminant $\mathfrak{g}$ and φ its norm form, where F is a totally real field of even degree. Taking a nonzero element h of V and a $\mathfrak{g}$-maximal lattice L in (V, φ), we have the complement (W, ψ) of Fh and the lattice $L \cap W$. We see that $A^+(W)$ is isomorphic to the present V as quaternion algebras. Our order $\mathfrak{O}$ is then $A^+(L \cap W)$ and has discriminant $q\mathfrak{b}^{-2}$ with $q = \varphi[h]$ and $\mathfrak{b} = 2\varphi(h, L)$. Let $c(\mathfrak{O})$ denote the class number of $\mathfrak{O}$ as before.

PROPOSITION 5.3. *In the above setting with $h \in L[q, 2^{-1}\mathfrak{b}]$ assume that F has class number* 1. *Then there exists an order $\mathfrak{O}$ of discriminant $q\mathfrak{b}^{-2}$ in V such that $\sum_{i\in I} \#\left\{L_i[q, 2^{-1}\mathfrak{b}]/\Gamma(L_i)\right\} = c(\mathfrak{O})$, where $\{L_i\}_{i\in I}$ is a set of representatives for the SO^φ-classes in the SO^φ-genus of L for which $L_i[q, 2^{-1}\mathfrak{b}] \neq \emptyset$.*

We first note by [8, Proposition 1.8] that, for every totally positive integer q of F, there is a $\mathfrak{g}$-maximal lattice L in (V, φ) such that $L[q] = \{x \in L \mid \varphi[x] = q\} \neq \emptyset$. Moreover if $q\mathfrak{g}$ is squarefree, then $L[q] = L[q, 2^{-1}\mathfrak{g}]$ because of $\mathfrak{b}(q) = \mathfrak{g}$.

Proof. Clearly formula (4.5) is applicable to $h \in L[q, 2^{-1}\mathfrak{b}]$. Since $C(L) = \tau(J_V)$ and F has class number 1, τ of (1.2) gives a bijection of $G^+(V) \setminus G^+(V)_{\mathbf{A}}/J_V$ onto $SO^\varphi \setminus SO^\varphi_{\mathbf{A}}/C(L)$. Furthermore we have $\tau(G^+(V) \cap yJ_Vy^{-1}) = \Gamma(L\tau(y)^{-1})$ for every $y \in G^+(V)_{\mathbf{A}}$. The assertion follows from these combined with (4.5). ☐

For example, take (V, φ) as in Proposition 5.3 over $F = \mathbf{Q}(\sqrt{d})$ with $d = 5$, 13, or 101. It is known that $\#\{SO^\varphi \setminus SO^\varphi_{\mathbf{A}}/C(L)\} = 1$ when $d = 5$, 13. As noted above, there is a maximal lattice L in (V, φ) such that $L[q] = L[q, 2^{-1}\mathfrak{g}] \neq \emptyset$ for a given totally-positive squarefree integer q of F. Applying Proposition 5.3 to $h \in L[q]$, we have an order $\mathfrak{O}$ in V of discriminant $q\mathfrak{g}$. Now suppose $q \in \mathfrak{g}^\times$. Then $\mathfrak{O}$ is maximal as $d(\mathfrak{O}) = \mathfrak{g}$. Its class number is 1 if $d = 5, 13$ and is 5 if $d = 101$. These results can be found in [5, Tabelle 2] due to Peters. Therefore by the same proposition, $\#\{L[q, 2^{-1}\mathfrak{g}]/\Gamma(L)\} = 1$ if $d = 5, 13$ and $\sum_{i\in I} \#\{L_i[q, 2^{-1}\mathfrak{g}]/\Gamma(L_i)\} = 5$ if $d = 101$, where $\{L_i\}_{i\in I}$ is a set of representatives of the SO^φ-classes in the genus of L for which $L_i[q, 2^{-1}\mathfrak{g}] \neq \emptyset$. We mention that there is a previous result [10, Theorem 1.11] concerning the application of [10, Theorem 1.6] to the norm forms of definite quaternion algebras over $\mathbf{Q}$.

5.3 NUMERICAL TABLES FOR $\{4,\ \mathbf{Q}(\sqrt{d}),\ B_{2,\infty},\ 4\}$

Let d be a prime number such that $d \equiv 1 \pmod 4$. We take a quadratic space (V, φ) over $\mathbf{Q}$ of invariants $\{4, \mathbf{Q}(\sqrt{d}), B_{2,\infty}, 4\}$ and a complete set $\{L_i\}_{i\in J}$ of representatives for the O^φ-classes in the O^φ-genus of maximal lattices in (V, φ). By (5.4) the number $\sum_{i\in J} \#\left\{L_i[dp^n, 2^{-1}d\mathbf{Z}]/\Gamma^\cdot(L_i)\right\}$ is given by the type number $t(\mathfrak{O})$ of some order $\mathfrak{O}$ in $B_{2,\infty}$ of discriminant $2p^n\mathbf{Z}$ for an odd prime number p prime to d and $0 \leq n \in \mathbf{Z}$, where we assume $\left(\frac{d}{p}\right) = 1$ if n is odd and remark that $L_i[dp^n, 2^{-1}d\mathbf{Z}]$ may be empty for some $i \in J$. We put $\mathfrak{c}(dp^n) = \sum_{i\in J} \#\left\{L_i[dp^n, 2^{-1}d\mathbf{Z}]/\Gamma^\cdot(L_i)\right\}$ for convenience. We restrict ourselves to the case $n = 0$ or 1. In this section we shall not only give the numbers $\mathfrak{c}(dp^n)$ by quoting $t(\mathfrak{O})$, but also present $\#L_i[dp^n, 2^{-1}d\mathbf{Z}]$ for $i \in J$ by taking $\{L_i\}_{i\in J}$ in the case of $d = 5,\ 13,\ 17,$ or 29.

To obtain $\{L_i\}_{i\in J}$ for these primes d, we proceed according to the viewpoint explained at the last part of §4.1. Let (X, Φ) be as in that section. We set

$$\Lambda = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}g + \mathbf{Z}e_5,$$

where $\{e_i\}$ is the standard basis of $\mathbf{Q}_5^1$ and $g = 2^{-1}(e_1 + e_2 + e_3 + e_4)$. Then Λ is a $\mathbf{Z}$-maximal lattice in (X, Φ). By (4.3) we have a bijection

$$k_i\Gamma^\cdot(\Lambda) \longmapsto (\Lambda \cap (\mathbf{Q}k_i)^\perp)\gamma_i^{-1}O^\varphi$$

of $\Lambda[d, \mathbf{Z}]/\Gamma^\cdot(\Lambda)$ onto the O^φ-classes in the O^φ-genus of maximal lattices in (V, φ) with some $\gamma_i \in O^\Phi$ so that $k_i = k_0\gamma$ for $i \in J$, where $\{k_i\}_{i\in J}$ is a complete set of representatives for $\Lambda[d, \mathbf{Z}]/\Gamma^\cdot(\Lambda)$ and k_0 is an arbitrarily fixed element of $\Lambda[d, \mathbf{Z}]$; we put $V = (\mathbf{Q}k_0)^\perp$ and $\varphi = \Phi|_V$. Hence the desired representatives $\{L_i\}_{i\in J}$ can be obtained from explicit elements k_i for $i \in J$ by taking $(\Lambda \cap (\mathbf{Q}k_i)^\perp)\gamma_i^{-1}$ as L_i. A method of determining $\Lambda[d, \mathbf{Z}]/\Gamma^\cdot(\Lambda)$ is explained in [9, §12.15]; in which $\{k_i\}_{i\in J}$ was found for the case of $d = 29$. We employ that method for our purpose. Once such a set $\{k_i\}_{i\in J}$ is obtained, using the lattice $\Lambda \cap (\mathbf{Q}k_i)^\perp$, we can compute the number $\#L_i[dp^n, 2^{-1}d\mathbf{Z}]$ for every $i \in J$.

Here is a list of the representatives k_i for $\Lambda[d,\,\mathbf{Z}]/\Gamma^{\cdot}(\Lambda)$ and the corresponding lattices $\Lambda \cap (\mathbf{Q}k_i)^{\perp}$ for $i \in J$ and $d = 5,\ 13,\ 17,\ 29$:

(1) $d = 5$.

$$\begin{aligned} &k_1 = 2e_1 + e_5. \\ &\Lambda \cap (\mathbf{Q}k_1)^{\perp} = \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}e_4 + \mathbf{Z}(g - e_5). \end{aligned}$$

(2) $d = 13$.

$$\begin{aligned} &k_1 = 2e_1 + 3e_5, \quad k_2 = 2(e_2 + e_3 + e_4) + e_5. \\ &\Lambda \cap (\mathbf{Q}k_1)^{\perp} = \mathbf{Z}(e_2 + e_3 + e_4) + \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}(3g - e_5), \\ &\Lambda \cap (\mathbf{Q}k_2)^{\perp} = \mathbf{Z}e_1 + \mathbf{Z}(e_2 - 2e_5) + \mathbf{Z}(e_3 - 2e_5) + \mathbf{Z}(g - 3e_5). \end{aligned}$$

(3) $d = 17$.

$$\begin{aligned} &k_1 = 4e_4 + e_5, \quad k_2 = 2(e_3 + e_4) + 3e_5. \\ &\Lambda \cap (\mathbf{Q}k_1)^{\perp} = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}(g - 2e_5), \\ &\Lambda \cap (\mathbf{Q}k_2)^{\perp} = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}(3e_3 - 2e_5) + \mathbf{Z}(g - e_3). \end{aligned}$$

(4) $d = 29$.

$$\begin{aligned} &k_1 = 2e_4 + 5e_5, \quad k_2 = 2(e_3 + 2e_4) + 3e_5, \\ &\qquad\qquad k_3 = 2(e_1 + e_2 + e_3 + 2e_4) + e_5. \\ &\Lambda \cap (\mathbf{Q}k_1)^{\perp} = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}(5g - e_5), \\ &\Lambda \cap (\mathbf{Q}k_2)^{\perp} = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}(3e_3 - 2e_5) + \mathbf{Z}(g - e_5), \\ &\Lambda \cap (\mathbf{Q}k_3)^{\perp} = \mathbf{Z}(e_1 - 2e_5) + \mathbf{Z}(e_2 - 2e_5) + \\ &\qquad\qquad\qquad + \mathbf{Z}(e_3 - 2e_5) + \mathbf{Z}(g - 5e_5). \end{aligned}$$

Here we note that the case $d = 5$ can be seen from [10, §4.4, (4.12c)]. It can also be verified that these k_i for $i \in J$ form a complete set of representatives for $\Lambda[d,\,\mathbf{Z}]/\Gamma(\Lambda)$ for $d = 5,\ 13,\ 17,\ 29$. Since [9, Proposition 11.13 (ii)] is also applicable to $k_0 \in \Lambda[d,\,\mathbf{Z}]$, by Lemma 4.1, $\{L_i\}_{i \in J}$ gives a complete set of representatives for the SO^{φ}-classes in the SO^{φ}-genus of maximal lattices in $(V,\,\varphi)$.

We can further determine $[\Gamma(L_i) : 1]$ for $i \in J$. In fact, by Theorem 5.1 we have an explicit formula (5.2) for $R[dp^n,\,2^{-1}d\mathbf{Z}]$ with the notation of (5.7); then $\#\Gamma(L_i)$ is computable in an elementary way by using this formula combined with the numerical data of $\#L_i[dp^n,\,2^{-1}d\mathbf{Z}]$ in our tables. For example, if $d = 29$, then we have three maximal lattices $\{L_1,\,L_2,\,L_3\}$ given above. Looking at Table 1 for $d = 29$ and at Table 3 for $d = 29$, $p = 5$ and 7, we have $2 \cdot \#\Gamma(L_1)^{-1} = 24^{-1}$, $2 \cdot \#\Gamma(L_2)^{-1} = 4^{-1}$, and $2 \cdot \#\Gamma(L_3)^{-1} = 3^{-1}$ by (5.2). From these we get $\#\Gamma(L_1) = 48$, $\#\Gamma(L_2) = 8$, and $\#\Gamma(L_3) = 6$. Moreover the mass of the genus with respect to SO^φ is $5/16$, which indeed coincides with the mass derived from the exact formula of [7, Theorem 5.8]. Similarly for $d = 5,\ 13,\ 17$, we have $\#\Gamma(L_1) = 48$ if $d = 5$; $\#\Gamma(L_1) = 48$ and $\#\Gamma(L_2) = 12$ if $d = 13$; $\#\Gamma(L_1) = \#\Gamma(L_2) = 48$ if $d = 17$.

In the numerical tables below, we put $N_i(dp^n) = \#L_i[dp^n,\, 2^{-1}d\mathbf{Z}]$ and denote by $t(2,\, p^n)$ (resp. $c(2,\, p^n)$) the type number (resp. the class number) of $\mathfrak{O}$ in $B_{2,\,\infty}$ of discriminant $2p^n\mathbf{Z}$. We quote $t(2,\, p^n)$ and $c(2,\, p^n)$ from [6, Table 1] due to Pizer. It is noted by Corollary 4.2 that the number $\sum_{i\in J} \#\left\{L_i[dp^n,\, 2^{-1}d\mathbf{Z}]/\Gamma(L_i)\right\}$ coincides with $c(2,\, p^n)$ if $t(2,\, p^n) = c(2,\, p^n)$.

d	$N_1(d)$	$N_2(d)$	$N_3(d)$	$t(2,\,1)$	$c(2,\,1)$	$\mathfrak{c}(d)$
5	2	$*$	$*$	1	1	1
13	2	0	$*$	1	1	1
17	2	0	$*$	1	1	1
29	2	0	0	1	1	1

Table 1: $\mathfrak{c}(d)$ for $d = 5,\ 13,\ 17,\ 29$

Let us verify our numerical results for $\mathfrak{c}(dp)$ in a straightforward way by using the lattices listed above. As an example, we take up the case of $d = 13$ and $p = 23$. We begin with the 5-dimensional space $(X,\, \Phi)$ and Λ as above. Put $k_1 = 2e_1 + 3e_5$ and $k_2 = 2(e_2 + e_3 + e_4) + e_5$. In our list with $d = 13$, k_1 and k_2 form a complete set of representatives for $\Lambda[13,\, \mathbf{Z}]/\Gamma^{\cdot}(\Lambda)$ (and it is true for $\Gamma(\Lambda)$ in place of $\Gamma^{\cdot}(\Lambda)$). Set $V = (\mathbf{Q}k_2)^{\perp}$ and let φ be the restriction of Φ to V. Then $(V,\, \varphi)$ has invariants $\{4,\, \mathbf{Q}(\sqrt{13}),\, B_{2,\,\infty},\, 4\}$ and $L_2 = \Lambda \cap V$ is $\mathbf{Z}$-maximal in $(V,\, \varphi)$. Since $\{e_1,\, e_2 - 2e_5,\, e_3 - 2e_5,\, g - 3e_5\}$ is a $\mathbf{Z}$-basis of L_2, representing φ by this basis, we may put $V = \mathbf{Q}_4^1$,

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 5 & 4 & 13/2 \\ 0 & 4 & 5 & 13/2 \\ 1/2 & 13/2 & 13/2 & 10 \end{bmatrix},$$

and $L_2 = \mathbf{Z}_4^1$. Under this identification, $\Gamma^{\cdot}(L_2)$ is the subgroup $\{\gamma \in GL_4(\mathbf{Z}) \mid$

d	p	$N_1(dp)$	$N_2(dp)$	$t(2,\ p)$	$c(2,\ p)$	$\mathfrak{c}(dp)$
5	11	24	$*$	1	1	1
5	19	40	$*$	2	3	2
5	29	60	$*$	2	3	2
5	31	64	$*$	2	4	2
5	41	84	$*$	3	4	3
5	59	120	$*$	3	5	3
5	61	124	$*$	4	7	4
5	71	144	$*$	2	6	2
5	79	160	$*$	3	8	3
5	89	180	$*$	5	8	5
5	101	204	$*$	5	9	5
13	3	0	2	1	1	1
13	17	12	6	2	2	2
13	23	0	12	1	2	1
13	29	12	12	2	3	2
13	43	16	18	3	5	3
13	53	12	24	3	5	3
13	61	12	28	4	7	4
13	79	48	28	3	8	3
13	101	60	36	5	9	5

Table 2: $\mathfrak{c}(dp)$ for $d = 5,\ 13$

d	p	$N_1(dp)$	$N_2(dp)$	$N_3(dp)$	$t(2, p)$	$c(2, p)$	$\mathfrak{c}(dp)$
17	13	16	12	$*$	2	3	2
17	19	16	24	$*$	2	3	2
17	43	40	48	$*$	3	5	3
17	47	48	48	$*$	2	4	2
17	53	60	48	$*$	3	5	3
17	59	72	48	$*$	3	5	3
17	67	64	72	$*$	4	7	4
17	83	72	96	$*$	4	7	4
17	89	96	84	$*$	5	8	5
17	101	96	108	$*$	5	9	5
29	5	0	2	0	1	1	1
29	7	0	0	2	1	2	1
29	13	0	2	2	2	3	2
29	23	0	0	6	1	2	1
29	53	0	10	6	3	5	3
29	59	24	8	6	3	5	3
29	67	24	8	8	4	7	4
29	71	0	8	12	2	6	2
29	83	24	16	6	4	7	4
29	103	16	16	12	5	10	5

Table 3: $\mathfrak{c}(dp)$ for $d = 17,\ 29$

$\gamma\varphi \cdot {}^t\gamma = \varphi\}$ of $GL_4(\mathbf{Z})$. Then $L_2[13p^n,\ 2^{-1}\cdot 13\mathbf{Z}]$ is given by

$$\begin{aligned}
&L_2[13p^n,\ 2^{-1}\cdot 13\mathbf{Z}] = \{[x_1\ x_2\ x_3\ x_4] \in \mathbf{Z}_4^1 \mid \\
&x_1^2 + 5x_2^2 + 5x_3^2 + 10x_4^2 + x_1x_4 + 8x_2x_3 + 13x_2x_4 + 13x_3x_4 = 13p^n, \\
&(2x_1 + x_4)\mathbf{Z} + (10x_2 + 8x_3 + 13x_4)\mathbf{Z} + (8x_2 + 10x_3 + 13x_4)\mathbf{Z} \\
&+(x_1 + 13x_2 + 13x_3 + 20x_4)\mathbf{Z} = 13\mathbf{Z}\}.
\end{aligned}$$

Now for $p = 23$ and $n = 1$ we have all solutions in $L_2[13\cdot 23,\ 2^{-1}\cdot 13\mathbf{Z}]$:

$$[\pm 5\ \ \mp 13\ \ \mp 13\ \ \pm 16],\ [\pm 8\ 0\ \ \mp 13\ \ \pm 10],\ [\pm 8\ \ \mp 13\ 0\ \ \pm 10],$$
$$[\pm 18\ 0\ \ \pm 13\ \ \mp 10],\ [\pm 18\ \ \pm 13\ 0\ \ \mp 10],\ [\pm 21\ \ \pm 13\ \ \pm 13\ \ \mp 16].$$

We put

$$\gamma_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix},\quad \gamma_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & -2 \\ 1 & 0 & 0 & -1 \end{bmatrix},\quad \gamma_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

These matrices belong to $\Gamma^{\cdot}(L_2)$. Consider the subgroup U of $\Gamma^{\cdot}(L_2)$ generated by $\gamma_1, \gamma_2, \gamma_3$, and -1_4, where 1_4 is the identity matrix of size 4. Put $x = [5\ \ -13\ \ -13\ 16]$. Then it can be seen that xU contains all elements of $L_2[13\cdot 23,\ 2^{-1}\cdot 13\mathbf{Z}]$. Thus we have $L_2[13\cdot 23,\ 2^{-1}\cdot 13\mathbf{Z}] = x\Gamma^{\cdot}(L_2)$.
Similarly for k_1, we can consider a $\mathbf{Z}$-lattice $\Lambda \cap (\mathbf{Q}k_1)^{\perp}$. Denoting by φ_1 the restriction of Φ to $(\mathbf{Q}k_1)^{\perp}$, we may put $(\mathbf{Q}k_1)^{\perp} = \mathbf{Q}_4^1$,

$$\varphi_1 = \begin{bmatrix} 3 & 1 & 1 & 9/2 \\ 1 & 1 & 0 & 3/2 \\ 1 & 0 & 1 & 3/2 \\ 9/2 & 3/2 & 3/2 & 10 \end{bmatrix},$$

$\Lambda \cap (\mathbf{Q}k_1)^{\perp} = \mathbf{Z}_4^1$, and $\Gamma^{\cdot}(\Lambda \cap (\mathbf{Q}k_1)^{\perp}) = \{\gamma \in GL_4(\mathbf{Z}) \mid \gamma\varphi_1 \cdot {}^t\gamma = \varphi_1\}$ under the identification with respect to a $\mathbf{Z}$-basis $\{e_2+e_3+e_4,\ e_2,\ e_3,\ 3g-e_5\}$ of $\Lambda \cap (\mathbf{Q}k_1)^{\perp}$. Let L_1 be the lattice in $(V,\ \varphi)$ corresponding to $\Lambda \cap (\mathbf{Q}k_1)^{\perp}$ under some isomorphism of $(V,\ \varphi)$ onto $((\mathbf{Q}k_1)^{\perp},\ \varphi_1)$. Then the number $\#L_1[13p^n,\ 2^{-1}\cdot 13\mathbf{Z}]$ is equal to

$$\begin{aligned}
&\#\{[x_1\ x_2\ x_3\ x_4] \in \mathbf{Z}_4^1 \mid \\
&3x_1^2 + x_2^2 + x_3^2 + 10x_4^2 + 2x_1x_2 + 2x_1x_3 + 9x_1x_4 + 3x_2x_4 + 3x_3x_4 = 13p^n, \\
&(6x_1 + 2x_2 + 2x_3 + 9x_4)\mathbf{Z} + (2x_1 + 2x_2 + 3x_4)\mathbf{Z} + (2x_1 + 2x_3 + 3x_4)\mathbf{Z} \\
&+(9x_1 + 3x_2 + 3x_3 + 20x_4)\mathbf{Z} = 13\mathbf{Z}\}.
\end{aligned}$$

For $p = 23$ and $n = 1$ there is no elements of $(\Lambda \cap (\mathbf{Q}k_1)^{\perp})[13\cdot 23,\ 2^{-1}\cdot 13\mathbf{Z}]$. Hence $\#L_1[13\cdot 23,\ 2^{-1}\cdot 13\mathbf{Z}] = 0$. Because L_1 and L_2 are not in the same O^{φ}-class as $k_1\Gamma^{\cdot}(\Lambda) \neq k_2\Gamma^{\cdot}(\Lambda)$, we have therefore $\mathfrak{c}(13\cdot 23) = \#\{L_2[13\cdot 23,\ 2^{-1}\cdot$

$13\mathbf{Z}]/\Gamma^{\cdot}(L_2)\} = 1$. This coincides with our result in the case of $d = 13$ and $p = 23$ in Table 2.
We note that $\#U = 24$, $\Gamma^{\cdot}(L_2) = U$, and $\Gamma(L_2)$ is generated by γ_1, γ_2, -1_4; furthermore we have $x\Gamma(L_2) = L_2[13 \cdot 23,\ 2^{-1} \cdot 13\mathbf{Z}]$, that is, $\#\{L_2[13 \cdot 23,\ 2^{-1} \cdot 13\mathbf{Z}]/\Gamma(L_2)\} = 1$. As for $\Lambda \cap (\mathbf{Q}k_1)^{\perp}$, four elements $\delta_1, \cdots, \delta_4$ and -1_4 generate $\Gamma^{\cdot}(\Lambda \cap (\mathbf{Q}k_1)^{\perp})$ and then $\Gamma(\Lambda \cap (\mathbf{Q}k_1)^{\perp})$ is generated by $\delta_1\delta_2$, $\delta_2\delta_3$, δ_4, -1_4, where

$$\delta_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & -1 \end{bmatrix}, \qquad \delta_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
$$\delta_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \delta_4 = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}.$$

We shall show one more example for $d = 13$ and $p = 79$ obtained in the same manner:

$$\#\left\{L_1[13 \cdot 79,\ 2^{-1} \cdot 13\mathbf{Z}]/\Gamma^{\cdot}(L_1)\right\} = 1,\ \#\left\{L_2[13 \cdot 79,\ 2^{-1} \cdot 13\mathbf{Z}]/\Gamma^{\cdot}(L_2)\right\} = 2,$$
$$\#\left\{L_1[13 \cdot 79,\ 2^{-1} \cdot 13\mathbf{Z}]/\Gamma(L_1)\right\} = 1,\ \#\left\{L_2[13 \cdot 79,\ 2^{-1} \cdot 13\mathbf{Z}]/\Gamma(L_2)\right\} = 3.$$

Here $L_1'[13 \cdot 79,\ 2^{-1} \cdot 13\mathbf{Z}]$, with $L_1' = \Lambda \cap (\mathbf{Q}k_1)^{\perp} \cong L_1$, consists of 48 solutions

$$[\pm 10\ \mp 39\ \mp 26\ \pm 2],\ \cdots,\ [\pm 29\ \mp 13\ \mp 13\ \mp 2]$$

and $L_2[13 \cdot 79,\ 2^{-1} \cdot 13\mathbf{Z}]$ of 28 solutions

$$[\pm 3\ \pm 13\ \pm 39\ \mp 32],\ \cdots,\ [\pm 36\ \pm 39\ \pm 26\ \mp 46].$$

Accordingly $\sum_{i=1}^{2} \#\{L_i[13 \cdot 79,\ 2^{-1} \cdot 13\mathbf{Z}]/\Gamma(L_i)\}$ is a quantity that differs from both the type number and the class number of $\mathfrak{O}$.

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