An Equivariant Lefschetz Fixed-Point Formula for Correspondences

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Abstract. We compute the trace of an endomorphism in equivariant bivariant K-theory for a compact group G in several ways: geometrically using geometric correspondences, algebraically using localisation, and as a Hattori–Stallings trace. This results in an equivariant version of the classical Lefschetz fixed-point theorem, which applies to arbitrary equivariant correspondences, not just maps.

We dedicate this article to Tamaz Kandelaki, who was a coauthor in an earlier version of this article, and passed away in 2012. We will remember him for his warm character and his perseverance in doing mathematics in difficult circumstances.

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1 Introduction

Here we continue a series of articles by the last two authors about Euler characteristics and Lefschetz invariants in equivariant bivariant K-theory. These invariants were introduced in [11, 13–16]. The goal is to compute Lefschetz invariants explicitly in a way that generalises the Lefschetz–Hopf fixed-point formula.

Let X be a smooth compact manifold and $f\colon X \to X$ a self-map with simple isolated fixed points. The Lefschetz–Hopf fixed-point formula identifies

1. the sum over the fixed points of f, where each fixed point contributes ± 1 depending on its index;

2. the supertrace of the $\mathbb{Q}$-linear, grading-preserving map on $K^*(X) \otimes \mathbb{Q}$ induced by f.

It makes no difference in (2) whether we use rational cohomology or K-theory because the Chern character is an isomorphism between them.

We will generalise this result in two ways. First, we allow a compact group G to act on X and get elements of the representation ring $R(G)$ instead of numbers. Secondly, we replace self-maps by self-correspondences in the sense of [15]. Sections 2 and 3 generalise the invariants (1) and (2) respectively to this setting. The invariant of Section 2 is local and geometric and generalises (1) above; the formulas in Sections 3 and 4 are global and homological and generalise (2) (in two different ways.) The equality of the geometric and homological invariants is our generalisation of the Lefschetz fixed-point theorem.

A first step is to interpret the invariants (1) or (2) in a category-theoretic way in terms of the trace of an endomorphism of a dualisable object in a symmetric monoidal category.

Let $\mathcal{C}$ be a symmetric monoidal category with tensor product $\otimes$ and tensor unit $\mathbb{1}$. An object A of $\mathcal{C}$ is called *dualisable* if there is an object A^*, called its *dual*, and a natural isomorphism

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(B, A^* \otimes C)$$

for all objects B and C of $\mathcal{C}$. Such duality isomorphisms exist if and only if there are two morphisms $\eta \colon \mathbb{1} \to A \otimes A^*$ and $\varepsilon \colon A^* \otimes A \to \mathbb{1}$, called unit and counit of the duality, that satisfy two appropriate conditions. Let $f \colon A \to A$ be an endomorphism in $\mathcal{C}$. Then the *trace* of f is the composite endomorphism

$$\mathbb{1} \xrightarrow{\eta} A \otimes A^* \xrightarrow{\mathsf{sym}} A^* \otimes A \xrightarrow{\mathrm{id}_{A^*} \otimes f} A^* \otimes A \xrightarrow{\varepsilon} \mathbb{1},$$

where sym denotes the symmetry (or braiding) isomorphism. In this article we also call the trace the *Lefschetz index* of the morphism. This is justified by the following example.

Let $\mathcal{C}$ be the Kasparov category KK with its usual tensor product structure, $A = C(X)$ for a smooth compact manifold X, and $\hat{f} \in KK_0(A, A)$ for some morphism. We may construct a dual A^* from the tangent bundle or the stable normal bundle of X. In the case of a smooth self-map of X, and assuming a certain transversality condition, the trace of the morphism $\hat{f}$ induced by the self-map equals the invariant (1), that is, equals the number of fixed-points of the map, counted with appropriate signs. This is checked by direct computation in Kasparov theory, see [13] for more general results.

This paper springs in part from the reference [13]. A similar invariant to the Lefschetz index was introduced there, called the *Lefschetz class* (of the morphism). The Lefschetz class for an equivariant Kasparov endomorphism

of X was defined as an equivariant K-homology class for X. The Lefschetz *index*, that is, the categorical trace, discussed above, is the Atiyah–Singer index of the Lefschetz class of [13].

The main goal of this article is to give a global, homological formula for the Lefschetz index generalising the invariant (2) for a non-equivariant self-map. The formulation and proof of our homological formula works best for Hodgkin Lie groups. A more complicated form applies to all compact groups. The article [13] also provides two formulas for the equivariant Lefschetz class whose equality generalises that of the invariants (1) and (2), but the methods there are completely different.

The other main contribution of this article is to compute the geometric expression for the Lefschetz index in the category $\widehat{\mathrm{KK}}^G$ of geometric correspondences introduced in [15]. This simplifies the computation in Kasparov's analytic theory in [13] and also gives a more general result, since we can work with general smooth correspondences rather than just maps. Furthermore, using an idea of Baum and Block in [4], we give a recipe for composing two smooth equivariant correspondences under a weakening of the usual transversality assumption (of [6]). This technique is important for computing the Lefschetz index in the case of continuous group actions, where transversality is sometimes difficult to achieve, and in particular, aids in describing equivariant Euler characteristics in our framework.

Section 2 contains our geometric formula for the Lefschetz index of an equivariant self-correspondence. Why is there a nice geometric formula for the Lefschetz index of a self-map in Kasparov theory? A good explanation is that Connes and Skandalis [6] describe KK-theory for commutative C*-algebras geometrically, including the Kasparov product; furthermore, the unit and counit of the KK-duality for smooth manifolds have a simple form in this geometric variant of KK. An equivariant version of the theory in [6] is developed in [15]. In Section 2, we also recall some basic results about the geometric KK-theory introduced in [15]. If X is a smooth compact G-manifold for a compact group G, then $\mathrm{KK}^G_*(\mathrm{C}(X), \mathrm{C}(X))$ is isomorphic to the geometrically defined group $\widehat{\mathrm{KK}}^G_*(X, X)$. Its elements are smooth *correspondences*

$$X \xleftarrow{b} (M, \xi) \xrightarrow{f} X \qquad (1.1)$$

consisting of a smooth G-map b, a K_G-oriented smooth G-map f, and $\xi \in \mathrm{K}^*_G(M)$. Theorem 2.18 computes the categorical trace, or Lefschetz index, of such a correspondence under suitable assumptions on b and f.

Assume first that X has no boundary and that b and f are transverse; equivalently, for all $m \in M$ with $f(m) = b(m)$ the linear map $Db - Df \colon \mathrm{T}_m M \to \mathrm{T}_{f(m)} X$ is surjective. Then

$$Q := \{m \in M \mid b(m) = f(m)\} \qquad (1.2)$$

is naturally a K_G-oriented smooth manifold. We show that the Lefschetz index is the G-index of the Dirac operator on Q twisted by $\xi|_Q \in \mathrm{K}^*_G(Q)$ (Theo-

rem 2.18). More generally, suppose that the coincidence space Q as defined above is merely assumed to be a smooth submanifold of M, and that $x \in TX$ and $Df(\xi) = Db(\xi)$ implies that $\xi \in TQ$. Then we say that f and b *intersect smoothly*. For example, the identity correspondence, where f and b are the identity maps on X, does not satisfy the above transversality hypothesis, but f and b clearly intersect smoothly. In the case of a smooth intersection, the cokernels of the map $Df - Db$ form a vector bundle on Q which we call the *excess intersection bundle η*. This bundle measures the failure of transversality of f and b. Let η be K_G-oriented. Then TQ also inherits a canonical K_G-orientation. The restriction of the Thom class of η to the zero section gives a class $e(\eta) \in K_G^*(Q)$.

Then Theorem 2.18 asserts that the Lefschetz index of the correspondence (1.1) with smoothly intersecting f and b is the index of the Dirac operator on the coincidence manifold Q twisted by $\xi \otimes e(\eta)$. This is the main result of Section 2. In Section 3 we generalise the global homological formula involved in the classical Lefschetz fixed-point theorem, to the equivalent situation. This involves completely different ideas. The basic idea to use Künneth and Universal Coefficient theorems for such a formula already appears in [9]. In the equivariant case, these theorems become much more complicated, however. The new idea that we need here is to first localise KK^G and compute the Lefschetz index in the localisation.

In the introduction, we only state our result in the simpler case of a Hodgkin Lie group G. Then $R(G)$ is an integral domain and thus embeds into its field of fractions F. For any G-C*-algebra A, $K_*^G(A)$ is a $\mathbb{Z}/2$-graded $R(G)$-module. Thus $K_*^G(A; F) := K_*^G(A) \otimes_{R(G)} F$ becomes a $\mathbb{Z}/2$-graded F-vector space. Assume that A is dualisable and belongs to the bootstrap class in KK^G. Then $K_*^G(A; F)$ is finite-dimensional, so that the map on $K_*^G(A; F)$ induced by an endomorphism $\varphi \in KK_0^G(A, A)$ has a well-defined (super)trace in F. Theorem 3.4 asserts that this supertrace belongs to $R(G) \subseteq F$ and is equal to the Lefschetz index of φ. In particular, this applies if $A = C(X)$ for a compact G-manifold.

The results of Sections 2 and 3 together thus prove the following:

THEOREM 1.1. *Let G be a Hodgkin Lie group, let F be the field of fractions of $R(G)$. Let X be a closed G-manifold. Let $X \xleftarrow{b} (M, \xi) \xrightarrow{f} X$ be a smooth G-equivariant correspondence from X to X with $\xi \in K_G^{\dim M - \dim X}(X)$; it represents a class $\varphi \in \widehat{KK}_0^G(X, X)$. Assume that b and f intersect smoothly with K_G-oriented excess intersection bundle η. Equip $Q := \{m \in M \mid b(m) = f(m)\}$ with its induced K_G-orientation.*

Then the $R(G)$-valued index of the Dirac operator on Q twisted by $\xi|_Q \otimes e(\eta)$ is equal to the supertrace of the F-linear map on $K_G^*(X) \otimes_{R(G)} F$ induced by φ.

If G is a connected Lie group, then there is a finite covering $\hat{G} \twoheadrightarrow G$ that is a Hodgkin Lie group. We may turn G-actions into $\hat{G}$-actions using the projection map, and get a symmetric monoidal functor $KK^G \to KK^{\hat{G}}$. Since

the map $\mathrm{R}(G) \to \mathrm{R}(\hat{G})$ is clearly injective, we may compute the Lefschetz index of $\varphi \in \mathrm{KK}_0^G(A, A)$ by computing instead the Lefschetz index of the image of φ in $\mathrm{KK}_0^{\hat{G}}(A, A)$. By the result mentioned above, this uses the induced map on $\mathrm{K}_*^{\hat{G}}(A) \otimes_{\mathrm{R}(\hat{G})} \hat{F}$, where $\hat{F}$ is the field of fractions of $\mathrm{R}(\hat{G})$. Thus we get a satisfactory trace formula for all connected Lie groups. But the result may be quite different from the trace of the induced map on $\mathrm{K}_*^G(A) \otimes_{\mathrm{R}(G)} F$.

If G is not connected, then the total ring of fractions of G is a product of finitely many fields. Its factors correspond to conjugacy classes of Cartan subgroups in G. Each Cartan subgroup $H \subseteq G$ corresponds to a minimal prime ideal $\mathfrak{p}_H$ in $\mathrm{R}(G)$. The quotient $\mathrm{R}(G)/\mathfrak{p}_H$ is an integral domain and embeds into a field of fractions F_H. We show that the map $\mathrm{R}(G) \to F_H$ maps the Lefschetz index of φ to the supertrace of $\mathrm{K}_*^H(\varphi; F_H)$ (Theorem 3.23). It is crucial to use H-equivariant K-theory here. The very simple counterexample 3.7 shows that there may be two elements $\varphi_1, \varphi_2 \in \mathrm{KK}_0^G(A, A)$ with different Lefschetz index but inducing the same map on $\mathrm{K}_*^G(A)$.

Thus the generalisation of Theorem 1.1 to disconnected G identifies the image of the index of the Dirac operator on Q twisted by $\xi|_Q \otimes e(\eta)$ under the canonical map $\mathrm{R}(G) \to F_H$ with the supertrace of the F_H-linear map on $\mathrm{K}_G^*(X) \otimes_{\mathrm{R}(G)} F_H$ induced by φ, for each Cartan subgroup H.

The trace formulas in Section 3 require the algebra A on which we compute the trace to be dualisable and to belong to an appropriate bootstrap class, namely, the class of all G-C^*-algebras that are KK^G-equivalent to a type I G-C^*-algebra. This is strictly larger than the class of G-C^*-algebras that are KK^G-equivalent to a commutative one, already if G is the circle group (see [10]). We describe the bootstrap class as being generated by so-called elementary G-C^*-algebras in Section 3.1. This list of generators is rather long, but for the purpose of the trace computations, we may localise KK^G at the multiplicatively closed subset of non-zero divisors in $\mathrm{R}(G)$. The image of the bootstrap class in this localisation has a very simple structure, which is described in Section 3.2. The homological formula for the Lefschetz index follows easily from this description of the localised bootstrap category.

In Section 4, we give a variant of the global homological formula for the trace for a Hodgkin Lie group G. Given a commutative ring R and an R-module M with a projective resolution of finite type, we may define a Hattori–Stallings trace for endomorphisms of M by lifting the endomorphism to a finite type projective resolution and using the standard trace for endomorphisms of finitely generated projective resolutions. This defines the trace of the $\mathrm{R}(G)$-module homomorphism $\mathrm{K}_*^G(\varphi) \colon \mathrm{K}_*^G(A) \to \mathrm{K}_*^G(A)$ in $\mathrm{R}(G)$ without passing through a field of fractions.

2 Lefschetz indices in geometric bivariant K-theory

The category $\widehat{\mathrm{KK}}^G$ introduced in [15] provides a geometric analogue of Kasparov theory. We first recall some basic facts about this category and duality in

bivariant K-theory from [14–16] and then compute Lefschetz indices in it as intersection products. Later we are going to compare this with other formulas for Lefschetz indices. We also prove an excess intersection formula to compute the composition of geometric correspondences under a weaker assumption than transversality. This formula goes back to Baum and Block [4].

All results in this section extend to the case where G is a proper Lie groupoid with enough G-vector bundles in the sense of [14, Definition 3.1] because the theory in [14–16] is already developed in this generality. For the sake of concreteness, we limit our treatment here to compact Lie groups acting on smooth manifolds.

The results in this section work both for real and complex K-theory. For concreteness, we assume in our notation that we are dealing with the complex case. In the real case, K must be replaced by KO throughout. In particular, K_G-orientations (that is, G-equivariant Spin^c-structures) must be replaced by KO^G-orientations (that is, G-equivariant Spin structures). In some examples, we use the isomorphisms $\widehat{KK}_{2n}^G(\mathrm{pt}, \mathrm{pt}) = \mathrm{R}(G)$ and $\widehat{KK}_{2n+1}^G(\mathrm{pt}, \mathrm{pt}) = 0$ for all $n \in \mathbb{Z}$. Here $\mathrm{R}(G)$ denotes the representation ring of G. Of course, this is true only in complex K-theory.

2.1   Geometric bivariant K-theory

Like Kasparov theory, geometric bivariant K-theory yields a category $\widehat{KK}^G$. Its objects are (Hausdorff) locally compact G-spaces; arrows from X to Y are *geometric correspondences* from X to Y in the sense of [15, Definition 2.3]. These consist of

- M: a G-space;

- b: a G-map (that is, a continuous G-equivariant map) $b: M \to X$;

- ξ: a G-equivariant K-theory class on M with X-compact support (where we view M as a space over X via the map b); we write $\xi \in \mathrm{RK}_{G,X}^*(M)$;

- f: a K_G-oriented normally non-singular G-map $f: M \to Y$.

Equivariant K-theory with X-compact support and equivariant vector bundles are defined in [12, Definitions 2.5 and 2.6]. If b is a proper map, in particular if M is compact, then $\mathrm{RK}_{G,X}^*(M)$ is the ordinary G-equivariant (compactly supported) K-theory $\mathrm{K}_G^*(M)$ of M.

A K_G-*oriented normally non-singular map* from M to Y consists of

- V: a K_G-oriented G-vector bundle on M,

- E: a K_G-oriented finite-dimensional linear G-representation, giving rise to a trivial K_G-oriented G-vector bundle $Y \times E$ on Y,

- $\hat{f}$: a G-equivariant homeomorphism from the total space of V to an open subset in the total space of $Y \times E$, $\hat{f}: V \hookrightarrow Y \times E$.

We will not distinguish between a vector bundle and its total space in our notation.

A normally non-singular map $f = (V, E, \hat{f})$ has an *underlying map*

$$M \rightarrowtail V \xhookrightarrow{\hat{f}} Y \times E \twoheadrightarrow Y,$$

where the first map is the zero section of the vector bundle V and the third map is the coordinate projection. This map is called its "trace" in [14], but we avoid this name here because we use "trace" in a different sense. The *degree* of f is $d = \dim V - \dim E$. A wrong-way element $f_! \in \mathrm{KK}_d^G(\mathrm{C}_0(M), \mathrm{C}_0(Y))$ induced by f is defined in [14, Section 5.3]).

Our geometric correspondences are variants of those introduced by Alain Connes and Georges Skandalis in [6]. The changes in the definition avoid technical problems with the usual definition in the equivariant case.

The ($\mathbb{Z}/2$-graded) geometric KK-group $\widehat{\mathrm{KK}}_*^G(X, Y)$ is defined as the quotient of the set of geometric correspondences from X to Y by an appropriate equivalence relation, generated by bordism, Thom modification, and equivalence of normally non-singular maps. Bordism includes homotopies for the maps b and f by [15, Lemma 2.12]. We will use this several times below. The Thom modification allows to replace the space M by the total space of a K_G-oriented vector bundle on M. In particular, we could take the K_G-oriented vector bundle from the normally non-singular map f. This results in an equivalent normally non-singular map where $f \colon M \to Y$ is a *special submersion*, that is, an open embedding followed by a coordinate projection $Y \times E \twoheadrightarrow Y$ for some linear G-representation E. Correspondences with this property are called *special*.

The composition in $\widehat{\mathrm{KK}}^G$ is defined as an intersection product (see Section 2.2) if the map $f \colon M \to Y$ is such a special submersion. This turns $\widehat{\mathrm{KK}}^G$ into a category; the identity map on X is the correspondence with $f = b = \mathrm{id}_X$ and $\xi = 1$. The product of G-spaces provides a symmetric monoidal structure in $\widehat{\mathrm{KK}}^G$ (see [15, Theorem 2.27]).

There is an additive, grading-preserving, symmetric monoidal functor

$$\widehat{\mathrm{KK}}_*^G(X, Y) \to \mathrm{KK}_*^G(\mathrm{C}_0(X), \mathrm{C}_0(Y)).$$

This is an isomorphism if X is *normally non-singular* by [15, Corollary 4.3], that is, if there is a normally non-singular map $X \to \mathrm{pt}$. This means that there is a G-vector bundle V over X whose total space is G-equivariantly homeomorphic to an open G-invariant subset of some linear G-space. In particular, by Mostow's Embedding Theorem smooth G-manifolds of finite orbit type are normally non-singular (see [14, Theorem 3.22]).

Stable K_G-orientations play an important technical role in our trace formulas and should therefore be treated with care. A K_G-*orientation* on a G-vector bundle V is, by definition, a G-equivariant complex spinor bundle for V. (This is equivalent to a reduction of the structure group to Spin^c.) Given such

K_G-orientations on V_1 and V_2, we get an induced K_G-orientation on $V_1 \oplus V_2$; conversely, K_G-orientations on $V_1 \oplus V_2$ and V_1 induce one on V_2.

Let $\xi \in \mathrm{RK}_G^0(M)$ be represented by the formal difference $[V_1] - [V_2]$ of two G-vector bundles. A *stable K_G-orientation* on ξ means that we are given another G-vector bundle V_3 and K_G-orientations on both $V_1 \oplus V_3$ and $V_2 \oplus V_3$. Since $\xi = [V_1 \oplus V_3] - [V_2 \oplus V_3]$, this implies that ξ is a formal difference of two K_G-oriented G-vector bundles. Conversely, assume that $\xi = [W_1] - [W_2]$ with two K_G-oriented G-vector bundles; then there are G-vector bundles V_3 and W_3 such that $V_i \oplus V_3 \cong W_i \oplus W_3$ for $i = 1, 2$; since W_3 is a direct summand in a K_G-oriented G-vector bundle, we may enlarge V_3 and W_3 so that W_3 itself is K_G-oriented. Then $V_i \oplus V_3 \cong W_i \oplus W_3$ for $i = 1, 2$ inherit K_G-orientations. Roughly speaking, stably K_G-oriented K-theory classes are equivalent to formal differences of K_G-oriented G-vector bundles.

A K_G-orientation on a normally non-singular map $f = (V, E, \hat{f})$ from M to Y means that both V and E are K_G-oriented. Since "lifting" allows us to replace E by $E \oplus E'$ and V by $V \oplus (M \times E')$, we may assume without loss of generality that E is already K_G-oriented. Thus a K_G-orientation on f becomes equivalent to one on V. But the chosen K_G-orientation on E remains part of the data: changing it changes the K_G-orientation on f. By [14, Lemma 5.13], all essential information is contained in a K_G-orientation on the formal difference $[V] - [M \times E] \in \mathrm{RK}_G^0(M)$, which we call the *stable normal bundle* of the normally non-singular map f. If $[V] - [M \times E]$ is K_G-oriented, then we may find a G-vector bundle V_3 such that $V \oplus V_3$ and $(M \times E) \oplus V_3$ are K_G-oriented. Since $(M \times E) \oplus V_3$ is a direct summand in a K_G-oriented trivial G-vector bundle, we may assume without loss of generality that V_3 itself is trivial, $V_3 = M \times E'$, and that already $E \oplus E'$ is K_G-oriented. Lifting f along E' then gives a normally non-singular map $(V \oplus (M \times E'), E \oplus E', \hat{f} \times \mathrm{id}_{E'})$, where both $V \oplus (M \times E')$ and $E \oplus E'$ are K_G-oriented. Thus a K_G-orientation on f is equivalent to a stable K_G-orientation on the stable normal bundle of f.

LEMMA 2.1. *If $f = (V, E, \hat{f})$ is a smooth normally non-singular map with underlying map $\bar{f}\colon M \to Y$, then its stable normal bundle is equal to $\bar{f}^*[\mathrm{T}Y] - [\mathrm{T}M] \in \mathrm{RK}_G^0(M)$.*

Proof. The tangent bundles of the total spaces of V and $Y \times E$ are $\mathrm{T}M \oplus V$ and $\mathrm{T}Y \oplus E$, respectively. Since $\hat{f}$ is an open embedding, $\hat{f}^*(\mathrm{T}Y \oplus E) \cong \mathrm{T}M \oplus V$. This implies $\bar{f}^*(\mathrm{T}Y) \oplus (M \times E) \cong \mathrm{T}M \oplus V$. Thus $[V] - [M \times E] = \bar{f}^*[\mathrm{T}Y] - [\mathrm{T}M]$. $\square$

This lemma also shows that the stable normal bundle of f and hence the orientability assumption depend only on the equivalence class of f.

Another equivalent way to describe stable K_G-orientations is the following. Suppose we are already given a G-vector bundle V on Y such that $\mathrm{T}Y \oplus V$ is K_G-oriented. Then a stable K_G-orientation on f is equivalent to one on

$$[\bar{f}^*V \oplus \mathrm{T}M] = \bar{f}^*[\mathrm{T}Y \oplus V] - (\bar{f}^*[\mathrm{T}Y] - [\mathrm{T}M]),$$

which is equivalent to a K_G-orientation on $\bar{f}^*V \oplus TM$ in the usual sense.

If X and Y are smooth G-manifolds (without boundary), we may require the maps b and $\hat{f}$ and the vector bundles V and E to be smooth. This leads to a smooth variant of $\widehat{KK}^G$. This variant is isomorphic to the one defined above by [15, Theorem 4.8] provided X is of finite orbit type and hence normally non-singular.

Working in the smooth setting has two advantages.

First, assuming M to be of finite orbit type, [14, Theorem 3.22] shows that any smooth G-map $f \colon M \to Y$ lifts to a smooth normally non-singular map that is unique up to equivalence. Thus we may replace normally non-singular maps by smooth maps in the usual sense in the definition of a geometric correspondence. Moreover, $Nf = f^*[TY] - [TM]$, so f is K_G-oriented if and only if there are K_G-oriented G-vector bundles V_1 and V_2 over M with $f^*[TY] \oplus V_1 \cong TM \oplus V_2$ (compare [14, Corollary 5.15]).

Secondly, in the smooth setting there is a particularly elegant way of composing correspondences when they satisfy a suitable transversality condition, see [15, Corollary 2.39]. This description of the composition is due to Connes and Skandalis [6].

2.2 Composition of geometric correspondences

By [15, Theorem 2.38], a smooth normally non-singular map lifting $f \colon M_1 \to Y$ and a smooth map $b \colon M_2 \to Y$ are *transverse* if

$$D_{m_1}f(T_{m_1}M_1) + D_{m_2}b(T_{m_2}M_2) = T_yY$$

for all $m_1 \in M_1$, $m_2 \in M_2$ with $y := f(m_1) = b(m_2)$. Equivalently, the map

$$Df - Db \colon \mathrm{pr}_1^*(TM_1) \oplus \mathrm{pr}_2^*(TM_2) \to (f \circ \mathrm{pr}_1)^*(TY)$$

is surjective; this is a bundle map of vector bundles over

$$M_1 \times_Y M_2 := \{(m_1, m_2) \mid f(m_1) = b(m_2)\},$$

where $\mathrm{pr}_1 \colon M_1 \times_Y M_2 \to M_1$ and $\mathrm{pr}_2 \colon M_1 \times_Y M_2 \to M_2$ denote the restrictions to $M_1 \times_Y M_2$ of the coordinate projections. (We shall always use this notation for restrictions of coordinate projections.)

A commuting square diagram of smooth manifolds is called *Cartesian* if it is isomorphic (as a diagram) to a square

$$
\begin{array}{ccc}
M_1 \times_Y M_2 & \xrightarrow{\ \mathrm{pr}_2\ } & M_2 \\
{\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
M_1 & \xrightarrow{\ \ b\ \ } & Y
\end{array}
$$

where f and b are transverse smooth maps in the sense above; then $M_1 \times_Y M_2$ is again a smooth manifold and pr_1 and pr_2 are smooth maps.

The tangent bundles of these four manifolds are related by an exact sequence

$$0 \to \mathrm{T}(M_1 \times_Y M_2) \xrightarrow{(D\mathrm{pr}_1, D\mathrm{pr}_2)} \mathrm{pr}_1^*(\mathrm{T}M_1) \oplus \mathrm{pr}_2^*(\mathrm{T}M_2)$$
$$\xrightarrow{Df - Db} (f \circ \mathrm{pr}_1)^*\mathrm{T}Y \to 0. \quad (2.1)$$

That is, $\mathrm{T}(M_1 \times_Y M_2)$ is the sub-bundle of $\mathrm{pr}_1^*(\mathrm{T}M_1) \oplus \mathrm{pr}_2^*(\mathrm{T}M_2)$ consisting of those vectors $(m_1, \xi, m_2, \eta) \in \mathrm{T}M_1 \oplus \mathrm{T}M_2$ (where $f(m_1) = b(m_2)$) with $D_{m_1}f(\xi) = D_{m_2}b(\eta)$. We may denote this bundle briefly by $\mathrm{T}M_1 \oplus_{\mathrm{T}Y} \mathrm{T}M_2$. Furthermore, from (2.1),

$$\mathrm{T}(M_1 \times_Y M_2) - \mathrm{pr}_2^*(\mathrm{T}M_2) = \mathrm{pr}_1^*(\mathrm{T}M_1 - f^*(\mathrm{T}Y)) \quad (2.2)$$

as stable G-vector bundles. Thus a stable K_G-orientation for $\mathrm{T}M_1 - f^*(\mathrm{T}Y)$ may be pulled back to one for $\mathrm{T}(M_1 \times_Y M_2) - \mathrm{pr}_2^*(\mathrm{T}M_2)$. More succinctly, a K_G-orientation for the map f induces one for pr_2.

Now consider two composable smooth correspondences

$$
\begin{array}{ccccc}
& M_1 & & M_2 & \\
{}^{b_1}\swarrow & & \searrow{}^{f_1} \quad {}^{b_2}\swarrow & & \searrow{}^{f_2} \\
X & & Y & & Z,
\end{array}
\qquad (2.3)
$$

with K-theory classes $\xi_1 \in \mathrm{RK}^G_{*,X}(M_1)$ and $\xi_2 \in \mathrm{RK}^G_{*,Y}(M_2)$. We assume that the pair of smooth maps (f_1, b_2) is transverse. Then there is an essentially unique commuting diagram

$$
\begin{array}{ccc}
& M_1 \times_Y M_2 & \\
{}^{\mathrm{pr}_1}\swarrow & & \searrow{}^{\mathrm{pr}_2} \\
M_1 & & M_2 \\
{}^{b_1}\swarrow \quad \searrow{}^{f_1} & {}^{b_2}\swarrow & \searrow{}^{f_2} \\
X \qquad Y & & Z,
\end{array}
\qquad (2.4)
$$

where the square is Cartesian. We briefly call such a diagram an *intersection diagram* for the two given correspondences.

By the discussion above, the map pr_2 inherits a K_G-orientation from f_1, so that the map $f := f_2 \circ \mathrm{pr}_2$ is also K_G-oriented. Let $M := M_1 \times_Y M_2$ and $b := b_1 \circ \mathrm{pr}_1$. The product $\xi := \mathrm{pr}_1^*(\xi_1) \otimes \mathrm{pr}_2^*(\xi_2)$ belongs to $\mathrm{RK}^G_{*,X}(M)$, that is, it has X-compact support with respect to the map $b \colon M \to X$. Thus we get a G-equivariant correspondence (M, b, f, ξ) from X to Y. The assertion of [15, Corollary 2.39] – following [6] – is that this represents the composition of the two given correspondences. It is called their *intersection product*.

Example 2.2. Consider the diagonal embedding $\delta\colon X \to X \times X$ and the graph embedding $\bar{f}\colon X \to X \times X$, $x \mapsto (x, f(x))$, for a smooth map $f\colon X \to X$. These two maps are transverse if and only if f has simple fixed points. If this is the case, then the intersection space is the set of fixed points of f. If, say, $f = \mathrm{id}_X$, then δ and $\bar{f}$ are not transverse.

To define the composition also in the non-transverse case, a Thom modification is used in [15] to achieve transversality (see [15, Theorem 2.32]). Take two composable (smooth) correspondences as in (2.3), and let $f_1 = (V_1, E_1, \hat{f}_1)$ as a normally non-singular map. By a Thom modification, the geometric correspondence $X \xleftarrow{b_1} (M_1, \xi) \xrightarrow{f_1} Y$ is equivalent to

$$X \xleftarrow{b_1 \circ \pi_{V_1}} (V_1, \tau_{V_1} \otimes \pi_{V_1}^* \xi) \xrightarrow{\pi_{E_1} \circ \hat{f}_1} Y, \tag{2.5}$$

where $\pi_{V_1}\colon V_1 \to M_1$ and $\pi_{E_1}\colon Y \times E \to Y$ are the bundle projections, and $\tau_{V_1} \in \mathrm{RK}_{G,M_1}^*(V_1)$ is the Thom class of V_1. We write $\otimes$ for the multiplication of K-theory classes. The support of such a product is the intersection of the supports of the factors. Hence the support of $\tau_{V_1} \otimes \pi_{V_1}^* \xi$ is an X-compact subset of V_1.

The forward map $V_1 \to Y$ in (2.5) is a special submersion and, in particular, a submersion. As such it is transverse to any other map $b_2\colon M_2 \to Y$. Hence after the Thom modification we may compute the composition of correspondences as an intersection product of the correspondence (2.5) with the correspondence $Y \xleftarrow{b_2} M_2 \xrightarrow{f_2} Y$. This yields

$$X \xleftarrow{b_1 \circ \pi_{V_1} \circ \mathrm{pr}_1} \left(V_1 \times_Y M_2, \mathrm{pr}_{V_1}^*(\tau_{V_1} \otimes \pi_{V_1}^*(\xi))\right) \xrightarrow{f_2 \circ \mathrm{pr}_2} Z, \tag{2.6}$$

where

$$V_1 \times_Y M_2 := \{(x, v, m_2) \in V_1 \times M_2 \mid (\pi_{E_1} \circ \hat{f}_1)(x, v) = b_2(m_2)\}$$

and $\mathrm{pr}_1\colon V_1 \times_Y M_2 \to V_1$ and $\mathrm{pr}_2\colon V_1 \times_Y M_2 \to M_2$ are the coordinate projections. The intersection space $V_1 \times_Y M_2$ is a smooth manifold with tangent bundle

$$\mathrm{T}V_1 \oplus_{\mathrm{T}Y} \mathrm{T}M_2 := \mathrm{pr}_1^*(\mathrm{T}V_1) \oplus_{(\pi_{E_1} \circ \hat{f}_1)^*(\mathrm{T}Y)} \mathrm{pr}_2^*(\mathrm{T}M_2),$$

and the map pr_2 is a submersion with fibres tangent to E_1. Thus it is K_G-oriented.

This recipe to define the composition product for all geometric correspondences is introduced in [15]. It is shown there that it is equivalent to the intersection product if f_1 and b_2 are transverse. But the space $V_1 \times_Y M_2$ has high dimension, making it inefficient to compute with this formula. And we are usually given only the underlying map $f_1\colon M_1 \to Y$, not its factorisation as a normally non-singular map – and the latter is difficult to compute. We will weaken the transversality requirement in Section 2.5. The more general condition still applies, say, if $f_1 = b_2$. This is particularly useful for computing Euler characteristics.

2.3   Duality and the Lefschetz index

Duality plays a crucial role in [15] in order to compare the geometric and analytic models of equivariant Kasparov theory. Duality is also used in [16, Definition 4.26] to construct a Lefschetz map

$$\mathcal{L}\colon \mathrm{KK}^G_*\big(\mathrm{C}(X),\mathrm{C}(X)\big) \to \mathrm{KK}^G_*(\mathrm{C}(X),\mathbb{C}), \tag{2.7}$$

for a compact smooth G-manifold X. We may compose $\mathcal{L}$ with the index map $\mathrm{KK}^G_*(\mathrm{C}(X),\mathbb{C}) \to \mathrm{KK}^G_*(\mathbb{C},\mathbb{C}) \cong \mathrm{R}(G)$ to get a Lefschetz index $\mathrm{L\text{-}ind}(f) \in \mathrm{R}(G)$ for any $f \in \mathrm{KK}^G_*\big(\mathrm{C}(X),\mathrm{C}(X)\big)$. This is the invariant we will be studying in this paper.

This Lefschetz map $\mathcal{L}$ is a special case of a very general construction. Let $\mathcal{C}$ be a symmetric monoidal category. Let A be a dualisable object of $\mathcal{C}$ with a dual A^*. Let $\eta\colon \mathbb{1} \to A \otimes A^*$ and $\varepsilon\colon A^* \otimes A \to \mathbb{1}$ be the unit and counit of the duality. Being unit and counit of a duality means that they satisfy the zigzag equations: the composition

$$A \xrightarrow{\eta\otimes\mathrm{id}_A} A \otimes A^* \otimes A \xrightarrow{\mathrm{id}_A\otimes\varepsilon} A \tag{2.8}$$

is equal to the identity $\mathrm{id}_A\colon A \to A$, and similarly for the composition

$$A^* \xrightarrow{\mathrm{id}_{A^*}\otimes\eta} A^* \otimes A \otimes A^* \xrightarrow{\varepsilon\otimes\mathrm{id}_{A^*}} A^*. \tag{2.9}$$

If $\mathcal{C}$ is $\mathbb{Z}$-graded, then we may allow dualities to shift degrees. Then some signs are necessary in the zigzag equations, see [16, Theorem 5.5].

Given a multiplication map $m\colon A \otimes A \to A$, we define the *Lefschetz map*

$$\mathcal{L}\colon \mathcal{C}(A,A) \to \mathcal{C}(A,\mathbb{1})$$

by sending an endomorphism $f\colon A \to A$ to the composite morphism

$$A \cong A\otimes\mathbb{1} \xrightarrow{\mathrm{id}_A\otimes\eta} A\otimes A\otimes A^* \xrightarrow{m\otimes\mathrm{id}_{A^*}} A\otimes A^* \xrightarrow{f\otimes\mathrm{id}_{A^*}} A\otimes A^* \xrightarrow{\mathsf{sym}} A^*\otimes A \xrightarrow{\varepsilon} \mathbb{1}.$$

This depends only on m and f, not on the choices of the dual, unit and counit. For $f = \mathrm{id}_A$ we get the *higher Euler characteristic* of A in $\mathcal{C}(A,\mathbb{1})$.

While the geometric computations below give the Lefschetz map as defined above, the global homological computations in Sections 3 and 4 only apply to the following coarser invariant:

Definition 2.3. The *Lefschetz index* $\mathrm{L\text{-}ind}(f)$ (or *trace* $\mathrm{tr}(f)$) of an endomorphism $f\colon A \to A$ is the composite

$$\mathbb{1} \xrightarrow{\eta} A \otimes A^* \xrightarrow{\mathsf{sym}} A^* \otimes A \xrightarrow{\mathrm{id}_{A^*}\otimes f} A^* \otimes A \xrightarrow{\varepsilon} \mathbb{1}, \tag{2.10}$$

where sym denotes the symmetry isomorphism. The Lefschetz index of id_A is called the *Euler characteristic* of A.

If A is a unital algebra object in $\mathcal{C}$ with multiplication $m\colon A \otimes A \to A$ and unit $u\colon \mathbb{1} \to A$, then $\text{L-ind}(f) = \mathcal{L}(f) \circ u$. In particular, the Euler characteristic is the composite of the higher Euler characteristic with u.

In this section, we work in $\mathcal{C} = \widehat{\mathrm{KK}}^G$ for a compact group G with $\mathbb{1} = \mathrm{pt}$ and $\otimes = \times$. In Section 3, we work in the related analytic category $\mathcal{C} = \mathrm{KK}^G$ with $\mathbb{1} = \mathbb{C}$ and the usual tensor product.

We will show below that any compact smooth G-manifold X is dualisable in $\widehat{\mathrm{KK}}^G$. The multiplication $m\colon X \times X \to X$ and unit $u\colon \mathrm{pt} \to X$ are given by the geometric correspondences

$$X \times X \xleftarrow{\Delta} X \xrightarrow[=]{\mathrm{id}_X} X, \qquad \mathrm{pt} \leftarrow X \xrightarrow[=]{\mathrm{id}_X} X$$

with $\Delta(x) = (x,x)$; these induce the multiplication *-homomorphism

$$m\colon \mathrm{C}(X \times X) \cong \mathrm{C}(X) \otimes \mathrm{C}(X) \to \mathrm{C}(X)$$

and the embedding $\mathbb{C} \to \mathrm{C}(X)$ of constant functions. Composing with u corresponds to taking the *index* of a K-homology class.

Remark 2.4. In [11, 13, 16] Lefschetz maps are also studied for non-compact spaces X, equipped with group actions of possibly non-compact groups. A non-compact G-manifold X is usually not dualisable in $\widehat{\mathrm{KK}}^G$, and even if it were, the Lefschetz map that we would get from this duality would not be the one studied in [11, 13, 16].

2.4 Duality for smooth compact manifolds

We are going to show that compact smooth G-manifolds are dualisable in the equivariant correspondence theory $\widehat{\mathrm{KK}}^G$. This was already proved in [15], but since we need to know the unit and counit to compute Lefschetz indices, we recall the proof in detail. It is of some interest to treat duality for smooth manifolds with boundary because any finite CW-complex is homotopy equivalent to a manifold with boundary.

In case X has a boundary ∂X, let $\mathring{X} := X \setminus \partial X$ denote its interior and let $\iota\colon \mathring{X} \to X$ denote the inclusion map. The boundary $\partial X \subseteq X$ admits a G-equivariant collar, that is, the embedding $\partial X \to X$ extends to a G-equivariant diffeomorphism from $\partial X \times [0,1)$ onto an open neighbourhood of ∂X in X (see also [16, Lemma 7.6] for this standard result). This collar neighbourhood together with a smooth map $[0,1) \to (0,1)$ that is the identity near 1 provides a smooth G-equivariant map $\rho\colon X \to \mathring{X}$ that is inverse to ι up to smooth G-homotopy. Furthermore, we may assume that ρ is a diffeomorphism onto its image.

If X has no boundary, then $\mathring{X} = X$, $\iota = \mathrm{id}$, and $\rho = \mathrm{id}$.

The results about smooth normally non-singular maps in [14] extend to smooth manifolds with boundary if we add suitable assumptions about the behaviour near the boundary. We mention one result of this type and a counterexample.

Proposition 2.5. *Let X and Y be smooth G-manifolds with X of finite orbit type and let $f : X \to Y$ be a smooth map with $f(\partial X) \subseteq \partial Y$ and f transverse to ∂Y. Then f lifts to a normally non-singular map, and any two such normally non-singular liftings of f are equivalent.*

Proof. Since X has finite orbit type, we may smoothly embed X into a finite-dimensional linear G-representation E. Our assumptions ensure that the resulting map $X \to Y \times E$ is a smooth embedding between G-manifolds with boundary in the sense of [14, Definition 3.17] and hence has a tubular neighbourhood by [14, Theorem 3.18]. This provides a normally non-singular map $X \to Y$ lifting f. The uniqueness up to equivalence is proved as in the proof of [14, Theorem 4.36]. □

Example 2.6. The inclusion map $\{0\} \to [0, 1)$ is a smooth map between manifolds with boundary, but it does not lift to a smooth normally non-singular map.

Let X be a smooth compact G-manifold. Since X has finite orbit type, it embeds into some linear G-representation E. We may choose this G-representation to be K_G-oriented and even-dimensional by a further stabilisation. Let $NX \twoheadrightarrow X$ be the *normal bundle* for such an embedding $X \to E$. Thus $TX \oplus NX \cong X \times E$ is G-equivariantly isomorphic to a K_G-oriented trivial G-vector bundle.

Theorem 2.7. *Let X be a smooth compact G-manifold, possibly with boundary. Then X is dualisable in $\widehat{\mathrm{KK}}_*^G$ with dual $N\mathring{X}$, and the unit and counit for the duality are the geometric correspondences*

$$\mathrm{pt} \leftarrow X \xrightarrow{(\mathrm{id},\zeta\rho)} X \times N\mathring{X}, \qquad N\mathring{X} \times X \xleftarrow{(\mathrm{id},\iota\pi)} N\mathring{X} \to \mathrm{pt},$$

where $\zeta : \mathring{X} \to N\mathring{X}$ is the zero section, $\rho : X \to \mathring{X}$ is some G-equivariant collar retraction, $\pi : N\mathring{X} \to \mathring{X}$ is the bundle projection, and $\iota : \mathring{X} \to X$ the identical inclusion. The K-theory classes on the space in the middle are the trivial rank-one vector bundles for both correspondences.

Proof. First we must check that the purported unit and counit above are indeed geometric correspondences; this contains describing the K_G-orientations on the forward maps, which is part of the data of the geometric correspondences.
The maps $X \to \mathrm{pt}$ and $N\mathring{X} \to N\mathring{X} \times X$ above are proper. Hence there is no support restriction for the K-theory class on the middle space, and the trivial rank-one vector bundle is allowed.
By the Tubular Neighbourhood Theorem, the normal bundle $N\mathring{X}$ of the embedding $\mathring{X} \to E$ is diffeomorphic to an open subset of E. This gives a canonical isomorphism between the tangent bundle of $N\mathring{X}$ and E. We choose this isomorphism and the given K_G-orientation on the linear G-representation E to K_G-orient $N\mathring{X}$ and thus the projection $N\mathring{X} \to \mathrm{pt}$. With this K_G-orientation,

the counit $N\mathring{X} \times X \xleftarrow{(\mathrm{id}, \iota\pi)} N\mathring{X} \to \mathrm{pt}$ is a G-equivariant geometric correspondence – even a special one in the sense of [15].

We identify the tangent bundle of $X \times N\mathring{X}$ with $TX \times T\mathring{X} \oplus N\mathring{X}$ in the obvious way. The normal bundle of the embedding $(\mathrm{id}, \zeta\rho) \colon X \to X \times N\mathring{X}$ is isomorphic to the quotient of $TX \oplus \rho^*(T\mathring{X}) \oplus \rho^*(N\mathring{X})$ by the relation $(\xi, D\rho(\xi), 0) \sim 0$ for $\xi \in TX$. We identify this with $TX \oplus NX \cong X \times E$ by $(\xi_1, \xi_2, \eta) \mapsto (D\rho^{-1}(\xi_2) - \xi_1, D\rho^{-1}(\eta))$ for $\xi_1 \in T_x X$, $\xi_2 \in T_{\rho(x)}X$, $\eta \in \rho^*(N\mathring{X})_x = N_{\rho(x)}X$. With this K_G-orientation on $(\mathrm{id}, \zeta\rho)$, the unit above is a G-equivariant geometric correspondence. A boundary of X, if present, causes no problems here. The same goes for the computations below: although the results in [15] are formulated for smooth manifolds without boundary, they continue to hold in the cases we need.

We establish the duality isomorphism by checking the zigzag equations as in [16, Theorem 5.5]. This amounts to composing geometric correspondences. In the case at hand, the correspondences we want to compose are transverse, so that they may be composed by intersections as in Section 2.2. Actually, we are dealing with manifolds with boundary, but the argument goes through nevertheless. We write down the diagrams together with the relevant Cartesian square.

The intersection diagram for the first zigzag equation is

$$(2.11)$$

The square is Cartesian because $(x, y, z, (w, \nu)) \in X^3 \times N\mathring{X}$ satisfies

$$(x, (\rho(x), 0), y) = (z, (w, \nu), w)$$

if and only if $y = \rho(x)$, $z = x$, $w = \rho(x)$, and $\nu = 0$ for some $x \in X$. The K_G-orientation on the map $(\mathrm{id}, \zeta\rho)$ described above is chosen such that the composite map $f := \mathrm{pr}_1 \circ (\mathrm{id}, \zeta\rho) = \mathrm{id}$ carries the standard K_G-orientation. The map $b := \mathrm{pr}_2 \circ (\mathrm{id}, \iota\rho) = \iota\rho$ is properly homotopic to the identity map. Hence the composition above gives the identity map on X as required.

The intersection diagram for the second zigzag equation is

$$
\begin{array}{c}
\mathrm{N}\mathring{X} \\
\end{array}
$$

$$(2.12)$$

because $((x,\nu), y, (w,\mu), (z,\kappa)) \in \mathrm{N}\mathring{X} \times X \times (\mathrm{N}\mathring{X})^2$ satisfy

$$((x,\nu), y, (\rho(y), 0)) = ((w,\mu), w, (z,\kappa))$$

if and only if $(w,\mu) = (x,\nu)$, $y = x$, $z = \rho(x)$, $\kappa = 0$ for some $(x,\nu) \in \mathrm{N}\mathring{X}$. The map $(\mathrm{id}, \zeta\rho\pi)$ is smoothly homotopic to the diagonal embedding $\delta \colon \mathrm{N}\mathring{X} \to \mathrm{N}\mathring{X} \times \mathrm{N}\mathring{X}$. Replacing $(\mathrm{id}, \zeta\rho\pi)$ by δ gives an equivalent geometric correspondence. The K_G-orientation on the normal bundle of $(\mathrm{id}, \zeta\rho\pi)$ that comes with the composition product is transformed by this homotopy to the K_G-orientation on the normal bundle of the diagonal embedding that we get by identifying the latter with the pull-back of E by mapping

$$(\xi_1, \eta_1, \xi_2, \eta_2) \in \mathrm{T}_{(x,\zeta,x,\zeta)}(\mathrm{N}\mathring{X} \times \mathrm{N}\mathring{X}) \cong \mathrm{T}_x\mathring{X} \oplus \mathrm{N}_x\mathring{X} \times \mathrm{T}_x\mathring{X} \times \mathrm{N}_x\mathring{X} \cong E_x \times E_x$$

to $(\xi_2 - \xi_1, \eta_2 - \eta_1) \in E_x$. Since E has even dimension, changing this to $(\xi_1 - \xi_2, \eta_1 - \eta_2)$ does not change the K_G-orientation. Hence the induced K_G-orientation on the fibres of $D\mathrm{pr}_2$ is the same one that we used to K_G-orient pr_2. The induced K_G-orientation on $\mathrm{pr}_2 \circ \delta = \mathrm{id}$ is the standard one. Thus the composition in (2.12) is the identity on $\mathrm{N}\mathring{X}$. $\square$

Corollary 2.8. *Let X be a compact smooth G-manifold and let Y be any locally compact G-space. Then every element of $\widehat{\mathrm{KK}}_*^G(X, Y)$ is represented by a geometric correspondence of the form*

$$X \xleftarrow{\iota\circ\pi\circ\mathrm{pr}_1} \mathrm{N}\mathring{X} \times Y \xrightarrow{\mathrm{pr}_2} Y, \qquad \xi \in \mathrm{K}_G^*(\mathrm{N}\mathring{X} \times Y),$$

and two such correspondences for $\xi_1, \xi_2 \in \mathrm{K}_G^*(\mathrm{N}\mathring{X} \times Y)$ give the same element of $\widehat{\mathrm{KK}}_*^G(X, Y)$ if and only if $\xi_1 = \xi_2$. Here $\mathrm{pr}_1 \colon \mathrm{N}\mathring{X} \times Y \to \mathrm{N}\mathring{X}$ and $\mathrm{pr}_2 \colon \mathrm{N}\mathring{X} \times Y \to Y$ are the coordinate projections and $\iota \circ \pi \colon \mathrm{N}\mathring{X} \to \mathring{X} \subseteq X$ is as above.

Proof. Duality provides a canonical isomorphism

$$\mathrm{K}_G^*(\mathrm{N}\mathring{X} \times Y) \cong \widehat{\mathrm{KK}}_*^G(\mathrm{pt}, \mathrm{N}\mathring{X} \times Y) \cong \widehat{\mathrm{KK}}_*^G(X, Y).$$

It maps $\xi \in \mathrm{K}_G^*(\mathrm{N}\mathring{X} \times Y)$ to the composition of correspondences described by the following intersection diagram:

$$
\begin{array}{ccccccc}
& & \mathrm{N}\mathring{X} \times Y & & & & \\
& {\scriptstyle (\iota\pi,\mathrm{id})\times\mathrm{id}}\swarrow & & \searrow{\scriptstyle id} & & & \\
& X \times \mathrm{N}\mathring{X} \times Y & & & \mathrm{N}\mathring{X} \times Y & & \\
{\scriptstyle \mathrm{pr}_1}\swarrow & & \searrow{\scriptstyle id} & {\scriptstyle (\iota\pi,\mathrm{id})\times\mathrm{id}}\swarrow & & \searrow{\scriptstyle \mathrm{pr}_2} & \\
X & & & X \times \mathrm{N}\mathring{X} \times Y & & & Y,
\end{array}
$$
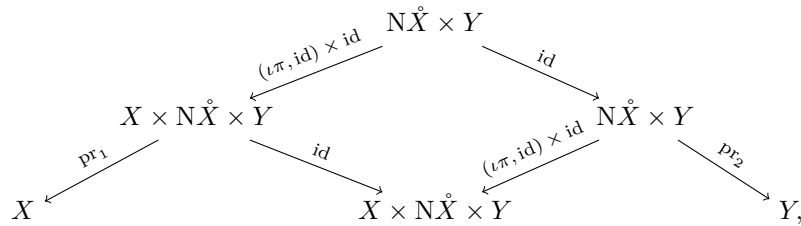
with the K-theory class ξ on $\mathrm{N}\mathring{X} \times Y$. Hence it involves the maps $\iota\pi \colon \mathrm{N}\mathring{X} \times Y \to X$ and $\mathrm{pr}_2 \colon \mathrm{N}\mathring{X} \times Y \to Y$. $\qquad\square$

If X is, in addition, K_G-oriented, then the Thom isomorphism provides an isomorphism $\mathrm{N}\mathring{X} \cong \mathring{X}$ in $\widehat{\mathrm{KK}}_*^G$ (which has odd parity if the dimension of X is odd). A variant of Corollary 2.8 yields a duality isomorphism

$$
\mathrm{K}_G^{*+\dim(X)}(\mathring{X} \times Y) \cong \widehat{\mathrm{KK}}_*^G(X,Y),
$$

which maps $\xi \in \mathrm{K}_G^*(\mathring{X} \times Y)$ to the geometric correspondence

$$
X \xleftarrow{\iota \circ \mathrm{pr}_1} \mathring{X} \times Y \xrightarrow{\mathrm{pr}_2} Y, \qquad \xi \in \mathrm{K}_G^*(\mathring{X} \times Y).
$$

Hence any element of $\widehat{\mathrm{KK}}_*^G(X,Y)$ is represented by a correspondence of this form.

If X is K_G-oriented and has no boundary, this becomes

$$
X \xleftarrow{\mathrm{pr}_1} X \times Y \xrightarrow{\mathrm{pr}_2} Y, \qquad \xi \in \mathrm{K}_G^*(X \times Y).
$$

These standard forms for correspondences are less useful than one may hope at first because their intersection products are no longer in this standard form.

2.5 More on composition of geometric correspondences

With our geometric formulas for the unit and counit of the duality, we could now compute Lefschetz indices geometrically, assuming the necessary intersections are transverse. While this works well, say, for self-maps with regular non-degenerate fixed points, it fails badly for the identity correspondence, whose Lefschetz index is the Euler characteristic. Building on work of Baum and Block [4], we now describe the composition as a modified intersection product under a much weaker assumption than transversality that still covers the computation of Euler characteristics.

Definition 2.9. We say that the smooth maps $f_1 \colon M_1 \to Y$ and $b_2 \colon M_2 \to Y$ *intersect smoothly* if

$$
M := M_1 \times_Y M_2
$$

is a smooth submanifold of $M_1 \times M_2$ and any $(\xi_1, \xi_2) \in \mathrm{T}M_1 \times \mathrm{T}M_2$ with $Df_1(\xi_1) = Db_2(\xi_2) \in \mathrm{T}Y$ is tangent to M.

If f_1 and b_2 intersect smoothly, then we define the *excess intersection bundle* $\eta(f_1, b_2)$ on M as the cokernel of the vector bundle map

$$(Df_1, -Db_2)\colon \mathrm{pr}_1^*(\mathrm{T}M_1) \oplus \mathrm{pr}_2^*(\mathrm{T}M_2) \to f^*(\mathrm{T}Y), \qquad (2.13)$$

where $f := f_1 \circ \mathrm{pr}_1 = b_2 \circ \mathrm{pr}_2\colon M \to Y$.

If the maps f_1 and b_2 are G-equivariant with respect to a compact group G, then the excess intersection bundle is a G-vector bundle.

We call the square

$$
\begin{array}{ccc}
M & \xrightarrow{\ \mathrm{pr}_2\ } & M_2 \\
{\scriptstyle \mathrm{pr}_1} \downarrow & & \downarrow {\scriptstyle f_1} \\
M_1 & \xrightarrow{\ b_2\ } & Y
\end{array}
$$

η-Cartesian if f_1 and b_2 intersect smoothly with excess intersection bundle η.

If M is a smooth submanifold of $M_1 \times M_2$, then $\mathrm{T}M \subseteq \mathrm{T}(M_1 \times M_2)$; and if $(\xi_1, \xi_2) \in \mathrm{T}(M_1 \times M_2)$ is tangent to M, then $Df_1(\xi_1) = Db_2(\xi_2)$ in $\mathrm{T}Y$. These pairs (ξ_1, ξ_2) form a subspace of $\mathrm{T}(M_1 \times M_2)|_M = \mathrm{pr}_1^*\mathrm{T}M_1 \oplus \mathrm{pr}_2^*\mathrm{T}M_2$, which in general need not be a vector bundle, that is, its rank need not be locally constant. The smooth intersection assumption forces it to be a subbundle: the kernel of the map in (2.13). Hence the excess intersection bundle is a vector bundle over M, and there is the following exact sequence of vector bundles over M:

$$0 \to \mathrm{T}M \to \mathrm{pr}_1^*(\mathrm{T}M_1) \oplus \mathrm{pr}_2^*(\mathrm{T}M_2) \xrightarrow{(Df_1, -Db_2)} (f_1 \circ \mathrm{pr}_1)^*(\mathrm{T}Y) \to \eta \to 0.$$
$$(2.14)$$

Example 2.10. Let $M_1 = M_2 = X$ and let $f_1 = b_2 = i\colon X \to Y$ be an injective immersion. Then $M_1 \times_Y M_2 \cong X$ is the diagonal in $M_1 \times M_2 = X^2$, which is a smooth submanifold. Furthermore, if $(\xi_1, \xi_2) \in \mathrm{T}M_1 \times \mathrm{T}M_2$ satisfy $Df_1(\xi_1) = Db_2(\xi_2)$, then $\xi_1 = \xi_2$ because $Di\colon \mathrm{T}M \to \mathrm{T}Y$ is assumed injective. Hence M_1 and M_2 intersect smoothly, and the excess intersection bundle is the normal bundle of the immersion i.

Example 2.11. Let M_1 and M_2 be two circles embedded in $Y = \mathbb{R}^2$. The four possible configurations are illustrated in Figure 1.

1. The circles meet in two points. Then $M = \{p_1, p_2\}$ and the intersection is transverse.

2. The two circles are disjoint. Then $M = \emptyset$ and the intersection is transverse.

3. The two circles are identical. Then $M = M_1 = M_2$. The intersection is not transverse, but smooth by Example 2.10; the excess intersection bundle is the normal bundle of the circle, which is trivial of rank 1.
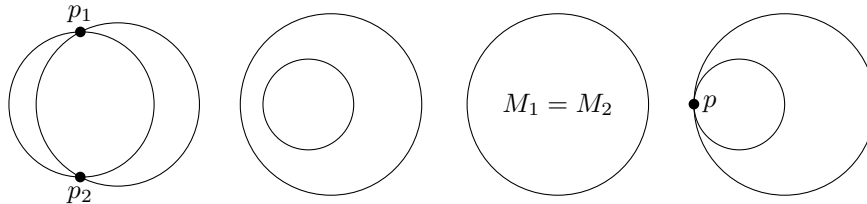
Figure 1: Four possible configurations of two circles in the plane

4. The two circles touch in one point. Then $M := M_1 \times_Y M_2 = \{p\}$, so that the tangent bundle of M is zero-dimensional. But $\mathrm{T}_p M_1 \cap \mathrm{T}_p M_2$ is one-dimensional because $\mathrm{T}_p M_1 = \mathrm{T}_p M_2$. Hence the embeddings do *not* intersect smoothly.

Remark 2.12. The maps $f \colon M_1 \to Y$ and $b \colon M_2 \to Y$ intersect smoothly if and only if $f \times b \colon M_1 \times M_2 \to Y \times Y$ and the diagonal embedding $Y \to Y \times Y$ intersect smoothly; both pairs of maps have the same excess intersection bundle. Thus we may always normalise intersections to the case where one map is a diagonal embedding and thus an embedding.

Example 2.13. Let η be a K_G-oriented vector bundle over X. Let $M_1 = M_2 = X$, $Y = \eta$, and let $f_1 = b_2 = \zeta \colon Y \to \eta$ be the zero section of η. This is a special case of Example 2.10. The maps f_1 and b_2 intersect smoothly with excess intersection bundle η.

In this example it is easy to compose the geometric correspondences $X = X \to \eta$ and $\eta \leftarrow X = X$. A Thom modification of the first one along the K_G-oriented vector bundle η gives the special correspondence

$$X \leftarrow (\eta, \tau_\eta) = \eta,$$

where $\tau_\eta \in \mathrm{RK}^*_{G,X}(\eta)$ is the Thom class of η. The intersection product of this with $\eta \leftarrow X = X$ is $X = (X, \zeta^*(\tau_\eta)) = X$, that is, it is the class in $\widehat{\mathrm{KK}}^G_*(X, X)$ of $\zeta^*(\tau_\eta) \in \mathrm{RK}^*_G(X)$. This K-theory class is the restriction of τ_η to the zero section of η. By the construction of the Thom class, it is the K-theory class of the spinor bundle of η.

DEFINITION 2.14. Let η be a K_G-oriented G-vector bundle over a G-space X. Let $\zeta \colon X \to \eta$ be the zero section and let $\tau_\eta \in \mathrm{RK}^*_{G,X}(\eta)$ be the Thom class. The *Euler class* of η is $\zeta^*(\tau_\eta)$, the restriction of τ_η to the zero section.

By definition, the Euler class is the composition of the correspondences $\mathrm{pt} \leftarrow X \to \eta$ and $\eta \leftarrow X = X$ involving the zero section $\zeta \colon X \to \eta$ in both cases.

Example 2.15. Assume that there is a G-equivariant section $s \colon X \to \eta$ of η with isolated simple zeros; that is, s and ζ are transverse. The linear homotopy connects s to the zero section and hence gives an equivalent correspondence

$\eta \xleftarrow{s} X = X$. Since s and ζ are transverse by assumption, the composition is $X \leftarrow Z \rightarrow X$, where Z is the zero set of s and the maps $Z \rightarrow X$ are the inclusion map, suitably K_G-oriented.

Example 2.16. Let $M_1 = S^1$, $M_2 = S^2$, $Y = \mathbb{R}^3$, $b_2 \colon M_2 \rightarrow \mathbb{R}^3$ be the standard embedding of the 2-sphere in $\mathbb{R}^3$, and let $f_1 \colon M_1 \rightarrow M_2 \rightarrow \mathbb{R}^3$ be the embedding corresponding to the equator of the circle. Then $M_1 \times_Y M_2 = M_1 \times_{M_2} M_2 = M_1$, embedded diagonally into $M_1 \times M_1 \subset M_1 \times M_2$. This is a case of smooth intersection. The excess intersection bundle is the restriction to the equator of the normal bundle of the embedding b_2. This is isomorphic to the rank-one trivial bundle on S^2. Hence the Euler class $e(\eta)$ is zero in this case.

Theorem 2.17. *Let*

$$X \xleftarrow{b_1} (M_1, \xi_1) \xrightarrow{f_1} Y \xleftarrow{b_2} (M_2, \xi_2) \xrightarrow{f_2} Z \qquad (2.15)$$

be a pair of G-equivariant correspondences as in (2.3). *Assume that b_2 and f_1 intersect smoothly and with a K_G-oriented excess intersection bundle η. Then the composition of* (2.15) *is represented by the G-equivariant correspondence*

$$X \xleftarrow{b_1 \circ \mathrm{pr}_1} \left(M_1 \times_Y M_2, e(\eta) \otimes \mathrm{pr}_1^*(\xi_1) \otimes \mathrm{pr}_2^*(\xi_2) \right) \xrightarrow{f_2 \circ \mathrm{pr}_2} Z, \qquad (2.16)$$

where $e(\eta)$ is the Euler class and the projection $\mathrm{pr}_2 \colon M_1 \times_Y M_2 \rightarrow M_2$ carries the K_G-orientation induced by the K_G-orientations on f_1 and η (explained below).

In the above situation of smooth intersection, we call the diagram (2.4) an *η-intersection diagram*. It still computes the composition, but we need the Euler class of the excess intersection bundle η to compensate the lack of transversality.
We describe the canonical K_G-orientation of $\mathrm{pr}_2 \colon M_1 \times_Y M_2 \rightarrow M_2$. The excess intersection bundle η is defined so as to give an exact sequence of vector bundles (2.14). From this it follows that

$$[\eta] = (f_1 \circ \mathrm{pr}_1)^*[\mathrm{T}Y] + \mathrm{T}M - \mathrm{pr}_1^*[\mathrm{T}M_1] - \mathrm{pr}_2^*[\mathrm{T}M_2].$$

On the other hand, the stable normal bundle $N\mathrm{pr}_2$ of pr_2 is equal to $\mathrm{pr}_2^*[\mathrm{T}M_2] - [\mathrm{T}M]$. Hence

$$[\eta] = \mathrm{pr}_1^*\big(f_1^*[\mathrm{T}Y] - [\mathrm{T}M_1]\big) - N\mathrm{pr}_2.$$

A K_G-orientation on f_1 means a stable K_G-orientation on $Nf_1 = f_1^*[\mathrm{T}Y] - [\mathrm{T}M_1]$. If such an orientation is given, it pulls back to one on $\mathrm{pr}_1^*\big(f_1^*[\mathrm{T}Y] - [\mathrm{T}M_1]\big)$, and then (stable) K_G-orientations on $[\eta]$ and on $N\mathrm{pr}_2$ are in 1-to-1 correspondence. In particular, a K_G-orientation on the bundle η induces one on the normal bundle of pr_2. This induced K_G-orientation on pr_2 is used in (2.16). ([14, Lemma 5.13] justifies working with K_G-orientations on stable normal bundles.)

Proof of Theorem 2.17. Lift f_1 to a G-equivariant smooth normally non-singular map $(V_1, E_1, \hat{f}_1)$. The composition of (2.15) is defined in [15, Section 2.5] as the intersection product

$$X \xleftarrow{b_1 \circ \pi_{V_1} \circ \mathrm{pr}_{V_1}} V_1 \times_Y M_2 \xrightarrow{f_2 \circ \mathrm{pr}_2} Z \qquad (2.17)$$

with K-theory datum $\mathrm{pr}_{V_1}^*(\tau_{V_1}) \otimes \pi_{V_1}^*(\xi_1) \otimes \mathrm{pr}_2^*(\xi_2) \in \mathrm{RK}_{G,X}^*(V_1 \times_Y M_2)$. We define the manifold $V_1 \times_Y M_2$ using the (transverse) maps $\pi_{E_1} \circ \hat{f}_1 \colon V_1 \to Y$ and $b_2 \colon M_2 \to Y$. We must compare this with the correspondence in the statement of the theorem.

We have a commuting square of embeddings of smooth manifolds

$$
\begin{array}{ccc}
M_1 \times_Y M_2 & \xrightarrow{\iota_0} & M_1 \times M_2 \\
\downarrow{\scriptstyle \zeta_0} & & \downarrow{\scriptstyle \zeta_1} \\
V_1 \times_Y M_2 & \xrightarrow{\iota_1} & V_1 \times M_2
\end{array}
\qquad (2.18)
$$

where the vertical maps are induced by the zero section $M_1 \to V_1$ and the horizontal ones are the obvious inclusion maps. The map ζ_0 is a smooth embedding because the other three maps in the square are so.

Let $N\iota_0$ and $\nu := N\zeta_0$ denote the normal bundles of the maps ι_0 and ζ_0 in (2.18). The normal bundle of ι_1 is isomorphic to the pull-back of $\mathrm{T}Y$ because $V_1 \to Y$ is submersive. Since $M_1 \times M_2 \to V_1 \times M_2$ is the zero section of the pull back of the vector bundle V_1 to $M_1 \times M_2$, the normal bundle of ζ_1 is isomorphic to $\mathrm{pr}_1^*(V_1)$. Recall that $M := M_1 \times_Y M_2$. We get a diagram of vector bundles over M:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{T}M & \xrightarrow{D\iota_0} & \mathrm{T}(M_1 \times M_2)|_M & \longrightarrow & N\iota_0 & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle D\zeta_0} & & \downarrow{\scriptstyle D\zeta_1} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{T}(V_1 \times_Y M_2)|_M & \xrightarrow{D\iota_1} & \mathrm{T}(V_1 \times M_2)|_M & \longrightarrow & f^*(\mathrm{T}Y) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \nu & \longrightarrow & \mathrm{pr}_1^*(V_1) & \dashrightarrow & \eta & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

The first two rows and the first two columns are exact by definition or by our description of the normal bundles of ζ_1 and ι_1. The third row is exact with the excess intersection bundle η by (2.14). Hence the dotted arrow exists and

makes the third row exact. Since extensions of G-vector bundles always split, we get

$$\nu \oplus \eta \cong \mathrm{pr}_1^*(V_1).$$

Since η and V_1 are K_G-oriented, the bundle ν inherits a K_G-orientation. We apply Thom modification with the K_G-oriented G-vector bundle ν to the correspondence in (2.16). This gives the geometric correspondence

$$X \xleftarrow{b_1 \circ \mathrm{pr}_1 \circ \pi_\nu} \nu \xrightarrow{f_2 \circ \mathrm{pr}_2 \circ \pi_\nu} Z \qquad (2.19)$$

with K-theory datum

$$\xi := \tau_\nu \otimes \pi_\nu^* \big(e(\eta) \otimes \mathrm{pr}_1^*(\xi_1) \otimes \mathrm{pr}_2^*(\xi_2) \big) \in \mathrm{RK}_{G,X}^*(\nu).$$

The Tubular Neighbourhood Theorem gives a G-equivariant open embedding $\hat{\zeta}_0 \colon \nu \to V_1 \times_Y M_2$ onto some G-invariant open neighbourhood of M (see [14, Theorem 3.18]).
We may find an open G-invariant neighbourhood U of the zero section in V_1 such that $U \times_Y M_2 \subseteq V_1 \times_Y M_2$ is contained in the image of $\hat{\zeta}_0$ and relatively M-compact. We may choose the Thom class $\tau_{V_1} \in \mathrm{K}_G^{\dim V_1}(V_1)$ to be supported in U. Hence we may assume that $\mathrm{pr}_1^*(\tau_{V_1})$, the pull-back of τ_{V_1} along the coordinate projection $\mathrm{pr}_1^* \colon V_1 \times_Y M_2 \to V_1$, is supported inside a relatively M-compact subset of $\hat{\zeta}_0(\nu)$.
Then [15, Example 2.14] provides a bordism between the cycle in (2.17) and

$$X \xleftarrow{b_1 \circ \pi_{V_1} \circ \mathrm{pr}_1 \circ \hat{\zeta}_0} \nu \xrightarrow{f_2 \circ \mathrm{pr}_2 \circ \hat{\zeta}_0} Z, \qquad (2.20)$$

with K-theory class $\mathrm{pr}_1^*(\tau_{V_1}) \otimes \hat{\zeta}_0^* \mathrm{pr}_1^* \pi_{V_1}^*(\xi_1) \otimes \hat{\zeta}_0^* \mathrm{pr}_2^*(\xi_2)$.
Let $s_t \colon \nu \to \nu$ be the scalar multiplication by $t \in [0,1]$. Composition with s_t is a G-equivariant homotopy

$$\pi_{V_1} \mathrm{pr}_1 \hat{\zeta}_0 \sim \mathrm{pr}_1 \pi_\nu \colon \nu \to M_1, \qquad \mathrm{pr}_2 \hat{\zeta}_0 \sim \mathrm{pr}_2 \pi_\nu \colon \nu \to M_2.$$

Hence $s_t^*(\hat{\zeta}_0^* \mathrm{pr}_1^* \pi_{V_1}^*(\xi_1) \otimes \hat{\zeta}_0^* \mathrm{pr}_2^*(\xi_2))$ is a G-equivariant homotopy

$$\hat{\zeta}_0^* \mathrm{pr}_1^* \pi_{V_1}^*(\xi_1) \otimes \hat{\zeta}_0^* \mathrm{pr}_2^*(\xi_2) \sim \pi_\nu^* \big(\mathrm{pr}_1^*(\xi_1) \otimes \mathrm{pr}_2^*(\xi_2) \big).$$

When we tensor with $\mathrm{pr}_1^*(\tau_{V_1})$, this homotopy has X-compact support because the support of $\mathrm{pr}_1^*(\tau_{V_1})$ is relatively M-compact.
This gives a homotopy of geometric correspondences between (2.17) and the variant of (2.19) with K-theory datum

$$\mathrm{pr}_1^*(\tau_{V_1}) \otimes \pi_\nu^* \mathrm{pr}_1^*(\xi_1) \otimes \pi_\nu^* \mathrm{pr}_2^*(\xi_2);$$

the relative M-compactness of the support of $\mathrm{pr}_{V_1}^*(\tau_{V_1})$ ensures that the homotopy of K_G-cycles implicit here has X-compact support. (We use [15, Lemma

2.12] here, but the statement of the lemma is unclear about the necessary compatibility between the homotopy and the support of ξ.)

The K-theory class $\mathrm{pr}_1^*(\tau_{V_1})$ in this formula is the restriction of the Thom class for the vector bundle $\mathrm{pr}_1^*(V_1)$ over M to ν. Since $\mathrm{pr}_1^*(V_1) \cong \nu \oplus \eta$ and the Thom isomorphism for a direct sum bundle is the composition of the Thom isomorphisms for the factors, the Thom class of $\mathrm{pr}_1^*(V_1)$ is $\mathrm{pr}_1^*(\tau_{V_1}) = \tau_\nu \otimes \tau_\eta$. Restricting this to the subbundle ν gives $\tau_\nu \otimes \pi_\nu^*(e(\eta))$. Hence the K-theory classes that come from (2.17) and (2.19) are equal. This finishes the proof. $\square$

2.6   The geometric Lefschetz index formula

In this section we compute Lefschetz indices in the symmetric monoidal category $\widehat{\mathrm{KK}}^G$ for smooth G-manifolds with boundary. Our computation is geometric and uses the intersection theory of equivariant correspondences discussed in Sections 2.2 and 2.5.

Let X be a smooth compact G-manifold, possibly with boundary. Let $\mathring{X}$ be its interior. Let

$$X \xleftarrow{b} M \xrightarrow{f} X, \qquad \xi \in \mathrm{RK}_{G,X}^*(M) \tag{2.21}$$

be a K_G-oriented smooth geometric correspondence from X to itself, with M of finite orbit type to ensure that $f\colon M \to X$ lifts to an essentially unique normally non-singular map. Since X is compact, $\mathrm{RK}_{G,X}^*(M) = \mathrm{K}_G^*(M)$ is the usual K-theory with compact support. The K_G-orientation for (2.21) means a K_G-orientation on the stable normal bundle of f. This is equivalent to giving a G-vector bundle V over X and K_G-orientations on $\mathrm{T}M \oplus f^*(V)$ and $\mathrm{T}X \oplus V$. If X has a boundary, then the requirements for a smooth correspondence are that M be a smooth manifold with boundary of finite orbit type, such that $f(\partial M) \subseteq \partial X$ and f is transverse to ∂X. This ensures that f has an essentially unique lift to a normally non-singular map from M to X by Proposition 2.5. Recall the map $\rho\colon X \to \mathring{X}$, which is shrinking the collar around ∂X.

THEOREM 2.18. *Let $\alpha \in \widehat{\mathrm{KK}}_i^G(X, X)$ be represented by a K_G-oriented smooth geometric correspondence as in (2.21). Assume that $(\rho b, f)\colon M \to X \times X$ and the diagonal embedding $X \to X \times X$ intersect smoothly with a K_G-oriented excess intersection bundle η. Then $Q_{\rho b,f} := \{m \in M \mid \rho b(m) = f(m)\}$ is a smooth manifold without boundary. For a certain canonical K_G-orientation on $Q_{\rho b,f}$, $\mathcal{L}(\alpha) \in \widehat{\mathrm{KK}}_i^G(X, \mathrm{pt})$ is represented by the geometric correspondence $X \leftarrow Q_{\rho b,f} \to \mathrm{pt}$ with K-theory class $\xi|_{Q_{\rho b,f}} \otimes e(\eta)$ on $Q_{\rho b,f}$; here the map $Q_{\rho b,f} \to X$ is given by $m \mapsto \rho b(m) = f(m)$.*

The Lefschetz index of α in $\widehat{\mathrm{KK}}_i^G(\mathrm{pt}, \mathrm{pt})$ is represented by the geometric correspondence $\mathrm{pt} \leftarrow Q_{\rho b,f} \to \mathrm{pt}$ with K_G-theory class $\xi|_{Q_{\rho b,f}} \otimes e(\eta)$ on $Q_{\rho b,f}$.

The Lefschetz index of α is the index of the Dirac operator on $Q_{\rho b,f}$ with coefficients in $\xi|_{Q_{\rho b,f}} \otimes e(\eta)$.

Proof. We abbreviate $Q := Q_{\rho b,f}$ throughout the proof. We have $Q \subseteq \mathring{M}$ because $\rho b(M) \subseteq \rho(X) \subseteq \mathring{X}$ and $f(\partial M) \subseteq \partial X$. The intersection $\mathring{M} \times_{\mathring{X} \times \mathring{X}} \mathring{X}$
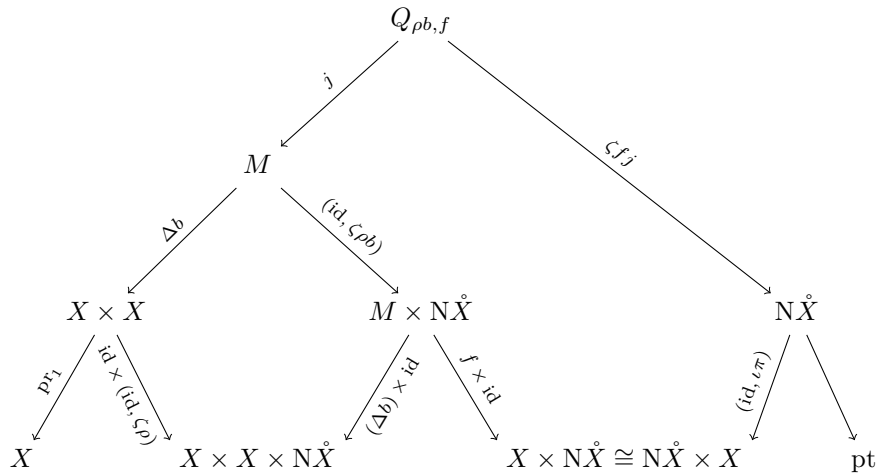
Figure 2: The intersection diagram for the computation of $\mathcal{L}(\alpha)$ in the proof of Theorem 2.18. Here $j\colon Q_{\rho b,f} \to M$ denotes the inclusion map; ζ the zero section $X \to \mathrm{N}X$ or $\mathring{X} \to \mathrm{N}\mathring{X}$; $\pi\colon \mathrm{N}\mathring{X} \to \mathring{X}$ the bundle projection; $\iota\colon \mathring{X} \to X$ the inclusion; $\Delta\colon X \to X \times X$ the diagonal embedding; $\mathrm{pr}_1\colon X \times X \to X$ the projection onto the first factor.

is Q and hence a smooth submanifold of $\mathring{M}$.

We compute $\mathcal{L}(\alpha)$ using the dual of X constructed in Theorem 2.7. This involves a G-vector bundle $\mathrm{N}X$ such that $\mathrm{T}X \oplus \mathrm{N}X \cong X \times E$ for a K_G-oriented G-vector space E.

With the unit and counit from Theorem 2.7, $\mathcal{L}(\alpha)$ becomes the composition of the three geometric correspondences in the bottom zigzag in Figure 2; here we already composed α with the multiplication correspondence, which simply composes b with Δ.

We first consider the small left square. Computing its intersection space naively gives M, which is a manifold with boundary. We would hope that this square is Cartesian. But $X \times X$ is only a manifold with corners if X has a boundary, and we we did not discuss smooth correspondences in this generality. Hence we check directly that the composition of the correspondences from X to $X \times X \times \mathrm{N}\mathring{X}$ and on to $M \times \mathrm{N}\mathring{X}$ is represented by $X \leftarrow M \to M \times \mathrm{N}\mathring{X}$.

The manifold $\mathrm{N}\mathring{X}$ is an open subset of E by construction. Hence the map

$$\mathrm{id} \times (\mathrm{id}, \zeta\rho b)\colon X \times X \to X \times X \times \mathrm{N}\mathring{X}$$

extends to an open embedding

$$\psi\colon X \times X \times E \to X \times X \times \mathrm{N}\mathring{X}, \qquad (x_1, x_2, e) \mapsto \bigl(x_1, x_2, \zeta\rho(x_2) + h_{x_2}(\|e\|^2)\cdot e\bigr),$$

where $h_{x_2}\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a diffeomorphism onto a bounded interval $[0,t)$ depending smoothly and G-invariantly on x_2, such that the t-ball in E around

$\zeta\rho(x_2) \in \mathrm{N}\mathring{X}$ is contained in $\mathrm{N}\mathring{X}$.

The map ψ gives a special correspondence

$$X \xleftarrow{\mathrm{pr}_1 \circ \pi_E} X \times X \times E \xrightarrow{\psi} X \times X \times \mathrm{N}\mathring{X}$$

with K-theory class the pull-back of the Thom class of E. This is equivalent to the given correspondence from X to $X \times X \times \mathrm{N}\mathring{X}$ because of a Thom modification for the trivial vector bundle E and a homotopy. In particular, the K_G-orientation of $\mathrm{id} \times (\mathrm{id}, \zeta\rho)$ that is implicit here is the one that we get from the K_G-orientation in the proof of Theorem 2.7.

For a special correspondence, the intersection always gives the composition product. Here we get the space

$$\big\{ (x_1, x_2, e, m, y, \mu) \in X \times X \times E \times M \times \mathrm{N}\mathring{X} \mid$$
$$(x_1, x_2, \zeta\rho(x_2) + h_{x_2}(\|e\|^2) \cdot e)) = (b(m), b(m), y, \mu) \big\}.$$

That is, $x_1 = x_2 = b(m)$, $(y, \mu) = \rho b(m) + h_{b(m)}(\|e\|^2) \cdot e$. Since $m \in M$ and $e \in E$ may be arbitrary and determine the other variables, we may identify this space with $M \times E$.

In the same way, we may replace

$$X \xleftarrow{b} M \xrightarrow{(\mathrm{id}, \zeta\rho b)} M \times \mathrm{N}\mathring{X} \tag{2.22}$$

by an equivalent special correspondence with space $M \times E$ in the middle. This gives exactly the composition computed above. Hence (2.22) also represents the composition of the correspondences from X to $M \times \mathrm{N}\mathring{X}$ in Figure 2.

Composing further with $f \times \mathrm{id}$ simply composes K_G-oriented normally non-singular maps. Since we are now in the world of manifolds with boundary, we may identify smooth maps and smooth normally non-singular maps. The large right square contains the G-maps

$$(f, \zeta\rho b) = (f \times \mathrm{id}) \circ (\mathrm{id}, \zeta\rho b) \colon M \to X \times \mathrm{N}\mathring{X},$$
$$(\iota\pi, \mathrm{id}) \colon \mathrm{N}\mathring{X} \to X \times \mathrm{N}\mathring{X}.$$

The pull-back contains those $(m, x, \mu) \in M \times \mathrm{N}\mathring{X}$ with $(f(m), \rho b(x), 0) = (x, x, \mu)$ in $X \times \mathrm{N}\mathring{X}$. This is equivalent to $x = f(m) = \rho b(m)$ and $\mu = 0$, so that the pull-back is Q. Since all vectors tangent to the fibres of $\mathrm{N}\mathring{X}$ are in the image of $D(\iota\pi, \mathrm{id})$, the intersection is smooth and the excess intersection bundle is the same bundle η as for $(f, \rho b) \colon \mathring{M} \to \mathring{X} \times \mathring{X}$ and $\delta \colon \mathring{X} \to \mathring{X} \times \mathring{X}$. Hence the right square is η-Cartesian.

Theorem 2.17 shows that $\mathcal{L}(\alpha)$ is represented by a correspondence of the form $X \xleftarrow{bj} Q \to \mathrm{pt}$, with a suitable class in $\mathrm{K}_G^*(Q)$ and a suitable K_G-orientation on the map $Q \to \mathrm{pt}$ or, equivalently, the manifold Q. Here we may replace bj by the properly homotopic map $\rho bj = fj$. It remains to describe the K-theory and orientation data.

First, the given K-theory class ξ on M is pulled back to $\xi \otimes 1$ on $M \times \mathring{NX}$ when we take the exterior product with $\mathring{NX}$. In the intersection product, this is pulled back to M along $(\mathrm{id}, \zeta \rho b)$, giving ξ again, and then to Q along j, giving the restriction of ξ to $Q \subseteq M$. The unit and counit have 1 as its K-theory datum. Thus the Lefschetz index has $\xi|_Q \otimes e(\eta) \in \mathrm{K}_G^*(Q)$ as its K-theory datum by Theorem 2.17.

The given K_G-orientations on E, f and η induce K_G-orientations on all maps in Figure 2 that point to the right. This is the K_G-orientation on the map $Q \to \mathrm{pt}$ that we need. We describe it in greater detail after the proof of the theorem.

The K_G-orientation on the map $Q \to \mathrm{pt}$ is equivalent to a G-equivariant Spin^c-structure on Q. The isomorphism

$$\widehat{\mathrm{KK}}_*^G(\mathrm{pt}, \mathrm{pt}) \to \widehat{\mathrm{KK}}_*^G(\mathrm{C}(\mathrm{pt}), \mathrm{C}(\mathrm{pt}))$$

described in [15, Theorem 4.2] maps the geometric correspondence just described to the index of the Dirac operator on Q for the chosen Spin^c-structure twisted by $\xi|_Q \otimes e(\eta)$. This gives the last assertion of the theorem. $\qquad\square$

Since the K_G-orientation on $Q_{\rho b, f}$ is necessary for computations, we describe it more explicitly now. We still use the notation from the previous proof.

We are given K_G-orientations on E, f and η. The K_G-orientation on f is equivalent to one on the G-vector bundle $\mathrm{T}M \oplus f^*(\mathrm{N}X)$ over M because

$$\mathrm{T}X \oplus \mathrm{N}X \cong X \times E$$

is a K_G-oriented G-vector bundle on X.

We already discussed during the proof of the theorem that $\mathrm{id} \times (\mathrm{id}, \zeta \rho)$ and $(\mathrm{id}, \zeta \rho)$ are normally non-singular embeddings with normal bundle E; this gives the correct K_G-orientation for these maps as well.

A K_G-orientation on the map $(f, \zeta \rho b) \colon M \to X \times \mathring{NX}$ is equivalent to one for $\mathrm{T}M \oplus f^*(\mathrm{N}X)$ because the bundle $\mathrm{T}(X \times \mathring{NX}) \oplus \mathrm{pr}_1^*(\mathrm{N}X)$ over $X \times \mathring{NX}$ is isomorphic to the trivial bundle with fibre $E \oplus E$ and $(f, \zeta \rho b)^* \mathrm{pr}_1^*(\mathrm{N}X) = f^*(\mathrm{N}X)$. We are already given such a K_G-orientation from the K_G-orientation of f.

LEMMA 2.19. *The given* K_G-*orientation on* $\mathrm{T}M \oplus f^*(\mathrm{N}X)$ *is also the one that we get by inducing* K_G-*orientations on* $(\mathrm{id}, \zeta \rho b)$ *from* $(\mathrm{id}, \zeta \rho)$ *and on* $f \times \mathrm{id}$ *from* f *and then composing.*

Proof. The K_G-orientation of f induces one for $f \times \mathrm{id}$, which is equivalent to a K_G-orientation for

$$\mathrm{T}(M \times \mathring{NX}) \oplus (f\mathrm{pr}_1)^*(\mathrm{N}X) \cong \big(\mathrm{T}M \oplus f^*(\mathrm{N}X)\big) \times (\mathring{NX} \times E).$$

This K_G-orientation is exactly the direct sum orientation from $\mathrm{T}M \oplus f^*(\mathrm{N}X)$ and E; no sign appears in changing the order because E has even dimension.

The map $h = (\mathrm{id}, \zeta\rho)$ is a smooth embedding with normal bundle E. Hence we get an extension of vector bundles

$$\mathrm{T}M \oplus f^*(\mathrm{N}X) \rightarrowtail h^*\big(\mathrm{T}(M \times \mathrm{N}\mathring{X}) \oplus (f\mathrm{pr}_1)^*(\mathrm{N}X)\big) \twoheadrightarrow E.$$

The given K_G-orientations on $\mathrm{T}M \oplus f^*(\mathrm{N}X)$ and E induce one on the vector bundle in the middle. This is the same one as the pull-back of the one constructed above. This means that the K_G-orientation on $\mathrm{T}M \oplus f^*(\mathrm{N}X)$ induced by h is the given one. $\qquad\square$

Equation (2.14) provides the following exact sequence of vector bundles over Q:

$$0 \to \mathrm{T}Q \xrightarrow{Dj, D(\zeta f j)} j^*(\mathrm{T}M) \oplus (\zeta f j)^*\mathrm{T}(\mathrm{N}\mathring{X})$$

$$\xrightarrow{D(f, \zeta\rho b), -D(\iota\pi, \mathrm{id})} (f, \zeta\rho b)^*\mathrm{T}(X \times \mathrm{N}\mathring{X}) \to \eta \to 0.$$

Since $-D(\iota\pi, \mathrm{id})$ is injective, we may divide out $\mathrm{T}(\mathrm{N}\mathring{X})$ and its image to get the simpler short exact sequence

$$0 \to \mathrm{T}Q \xrightarrow{Dj} j^*\mathrm{T}M \xrightarrow{Df - D(\rho b)} f^*\mathrm{T}X \to \eta \to 0.$$

Then we add the identity map on $j^*f^*(\mathrm{N}X)$ to get

$$0 \to \mathrm{T}Q \xrightarrow{(Dj, 0)} j^*(\mathrm{T}M \oplus f^*\mathrm{N}X) \xrightarrow{(Df - D(\rho b), \mathrm{id})} f^*(\mathrm{T}X \oplus \mathrm{N}X) \to \eta \to 0. \tag{2.23}$$

In the last long exact sequence, the vector bundles $j^*(\mathrm{T}M \oplus f^*\mathrm{N}X)$, $f^*(\mathrm{T}X \oplus \mathrm{N}X) \cong Q \times E$ and η carry K_G-orientations. These together induce one on $\mathrm{T}Q$. This is the K_G-orientation that appears in Theorem 2.18.

Of course, the resulting geometric cycle should not depend on the auxiliary choice of a K_G-orientation on η. Indeed, if we change it, then we change both $e(\eta)$ and the K_G-orientation on $\mathrm{T}Q$, and these changes cancel each other.

We now consider some examples of Theorem 2.18.

2.6.1 Self-maps transverse to the identity map

Let X be a compact G-manifold with boundary and let $b \colon X \to X$ be a smooth G-map that is transverse to the identity map. Thus b has only finitely many isolated fixed points and $1 - D_x b \colon \mathrm{T}_x X \to \mathrm{T}_x X$ is invertible for all fixed points x of b. We turn b into a geometric correspondence α from X to itself by taking $M = X$, $f = \mathrm{id}$ (with standard K_G-orientation) and $\xi = 1$.

Since b has only finitely many fixed points, we may choose the collar neighbourhood so small that all fixed points that do not lie on ∂X lie outside the collar neighbourhood, and such that the fixed points of ρb are precisely the fixed points of b not on the boundary of X. Hence $\rho b = b$ near all fixed points. Then ρb is also transverse to the diagonal map and Theorem 2.18 applies. The intersection space in Theorem 2.18 is

$$Q = Q_{\rho b, \mathrm{id}} = \{x \in X \mid \rho b(x) = x\} = \{x \in \mathring{X} \mid b(x) = x\},$$

the set of fixed points of b in $\mathring{X}$. The K-theory class on Q is 1 because $\xi = 1$ and the intersection is transverse. More precisely, the bundle η is zero-dimensional, and we may give it a trivial K_G-orientation for which $e(\eta) = 1$.

Although Q is discrete, the K_G-orientation of the map $Q \to \mathrm{pt}$ is important extra information: it provides the signs that appear in the familiar Lefschetz fixed-point formula. Equation (2.23) simplifies to

$$0 \to \mathrm{T}Q \to (\mathrm{T}X \oplus \mathrm{N}X)|_Q \xrightarrow{(\mathrm{id} - Db, \mathrm{id})} (\mathrm{T}X \oplus \mathrm{N}X)|_Q \to 0.$$

We left out η because it is zero-dimensional and carries the trivial K_G-orientation to ensure that $e(\eta) = 1$. The bundle $\mathrm{T}Q$ is also zero-dimensional. But a zero-dimensional bundle has non-trivial K_G-orientations. The Clifford algebra bundle of a zero-dimensional bundle is the trivial, trivially graded one-dimensional bundle spanned by the unit section. Thus an irreducible Clifford module (spinor bundle) for it is the same as a $\mathbb{Z}/2$-graded G-equivariant complex line bundle.

Let S be the spinor bundle associated to the given K_G-orientation on $\mathrm{T}X \oplus \mathrm{N}X \cong E$. The exact sequence (2.23) says that the K_G-orientation of Q is the $\mathbb{Z}/2$-graded G-equivariant complex line bundle ℓ such that $(\mathrm{id} - Db)^*(S|_Q) \otimes \ell \cong S|_Q$ as Clifford modules. This uniquely determines ℓ. Thus ℓ measures whether Db changes orientation or not. This is exactly the *sign* of the G-equivariant vector bundle automorphism $1 - Db$ on $\mathrm{T}X|_Q$, which is studied in detail in [13]. In particular, it is shown in [13] that ℓ is the complexification of a $\mathbb{Z}/2$-graded G-equivariant *real* line bundle. The $\mathbb{Z}/2$-grading gives one sign for each G-orbit in Q, namely, the index of $\mathrm{id} - Db_x$. In addition, the sign gives a real character $G_x \to \{-1, +1\}$ for each orbit, where G_x denotes the stabiliser of a point in the orbit.

Twisting the K_G-orientation by a line bundle over Q has the same effect as taking the trivial K_G-orientation and putting this line bundle on Q. Thus $\mathcal{L}(\alpha)$ is represented by the geometric correspondence

$$X \leftarrow (Q, \mathrm{sign}(1 - Db|_Q)) \to \mathrm{pt}$$

with the trivial K_G-orientation on the map $Q \to \mathrm{pt}$.

The Lefschetz index of α is the index of the Dirac operator on Q with coefficients in the line bundle $\mathrm{sign}(1 - Db)|_Q$; this is simply the $\mathbb{Z}/2$-graded G-representation on the space of sections of $\mathrm{sign}(1 - Db)|_Q$, which is a certain finite-dimensional $\mathbb{Z}/2$-graded, real G-representation.

If the group G is trivial, then the Lefschetz index is a number and $\mathrm{sign}(1 - Db)$ is the family of $\mathrm{sign}(1 - D_x b) \in \{\pm 1\}$ for $x \in Q$. If X is connected, then all maps $X \leftarrow \mathrm{pt}$ give the same element in $\widehat{\mathrm{KK}}$. Thus $\mathcal{L}(\alpha)$ is $\mathrm{L\text{-}ind}(\alpha)$ times the point evaluation class $[X \leftarrow \mathrm{pt} = \mathrm{pt}]$, and $\mathrm{L\text{-}ind}(\alpha)$ is the sum of the indices of all fixed points of b in $\mathring{X}$.

2.6.2 EULER CHARACTERISTICS

Now let $\xi \in \mathrm{K}_G^*(X)$ and consider the correspondence with $M = X$, $b = f = \mathrm{id}$, and the above class ξ. We want to compute the Lefschetz index of the geometric correspondence α associated to ξ. In particular, for $\xi = 1$ we get the Lefschetz index of the identity element in $\widehat{\mathrm{KK}}_0^G(X, X)$, which is the Euler characteristic of X.

We only compute the Lefschetz index of $\xi \in \mathrm{K}_G^*(X)$ for X with trivial boundary. Then the map ρ in Theorem 2.18 is the identity map, and id_X intersects itself smoothly. The intersection space is $Q = X$, embedded diagonally into $X \times X$. The excess intersection bundle η is $\mathrm{T}X$. To apply Theorem 2.18, we also assume that X is K_G-oriented. Then $\mathcal{L}(\alpha)$ is represented by the geometric correspondence

$$X \xleftarrow{\mathrm{id}_X} (X, \xi \otimes e(\mathrm{T}X)) \to \mathrm{pt}.$$

Here $e(\mathrm{T}X)$ and the map $X \to \mathrm{pt}$ both use the same K_G-orientation on X. The Lefschetz index of α is represented by

$$\mathrm{pt} \leftarrow (X, \xi \otimes e(\mathrm{T}X)) \to \mathrm{pt}.$$

By Theorem 2.18, this is the index of the Dirac operator of X with coefficients in $\xi \otimes e(\mathrm{T}X)$.

Twisting the Dirac operator by $e(\mathrm{T}X)$ gives the de Rham operator: this is the operator $d + d^*$ on differential forms with usual $\mathbb{Z}/2$-grading, so that its index is the Euler characteristic of X. Thus (the analytic version of) $\mathcal{L}(\alpha)$ is the class in $\mathrm{KK}_0^G(\mathrm{C}(X), \mathbb{C})$ of the de Rham operator with coefficients in ξ. This was proved already in [11] by computations in Kasparov's analytic KK-theory. Now we have a purely geometric proof of this fact, at least if X is K_G-oriented. Theorem 2.18 no longer works for X without K_G-orientation because there is no K_G-orientation on the excess intersection bundle. A way around this restriction would be to use twisted K-theory throughout. We shall not pursue this here, however.

We can now clarify the relationship between the Euler class $e(\mathrm{T}X) \in \mathrm{K}_G^{\dim(X)}(X)$ and the higher Euler characteristic $\mathrm{Eul}_X \in \mathrm{KK}_0^G(\mathrm{C}(X), \mathbb{C})$ introduced already in [11]. Since we assume X K_G-oriented and without boundary, there is a duality isomorphism $\mathrm{K}_G^{\dim(X)}(X) \cong \mathrm{K}_0^G(X) = \mathrm{KK}_0^G(\mathrm{C}(X), \mathbb{C})$. This duality isomorphism maps $e(\mathrm{T}X)$ to Eul_X.

2.6.3 SELF-MAPS WITHOUT TRANSVERSALITY

Let X be a compact G-manifold and let $b \colon X \to X$ be a smooth G-map. We want to compute the Lefschetz map on the geometric correspondence

$$X \xleftarrow{b} X \xrightarrow{\mathrm{id}_X} X$$

with K_G-theory class 1 on X.

If b is transverse to the identity map, then this is done already in Section 2.6.1. The case $b = \mathrm{id}_X$ is done already in Section 2.6.2. Now we assume that b and id_X intersect smoothly. We also assume that b has no fixed points on the boundary; then we may choose the collar neighbourhood of ∂X to contain no fixed points of b, so that $\rho(x) = x$ in a neighbourhood of the fixed point subset of b. Furthermore, all fixed points of ρb are already fixed points of b. That b and id_X intersect smoothly and away from ∂X means that

$$Q := \{x \in X \mid b(x) = x\} = \{x \in X \mid \rho b(x) = x\}$$

is a smooth submanifold of $\mathring{X}$ and that there is an exact sequence of G-vector bundles over Q:

$$0 \to \mathrm{T}Q \to \mathrm{T}X|_Q \xrightarrow{1 - D(\rho b)} \mathrm{T}X|_Q \to \eta \to 0,$$

where η is the excess intersection bundle.

Remark 2.20. The maps b and id_X always intersect smoothly if $b\colon X \to X$ is isometric with respect to a Riemannian metric on X; the reason is that if Db fixes a vector (x, ξ) at a fixed point of b, then b fixes the entire geodesic through x in direction ξ.

The vector bundles $\mathrm{T}Q$ and η are the kernel and cokernel of the vector bundle endomorphism $1 - D(\rho b)$ on $\mathrm{T}X|_Q$. Since both are vector bundles, $1 - D(\rho b)$ has locally constant rank. We may split

$$\mathrm{T}X|_Q \cong \ker(\mathrm{id} - D(\rho b)) \oplus \mathrm{im}(\mathrm{id} - D(\rho b)) = \mathrm{T}Q \oplus \mathrm{im}(\mathrm{id} - D(\rho b)),$$
$$\mathrm{T}X|_Q \cong \mathrm{coker}(\mathrm{id} - D(\rho b)) \oplus \mathrm{coim}(\mathrm{id} - D(\rho b)) = \eta \oplus \mathrm{coim}(\mathrm{id} - D(\rho b)).$$

Since $\mathrm{im}(\varphi) \cong \mathrm{coim}(\varphi)$ for any vector bundle homomorphism, it follows that η and $\mathrm{T}Q$ are stably isomorphic as G-vector bundles. Thus K_G-orientations on one of them translate to K_G-orientations on the other.

Remark 2.21. Given two stably isomorphic vector bundles, there is always a vector bundle endomorphism with these two as kernel and cokernel. Hence we cannot expect η and $\mathrm{T}Q$ to be isomorphic.

COROLLARY 2.22. *Let X be a compact G-manifold. Let $b\colon X \to X$ be a smooth G-map without fixed points on ∂X, such that b and id_X intersect smoothly. Let the fixed point submanifold Q of b be K_G-oriented, and equip the excess intersection bundle with the induced K_G-orientation. Then the Lefschetz index of the geometric correspondence*

$$X \xleftarrow{b} X \xrightarrow{\mathrm{id}_X} X$$

with K_G-theory class 1 on X is the index of the Dirac operator on Q twisted by $e(\eta)$.

The Lefschetz map sends the correspondence above to

$$X \xleftarrow{bj} Q \to \mathrm{pt}$$

with K-theory class $e(\eta)$ on Q.

2.6.4   Trace computation for standard correspondences

By Corollary 2.8, any element of $\widehat{\mathrm{KK}}_*^G(X, X)$ is represented by a correspondence of the form

$$X \xleftarrow{\iota \circ \pi \circ \mathrm{pr}_1} \mathrm{N}\mathring{X} \times X \xrightarrow{\mathrm{pr}_2} X$$

for a unique $\xi \in \mathrm{K}_G^*(\mathrm{N}\mathring{X} \times X)$. We may view this as a standard form for an element in $\widehat{\mathrm{KK}}_*^G(X, X)$.

The map $(\rho \circ \iota \circ \pi \circ \mathrm{pr}_1, \mathrm{pr}_2) = (\rho \circ \pi) \times \mathrm{id} \colon \mathrm{N}\mathring{X} \times X \to X \times X$ is a submersion and hence transverse to the diagonal. Thus Theorem 2.18 applies. The space $Q_{\rho b, f}$ is the graph of $\rho \pi \colon \mathrm{N}\mathring{X} \to X$. Thus the Lefschetz map gives the geometric correspondence

$$X \leftarrow \mathrm{N}\mathring{X} \to \mathrm{pt}, \qquad \xi|_{\mathrm{N}\mathring{X}} \in \mathrm{K}_G^*(\mathrm{N}\mathring{X}),$$

where we embed $\mathrm{N}\mathring{X} \to \mathrm{N}\mathring{X} \times X$ via $(\mathrm{id}, \rho \pi)$ and use the canonical K_G-orientation on $\mathrm{N}\mathring{X}$. The Lefschetz index in $\widehat{\mathrm{KK}}_*^G(\mathrm{pt}, \mathrm{pt}) \cong \mathrm{K}_G^*(\mathrm{pt})$ is computed analytically as the G-equivariant index of the Dirac operator on $\mathrm{N}\mathring{X}$ twisted by $\xi|_{\mathrm{N}\mathring{X}}$.

2.6.5   Trace computation for another standard form

Assume now that X has no boundary and is K_G-oriented. As we remarked at the end of Section 2.4, any element of $\widehat{\mathrm{KK}}_*^G(X, X)$ is represented by a correspondence

$$X \xleftarrow{\mathrm{pr}_1} X \times X \xrightarrow{\mathrm{pr}_2} X, \qquad \xi \in \mathrm{K}_G^*(X \times X).$$

The same computation as in Section 2.6.4 shows that the Lefschetz map sends this to

$$X = X \to \mathrm{pt}, \qquad \xi|_X \in \mathrm{K}_G^*(X),$$

where $\xi|_X$ is for the diagonal embedding $X \to X \times X$. Analytically, this is the K_G-homology class of the Dirac operator on X with coefficients $\xi|_X$.

2.6.6   Homogeneous correspondences

We call a self-correspondence $X \xleftarrow{b} M \xrightarrow{f} X$ *homogeneous* if X and M are homogeneous G-spaces. That is, $X := G/H$ and $M := G/L$ for closed subgroups $H, L \subseteq G$. Then there are elements $t_b, t_f \in G$ with $b(gL) := gt_bH$, $f(gL) := gt_fH$; we need $L \subseteq t_bHt_b^{-1} \cap t_fHt_f^{-1}$ for this to be well-defined. Since $G/L \cong G/t_f^{-1}Lt_f$ by $gL \mapsto gLt_f$, any homogeneous correspondence is isomorphic to one with $t_f = 1$, so that $L \subseteq H$. We assume this from now on and abbreviate $t = t_b$.

Since M and X are compact, the relevant K-theory group $\mathrm{RK}_{G,X}^*(M)$ for a homogeneous correspondence is just $\mathrm{K}_G^*(M)$. The induction isomorphism gives $\mathrm{RK}_{G,X}^*(M) = \mathrm{K}_G^*(G/L) \cong \mathrm{K}_L^*(\mathrm{pt})$.

A K_G-orientation for $f \colon G/L \to G/H$ is equivalent to a K^H-orientation for the projection map $H/L \to \mathrm{pt}$ because f is obtained from this H-map by

induction. Thus we must assume a K^H-orientation on H/L. Equivalently, the representation of L on $\mathrm{T}_{1L}(H/L)$ factors through Spin^c. This tangent space is the quotient $\mathfrak{h}/\mathfrak{l}$, where $\mathfrak{h}$ and $\mathfrak{l}$ denote the Lie algebras of H and L, respectively. Let $L' := H \cap tHt^{-1}$. Then $L \subseteq L'$ and both maps $f, b\colon G/L \to G/H$ factor through the quotient map $p\colon G/L \to G/L'$. The geometric correspondence

$$G/H \xleftarrow{b} G/L \xrightarrow{f} G/H, \qquad \xi \in \mathrm{K}_G^*(G/L)$$

is equivalent to the geometric correspondence

$$G/H \xleftarrow{b'} G/L' \xrightarrow{f'} G/H, \qquad \xi' \in \mathrm{K}_G^*(G/L')$$

with $\xi' := p_!(\xi)$ and $b'(gL') = gtH$ and $f'(gL') = gH$. (To construct the equivalence, we first need a normally non-singular map lifting p; then we apply vector bundle modifications on the domain and target of p to replace p by an open embedding; finally, for an open embedding we may construct a bordism as in [15, Example 2.14].)
Thus we may further normalise a homogeneous geometric self-correspondence to one with $L = H \cap tHt^{-1}$.
Now we compute the Lefschetz map for a such a normalised homogeneous self-correspondence.
First let $t \notin H$. Then the image of the map $(f, b)\colon G/L \to G/H \times G/H$ does not intersect the diagonal. Hence (f, b) is transverse to the diagonal and the coincidence space $Q_{b,f}$ is empty. Thus the Lefschetz map vanishes on a homogeneous correspondence with $t \notin H$ by Theorem 2.18.
Now let $t \in H$. Then $b = f\colon G/L \to G/H$ is the canonical projection map. Our normalisation condition yields $L = H$ and $b = f = \mathrm{id}$ in this case; that is, our geometric correspondence is the class in $\widehat{\mathrm{kk}}_*^G(G/H, G/H)$ of some $\xi \in \mathrm{K}_G^*(G/H)$. Thus we have a special case of the Euler characteristic computation in Section 2.6.2. The Lefschetz map gives the class of the geometric correspondence

$$G/H \xleftarrow[=]{\mathrm{id}} (G/H, e(\mathrm{T}G/H) \otimes \xi) \to \mathrm{pt},$$

provided G/H is K_G-oriented. The Lefschetz index is the index of the de Rham operator with coefficients in ξ.
When we identify $\mathrm{K}_G^*(G/H) \cong \mathrm{K}_H^*(\mathrm{pt})$, the Lefschetz index becomes a map

$$\mathrm{K}_H^*(\mathrm{pt}) \to \mathrm{K}_G^*(\mathrm{pt}).$$

In complex K-theory, this is a map $\mathrm{R}(H) \to \mathrm{R}(G)$. Graeme Segal studied this map in [28, Section 2], where it was denoted by $i_!$.
For instance, assume G to be connected and let $H = L$ be its maximal torus. Let $t \in W := N_G H/H$, the Weyl group of G. Assume that we are working with complex K-theory, so that $\mathrm{K}_G^*(G/H) \cong \mathrm{K}_H^*(\mathrm{pt}) \cong \mathrm{R}(H)$. The Weyl group W acts on G/H by right translations; these are G-equivariant maps.

Taking the correspondences $X \xleftarrow{w^{-1}} X = X$, this gives a representation $W \to \widehat{\mathrm{KK}}_0^G(G/H, G/H)$. We also map $\mathrm{R}(H) \cong \mathrm{K}_G^0(G/H) \to \widehat{\mathrm{KK}}_0^G(G/H, G/H)$ using the correspondences $X = (X, \xi) = X$. These representations of W and $\mathrm{R}(H)$ are a covariant pair of representations with respect to the canonical action of W on $\mathrm{R}(H)$ induced by the automorphisms $h \mapsto whw^{-1}$ of H for $w \in W$. Hence we map

$$\mathrm{R}(H) \rtimes W \to \widehat{\mathrm{KK}}_0^G(G/H, G/H).$$

The Lefschetz index $\mathrm{R}(H) \rtimes W \to \mathrm{R}(G)$ maps $a \cdot t \mapsto 0$ for $t \in W \setminus \{1\}$ and $a \cdot 1 \mapsto \mathrm{ind}_G \Lambda_a$, where Λ_a means the de Rham operator on G/H twisted by a.

2.7   Fixed points submanifolds for torus actions

As another application of our excess intersection formula, we reprove a result that is used in a recent article by Block and Higson [5] to reformulate the Weyl Character Formula in KK-theory.

Block and Higson also develop a more geometric framework for equivariant KK-theory for a compact group. For two locally compact G-spaces X and Y, they identify $\mathrm{KK}_*^G(X, Y)$ with the group of continuous natural transformations $\Phi_Z \colon \mathrm{K}_G^*(X \times Z) \to \mathrm{K}_G^*(Y \times Z)$ for all compact G-spaces Z; here continuity means that each Φ_Z is a $\mathrm{K}_G^*(Z)$-module homomorphism. The Kasparov product then becomes the composition of natural transformations. This reduces Kasparov's equivariant KK-theory to equivariant K-theory.

The theory $\widehat{\mathrm{KK}}^G$ does more: it contains the knowledge that all such natural transformations come from geometric correspondences, when geometric correspondences give the same natural transformation, and how to compose geometric correspondences. Thus we get a more concrete KK-theory.

THEOREM 2.23. *Let T be a compact torus and let X be a smooth, K_T-oriented T-manifold with boundary. Let $e(\mathrm{T}X) \in \mathrm{K}_T^0(X)$ be the Euler class of X for the chosen K_T-orientation. Let $F \subseteq X$ be the fixed-point subset of the T-action on X and let $j \colon F \to X$ be the inclusion map. Then F is again a smooth K-oriented manifold with boundary, with trivial T-action, so that the inclusion map j is K_T-oriented. Let $e(\mathrm{T}F) \in \mathrm{K}^0(F) \subseteq \mathrm{K}_T^0(F)$ be the Euler class of F. The two geometric correspondences*

$$X \xleftarrow{\mathrm{id}_X} (X, e(\mathrm{T}X)) \xrightarrow{\mathrm{id}_X} X,$$
$$X \xleftarrow{j} (F, e(\mathrm{T}F)) \xrightarrow{j} X$$

represent the same element in $\widehat{\mathrm{KK}}_0^T(X, X)$.

This is a generalisation of [5, Lemma 3.1]. We allow Spin^c-manifolds instead of complex manifolds. For a Spin^c-structure coming from a complex structure, the Euler class is $[\Lambda^* \mathrm{T}^* X] \in \mathrm{K}_T^0(X)$, which appears in [5]. The following proof is a translation of the proof in [5] into the category $\widehat{\mathrm{KK}}^G$.

Proof. The first geometric correspondence above, involving the Euler class of X, is represented by the composition of geometric correspondences

$$X \xleftarrow[=]{\mathrm{id_X}} X \xrightarrow{\zeta} \mathrm{T}X \xleftarrow{\zeta} X \xrightarrow[=]{\mathrm{id_X}} X$$

by Example 2.13; here ζ denotes the zero section, which is K_T-oriented using the given K_T-orientation on the T-vector bundle $\mathrm{T}X$.

Choose a generic element ξ in the Lie algebra of T, that is, the one-parameter group $\exp(s\xi)$, $s \in \mathbb{R}$, is dense in T. Let $\alpha_t \colon X \to X$ denote the action of $t \in T$ on X. The action of T maps ξ to a vector field $\alpha_\xi \colon X \to \mathrm{T}X$. There is a homotopy of geometric correspondences

$$X \xleftarrow[=]{\mathrm{id_X}} X \xrightarrow{\alpha_{s\xi}} \mathrm{T}X \xleftarrow{\zeta} X \xrightarrow[=]{\mathrm{id_X}} X$$

for $s \in [0,1]$. For $t = 0$ we get the composition above, involving $e(\mathrm{T}X)$. We claim that for $s = 1$, the two correspondences intersect smoothly and that the intersection product is the second geometric correspondence in the theorem, involving F and its Euler class.

First we show that the fixed-point submanifold F is a closed submanifold. Equip X with a T-invariant Riemannian metric. Let $x \in F$, that is, $\alpha_t(x) = x$ for all $t \in T$. Split $\mathrm{T}_x X$ into

$$V = \{v \in \mathrm{T}_x X \mid D\alpha_t(x, v) = (x, v) \text{ for all } t \in T\}$$

and its orthogonal complement $V^\perp$. Since the metric is T-invariant,

$$\alpha_t(\exp(x, v)) = \exp(D\alpha_t(x, v))$$

for all $v \in \mathrm{T}_x X$. Since the exponential mapping restricts to a diffeomorphism between a neighbourhood of 0 in $\mathrm{T}_x X$ and a neighbourhood of x in X, we have $\exp(x, v) \in F$ if $v \in V$, and the converse holds for v in a suitable neighbourhood of 0. Thus we get a closed submanifold chart for F near x with $\mathrm{T}_x F = V$. Hence F is a closed submanifold with

$$\mathrm{T}F = \{(x, v) \in \mathrm{T}X \mid D\alpha_t(x, v) = (x, v) \text{ for all } t \in T\}.$$

Since ξ is generic, $\alpha_\xi(x) = 0$ in $\mathrm{T}_x X$ if and only if $x \in F$. Thus F is the coincidence space of the pair of maps $\zeta, \alpha_\xi \colon X \to \mathrm{T}X$. Let $x \in F$ and let $v_1, v_2 \in \mathrm{T}_x X$ satisfy $D\zeta(x, v_1) = D\alpha_\xi(x, v_2)$. Then $v_1 = v_2$ by taking the horizontal components; and the vertical component of $D\alpha_\xi(x, v_2)$ vanishes, which means that $D\alpha_{\exp(s\xi)}(x, v_2) = (x, v_2)$ for all $s \in \mathbb{R}$. Hence $v_2 \in \mathrm{T}_x F$. This proves that ζ and α_ξ intersect smoothly. The excess intersection bundle is the cokernel of $D\alpha_{\exp(s\xi)} - \mathrm{id}$; since the action of T is by isometries, $D\alpha_{\exp(s\xi)} - \mathrm{id}$ is normal in each fibre, so that its image and kernel are orthogonal complements. Hence the cokernel is canonically isomorphic to the kernel of $D\alpha_{\exp(s\xi)} - \mathrm{id}$. Thus the excess intersection bundle is canonically isomorphic to $\mathrm{T}F$.

Hence Theorem 2.17 gives the geometric correspondence $X \xleftarrow{j} (F, e(\mathrm{T}F)) \xrightarrow{j} X$ as the composition, as desired. $\qquad\square$

3   The homological Lefschetz index of a Kasparov morphism

The example in Section 2.6.1 shows in what sense the geometric Lefschetz index computations in Section 2 generalise the local fixed-point formula for the Lefschetz index of a self-map. Now we turn to generalisations of the global homological formula for the Lefschetz index.

The classical Lefschetz fixed-point formula for a self-map $f\colon X \to X$ contains the (super)trace of the map on the cohomology of X with rational coefficients induced by f. We take rational coefficients in order to get vector spaces over a field, where there is a good notion of trace for endomorphisms. By the Chern character, we may as well take $\mathrm{K}^*(X) \otimes \mathbb{Q}$ instead of rational cohomology. It is checked in [9] that the Lefschetz index of $f \in \mathrm{KK}_0(A, A)$ for a dualisable C^*-algebra A in the bootstrap class is equal to the supertrace of the map on $\mathrm{K}_*(A) \otimes \mathbb{Q}$ induced by f.

We are going to generalise this result to the equivariant situation for a compact Lie group G. We assume that we are working with complex C^*-algebras, so that $\widehat{\mathrm{KK}}_*^G(\mathrm{pt}, \mathrm{pt}) = \mathrm{KK}_*^G(\mathbb{C}, \mathbb{C})$ vanishes in odd degrees and is the representation ring $\mathrm{R}(G)$ in even degrees. Our methods do not apply to the torsion invariants in $\mathrm{KK}_d^G(\mathbb{R}, \mathbb{R})$ for $d \neq 0$ in the real case because we (implicitly) tensor everything with $\mathbb{Q}$ to simplify the Lefschetz index.

Furthermore, we work in KK^G instead of $\widehat{\mathrm{KK}}^G$ in this section because the category KK^G is triangulated, unlike $\widehat{\mathrm{KK}}^G$. We explain in Remark 3.11 why $\widehat{\mathrm{KK}}^G$ is not triangulated; the triangulated structure on KK^G is introduced in [21].

Let $S \subseteq \mathrm{R}(G)$ be the set of all elements that are not zero divisors. This is a saturated, multiplicatively closed subset; even more, it is the largest multiplicatively closed subset for which the canonical map $\mathrm{R}(G) \to S^{-1}\mathrm{R}(G)$ to the ring of fractions is injective (see [1, Exercise 9 on p. 44]). The localisation $S^{-1}\mathrm{R}(G)$ is also called the *total ring of fractions* of $\mathrm{R}(G)$.

Since KK^G is symmetric monoidal with unit $\mathbb{1} = \mathbb{C}$ and $\mathrm{R}(G) = \mathrm{KK}_0^G(\mathbb{C}, \mathbb{C})$, the category KK^G is $\mathrm{R}(G)$-linear. Hence we may localise it at S as in [17]. The resulting category $\mathcal{T} := S^{-1}\mathrm{KK}^G$ has the same objects as KK^G and arrows

$$\mathcal{T}_*(A, B) := S^{-1}\mathrm{KK}_*^G(A, B) = S^{-1}\mathrm{R}(G) \otimes_{\mathrm{R}(G)} \mathrm{KK}_*^G(A, B).$$

The category $\mathcal{T}$ is $S^{-1}\mathrm{R}(G)$-linear. There is an obvious functor $\natural\colon \mathrm{KK}^G \to \mathcal{T}$. If A is a separable G-C^*-algebra, then

$$\mathcal{T}_*(\mathbb{C}, A) = S^{-1}\mathrm{KK}_*^G(\mathbb{C}, A) \cong S^{-1}\mathrm{R}(G) \otimes_{\mathrm{R}(G)} \mathrm{K}_*^G(A),$$

where we use the usual $\mathrm{R}(G)$-module structure on $\mathrm{K}_*^G(A) \cong \mathrm{KK}_*^G(\mathbb{C}, A)$.

There is a unique symmetric monoidal structure on $\mathcal{T}$ for which $\natural$ is a strict symmetric monoidal functor: simply extend the exterior tensor product on KK^G $S^{-1}\mathrm{R}(G)$-linearly. Hence if A is dualisable in KK^G, then its image in $\mathcal{T}$ is dualisable as well, and

$$\natural(\operatorname{tr} f) = \operatorname{tr}(\natural f) \qquad \text{for all } f \in \mathrm{KK}_*^G(A, A).$$

The crucial point for us is that $\natural \operatorname{tr}(f) = \operatorname{tr}(\natural f)$ uniquely determines $\operatorname{tr} f$ because the map

$$\mathrm{R}(G) \cong \mathrm{KK}_0^G(\mathbb{1}, \mathbb{1}) \xrightarrow{\natural} \mathcal{T}_0(\mathbb{1}, \mathbb{1}) \cong S^{-1}\mathrm{R}(G)$$

is injective. Thus it suffices to compute Lefschetz indices in $\mathcal{T}$. This may be easier because $\mathcal{T}$ has more isomorphisms and thus fewer isomorphism classes of objects. Furthermore, the endomorphism ring of the unit $\mathcal{T}_*(\mathbb{1}, \mathbb{1}) = S^{-1}\mathrm{R}(G)$ has a rather simple structure:

LEMMA 3.1. *The ring $S^{-1}\mathrm{R}(G)$ is a product of finitely many fields.*

Proof. Let $G/\operatorname{Ad}G$ be the space of conjugacy classes in G and let $\mathrm{C}(G/\operatorname{Ad}G)$ be the algebra of continuous functions on $G/\operatorname{Ad}G$. Taking characters provides a ring homomorphism $\chi\colon \mathrm{R}(G) \to \mathrm{C}(G/\operatorname{Ad}G)$, which is well-known to be injective. Hence $\mathrm{R}(G)$ is torsion-free as an Abelian group and has no nilpotent elements. Since G is a compact Lie group, $\mathrm{R}(G)$ is a finitely generated commutative ring by [28, Corollary 3.3]. Thus $\mathrm{R}(G)$ is Noetherian and reduced. This implies that its total ring of fractions is a finite product of fields (see [18, Exercise 6.5]). $\qquad\square$

The fields in this product decomposition correspond bijectively to minimal prime ideals in $\mathrm{R}(G)$. By [28, Proposition 3.7.iii], these correspond bijectively to cyclic subgroups of G/G^0, where G^0 denotes the connected component of the identity element. In particular, $S^{-1}\mathrm{R}(G)$ is a field if and only if G is connected.

Example 3.2. Let G be a connected compact Lie group. Let T be a maximal torus in G and let W be the Weyl group, $W := N_G(T)/T$. Highest weight theory provides an isomorphism $\mathrm{R}(G) \cong \mathrm{R}(T)^W$. Here $\mathrm{R}(T)$ is a ring of integral Laurent polynomials in r variables, where r is the rank of T. Since elements of $\mathbb{N}_{\geq 1}$ are not zero divisors in $\mathrm{R}(G)$, the total ring of fractions of $\mathrm{R}(G)$ is equal to the total ring of fractions of $\mathrm{R}(G) \otimes \mathbb{Q}$. The latter is the $\mathbb{Q}$-algebra of W-invariant elements in $\mathbb{Q}[x_1, \ldots, x_r, (x_1 \cdots x_r)^{-1}]$. This is the algebra of polynomial functions on the algebraic $\mathbb{Q}$-variety $(\mathbb{Q}^\times)^r$, and the W-invariants give the algebra of polynomials on the quotient variety $(\mathbb{Q}^\times)^r/W$. This variety is connected, so that the total ring of fractions $S^{-1}\mathrm{R}(G)$ in this case is the field of rational functions on the algebraic $\mathbb{Q}$-variety $(\mathbb{Q}^\times)^r/W$.

Now we can define an equivariant analogue of the trace of the map on $\mathrm{K}_*(A) \otimes \mathbb{Q}$ induced by $f \in \mathrm{KK}_0(A, A)$:

DEFINITION 3.3. Let $S^{-1}\mathrm{R}(G) = \prod_{i=1}^n F_i$ with fields F_i. A module over $S^{-1}\mathrm{R}(G)$ is a product $\prod_{i=1}^n V_i$, where each V_i is an F_i-vector space. In particular, if A is a G-C^*-algebra, then $\mathcal{T}_*(\mathbb{C}, A) = S^{-1}\mathrm{K}_*^G(A) = \prod_{i=1}^n \mathrm{K}_{*,i}^G(A)$ for certain $\mathbb{Z}/2$-graded F_i-vector spaces $\mathrm{K}_{*,i}^G(A)$. An endomorphism $f \in \mathcal{T}_0(A, A)$ induces grading-preserving endomorphisms $\mathrm{K}_{*,i}^G(f)\colon \mathrm{K}_{*,i}^G(A) \to \mathrm{K}_{*,i}^G(A)$.

If the vector spaces $K^G_{*,i}(A)$ are all finite-dimensional, then the (super)trace of $K^G_{*,i}(f)$ is defined to be $\operatorname{tr} K^G_{0,i}(f) - \operatorname{tr} K^G_{1,i}(f) \in F_i$, and

$$\operatorname{tr} S^{-1} K^G_*(f) := (\operatorname{tr} K^G_{*,i}(f))^n_{i=1} \in \prod_{i=1}^n F_i = S^{-1} R(G).$$

We will see below that dualisability for objects in appropriate bootstrap classes already implies that $K^G_*(A)$ is a finitely generated $R(G)$-module, and then each $K^G_{*,i}(A)$ must be a finite-dimensional F_i-vector space.

THEOREM 3.4. *Let A belong to the thick subcategory of KK^G generated by $\mathbb{C}$ and let $f \in KK^G_0(A, A)$. Then A is dualisable in KK^G, so that $\operatorname{tr} f$ is defined, and*

$$\natural(\operatorname{tr} f) = \operatorname{tr} S^{-1} K^G_*(f) \in S^{-1} R(G).$$

Thick subcategories are defined in [26, Definition 2.1.6]. The thick subcategory generated by $\mathbb{C}$ is, of course, the smallest thick subcategory that contains the object $\mathbb{C}$. We denote the thick subcategory generated by a set A of objects or a single object by $\langle A \rangle$.

As we remarked above, $\natural(\operatorname{tr} f)$ uniquely determines $\operatorname{tr} f \in R(G)$ because the canonical embedding $\natural \colon R(G) \to S^{-1} R(G)$ is injective.

We will prove Theorem 3.4 in Section 3.3.

How restrictive is the assumption that X should belong to the thick subcategory of KK^G generated by $\mathbb{C}$? The answer depends on the group G.

We consider the two extreme cases: *Hodgkin Lie groups* and finite groups.

A Hodgkin Lie group is, by definition, a connected Lie group with simply connected fundamental group; they are the groups to which the Universal Coefficient Theorem and the Künneth Theorem in [27] apply.

THEOREM 3.5. *Let G be a compact connected Lie group with torsion-free fundamental group. Then a G-C^*-algebra A belongs to the thick subcategory generated by $\mathbb{C}$ if and only if*

- *A, without the G-action, belongs to the bootstrap category in KK, and*

- *A is dualisable.*

We postpone the proof of this theorem until after the proof of Proposition 3.13, which generalises part of this theorem to arbitrary compact Lie groups.

The first condition in Theorem 3.5 is automatic for commutative C^*-algebras because the non-equivariant bootstrap category is the class of all separable C^*-algebras that are KK-equivalent to a commutative separable C^*-algebra. Hence Theorem 3.5 verifies the assumptions needed for Theorem 3.4 if $A = C_0(X)$ and $C_0(X)$ is dualisable in KK^G; the latter is necessary for the Lefschetz index to be defined, anyway.

In particular, let X be a compact smooth G-manifold with boundary, for a Hodgkin Lie group G. Then X is dualisable in $\widehat{KK}^G$ by Theorem 2.7, and

hence $C(X)$ is dualisable in KK^G because the functor $\widehat{KK}^G \to KK^G$ is symmetric monoidal. Furthermore, $\widehat{KK}^G_*(X,X) \cong KK^G_*(C(X), C(X))$ in this case, so that any endomorphism $f \in KK^G_0(C(X), C(X))$ comes from some self-correspondence in $\widehat{KK}^G_0(X,X)$. We get the following generalisation of the Lefschetz fixed-point formula:

COROLLARY 3.6. *Let G be a Hodgkin Lie group, X a smooth compact G-manifold, possibly with boundary, and $f \in \widehat{KK}^G_0(X,X)$. Then $\mathrm{tr}(f) \in R(G) \subseteq S^{-1} R(G)$ is equal to the supertrace of $S^{-1} K^*_G(f)$, acting on the $S^{-1} R(G)$-vector space $S^{-1} K^*_G(X)$.*

Notice that $S^{-1} R(G)$ for a Hodgkin Lie group is a field, not just a product of fields.

In particular, Corollary 3.6 for the trivial group gives the Lefschetz index formula in [9].

Whereas Theorem 3.4 yields quite satisfactory results for Hodgkin Lie groups, its scope for a finite group G is quite limited:

Example 3.7. For $G = \mathbb{Z}/2$ there is a locally compact G-space X with $K^*_G(X) = 0$ but $K^*(X) \neq 0$. Equivalently, $KK^G_*(\mathbb{C}, C_0(X)) = 0$ and $KK^G_*(C(G), C_0(X)) \neq 0$. This shows that $C(G)$ does not belong to $\langle \mathbb{C} \rangle$.

Worse, the Lefschetz index formula in Theorem 3.4 is false for endomorphisms of $C(G)$. We have $\widehat{KK}^G_*(G, G) \cong \mathbb{Z}[G]$, spanned by the classes of the translation maps $G \to G$, $x \mapsto x \cdot g$, for $g \in G$, and these are homogeneous correspondences as in Section 2.6.6.

Translation by $g = 1$ is the identity map, and its Lefschetz index is the class of the regular representation of G in $R(G)$. For $g \neq 1$, the Lefschetz index is zero because the fixed point subset is empty. However, $K^*_G(G) = K^*(\mathrm{pt}) = \mathbb{Z}[0]$ and all translation maps induce the identity map on $K^*_G(G)$. Thus the induced map on $K^*_G(G)$ is not enough information to compute the Lefschetz index of an endomorphism of G in $\widehat{KK}^G$.

3.1   The equivariant bootstrap category

A reasonable Lefschetz index formula should apply at least to KK^G-endomorphisms of $C(X)$ for all smooth compact G-manifolds and thus, in particular, for finite G-sets X. Example 3.7 shows that Theorem 3.4 fails on such a larger category. This leads us to improve the Lefschetz index formula. First we discuss the class of G-C^*-algebras where we expect it to hold.

We are going to describe an equivariant analogue of the bootstrap class in KK^G. Our class is larger than the class of C^*-algebras that are KK^G-equivalent to a commutative C^*-algebra. The latter subcategory is too small because it is not thick. The thick (or localising) subcategory of KK^G generated by commutative C^*-algebras is a better choice, but such a definition is not very intrinsic. We will choose an even larger subcategory of KK^G because it is not more difficult to treat and has a nicer characterisation.

The category KK^G only has countable coproducts because we need C*-algebras to be separable. Hence the standard notions of compact objects and localising subcategories have to be modified so that they only involve countable coproducts. As in [7, Definition 2.1], we speak of *compact*$_{\aleph_1}$ objects, *localising*$_{\aleph_1}$ subcategories, and *compactly*$_{\aleph_1}$ generated subcategories.

DEFINITION 3.8. Call a G-C*-algebra A *elementary* if it is of the form $\mathrm{Ind}_H^G \mathbb{M}_n \mathbb{C} = \mathrm{C}(G, \mathbb{M}_n \mathbb{C})^H$ for some closed subgroup $H \subseteq G$ and some action of H on $\mathbb{M}_n \mathbb{C}$ by automorphisms; the superscript H means the fixed points for the diagonal action of H.

DEFINITION 3.9. Let $\mathcal{B}^G \subseteq \mathrm{KK}^G$ be the localising$_{\aleph_1}$ subcategory generated by all elementary G-C*-algebras. We call $\mathcal{B}^G$ the G-*equivariant bootstrap category*.

An action of H on $\mathbb{M}_n \mathbb{C}$ comes from a projective representation of H on $\mathbb{C}^n$. Such a projective representation is a representation of an extension of H by the circle group. The extension is classified by a cohomology class in $\mathrm{H}^2(H, \mathrm{U}(1))$. Two actions on $\mathbb{M}_n \mathbb{C}$ are H-equivariantly Morita equivalent if and only if they belong to the same class in $\mathrm{H}^2(H, \mathrm{U}(1))$. The G-C*-algebras $\mathrm{Ind}_H^G \mathbb{M}_n \mathbb{C}$ for actions of H on $\mathbb{M}_n \mathbb{C}$ with different cohomology classes need not be KK^G-equivalent.

THEOREM 3.10. *A G-C*-algebra belongs to the localising$_{\aleph_1}$ subcategory generated by the elementary G-C*-algebras if and only if it is KK^G-equivalent to a G-action on a type I C*-algebra.*

Proof. It is already shown in [27, Theorem 2.8] that all G-actions on type I C*-algebras belong to the localising$_{\aleph_1}$ subcategory generated by the elementary G-C*-algebras. By definition, localising$_{\aleph_1}$ subcategories are closed under KK^G-equivalence. Elementary G-C*-algebras are type I C*-algebras, even continuous trace C*-algebras. To finish the proof we must show that the G-C*-algebras that are KK^G-equivalent to type I G-C*-algebras form a localising$_{\aleph_1}$ subcategory of KK^G.

Let $\mathcal{T}_1 \subseteq \mathrm{KK}^G$ be the full subcategory of type I, separable G-C*-algebras. If $A \in \mathcal{T}_1$, then $\mathrm{C}_0(\mathbb{R}, A) \in \mathcal{T}_1$, so that $\mathcal{T}_1$ is closed under suspension and desuspension. Let $A, B \in \mathcal{T}_1$ and $f \in \mathrm{KK}_0^G(A, B)$. We have $\mathrm{KK}_0^G(A, B) \cong \mathrm{KK}_1^G(A, \mathrm{C}_0(\mathbb{R}, B))$, and cycles for the latter group correspond to (equivariantly) semisplit extensions of G-C*-algebras

$$\mathrm{C}_0(\mathbb{R}, B) \otimes \mathcal{K} \rightarrowtail D \twoheadrightarrow A$$

with $\mathcal{K} := \mathcal{K}(L^2(G \times \mathbb{N}))$. Since B and A are type I, so are $\mathrm{C}_0(\mathbb{R}, B) \otimes \mathcal{K}$ and D because the property of being type I is inherited by extensions. The semisplit extension above provides an exact triangle isomorphic to

$$B[-1] \to D \to A \xrightarrow{f} B.$$

Thus there is an exact triangle containing f with all three entries in $\mathcal{T}_1$. Furthermore, countable direct sums of type I C*-algebras are again type I. This

implies that the G-C*-algebras KK^G-equivalent to one in $\mathcal{T}_1$ form a localising$_{\aleph_1}$ subcategory of KK^G. $\square$

Remark 3.11. In the non-equivariant case, any C*-algebra in the bootstrap class is KK-equivalent to a commutative one. This criterion fails already for $G = \mathrm{U}(1)$, as shown by a counterexample in [10]. Since the bootstrap class is the smallest localising subcategory containing $\mathbb{C}$, it follows that the commutative C*-algebras do not form a localising subcategory. Thus $\widehat{\mathrm{KK}}^G$ is not triangulated: it lacks cones for some maps.

In this case, the equivariant bootstrap class is already generated by $\mathbb{C}$ and contains all $\mathrm{U}(1)$-actions on C*-algebras in the non-equivariant bootstrap category. It is shown in [10] that the $\mathrm{U}(1)$-equivariant K-theory of a suitable Cuntz–Krieger algebra with its natural gauge action cannot arise from any $\mathrm{U}(1)$-action on a locally compact space.

Corollary 3.12. *The restriction and induction functors* KK$^G \to$ KK^H *and* KK$^H \to$ KK^G *for a closed subgroup H in a compact Lie group G restrict to functors between the bootstrap classes in* KK^G *and* KK^H.

Proof. Restriction does not change the underlying C*-algebra and thus preserves the property of being type I. Induction maps elementary H-C*-algebras to elementary G-C*-algebras, is triangulated, and commutes with direct sums. Hence it maps $\mathcal{B}^H$ to $\mathcal{B}^G$. $\square$

Proposition 3.13. *An object of $\mathcal{B}^G$ is compact$_{\aleph_1}$ if and only if it is dualisable, if and only if it belongs to the thick subcategory of $\mathcal{B}^G$ (or of* KK^G*) generated by the elementary G-C*-algebras.*

Proof. The tensor unit $\mathbb{C}$ is compact$_{\aleph_1}$ because KK$^G_*(\mathbb{C}, A) \cong \mathrm{K}^G_*(A) \cong \mathrm{K}_*(G \ltimes A)$ is countable for all G-C*-algebras A, and the functors $A \mapsto G \ltimes A$ and K_* are well-known to commute with coproducts. Furthermore, the tensor product in KK^G commutes with coproducts in both variables.

Using this, we show that dualisable objects of $\mathcal{B}^G$ are compact$_{\aleph_1}$. If A is dualisable with dual A^*, then KK$^G(A, B) \cong$ KK$^G(\mathbb{C}, A^* \otimes B)$, and since $\mathbb{C}$ is compact$_{\aleph_1}$ and $\otimes$ commutes with countable direct sums, it follows that A is compact$_{\aleph_1}$.

It follows from [8, Corollary 2.2] that elementary G-C*-algebras are dualisable and hence compact$_{\aleph_1}$. A compact group has only at most countably many compact subgroups by Lemma 3.14 below; and any of them has at most finitely many projective representations. Hence the set of elementary G-C*-algebras is at most countable. Therefore, $\mathcal{B}^G$ is compactly$_{\aleph_1}$ generated in the sense of [7, Definition 2.1]. By [7, Corollary 2.4] an object of $\mathcal{B}^G$ is compact$_{\aleph_1}$ if and only if it belongs to the thick subcategory generated by the elementary G-C*-algebras.

The Brown Representability Theorem [7, Corollary 2.2] shows that for every compact$_{\aleph_1}$ object A of $\mathcal{B}^G$ there is a functor $\mathrm{Hom}(A, {}_{\sqcup})$ from $\mathcal{B}^G$ to $\mathcal{B}^G$ such

that

$$\mathrm{KK}^G(A \otimes B, D) \cong \mathrm{KK}^G(B, \mathrm{Hom}(A, D))$$

for all B, D in $\mathcal{B}^G$. Using exactness properties of the internal Hom functor in the first variable, we then show that the class of dualisable objects in $\mathcal{B}^G$ is thick (see [7, Section 2.3]). Thus all objects of the thick subcategory generated by the elementary G-C*-algebras are dualisable. $\square$

The following lemma is well-known, see [25].

LEMMA 3.14. *A compact Lie group has at most countably many conjugacy classes of closed subgroups.*

Proof. Let H be a closed subgroup of a compact Lie group G. By the Mostow Embedding Theorem, G/H embeds into a linear representation of G, that is, H is a stabiliser of a point in some linear representation of G. Up to isomorphism, there are only countably many linear representations of G. Each linear representation has finite orbit type, that is, it admits only finitely many different conjugacy classes of stabilisers. Hence there are altogether at most countably many conjugacy classes of closed subgroups in G. $\square$

Proof of Theorem 3.5. Let G be a Hodgkin Lie group. The main result of [23] says that A belongs to the localising subcategory of KK^G generated by $\mathbb{C}$ if and only if $A \rtimes G$ belongs to the non-equivariant bootstrap category (this is special for Hodgkin Lie groups). Since this covers all elementary G-C*-algebras, we conclude that the localising subcategory generated by $\mathbb{C}$ contains $\mathcal{B}^G$ and is, therefore, equal to $\mathcal{B}^G$.

The same argument as in the proof of Proposition 3.13 shows that the following are equivalent for an object A of $\mathcal{B}^G$:

- A is dualisable;

- A is compact$_{\aleph_1}$;

- A belongs to the thick subcategory generated by $\mathbb{C}$.

This finishes the proof of Theorem 3.5. $\square$

So far we always used the bootstrap class, which is the domain where a Universal Coefficient Theorem holds. The next proposition is a side remark showing that we may also use the domain where a Künneth formula holds.

DEFINITION 3.15. An object $A \in \mathrm{KK}^G$ satisfies the Künneth formula if $\mathrm{K}^G_*(A \otimes B) = 0$ for all B that satisfy $\mathrm{K}^G_*(C \otimes B) = 0$ for all elementary G-C*-algebras C.

By results of [20, 24], the assumption in Definition 3.15 is necessary and sufficient for a certain natural spectral sequence that computes $\mathrm{K}^G_*(A \otimes B)$ from $\mathrm{KK}^G_*(C, A)$ and $\mathrm{KK}^G_*(C, B)$ for elementary C to converge for all B; we have no need to describe this spectral sequence.

Proposition 3.16. *Let $A \in \mathrm{KK}^G$ be dualisable with dual A^*. If A or A^* satisfies a Künneth formula, then both A and A^* belong to $\mathcal{B}^G$, and vice versa.*

Proof. Since $\mathcal{B}^G$ is generated by the elementary G-C*-algebras, $\mathrm{KK}^G_*(C, B) = 0$ for all elementary G-C*-algebras C if and only if $\mathrm{KK}^G_*(C, B) = 0$ for all $C \in \mathcal{B}^G$. Any elementary G-C*-algebra C is dualisable with a dual in $\mathcal{B}^G$. Hence $\mathrm{K}^G_*(C \otimes B) \cong \mathrm{KK}^G_*(C^*, B) = 0$ for elementary C if $\mathrm{KK}^G_*(C', B) = 0$ for all elementary G-C*-algebras C'; conversely $\mathrm{KK}^G_*(C, B) \cong \mathrm{K}^G_*(C^* \otimes B) = 0$ for elementary C if $\mathrm{K}^G_*(C' \otimes B) = 0$ for all elementary G-C*-algebras C'. Let us denote the class of G-C*-algebras with these equivalent properties $\mathcal{B}^{G,\perp}$.

It follows from [20, Theorem 3.16] that $(\mathcal{B}^G, \mathcal{B}^{G,\perp})$ is a complementary pair of localising subcategories. In particular, if $\mathrm{KK}^G_*(A, B) = 0$ for all $B \in \mathcal{B}^{G,\perp}$, then $A \in \mathcal{B}^G$.

Now assume, say, that A satisfies a Künneth formula. Then $\mathrm{KK}^G_*(A^*, B) \cong \mathrm{K}^G_*(A \otimes B) = 0$ for all $B \in \mathcal{B}^{G,\perp}$. Thus $A^* \in \mathcal{B}^G$. Then $\mathrm{K}^G_*(A^* \otimes B) = 0$ for all $B \in \mathcal{B}^{G,\perp}$ because the class of C with $\mathrm{K}^G_*(C \otimes B) = 0$ is localising and contains all elementary C if $B \in \mathcal{B}^{G,\perp}$. As above, this implies $(A^*)^* = A \in \mathcal{B}^G$. ∎

The proof of Theorem 3.5 above used that, for a Hodgkin Lie group, $\mathcal{B}^G$ is already generated by $\mathbb{C}$. For more general groups, we also expect that fewer generators suffice to generate $\mathcal{B}^G$. But we only need and only prove a result about topologically cyclic groups here.

A locally compact group G is called *topologically cyclic* if there is an element $g \in G$ that generates a dense subgroup of G. A topologically cyclic group is necessarily Abelian. We are interested in topologically cyclic, compact Lie groups here. A compact Lie group is topologically cyclic if and only if it is isomorphic to $\mathbb{T}^r \times F$ for some $r \geq 0$ and some finite cyclic group F (possibly the trivial group), where $\mathbb{T} = \mathbb{R}/\mathbb{Z} \cong \mathrm{U}(1)$. Here we use that any extension $\mathbb{T}^r \rightarrowtail E \twoheadrightarrow F$ for a finite cyclic group F splits. This also implies that any projective representation of a finite cyclic groups is a representation.

Theorem 3.17. *Let G be a topologically cyclic, compact Lie group. Then the bootstrap class $\mathcal{B}^G \subseteq \mathrm{KK}^G$ is already generated by the finitely many G-C*-algebras $\mathrm{C}(G/H)$ for all open subgroups $H \subseteq G$.*
Furthermore, an object of $\mathcal{B}^G$ is compact$_{\aleph_1}$ if and only if it is dualisable if and only if it belongs to the thick subcategory generated by $\mathrm{C}(G/H)$ for open subgroups $H \subseteq G$.

Proof. The second statement about compact$_{\aleph_1}$ objects in $\mathcal{B}^G$ follows from the first one and [7, Corollary 2.4], compare the proof of Proposition 3.13. Thus it suffices to prove that the objects $\mathrm{C}(G/H)$ for open subgroups already generate $\mathcal{B}^G$. For this, we use an isomorphism $G \cong \mathbb{T}^r \times F$ for some $r \geq 0$ and some finite cyclic subgroup F.

Let us first consider the special case $r = 0$, that is, G is a finite cyclic group. In this case, any subgroup of G is open and again cyclic. We observed above that cyclic groups have no non-trivial projective representations. Thus any elementary G-C*-algebra is Morita equivalent to $\mathrm{C}(G/H)$ for some open subgroup H

in G. Hence the assertion of the theorem is just the definition of $\mathcal{B}^G$ in this case.

If F is trivial, then the assertion follows from Theorem 3.5. Now we consider the general case where both F and $\mathbb{T}^r$ are non-trivial.

The Pontryagin dual $\hat{G}$ of G is isomorphic to the discrete group $\mathbb{Z}^r \times F$. If A is a G-C*-algebra, then $G \ltimes A$ carries a canonical action of $\hat{G}$ called the dual action. Similarly, $\hat{G} \ltimes A$ for a $\hat{G}$-C*-algebra A carries a canonical dual action of G. This provides functors $\mathrm{KK}^G \to \mathrm{KK}^{\hat{G}}$ and $\mathrm{KK}^{\hat{G}} \to \mathrm{KK}^G$. Baaj–Skandalis duality says that they are inverse to each other up to natural equivalence (see [2, Section 6]). Since both functors are triangulated, this is an equivalence of triangulated categories.

If A is type I, then so is $G \ltimes A$. Hence all objects in $\mathcal{B}^{\hat{G}} \subseteq \mathrm{KK}^{\hat{G}}$ are $\mathrm{KK}^{\hat{G}}$-equivalent to a $\hat{G}$-action on a type I C*-algebra by Theorem 3.10.

The group $\hat{G}$ is Abelian and hence satisfies a very strong form of the Baum–Connes conjecture: it has a dual Dirac morphism and $\gamma = 1$ in the sense of [22, Definition 8.1]. From this it follows that any $\hat{G}$-C*-algebra A belongs to the localising subcategory of $\mathrm{KK}^{\hat{G}}$ that is generated by $\mathrm{Ind}_{\hat{H}}^{\hat{G}} A$ for finite subgroups $\hat{H} \subseteq \hat{G}$ (this is shown as in the proof of [22, Theorem 9.3]).

The finite subgroups in $\mathbb{Z}^r \times \hat{F}$ are exactly the subgroups of $\hat{F}$, of course. Since we have induction in stages, we may assume $\hat{H} = \hat{F}$. Thus the subcategory of type I $\hat{G}$-C*-algebras is already generated by $\mathrm{Ind}_{\hat{F}}^{\hat{G}} A$ for type I $\hat{F}$-C*-algebras A. Since $\hat{F}$ is a finite cyclic group, the discussion above shows that the category of type I $\hat{F}$-C*-algebras A is already generated by $\mathrm{C}_0(\hat{F}/\hat{H})$ for subgroups $\hat{H} \subseteq \hat{F}$. Thus $\mathcal{B}^{\hat{G}}$ is generated by the $\hat{G}$-C*-algebras $\mathrm{Ind}_{\hat{F}}^{\hat{G}} \mathrm{C}_0(\hat{F}/\hat{H}) \cong \mathrm{C}_0(\hat{G}/\hat{H})$. The finite subgroups $\hat{H} \subseteq \hat{G}$ are exactly the orthogonal complements of (finite-index) open subgroups $H \subseteq G$.

Now $G \ltimes \mathrm{C}_0(G/H)$ is Morita equivalent to $\mathrm{C}^*(H) \cong \mathrm{C}_0(\hat{G}/\hat{H})$ for any open subgroup $H \subseteq G$, where $\hat{H} \subseteq \hat{G}$ denotes the orthogonal complement of H in $\hat{G}$. The dual action on $\mathrm{C}_0(\hat{G}/\hat{H})$ comes from the translation action of $\hat{G}$. Thus the G- and $\hat{G}$-C*-algebras $\mathrm{C}_0(G/H)$ and $\mathrm{C}_0(\hat{G}/\hat{H})$ correspond to each other via Baaj–Skandalis duality. We conclude that the G-C*-algebras $\mathrm{C}_0(G/H)$ for open subgroups $H \subseteq G$ generate $\mathcal{B}^G$. $\qquad\square$

Let G be topologically cyclic, say, $G \cong \mathbb{T}^r \times \mathbb{Z}/k$ for some $r \geq 0$, $k \geq 1$. Then open subgroups of G correspond to subgroups of $\mathbb{Z}/k$ and thus to divisors d of k. The representation ring of G is

$$\mathrm{R}(G) \cong \mathrm{R}(\mathbb{T}^r) \otimes \mathrm{R}(\mathbb{Z}/k) \cong \mathbb{Z}[x_1, \ldots, x_r, (x_1 \cdots x_r)^{-1}] \otimes \mathbb{Z}[t]/(t^k - 1). \quad (3.1)$$

Let

$$t^k - 1 = \prod_{d|k} \Phi_d(t)$$

be the decomposition into cyclotomic polynomials. Each factor Φ_d generates a minimal prime ideal of $\mathrm{R}(G)$, and these are all minimal prime ideals of $\mathrm{R}(G)$.

The localisation at (Φ_d) gives the field $\mathbb{Q}(\theta_d)(x_1, \ldots, x_r)$ of rational functions in r variables over the cyclotomic field $\mathbb{Q}(\theta_d)$, and the product of these localisations is the total ring of fractions of $\mathrm{R}(G)$,

$$S^{-1}\,\mathrm{R}(G) = \prod_{d|k} \mathbb{Q}(\theta_d)(x_1, \ldots, x_r).$$

(Compare Lemma 3.1.)

LEMMA 3.18. *Let $H \subsetneq G$ be a proper open subgroup. The canonical map*

$$\mathrm{R}(G) \to \mathrm{KK}_0^G(\mathrm{C}(G/H), \mathrm{C}(G/H))$$

from the exterior product in KK^G factors through the restriction map $\mathrm{R}(G) \to \mathrm{R}(H)$. The image of $\mathrm{C}(G/H)$ in the localisation of KK^G at the prime ideal (Φ_k) vanishes.

Proof. The exterior product of the identity map on $\mathrm{C}(G/H)$ and $\xi \in \mathrm{R}(G) \cong \mathrm{KK}_0^G(\mathbb{C}, \mathbb{C})$ is given by the geometric correspondence $G/H = G/H = G/H$ with the class $p^*(\xi) \in \mathrm{K}_G^0(G/H)$, where $p\colon G/H \to \mathrm{pt}$ is the constant map. Now identify $\mathrm{K}_G^0(G/H) \cong \mathrm{K}_H^0(\mathrm{pt}) \cong \mathrm{R}(H)$ and p^* with the restricton map $\mathrm{R}(G) \to \mathrm{R}(H)$ to get the first statement.
We have $H \cong \mathbb{T}^r \times \mathbb{Z}/d$ embedded via $(x, j) \mapsto (x, jk/d)$ into $G \cong \mathbb{T}^r \times \mathbb{Z}/k$. If $H \neq G$, then $d \neq k$. The restriction map $\mathrm{R}(G) \to \mathrm{R}(H)$ annihilates the polynomial $(t^k - 1)/\Phi_k = \prod_{d|k, d\neq k} \Phi_d$. This polynomial does not belong to the prime ideal (Φ_k) and hence becomes invertible in the localisation of $\mathrm{R}(G)$ at (Φ_k). Since an invertible endomorphism can only be zero on the zero object, $\mathrm{C}(G/H)$ becomes zero in the localisation of KK^G at (Φ_k).   □

3.2   Localisation of the bootstrap class

PROPOSITION 3.19. *Let $G \cong \mathbb{T}^r \times \mathbb{Z}/k$ be topologically cyclic. Let $\mathcal{B}_d^G$ be the thick subcategory of dualisable objects in the bootstrap class $\mathcal{B}^G \subseteq \mathrm{KK}^G$. Any object in the localisation of $\mathcal{B}_d^G$ at the prime ideal (Φ_k) in $\mathrm{R}(G)$ is isomorphic to a finite direct sum of suspensions of $\mathbb{C}$.*

Proof. By Theorem 3.17 an object of $\mathcal{B}^G$ is dualisable if and only if it belongs to the thick subcategory generated by $\mathrm{C}(G/H)$ for open subgroups $H \subseteq G$. Lemma 3.18 shows that all of them except $\mathbb{C} = \mathrm{C}(G/G)$ become zero when we localise at (Φ_k). Hence the image of $\mathcal{B}_d^G$ in the localisation is contained in the thick subcategory generated by $\mathbb{C}$. We must show that the objects isomorphic to a direct sum of suspensions of $\mathbb{C}$ already form a thick subcategory in the localisation of KK^G at (Φ_k).
The graded endomorphism ring of $\mathbb{C}$ in this localisation is

$$\mathrm{KK}_*^G(\mathbb{C}, \mathbb{C}) \otimes_{\mathrm{R}(G)} \mathrm{R}(G)_{(\Phi_k)} \cong \mathbb{Q}(\theta_k)(x_1, \ldots, x_r)[\beta, \beta^{-1}]$$

with β of degree two generating Bott periodicity. It is crucial that $\mathrm{KK}^G_*(\mathbb{C}, \mathbb{C}) \cong F[\beta, \beta^{-1}]$ for a field $F := \mathbb{Q}(\theta_k)(x_1, \ldots, x_r)$. The following argument only uses this fact.

We map a finite direct sum $A = \bigoplus_{i \in I} \mathbb{C}[\varepsilon_i]$ of suspensions of $\mathbb{C}$ to the $\mathbb{Z}/2$-graded F-vector spaces $V(A)$ with basis I and generators of degree ε_i. For two such direct sums, $\mathrm{KK}^G_0(A, B)$ is isomorphic to the space of grading-preserving F-linear maps $V(A) \to V(B)$ because this clearly holds for a single summand.

Now let $f \in \mathrm{KK}^G_0(A, B)$ and consider the associated linear map $V(f): V(A) \to V(B)$. Choose a basis for the kernel of $V(f)$ of homogeneous elements and extend it to a homogeneous basis for $V(A)$, and extend the resulting basis for the image of $V(f)$ to a homogeneous basis of $V(B)$. This provides isomorphisms $V(A) \cong V_0 \oplus V_1$, $V(B) \cong W_1 \oplus W_2$ such that $f|_{V_0} = 0$, $f(V_1) = W_1$ and $f|_{V_1}: V_1 \to W_1$ is an isomorphism. The chosen bases describe how to lift the $\mathbb{Z}/2$-graded vector spaces V_i and W_i to direct sums of suspensions of $\mathbb{C}$. Thus the map f is equivalent to a direct sum of three maps $f_0 \oplus f_1 \oplus f_2$ with $f_0: A_0 \to 0$ mapping to the zero object, f_1 invertible, and $f_2: 0 \to B_2$ with domain the zero object. The mapping cone of f_0 is the suspension of A_0, the cone of f_2 is B_2, and the cone of f_1 is zero. Hence the cone is again a direct sum of suspensions of $\mathbb{C}$. Furthermore, any idempotent endomorphism has a range object.

Thus the direct sums of suspensions of $\mathbb{C}$ already form an idempotent complete triangulated category. As a consequence, any object in the thick subcategory generated by $\mathbb{C}$ is isomorphic to a direct sum of copies of $\mathbb{C}$. $\qquad\square$

PROPOSITION 3.20. *Let G be a Hodgkin Lie group. Let $\mathcal{B}^G_d$ be the thick subcategory of dualisable objects in the bootstrap class $\mathcal{B}^G \subseteq \mathrm{KK}^G$. Any object in the localisation of $\mathcal{B}^G_d$ at S is isomorphic to a finite direct sum of suspensions of $\mathbb{C}$.*

Proof. Theorem 3.5 shows that $\mathcal{B}^G_d$ is the thick subcategory of KK^G generated by $\mathbb{C}$. The localisation $F := S^{-1}\mathrm{R}(G)$ is a field because G is connected, and the graded endomorphism ring of $\mathbb{C}$ in the localisation of KK^G at S is $F[\beta, \beta^{-1}]$ with β the generator of Bott periodicity. Now the argument is finished as in the proof of Proposition 3.19. $\qquad\square$

Remark 3.21. The localisations above use the groups $\mathrm{KK}^G(A, B) \otimes_{\mathrm{R}(G)} S^{-1}\mathrm{R}(G)$ for some multiplicatively closed subset $S \subseteq \mathrm{End}(\mathbb{1}) = R(G)$, following [17]. A drawback of this localisation is that the canonical functor $\mathrm{KK}^G \to S^{-1}\mathrm{KK}^G$ does not commute with (countable) coproducts. This is why Propositions 3.19 and 3.20 are formulated only for $\mathcal{B}^G_d$ and not for all of $\mathcal{B}^G$.

Another way to localise $\mathcal{B}^G$ at S is described in [7, Theorem 2.33]. Both localisations agree on $\mathcal{B}^G_d$ by [7, Theorem 2.33.h]. The construction in [7] has the advantage that the canonical functor from KK^G to this localisation commutes with $\mathrm{small}_{\aleph_1}$ (that is, countable) coproducts. Hence analogues of Propositions

3.19 and 3.20 hold for the whole bootstrap category $\mathcal{B}^G$, with small$_{\aleph_1}$ coproducts of suspensions of $\mathbb{C}$ instead of finite direct sums of suspensions of $\mathbb{C}$.

3.3   The Lefschetz index computation using localisation

Now we have all the tools available to formulate and prove a Lefschetz index formula for general compact Lie groups. We first prove Theorem 3.4, which deals with endomorphisms of objects in the thick subcategory generated by $\mathbb{C}$. Then we formulate and prove the general Lefschetz index formula.

Proof of Theorem 3.4. Since A belongs to the thick subcategory generated by $\mathbb{C}$, it is dualisable in KK^G by Proposition 3.13. Hence $\mathrm{tr}(f) \in \mathrm{R}(G)$ is defined for $f \in \mathrm{KK}_0^G(A, A)$.

The image of $\mathrm{tr}(f)$ in $S^{-1}\mathrm{R}(G)$ is the Lefschetz index of the image of f in the localisation of KK^G at S. The localisation $S^{-1}\mathrm{R}(G)$ is a product of fields. It is more convenient to compute each component separately. This means that we localise at larger multiplicatively closed subsets $\bar{S}$ such that $\bar{S}^{-1}\mathrm{R}(G)$ is one of the factors of $S^{-1}\mathrm{R}(G)$. In this localisation, the endomorphisms of $\mathbb{C}$ form a field again, not a product of fields. If our trace formula holds for all these localisations, it also holds for $S^{-1}\mathrm{R}(G)$.

Since the endomorphisms of $\mathbb{C}$ form a field, the same argument as in the proof of Proposition 3.19 show that, in this localisation, A is isomorphic to a finite sum of copies of suspensions of $\mathbb{C}$. Write $A \cong \bigoplus_{i=1}^n A_i$ with $A_i \cong \mathbb{C}[\varepsilon_i]$ in $S^{-1}\mathrm{KK}^G$ for some $\varepsilon_i \in \mathbb{Z}/2$. Then f becomes a matrix (f_{ij}) with $f_{ij} \in S^{-1}\mathrm{KK}_0^G(A_j, A_i)$. The dual of $A_i \cong \mathbb{C}[\varepsilon_i]$ is $A_i^* \cong \mathbb{C}[\varepsilon_i] \cong A_i$, and the unit and counit of adjunction $\mathbb{C} \leftrightarrows \mathbb{C}[\varepsilon_i] \otimes \mathbb{C}[\varepsilon_i]$ are the canonical isomorphism and its inverse with sign $(-1)^{\varepsilon_i}$, respectively; the sign is necessary because the exterior product is *graded* commutative. Hence the dual of A is isomorphic to A, with unit and counit

$$\mathbb{C} \leftrightarrows A \otimes A \cong \bigoplus_{i,j=1}^n A_i \otimes A_j$$

the sum of the canonical isomorphisms $\mathbb{C} \leftrightarrows A_i \oplus A_i$, up to signs, and the zero maps $\mathbb{C} \leftrightarrows A_i \oplus A_j$ for $i \neq j$. Thus the Lefschetz index of f is the sum $\sum_{i=1}^n (-1)^{\varepsilon_i} f_{ii}[\varepsilon_i]$ as an element in $S^{-1}\mathrm{KK}_0^G(\mathbb{C}, \mathbb{C})$. This is exactly the supertrace of f acting on $S^{-1}\mathrm{K}_*^G(A) \cong \bigoplus_{i=1}^n S^{-1}\mathrm{R}(G)[\varepsilon_i]$. $\qquad\square$

Let G be a general compact Lie group. Let C_G denote the set of conjugacy classes of Cartan subgroups of G in the sense of [28, Definition 1.1]. Such subgroups correspond bijectively to conjugacy classes of cyclic subgroups in the finite group G/G^0, where G^0 denotes the connected component of the identity element in G. Thus C_G is a non-empty, finite set, and it has a single element if and only if G is connected.

The support of a prime ideal $\mathfrak{p}$ in $\mathrm{R}(G)$ is defind in [28] as the smallest subgroup H such that $\mathfrak{p}$ comes from a prime ideal in $\mathrm{R}(H)$ via the restriction map $\mathrm{R}(G) \to \mathrm{R}(H)$. Given any Cartan subgroup H, there is a unique minimal

prime ideal with support H, and this gives a bijection between C_G and the set of minimal prime ideals in $\mathrm{R}(G)$ (see [28, Proposition 3.7]).

More precisely, if $H \subseteq G$ is a Cartan subgroup, then H is topologically cyclic and hence $H \cong \mathbb{T}^r \times \mathbb{Z}/k$ for some $r \geq 0$, $k \geq 1$. We described a prime ideal (Φ_k) in $\mathrm{R}(H)$ before Lemma 3.18, and its preimage in $\mathrm{R}(G)$ is a minimal prime ideal $\mathfrak{p}_H$ in $\mathrm{R}(G)$.

The total ring of fractions $S^{-1}\,\mathrm{R}(G)$ is a product of fields by Lemma 3.1. We can make this more explicit:

$$S^{-1}\,\mathrm{R}(G) \cong \prod_{H \in C_G} F(\mathrm{R}(G)/\mathfrak{p}_H),$$

where $F(\sqcup)$ denotes the field of fractions for an integral domain.

DEFINITION 3.22. Let A be dualisable in $\mathcal{B}^G \subseteq \mathrm{KK}^G$, let $\varphi \in \mathrm{KK}^G_0(A, A)$, and let $H \in C_G$. Let $F := F(\mathrm{R}(G)/\mathfrak{p}_H)$ and let $\mathrm{K}_H(A) := \mathrm{K}^H_*(A) \otimes_{\mathrm{R}(H)} F$, considered as a $\mathbb{Z}/2$-graded F-vector space. Let $\mathrm{K}_H(\varphi)$ be the grading-preserving F-linear endomorphism of $\mathrm{K}_H(A)$ induced by φ.

THEOREM 3.23. *Let A be dualisable in $\mathcal{B}^G \subseteq \mathrm{KK}^G$, let $\varphi \in \mathrm{KK}^G_0(A, A)$, and let $H \in C_G$. Then the image of $\mathrm{tr}(\varphi)$ in $F(\mathrm{R}(G)/\mathfrak{p}_H)$ is the supertrace of $\mathrm{K}_H(\varphi)$.*

Proof. The map $\mathrm{R}(G) \to F(\mathrm{R}(G)/\mathfrak{p}_H)$ factors through the restriction homomorphism $\mathrm{R}(G) \to \mathrm{R}(H)$ because $\mathfrak{p}_H$ is supported in H. Restricting the group action to H maps the bootstrap category in KK^G into the bootstrap category in KK^H by Corollary 3.12, and commutes with taking Lefschetz indices because restriction is a tensor functor. Hence we may replace G by H and take $\varphi \in \mathrm{KK}^H_0(A, A)$ throughout.

Since H is topologically cyclic, Proposition 3.19 applies. It shows that in the localisation of KK^H at $\mathfrak{p}_H$, any dualisable object in $\mathcal{B}^G$ becomes isomorphic to a finite direct sum of suspensions of $\mathbb{C}$. Now the argument continues as in the proof of Theorem 3.4 above. $\qquad\square$

4   HATTORI–STALLINGS TRACES

Before we found the above approach through localisation, we developed a different trace formula where, in the case of a Hodgkin Lie group, the trace is identified with the Hattori–Stallings trace of the $\mathrm{R}(G)$-module map $\mathrm{K}^G_*(f)$ on $\mathrm{K}^G_*(A)$. We briefly sketch this alternative formula here, although the localisation approach above seems much more useful for computations. The Hattori–Stallings trace has the advantage that it obviously belongs to $\mathrm{R}(G)$.

We work in the general setting of a tensor triangulated category $(\mathcal{T}, \otimes, \mathbb{1})$. We assume that $\mathcal{T}$ satisfies additivity of traces, that is:

ASSUMPTION 4.1. *Let $A \to B \to C \to A[1]$ be an exact triangle in $\mathcal{T}$ and assume that A and B are dualisable. Assume also that the left square in the*

following diagram

$$
\begin{array}{ccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\
\downarrow{\scriptstyle f_A} & & \downarrow{\scriptstyle f_B} & & \vdots{\scriptstyle f_C} & & \downarrow{\scriptstyle f_A[1]} \\
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1]
\end{array}
$$

commutes. Then C is dualisable and there is an arrow $f_C \colon C \to C$ such that the whole diagram commutes and $\operatorname{tr}(f_C) - \operatorname{tr}(f_B) + \operatorname{tr}(f_A) = 0$.

Additivity of traces holds in the bootstrap category $\mathcal{B}^G \subseteq \mathrm{KK}^G$. The quickest way to check this is the localisation formula for the trace in Theorem 3.23. It shows that $\mathcal{B}^G$ satisfies even more: $\operatorname{tr}(f_C) - \operatorname{tr}(f_B) + \operatorname{tr}(f_A) = 0$ holds for *any* arrow f_C that makes the diagram commute.

There are several more direct ways to verify additivity of traces, but all require significant work which we do not want to get into here. The axioms worked out by J. Peter May in [19] are lengthy and therefore rather unpleasant to check by hand. In a previous manuscript we embedded the localising subcategory of KK^G generated by $\mathbb{C}$ into a category of module spectra. Since additivity is known for categories of module spectra, this implies the required additivity result at least for this smaller subcategory. Another way would be to show that additivity of traces follows from the derivator axioms and to embed KK^G into a triangulated derivator.

In the following, we will just assume additivity of traces and use it to compute the trace. Let

$$
R := \mathcal{T}_*(\mathbb{1}, \mathbb{1}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{T}_n(\mathbb{1}, \mathbb{1})
$$

be the graded endomorphism ring of the tensor unit. It is graded-commutative provided $\mathcal{T}$ satisfies some very basic compatibility axioms; see [29] for details. If A is any object of $\mathcal{T}$, then $M(A) := \mathcal{T}_*(\mathbb{1}, A) = \bigoplus_{n \in \mathbb{Z}} \mathcal{T}_n(\mathbb{1}, A)$ is an R-module in a canonical way, and an endomorphism $f \in \mathcal{T}_n(A, A)$ yields a degree-n endomorphism $M(f)$ of $M(A)$. We will prove in Theorem 4.2 below that, under some assumptions, the trace of f equals the Hattori–Stallings trace of $M(f)$ and, in particular, depends only on $M(f)$.

Before we can state our theorem, we must define the Hattori–Stallings trace for endomorphisms of graded modules over graded rings. This is well-known for ungraded rings (see [3]). The grading causes some notational overhead. Let R be a (unital) graded-commutative graded ring. A finitely generated free R-module is a direct sum of copies of $R[n]$, where $R[n]$ denotes R with degree shifted by n, that is $R[n]_i = R_{n+i}$. Let $F \colon P \to P$ be a module endomorphism of such a free module, let us assume that F is homogeneous of degree d. We use an isomorphism

$$
P \cong \bigoplus_{i=1}^{r} R[n_i] \tag{4.1}
$$

to rewrite F as a matrix $(f_{ij})_{1 \le i,j \le r}$, where $f_{ij} \colon R[n_j] \to R[n_i]$ are R-module homomorphisms of degree d. The entry f_{ij} is given by right multiplication by some element of R of degree $n_i - n_j + d$. The (super)trace $\operatorname{tr} F$ is defined as

$$\operatorname{tr} F := \sum_{i=1}^{r} (-1)^{n_i} \operatorname{tr} f_{ii};$$

this is an element of R of degree d.

It is straightforward to check that $\operatorname{tr} F$ is well-defined, that is, independent of the choice of the isomorphism in (4.1). Here we use that the degree-zero part of R is central in R (otherwise, we still get a well-defined element in the commutator quotient $R_d/[R_d, R_0]$). Furthermore, if we shift the grading on P by n, then the trace is multiplied by the sign $(-1)^n$ – it is a supertrace.

If P is a finitely generated projective graded R-module, then $P \oplus Q$ is finitely generated and free for some Q, and for an endomorphism F of P we let

$$\operatorname{tr} F := \operatorname{tr}(F \oplus 0 \colon P \oplus Q \to P \oplus Q).$$

This does not depend on the choice of Q.

A *finite projective resolution* of a graded R-module M is a resolution

$$\cdots \to P_\ell \xrightarrow{d_\ell} P_{\ell-1} \xrightarrow{d_{\ell-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \tag{4.2}$$

of finite length by finitely generated projective graded R-modules P_j. *We assume that the maps d_j have degree one* (or at least odd degree). Assume that M has such a resolution and let $f \colon M \to M$ be a module homomorphism. Lift f to a chain map $f_j \colon P_j \to P_j$, $j = 0, \ldots, \ell$. We define the *Hattori–Stallings trace* of f as

$$\operatorname{tr}(f) = \sum_{j=0}^{\ell} \operatorname{tr}(f_j).$$

It may be shown that this trace does not depend on the choice of resolution. It is important for this that we choose d_j of degree one. Since shifting the degree by one alters the sign of the trace of an endomorphism, the sum in the definition of the trace becomes an *alternating* sum when we change conventions to have even-degree boundary maps d_j. Still the trace changes sign when we shift the degree of M.

THEOREM 4.2. *Let $F \in \mathcal{T}(A, A)$ be an endomorphism of some object A of $\mathcal{T}$. Assume that A belongs to the localising subcategory of $\mathcal{T}$ generated by $\mathbb{1}$. If the graded R-module $M(A) := \mathcal{T}_*(\mathbb{1}, A)$ has a finite projective resolution, then A is dualisable in $\mathcal{T}$ and the trace of F is equal to the Hattori–Stallings trace of the induced module endomorphism $\mathcal{T}_*(\mathbb{1}, f)$ of $M(A)$.*

Proof. Our main tool is the phantom tower over A, which is constructed in [20]. We recall some details of this construction.

Let $M^\perp$ be the functor from finitely generated projective R-modules to $\mathcal{T}$ defined by the adjointness property $\mathcal{T}(M^\perp(P), B) \cong \mathcal{T}(P, M(B))$ for all $B \in \mathcal{T}$. The functor $M^\perp$ maps the free rank-one module R to $\mathbb{1}$, is additive, and commutes with suspensions; this determines $M^\perp$ on objects. Since $R = \mathcal{T}_*(\mathbb{1}, \mathbb{1})$, $\mathcal{T}_*(M^\perp(P_1), M^\perp(P_2))$ is isomorphic (as a graded Abelian group) to the space of R-module homomorphisms $P_1 \to P_2$. Furthermore, we have canonical isomorphisms $M\big(M^\perp(P)\big) \cong P$ for all finitely generated projective R-modules P. By assumption, $M(A)$ has a finite projective resolution as in (4.2). Using $M^\perp$, we lift it to a chain complex in $\mathcal{T}$, with entries $\hat{P}_j := M^\perp(P_j)$ and boundary maps $\hat{d}_j := M^\perp(d_j)$ for $j \geq 1$. The map $\hat{d}_0 \colon \hat{P}_0 \to A$ is the pre-image of d_0 under the adjointness isomorphism $\mathcal{T}(M^\perp(P), B) \cong \mathcal{T}(P, M(B))$. We get back the resolution of modules by applying M to the chain complex $(\hat{P}_j, \hat{d}_j)$.

Next, it is shown in [20] that we may embed this chain complex into a diagram

$$A = N_0 \xrightarrow{\;\iota_0^1\;} N_1 \xrightarrow{\;\iota_1^2\;} N_2 \xrightarrow{\;\iota_2^3\;} N_3 \xrightarrow{\hspace{1.5cm}} \cdots$$

$$(4.3)$$

where the wriggly lines are maps of degree one; the triangles involving $\hat{d}_j$ commute; and the other triangles are exact. This diagram is called the *phantom tower* in [20].

Since $\hat{P}_j = 0$ for $j > \ell$, the maps ι_j^{j+1} are invertible for $j > \ell$. Furthermore, a crucial property of the phantom tower is that these maps ι_j^{j+1} are *phantom maps*, that is, they induce the zero map on $\mathcal{T}_*(\mathbb{1}, \sqcup)$. Together, these facts imply that $M(N_j) = 0$ for $j > \ell$. Since we assumed $\mathbb{1}$ to be a generator of $\mathcal{T}$, this further implies $N_j = 0$ for $j > \ell$. Therefore, $A \in \langle \mathbb{1} \rangle$, so that A is dualisable as claimed.

Next we recursively extend the endomorphism F of $A = N_0$ to an endomorphism of the phantom tower. We start with $F_0 = F \colon N_0 \to N_0$. Assume $F_j \colon N_j \to N_j$ has been constructed. As in [20], we may then lift F_j to a map $\hat{F}_j \colon \hat{P}_j \to \hat{P}_j$ such that the square

$$\begin{array}{ccc} \hat{P}_j & \xrightarrow{\;\pi_j\;} & N_j \\ {\scriptstyle \hat{F}_j} \downarrow & & \downarrow {\scriptstyle F_j} \\ \hat{P}_j & \xrightarrow{\;\pi_j\;} & N_j \end{array}$$

commutes. Now we apply additivity of traces (Assumption 4.1) to construct an endomorphism $F_{j+1} \colon N_{j+1} \to N_{j+1}$ such that $(\hat{F}_j, F_j, F_{j+1})$ is a triangle morphism and $\mathrm{tr}(F_j) = \mathrm{tr}(\hat{F}_j) + \mathrm{tr}(F_{j+1})$. Then we repeat the recursion step with F_{j+1} and thus construct a sequence of maps F_j. We get

$$\mathrm{tr}(F) = \mathrm{tr}(F_0) = \mathrm{tr}(\hat{F}_0) + \mathrm{tr}(F_1) = \cdots = \mathrm{tr}(\hat{F}_0) + \cdots + \mathrm{tr}(\hat{F}_\ell) + \mathrm{tr}(F_{\ell+1}).$$

Since $N_{\ell+1} = 0$, we may leave out the last term.

Finally, it remains to observe that the trace of $\hat{F}_j$ as an endomorphism of $\hat{P}_j$ agrees with the trace of the induced map on the projective module P_j. Since both traces are additive with respect to direct sums of maps, the case of general finitely generated projective modules reduces first to free modules and then to free modules of rank one. Both traces change by a sign if we suspend or desuspend once, hence we reduce to the case of endomorphisms of $\mathbb{1}$, which is trivial. Hence the computation above does indeed yield the Hattori–Stallings trace of $M(A)$ as asserted. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.3. Note that if a module has a finite projective resolution, then it must be finitely generated. Conversely, if the graded ring R is *coherent* and *regular*, then any finitely generated module has a finite projective resolution. (Regular means that every finitely generated module has a finite *length* projective resolution; coherent means that every finitely generated homogeneous ideal is finitely presented – for instance, this holds if R is (graded) Noetherian; coherence implies that any finitely generated graded module has a resolution by finitely generated projectives.)

Moreover, if R is coherent then the finitely presented R-modules form an abelian category, and this implies (by an easy induction on the triangular length of A) that for every $A \in \langle \mathbb{1} \rangle = (\langle \mathbb{1} \rangle_{\mathrm{loc}})_d$ the module $M(A)$ is finitely presented and thus *a fortiori* finitely generated. If R is also regular, each such $M(A)$ has a finite projective resolution.

In conclusion: if R is regular and coherent, an object $A \in \langle \mathbb{1} \rangle_{\mathrm{loc}}$ is dualisable if and only if the graded R-module $M(A)$ has a finite projective resolution.

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