Eisenstein Series and the Trace Formula for GL(2) over a Function Field

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Abstract. We write out and prove the trace formula for a convolution operator on the space of cusp forms on GL(2) over the function field F of a smooth projective absolutely irreducible curve over a finite field. The proof – which follows Drinfeld – is complete and all terms in the formula are explicitly computed. The structure of the homogeneous space $\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A})$ is studied in section 2 by means of locally free sheaves of $\mathcal{O}_X$-modules. Section 3 deals with the regularization and computation of the geometric terms, over conjugacy classes. Section 4 develops the theory of intertwining operators and Eisenstein Series, and the trace formula is proven in section 5.

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1 Introduction and statement of the Trace Formula

1.1 Introduction

The (non-invariant) trace formula for GL(2) over a number field was stated and its proof sketched in chapter 15 of the influential book of Jacquet and Langlands [JL70] of 1970. It was used there for comparison of automorphic representations of the multiplicative group of a quaternion algebra, with automorphic representations of GL(2).

Drinfeld used the trace formula for GL(2) over a function field F to prove Langlands' conjecture for $\mathrm{GL}(2,F)$, and to count in [D81] the number of two

dimensional irreducible representations of the fundamental group of a smooth projective geometrically irreducible curve X over a finite field. To check the statement of the trace formula of [JL70] in the function field case, Drinfeld gave a detailed (but unpublished) proof. It differs from the one sketched in [JL70]. It is this proof of Drinfeld which is given in this paper.
The main reason why this proof is still interesting is the elementary and unconventional treatment of Eisenstein series (see subsections 4.7-4.8 below), and the computation of traces in the spirit of Tate [T68], see subsection 5.2. In both cases it is based on a "baby model" (see Proposition 4.31, Corollary 4.32, Lemma 5.11), which cries out for generalization.

Let us describe the contents of this article.
The trace formula itself is stated in subsection 1.2 with a few comments. More comments, including informal ones, are given in section 3.
Section 2 contains a dictionary between the language of adèles and the language of vector bundles on the smooth projective curve X corresponding to F. In particular, the set of rank n vector bundles on X is identified with $\mathrm{GL}(n,F)\backslash \mathrm{GL}(n,\mathbb{A})/\mathrm{GL}(n,O_{\mathbb{A}})$, where $O_{\mathbb{A}} \subset \mathbb{A}$ is the ring of integral adèles. This dictionary goes back to A. Weil [W38], although in an older language. It underlies Drinfeld's Geometric Langlands program [BD].
The terms which appear in the geometric part of the trace formula – orbital integrals and weighted orbital integrals – are estimated and regularized in section 3.
In section 4 intertwining operators, Eisenstein series, and L-functions are introduced. The rationality of the intertwining operator $M(\mu_1,\mu_2,t)$ and the functional equation $M^2 = 1$ are first proven using local computations: normalization of the intertwining operators by L-functions and ε-factors, and the functional equation of the L-functions.
In subsections 4.7-4.8 these facts are proven using an alternative, global approach. The ideas might go back to Selberg. But technically the exposition is quite different and more elementary: in the case of function fields the analytic problems disappear.
The trace formula is proven in section 5. The logarithmic derivative of the intertwining operator appears as a result of a computation of the trace of some operator in a power series space, see Lemma 5.11. This computation is probably related to Tate's article [T68].

Here are some questions.
1. Could the methods of subsections 4.7-4.8 and section 5 be extended to prove the functional equation for Eisenstein series, and the trace formula, for an arbitrary reductive group over a function field?
2. Is there a modification of the technique from subsections 4.7-4.8 that would work in the case of number fields, e.g., for $\mathrm{GL}(2,\mathbb{Q})$? One could try to replace the space of formal power series used in subsections 4.7-4.8 by some space of holomorphic functions.
3. What is the precise relationship between Lemma 5.11 and Tate's [T68]?

4. What is the relationship between the approach to Eisenstein series of subsections 4.7-4.8, and the classical approaches: that of Selberg-Langlands-Arthur, and that of scattering theory (see [FP72] or [LP76])?

This author's initial motivation to write out Drinfeld's expression and proof of the trace formula for GL(2) over a function field stems from his search for higher rank analogues of Drinfeld's formula [D81]. This led us to count with Deligne [DF13] the number of rank n (≥ 2) local systems with principal unipotent local monodromy at least at two places. There we use the trace formula in the compact quotient case, and the transfer of automorphic representations from a compact form to GL(n). This explains the condition: "at least at two places". The case of [D81] is rank $n = 2$, no monodromy. To complete the study of [D81] and of [DF13] in rank two one has to consider the case of principal unipotent local monodromy at a single place. This is done in [F], using the explicit computations of the trace formula for GL(2) over a function field of the present work. This was our initial motivation to write out this formula. Drinfeld's proof in the case of rank two, no ramification, is also given in [F].

Of course there are numerous expositions of the trace formula of [JL70], e.g. [GJ79], geared to explain the lifting application of [JL70], mainly in the number field case. But none computes explicitly (and accurately, cf. [D81]) all the terms which appear in the trace formula. The latter is precisely what is needed for the counting applications of [D81] and [F]. An attempt at a complete exposition of the computations for GL(2) in the number field case is at [AFOO].

Of course the trace formula of [JL70] was generalized to the higher rank case by Arthur, see e.g. [A05], in the number field case, and by Lafforgue, see e.g. [Lf97], in the function field case. But the important applications of these works did not require explicit evaluation of all the terms which appear in the trace formula, so our results are not included in those of [Lf97], even in the case of GL(2) considered here.

In the number field case, the Remark on p. 112 of [A05] states: "As a matter of fact, it is only in the case of GL(2) that the general coefficients have been evaluated. It would be very interesting to understand them better in other examples, although this does not seem to be necessary for presently conceived applications of the trace formula". Indeed the applications of [D81], [DF13], [F] – counting rather than comparing – are of different nature than those of [JL70], [A05], [Lf97], where most terms can be erased a-priori in the comparison so they need not be computed.

To repeat what is explained above, we also think the approach of subsections 4.7-4.8 and section 5 is original, substantially different from the currently known methods (which are developed in [A05], [Lf97]), interesting and warrants further development.

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1.2 Statement of the Trace Formula

Let us write the trace formula for GL(2) over a function field F of a smooth projective geometrically connected curve X over a finite field $\mathbb{F}_q$, and a test function f in $C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$ (subscript c for "compactly supported", superscript ∞ for "locally constant", $\mathbb{A}$ denotes the ring of adèles of F).

Let r_0 be the representation of $\mathrm{GL}(2,\mathbb{A})$ by right translation on the space $A_{0,\alpha}$ of cusp forms on $\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$, and $r_0(f) = \int f(g)r_0(g)dg$ ($g \in \mathrm{GL}(2,\mathbb{A})$) the convolution operator; $dg = \otimes_v dg_v$ is a Haar measure. Here α is a fixed idèle of degree 1, whose components are almost all equal to 1.

A cusp form is a function $\phi : \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}) \to E$ (E is a fixed algebraically closed subfield of $\mathbb{C}$) which is invariant on the right by some open compact subgroup of $\mathrm{GL}(2,\mathbb{A})$, and $\int_{N(F)\backslash N(\mathbb{A})} \phi(nx)dn = 0$ for all x in $\mathrm{GL}(2,\mathbb{A})$. Here N denotes the unipotent upper triangular subgroup of GL(2). We also write A for the diagonal subgroup, and $A' = A - Z$ where Z is the center of GL(2). By a well known result of G. Harder, when F is a function field (but not a number field) a cusp form is compactly supported modulo $Z(\mathbb{A})$.

Theorem 1.1. *For any $f \in C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$ we have* $\mathrm{tr}\, r_0(f) = \sum_{1\le i\le 8} S_i(f)$. *The terms are:*

$$S_1(f) = \left|\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})\right| \sum_{\gamma\in\alpha^{\mathbb{Z}}\cdot F^\times} f(\gamma).$$

$$S_2(f) = \sum_{F_2} S_{2,F_2}(f),$$

$$S_{2,F_2}(f) = |\operatorname{Aut}_F F_2|^{-1} \sum_{\gamma\in\alpha^{\mathbb{Z}}(F_2-F)} \int_{\mathrm{GL}(2,\mathbb{A})/\alpha^{\mathbb{Z}}\cdot F_2^\times} f(x\gamma x^{-1})dx.$$

Here F_2 ranges over the set of isomorphism classes of quadratic extensions of the field F. For each F_2 we fix an embedding $F_2 \hookrightarrow M(2,F)$ into the ring of 2×2 matrices over F.

$$S_3(f) = \sum_{\gamma\in\alpha^{\mathbb{Z}}\cdot A'(F)} \int_{A(\mathbb{A})\backslash \mathrm{GL}(2,\mathbb{A})} f(x^{-1}\gamma x)v(x)dx.$$

Any $x \in \mathrm{GL}(2,\mathbb{A})$ can be written in the form ank, $a \in A(\mathbb{A})$, $k \in \mathrm{GL}(2,O_{\mathbb{A}})$, $n = \left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right)$, b is determined uniquely by x up to $b \mapsto ub + w$, $u \in O_{\mathbb{A}}^\times$, $w \in O_{\mathbb{A}}$. Put $v(x) = \sum_v \log_q(\max(1, |b_v|_v))$.

$$S_4(f) = \sum_{a\in F^\times\alpha^{\mathbb{Z}}} \tilde\theta_{a,f}(1), \qquad \tilde\theta_{a,f}(t) = \frac{1}{2}(\theta_{a,f}(t) + \theta_{a,f}(t^{-1})),$$

$$\theta_{a,f}(t) = \int_{F^\times\alpha^{\mathbb{Z}}N(F)\backslash \mathrm{GL}(2,\mathbb{A})} f\left(x^{-1}\left(\begin{smallmatrix}a & a\\ 0 & a\end{smallmatrix}\right)x\right) t^{\mathrm{ht}^+(x)}dx,$$

$\mathrm{ht}^+ : \mathrm{GL}(2,\mathbb{A}) \to \mathbb{Z}$ *is defined by* $\mathrm{ht}^+\left(\left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right) k\right) = \deg a - \deg b$ $(k \in \mathrm{GL}(2, O_{\mathbb{A}})$; $a, b \in \mathbb{A}^\times$; $c \in \mathbb{A})$.

$$S_5(f) = \frac{-1}{4\pi i} \sum_{\mu_1,\mu_2} \oint_{|z|=1} \mathrm{tr}\, I(\mu_1\nu_z, \mu_2\nu_{z^{-1}}, f) \frac{m'(\mu_1/\mu_2, z)}{m(\mu_1/\mu_2, z)} 2z dz.$$

Here $m(\mu, z) = L(\mu, z)/L(\mu, z/q)$. *The* μ_1, μ_2 *range over the set of characters of* $\mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}}$, $\nu_z(x) = z^{\deg(x)}$. *Also* $I(\mu_1, \mu_2)$ *is the space of right locally constant functions* ϕ *on* $\mathrm{GL}(2,\mathbb{A})$ *with*

$$\phi\left(\left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right) x\right) = |a/b|^{1/2} \mu_1(a)\mu_2(b)\phi(x) \qquad (x \in \mathrm{GL}(2,\mathbb{A});\ \ a, b \in \mathbb{A}^\times;\ \ c \in \mathbb{A}).$$

It is a $\mathrm{GL}(2,\mathbb{A})$*-module by right translation, and* $\mathrm{tr}\, I(\mu_1\nu_z, \mu_2\nu_{z^{-1}}, f)$ *is the trace of the indicated convolution operator.*

$$S_6(f) = \frac{-1}{4\pi i} \sum_{\mu_1,\mu_2} \oint_{|z|=1} \mathrm{tr}[I(\mu_1\nu_z, \mu_2\nu_{z^{-1}}, f) \cdot R(\mu_1, \mu_2, z)^{-1} \frac{d}{dz} R(\mu_1, \mu_2, z)] dz.$$

Notations are as in $S_5(f)$, *and* $R(\mu_1, \mu_2, z) : I(\mu_1\nu_z, \mu_2\nu_{z^{-1}}) \to I(\mu_2\nu_{z^{-1}}, \mu_1\nu_z)$ *is an operator, rational in* z, *defined as a product* $\otimes_v R(\mu_{1v}, \mu_{2v}, z_v)$, $z_v = z^{\deg(v)}$. *The product is well defined as the local operator maps the function in the source whose restriction to* $\mathrm{GL}(2, O_v)$ *is* 1 *to such function in the target. Further,* $R(\mu_{1v}, \mu_{2v}, z)$ *is defined to be*

$$[L(\mu_{1v}/\mu_{2v}, z^2/q_v)/L(\mu_{1v}/\mu_{2v}, z^2)] M(\mu_{1v}, \mu_{2v}, z).$$

The operator $M(\mu_{1v}, \mu_{2v}, z) = M(\mu_{1v}\nu_z, \mu_{2v}\nu_{z^{-1}})$ *is defined first by an integral*

$$\phi \mapsto \int \phi\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) x\right) dy \quad \textit{if} \quad |(\mu_{1v}/\mu_{2v})(\boldsymbol{\pi}_v) z^2| < 1,$$

then by analytic continuation, as it is a rational function in z. *The operators* $I(\mu_1\nu_z, \mu_2\nu_{z^{-1}}, f)$ *and* $R(\mu_1, \mu_2, z)$ *are considered as operators on*

$$I_0(\mu_1, \mu_2) = \{\phi \in C^\infty(\mathrm{GL}(2, O_{\mathbb{A}}));\ \phi\left(\left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right) x\right) = \mu_1(a)\mu_2(b)\phi(x);$$
$$x \in \mathrm{GL}(2, O_{\mathbb{A}}),\ \ a, b \in O_{\mathbb{A}}^\times;\ \ c \in O_{\mathbb{A}}\}.$$

$$S_7(f) = \frac{1}{4} \sum_{\mu} \mathrm{tr}\, I(\mu, \mu, f), \qquad S_8(f) = -\sum_{\mu} \int_{\mathrm{GL}(2,\mathbb{A})} f(x)\mu(\det x) dx.$$

Both sums range over all characters μ *of* $\mathbb{A}^\times/F^\times \cdot \alpha^{2\mathbb{Z}}$. *The sum of* S_8 *is over all automorphic one dimensional representations* $(\mu \circ \det)$ *of* $\alpha^{\mathbb{Z}} \backslash \mathrm{GL}(2,\mathbb{A})$. *The integral there represents the trace of the convolution operator associated with* f.

The terms $S_1(f)$ and $S_2(f)$ are finite by Proposition 3.5, 3.6, 3.9. The argument used in the proof of Proposition 3.9 shows that for any $\gamma \in \alpha^{\mathbb{Z}}(A(F) - Z(F))$ the function $x \mapsto f(x^{-1}\gamma x)$ on $A(\mathbb{A})\backslash \mathrm{GL}(2,\mathbb{A})$ has compact support, hence the integral in $S_3(f)$ converges.

By Proposition 3.11 the function $\theta_{a,f}(t)$ is rational and may have at $t = 1$ a pole of order at most 1, for each $a \in \mathbb{A}^\times$. Hence $\tilde{\theta}_{a,f}(t)$ is regular at $t = 1$. From Proposition 3.5 it follows that the sums in $S_3(f)$ and $S_4(f)$ are finite, so these terms are well defined.

For any $f = \otimes f_v$ in $C_c^\infty(\mathrm{GL}(2, \mathbb{A}))$, the operator $I(\mu_1, \mu_2, f)$ is zero unless μ_i are unramified at each v where f_v is $\mathrm{GL}(2, O_v)$ biinvariant. This implies that the sums in $S_i(f)$ $(5 \le i \le 8)$ are finite, for a given f. To see that $S_5(f)$ and $S_6(f)$ are well defined, note that the rational functions $m(\mu, t)$, $R(\mu_1, \mu_2, t)$, $R(\mu_1, \mu_2, t)^{-1}$ are regular on $|t| = 1$ for all characters μ, μ_1, μ_2 of $\mathbb{A}^\times / F^\times \cdot \alpha^{\mathbb{Z}}$. For $m(\mu, t)$ this follows from Proposition 4.11, for R and R^{-1} from Corollary 4.28.

The distributions [linear forms on $C_c^\infty(\mathrm{GL}(2, \mathbb{A}))$] $f \mapsto \operatorname{tr} r_0(f)$, $S_i(f)$ $(i = 1, 2, 5, 7, 8)$ are invariant, namely take the same value at f and $f^h(x) = f(h^{-1}xh)$, $h \in \mathrm{GL}(2, \mathbb{A})$. For $i = 3, 4, 6$ we have $S_i(f^h) = S_i(f)$ if $h \in \mathrm{GL}(2, O_{\mathbb{A}})$, but S_i is not invariant.

If $f \in C_c^\infty(\mathrm{GL}(2, \mathbb{A}))$ takes values in $\mathbb{Q}$ then $\operatorname{tr} r_0(f) \in \mathbb{Q}$, since the representation r_0 is defined over $\mathbb{Q}$. For $i = 1, 2, 3, 4, 8$ it is clear that $S_i(f) \in \mathbb{Q}$. For $i = 7$ the integrand contains the factor $\mu(ab)|a/b|^{1/2}$ which involves $\sqrt{q}$. However the sum includes with μ also $\mu\varepsilon$, $\varepsilon(\alpha) = -1$, and so the sum of the terms indexed by μ and $\mu\varepsilon$ can be written as an integral over the domain where $|a/b|$ is in $q^{2\mathbb{Z}}$.

To see that $S_5(f)$ is rational, we put $a(\mu_1, \mu_2) = \frac{1}{2\pi i} \oint_{|t|=1} f(\mu_1, \mu_2, t) dt$ where

$$f(\mu_1, \mu_2, t) = \operatorname{tr} I(\mu_1 \nu_t, \mu_2 \nu_{t^{-1}}, f) \cdot \frac{d}{dt} \ln m(\mu_1/\mu_2, t^2),$$

and claim that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ one has $\sigma(a(\mu_1, \mu_2)) = a({}^\sigma\mu_1, {}^\sigma\mu_2)$. Note that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the group of characters on $\mathbb{A}^\times / F^\times \cdot \alpha^{\mathbb{Z}}$ as they are all $\mathbb{Q}$-valued. Now $a(\mu_1, \mu_2)$ is the sum of the residues of $f(\mu_1, \mu_2, t)$ at the points of the unit disc. We have that $\sigma(f(\mu_1, \mu_2, t)) = f({}^\sigma\mu_1, {}^\sigma\mu_2, \varepsilon(\sigma) \cdot {}^\sigma t)$ with $\varepsilon(\sigma) = \sigma(\sqrt{q})/\sqrt{q}$. However, if $f(\mu_1, \mu_2, t)$ has a pole at $t = t_0$ and $|t_0| < 1$, then by Proposition 4.11, $|\sigma(t_0)| < 1$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Hence $S_5(f) \in \mathbb{Q}$. To see that $S_6(f) \in \mathbb{Q}$ one proceeds similarly, using the results of Corollary 4.28 on the poles of $R(\mu_1, \mu_2, t)$ and $R(\mu_1, \mu_2, t)^{-1}$.

2 Locally free sheaves of $\mathcal{O}_X$-modules

2.1 Stable bundles

Let X be a smooth geometrically connected projective curve over $\mathbb{F}_q$ (we take minimal q). Denote by $\mathcal{O}_X$ the structure sheaf of X. Denote by Bun_n the set of isomorphism classes of rank n locally free sheaves of $\mathcal{O}_X$-modules. By a (vector) bundle we mean here simply a locally free sheaf. In particular, $\mathrm{Bun}_1 = \operatorname{Pic} X$. The Picard group $\operatorname{Pic} X$ of invertible, or rank 1, locally free sheaves $\mathcal{L}$ of $\mathcal{O}_X$-modules, is naturally isomorphic to the group of classes $\overline{D}$ of (Weil) divisors $D = \sum_v n_v v$ $(n_v \in \mathbb{Z},\ v \in |X|)$. Here $|X|$ is the set of closed points of X, and the divisors D, D' lie in the same class (are linearly equivalent) if their difference is the (principal) divisor $(f) = \sum_v \operatorname{ord}_v(f)v$ where f is a nonzero rational function on X and $\operatorname{ord}_v(f)$ is the order of f at $v \in |X|$ ($\operatorname{ord}_v(f) > 0$ if v is a zero, $\operatorname{ord}_v(f) < 0$ if v is a pole, $\operatorname{ord}_v(f) = 0$ otherwise). If $\mathcal{L}$, $\mathcal{M} \in \operatorname{Pic} X$ correspond to the divisors D, D' then $\mathcal{L} \otimes \mathcal{M}$ corresponds to $D + D'$.

There is a degree map deg on $\operatorname{Pic} X$: $\deg(\sum_v n_v v) = \sum_v n_v \deg(v)$ defines $\deg(\mathcal{L}) = \deg(D)$, where $\deg(v) = [k_v : \mathbb{F}_q]$. Here k_v is the residue field of the function field $F = \mathbb{F}_q(X)$ of X over $\mathbb{F}_q$ at v; assume $\mathbb{F}_q$ is algebraically closed in F. We write F_v for the completion of F at v, O_v for its ring of integers. The cardinality of the residue field $k_v = \mathbb{F}_{q_v}$ at v is denoted by q_v, thus $q_v = q^{\deg(v)}$. We also write $\deg(\overline{D})$ for $\deg(D)$, as the degree of a principal divisor is 0; recall that $\overline{D}$ denotes the class of D.

Denote by $\chi(\mathcal{L}) = \dim_{\mathbb{F}_q} H^0(X, \mathcal{L}) - \dim_{\mathbb{F}_q} H^1(X, \mathcal{L})$ the Euler-Poincaré characteristic of $\mathcal{L} \in \operatorname{Pic} X$. Here $H^i(X, \mathcal{L})$ are finite dimensional vector spaces over $\mathbb{F}_q$. Then $\chi(\mathcal{O}_X) = 1 - g$ where $g = \dim_{\mathbb{F}_q} H^1(X, \mathcal{O}_X)$ is named the genus of X. The Riemann-Roch theorem asserts that $\chi(\mathcal{L}) - \deg(\mathcal{L}) = \chi(\mathcal{O}_X)$ is independent of $\mathcal{L} \in \operatorname{Pic} X$.

Define the *degree* of a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules of rank n to be $\deg \mathcal{E} = \chi(\mathcal{E}) - n\chi(\mathcal{O}_X)$. The *determinant* of $\mathcal{E}$ is $\det \mathcal{E} = \bigwedge^n \mathcal{E} \in \operatorname{Pic} X$. We have $\deg \mathcal{E} = \deg \det \mathcal{E}$. This gives an alternative definition of the degree. A proof of this equality is as follows. If $\mathcal{E}$ is a line bundle, then there is nothing to prove. In the general case, use the fact that both $\deg \mathcal{E}$ and $\deg \det \mathcal{E}$ are additive (if $\mathcal{E}' \subset \mathcal{E}$ is a subbundle, then $\deg \mathcal{E} = \deg \mathcal{E}' + \deg(\mathcal{E}/\mathcal{E}')$ and similarly for $\deg \det \mathcal{E}$), and that each vector bundle has a flag, $\mathcal{E}_i$, such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ are line bundles.

The *height* of a rank two locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules is the integer $\operatorname{ht}(\mathcal{E}) = \max_{\mathcal{L}}(2 \deg \mathcal{L} - \deg \mathcal{E})$, $\mathcal{L}$ ranges over all invertible subsheaves of $\mathcal{E}$.

Proposition 2.1. *We have* $-2g \leq \operatorname{ht}(\mathcal{E}) < \infty$.

Proof. Let $\mathcal{L}$ be an invertible subsheaf of $\mathcal{E}$. From the Riemann-Roch theorem $\chi(\mathcal{L}) = \deg \mathcal{L} + 1 - g$ we obtain $\dim_{\mathbb{F}_q} H^0(X, \mathcal{L}) \geq \deg \mathcal{L} + 1 - g$, whence $\deg \mathcal{L} \leq \dim_{\mathbb{F}_q} H^0(X, \mathcal{L}) + g - 1 \leq \dim_{\mathbb{F}_q} H^0(X, \mathcal{E}) + g - 1$, so $\operatorname{ht}(\mathcal{E})$ is finite.

Let $\mathcal{L}$ be an invertible subsheaf of $\mathcal{E}$ of maximal degree. Let $\mathcal{M}$ be an invertible sheaf with $\deg \mathcal{M} = \deg \mathcal{L} + 1$. Then $\operatorname{Hom}(\mathcal{M}, \mathcal{E}) = 0$. Also, by Riemann-Roch for the rank 2 sheaf $\mathcal{E}$, $\dim_{\mathbb{F}_q} \operatorname{Hom}(\mathcal{M}, \mathcal{E}) = \dim_{\mathbb{F}_q} H^0(X, \mathcal{M}^{-1}\mathcal{E}) \geq \deg(\mathcal{M}^{-1}\mathcal{E}) + 2 - 2g = \deg \mathcal{E} - 2\deg \mathcal{M} + 2 - 2g = \deg \mathcal{E} - 2\deg \mathcal{L} - 2g$, so $2\deg \mathcal{L} - \deg \mathcal{E} \geq -2g$. □

A rank two locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules is called *stable* if $\operatorname{ht}(\mathcal{E}) < 0$ and *semistable* if $\operatorname{ht}(\mathcal{E}) \leq 0$. In general, the *slope* $\mu(\mathcal{E})$ of a locally free sheaf $\mathcal{E}$ over an algebraic curve is defined to be $\deg \mathcal{E} / \operatorname{rk} \mathcal{E}$, and $\mathcal{E}$ is called stable if $\mu(\mathcal{F}) < \mu(\mathcal{E})$ for all proper nonzero subbundles $\mathcal{F}$ of $\mathcal{E}$ (semistable if $\leq$). A locally free sheaf $\mathcal{E}$ of rank two is called *almost stable* if $\operatorname{ht}(\mathcal{E}) < 2g - 1$, and *very unstable* if $\operatorname{ht}(\mathcal{E}) \geq 2g - 1$. If $g = 0$, every $\mathcal{E}$ is very unstable.

Remark 1. A very unstable vector bundle $\mathcal{E}$ of rank 2 splits into the direct sum of two line bundles. We give here a relatively elementary treatment. An extension can be found in the work of Harder and Narasimhan. If $\mathcal{E}$ is very unstable, $\mathcal{L}$ is an invertible subsheaf of $\mathcal{E}$ of maximal degree, and $\mathcal{M} = \mathcal{E}/\mathcal{L}$, then $\mathcal{M}$ is invertible and $\mathrm{Ext}(\mathcal{M}, \mathcal{L}) = H^1(X, \mathcal{M}^{-1}\mathcal{L})$ is 0 since

$$\deg \mathcal{M}^{-1}\mathcal{L} = \deg \mathcal{L} - \deg \mathcal{M} = 2\deg \mathcal{L} - \deg \mathcal{E} = \mathrm{ht}\, \mathcal{E} \geq 2g - 1.$$

Indeed, by Serre duality $H^1(X, \mathcal{M}^{-1}\mathcal{L}) = H^0(X, \mathcal{L}^{-1}\mathcal{M}\omega)$ where ω denotes the canonical bundle. But $\deg \mathcal{L}^{-1}\mathcal{M}\omega \leq 2g-2-(2g-1) < 0$, and $H^0(X, \mathcal{F}) = 0$ for an invertible sheaf $\mathcal{F}$ with negative degree.

Proposition 2.2. *The number of isomorphism classes of almost stable rank two locally free sheaves $\mathcal{E}$ of $\mathcal{O}_X$-modules with a fixed degree is finite.*

Proof. The height of an almost stable sheaf lies in $[-2g, 2g-2]$. Hence it suffices to show the finiteness for $\mathcal{E}$ with a fixed degree n and height h. Every such sheaf lies in an exact sequence $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \to 0$, where $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves and $2\deg \mathcal{L} - \deg \mathcal{E} = h$. Then $\deg \mathcal{L} = (n+h)/2$, $\deg \mathcal{M} = (n-h)/2$. Since the degrees of $\mathcal{L}$ and $\mathcal{M}$ are fixed, there are only finitely many possibilities for $\mathcal{L}$ and $\mathcal{M}$ (set of cardinality of the $\mathbb{F}_q$-points on the abelian variety $\mathrm{Pic}^0(X)$). With $\mathcal{L}$ and $\mathcal{M}$ fixed there are only finitely many choices for $\mathcal{E}$ as $\mathrm{Ext}(\mathcal{L}, \mathcal{M})$ is finite. ☐

The group $\mathrm{Pic}\, X$ acts on $\mathrm{Bun}_2 : (\mathcal{L} \in \mathrm{Pic}\, X, \mathcal{E} \in \mathrm{Bun}_2) \mapsto \mathcal{L} \otimes \mathcal{E}$. As

$$\deg(\mathcal{L} \otimes \mathcal{E}) = 2\deg(\mathcal{L}) + \deg(\mathcal{E}),$$

the set of almost stable sheaves is invariant under this action. In a $\mathrm{Pic}\, X$-orbit we may choose $\mathcal{E}$ to have $\deg(\mathcal{E})$ in $\{0, 1\}$. Hence we deduce

Corollary 2.3. *The number of* $\mathrm{Pic}\, X$*-orbits on the set of isomorphism classes of almost stable rank two locally free sheaves of $\mathcal{O}_X$-modules is finite.*

2.2 Bundles and lattices

Let $\mathcal{E}$ be a rank n locally free sheaf of $\mathcal{O}_X$-modules. Denote by $\mathcal{E}_\eta$ the fiber (= stalk) of $\mathcal{E}$ over the generic point η of X. Let $\mathcal{E}_{(v)}$ be the stalk of $\mathcal{E}$ at the closed point $v \in |X|$. Let $O_{(v)}$ be the local ring of X at v. Then $\mathcal{E}_\eta$ is an n-dimensional vector space over F, and $\mathcal{E}_{(v)}$ is an $O_{(v)}$-lattice in $\mathcal{E}_\eta$, namely a rank n free $O_{(v)}$-submodule of $\mathcal{E}_\eta$.

A set M of $O_{(v)}$-lattices $M_{(v)}$ in a finite dimensional vector space V over F, v ranges over the set $|X|$ of closed points in X, is called *adelic* if there exists a basis $\{e_1, \dots, e_n\}$ in V such that $M_{(v)} = O_{(v)}e_1 + \cdots + O_{(v)}e_n$ for almost all v in $|X|$. "Almost all" means "with at most finitely many exceptions". If M is adelic then it is adelic with respect to any basis $\{e_1, \dots, e_n\}$ of V.

The set of stalks $\{\mathcal{E}_{(v)}; v \in |X|\}$ of a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules is adelic. Conversely, an adelic set of lattices $M = \{M_{(v)}; v \in |X|\}$ in a finite dimensional vector space V over F is the set of stalks of the locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules defined by

$$H^0(\mathrm{U}, \mathcal{E}) = \{s \in V; \forall v \in \mathrm{U}, s \in M_{(v)}\}$$

for any open subset U of X. Obtained is an equivalence of the category of finite rank locally free sheaves of $\mathcal{O}_X$-modules, with the category of finite dimensional vector spaces over F with adelic sets of $O_{(v)}$-lattices.
Let O_v be the completion of $O_{(v)}$. The completion of F at v is denoted F_v. Let V be a finite dimensional vector space over F. Put $V_v = V \otimes_F F_v$. There is a natural bijection between the set of $\mathcal{O}_{(v)}$-lattices in V, and O_v-lattices in V_v: an $O_{(v)}$-lattice $M \subset V$ corresponds to the lattice $M \otimes_{\mathcal{O}_{(v)}} \mathcal{O}_v$ in V_v; an O_v-lattice $N \subset V_v$ corresponds to the $O_{(v)}$-lattice $N \cap V$.
The category $\mathcal{C}$ whose objects are finite dimensional F-vector spaces V with adelic sets $\{M_v; v \in |X|\}$ of O_v-lattices M_v in V_v is equivalent to the category of finite rank locally free sheaves of $\mathcal{O}_X$-modules $\mathcal{E}$, by $\mathcal{E} \mapsto (\mathcal{E}_\eta, \{\mathcal{E}_v\})$, where $\mathcal{E}_\eta$ is the generic fiber of $\mathcal{E}$ and $\mathcal{E}_v$ is the completion of the stalk of $\mathcal{E}$ at the closed point $v \in |X|$.
Let R_n be the set of isomorphism classes of pairs $(\mathcal{E}, i)$ where $\mathcal{E}$ is a rank n locally free sheaf of $\mathcal{O}_X$-modules, and i is an isomorphism from the generic fiber of $\mathcal{E}$ to F^n. The pairs $(\mathcal{E}, i)$ and $(\mathcal{E}', i')$ are isomorphic if there is an isomorphism $\mathcal{E} \tilde{\to} \mathcal{E}'$ which induces a commutative diagram when restricted to the generic fiber with sides i and i' and the identity $F^n \to F^n$. The group $\mathrm{GL}(n, F)$ acts on R_n by $g : (\mathcal{E}, i) \mapsto (\mathcal{E}, g \circ i)$. Then $\mathrm{GL}(n, F) \backslash R_n = \mathrm{Bun}_n$ is the set of isomorphism classes of rank n locally free sheaves of $\mathcal{O}_X$-modules.
The set R_n is the set of adelic collections of O_v-lattices $M_v \subset F_v^n$, $v \in |X|$. The group $\mathrm{GL}(n, F_v)$ acts transitively on the set of O_v-lattices in F_v^n. The stabilizer of the standard lattice O_v^n in F_v^n is $\mathrm{GL}(n, O_v)$. Thus the set of O_v-lattices in F_v^n is $\mathrm{GL}(n, F_v)/\mathrm{GL}(n, O_v)$, and R_n is $\mathrm{GL}(n, \mathbb{A})/\mathrm{GL}(n, O_\mathbb{A})$, where $\mathbb{A}$ is the ring of adèles in F and $O_\mathbb{A} = \prod_{v \in |X|} O_v$. Thus

$$\mathrm{Bun}_n = \mathrm{GL}(n, F) \backslash \mathrm{GL}(n, \mathbb{A}) / \mathrm{GL}(n, O_\mathbb{A}).$$

The elements of $\mathrm{GL}(n, \mathbb{A})/\mathrm{GL}(n, O_\mathbb{A})$ are called matrix divisors, and the elements of $\mathrm{GL}(n, F)\backslash \mathrm{GL}(n, \mathbb{A})/\mathrm{GL}(n, O_\mathbb{A})$ classes of matrix divisors. For $n = 1$, the identification of $\mathrm{GL}(n, F)\backslash \mathrm{GL}(n, \mathbb{A})/\mathrm{GL}(n, O_\mathbb{A})$ with Bun_n is the identification of classes of divisors with invertible sheaves.
The group $\mathrm{GL}(n, \mathbb{A})$ can be identified with the set of triples

$$(\mathcal{E}, i_\eta : \mathcal{E}_\eta \xrightarrow{\sim} F^n, (i_v : \mathcal{E}_v \xrightarrow{\sim} O_v^n)).$$

Given a rank n locally free sheaf $\mathcal{E}$, an isomorphism $i_\eta : \mathcal{E}_\eta \xrightarrow{\sim} F^n$, and for each closed point v in $|X|$ an isomorphism $i_v : \mathcal{E}_v \xrightarrow{\sim} O_v^n$ of the completion $\mathcal{E}_v$ of the stalk $\mathcal{E}_{(v)}$ at v with O_v^n, let us define the corresponding $g = (g_v)$ in $\mathrm{GL}(n, \mathbb{A})$. Each g_v has to be an automorphism $F_v^n \to F_v^n$, with $g_v(O_v^n) = O_v^n$ for almost all v. Construct g_v as the composition $i_v \circ i_\eta^{-1}$:

$$F_v^n = F^n \otimes_F F_v \overset{i_\eta}{\tilde{\leftarrow}} \mathcal{E}_\eta \otimes_F F_v = \mathcal{E}_{F_v} = \mathcal{E}_v \otimes_{O_v} F_v \overset{i_v}{\tilde{\rightarrow}} O_v^n \otimes_{O_v} F_v = F_v^n.$$

Note that since $\mathcal{E}$ is locally free, for almost all v the map $g_v = i_v \circ i_\eta^{-1}$ takes $O_v^n \subset F_v^n$ to $\mathcal{E}_v \subset \mathcal{E}_\eta \otimes_F F_v$ via i_η^{-1}, and then to O_v^n via i_v. To show that the map $\{(\mathcal{E}, i_\eta, (i_v))\} \to \mathrm{GL}(n, \mathbb{A})$ is bijective one shows that $\mathrm{GL}(n, \mathbb{A})$ acts on the set of triples, simply transitively. Viewing the trivial locally free sheaf as $O_\mathbb{A}^n$ (space of columns), $g(\mathcal{E}, i_\eta, (i_v))$ is defined to be $(g\mathcal{E}, i_\eta, (i_v \circ g_v^{-1}))$, where $i_v \circ g_v^{-1}$ maps the stalk $g_v\mathcal{E}_v$ of $g\mathcal{E}$ at v to O_v^n. The set of pairs $\{(\mathcal{E}, i_\eta)\}$ then corresponds to $\mathrm{GL}(n, \mathbb{A})/\mathrm{GL}(n, O_\mathbb{A})$, the set of pairs $\{(\mathcal{E}, (i_v))\}$ to $\mathrm{GL}(n, F)\backslash \mathrm{GL}(n, \mathbb{A})$, and the set $\{\mathcal{E}\}$ to $\mathrm{GL}(n, F)\backslash \mathrm{GL}(n, \mathbb{A})/\mathrm{GL}(n, O_\mathbb{A})$.

To an idèle $a = (\boldsymbol{\pi}_v^{-n_v} u_v; v \in |X|)$, where $\boldsymbol{\pi}_v$ denotes a generator of the maximal ideal in the ring O_v of integers in F_v, $u_v \in O_v^\times$ and $n_v \in \mathbb{Z}$, we associate the divisor $D = \sum_v n_v v$, and the degree

$$\deg(a) = \deg(D) = \sum_v n_v \deg(v), \qquad \deg(v) = [\mathbb{F}_v : \mathbb{F}_q],$$

where $\mathbb{F}_v$ is the residue field of F at v, a finite field of $q_v = q^{\deg(v)}$ elements. For $g \in \mathrm{GL}(2, \mathbb{A})$ write $\deg g$ for $\deg \det g$. Recall that $O_\mathbb{A} = \prod_v O_v$ ($v \in |X|$). For $t \in \mathbb{C}^\times$ we write

$$\nu_t(a) = t^{-\deg(a)} = \prod_v t_v^{-n_v}$$

where $t_v = t^{\deg(v)}$. Then $\nu_{q^{-1}}(a) = \prod_v q_v^{n_v} = |a|$ is equal to $\nu(a) = q^{\deg(a)}$. Also $\nu_t(\boldsymbol{\pi}_v) = t_v$, $\nu_{q^{-1}}(\boldsymbol{\pi}_v) = |\boldsymbol{\pi}_v|$.

Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves. Fix isomorphisms $i_\mathcal{L}$, $i_\mathcal{M}$ of their generic fibers with F. Each of $(\mathcal{L}, i_\mathcal{L})$ and $(\mathcal{M}, i_\mathcal{M})$ defines an element of $\mathbb{A}^\times/O_\mathbb{A}^\times$, namely a divisor on X. Choose representatives a, b in $\mathbb{A}^\times$, for example $\sum_v n_v v$ is represented by $(\boldsymbol{\pi}_v^{-n_v})$. Given an exact sequence $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \to 0$ of locally free sheaves, choose an isomorphism φ between the generic fiber of $\mathcal{E}$ and F^2 so that the induced exact sequence of generic fibers $0 \to F \to F^2 \to F \to 0$ is standard $\big(x \mapsto \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto y\big)$. The isomorphism φ is defined uniquely up to left multiplication by an automorphism of F^2 of the form $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$, $t \in F$. The pair $(\mathcal{E}, \varphi)$ determines an element of $\mathrm{GL}(2, \mathbb{A})/\mathrm{GL}(2, O_\mathbb{A})$, of the form $u = \left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, with z in $\mathbb{A}$. Since u is defined up to right multiplication by an element of $\mathrm{GL}(2, O)$, z is uniquely defined up to addition of an element of $\frac{a}{b}O_\mathbb{A}$. Replacing φ by $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)\varphi$ with $t \in F$ replaces z by $z + t$. Thus we get a bijection

$$\mathrm{Ext}(\mathcal{M}, \mathcal{L}) \to \mathbb{A}/(F + \frac{a}{b}O_\mathbb{A}).$$

This is an isomorphism of $\mathbb{F}_q$-vector spaces.

In summary, if the invertible sheaves $\mathcal{L}$ and $\mathcal{M}$ correspond to idèles a and b, then $\mathrm{Ext}(\mathcal{M}, \mathcal{L}) \simeq \mathbb{A}/(F + \frac{a}{b}O_\mathbb{A})$, and the map $\mathrm{Ext}(\mathcal{M}, \mathcal{L}) \to \mathrm{Bun}_2$ which associates to the exact sequence $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \to 0$ its middle term, coincides with the map $\mathbb{A}/(F+\frac{a}{b}O_{\mathbb{A}}) \simeq H^1(X,\mathcal{M}^{-1}\mathcal{L})$, see [S97], II. 5. The isomorphism $\mathbb{A}/(F+\frac{a}{b}O_{\mathbb{A}}) \tilde{\to} \operatorname{Ext}(\mathcal{M},\mathcal{L})$ is $H^1(X,\mathcal{M}^{-1}\mathcal{L}) \tilde{\to} \operatorname{Ext}(\mathcal{M},\mathcal{L})$.

2.3 The space $\mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$

Proposition 2.4. *Given $a \in \mathbb{A}^\times$, $\deg a \geq 2g-1$, then $aO_{\mathbb{A}} + F = \mathbb{A}$.*

Proof. If $\mathcal{L}$ is an invertible sheaf on X associated with a, then $\mathbb{A}/(F+aO_{\mathbb{A}}) = H^1(X,\mathcal{L})$. By Serre duality $H^1(X,\mathcal{L}) \simeq H^0(X,\mathcal{L}^{-1}\omega)$, where ω is the canonical bundle of degree $2g-2$. Then $\deg(\mathcal{L}^{-1}\omega) \leq (2g-2)-(2g-1) = -1 < 0$, hence $H^0(X,\mathcal{L}^{-1}\omega) = \{0\}$. □

Define a function

$$\mathrm{ht}^+ : \mathrm{GL}(2,\mathbb{A}) \to \mathbb{Z} \qquad \text{by} \qquad \mathrm{ht}^+\left(\left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right) k\right) = \deg a - \deg b$$

for all $a, b \in \mathbb{A}^\times$, $c \in \mathbb{A}$, $k \in \mathrm{GL}(2,O_{\mathbb{A}})$. It is clearly a well defined function on $B(F)\backslash \mathrm{GL}(2,\mathbb{A})$. For $x \in \mathrm{GL}(2,\mathbb{A})$, put

$$\mathrm{ht}(x) = \max_{\gamma \in \mathrm{GL}(2,F)} \mathrm{ht}^+(\gamma x).$$

On $\mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$ it is well defined.

Proposition 2.5. *For any $x \in \mathrm{GL}(2,\mathbb{A})$ we have $-2g \leq \mathrm{ht}(x) < \infty$.*

Proof. This follows from Proposition 2.1 as if $\mathcal{E}$ is a rank two locally free sheaf of $\mathcal{O}_X$-modules associated to the image of x in $\mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})/\mathrm{GL}(2,O_{\mathbb{A}})$, then $\mathrm{ht}(x) = \mathrm{ht}(\mathcal{E})$. □

Put $H_B = \{x \in B(F)\backslash \mathrm{GL}(2,\mathbb{A}); \mathrm{ht}^+(x) > 0\}$ and

$$H = \{x \in \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}); \mathrm{ht}(x) > 0\}.$$

Proposition 2.6. (1) *The restriction p to H_B of the natural projection p' : $B(F)\backslash \mathrm{GL}(2,\mathbb{A}) \to \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$ is a homeomorphism $H_B \to H$.*
(2) *The set $\{x \in \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}); \mathrm{ht}(x) \leq n\}$ is compact modulo the center $Z(\mathbb{A})$ of $\mathrm{GL}(2,\mathbb{A})$ for every integer n.*

Proof. (1) The map p is clearly onto. To show that p is injective it suffices to show for any x in $\mathrm{GL}(2,\mathbb{A}), \gamma \in \mathrm{GL}(2,F)$, that $\mathrm{ht}^+(x) > 0$ and $\mathrm{ht}^+(\gamma x) > 0$ implies $\gamma \in B(F)$. This is a typical application of the Harder-Narasimhan filtration. In simple, explicit terms, this follows from

Lemma 2.7. *If $g \in \mathrm{GL}(2,F) - B(F)$ then $\mathrm{ht}^+(x) + \mathrm{ht}^+(gx) \leq 0$.*

Proof. Write g as $g_1 w g_2$ with g_1, g_2 in $B(F)$, $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Put $x' = g_2 x$. Then $\mathrm{ht}^+(x) = \mathrm{ht}^+(x')$, $\mathrm{ht}^+(gx) = \mathrm{ht}^+(wx')$. Thus we need to show that $\mathrm{ht}^+(x') + \mathrm{ht}^+(wx') \leq 0$. Suppose $x' = \left(\begin{smallmatrix} a_1 & c_1 \\ 0 & b_1 \end{smallmatrix}\right) k_1$, $wx' = \left(\begin{smallmatrix} a_2 & c_2 \\ 0 & b_2 \end{smallmatrix}\right) k_2$ with

$k_1, k_2 \in \mathrm{GL}(2, O_{\mathbb{A}})$. Put $k_2 k_1^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then $\left(\begin{smallmatrix} a_2 & c_2 \\ 0 & b_2 \end{smallmatrix}\right) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = w \left(\begin{smallmatrix} a_1 & c_1 \\ 0 & b_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & b_1 \\ a_1 & c_1 \end{smallmatrix}\right)$. Hence $b_2 \gamma = a_1$. Thus $\deg a_1 \leq \deg b_2$ (as $\deg \gamma \leq 0$, since $\gamma \in O_{\mathbb{A}}$). But $\deg a_2 b_2 = \deg a_1 b_1$. Hence $\deg a_2 \leq \deg b_1$. Then $\mathrm{ht}^+(x') + \mathrm{ht}^+(wx') = \deg a_1 - \deg b_1 + \deg a_2 - \deg b_2 \leq 0$. □

Now the natural projection $p' : B(F)\backslash \mathrm{GL}(2, \mathbb{A}) \to \mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A})$ is open and H_B is an open subset of $B(F)\backslash \mathrm{GL}(2, \mathbb{A})$. Hence the bijection $p = p'|H_B : H_B \to H$ is open. Since it is also continuous, p is a homeomorphism.

(2) The image of the set $S = \{x \in B(F)\backslash \mathrm{GL}(2, \mathbb{A}); -2g \leq \mathrm{ht}^+(x) \leq n\}$ in $\mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A})$ under p' contains the set $\{x \in \mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A}); \mathrm{ht}(x) \leq n\}$. So it suffices to show that S is compact mod $Z(\mathbb{A})$. Choose a compact C in $\mathbb{A}^\times$ with

$$CF^\times = \{t \in \mathbb{A}^\times; -2g \leq \deg t \leq n\}.$$

Choose an idèle d with $\deg d \geq 2g - 1$. Put

$$Y = \left\{ \left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) k; \quad k \in \mathrm{GL}(2, O_{\mathbb{A}}),\ a, b \in \mathbb{A}^\times,\ \frac{a}{b} \in C,\ c \in dO_{\mathbb{A}} \right\}.$$

Lemma 2.8. *The map $Y \to S$ is surjective.*

Proof. Let $x \in \mathrm{GL}(2, \mathbb{A}), -2g \leq \mathrm{ht}^+(x) \leq n$. We need to show that x can be written as hy with $y \in Y$ and $h \in B(F)$. Write x as $\left(\begin{smallmatrix} r & s \\ 0 & t \end{smallmatrix}\right) K$ with $k \in \mathrm{GL}(2, O_{\mathbb{A}}), r, t \in \mathbb{A}^\times, s \in \mathbb{A}$. It remains to show that $\left(\begin{smallmatrix} r & s \\ 0 & t \end{smallmatrix}\right)$ can be expressed as $\left(\begin{smallmatrix} \alpha & \gamma \\ 0 & \beta \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ with $a, b \in \mathbb{A}^\times$, $\frac{a}{b} \in C$, $c \in dO_{\mathbb{A}}$, $\alpha, \beta \in F^\times$, $\gamma \in F$. Thus we need to show the existence of $a, b, c, \alpha, \beta, \gamma$ such that

(*) $\quad a\alpha = r, \quad \beta b = t, \quad a, b \in \mathbb{A}^\times, \quad \alpha, \beta \in F^\times, \quad \frac{a}{b} \in C,$

(**) $\quad b(\alpha c + \gamma) = s, \quad c \in dO_{\mathbb{A}}, \quad \gamma \in F.$

By definition of x, $\deg r - \deg t$ lies in $[-2g, n]$, so the existence of a, b, α, β satisfying (*) follows from the definition of C. The existence of $c \in dO_{\mathbb{A}}$ and $\gamma \in F$ satisfying $\alpha c + \gamma = s/b$ follows from: $cO_{\mathbb{A}} + F = \mathbb{A}$ if $\deg c \geq 2g - 1$. □

Since Y is compact mod $Z(\mathbb{A})$, so is S, and (2) follows. □

In summary, the homogeneous space $\mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A})$ is the union of the compact mod $Z(\mathbb{A})$ set $\{x \in \mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A});\ \mathrm{ht}(x) \leq 0\}$, and the set $H = \{x \in \mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A}); \mathrm{ht}(x) > 0\}$, whose structure is simpler. The set H_B, hence also the sets H and $\mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A})$, are noncompact modulo $Z(\mathbb{A})$. Indeed the function ht^+ takes arbitrary large values.

The image of H in $\mathrm{Bun}_2 = \mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A})/\mathrm{GL}(2, O_{\mathbb{A}})$ is the set of non-semistable locally free sheaves.

The set $\mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A})/\mathrm{GL}(2, O_{\mathbb{A}})$ is analogous to the set

$$\mathrm{SL}(2, \mathbb{Z})\backslash \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2) = \mathrm{SL}(2, \mathbb{Z})\backslash \mathfrak{h},$$

where $\mathfrak{h} = \{z \in \mathbb{C}; \mathrm{Im}\, z > 0\}$, the upper half plane, is isomorphic to $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$, by

$$g \mapsto g(i) = (ai + b)/(ci + d).$$

The set $B(F)\backslash \mathrm{GL}(2,\mathbb{A})/\mathrm{GL}(2,O_{\mathbb{A}})$ is analogous to $N\backslash\mathfrak{h}$ where N is the group of transformations $z\mapsto z+n$ ($n\in\mathbb{Z}$) on $\mathfrak{h}$. The function ht^+ is analogous to the function $z\mapsto \ln \mathrm{Im}\, z$ on $N\backslash\mathfrak{h}$. The statement $-2g\le \mathrm{ht}(x)<\infty$ corresponds to the statement that the natural map from the half plane $\{z\in\mathbb{C};\mathrm{Im}\, z\ge\sqrt{3}/2\}$ to $\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}$ is onto. The statement that $p: H_B\to H$ is homeomorphism corresponds to the statement that the map $N\backslash\{z\in\mathbb{C};\mathrm{Im}\, z>1\}\to \mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}$ is injective, and the compactness of $\{x\in \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A});\ \mathrm{ht}(x)\le n\}$ corresponds to the statement that the complement in $\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}$ of the image of the half plane $\{z\in\mathbb{C};\mathrm{Im}\, z>h\}$ is compact.

2.4 ℓ-groups

An *ℓ-space* is a Hausdorff topological space such that each of its points has a fundamental system of open compact neighborhoods.
We shall consider on ℓ-spaces only measures for which every open compact subset is measurable, and its volume is a rational number. If dx is such a measure on an ℓ-space Y, and f is a locally constant compactly supported function on Y with values in a field E of characteristic zero, then $\int_Y f(x)dx$ reduces to a finite sum, and it is well defined.
On topological groups we consider only left- or right-invariant measures.
An ℓ-group is a topological group with an ℓ-space structure.

Proposition 2.9. *Let G be an ℓ-group. Then* (1) *there exists a fundamental system of neighborhoods of the identity in G consisting of open compact subgroups*;
(2) *there exists a left Haar measure on G such that the volume of each open compact set is a rational number.*

Proof. (1) Let U be a neighborhood of the identity in G. We shall show that U contains an open compact subgroup. Since G is ℓ-space, we may assume that U is open and compact. Put $V=\{x\in G; xU\subset U\}$. Then $V=\cap_{u\in U}Uu^{-1}$, hence it is compact. Now for each v in V and u in U, by continuity of multiplication m there exists an open subset W_u containing v, and U_u in U containing u, such that $m(W_u,U_u)\subset U$. As U is compact and $U=\cup_{u\in U}U_u$, there are finitely many $u_1,\dots,u_n$ in U with $U=\cup_{1\le i\le n}U_{u_i}$. Then $W=\cap_{1\le i\le n}W_{u_i}$ is open in V and it contains v. Thus V is an open neighborhood of the identity, and $V\cdot V=V$. Then $V\cap V^{-1}$ is an open compact subgroup in U.
(2) Fix some left Haar measure on G. Denote the volume of an open compact subgroup U by $|U|$. For two such groups, U_1 and U_2 we have

$$\frac{|U_1|}{|U_2|}=\frac{|U_1|}{|U_1\cap U_2|}\Big/\frac{|U_2|}{|U_1\cap U_2|}=\frac{[U_1:U_1\cap U_2]}{[U_2:U_1\cap U_2]}\in\mathbb{Q}.$$

Consequently the Haar measure on G can be chosen to assign rational volume to every open compact subgroup of G. But then the volume of every open compact subset K in G is rational, since as in (1) for such K there is a compact open subgroup U of G with $KU \subset K$, and then $|K| = [K:U]|U|$ is rational, where K is a disjoint union of $[K:U]$ translates of U. □

Fix an ℓ-group G and a left Haar measure on G such that the volume of any open compact set is a rational number. Fix a field E of characteristic zero. The E-vector space H_G of compactly supported locally constant functions $f : G \to E$ is an algebra under the convolultion $(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1}g)dh$. For an open compact subgroup U in G the set of U-biinvariant functions in H_G is a subalgebra H_G^U, called the *Hecke algebra* of (G,U). Although H_G has no unit (unless G is discrete, when the δ-function is in H_G), H_G^U does: it is $\delta_U : G \to \mathbb{Q}$, the characteristic function of U divided by $|U|$.
A representation π of the group G on a vector space V is called *smooth* if the stabilizer of any vector of V is open, and *admissible* if it is smooth and for any open subgroup U of G the space V^U of U-fixed vectors in V is finite dimensional.
If π is a smooth representation of an ℓ-group G on a vector space V over E, for each $f \in H_G$ define the operator $\pi(f) : V \to V$ by $\pi(f)v = \int_G f(g)\pi(g)vdg$. This integral reduces to a finite sum since π is smooth, and $\pi(f_1 * f_2) = \pi(f_1) \circ \pi(f_2)$. Then V is naturally an H_G-module, and for any open compact subgroup U of G, the space V^U is a unital module over H_G^U.

Proposition 2.10. (1) *A smooth G-module $V \neq \{0\}$ is irreducible iff for every open compact subgroup U of G either $V^U = 0$ or V^U is an irreducible H_G^U-module.*
(2) *Given an open compact subgroup U of G and an irreducible unital H_G^U-module M, there exists a smooth irreducible G-module V such that V^U is isomorphic to M as an H_G^U-module, and V is determined by this property up to isomorphism.*

For a proof see [BZ76], 2.10. See [BZ76], 2.11 for
Schur's Lemma. *Let π be an irreducible admissible representation of G in a vector space V over an algebraically closed field E. Then any nonzero G-module morphism (intertwining operator) $V \to V$ is a scalar.*

Proposition 2.11. *Let π be an irreducible admissible representation of G in a vector space V over an algebraically closed field E. For any field extension E' of E, the representation of G in $V \otimes_E E'$ is also irreducible.*

Proof. By Proposition 2.10, the statement reduces to a similar statement for finite dimensional algebras, since π is assumed to be admissible. □

Let E be a subfield of $\mathbb{C}$ invariant with respect to complex conjugation. A representation of G on a vector space V over E is *unitary* if there is a G-invariant scalar product on V (thus a bilinear function $(\cdot,\cdot) : V \times V \to E$ with $\overline{(v,w)} = (w,v)$ and $(v,v) = 0$ iff $v = 0$, and $(gv, gw) = (v,w)$ for all v,w in V and g in G).

Note that we do not require V to be complete with respect to the scalar product, even in the case $E = \mathbb{C}$. If E is algebraically closed and the representation of G in E is irreducible and admissible, then the G-invariant inner product on V is unique up to a scalar multiple, if it exists.

PROPOSITION 2.12. *Let π be an admissible unitary representation of G in the E-space V. Fix a G-invariant scalar product on V. Let L be an invariant subspace of V, and $L^\perp$ its orthogonal complement. Then $V = L \oplus L^\perp$.*

Proof. Given $x \in V$, we need to express it as $x_1 + x_2$ with $x_1 \in L$ and $x_2 \in L^\perp$. Since π is smooth there exists a compact open subgroup U of G with $x \in V^U$. Since π is admissible, $\dim_E V^U$ is finite. Thus $x = x_1 + x_2$ for some $x_1 \in L^U$, $x_2 \in V^U$, x_2 orthogonal to L^U. It remains to show that x_2 is orthogonal to the entire space L. Let δ_U be the unit in H_G^U. Then $\pi(\delta_U)$ is the orthogonal projector $V \mapsto V^U$. Hence for every y in L, $(x_2, y) = (\pi(\delta_U)x_2, y) = (x_2, \pi(\delta_U)y) = 0$ since $\pi(\delta_U)y \in L^U$. $\square$

It follows that every admissible unitary representation of G is a direct sum of irreducible representations. This sum is not necessarily finite. However, given an open compact subgroup U of G, only finitely many summands contain nonzero U-invariant vectors.

2.5 AUTOMORPHIC FORMS

Let E be an algebraically closed field of characteristic zero. An *automorphic form* is a smooth function $\phi : \mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A}) \to E$, where by *smooth* we mean that there is an open subgroup U_ϕ of $\mathrm{GL}(2, \mathbb{A})$ such that $\phi(xu) = \phi(x)$ for all $u \in \mathrm{U}_\phi$ and $x \in \mathrm{GL}(2, \mathbb{A})$. A *cusp form* is an automorphic form ϕ with $\int_{\mathbb{A}/F} \phi\left(\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) x\right) dz = 0$ for all $x \in \mathrm{GL}(2, \mathbb{A})$.

Since ϕ is right locally constant (= smooth) and $\mathbb{A}/F$ is compact, the integral here is well defined and reduces to a finite sum.

Let A_0^E be the space of cusp forms $\phi : \mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A}) \to E$. The group $\mathrm{GL}(2, \mathbb{A})$ acts on A_0^E by right translation: $(r(h)\phi)(g) = \phi(gh)$. By a *character* of an ℓ-group G with values in E we mean a locally constant homomorphism $\chi : G \to E^\times$. If $E \subset \mathbb{C}$ such χ is called a *unitary* character if $|\chi(g)| = 1$ for all g in G.

Denote by $A_0^E(\chi)$ the space of $\phi \in A_0^E$ with $\phi(ax) = \chi(a)\phi(x), a \in \mathbb{A}^\times$ (identified with the center of $\mathrm{GL}(2, \mathbb{A})), x \in \mathrm{GL}(2, F)\backslash \mathrm{GL}(2, \mathbb{A})$. The space $A_0^E(\chi)$ is invariant under the $\mathrm{GL}(2, \mathbb{A})$-action.

Let π be an irreducible representation of $\mathrm{GL}(2, \mathbb{A})$ over E. By Schur's lemma, there is a character $\chi : \mathbb{A}^\times \to E^\times$ such that for every a in $\mathbb{A}^\times, \pi(a)$ is multiplication by $\chi(a)$. This χ is called the *central character* of π.

If $V \subset A_0^E$ is an irreducible admissible representation π of $\mathrm{GL}(2, \mathbb{A})$ and χ is the central character of V, then $V \subset A_0^E(\chi)$. Since the center of $\mathrm{GL}(2, F)$ acts trivially on A_0^E, χ is trivial on $F^\times$. Thus every irreducible admissible $\pi \subset A_0^E$ lies in $A_0^E(\chi)$, where χ is the central character of π, which is a character of $\mathbb{A}^\times/F^\times$. The following is known also e.g. for $\mathrm{GL}(n)$.

Proposition 2.13. *Fix an open subgroup U of* $\mathrm{GL}(2,\mathbb{A})$. *There exists a compact mod $Z(\mathbb{A})$ subset K of* $\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A})$ *such that the support of any U-invariant cusp form is contained in K.*

Proof. We first show that there is an integer n such that given $z\in\mathbb{A}$ and $x\in\mathrm{GL}(2,\mathbb{A})$ with $\mathrm{ht}^{+}(x)\geq n$, there exist $u\in U$ and $\beta\in F$ with $\left(\begin{smallmatrix}1&z\\0&1\end{smallmatrix}\right)x=\left(\begin{smallmatrix}1&\beta\\0&1\end{smallmatrix}\right)xu$.
To see this, fix an effective divisor $-D=\sum_{v\in|X|}n_v v$ on X, put $d=(\boldsymbol{\pi}_v^{n_v})$ and let $J_D=dO_{\mathbb{A}}$ be the corresponding ideal in $O_{\mathbb{A}}$. The groups $\Gamma(D)=\{\gamma\in\mathrm{GL}(2,O_{\mathbb{A}});\gamma\equiv I\mod J_D\}$ make a basis of neighborhoods of the identity in $\mathrm{GL}(2,\mathbb{A})$. Thus we may assume in this proof that $U=\Gamma(D)$. In this case we shall show that $n=2g-1-\deg(d)$. Indeed, fix $z\in\mathbb{A}$ and $x=\left(\begin{smallmatrix}a&c\\0&b\end{smallmatrix}\right)k$ with $k\in\mathrm{GL}(2,O_{\mathbb{A}})$ and $\mathrm{ht}^{+}(x)=\deg a-\deg b\geq 2g-1-\deg(d)$ (note: $\deg(d)=-\deg D=\sum_v n_v\deg v$). Then $\frac{ad}{b}O_{\mathbb{A}}+F=\mathbb{A}$ and $z=\frac{ad}{b}t+\beta$ for some $\beta\in F$ and $t\in O_{\mathbb{A}}$. Put $u=k^{-1}\left(\begin{smallmatrix}1&td\\0&1\end{smallmatrix}\right)k$. Then $u\in\Gamma(D)$ and $\left(\begin{smallmatrix}1&z\\0&1\end{smallmatrix}\right)x=\left(\begin{smallmatrix}1&\beta\\0&1\end{smallmatrix}\right)xu$.
We claim the proposition holds with $K=\{x\in\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A});\mathrm{ht}(x)<n\}$. This K is compact modulo $Z(\mathbb{A})$. Let ϕ be a U-invariant cusp form, $x\in\mathrm{GL}(2,\mathbb{A})$, $\mathrm{ht}(x)\geq n$. We shall show that $\phi(x)=0$. Replacing x by γx for suitable $\gamma\in\mathrm{GL}(2,F)$, we assume that $\mathrm{ht}^{+}(x)\geq n$. By our choice of n, $\phi((\left(\begin{smallmatrix}1&z\\0&1\end{smallmatrix}\right)x)=\phi(x)$ for all z in $\mathbb{A}$. Since ϕ is a cusp form, $\phi(x)=0$. $\square$

Corollary 2.14. *The representation of* $\mathrm{GL}(2,\mathbb{A})$ *in* $A_0^E(\chi)$ *is admissible.*

Proposition 2.15. *Let E' be an extension of E, and $\chi:\mathbb{A}^\times/F^\times\to E^\times$ a character. Then $A_0^{E'}(\chi)=A_0^E(\chi)\otimes_E E'$.*

Proof. The space $A_0^E(\chi)\otimes_E E'$ consists of the functions ϕ in $A_0^{E'}(\chi)$ whose values span a finite dimensional space over E, since $\phi\in A_0^E(\chi)$ takes finite number of values times the set Γ of values of χ. But every ϕ in $A_0^{E'}(\chi)$ has this property, since the set of its values lies in finitely many cosets of Γ. $\square$

Given a representation π of $\mathrm{GL}(2,\mathbb{A})$ over E and a character $\omega:\mathbb{A}^\times\to E^\times$, write $\omega\pi$ or $\pi\omega$ or $\omega\otimes\pi$ or $\pi\otimes\omega$ for the representation $(\pi\omega)(x)=\omega(\det x)\pi(x)$ in the space of π.

Proposition 2.16. *For any characters χ, $\omega:\mathbb{A}^\times/F^\times\to E^\times$, we have* $A_0^E(\chi)\otimes\omega=A_0^E(\chi\omega^2)$.

Proof. We need to construct an invertible linear map $L:A_0^E(\chi)\to A_0^E(\chi\omega^2)$ such that for every $\phi\in A_0^E(\chi)$ and $h\in\mathrm{GL}(2,\mathbb{A})$ we have $r(h)L(\phi)=\omega(\det h)L(r(h)\phi)$, where $(r(h)\phi)(x)=\phi(xh)$.
Such L is $(L\phi)(x)=\phi(x)\omega(\det x)$. $\square$

Proposition 2.17. *Given a character $\chi:\mathbb{A}^\times/F^\times\to E^\times$ there exists a character $\omega:\mathbb{A}^\times/F^\times\to E^\times$ such that $\chi(x)\omega(x)^2$ is a root of unity for every x in $\mathbb{A}^\times/F^\times$.*

Proof. Fix $\alpha \in \mathbb{A}^\times/F^\times$ with $\deg \alpha = 1$. Such α exists since in the finite field extension $F/\mathbb{F}_q(t)$, where $t \in F$ is transcendental over $\mathbb{F}_q$, there are always primes which split completely. Fix c in the algebraically closed field E with $c^2 = \chi(\alpha)$. Define $\omega : \mathbb{A}^\times/F^\times \to E^\times$ by $\omega(x) = c^{-\deg(x)}$, put $\chi_1(x) = \chi(x)\omega^2(x)$, put $\alpha^{\mathbb{Z}} = \{\alpha^n; n \in \mathbb{Z}\}$. Then χ_1 is a character of the profinite group $\mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}}$, hence the values of χ_1 are roots of 1. □

Proposition 2.18. *Let E be a subfield of $\mathbb{C}$ invariant under complex conjugation, χ an $E^\times$-valued unitary character of $\mathbb{A}^\times/F^\times$. Then the representation of $\mathrm{GL}(2,\mathbb{A})$ in $A_0^E(\chi)$ is unitary.*

Proof. The function $x \mapsto \phi_1(x)\overline{\phi}_2(x)$ on $\mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$, where ϕ_1, $\phi_2 \in A_0^E(\chi)$, is invariant under $Z(\mathbb{A})$ and is compactly supported as a function on $\mathrm{PGL}(2,F)\backslash \mathrm{PGL}(2,\mathbb{A})$. Let dx be an invariant measure on $\mathrm{PGL}(2,F)\backslash \mathrm{PGL}(2,\mathbb{A})$. It exists since $\mathrm{PGL}(2,F)$ is a discrete subgroup of $\mathrm{PGL}(2,\mathbb{A})$, a group with a two-sided invariant measure. Then

$$(\phi_1,\phi_2) = \int \phi_1(x)\overline{\phi}_2(x)dx \qquad (x \in \mathrm{PGL}(2,F)\backslash \mathrm{PGL}(2,\mathbb{A}))$$

is an invariant scalar product on $A_0^E(\chi)$. □

Corollary 2.19. *The representation of $\mathrm{GL}(2,\mathbb{A})$ in $A_0^E(\chi)$ is a direct sum of irreducible subrepresentations.*

Note that we may assume that all values of χ are roots of unity, and that $E = \overline{\mathbb{Q}}$.
The *multiplicity one theorem* asserts that in $A_0^E(\chi)$ any irreducible representation of $\mathrm{GL}(2,\mathbb{A})$ occurs with multiplicity one.
An irreducible representation of $\mathrm{GL}(2,\mathbb{A})$ over an algebraically closed field E is called *cuspidal* if it is isomorphic to a subrepresentation of A_0^E.

2.6 Factorizability

Irreducible admissible representations of $\mathrm{GL}(2,\mathbb{A})$ are factorizable, as we proceed to show. Let E denote an algebraically closed subfield of $\mathbb{C}$. An irreducible representation of $\mathrm{GL}(2,F_v)$ in an E-space V is *unramified* if V contains a nonzero $\mathrm{GL}(2,O_v)$-invariant vector.

Proposition 2.20. *The space of $\mathrm{GL}(2,O_v)$-invariant vectors $V^{\mathrm{GL}(2,O_v)}$ in an unramified representation (π,V) of $\mathrm{GL}(2,F_v)$ is one dimensional.*

Proof. Denote by $H_v = C_c(\mathrm{GL}(2,O_v)\backslash \mathrm{GL}(2,F_v)/\mathrm{GL}(2,O_v))$ the Hecke convolution algebra of compactly supported $\mathrm{GL}(2,O_v)$-biinvariant E-valued functions on $\mathrm{GL}(2,F_v)$. We claim it is a commutative algebra. Indeed, for any $f \in H_v$, the function ${}^tf(x) = f({}^tx)$, where ${}^tx$ is the transpose of x, is also in H_v. Since ${}^t(xy) = {}^ty{}^tx$, we have ${}^t(f_1 * f_2) = {}^tf_2 * {}^tf_1$ for all $f_1, f_2 \in H_v$. By Cartan decomposition every $\mathrm{GL}(2,O_v)$-double coset in $\mathrm{GL}(2,F_v)$ contains a diagonal matrix. Hence ${}^t f = f$ for all $f \in H_v$, and $f_1 * f_2 = {}^t(f_1 * f_2) = {}^t f_2 * {}^t f_1 = f_2 * f_1$ for all $f_1,\ f_2 \in H_v$. If V is unramified, $V^{\mathrm{GL}(2,O_v)}$ is a nonzero irreducible H_v-module. But H_v is commutative, so $\dim_E V^{\mathrm{GL}(2,O_v)}$ is 1. □

Given an irreducible admissible representation π_v of $\mathrm{GL}(2, F_v)$ in a space V_v for every closed point $v \in |X|$ such that π_v is unramified for all $v \in S$, $S \subset |X|$ finite, construct a representation $\pi = \otimes \pi_v$ of $\mathrm{GL}(2, \mathbb{A})$ as follows. For each $v \in |X| - S$ choose a nonzero vector $\xi_v^0 \in V_v^{\mathrm{GL}(2,O_v)}$. For any finite set $S' \supset S$ of closed points of X put $V_{S'} = \otimes_{v \in S'} V_v$. If $S'' \supset S' \supset S$, define an inclusion $V_{S'} \hookrightarrow V_{S''}$ by $x \mapsto (\otimes_{v \in S''-S'} \xi_v^0) \otimes x$. Put $V = \varinjlim_{S' \supset S} V_{S'}$. It is the span of the vectors $\otimes_{v \in |X|} \xi_v$, $\xi_v = \xi_v^0$ for almost all v, and $\xi_v \in V_v$ for all $v \in |X|$. Then V is a $\mathrm{GL}(2, \mathbb{A})$-module in a natural way; denote by π the corresponding representation of $\mathrm{GL}(2, \mathbb{A})$. The vectors ξ_v^0 are determined uniquely up to a scalar multiple, hence π is uniquely determined by the π_v for all $v \in |X|$.
Reducing to irreducible finite dimensional representations of tensor products of algebras, we have

Proposition 2.21. *Given an irreducible admissible representation π_v of $\mathrm{GL}(2, F_v)$ for every v in $|X|$ which is unramified for almost all v, $\pi = \otimes_v \pi_v$ is an irreducible admissible representation of $\mathrm{GL}(2, \mathbb{A})$. Every irreducible admissible representation π of $\mathrm{GL}(2, \mathbb{A})$ equals $\otimes_v \pi_v$ for some irreducible admissible representations π_v of $\mathrm{GL}(2, F_v)$ which are almost all unramified. The representations π_v are determined by π uniquely up to isomorphism.*

3 Looking for a trace formula

3.1 Trace formula in the compact case

Let X be an ℓ-space. Denote by $C^\infty(X)$ the space of E-valued locally constant (= smooth) functions on X. Here E is a fixed algebraically closed subfield of $\mathbb{C}$. Let $C_c^\infty(X)$ be the space of smooth compactly supported E-valued functions on X. Let r be an admissible representation of an ℓ-group G in an E-space V. Fix a Haar measure dx on G. Given $f \in C_c^\infty(G)$, define $r(f) = \int_G f(x) r(x) dx$, an endomorphism of V. Since f is C^∞, that is smooth, it is right invariant under an open subgroup U of G. Then $\operatorname{Im} r(f) \subset V^U$, so $\operatorname{Im} r(f)$ is finite dimensional, and the trace $\operatorname{tr} r(f)$ is well defined. Let r be now the representation of G on $C^\infty(\Gamma \backslash G)$ by right translation, where Γ is a discrete cocompact subgroup of G. Since r is admissible, $\operatorname{tr} r(f)$ is defined.

Proposition 3.1. *Let G be an ℓ-group. Let Γ be a discrete cocompact sugroup of G. Then G has a two sided invariant measure and $\Gamma \backslash G$ has a G-invariant measure.*

Proof. Since (see [BZ76]) $\Gamma \backslash G$ admits a measure which when translated by x in G is multiplied by $\Delta(x)$, where Δ is the modulus of G, we have $|\Gamma \backslash G| = \Delta(x) |\Gamma \backslash G|$, thus $\Delta = 1$. □

Proposition 3.2. *Let X be an ℓ-space, dx a measure on X, $K \in C_c^\infty(X \times X)$. Define a linear endomorphism A of $C^\infty(X)$ by $(A\phi)(y) = \int_X K(x,y)\phi(x)dx$. Then the image of A is finite dimensional and* $\operatorname{tr} A = \int_X K(x,x)dx$.

Proof. We may assume that $K(x,y)$ is of the form $\varphi(x)\psi(y)$, as such functions span $C_c^\infty(X \times X)$. In this case the claim is clear. □

Proposition 3.3. *Let G be an ℓ-group, Γ a discrete cocompact subgroup, r the representation of G in $C^\infty(\Gamma\backslash G)$ by right translation, dx a Haar measure on $G, f \in C_c^\infty(G), S$ a set of representatives of the conjugacy classes in $\Gamma, Z_\Gamma(\gamma)$ the centralizer of γ in Γ. Then* $\operatorname{tr} r(f) = \sum_{\gamma\in S} \int_{G/Z_\Gamma(\gamma)} f(x\gamma x^{-1})dx$.

Proof. We first show that for each $\gamma \in \Gamma$ the function $x \mapsto f(x\gamma x^{-1})$ on $G/Z_\Gamma(\gamma)$ is compactly supported, and that there are at most finitely many $\gamma \in S$ for which $x \mapsto f(x\gamma x^{-1})$ is not identically zero. For this, fix a compact subset K in G with $K\Gamma = G$. Given $x \in G$ there are $k \in K, \delta \in \Gamma$, with $x = k\delta$. Fix $\gamma \in \Gamma$. If $f(x\gamma x^{-1}) \neq 0$ then $k\delta\gamma\delta^{-1}k^{-1}$ lies in $\operatorname{supp} f$, thus $\delta\gamma\delta^{-1} \in K_f = K{\cdot}\operatorname{supp} f \cdot K$. Since K_f is compact $K_f \cap \Gamma$ is finite, and there are only finite number of possibilities for $\delta\gamma\delta^{-1}$. Hence there are only a finite number of possibilities $\delta_1, \dots, \delta_n$ for δ modulo $Z_\Gamma(\gamma)$. Then $f(x\gamma x^{-1}) \neq 0$ implies that $x \in K' Z_\Gamma(\gamma)$, where $K' = \cup_{1\le i\le n} K\delta_i$ is compact. If $f(x\gamma x^{-1}) \neq 0$, the conjugacy class of γ in Γ intersects the finite set $K_f \cap \Gamma$. The number of such classes is finite. Thus the sum is finite and the integrals converge.
Now given ϕ in $C^\infty(\Gamma\backslash G)$, for any y in G we have

$$(r(f)\phi)(y) = \int_G f(x)\phi(yx)dx = \int_G f(y^{-1}x)\phi(x)dx = \int_{\Gamma\backslash G} K_f(x,y)\phi(x)dx$$

where $K_f(x,y) = \sum_{\gamma\in\Gamma} f(y^{-1}\gamma x)$. Then

$$\operatorname{tr} r(f) = \int_{\Gamma\backslash G} K_f(x,x)dx = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)dx$$

$$= \int_{\Gamma\backslash G} \sum_{\gamma\in S} \sum_{\delta\in Z_\Gamma(\gamma)\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)dx = \sum_{\gamma\in S} \int_{\Gamma\backslash G} \sum_{\delta\in Z_\Gamma(\gamma)\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)dx$$

$$= \sum_{\gamma\in S} \int_{Z_\Gamma(\gamma)\backslash G} f(x^{-1}\gamma x)dx.$$

□

3.2 Case of GL(2), oversimplified

Let now A_0^E denote the space of E-valued cusp forms on $\mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$. The right-shifts representation of $\mathrm{GL}(2,\mathbb{A})$ on A_0^E is not admissible since the center $Z(\mathbb{A})$ of $\mathrm{GL}(2,\mathbb{A})$ is not compact. Fix a degree-one idèle α and put $\alpha^{\mathbb{Z}} = \{\alpha^n; n \in \mathbb{Z}\}$. It is a cyclic subgroup of $\mathbb{A}^\times$, and we view $\mathbb{A}^\times$ as the center of $\mathrm{GL}(2,\mathbb{A})$. Denote by $A^E_{0,\alpha}$ the space of cusp forms in A^E_0 invariant under α, and by r_0 the representation of $\mathrm{GL}(2,\mathbb{A})$ on $A^E_{0,\alpha}$ by right translation. Since $\mathbb{A}^\times/F^\times\alpha^{\mathbb{Z}}$ is compact and every U-invariant cusp form – where U is an open subgroup of $\mathrm{GL}(2,\mathbb{A})$ – is supported on some compact modulo $Z(\mathbb{A})$ set $K\subset \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$, the representation r_0 is admissible. Hence $\operatorname{tr} r_0(f)$ is defined for every $f\in C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$.

Put $A_{c,\alpha}=C_c^\infty(\alpha^{\mathbb{Z}}\cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}))$. Fix $f\in C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$. Let r be the right representation of $\mathrm{GL}(2,\mathbb{A})$ on $A_{c,\alpha}$. We proceed to compute $\operatorname{tr} r(f)$ as if the space $\alpha^{\mathbb{Z}}\cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$ were compact, to see what needs to be corrected. This space is not compact and r is not admissible, so that in fact $\operatorname{tr} r(f)$ makes no sense.

For any ring R define $A(R)=\{\mathrm{diag}(a,b); a,b\in R^\times\}$, $A'(R)=\{\mathrm{diag}(a,b); a,b\in R^\times,\ a\neq b\}$, $N(R)=\{\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right); a\in R\}$. Let Q be the set of quadratic extensions of the field F. For each $L\in Q$ choose an embedding $L\hookrightarrow M(2,F)$; it exists and is unique up to an automorphism of $M(2,F)$; all automorphisms of $M(2,F)$ are inner. Given $\gamma\in\alpha^{\mathbb{Z}}\cdot \mathrm{GL}(2,F)$, denote by $Z(\gamma)$ the centralizer of γ in $\alpha^{\mathbb{Z}}\,\mathrm{GL}(2,F)$.

Proposition 3.4. *Every conjugacy class of $\alpha^{\mathbb{Z}}\cdot\mathrm{GL}(2,F)$ intersects precisely one of : $F^\times\cdot\alpha^{\mathbb{Z}}$; $a\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $a\in F^\times\cdot\alpha^{\mathbb{Z}}$; $\alpha^{\mathbb{Z}}\cdot A'(F)$; $\alpha^{\mathbb{Z}}\cdot(L^\times-F^\times)$ for some $L\in Q$. In the first two cases the number of intersection points is 1, in the 3rd case 2, in the 4th case: the number of automorphisms of L over F. The centralizers $Z(\gamma)$ are $\alpha^{\mathbb{Z}}\cdot\mathrm{GL}(2,F)$, $\alpha^{\mathbb{Z}}F^\times N(F)$, $\alpha^{\mathbb{Z}}\cdot A(F)$, $\alpha^{\mathbb{Z}}L^\times$, respectively.*

Immitating the trace formula in the compact case, one may expect

$$\operatorname{tr} r(f)=S_1(f)+\sum_{L\in Q}S_{2,L}(f)+S_3(f)+S_4(f)$$

with

$$S_1(f)=|\alpha^{\mathbb{Z}}\cdot\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A})|,$$

$$S_{2,L}(f)=|\operatorname{Aut}_F(L)|^{-1}\sum_{\gamma\in\alpha^{\mathbb{Z}}\cdot(L^\times-F^\times)}\int_{\alpha^{\mathbb{Z}}\cdot L^\times\backslash\mathrm{GL}(2,\mathbb{A})}f(x^{-1}\gamma x)dx,$$

$$S_3(f)=\frac{1}{2}\sum_{\gamma\in\alpha^{\mathbb{Z}}A'(F)}\int_{\alpha^{\mathbb{Z}}A(F)\backslash\mathrm{GL}(2,\mathbb{A})}f(x^{-1}\gamma x)dx,$$

$$S_4(f)=\sum_{a\in\alpha^{\mathbb{Z}}\cdot F^\times}\int_{\alpha^{\mathbb{Z}}F^\times N(F)\backslash\mathrm{GL}(2,\mathbb{A})}f(x^{-1}a\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)x)dx.$$

The left side of this wrong trace formula is divergent. So is $S_3(f)$, since the homogeneous space $A(\mathbb{A})/\alpha^{\mathbb{Z}}\cdot A(F)$ is not compact. We shall show that $S_1(f)$ and $\sum_{L\in Q}S_{2,L}(f)$ converge, and although $S_4(f)$ diverges, we shall show in which way it does.

Proposition 3.5. *Given $f\in C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$, the number of conjugacy classes of $\gamma\in\alpha^{\mathbb{Z}}\cdot\mathrm{GL}(2,F)$ with $x\in\mathrm{GL}(2,\mathbb{A})$ and $f(x\gamma x^{-1})\neq 0$ is finite.*

Proof. The sets $K_1 = \{\operatorname{tr} h; h \in \operatorname{supp} f\} \subset \mathbb{A}$, $K_2 = \{\det h; h \in \operatorname{supp} f\} \subset \mathbb{A}^\times$ are compact. It suffices to show that the set $\{\gamma \in \alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F); \operatorname{tr}\gamma \in K_1, \det\gamma \in K_2\}$ is a union of finitely many conjugacy classes. Put $\gamma = \alpha^n x$ for some $x \in \mathrm{GL}(2,F)$. Then $2n = \deg\gamma$, so n lies in a finite set. Fix n. Then $\operatorname{tr} x \in \alpha^{-n}K_1, \det x \in \alpha^{-2n}K_2$. But the sets $F \cap \alpha^{-n}K_1$ and $F^\times \cap \alpha^{-2n}K_2$ are finite. Hence the trace and determinant of x can take only finitely many values. As the number of conjugacy classes of elements in $\mathrm{GL}(2,F)$ with fixed trace and determinant is at most two, we are done. □

3.3 Central elements

Proposition 3.6. *The volume* $|\mathrm{GL}(2,F)\cdot\alpha^{\mathbb{Z}}\backslash \mathrm{GL}(2,\mathbb{A})|$ *is finite.*

Proof. This volume is equal to (below $x \in \alpha^{\mathbb{Z}}\,\mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})/\mathrm{GL}(2,O_{\mathbb{A}})$)

$$\begin{aligned}&\textstyle\sum_x |\alpha^{\mathbb{Z}}\,\mathrm{GL}(2,F)\cap x\,\mathrm{GL}(2,O_{\mathbb{A}})x^{-1}\backslash x\,\mathrm{GL}(2,O_{\mathbb{A}})|\\ &= |\mathrm{GL}(2,O_{\mathbb{A}})|\textstyle\sum_x |\alpha^{\mathbb{Z}}\,\mathrm{GL}(2,F)\cap x\,\mathrm{GL}(2,O_{\mathbb{A}})x^{-1}|^{-1}.\end{aligned}$$

For x in $\mathrm{GL}(2,\mathbb{A})/\mathrm{GL}(2,O_{\mathbb{A}})$, let $\mathcal{E} = xO_{\mathbb{A}}^2$ be the associated rank 2 locally free sheaf on X. Then $\mathrm{Aut}(\mathcal{E})$ consists of the $g \in \mathrm{GL}(2,\mathbb{A})$ which map $(\mathcal{E} =)xO_{\mathbb{A}}^2$ to $xO_{\mathbb{A}}^2$ and the generic fiber F^2 to itself, thus $\operatorname{Aut}\mathcal{E}$ is

$$\mathrm{GL}(2,F)\cap x\,\mathrm{GL}(2,O_{\mathbb{A}})x^{-1} = \alpha^{\mathbb{Z}}\,\mathrm{GL}(2,F)\cap x\,\mathrm{GL}(2,O_{\mathbb{A}})x^{-1}.$$

We then need to show the convergence of

$$\sum_{\mathcal{E}\in \mathrm{Bun}_2/J} |\operatorname{Aut}\mathcal{E}|^{-1},$$

J being the image of $\alpha^{\mathbb{Z}}$ under the natural homomorphism $\mathbb{A}^\times \to \operatorname{Pic} X$. The number of J-orbits on the set of stable rank two locally free sheaves on X is finite, so it remains to show that the sum of $|\operatorname{Aut}\mathcal{E}|^{-1}$ over the set $\mathrm{Bun}_2^{\mathrm{un}}$ of J-orbits of unstable rank two locally free sheaves on X is convergent.

Lemma 3.7. (1) *A rank two locally free sheaf* $\mathcal{E}$ *on* X *is very unstable* $(\mathrm{ht}(\mathcal{E}) \geq 2g-1)$ *iff* $\mathcal{E} \simeq \mathcal{L}\oplus\mathcal{M}$ *where* $\mathcal{L}$, $\mathcal{M}$ *are invertible sheaves with* $\deg\mathcal{L}-\deg\mathcal{M} \geq 2g-1$.
(2) *If* $\mathcal{L},\mathcal{M} \in \operatorname{Pic} X$ *and* $\deg\mathcal{L}-\deg\mathcal{M} \geq \max(2g-1,1)$ *then*

$$|\operatorname{Aut}(\mathcal{L}\oplus\mathcal{M})| = (q-1)^2 q^{\deg\mathcal{L}-\deg\mathcal{M}+1-g}.$$

(3) *If* $\mathcal{L}\oplus\mathcal{M} \simeq \mathcal{L}'\oplus\mathcal{M}'$ *with* $\deg\mathcal{L} > \deg\mathcal{M}$, $\deg\mathcal{L}' > \deg\mathcal{M}'$ *then* $\mathcal{L} \simeq \mathcal{L}', \mathcal{M} \simeq \mathcal{M}'$.

Proof. (1) If $\mathcal{L}$ is an invertible sheaf of $\mathcal{E}$ of maximal degree and $\mathcal{M} = \mathcal{E}/\mathcal{L}$, then $\mathcal{M}$ is invertible, and $\mathrm{Ext}(\mathcal{M},\mathcal{L}) = H^1(X,\mathcal{M}^{-1}\mathcal{L})$ is 0 (by Serre duality) as

$$\deg\mathcal{M}^{-1}\mathcal{L} = \deg\mathcal{L}-\deg\mathcal{M} = 2\deg\mathcal{L}-\deg\mathcal{E} = \mathrm{ht}(\mathcal{E}) \geq 2g-1.$$

The exact sequence

$$0 \to \mathrm{Hom}(\mathcal{M},\mathcal{L}) \to \mathrm{Aut}(\mathcal{L}\oplus\mathcal{M}) \to \mathrm{Aut}\,\mathcal{L}\times\mathrm{Aut}\,\mathcal{M} \to 0$$

implies (2) since $\mathrm{Hom}(\mathcal{M},\mathcal{L}) = H^0(X,\mathcal{M}^{-1}\mathcal{L})$ and $H^1(X,\mathcal{M}^{-1}\mathcal{L}) = \{0\}$, so the Riemann-Roch theorem implies that $\dim H^0(X,\mathcal{M}^{-1}\mathcal{L}) = \deg(\mathcal{M}^{-1}\mathcal{L}) + 1-g$. If the invertible sheaf $\mathcal{L}$ corrsponds to $aO_{\mathbb{A}}$, then $\mathrm{Aut}\,\mathcal{L}$ consists of $g \in \mathbb{A}^\times$ which map the generic fiber F onto itself (thus $g \in F^\times$) and map $aO_{\mathbb{A}}$ onto itself (thus $g \in O_{\mathbb{A}}^\times$). Then $\mathrm{Aut}\,\mathcal{L} = F^\times \cap O_{\mathbb{A}}^\times = \mathbb{F}_q^\times$ has cardinality $q-1$.
For (3), put $\mathcal{E} = \mathcal{L}\oplus\mathcal{M} \xrightarrow{\sim} \mathcal{L}'\oplus\mathcal{M}'$. Since $\deg\mathcal{L} > (\deg\mathcal{E})/2 > \deg\mathcal{M}'$, we have $\mathrm{Hom}(\mathcal{L},\mathcal{M}') = \{0\}$. Hence the image of $\mathcal{L}$ under the isomorphism $\mathcal{L}\oplus\mathcal{M} \xrightarrow{\sim} \mathcal{L}'\oplus\mathcal{M}'$ lies in $\mathcal{L}'$. Hence $\mathcal{L}\simeq\mathcal{L}'$ and $\mathcal{M}\simeq\mathcal{E}/\mathcal{L}\simeq\mathcal{E}/\mathcal{L}'\simeq\mathcal{M}'$. □

Assume $g \geq 1$, so that $2g-1 \geq 1$ (the case $g=0$ is similar). The lemma implies

$$\sum_{\mathcal{E}\in\mathrm{Bun}_2^{\mathrm{un}}/J} |\,\mathrm{Aut}\,\mathcal{E}|^{-1} = (q-1)^{-2}|\,\mathrm{Pic}^0(X)|\sum_{n\geq 2g-1} q^{g-1-n} < \infty.$$

□

Corollary 3.8. *If the Haar measure on* $\mathrm{GL}(2,\mathbb{A})$ *is normalized so that* $|\,\mathrm{GL}(2,O_{\mathbb{A}})|$ *is a rational number, then* $|\alpha^{\mathbb{Z}}\cdot\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A})| \in \mathbb{Q}$.

This follows from the proof of the last proposition.

3.4 Elliptic elements

Proposition 3.9. *Let L be a quadratic extension of F, $\gamma \in \alpha^{\mathbb{Z}}\cdot(L^\times - F^\times) \subset \mathrm{GL}(2,\mathbb{A})$, and $f \in C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$. Then the function $x \mapsto f(x\gamma x^{-1})$ on $\mathrm{GL}(2,\mathbb{A})/\alpha^{\mathbb{Z}}\cdot L^\times$ has compact support.*

Proof. We need to show that the map $x \mapsto x\gamma x^{-1}$ on $\mathrm{GL}(2,\mathbb{A})/\alpha^{\mathbb{Z}}\cdot L^\times$ is proper (the preimage of a compact is compact). Since $(L\otimes_F\mathbb{A})^\times/\alpha^{\mathbb{Z}}\cdot L^\times$ is compact, it suffices to show that the map $\psi(x) = x\gamma x^{-1}$, $\psi : \mathrm{GL}(2,\mathbb{A})/\mathbb{A}_L^\times \to \mathrm{GL}(2,\mathbb{A})$, is proper ($\mathbb{A}_L = L\otimes_F\mathbb{A}$ is the ring of adèles of L).

Lemma 3.10. *Let F be a local field in this lemma. Suppose $\gamma \in M(2,F)$ is regular, i.e. the subalgebra $E = F[\gamma]$ generated by γ is a field or is $F\times F$. Then the map $\psi : \mathrm{GL}(2,F)/E^\times \to \mathrm{GL}(2,F)$, $x \mapsto x\gamma x^{-1}$, is proper. Moreover, if $\gamma \in \mathrm{GL}(2,O)$ and the ring $O[\gamma]$ is integrally closed, then $\psi^{-1}(\mathrm{GL}(2,O)) = \mathrm{GL}(2,O)/E^\times \cap \mathrm{GL}(2,O)$.*

Proof. The conjugacy class C of γ is a closed subset of $\mathrm{GL}(2,F)$, since γ is regular. So it suffices to show that ψ maps $\mathrm{GL}(2,F)/E^\times$ homeomorphically onto C. It is clear that ψ is continuous, injective and $\mathrm{Im}\,\psi = C$. It remains to show that the map $\psi' : \mathrm{GL}(2,F) \to C$, $x \mapsto x\gamma x^{-1}$, is open. For this, it suffices to show that C is the set of F-points of a smooth variety $\mathbf{C}$ over F, and that ψ'

is smooth, that is its differential is everywhere onto. Since $\mathbf{C}$ is a homogeneous space under a connected group $\mathbf{G}$ is suffices to show that the tangent map $d\psi'$ of ψ' at the identity is onto. When verifying these properties of $\mathbf{C}$ and ψ', we may replace F with an extension, thus we may assume that γ is of the form $\mathrm{diag}(a,b)$ with $a \neq b$, or $\left(\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right)$ (if E is nonseparable over F). To compute the tangent map $d\psi' : \mathrm{Lie}\ G \to T_\gamma(\mathbf{C})$ of $\psi'(x) = x\gamma x^{-1}$ near the identity $x = 1$, let Y be in $\mathrm{Lie}\ G$, and put $x = 1 + \epsilon Y$, where $\epsilon^2 = 0$. Then $x^{-1} = 1 - \epsilon Y$ and $\psi'(x) = (1 + \epsilon Y)\gamma(1 - \epsilon Y) = 1 + \epsilon(Y\gamma - \gamma Y)$, so $d\psi'(Y) = Y\gamma - \gamma Y$ is onto the tangent space $T_\gamma(\mathbf{C})$ of $\mathbf{C}$ at γ, and ψ is proper.
If $x \in \mathrm{GL}(2,F)$ and $x\gamma x^{-1} \in \mathrm{GL}(2,O)$, put $M = x^{-1}O^2$. Then $\gamma M \subset M$. In addition, $\gamma \in \mathrm{GL}(2,O)$, so $\gamma O^2 \subset O^2$. Thus M and O^2 are $O[\gamma]$-submodules in F^2. Both modules are of finite type. As F^2 is a rank one free $E = F[\gamma]$-module, and we assume that $O[\gamma]$ is integrally closed, namely it is the ring of integers in $E = F[\gamma]$, both M and O^2 are rank one torsion free over the discrete valuation ring $O[\gamma]$ (being rank two over O). Hence there exists $a \in E^\times$ with $M = aO^2$. Thus $xaO^2 = O^2$, that is $xa \in \mathrm{GL}(2,O)$. □

Now for γ as in the proposition, for almost all closed points in X the component of α at v is 1, $\gamma \in \mathrm{GL}(2,O_v)$, and the ring $O_v[\gamma]$ is integrally closed. This and the lemma imply the proposition. □

3.5 Regularization of the unipotent terms

To study the integral which occurs in $S_4(f)$, we regularize it as

$$\theta_{a,f}(t) = \int_{\alpha^{\mathbb{Z}}\cdot F^\times N(F)\backslash \mathrm{GL}(2,F)} f(ax^{-1}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)x)t^{\mathrm{ht}^+(x)}dx.$$

Proposition 3.11. (1) *For every $f \in C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$ and $a \in \mathbb{A}^\times$, the integral $\theta_{a,f}(t)$ converges as an element of $\mathbb{C}((t))$, and $\zeta_F(q^{-1}t)^{-1}\theta_{a,f}(t) \in \mathbb{C}[t,t^{-1}]$, where $\zeta_F(t) = \prod_{v\in|X|}(1-t_v)^{-1}$, $t_v = t^{\deg v}$.*
(2) *If f is the characteristic function of $\mathrm{GL}(2,O_\mathbb{A})$ in $\mathrm{GL}(2,\mathbb{A})$, then*

$$\theta_{1,f}(t) = |\,\mathrm{GL}(2,O_\mathbb{A})| \cdot (q-1)^{-1}q^{g-1} \cdot |\,\mathrm{Pic}^0(X)|\zeta_F(q^{-1}t).$$

Proof. (1) It suffices to consider $f(x) = \prod_v f_v(x_v)$, $x = (x_v) \in \mathrm{GL}(2,\mathbb{A})$, where $f_v \in C_c^\infty(\mathrm{GL}(2,F_v))$ for all $v \in |X|$ and f_v is the characteristic function f_v^0 of $\mathrm{GL}(2,O_v)$ at almost all v, since such functions span $C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$. Normalize the measures on $F_v^\times$ and F_v so that $|O_v^\times| = 1 = |O_v|$. Denote by $\mathrm{val}_v(x_v)$ the valuation of $x_v \in F_v^\times$, normalized by $\mathrm{val}_v(\boldsymbol{\pi}_v) = 1$. Define a function

$$h_v^+ : \mathrm{GL}(2,F_v) \to \mathbb{Z} \text{ by } h_v^+(\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)k) = \mathrm{val}_v(a) - \mathrm{val}_v(c), \quad k \in \mathrm{GL}(2,O_v).$$

Then h_v^+ is well-defined and $\mathrm{ht}^+(x) = \sum_{v\in|X|} h_v^+(x_v)\deg(v)$. We have

$$\theta_{a,f}(t) = |\mathbb{A}^\times/\alpha^{\mathbb{Z}} \cdot F^\times| \cdot |\mathbb{A}/F| \prod_v \theta_{a_v,f_v}(t_v)$$

where

$$\theta_{a_v,f_v}(t_v)=\int_{F_v^\times N(F_v)\backslash \operatorname{GL}(2,F_v)} f_v(a_v x^{-1}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)x)t^{h_v^+(x)\deg v}dx$$

and $t_v=t^{\deg(v)}$. To compute it, note that $p_{n,v}=\operatorname{diag}(\boldsymbol{\pi}_v^n,1)$ $(n\in\mathbb{Z})$ make a set of representatives of the two sided coset space

$$F_v^\times N(F_v)\backslash \operatorname{GL}(2,F_v)/\operatorname{GL}(2,O_v).$$

Then

$$\theta_{a_v,f_v}(t_v)=\sum_{n\in\mathbb{Z}} t_v^n\int_{F_v^\times N(F_v)\cap p_{n,v}^{-1}\operatorname{GL}(2,O_v)p_{n,v}\backslash p_{n,v}^{-1}\operatorname{GL}(2,O_v)} f_v(a_v x^{-1}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)x)dx$$

$$=\sum_{n\in\mathbb{Z}} t_v^n|F_v^\times N(F_v)\cap p_{n,v}^{-1}\operatorname{GL}(2,O_v)p_{n,v}|^{-1}\int_{p_{n,v}^{-1}\operatorname{GL}(2,O_v)} f_v(a_v x^{-1}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)x)dx$$

$$=\sum_{n\in\mathbb{Z}} q_v^{-n}t_v^n\int_{\operatorname{GL}(2,O_v)} f_v(a_v y p_{n,v}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)p_{n,v}^{-1}y^{-1})dy=\sum_{n\in\mathbb{Z}}\tau_n(f_v)q_v^{-n}t_v^n,$$

where $\tau_n(f_v)=\int_{\operatorname{GL}(2,O_v)} f_v(a_v y\left(\begin{smallmatrix}1&\boldsymbol{\pi}_v^n\\0&1\end{smallmatrix}\right)y^{-1})dy$ is 0 if $n<<0$ and $\tau_n(f_v)=f_v(a_v)$ for $n>>0$.

If $a_v\in O_v^\times$ and f_v is the characteristic function of $\operatorname{GL}(2,O_v)$, then $\tau_n(f_v)=|\operatorname{GL}(2,O_v)|$ for $n\geq 0$ and $u_{n,v}=0$ for $n<0$, so

$$\theta_{a_v,f_v}(t_v)=|\operatorname{GL}(2,O_v)|(1-t_v/q_v)^{-1}.$$

(2) It remains to compute (note that $|O_{\mathbb{A}}^\times|=1$ and $|O_{\mathbb{A}}|=1$) :

$$|\mathbb{A}^\times N(\mathbb{A})/\alpha^{\mathbb{Z}}F^\times N(F)|=(|\mathbb{A}^\times/\alpha^{\mathbb{Z}}F^\times|/|O_{\mathbb{A}}^\times|)(|\mathbb{A}/F|/|O_{\mathbb{A}}|).$$

The exact sequence $1\to\mathbb{F}_q^\times\to O_{\mathbb{A}}^\times\to\mathbb{A}^\times/\alpha^{\mathbb{Z}}F^\times\to\operatorname{Pic}X/\alpha^{\mathbb{Z}}(=\operatorname{Pic}^0(X))\to 1$ implies that the first factor on the right is $|\operatorname{Pic}^0(X)|/(q-1)$. The exact sequence

$$0\to\mathbb{F}_q\to O_{\mathbb{A}}\to\mathbb{A}/F\to H^1(X,O_X)\to 0$$

implies that the second factor on the right is q^{g-1}. □

4 Intertwining operators and Eisenstein series

4.1 Intertwining operators

Let E be an algebraically closed field of characteristic zero, and $v\in|X|$ a closed point of X. Denote by $|a|_v$ the absolute value of $a\in F_v^\times$ normalized by $|\boldsymbol{\pi}_v|=q_v^{-1}$. It is an $E^\times$-valued character of $F_v^\times$. Fix a square root $\sqrt{q}=q^{1/2}$ of q in E. If $E\subset\mathbb{C}$ we choose $q^{1/2}>0$. For E-valued characters

μ_1, μ_2 of $F_v^\times$ *denote by* $I(\mu_1, \mu_2)$ *both the space of right locally constant functions* $\phi : \mathrm{GL}(2, F_v) \to E$ with $\phi(\left(\begin{smallmatrix} a_1 & b \\ 0 & a_2 \end{smallmatrix}\right) x) = |a_1/a_2|_v^{1/2} \mu_1(a_1)\mu_2(a_2)\phi(x)$ $(x \in \mathrm{GL}(2, F_v); a_1, a_2 \in F_v^\times; b \in F_v)$, *and the action of the group* $\mathrm{GL}(2, F_v)$ by right translation on $I(\mu_1, \mu_2)$. The induced representation $I(\mu_1, \mu_2)$ is admissible by the Iwasawa decomposition $G = BK$. It is unitarizable when μ_1, μ_2 are unitary. It is possible to work with $I(|\cdot|_v^{1/2}\mu_1, |\cdot|_v^{1/2}\mu_2)$, in whose definition the factor $|a_1/a_2|_v^{1/2}\mu_1(a_1)\mu_2(a_2)$ becomes $|a_1|_v \mu_1(a_1)\mu_2(a_2)$, but later we shall need to multiply back by $|\cdot|_v^{-1/2}$. The following is a standard basic result.

Proposition 4.1. *If* $\mu_1/\mu_2 \neq |\cdot|_v, |\cdot|_v^{-1}$, *then the representations of* $\mathrm{GL}(2, F_v)$ *in* $I(\mu_1, \mu_2)$ *and* $I(\mu_2, \mu_1)$ *are irreducible and isomorphic. If* $\mu_1/\mu_2 = |\cdot|_v$ *or* $|\cdot|_v^{-1}$ *then* $I(\mu_1, \mu_2)$ *contains a unique proper invariant subspace* $I'(\mu_1, \mu_2)$ *and there is a* $\mathrm{GL}(2, F_v)$*-isomorphism* $I'(\mu_1, \mu_2) \simeq I(\mu_2, \mu_1)/I'(\mu_2, \mu_1)$. *If* $\mu_2/\mu_1 = |\cdot|_v$, *the subspace* $I'(\mu_1|\cdot|_v^{-1/2}, \mu_1|\cdot|_v^{1/2})$ *is one dimensional;* $x \in \mathrm{GL}(2, F_v)$ *acts on* $I'(\mu_1|\cdot|_v^{-1/2}, \mu_1|\cdot|_v^{1/2})$ *via multiplication by* $\mu_1(x)$. *The subspace*

$$I'(\mu_2|\cdot|_v^{1/2}, \mu_2|\cdot|_v^{-1/2}) \quad \textit{is denoted by} \quad \mathrm{St}(\mu_2) = \mathrm{St}(\mu_2|\cdot|_v^{1/2}, \mu_2|\cdot|_v^{-1/2}).$$

It is isomorphic to $I(\mu_2|\cdot|_v^{-1/2}, \mu_2|\cdot|_v^{1/2})/I'(\mu_2|\cdot|_v^{-1/2}, \mu_2|\cdot|_v^{1/2})$. *It consists of*

$$\phi \in I(\mu_2|\cdot|_v^{1/2}, \mu_2|\cdot|_v^{-1/2}) \quad \textit{with} \quad \int_{\mathrm{GL}(2,O_v)} \mu_2(\det x)^{-1}\phi(x)dx = 0.$$

If $I(\mu_1, \mu_2) \simeq I(\mu_1', \mu_2')$ *then* $\{\mu_1, \mu_2\} = \{\mu_1', \mu_2'\}$, *the representations* $I(\mu_1, \mu_2)$ $(\mu_1/\mu_2 \neq |\cdot|_v$ *or* $|\cdot|_v^{-1})$ *and* $\mathrm{St}(\mu_2')$ *are infinite dimensional and inequivalent, and* $\mathrm{St}(\mu_1) \simeq \mathrm{St}(\mu_2)$ *implies* $\mu_1 = \mu_2$.

We proceed to describe the operator intertwining $I(\mu_1, \mu_2)$ and $I(\mu_2, \mu_1)$.

Proposition 4.2. *If* $|\mu_1(\boldsymbol{\pi}_v)/\mu_2(\boldsymbol{\pi}_v)| < 1$ *the integral*

$$(M\phi)(x) = \int_{F_v} \phi(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) x)dy$$

converges for each $\phi \in I(\mu_1, \mu_2)$ *and* $x \in \mathrm{GL}(2, F_v)$, *and* $M\phi \in I(\mu_2, \mu_1)$.

Proof. As $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) = \begin{pmatrix} y^{-1} & -1 \\ 0 & y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y^{-1} & 1 \end{pmatrix}$, the integrand is

$$\mu_2(y)\mu_1(y)^{-1}|y|_v^{-1}\phi\left(\begin{pmatrix} 1 & 0 \\ y^{-1} & 1 \end{pmatrix} x\right),$$

which is 0 if $|y|_v$ is small, and $\mu_2(y)\mu_1(y)^{-1}|y|_v^{-1}\phi(x)$ if $|y|_v$ is big enough. For sufficiently large n then the part of the integral over $|y|_v \geq q_v^n$ is bounded by $\phi(x)$ times

$$\int\limits_{|y|_v \geq q_v^n} |\mu_2(y)/\mu_1(y)| \cdot |y|_v^{-1}dy = |O_v^\times| \sum_{k \geq n} |\mu_1(\boldsymbol{\pi}_v)/\mu_2(\boldsymbol{\pi}_v)|^k < \infty.$$

It is clear that $(M\phi)((\begin{smallmatrix}1&c\\0&1\end{smallmatrix})x)=(M\phi)(x)$ $(c\in F_v)$ and $(M\phi)((\begin{smallmatrix}a&0\\0&b\end{smallmatrix})x)$ equals

$$\int_{F_v}\phi((\begin{smallmatrix}b&0\\0&a\end{smallmatrix})(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix})(\begin{smallmatrix}1&yb/a\\0&1\end{smallmatrix})x)dy=\mu_1(b)\mu_2(a)\left|\frac{b}{a}\right|_v^{1/2}\left|\frac{a}{b}\right|_v(M\phi)(x).$$

□

We obtained, if $|\mu_1(\boldsymbol{\pi}_v)/\mu_2(\boldsymbol{\pi}_v)|<1$, a $\mathrm{GL}(2,F_v)$-equivariant map

$$M=M(\mu_1,\mu_2):I(\mu_1,\mu_2)\to I(\mu_2,\mu_1).$$

Let ν_t *be the unramified character of* $F_v^\times$ with $\nu_t(\boldsymbol{\pi}_v)=t$. Put $M(\mu_1,\mu_2,t)=M(\mu_1\nu_t,\mu_2\nu_{t^{-1}})$. It converges for any μ_1,μ_2, provided $t\in\mathbb{C}$ is small enough in absolute value. To define $M(\mu_1,\mu_2)$ as the value at $t=1$ of the analytic continuation of $M(\mu_1,\mu_2,t)$, we need these operators to be defined on the same space, which we will take to be

$$\begin{aligned}I_0(\mu_1,\mu_2)=\quad&\{\phi\in C^\infty(\mathrm{GL}(2,O_v));\phi((\begin{smallmatrix}a_1&b\\0&a_2\end{smallmatrix})x)=\mu_1(a_1)\mu_2(a_2)\phi(x),\\&\qquad a_1,a_2\in O_v^\times,\ \ b\in O_v,\ \ x\in\mathrm{GL}(2,O_v)\}.\end{aligned}$$

By the Iwasawa decomposition $G=BK$, the restriction map $I(\mu_1\nu_t,\mu_2\nu_{t^{-1}})\to I_0(\mu_1,\mu_2)$ is bijective for any t. Identifying these spaces, the operator $M(\mu_1,\mu_2,t)$ becomes a map $I_0(\mu_1,\mu_2)\to I_0(\mu_2,\mu_1)$.
Write $L(\mu,t)$ for $(1-\mu(\boldsymbol{\pi}_v)t)^{-1}$ if μ *is unramified*, and $L(\mu,t)=1$ if μ *is a ramified character of* $F_v^\times$.

Proposition 4.3. *The operator valued function* $M(\mu_1,\mu_2,t)$ *is rational in* $t\in\mathbb{C}^\times$. *In fact the function* $t\mapsto L(\mu_1/\mu_2,t^2)^{-1}(M(\mu_1,\mu_2,t)\phi)(x)$ *is a polynomial in* t *for all* $\phi\in I_0(\mu_1,\mu_2)$, $x\in\mathrm{GL}(2,O_v)$. *If* μ_1,μ_2 *are unramified and the restrictions of* $\phi\in I(\mu_1\nu_t,\mu_2\nu_{t^{-1}})$ *and* $\psi\in I(\mu_2\nu_{t^{-1}},\mu_1\nu_t)$ *to* $\mathrm{GL}(2,O_v)$ *are* 1, *then* $M(\mu_1,\mu_2,t)\phi=\frac{L(\mu_1/\mu_2,t^2)}{L(\mu_1/\mu_2,q_v^{-1}t^2)}\psi$.

Proof. Put $\phi_t=M(\mu_1,\mu_2,t)\phi$ and $a_1=\int_{|y|_v\le1}\phi((\begin{smallmatrix}0&-1\\1&y\end{smallmatrix})x)dy$ where $x\in\mathrm{GL}(2,O_v)$. Then

$$\phi_t(x)=a_1+\int_{|y|_v>1}\mu_2(y)\mu_1(y)^{-1}|y|_v^{-1}\nu_t(y)^{-2}\phi\left(\left(\begin{smallmatrix}1&0\\y^{-1}&1\end{smallmatrix}\right)x\right)dy.$$

We shall show that this is the Taylor series of a rational function.
If n is large enough, $\phi\left(\left(\begin{smallmatrix}1&0\\y^{-1}&1\end{smallmatrix}\right)x\right)=\phi(x)$ for $|y|_v\ge q_v^n$. Then $\phi_t(x)=a_1+a_2(t)+a_3(t)$ with

$$\begin{aligned}a_2(t)&=\int_{1<|y|_v<q_v^n}\mu_2(y)\mu_1(y)^{-1}|y|_v^{-1}\nu_t(y)^{-2}\phi\left(\left(\begin{smallmatrix}1&0\\y^{-1}&1\end{smallmatrix}\right)x\right)dy,\\a_3(t)&=\phi(x)\int_{|y|_v\ge q_v^n}\mu_2(y)\mu_1(y)^{-1}|y|_v\nu_t(y)^{-2}dy.\end{aligned}$$

Clearly $a_2(t)$ is a polynomial in t (since $\nu_t(\boldsymbol{\pi}_v^{-1})^{-1} = t$) and $a_3(t) = ct^{2n}L(\mu_1/\mu_2, t^2)$.
If μ_1, μ_2 are unramified and $x \in \mathrm{GL}(2, O_v)$, $a_1 = 1$ and the expression for $\phi_t(x)$ is

$$\begin{aligned}\phi_t(x) = 1 \quad &+ \quad \int_{|y|_v > 1} \mu_2(y)\mu_1(y)^{-1}|y|_v^{-1}\nu_t(y)^{-2}dy \\ &= \quad 1 - (1 - q_v^{-1}) \sum_{k \geq 1} (\mu_1(\boldsymbol{\pi}_v)/\mu_2(\boldsymbol{\pi}_v))^k t^{2k} \\ &= \quad 1 + \frac{(1 - q_v^{-1})(\mu_1(\boldsymbol{\pi}_v)/\mu_2(\boldsymbol{\pi}_v))t^2}{1 - (\mu_1(\boldsymbol{\pi}_v)/\mu_2(\boldsymbol{\pi}_v))t^2} = \frac{L(\mu_1/\mu_2, t^2)}{L(\mu_1/\mu_2, q_v^{-1}t^2)}.\end{aligned}$$

□

The operator $M(\mu_1, \mu_2, t) : I(\mu_1\nu_t, \mu_2\nu_{t^{-1}}) \to I(\mu_2\nu_{t^{-1}}, \mu_1\nu_t)$ intertwines the $\mathrm{GL}(2, F_v)$-modules for every t where it is defined. It can be regarded as a rational function of t (in fact, of t^2) with values in the set of operators $I_0(\mu_1, \mu_2) \to I_0(\mu_2, \mu_1)$. Indeed,

$$M(\mu_1, \mu_2, t) = M(\mu_1\nu_t, \mu_2\nu_{t^{-1}}) = M(\mu_1\nu_{t^2}, \mu_2).$$

Define

$$R(\mu_1, \mu_2, t) = \frac{L(\mu_1/\mu_2, q_v^{-1}t^2)}{L(\mu_1/\mu_2, t^2)} M(\mu_1, \mu_2, t).$$

Corollary 4.4. *Suppose μ_1 and μ_2 are unramified and $\varphi \in I(\mu_1\nu_t, \mu_2\nu_{t^{-1}})$, $\psi \in I(\mu_2\nu_{t^{-1}}, \mu_1\nu_t)$ are the functions whose restrictions to $\mathrm{GL}(2, O_v)$ are one, then $R(\mu_1, \mu_2, t)\varphi = \psi$.* □

Given characters μ_1, μ_2 of $\mathbb{A}^\times$, write $I(\mu_1, \mu_2)$ *for the space of right locally constant functions* ϕ on $\mathrm{GL}(2, \mathbb{A})$ which satisfy

$$\phi\left(\left(\begin{smallmatrix} a_1 & b \\ 0 & a_2 \end{smallmatrix}\right) x\right) = \mu_1(a_1)\mu_2(a_2)|a_1/a_2|^{1/2}\phi(x). \quad \text{Put} \quad \nu(a) = q^{\deg(a)}.$$

Then $I(\mu_1, \mu_2)$ is the restricted tensor product of the spaces $I(\mu_{1v}, \mu_{2v})$ where μ_{iv} is the component of μ_i at v (the restriction of μ_i to $F_v^\times \hookrightarrow \mathbb{A}^\times$); it is spanned by $\otimes_v \phi_v$ with $\phi_v \in I(\mu_{1v}, \mu_{2v})$ for all v and $\phi_v|\,\mathrm{GL}(2, O_v) = 1$ for almost all v, where $\mu_{iv}|O_v^\times = 1$, i.e. μ_{iv} are unramified. *Define the character* ν_t of $\mathbb{A}^\times$ by $\nu_t(a) = t^{\deg(a)}$. Then the restriction of ν_t to $F_v^\times$ is ν_{t_v}, the unramified character of $F_v^\times$ with $\nu_{t_v}(\boldsymbol{\pi}_v) = t_v (= t^{\deg(v)})$. As in the local case, we identify the spaces $I(\mu_1\nu_t, \mu_2\nu_{t^{-1}})$ with $I_0(\mu_1, \mu_2)$ for all t. The operator $R(\mu_1, \mu_2, t)$ from $I(\mu_1\nu_t, \mu_2\nu_{t^{-1}})$ to $I(\mu_2\nu_{t^{-1}}, \mu_1\nu_t)$ *defined by* $R(\mu_1, \mu_2, t) = \otimes_v R(\mu_{1v}, \mu_{2v}, t_v)$ is rational in t. On any element in $I(\mu_1\nu_t, \mu_2\nu_{t^{-1}})$ at most finitely many components $R(\mu_{1v}, \mu_{2v}, t_v)$ do not act as the identity. Also *write* $m(\mu, t)$ for $L(\mu, t)/L(\mu, t/q)$.

4.2 Eisenstein series

Write $A_\alpha = C^\infty(\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}))$,

$$A_{c,\alpha} = C_c^\infty(\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})), \quad Y = A(F)N(\mathbb{A})\backslash \mathrm{GL}(2,\mathbb{A})$$

and $Y_\alpha = Y/\alpha^{\mathbb{Z}}$. Normalize the Haar measure on $N(\mathbb{A}) \simeq \mathbb{A}$ by $|N(\mathbb{A})/N(F)| = |\mathbb{A}/F| = 1$. The Haar measure on $N(\mathbb{A})$ is invariant with respect to conjugation by the elements of $A(F)$ by the product formula. So it extends to a two-sided invariant measure on the space $\alpha^{\mathbb{Z}} \cdot A(F)N(\mathbb{A})$. This, and the two-sided Haar measure on $\mathrm{GL}(2,\mathbb{A})$ induce an invariant measure on Y_α.
Let φ and ψ be locally constant functions on Y_α, at least one of which is compactly supported. Put $(\varphi,\psi) = \int_{Y_\alpha} \varphi(x)\overline{\psi}(x)dx$. On $\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$ a scalar product is similarly defined. *Define the map* $E^* : A_\alpha \to C^\infty(Y_\alpha)$ by

$$\phi \mapsto \phi_N, \quad \phi_N(x) = \int_{N(F)\backslash N(\mathbb{A})} \phi(nx)dn, \quad x \in \mathrm{GL}(2,\mathbb{A}).$$

Note that $N(F)\backslash N(\mathbb{A})$ is compact, so the integral converges. Note that $\ker E^*$ is the space $A_{0,\alpha}$ of cusp forms invariant under α. For any $f \in C_c^\infty(Y_\alpha)$ *define a function* Ef on $\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$ by

$$(Ef)(x) = \sum_{\gamma \in A(F)N(F)\backslash \mathrm{GL}(2,F)} f(\gamma x), \quad x \in \mathrm{GL}(2,\mathbb{A}).$$

Proposition 4.5. *The sum defining* $(Ef)(x)$ *converges. For* $f \in C_c^\infty(Y_\alpha)$ *and* $\phi \in A_\alpha$ *we have* $(Ef,\phi) = (f, E^*\phi)$.

Proof. Consider the diagram

$$Y_\alpha \xleftarrow{r} \alpha^{\mathbb{Z}} \cdot A(F)N(F)\backslash \mathrm{GL}(2,\mathbb{A}) \xrightarrow{s} \alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}).$$

Since $N(F)\backslash N(\mathbb{A})$ is compact, the map r is proper. Hence the natural embedding r^* maps $C_c^\infty(Y_\alpha)$ to $C_c^\infty(\alpha^{\mathbb{Z}} \cdot A(F)N(F)\backslash \mathrm{GL}(2,\mathbb{A}))$. Given

$$\psi \in C_c^\infty(\alpha^{\mathbb{Z}} A(F)N(F)\backslash \mathrm{GL}(2,\mathbb{A})),$$

define a function $s_*\psi$ on $\alpha^{\mathbb{Z}}\,\mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$ by

$$(s_*\psi)(x) = \sum_{\gamma \in A(F)N(F)\backslash \mathrm{GL}(2,F)} \psi(\gamma x), \quad x \in \mathrm{GL}(2,\mathbb{A}).$$

The sum is finite since ψ is compactly supported, and

$$s_*\psi \in C_c^\infty(\alpha^{\mathbb{Z}}\,\mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})).$$

The sum which defines $(Ef)(x)$ converges since $E = s_* r^*$.
Now define $E^* = r_* s^*$, where s^* is the natural embedding, and

$$r_* : C^\infty(\alpha^{\mathbb{Z}} A(F)N(F)\backslash \mathrm{GL}(2,\mathbb{A})) \to C^\infty(Y_\alpha)$$

is defined by $(r_*h)(x) = \int_{N(F)\backslash N(\mathbb{A})} h(nx)dn$, $x \in \mathrm{GL}(2,\mathbb{A})$. Since (r^*, r_*) and (s_*, s^*) are adjoint pairs, so is $(E = s_*r^*, E^* = r_*s^*)$. □

The image $A_{E,\alpha}$ of the *Eisenstein map* $E = s_* r^* : C_c^\infty(Y_\alpha) \to A_{c,\alpha}$ is called the *Eisenstein part* of $A_{c,\alpha}$. The maps E and E^* intertwine the $\mathrm{GL}(2,\mathbb{A})$-action; $A_{E,\alpha}$ is an invariant subspace of $A_{c,\alpha}$.

Proposition 4.6. *The space $A_{c,\alpha}$ is an orthogonal direct sum of the space $A_{0,\alpha}$ of cusp forms and of $A_{E,\alpha}$.*

Proof. Cusp forms are compactly supported. Since $A_{0,\alpha} = \ker E^*$ and $A_{E,\alpha} = \operatorname{im} E$, we have $A_{0,\alpha} \perp A_{E,\alpha}$. Given a compact open subgroup U in $\mathrm{GL}(2,\mathbb{A})$, put A_α^U for the space of U-invariant functions in A_α, and

$$A_{c,\alpha}^U = A_{c,\alpha} \cap A_\alpha^U, \quad A_{0,\alpha}^U = A_{0,\alpha} \cap A_\alpha^U, \quad A_{E,\alpha}^U = A_{E,\alpha} \cap A_\alpha^U.$$

It remains to show that $A_{0,\alpha}^U + A_{E,\alpha}^U = A_{c,\alpha}^U$. If not there exists a nonzero linear form $\ell : A_{c,\alpha}^U \to \mathbb{C}$ which is zero on $A_{0,\alpha}^U + A_{E,\alpha}^U$. There exists $f \in A_\alpha^U$ such that $\ell(\phi) = (\phi, f)$ for every $\phi \in A_{c,\alpha}^U$. For any U-invariant function $\psi \in C_c^\infty(Y_\alpha)$ we have $(\psi, E^* f) = (E\psi, f) = \ell(E\psi) = 0$. Hence $E^* f = 0$, thus $f \in A_{0,\alpha}^U$. This however is impossible since f is orthogonal to the space $A_{0,\alpha}^U$ of U-invariant cusp forms. $\square$

Given $\phi \in C_c^\infty(Y_\alpha)$ and $x \in \mathrm{GL}(2,\mathbb{A})$, put $(M\phi)(x) = \int_{N(\mathbb{A})} \phi(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx) dn$. The integral converges, by

Proposition 4.7. *The map $N(\mathbb{A}) \to Y_\alpha$, $n \mapsto \alpha^{\mathbb{Z}} A(F) N(\mathbb{A}) \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx$, is proper.*

Proof. It suffices to consider the case of $x = 1$. The function

$$\mathrm{ht}^+ : Y_\alpha \to \mathbb{Z}, \qquad \left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right) k \mapsto \deg a - \deg b,$$

is continuous. Thus it suffices to show that the map $\varphi(a) = \mathrm{ht}^+(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right))$, $\varphi : \mathbb{A} \to \mathbb{Z}$, is proper. But $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & a_v \\ 0 & 1 \end{smallmatrix}\right)$ is in $\mathrm{GL}(2, O_v)$ if $|a_v|_v \leq 1$; otherwise it is $= \begin{pmatrix} a_v^{-1} & -1 \\ 0 & a_v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_v^{-1} & 1 \end{pmatrix}$. If $a = (a_v)$, then $\varphi(a) = -2 \sum_v \max(0, \log_q |a_v|_v)$, as $\log_q |a_v|_v = -\operatorname{val}_v(a_v) \deg(v)$. Hence φ is proper. $\square$

By definition, $x \mapsto (M\phi)(x)$ is invariant under left translation by $N(\mathbb{A})$, and also by $\alpha^{\mathbb{Z}} \cdot A(F)$. Indeed,

$$(M\phi)(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) x) = \int_{\mathbb{A}} \phi(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) n \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) x) dy = \left|\frac{a}{b}\right| \int_{N(\mathbb{Z})} \phi(\left(\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx) dn$$

and $|a/b| = q^{\deg(a/b)}$. Thus M maps $C_c^\infty(Y_\alpha)$ to $C^\infty(Y_\alpha)$.

Proposition 4.8. *Denote by I the natural embedding of $C_c^\infty(Y_\alpha)$ in $C^\infty(Y_\alpha)$. Then*

$$E^* E = I + M.$$

Proof. By the Bruhat decomposition, an element of $\mathrm{GL}(2,F)$ which is not in $A(F)N(F)$ has a unique decomposition $n_1 a \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) n_2$ with $n_i \in N(F)$, $a \in A(F)$. Thus, for any $\phi \in C_c^\infty(Y_\alpha)$, $x \in \mathrm{GL}(2,\mathbb{A})$, we have

$$(E\phi)(x) = \sum_{\gamma \in A(F)N(F)\backslash \mathrm{GL}(2,F)} \phi(\gamma x) = \phi(x) + \sum_{\nu \in N(F)} \phi(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \nu x).$$

Hence

$$\begin{aligned}(E^* E\phi)(x) &= |N(\mathbb{A})/N(F)|\phi(x) + \int_{N(F)\backslash N(\mathbb{A})} \sum_{\nu \in N(F)} \phi(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \nu n x) dn \\ &= \phi(x) + \int_{N(\mathbb{A})} \phi(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) n x) dn \quad = \quad \phi(x) + (M\phi)(x).\end{aligned}$$

□

Proposition 4.9. *Let μ_1, μ_2 be characters of $\mathbb{A}^\times/F^\times$. If t is sufficiently small, for all $\phi \in I(\mu_1\nu_t, \mu_2\nu_{t^{-1}})$ and $x \in \mathrm{GL}(2,\mathbb{A})$, the integral $(M(\mu_1,\mu_2,t)\phi)(x) = \int_{N(\mathbb{A})} \phi(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx)dn$ converges and defines a function in $I(\mu_2\nu_{t^{-1}}, \mu_1\nu_t)$. Moreover, $M(\mu_1,\mu_2,t) = q^{1-g} m(\mu_1/\mu_2, t^2) R(\mu_1,\mu_2,t)$.*

Proof. Recall that $|a| = q^{\deg(a)}$ and that $I(\mu_1,\mu_2)$ consists of the ϕ in $C^\infty(\mathrm{GL}(2,\mathbb{A}))$ with

$$\phi(\left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right) x) = |a_1/a_2|^{1/2} \mu_1(a_1)\mu_2(a_2)\phi(x),$$

while $\nu_t(a) = t^{\deg a}$. We put $t_v = t^{\deg(v)}$. We may assume that $\phi(x) = \prod_v \phi_v(x_v)$ with $\phi_v \in I(\mu_{1v}\nu_{t_v}, \mu_{2v}\nu_{t_v^{-1}})$. For almost all v, the restriction of ϕ_v to $\mathrm{GL}(2,O_v)$ is 1. We may replace ϕ_v, μ_i, t by their complex absolute values to assume $t > 0$ and ϕ_v, μ_i take real nonnegative values. Then $(M(\mu_1,\mu_2,t)\phi)(x) = c\prod_v \tau_v$, with $\tau_v = \int_{N(F_v)} \phi_v(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx_v)dn = \int_{F_v} \phi_v(\left(\begin{smallmatrix} 0 & -1 \\ 1 & z \end{smallmatrix}\right) x_v)dz$. The measure dn_v on $N(F_v)$ is normalized by $|N(O_v)| = 1$, and $c = |N(\mathbb{A})/N(F)|$ in the measure $\otimes_v dn_v$ on $N(\mathbb{A})$.
We saw that for small enough t the integral which defines τ_v converges for all v. For almost all v we have $\tau_v = L(\mu_{1v}/\mu_{2v}, t_v^2)/L(\mu_{1v}/\mu_{2v}, q_v^{-1}t_v^2)$, so the product $\prod_v \tau_v$ converges for small t. Now $M(\mu_1,\mu_2,t) = c\prod_v M(\mu_{1v},\mu_{2v},t_v)$. Each factor here is $\frac{L(\mu_{1v}/\mu_{2v},t_v^2)}{L(\mu_{1v}/\mu_{2v},q_v^{-1}t_v^2)} R(\mu_{1v},\mu_{2v},t_v)$. Put $R(\mu_1,\mu_2,t) = \otimes_v R(\mu_{1v},\mu_{2v},t_v)$, and $m(\mu,t) = \frac{L(\mu,t)}{L(q^{-1}t,\mu)}$, where $L(\mu,t) = \prod_v L(\mu_v,t_v)$. Note that c is $|O| = q^{1-g}$, using $0 \to \mathbb{F}_q \to O \to \mathbb{A}/F \to H^1(X,O_X) \to 0$. □

It follows (since $L(\mu,t)$ is a rational function of t) that after identifying the spaces $I(\mu_1\nu_t, \mu_2\nu_{t^{-1}})$ for all t, the operator

$$M(\mu_1,\mu_2,t) : I(\mu_1\nu_t, \mu_2\nu_{t^{-1}}) \to I(\mu_2\nu_{t^{-1}}, \mu_1\nu_t)$$

(defined for small t) depends on t rationally. Hence $M(\mu_1,\mu_2,t)$ is defined for almost all t, and it commutes with the action of $\mathrm{GL}(2,\mathbb{A})$.

4.3 L-functions

Let us review the theory of L-functions for GL(2). Let E be an algebraically closed field of characteristic zero. The valuation $\mathrm{val}_v(a)$ of $a \in F_v^\times$ is the largest integer n with $a \in \boldsymbol{\pi}_v^n O_v$. For any character $\psi : F_v \to E^\times$, $\psi \neq 1$, let $r(\psi)$ be the largest n such that $\psi(\boldsymbol{\pi}_v^{-n} O_v) = 1$. Normalize the Haar measure on F_v by $|O_v| = 1$. The *conductor* of a character $\chi : F_v^\times \to E^\times$ is $n = 0$ if $\chi(O_v^\times) = 1$, i.e., χ is unramified; otherwise it is the smallest $n \geq 1$ such that $\chi(1 + \boldsymbol{\pi}_v^n O_v) = 1$. Given χ, put $L(t, \chi) = (1 - \chi(\boldsymbol{\pi}_v)t)^{-1}$ if χ is unramified, $L(t, \chi) = 1$ is χ is ramified. Given $\psi \neq 1$, put

$$\Gamma(\chi, \psi, t) = \int_{F_v^\times} \chi(x)^{-1} \psi(x) t^{-\mathrm{val}_v(x)} dx, \quad \psi : F_v \to E^\times.$$

This $\Gamma(\chi, \psi, t)$ is a formal power series in t which contains positive and negative powers of t. Tate's thesis (see [Lg94], VII, section 3-4) establishes

Proposition 4.10. *The formal series $\Gamma(\chi, \psi, t)$ has finitely many positive powers of t. It is a rational function of t, namely a Laurent series of a rational function of t at $t = \infty$. Put $\varepsilon(\chi, \psi, t) = \frac{L(\chi,t)\Gamma(\chi,\psi,t)}{L(\chi^{-1}, q_v^{-1}t^{-1})}$. It has the form $c(\chi, \psi)t^{n(\chi,\psi)}$. If $r(\psi) = 0$ then $n(\chi, \psi)$ is the conductor of χ. If in addition χ is unramified then $\varepsilon(\chi, \psi, t)$ is 1. If $a \in F_v^\times, \psi_a(x) = \psi(ax)$, then $\varepsilon(\chi, \psi_a, t) = \chi(a)(q_v t)^{\mathrm{val}_v(a)} \varepsilon(\chi, \psi, t)$.*

Note that L and ε are usually considered, in the case where $E = \mathbb{C}$, as functions of s, where $t = q_v^{-s}$, rather than of t. The Haar measure on F_v is usually normalized by $|O_v| = q_v^{-r(\psi)/2}$, as this measure is self-dual with respect to the pairing $F_v \times F_v \to E^\times, (x, y) \mapsto \psi(xy)$. This choice of measure is not convenient if $E \neq \mathbb{C}$ since E has no distinguished square root of q.
Given a character χ of $\mathbb{A}^\times$, denote its restriction to $F_v^\times$ by χ_v. The restriction to F_v of a character ψ of $\mathbb{A}$ is denoted ψ_v. For a closed point v of X, we write $\deg(v)$ for the dimension of the residue field at v over $\mathbb{F}_q$, and $q_v = q^{\deg(v)}$. Given a character $\chi : \mathbb{A}^\times / F^\times \to E^\times$, put $L(\chi, t) = \prod_v L(\chi_v, t_v)$, where $t_v = t^{\deg(v)}$; the product converges in $E[[t]]$. Let $\psi : \mathbb{A}/F \to E^\times$ be a character $\neq 1$. Then $\varepsilon(\chi, t) = q^{1-g} \prod_v \varepsilon(\chi_v, \psi_v, t_v)$ converges as almost all factors are 1, and $\varepsilon(\chi, t)$ is independent of ψ by Proposition 4.10.

Proposition 4.11. *For any character $\chi : \mathbb{A}^\times / F^\times \to E^\times$ the formal series $L(\chi, t)$ is rational in t, and $L(\chi, t) = \varepsilon(\chi, t) L(\chi^{-1}, q^{-1}t^{-1})$. If the restriction of χ to the group of $x \in \mathbb{A}^\times / F^\times$ with $\deg(x) = 0$ is nontrivial, then $L(\chi, t)$ is a polynomial. If the restriction is trivial, χ is given by $\chi(x) = u^{\deg(x)}$, and then $L(\chi, t)$ has precisely two poles: $t = u^{-1}$ and $t = q^{-1}u^{-1}$, both poles are simple. If $\chi : \mathbb{A}^\times / F^\times \to \mathbb{C}^\times$ is a unitary character ($|\chi(x)| = 1$ for all x) then the zeroes of $L(\chi, t)$ lie in the doughnut $\{t \in \mathbb{C}; q^{-1} < |t| < 1\}$.*

The proof of this is also in [Lg94], Chapter VII, sections 7-8. The following is due to [W45].

THEOREM 4.12. (*A. Weil*). *For any unitary character $\chi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$, all zeroes of $L(\chi, t)$ lie on the circle $|t| = q^{-1/2}$.*

Given a character $\psi : \mathbb{A}/F \to E^\times, \psi \neq 1$, let $W(\psi)$ be the space of locally constant functions $\phi : \mathrm{GL}(2, F_v) \to E$ with $\phi(\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) x) = \psi(z)\phi(x)$ for all $z \in F_v$, $x \in \mathrm{GL}(2, F_v)$. The group $\mathrm{GL}(2, F_v)$ acts on $W(\psi)$ by right translation. Fix a Haar measure $d^\times x$ on $F_v^\times$. For any $\phi \in W(\psi)$ put

$$\Lambda_\phi(t) = \int \phi(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right))(q_v t)^{\mathrm{val}_v(a)} d^\times a, \quad \tilde{\Lambda}_\phi(t) = \int \phi(\left(\begin{smallmatrix} 0 & 1 \\ a & 0 \end{smallmatrix}\right))(q_v t)^{\mathrm{val}_v(a)} d^\times a.$$

Both $\Lambda_\phi(t)$ and $\tilde{\Lambda}_\pi(t)$ are formal power series in t, containing positive and negative powers of t.

Let π be an irreducible admissible representation of $\mathrm{GL}(2, F_v)$ over E. Then $\pi(\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right))$ is the operator of multiplication by a scalar $\eta(a) \in E^\times$. The character $\eta : F_v^\times \to E^\times$ is called the *central character* of π.

PROPOSITION 4.13. *Let π be an irreducible admissible infinite dimensional representation over E of* $\mathrm{GL}(2, F_v)$. *Let η be the central character of π.* (1) *There exists a unique* $\mathrm{GL}(2, F_v)$*-invariant subspace $W(\pi, \psi)$ of $W(\psi)$ equivalent to π.* (2) *If $\phi \in W(\pi, \psi)$ then $\Lambda_\phi(t)$ is the Laurent series at $t = 0$ of a rational function, and $\tilde{\Lambda}_\phi(t)$ is the Laurent series at $t = \infty$ of a rational function.* (3) *There exists a nonzero polynomial $P \in E[t]$ such that for any $\phi \in W(\pi, \psi)$ we have $P(t)\Lambda_\phi(t) \in E[t, t^{-1}]$. There exists $\phi \in W(\pi, \psi)$ with $\Lambda_\phi(t) \neq 0$.* (4) *The quotient $\tilde{\Lambda}_\phi(t)/\Lambda_\phi(t)$ of rational functions in t does not depend on the choice of ϕ in $W(\pi, \psi)$ with $\Lambda_\phi(t) \neq 0$.* (5) *The lowest degree polynomial $P \in E[t]$ which satisfies* (3) *and $P(0) = 1$ is independent of ψ.* (6) *Put $\Gamma(\pi, \psi, t) = \tilde{\Lambda}_\phi(t)/\Lambda_\phi(t)$ and $\varepsilon(\pi, \psi, t) = \frac{\Gamma(\pi,\psi,t)L(\pi,t)}{L(\pi\otimes\eta^{-1}, q_v^{-2}t^{-1})}$ where $L(\pi, t) = P(t)^{-1}$ with P of* (5). *Then $\varepsilon(\pi, \psi, t)$ has the form $c(\pi, \psi)t^{n(\pi,\psi)}$, $c(\pi, \psi)$ in $E^\times$ and $n(\pi, \psi)$ in $\mathbb{Z}$.* (7) *If $\psi_a(x)$ is $\psi(ax)$ for $a \in F_v^\times$, then $\varepsilon(\pi, \psi_a, t) = \eta(a)(q_v t)^{2\,\mathrm{val}_v(a)}\varepsilon(\pi, \psi, t)$.*

This is [JL70], Theorem 2.18. Our L and ε relate to those L_{JL}, ε_{JL} of Jacquet-Langlands by $L_{JL}(\pi, s) = L(\pi, t_v)$, $t_v = q_v^{-s}$, $\varepsilon_{JL}(\pi, \psi, s) = \varepsilon(\pi, \psi, t_v)$. Note that the proof of [JL70], which claims that $\Lambda_\phi(t)$ is a Laurent series of a meromorphic function in $\mathbb{C} - \{0\}$, shows that $\Lambda_\phi(t)$ is rational. In general, the meromorpic functions of s over p-adic and global function fields are rational functions of q^s. Every smooth finite dimensional irreducible representation of $\mathrm{GL}(2, F_v)$ is one dimensional, of the form $x \mapsto \chi(\det x)$, where $\chi : F_v^\times \to E^\times$ is a character ([JL70], Proposition 2.7).

PROPOSITION 4.14. *Let π, π' be irreducible admissible infinite dimensional representations of* $\mathrm{GL}(2, F_v)$ *with equal central characters. If there is a character $\psi : F_v \to E^\times$ such that for every character $\omega : F_v^\times \to E^\times$ we have $\Gamma(\pi\omega, \psi, t) = \Gamma(\pi'\omega, \psi, t)$, then $\pi \simeq \pi'$.*

For a proof see [JL70], Corollary 2.19.

The *conductor* of an irreducible admissible infinite dimensional representation π of $\mathrm{GL}(2,F_v)$ is the integer $n(\pi,\psi)$, with ψ normalized by $r(\psi)=0$. It is well defined, as from (7) above, the integer $n(\pi,\psi)$ of (6) is not changed if ψ is replaced by $\psi_a : x \mapsto \psi(ax)$.

Proposition 4.15. *The conductor of π is the least integer n such that the representation space of π contains a nonzero vector invariant under the group $H_n = \{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{GL}(2,O_v);\ c \in \boldsymbol{\pi}_v^n O_v,\ d \in 1+\boldsymbol{\pi}_v^n O_v\}$. For this n, $\dim_E \boldsymbol{\pi}^{H_n} = 1$.*

For a proof see Casselman, Math. Ann. 201 (1973), 301-314.

Proposition 4.16. *Let π be an irreducible admissible infinite dimensional representation, with central character η, of $\mathrm{GL}(2,F_v)$. Let $\psi : F_v \to E^\times$ be a nontrivial character. Then there exists an integer m_π such that if $\chi : F_v^\times \to E^\times$ is any character with conductor $> m_\pi$, then $L(\pi\chi,t)=1$ and*

$$\varepsilon(\pi\chi,\psi,t) = \varepsilon(\chi,\psi,t)\varepsilon(\chi\eta,\psi,q_v t)q_v^{-r(\psi)}.$$

For a proof see [JL70], Proposition 3.8. See [JL70], Proposition 3.5, 3.6, for a proof of:

Proposition 4.17. *Let μ_1, μ_2 be characters of $F_v^\times$, and $\psi \neq 1$ a character of F_v. If $\mu_1/\mu_2 \neq |\cdot|_v^{\pm 1}$ then $L(I(\mu_1,\mu_2),t) = L(\mu_1,t)L(\mu_2,t)$ and*

$$\varepsilon(I(\mu_1,\mu_2),\psi,t) = \varepsilon(\mu_1,\psi,t)\varepsilon(\mu_2,\psi,t)q_v^{-r(\psi)}.$$

If $\mu_2/\mu_1 = |\cdot|_v$, then

$$L(\mathrm{St}(\mu_1|\cdot|_v^{-1/2},\mu_1|\cdot|_v^{1/2}),t) = L(\mu_1|\cdot|_v^{1/2},t),$$

$$\varepsilon(\mathrm{St}(\mu_1|\cdot|_v^{-1/2},\mu_1|\cdot|_v^{1/2}),\psi,t) = \frac{L(\mu_1^{-1},t^{-1})}{L(\mu_1,t)}\varepsilon(\mu_1,\psi,t)\varepsilon(\mu_1|\cdot|_v,\psi,t)q_v^{-r(\psi)}.$$

If π is a cuspidal representation of $\mathrm{GL}(2,F_v)$ then $L(\pi,t)$ is 1.

Recall that an irreducible admissible infinite dimensional representation π of $\mathrm{GL}(2,F_v)$ on a vector space V is called unramified if its space V^K of $K = \mathrm{GL}(2,O_v)$-fixed vectors is nonzero. In this case V^K is one dimensional, and $\pi = I(\mu_1,\mu_2)$ with unramified μ_1, μ_2 and $\mu_1/\mu_2 \neq |\cdot|^{\pm 1}$.

Corollary 4.18. *Let π be an unramified irreducible admissible infinite dimensional representation of $\mathrm{GL}(2,F_v)$ and $\psi \neq 1$ with $r(\psi)=0$. Then $\varepsilon(\pi,\psi,t)=1$.*

Proof. Here $\pi = I(\mu_1,\mu_2)$ with unramified μ_1,μ_2, so the claim follows from the last proposition and Tate's Thesis. ∎

Let π be an admissible irreducible representation of $\mathrm{GL}(2,\mathbb{A})$ whose local components are all infinite dimensional. Put $L(\pi,t)=\prod_v L(\pi_v,t_v)$, $t_v=t^{\deg(v)}$; the infinite product converges in $E[[t]]$. For any character $\psi:\mathbb{A}/F\to E^\times,\psi\neq 1$, put $\varepsilon(\pi,\psi,t)=\prod_v\varepsilon(\pi_v,\psi_v,t_v)$; almost all factors here are 1. From (7) it follows that if the central character of π is trivial on $F^\times$, then $\varepsilon(\pi,\psi,t)$ is independent of the choice of $\psi:\mathbb{A}/F\to E^\times$. We denote it in this case by $\varepsilon(\pi,t)$.

Theorems 11.1, 11.3 of [JL70] assert:

THEOREM 4.19. *Let π be an irreducible admissible representation of* $\mathrm{GL}(2,\mathbb{A})$ *over E. Denote by $\eta:\mathbb{A}^\times\to E^\times$ its central character. Then π is cuspidal iff* (1) *η is trivial on $F^\times$*; (2) *all local components of π are infinite dimensional*; (3) *for any character $\omega:\mathbb{A}^\times/F^\times\to E^\times$, the formal series $L(\pi\omega,t)$ is a polynomial in t, and* (4) $L(\pi\omega,t)=\varepsilon(\pi\omega,t)L(\pi\eta^{-1}\omega^{-1},q^{-2}t^{-1})$.

Note that (4) makes sense due to (3). In [JL70], (3) is formulated as stating that the product $\prod_v L(\pi_v\omega_v,t_v)$ converges absolutely for sufficiently small t, and its value has an analytic continuation to a holomorphic function in $\mathbb{C}-\{0\}$. But the argument of [JL70] can be modified to lead to (3) in our case of E which is not $\mathbb{C}$, over a function field F. Note that (4) is not $\prod_v\Gamma(\pi_v\omega_v,\psi_v,t_v)=1$; indeed the product here does not converge.

PROPOSITION 4.20. *If π, π' are cuspidal representations of* $\mathrm{GL}(2,\mathbb{A})$ *and $\pi_v\simeq\pi'_v$ for almost all v, then $\pi\simeq\pi'$.*

Proof. Let S be a finite set of closed points of X with $\pi_v\simeq\pi'_v$ at $v\notin S$. Let η, η' be the central characters of π, π', and η_v, η'_v their components at v (restrictions to $F_v^\times$). By our assumption, $\eta'_v=\eta_v$ for all $v\notin S$. But the groups $F_v^\times$, $v\notin S$, generate a dense subgroup of $\mathbb{A}^\times/F^\times$. Hence $\eta'=\eta$. By the Theorem 4.19, of [JL70], above, fixing a character $\psi:\mathbb{A}/F\to E^\times$, $\psi\neq 1$, for any character $\omega:\mathbb{A}^\times/F^\times\to E^\times$ one has

$$\begin{aligned}\prod_v L(\pi_v\omega_v,t_v) &= \prod_v\varepsilon(\pi_v\omega_v,\psi_v,t_v)L(\pi_v\eta_v^{-1}\omega_v^{-1},q_v^{-2}t_v^{-1}),\\ \prod_v L(\pi'_v\omega_v,t_v) &= \prod_v\varepsilon(\pi'_v\omega_v,\psi_v,t_v)L(\pi'_v{\eta'_v}^{-1}\omega_v^{-1},q_v^{-2}t_v^{-1}).\end{aligned}$$

Since $\pi_v\simeq\pi'_v$ at all $v\notin S$, we conclude

$$\begin{aligned}\prod_{v\in S}\Gamma(\pi_v\omega_v,\psi_v,t_v)&=\prod_{v\in S}\frac{\varepsilon(\pi_v\omega_v,\psi_v,t_v)L(\pi_v\eta_v^{-1}\omega_v^{-1},q_v^{-2}t_v^{-1})}{L(\pi_v\omega_v,t_v)}\\ =\prod_{v\in S}\frac{\varepsilon(\pi'_v\omega_v,\psi_v,t_v)L(\pi'_v{\eta'_v}^{-1}\omega_v^{-1},q_v^{-2}t_v^{-1})}{L(\pi'_v\omega_v,t_v)}&=\prod_{v\in S}\Gamma(\pi'_v\omega_v,\psi_v,t_v).\end{aligned}$$

Since $\eta=\eta'$, it follows from Proposition 4.16 that for each $v\in S$ there exists $m_v>0$ such that if $\chi:F_v^\times\to E^\times$ is any character whose conductor is $\geq m_v$, then $\Gamma(\pi_v\chi, \psi_v, t) = \Gamma(\pi'_v\chi, \psi_v, t)$. Fix $v \in S$ and a character χ of $F_v^\times$. By Proposition 4.14, it suffices to show $\Gamma(\pi_v\chi, \psi_v, t) = \Gamma(\pi'_v\chi, \psi_v, t)$. For this, it suffices to choose a character $\omega : \mathbb{A}^\times/F^\times \to E^\times$ in the last displayed equation with $\omega_v = \chi$ and such that for each $u \in S - \{v\}$, the conductor of ω_u is bigger than m_u. But the group $H = F_v^\times \prod_{u\in S-\{v\}} O_u^\times$ maps isomorphically and homeomorphically onto its image in $\mathbb{A}^\times/F^\times$. Hence any character of H extends to a character of $\mathbb{A}^\times/F^\times$. □

Proposition 4.21. *Let η be a character of $\mathbb{A}^\times/F^\times$, S a finite set of closed points of $X, \psi \neq 1$ a character of $\mathbb{A}/F$ with $r(\psi_u) = 0$ for all u in S. Suppose that for any closed point $v \in |X| - S$, π_v is an irreducible admissible infinite dimensional representation of $\mathrm{GL}(2, F_v)$ with central character η_v such that almost all π_v are unramified, there is no pair μ_1, μ_2 of characters of $\mathbb{A}^\times/F^\times$ with $\pi_v = \pi(\mu_{1v}, \mu_{2v})$ for almost all $v \in |X|-S$, and for any character ω of $\mathbb{A}^\times/F^\times$ which is unramified at all points of S, the formal series $\prod_{v\notin S} L(\pi_v\omega_v, t_v)$ and $\prod_{v\notin S} L(\pi_v\eta_v^{-1}\omega_v^{-1}, t_v)$ are polynomials, and there exists a number $c \in E^\times$ and integers $n_u > 0$ $(u \in S)$ such that*

$$\prod_{v\notin S} L(\pi_v\omega_v, t_v) = c \prod_{u\in S} (\omega(\boldsymbol{\pi}_u)t_u)^{n_u} \prod_{v\notin S} \varepsilon(\pi_v\omega_v, \psi_v, t_v) L(\pi_v\eta_v^{-1}\omega_v^{-1}, q_v^{-2}t_v^{-1}).$$

Then there exists a cuspidal representation π of $\mathrm{GL}(2, \mathbb{A})$ with central character η such that for every $v \in |X| - S$ the local component of π at v is π_v.

A proof is in [JL70], Theorem 11, Corollary 11.6, proof of Theorem 12.2. The representation π is unique by Proposition 4.20.

4.4 Intertwining again

We can now return to the study of the intertwining operators.

Proposition 4.22. *Let μ_1, μ_2 be characters of $F_v^\times$. Let $\psi \neq 1$ be a character of F_v. Then*

$$R(\mu_1, \mu_2, t)R(\mu_2, \mu_1, t^{-1}) = \varepsilon\left(\frac{\mu_1}{\mu_2}, \psi, q_v^{-1}t^2\right) \varepsilon\left(\frac{\mu_2}{\mu_1}, \psi, q_v^{-1}t^{-2}\right).$$

Proof. By the transformation formula for the ε-factors, the right hand side does not depend on ψ. We then choose ψ with $\ker\psi \supset O_v$ and $\ker\psi \not\supset \boldsymbol{\pi}_v^{-1}O_v$. We can rewrite the asserted equality as

$$M(\mu_1, \mu_2, t)M(\mu_2, \mu_1, t^{-1}) = \Gamma\left(\frac{\mu_2}{\mu_1}, \psi, q_v^{-1}t^2\right) \Gamma\left(\frac{\mu_2}{\mu_1}, \psi, q_v^{-1}t^{-2}\right).$$

The restriction map $I(\mu_1, \mu_2) \to I(\mu_1/\mu_2)$, where

$$I(\mu) = \{f \in C^\infty(\mathrm{SL}(2, F_v)); f\left(\left(\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right) x\right) = \mu(a)|a|_v f(x)\},$$

is an isomorphism ($\mu : F_v^\times \to E^\times$ is a character). The group $\mathrm{SL}(2, F_v)$ acts transitively on $F_v^2 - \{(0,0)\}$ on the right. The stabilizer of the vector $(0,1)$ is $N(F_v)$. Then $N(F_v)\backslash \mathrm{SL}(2, F_v)$ can be identified with $F_v^2 - \{(0,0)\}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (c,d) \in F_v^2 - \{(0,0)\}$. Using this we identify $I(\mu)$ with

$$V(\mu) = \{f \in C^\infty(F_v^2 - \{(0,0)\});$$
$$f(ax) = \mu(a)^{-1}|a|_v^{-1} f(x), a \in F_v^\times, x \in F_v^2 - \{(0,0)\}\},$$

so $I(\mu_1, \mu_2)$ with $V(\mu_1/\mu_2)$. The operator $M(\mu_1, \mu_2, t)$ corresponds to the operator $\overline{M}(\mu_1/\mu_2, t^2)$ where

$$\overline{M}(\mu, s) : V(\mu\nu_s) \to V(\mu^{-1}\nu_{s^{-1}}), \quad (\overline{M}(\mu, s)f)(x) = \int_{\{y; x\wedge y = 1\}} f(y)dy.$$

Here $\wedge$ denotes the symplectic form $(a,b)\wedge(c,d) = ad - bc$ on F_v^2. The measure on the line $\ell_x = \{y \in F_v^2; x \wedge y = 1\}$ is transferred from the Haar measure on F_v via the map $F_v \to \ell_x$ given by $a \mapsto y_0 + ax$ where y_0 is a fixed point on ℓ_x. So we need to show:

$$\overline{M}(\mu, s)\overline{M}(\mu^{-1}, s^{-1}) = \Gamma(\mu, \psi, q_v^{-1}s)\Gamma(\mu^{-1}, \psi, q_v^{-1}s^{-1}).$$

For sufficiently small $s \in \mathbb{C}^\times$ define operators $A_s : C_c^\infty(F_v^2) \to V(\mu\nu_s)$ and $B_s : C_c^\infty(F_v^2) \to V(\mu^{-1}\nu_s)$ by

$$(A_s f)(x) = \int_{F_v} f(ax)\mu(a)\nu_s(a)da, \quad (B_s f)(x) = \int_{F_v} f(ax)\mu(a)^{-1}\nu_s(a)da.$$

Restriction defines an isomorphism $V(\mu\nu_s) \to V_0(\mu)$, where

$$V_0(\mu) = \{f \in C^\infty(O_v^2 - \{(0,0)\});$$
$$f(ax) = \mu(a)^{-1}f(x),\ x \in O_v^2 - \{(0,0)\},\ a \in O_v^\times\},$$

so we can identify the spaces $V(\mu\nu_s)$ as s varies.
The operators A_s and B_s, defined above for small s, depend rationally on s. Hence they can be extended to all s.
Consider the Fourier transform

$$F : C_c^\infty(F_v^2) \to C_c^\infty(F_v^2), \quad (Ff)(y) = \int_{F_v^2} f(x)\psi(x \wedge y)dx.$$

Lemma 4.23. *We have* $\overline{M}(\mu, s)A_s = \Gamma(\mu^{-1}, \psi, q_v^{-1}s^{-1})B_{s^{-1}}F$,

$$\overline{M}(\mu^{-1}, s^{-1})B_{s^{-1}} = \Gamma(\mu, \psi, q_v^{-1}s)A_s F.$$

Proof. Given $f \in C_c^\infty(F_v^2)$, $x \in F_v^2 - \{(0,0)\}$, we first show

$$\Gamma(\mu^{-1}, \psi, q_v^{-1}s^{-1})(B_{s^{-1}}Ff)(x) = (\overline{M}(\mu, s)A_s f)(x).$$

The operators F, A_s, B_s commute with the action of $\mathrm{SL}(2,F_v)$. This action is transitive on $F_v^2-\{(0,0)\}$, so we may assume $x=(0,1)$. We compute

$$\begin{aligned}(B_{s^{-1}}Ff)((0,1)) &= \int_{F_v}(Ff)((0,a))\mu(a)^{-1}\nu_{s^{-1}}(a)da,\\ (Ff)((0,a)) &= \int_{F_v^2} f(y,z)\psi(ya)dydz=\hat\varphi(-a),\\ \hat\varphi(a) &= \int\varphi(y)\psi(-ya)dy, \qquad \varphi(y)=\int f(y,z)dz.\end{aligned}$$

Tate's functional equation (see [L], VII, section 3-4) is

$$\Gamma(\mu^{-1},\psi,q_v^{-1}s^{-1})\int\hat\varphi(a)\mu^{-1}(a)\nu_{s^{-1}}(a)da=\int\varphi(y)\mu(y)\nu_s(y)\frac{dy}{|y|}.$$

(Formally this can be deduced from the definition of the Γ-function and the inversion formula $\varphi(y)=\int\hat\varphi(a)\psi(ay)da$. However the left side converges for large $|s|$, while the right for small $|s|$, so one has to show both sides are rational in s).

We conclude that the left side of the equation to be shown is

$$\int\varphi(y)\mu(-y)\nu_s(y)|y|^{-1}dy=\int\int f(y,z)\mu(-y)\nu_s(y)|y|^{-1}dydz$$

while the right side is (recall: $x=(0,1)$, so $(0,1)\wedge(y,z)=-y$)

$$\int(A_sf)(-1,z)dz=\int\int f(-y,yz)\mu(y)\nu_s(y)dydz.$$

The proof of the second identity of the lemma is similar. $\square$

The inverse Fourier transform coincides with F since the form $(x,y)\mapsto x\wedge y$ in the definition of F is skew-symmetric. Hence $F^2=1$, and it follows from the Lemma that

$$\overline{M}(\mu,s)\overline{M}(\mu^{-1},s^{-1})B_{s^{-1}}=\Gamma(\mu,\psi,q_v^{-1}s)\Gamma(\mu^{-1},\psi,q_v^{-1}s^{-1})B_{s^{-1}}.$$

However, the operator $B_{s^{-1}}$ is onto for those s where it is defined (even its restriction to $C_c^\infty(F_v^2-\{(0,0)\})$ is onto), as $V(\mu\nu_s)$ is irreducible, so the proposition follows. $\square$

Proposition 4.24. *For any characters μ_1, μ_2 of $\mathbb{A}^\times/F^\times$ we have*

$$M(\mu_1,\mu_2,t)M(\mu_2,\mu_1,t^{-1})=1.$$

Proof. From Proposition 4.21, $M(\mu_1,\mu_2,t)M(\mu_2,\mu_2,t^{-1})$ is equal to

$$q^{2-2g}m(\mu_1/\mu_2,t^2)m(\mu_2/\mu_1,t^{-2})R(\mu_1,\mu_2,t)R(\mu_2,\mu_1,t^{-1}),$$

while Proposition 4.22 implies, for any character $\psi \neq 1$ of $\mathbb{A}/F$, that

$$R(\mu_1,\mu_2,t)R(\mu_2,\mu_1,t^{-1})$$

is

$$\prod_v [\varepsilon(\mu_{1v}/\mu_{2v},\psi_v,q_v^{-1}t_v^2)\varepsilon(\mu_{2v}/\mu_{1v},\psi_v,q_v^{-1}t_v^{-2})]$$
$$= q^{2g-2}\varepsilon(\mu_1/\mu_2,q^{-1}t^2)\varepsilon(\mu_2/\mu_1,q^{-1}t^{-2}).$$

As $\varepsilon(\chi,t) = q^{1-g}\prod_v \varepsilon(\chi_v,\psi_v,t_v)$ satisfies the functional equation $L(\chi,t) = \varepsilon(\chi,t)L(\chi^{-1},q^{-1}t^{-1})$, we have that

$$\varepsilon(\mu_1/\mu_2,q^{-1}t^2)\varepsilon(\mu_2/\mu_1,q^{-1}t^{-2})m(\mu_1/\mu_2,t^2)m(\mu_2/\mu_1,t^{-2}),$$

which is equal to

$$\frac{\varepsilon(\mu_1/\mu_2,q^{-1}t^2)L(\mu_2/\mu_1,t^2)}{L(\mu_1/\mu_2,q^{-1}t^2)}\cdot\frac{\varepsilon(\mu_1/\mu_2,q^{-1}t^{-2})L(\mu_1/\mu_2,t^2)}{L(\mu_2/\mu_1,q^{-1}t^{-2})}$$

is equal to 1. □

4.5 $M^2 = 1$ via Mellin transform

We shall next study the relationship between $M : C_c^\infty(Y_\alpha) \to C^\infty(Y_\alpha)$ and $M(\mu_1,\mu_2,t) : I(\mu_1\nu^t,\mu_2\nu^{-t}) \to I(\mu_2\nu^{-t},\mu_1\nu^t)$, and conclude that $M^2 = 1$. Both are defined by the same integral formula. Here μ_1, μ_2 are characters of $\mathbb{A}^\times/F^\times\cdot\alpha^{\mathbb{Z}}$. Put

$$\eta(\left(\begin{smallmatrix} a & 0\\ 0 & b\end{smallmatrix}\right)) = \mu_1(a)\mu_2(b)|a/b|^{1/2}\nu_t(a/b), \quad \eta : A(\mathbb{A})/A(F)\cdot\alpha^{\mathbb{Z}} \to E^\times.$$

It is a character. Recall that $Y_\alpha = \alpha^{\mathbb{Z}}N(\mathbb{A})A(F)\backslash \mathrm{GL}(2,\mathbb{A})$ and $(Mf)(x) = \int_{N(\mathbb{A})} f(\left(\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right)nx)dn$. Suppose that $f \in C_c^\infty(Y_\alpha)$, and $t \in E^\times$. Define a function $T(f,\mu_1,\mu_2,t) : \mathrm{GL}(2,\mathbb{A}) \to \mathbb{C}$ by

$$(T(f,\mu_1,\mu_2,t))(x) = \int_{\alpha^{\mathbb{Z}}A(F)\backslash A(\mathbb{A})} f(a^{-1}x)\eta(a)d^\times a.$$

Then $T(f,\mu_1,\mu_2,t) \in I(\mu_1\nu_t,\mu_2\nu_{-t})$ is called the *Mellin transform* of f. The notation T can be used also when $f \in C^\infty(Y_\alpha)$ is not compactly supported, whenever the integral converges.

Proposition 4.25. *For $\varphi \in C_c^\infty(Y_\alpha)$, characters $\mu_1,\mu_2 : \mathbb{A}^\times/F^\times\cdot\alpha^{\mathbb{Z}} \to E^\times$ and large enough $t \in \mathbb{C}^\times$, the integral defining T converges, and $T(M\varphi,\mu_1,\mu_2,t) = M(\mu_2,\mu_1,t^{-1})T(\varphi,\mu_2,\mu_1,t^{-1})$.*

Proof. By definition,

$$(T(f,\mu_1,\mu_2,t))(x) = \iint f(\left(\begin{smallmatrix} a & 0\\ 0 & b\end{smallmatrix}\right)^{-1}x)\mu_1(a)\mu_2(b)|a/b|^{1/2}\nu_t(a/b)d^\times a d^\times b.$$

Put $f = M\varphi$, so $f(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)^{-1} x) = |b/a| \int_{N(\mathbb{A})} \varphi(\left(\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx) dn$. Hence $(T(f, \mu_1, \mu_2, t))(x)$ equals

$$\begin{aligned} & \int\int\int \varphi(\left(\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx)\mu_1(a)\mu_2(b)|b/a|^{1/2}\nu_t(a/b) d^\times a d^\times b dn \\ = & \int_{N(\mathbb{A})} (T(\varphi, \mu_2, \mu_1, t^{-1}))(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx) dn \\ = & (M(\mu_2, \mu_1, t^{-1}) T(\varphi, \mu_2, \mu_1, t^{-1}))(x). \end{aligned}$$

If t is large enough, the integral which defines $M(\mu_2, \mu_1, t^{-1})$ converges, and so is the integral which defines $T(f, \mu_1, \mu_2, t)$, which justifies the computation. $\square$

Proposition 4.26. *If $\varphi \in C_c^\infty(Y_\alpha)$ then $M\varphi \in C^\infty(Y_\alpha)$. If $M\varphi \in C_c^\infty(Y_\alpha)$ then $M^2\varphi = \varphi$.*

Proof. Put $f = M\varphi$ and $h = Mf = M^2\varphi$ (h is defined if $f \in C_c^\infty(Y_\alpha)$). By Proposition 4.25,

$$\begin{aligned} T(h, \mu_1, \mu_2, t) &= M(\mu_2, \mu_1, t^{-1}) T(f, \mu_2, \mu_1, t^{-1}), \\ T(f, \mu_2, \mu_1, t^{-1}) &= M(\mu_1, \mu_2, t) T(\varphi, \mu_1, \mu_2, t). \end{aligned}$$

The first equation holds only for large enough t, and the second only for small enough t. However, both sides of the second equality depend rationally on t (for the left side, this is true since $f = M\varphi$ is compactly supported), hence it holds for all t in $\mathbb{C}^\times$. Hence for large enough t, by Proposition 4.24 $T(h, \mu_1, \mu_2, t) = T(\varphi, \mu_1, \mu_2, t)$ for all μ_1, μ_2. This implies $h = \varphi$. $\square$

4.6 Poles, zeroes and values of R and M

Recall that $\nu_t(x) = t^{\deg(x)}$ is a character of $\mathbb{A}^\times/F^\times$ with $\nu_t(\boldsymbol{\pi}_v) = t_v$ $(= t^{\deg(v)})$, and locally we write ν_t for the unramified character of $F_v^\times$ with $\nu_t(\boldsymbol{\pi}_v) = t$. Let μ_1, μ_2 be characters of $F_v^\times$. Recall:

$$R(\mu_1, \mu_2, t) = \frac{L(\mu_1/\mu_2, q_v^{-1} t^2)}{L(\mu_1/\mu_2, t^2)} M(\mu_1, \mu_2, t).$$

Proposition 4.27. (1) *The function $R(\mu_1, \mu_2, t)$ is regular at $t = 0$.*
It has a pole at $\tau \in \mathbb{C}^\times$ iff $\mu_2\nu_{\tau^{-1}}/\mu_1\nu_\tau = \nu$ (with $\nu(\boldsymbol{\pi}_v) = q_v^{-1}$). This pole has order 1.
The function $R(\mu_1, \mu_2, t)^{-1}$ has a pole at $\tau \in \mathbb{C}^\times$ iff $\mu_1\nu_\tau/\mu_2\nu_{\tau^{-1}} = \nu$. This pole has order 1.
(2) *Suppose $R(\mu_1, \mu_2, t)^{-1}$ has a pole at $\tau \in \mathbb{C}^\times$. Then the function $R(\mu_1, \mu_2, t)$ is regular at $t = \tau$. Put $L = \lim_{t\to\tau}(t - \tau) R(\mu_1, \mu_2, t)^{-1}$ and $Q = R(\mu_1, \mu_2, \tau)$. The operators $Q : I(\mu_1\nu_\tau, \mu_2\nu_{\tau^{-1}}) \to I(\mu_2\nu_{\tau^{-1}}, \mu_1\nu_\tau)$ and $L : I(\mu_2\nu_{\tau^{-1}}, \mu_1\nu_\tau) \to I(\mu_1\nu_\tau, \mu_2\nu_{\tau^{-1}})$ intertwine the* $\mathrm{GL}(2, F_v)$*-action. The representations of* $\mathrm{GL}(2, F_v)$ *in the spaces* $\ker Q$, $\operatorname{coker} Q$, $\operatorname{im} L$ *are isomorphic*

to the square integrable $\mathrm{St}(\mu_1\nu_\tau, \mu_2\nu_{\tau^{-1}})$. *The representations of* $\mathrm{GL}(2, F_v)$ *in the spaces* $\ker L$, $\operatorname{coker} L$, $\operatorname{im} Q$ *are isomorphic to the one dimensional* $x \mapsto \mu_2(x)(\nu\nu_{\tau^{-1}})(x) = \mu_1(x)\nu_\tau(x)$.
(3) *The statement* (2) *remains true with* $R(\mu_1, \mu_2, t)$ *replaced by* $R(\mu_1, \mu_2, t)^{-1}$.

Proof. From the first part of the proof of Proposition 4.3 it follows that

$$M(\mu_1, \mu_2, t)/L(\mu_1/\mu_2, t^2) = R(\mu_1, \mu_2, t)/L(\mu_1/\mu_2, q_v^{-1}t^2)$$

is regular. So $R(\mu_1, \mu_2, t)$ could have a pole at $t \in \mathbb{C}^\times$ only if $L(\mu_1/\mu_2, q_v^{-1}t^2)$ is ∞, that is $\mu_2\nu_{\tau^{-1}}/\mu_1\nu_\tau = \nu$ (recall: $\nu(x) = |x|$), and the order of the pole is at most 1.
A similar statement holds for $R(\mu_1, \mu_2, t)^{-1} = c(\mu_1, \mu_2)t^{n(\mu_1,\mu_2)}R(\mu_2, \mu_1, t^{-1})$. (The last equality follows from Proposition 4.22. In fact $n(\mu_1, \mu_2) = 0$, but we do not need this.) Namely $R(\mu_1, \mu_2, t)^{-1}$ has a pole at $\tau \in \mathbb{C}^\times$ iff $\mu_1\nu_\tau/\mu_2\nu_{\tau^{-1}} = \nu$. This pole has order 1.
Suppose $\mu_1\nu_\tau/\mu_2\nu_{\tau^{-1}} = \nu$. Then $\mu_2\nu_{\tau^{-1}}/\mu_1\nu_\tau \neq \nu$ so that $R(\mu_1, \mu_2, t)^{-1}$ is regular at $t = \tau$. With L, Q defined as in the proposition, it is clear they commute with the $\mathrm{GL}(2, F_v)$-action. If $L = 0$ then $Q = R(\mu_1, \mu_2, \tau)$ has no pole, in fact it is an isomorphism. If $Q = 0$ then L would be an isomorphism, as the operator $\lim_{t\to\tau} R(\mu_1, \mu_2, t)/(t - \tau)$ would be the inverse of L. However, the representations of $\mathrm{GL}(2, F_v)$ in $I(\mu_1\nu_\tau, \mu_2\nu_{\tau^{-1}})$ and $I(\mu_2\nu_{\tau^{-1}}, \mu_1\nu_\tau)$ are not equivalent, hence $L \neq 0$, $Q \neq 0$. As $L \neq 0$, the function $R(\mu_1, \mu_2, t)^{-1}$ does have a pole at $t = \tau$. From the description of the invariant subspaces of $I(\mu_1\nu_\tau, \mu_2\nu_{\tau^{-1}})$ and $I(\mu_2\nu_{\tau^{-1}}, \mu_1\nu_\tau)$ the claims in the proposition on the description of the action of $\mathrm{GL}(2, F_v)$ follow. The regularity of $R(\mu_1, \mu_2, t)$ at $t = 0$ follows from that of $L(\mu_1/\mu_2, q_v^{-1}t^2)^{-1}R(\mu_1, \mu_2, t)$. □

In conclusion, the representation of $\mathrm{GL}(2, F_v)$ in $I(\mu_1\nu_t, \mu_2\nu_{t^{-1}})$ is reducible iff $R(\mu_1, \mu_2, t)$ or $R(\mu_1, \mu_2, t)^{-1}$ has a pole at $t = \tau$. These last operators are regular at $t \in \mathbb{C}^\times$ if μ_1/μ_2 is ramified. If μ_1/μ_2 is unramified and $(\mu_1/\mu_2)(\boldsymbol{\pi}_v) = a$, then the poles of $R(\mu_1, \mu_2, t)$ are at $\pm\sqrt{q_v/a}$, and those of $R(\mu_1, \mu_2, t)^{-1}$ are at $\pm\sqrt{a/q_v}$.

Corollary 4.28. *Let* μ_1, μ_2 *be characters of* $\mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}}$. *If* $R(\mu_1, \mu_2, t)$ *has a pole at* $t = \tau \in \mathbb{C}^\times$, *then* $|\tau| = \sqrt{q}$. *If* $R(\mu_1, \mu_2, t)^{-1}$ *has a pole at* $t = \tau \in \mathbb{C}^\times$ *then* $|\tau| = q^{-1/2}$.

Indeed, a character of $\mathbb{A}^\times/F^\times$ which takes the value 1 at α is unitary, thus $|a| = 1$.

Proposition 4.29. *Let* μ_1, μ_2 *be characters of* $\mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}}$ *and* $\tau \in \mathbb{C}^\times$, $|\tau| \leq 1$. *If* $M(\mu_1, \mu_2, t)$ *has a pole at* $t = \tau$ *then* $\mu_1 = \mu_2$ *and* $\tau = \pm q^{-1/2}$. *If* $\mu_1 = \mu_2$ *is denoted* μ *and* $\tau = \pm q^{-1/2}$ *then* $M(\mu, \mu, t)$ *has an order* 1 *pole at* τ. *The image of the operator* $C = \lim_{t\to\tau}(t - \tau)M(\mu, \mu, t)$ *in this case is one dimensional and is spanned by the function* $f(x) = \mu(\det x)\nu_\tau(\det x)$ *in* $I(\mu\nu_{\tau^{-1}}, \mu\nu_\tau)$. *Further,* $M(\mu_1, \mu_2, t)$ *is regular at* $t = 0$.

Proof. Recall that $M(\mu_1,\mu_2,t) = q^{1-g}m(\mu_1/\mu_2,t^2)R(\mu_1,\mu_2,t)$ where $m(\mu,t) = L(\mu,t)/L(\mu,t/q)$. Let $\tau \in \mathbb{C}^\times$, $|\tau| \le 1$. By Corollary 4.28, the function $R(\mu_1,\mu_2,t)$ is regular at τ. By Proposition 4.11, the function $m(\mu_1/\mu_2,t^2)$ is not regular at τ only if $\mu_1 = \mu_2$ and $\tau = \pm q^{-1/2}$. In these cases it has a simple pole. Hence $M(\mu_1,\mu_2,t)$ is regular at $t = \tau$ $(0 < |\tau| \le 1)$ unless $\mu_1 = \mu_2$ and $\tau = \pm q^{-1/2}$ where the order of the pole is at most 1. When $\mu_1 = \mu_2 = \mu$ and $\tau = \pm q^{-1/2}$, the operator $C = \lim_{t\to\tau}(t-\tau)M(\mu,\mu,t)$ is a scalar multiple of $R(\mu,\mu,t) = \otimes_v R(\mu_v,\mu_v,\tau_v)$, $\tau_v = \tau^{\deg(v)}$.
From (1) in Proposition 4.27, the function $R(\mu_v,\mu_v,\tau_v)^{-1}$ has a pole at $t = \tau$ $(t_v = \tau_v)$. Its statement (2) implies that the image of $R(\mu_v,\mu_v,\tau_v)$ is one dimensional and $\mathrm{GL}(2,F_v)$ acts on it via the character $x \mapsto \mu_v(\det x)\nu_\tau(\det x)^{\deg v}$. This implies the proposition, except the final claim, which follows from the regularily of $R(\mu_1,\mu_2,t)$ at $t = 0$, and that of $m(\mu_1/\mu_2,t^2)$ at $t = 0$. □

Let μ_1, μ_2 be characters of $\mathbb{A}^\times/F^\times$. The operator $M(\mu_1,\mu_2,t)$ maps $I(\mu_1\nu_t,\mu_2\nu_{t^{-1}})$ into the space $I(\mu_2\nu_{t^{-1}},\mu_1\nu_t)$, which in general is different from $I(\mu_1\nu_t,\mu_2\nu_{t^{-1}})$. However, when $\mu_1 = \mu_2 = \mu$ and $t = \pm 1$, then $M(\mu_1,\mu_2,t)$ maps $I(\mu_1\nu_t,\mu_2\nu_{t^{-1}})$ to itself; $M(\mu,\mu,t)$ is regular at $t = \pm 1$. The representation of $\mathrm{GL}(2,\mathbb{A})$ in $I(\mu\nu_\tau,\mu\nu_{\tau^{-1}})$, $\tau = \pm 1$, is irreducible, and hence $M(\mu,\mu,\tau)$ is a scalar operator. Moreover, from Proposition 4.26, $M(\mu,\mu,\tau)^2 = 1$ at $\tau = \pm 1$.

Proposition 4.30. *If μ is a character of $\mathbb{A}^\times/F^\times$ and $\tau = \pm 1$, then $M(\mu,\mu,\tau) = -1$.*

Proof. In view of the relation between M and R, it suffices to verify that

$$\lim_{t\to 1}\frac{L(1,t)}{L(1,t/q)} = -q^{g-1} \qquad \text{and} \qquad R(\mu,\mu,\tau) = 1.$$

In fact, for any character ω of $F_v^\times$, $R(\omega,\omega,\tau)$ is 1 at $\tau = \pm 1$. Indeed, suppose first ω is unramified. Then there exists a function f in $I(\omega\nu_\tau,\omega\nu_\tau)$ whose restriction to $\mathrm{GL}(2,O_v)$ is 1. By the normalization of the intertwining operator (Proposition 4.3(2)), $R(\omega,\omega,\tau)f = f$. However, the representation of $\mathrm{GL}(2,F_v)$ on $I(\omega\nu_\tau,\omega\nu_\tau)$ is irreducible, so $R(\omega,\omega,\tau) = 1$ if ω is unramified. The general case reduces to the case where ω is unramified, or even $\omega = 1$, by the commutativity of the diagram

$$\begin{array}{ccc} I(\omega\nu_\tau,\omega\nu_\tau) & \xrightarrow{R(\omega,\omega,\tau)} & I(\omega\nu_\tau,\omega\nu_\tau) \\ \uparrow & & \uparrow \\ I(\nu_\tau,\nu_\tau)\otimes\omega & \xrightarrow{R(1,1,\tau)} & I(\nu_\tau,\nu_\tau)\otimes\omega \end{array}$$

To compute the limit of the ratio of L-functions, we use the functional equation $L(1,t/q) = \varepsilon(1,t/q)L(1,t^{-1})$. Then

$$\lim_{t\to 1} L(1,t)/L(1,t/q) = \varepsilon(1,1/q)^{-1}\lim_{t\to 1} L(1,t)/L(1,t^{-1}).$$

By the definition of the global ε-function and its properties (Proposition 6.1, 6.3), $\varepsilon(1,1/q) = q^{1-g}$. Since $L(1,t)$ has a pole of order one at $t = 1$, by L'Hôpital rule $\lim_{t\to 1} L(1,t)/L(1,t^{-1})$ is -1. □

4.7 Global Eisenstein approach

These proofs of $M^2 = 1$ and rationality of $M(\mu_1, \mu_2, t)$ are based on local computations (normalization of the intertwining operators by L-functions and ε-factors), and the functional equation of the L-function. The following alternative proof of these results is based on properties of the Eisenstein map.

The alternative approach of this subsection, the following subsection 4.8, and the computation of traces in subsection 5.2 are motivated by Tate [T68]. They are the newest part of this paper, which – as noted in the introduction – cries out for generalization from our context of GL(2), and for further study.

We shall use the maps $\mathrm{ht}^+ : Y_\alpha \to \mathbb{Z}$ and $\mathrm{ht} : \alpha^{\mathbb{Z}} \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}) \to \mathbb{Z}$. Both maps are proper. However, ht^+ is onto while the image of ht contains the positive integers but only finitely many negatives. So in some sense Y_α is less compact than $\alpha^{\mathbb{Z}} \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})$, so the map $E : C_c^\infty(Y_\alpha) \to C_c^\infty(\alpha^{\mathbb{Z}} \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}))$ should have a big kernel. For φ in $\ker E$ we have $(1+M)\varphi = E^*E\varphi = 0$. Hence $M^2\varphi = \varphi$. Unlike M, the operator M^2 commutes with the action of $A(\mathbb{A})$ on $C_c^\infty(Y_\alpha)$ by left translation. Hence $M^2\varphi = \varphi$ not only for $\varphi \in \ker E$ but also for φ in the span of $A(\mathbb{A})$-translates of φ in $\ker E$. The number of such linear combinations is already sufficiently large to imply $M^2 = 1$. We now turn to rigorous proofs.

Proposition 4.31. *Let $M : \mathbb{C}[z,z^{-1}]^n \to \mathbb{C}((z))^n$ be a $\mathbb{C}$-linear map with $M(zu) = z^{-1}M(u)$ for all $u \in \mathbb{C}[z,z^{-1}]^n$. Let I denote the natural embedding $\mathbb{C}[z,z^{-1}]^n \hookrightarrow \mathbb{C}((z))^n$. Put $B = I + M$. Suppose there is some $k \in \mathbb{Z}$ for which the vector space $(\mathrm{Im}\, B)/B(z^k\mathbb{C}[z^{-1}]^n)$ is finite dimensional. Then there is some*

$$P(z) \in \mathrm{GL}(n, \mathbb{C}(z)) \subset \mathrm{GL}(n, \mathbb{C}((z)))$$

with $P(z^{-1}) = P(z)^{-1}$ and $(Mu)(z) = P(z)u(z^{-1})$ for all $u(z) \in \mathbb{C}[z,z^{-1}]^n$.

Proof. Denote by e_i the column in $\mathbb{C}^n$ with nonzero entry only at the ith row, where it is 1. From $M(\sum_i(\sum_j c_{ij}z^j)e_i) = \sum_i(\sum_j c_{ij}z^{-j})Me_i$, we see that $(Mu)(z) = P(z)u(z^{-1})$ where $P(z)$ is the $n \times n$ matrix with columns $Me_1, \dots, Me_n$ whose entries are in $\mathbb{C}((z))$. If u is in the kernel of $B = I + M$, then $P(z)u(z^{-1}) = -u(z)$. Since $\mathrm{Im}\, B = \cup_{m\ge1} B(z^m\mathbb{C}[z^{-1}]^n)$ and there is some $k \ge 0$ such that $B(z^k\mathbb{C}[z^{-1}]^n)$ has finite codimension in $\mathrm{Im}\, B$, there is some ℓ with $B(z^\ell\mathbb{C}[z^{-1}]^n) = \mathrm{Im}\, B$. Then $\ker B + z^\ell\mathbb{C}[z^{-1}]^n = \mathbb{C}[z,z^{-1}]^n$. For each i $(1 \le i \le n)$, $z^{\ell+1}e_i \in \ker B + z^\ell\mathbb{C}[z^{-1}]^n$. Hence there is a matrix $W \in M(n, \mathbb{C}[z,z^{-1}])$ whose columnes are in $\ker B$ and $W - z^{\ell+1}\,\mathrm{Id} \in z^\ell M(n, \mathbb{C}[z^{-1}])$, where Id is the identity matrix. But then $W \in \mathrm{GL}(n, \mathbb{C}(z))$, and since the columns of W are in $\ker B$, we have $P(z)W(z^{-1}) = -W(z)$. Then $P(z) = -W(z)W(z^{-1})^{-1}$, and $P(z^{-1}) = -W(z^{-1})W(z)^{-1} = P(z)^{-1}$. □

Corollary 4.32. *A $\mathbb{C}$-linear map $M : \mathbb{C}[z,z^{-1}] \to \mathbb{C}[z,z^{-1}]$ which satisfies the conditions of Proposition 4.31 has $M^2 = \mathrm{Id}$.*

Recall that $Y_\alpha = \alpha^{\mathbb{Z}} A(F)N(\mathbb{A})\backslash \mathrm{GL}(2,\mathbb{A})$. Write $C_+^\infty(Y_\alpha)$ for the space of the E-valued functions f on Y_α with (1) $f(x) = 0$ if $\mathrm{ht}^+(x)$ is large enough, and (2) f is invariant under right translation by some open subgroup U of $\mathrm{GL}(2,\mathbb{A})$. Note that $C_c^\infty(Y_\alpha) \subset C_+^\infty(Y_\alpha) \subset C^\infty(Y_\alpha)$.

PROPOSITION 4.33. *The image of $C_c^\infty(Y_\alpha)$ under M lies in $C_+^\infty(Y_\alpha)$.*

Proof. For $f \in C_c^\infty(Y_\alpha)$ there exists an integer m such that $f(x) = 0$ if $\mathrm{ht}^+(x) < -m$. We shall show that for such f, $(Mf)(x) = \int_{N(\mathbb{A})} f\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx\right) dx$ is zero if $\mathrm{ht}^+(x) > m$. It suffices to show then that for $x \in \mathrm{GL}(2,\mathbb{A})$ with $\mathrm{ht}^+(x) > m$, and any $n \in N(\mathbb{A})$, we have $\mathrm{ht}^+\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx\right) < -m$. But by Lemma 2.7 we have

$$\mathrm{ht}^+(x) + \mathrm{ht}^+\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx\right) = \mathrm{ht}^+(nx) + \mathrm{ht}^+\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx\right) \le 0.$$

□

PROPOSITION 4.34. *Let U be an open subgroup of* $\mathrm{GL}(2,O)$*. For every integer $m \ge 1$ define*

$$W_m^U = \{\varphi \in C_c^\infty(Y_\alpha)^U;\ \varphi(x) = 0 \ \textit{ if } \ \mathrm{ht}^+(x) < m\},$$

$$Y_m^U = \{\varphi \in C_c^\infty(\alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}))^U;\ \varphi(x) = 0 \ \textit{ if } \ \mathrm{ht}^+(x) < m\}.$$

Then $E(W_m^U) = Y_m^U$ for large enough m.

Proof. Put

$$Z_m^U = \{\varphi \in C_c^\infty(\alpha^{\mathbb{Z}} \cdot A(F)N(F)\backslash \mathrm{GL}(2,\mathbb{A}))^U;\ \varphi(x) = 0 \ \text{ if } \ \mathrm{ht}^+(x) < m\}.$$

Recall that

$$E = s_* r^*, \qquad s_*(x) = \sum_\gamma \psi(\gamma x), \qquad \gamma \in A(F)N(F)\backslash \mathrm{GL}(2,F).$$

It is clear that $s_*(Z_m^U) = Y_m^U$. It suffices to show that $r^*(W_m^U) = Z_m^U$ for sufficiently large m. In fact, we showed, as the first claim in the proof of Proposition 2.13, that for an open subgroup U of $\mathrm{GL}(2,\mathbb{A})$, that there is an integer m with the property that if $z \in \mathbb{A}$, $x \in \mathrm{GL}(2,\mathbb{A})$, $\mathrm{ht}^+(x) \ge m$, then there is $u \in U$, $\beta \in F$, with $\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) x = \left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right) xu$. In other words, if $x \in \mathrm{GL}(2,\mathbb{A})$ and $\mathrm{ht}^+(x)$ is large enough, then $N(\mathbb{A})x \subset N(F)xU$. □

We shall now give a different proof of Proposition 4.26.

PROPOSITION 4.35. *If $\varphi \in C_c^\infty(Y_\alpha)$ and $M\varphi \in C_c^\infty(Y_\alpha)$ then $M^2\varphi = \varphi$.*

Proof. Let us introduce a structure of $\mathbb{C}[z,z^{-1}]$-module on $C^\infty(Y_\alpha)$ by

$$(zf)(x) = \frac{1}{\sqrt{q}} f\left(\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right) x\right), \quad f \in C^\infty(Y_\alpha), \quad x \in \mathrm{GL}(2,\mathbb{A}).$$

From

$$(M\phi)\left(\left(\begin{smallmatrix}a & 0\\ 0 & b\end{smallmatrix}\right) x\right) = \left|\frac{a}{b}\right| \int_{N(\mathbb{A})} \phi\left(\left(\begin{smallmatrix}b & 0\\ 0 & a\end{smallmatrix}\right)\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right) nx\right) dn$$

it follows that $M(zf) = z^{-1}M(f)$; recall that $|\alpha| = q$, and f is invariant under α. This is the reason for introducing the factor $\sqrt{q}$. Let U be an open subgroup of $\mathrm{GL}(2,O)$. Put

$$W_c^U = C_c^\infty(Y_\alpha)^U, \qquad W_+^U = C_+^\infty(Y_\alpha)^U.$$

Both are $\mathbb{C}[z,z^{-1}]$-submodules in $C^\infty(Y_\alpha)$. Denote by W_0^U the set of functions $f \in C^\infty(Y_\alpha)^U$ such that $f(x) = 0$ if $\mathrm{ht}^+(x) \neq 0$. Then the natural map $W_0^U \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}] \to W_c^U$ is an isomorphism. In the same way we have a canonical isomorphism $W_0^U \otimes_{\mathbb{C}} \mathbb{C}((z)) \to W_+^U$. The operator

$$M : W_c = C_c^\infty(Y_\alpha) \to W_+ = C_+^\infty(Y_\alpha)$$

maps W_c^U into W_+^U. Hence it defines a map

$$M : W_0^U \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}] \to W_0^U \otimes_{\mathbb{C}} \mathbb{C}((z))$$

satisfying the first condition of Proposition 4.31.
It remains to check the second condition of that Proposition. The space W_m^U can be identified with $W_0^U \otimes_{\mathbb{C}} z^{-m}\mathbb{C}[z^{-1}]$, and then the operator $B = I + M$ is just E^*E. Thus it suffices to show that for some $m \in \mathbb{Z}$, the space $E^*E(W_c^U)/E^*E(W_m^U)$ is finite dimensional. Since $E(W_m^U) = Y_m^U$ for large m, and $\{x \in \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}); \mathrm{ht}(x) \leq m\}$ is compact mod $Z(\mathbb{A})$, it follows that the subspace

$$E(W_m^U) \subset C_c^\infty(\alpha^{\mathbb{Z}} \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}))^U$$

has finite codimension. Thus M satisfies both conditions of Proposition 4.31, and our claim follows from Corollary 4.32. □

To use Proposition 4.31 to give another proof of the rationality of $M(\mu_1,\mu_2,t)$, we take a different view of the Mellin transform and the relationship between the operators M and $M(\mu_1,\mu_2,t)$. Let $I_c(\mu_1\nu_{z^{-1}},\mu_2\nu_z)$ be the space of locally constant functions $f : \mathrm{GL}(2,\mathbb{A}) \to \mathbb{C}[z,z^{-1}]$ with

$$f\left(\left(\begin{smallmatrix}a & c\\ 0 & b\end{smallmatrix}\right) x\right) = \mu_1(a)\mu_2(b)\nu_z(b/a)|a/b|^{1/2} f(x).$$

Let $I_+(\mu_1\nu_{z^{-1}},\mu_2\nu_z)$ be

$$I_c(\mu_1\nu_{z^{-1}},\mu_2\nu_z) \otimes_{\mathbb{C}[z,z^{-1}]} \mathbb{C}((z)).$$

The group $\alpha^{\mathbb{Z}} \subset \mathrm{GL}(2,\mathbb{A})$ acts trivially on these I_c and I_+. We put

$$I_c = \oplus I_c(\mu_1\nu_{z^{-1}}, \mu_2\nu_z), \qquad I_+ = \oplus I_+(\mu_1\nu_{z^{-1}}, \mu_2\nu_z),$$

where the sums range over all characters μ_1, μ_2 of $\mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}}$.

Proposition 4.36. *There exists an isomorphism of $\mathbb{C}((z))$-modules $I_+ \xrightarrow{\sim} C_+^\infty(Y_\alpha)$ which is $\mathrm{GL}(2,\mathbb{A})$-equivariant and maps I_c to $C_c^\infty(Y_\alpha)$.*

Proof. Define a map $F : I_+ \to C_+^\infty(Y_\alpha)$ by mapping

$$\varphi = \{\varphi_{\mu_1,\mu_2}\} \in I_+, \qquad \varphi_{\mu_1,\mu_2} \in I_c(\mu_1\nu_{z^{-1}}, \mu_2\nu_z),$$

to

$$(F\varphi)(x) = \text{constant term of the formal series} \sum_{\mu_1,\mu_2} \varphi_{\mu_1,\mu_2}(x) \in \mathbb{C}((z)),$$

for any $x \in \mathrm{GL}(2,\mathbb{A})$. The map F is well defined, commutes with the actions of $\mathbb{C}((z))$ and $\mathrm{GL}(2,\mathbb{A})$. The inverse of F exists, as follows. If $\psi \in C_+^\infty(Y_\alpha)$ then $F^{-1}(\psi) = \{\varphi_{\mu_1,\mu_2}\}$ with $\varphi_{\mu_1,\mu_2} \in I_+(\mu_1\nu_{z^{-1}}, \mu_2\nu_z)$ given by

$$\varphi_{\mu_1,\mu_2}(x) = \int_{A(\mathbb{A})/\alpha^{\mathbb{Z}}\cdot A(F)} \psi(h^{-1}x)\eta(h)dh,$$

where

$$\eta : A(\mathbb{A}) \to \mathbb{C}((z))^\times, \quad \eta(\mathrm{diag}(a,b)) = \mu_1(a)\mu_2(b)\nu_z(a/b).$$

The last integral converges in the field $\mathbb{C}((z))$. A base of the topology is given by $z^n\mathbb{C}[[z]]$, $n > 0$. The map F maps I_c to $C_c^\infty(Y_\alpha)$. □

Put $I_0 = \oplus_{\mu_1,\mu_2} I_0(\mu_1,\mu_2)$, with

$$I_0(\mu_1,\mu_2) = \{f \in C^\infty(\mathrm{GL}(2,O));\ f\left(\left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right)x\right) = \mu_1(a)\mu_2(b)f(x)\}.$$

Denote by $M(z)$ the map $I_0 \to I_0$ which takes $I_0(\mu_1,\mu_2)$ to $I_0(\mu_2,\mu_1)$ via $M(\mu_1,\mu_2,z)$. We use the isomorphism F to identify the spaces I_+ and $C_+^\infty(Y_\alpha)$, as well as I_c and $C_c^\infty(Y_\alpha)$. The natural isomorphism

$$I_c(\mu_1\nu_{z^{-1}}, \mu_2\nu_z) \xrightarrow{\sim} I_0(\mu_1,\mu_2) \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$$

and

$$I_+(\mu_1\nu_{z^{-1}}, \mu_2\nu_z) \xrightarrow{\sim} I_0(\mu_1,\mu_2) \otimes_{\mathbb{C}} \mathbb{C}((z))$$

permit us to identify I_c and $I_0 \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$ as well as I_+ and $I_0 \otimes_{\mathbb{C}} \mathbb{C}((z))$. Thus the map $M : C_c^\infty(Y_\alpha) \to C_+^\infty(Y_\alpha)$ induces an operator

$$M_0 : I_0 \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}] \to I_0 \otimes_{\mathbb{C}} \mathbb{C}((z)).$$

Proposition 4.37. *Regard the elements of $I_0 \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$ as functions of z with values in I_0 and the elements of $I_0 \otimes_{\mathbb{C}} \mathbb{C}((z))$ as formal series in z with coefficients in I_0. Then for any $u \in I_0 \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$ one has $(M_0 u)(z) = M(z)u(z^{-1})$, $M(z)$ is viewed as a formal series in z.*

Proof. Write ι for the automorphism of $\mathbb{C}[z,z^{-1}]$ which maps z to z^{-1}. Given a function $f : \mathrm{GL}(2,\mathbb{A}) \to \mathbb{C}((z))$, denote by f_0 the function $\mathrm{GL}(2,\mathbb{A}) \to \mathbb{C}$ such that $f_0(x)$ is the constant term of $f(x)$.
Define an operator

$$M'' : I_0 \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}] \to I_0 \otimes_{\mathbb{C}} \mathbb{C}((z)) \qquad \text{by} \qquad (M''u)(z) = M(z)u(z^{-1}).$$

We claim that $M_0 = M''$. Consider M'' as a map $I_c \to I_+$. We have to show that for every $f \in I_c$, we have $FM''f = MFf$, for the isomorphism $F : I_+ \xrightarrow{\sim} C_+^\infty(Y_\alpha)$. As I_c is the sum over μ_1, μ_2 of $I_c(\mu_1\nu_{z^{-1}}, \mu_2\nu_z)$, it suffices to consider f in one of these summands.
For $x \in \mathrm{GL}(2,\mathbb{A})$, we have $(M''f)(x) = \int_{N(\mathbb{A})} \iota f\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx\right) dn$. Then

$$(FM''f)(x) = (M''f)_0(x) = \int_{N(\mathbb{A})} f_0\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx\right) dn$$

$$(MFf)(x) = \int_{N(\mathbb{A})} Ff\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx\right) dn = \int_{N(\mathbb{A})} f_0\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) nx\right) dn$$

are equal, as required. □

4.8 A second proof of the rationality of $M(\mu_1,\mu_2,t)$ and of the functional equation $M(\mu_1,\mu_2,t)M(\mu_2,\mu_1,t^{-1}) = 1$

Let U, W^U, A be as in the proof of Proposition 4.35. Then $W^U = \oplus_{\mu_1,\mu_2} W^U_{\mu_1,\mu_2}$, where $W^U_{\mu_1,\mu_2}$ is the space of functions $f \in W^U$ with

$$f\left(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) x\right) = \mu_1(a)^{-1}\mu_2(b)^{-1} f(x)$$

whenever $\deg(a) = \deg(b) = 0$. The natural maps $I_0(\mu_2,\mu_1)^U \xrightarrow{\sim} W^U_{\mu_1,\mu_2}$ permit one to identify W^U and the space I_0^U. The map

$$M : W^U \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}] \to W^U \otimes_{\mathbb{C}} \mathbb{C}((z))$$

is induced by the operator

$$M_0 : I_0 \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}] \to I_0 \otimes_{\mathbb{C}} \mathbb{C}((z)).$$

The proof of Proposition 4.35 implies that the operator M satisfies the conditions of Proposition 4.31. Then M is given by a formula of the form $(Mu)(z) = P(z)u(z^{-1})$, where $P(z)$ is an automorphism of V which depends on z rationally, and $P(z^{-1}) = P(z)^{-1}$. From Proposition 4.37 it follows that

$P(z)$ is just the restriction of $M(z)$ to $I_0^U \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$. The group U may be arbitrarily small. Hence $M(z)$ is a rational function of z, and $M(z)M(z^{-1}) = 1$. Hence for any characters μ_1, μ_2, of $\alpha^{\mathbb{Z}} \cdot F^\times \backslash \mathbb{A}^\times$, the operator $M(\mu_1, \mu_2, z)$ depends rationally on z, and

$$M(\mu_1, \mu_2, z)M(\mu_1, \mu_2, z^{-1}) = 1.$$

The same is true for any characters μ_1, μ_2 of $\mathbb{A}^\times / F^\times$, which are not necessarily trivial at α. To see this, it suffices to use the identities $M(\mu_1\nu_t, \mu_2\nu_t, z) = M(\mu_1, \mu_2, z)$ and $M(\mu_1\nu_t, \mu_2\nu_{t^{-1}}, z) = M(\mu_1, \mu_2, tz)$. □

5 Proof of the trace formula

5.1 The geometric part

Our aim is to compute the trace $\operatorname{tr} r_0(f)$, where $f \in C_c^\infty(\mathrm{GL}(2, \mathbb{A}))$ and r_0 is the representation of $\mathrm{GL}(2, \mathbb{A})$ by right translation on the space $A_{0,\alpha}$ of cusp forms invariant under α. Recall that the space $A_{c,\alpha}$ of α-invariant automorphic forms is equal to the direct sum of $A_{0,\alpha}$ and $A_{E,\alpha} = \operatorname{Im}(E : C_c^\infty(Y_\alpha) \to A_{c,\alpha})$. The corresponding representations of $\mathrm{GL}(2, \mathbb{A})$ are denoted by r and r_E. Had r been admissible, we would have had $\operatorname{tr} r_0(f) = \operatorname{tr} r(f) - \operatorname{tr} r_E(f)$, and the computation of $\operatorname{tr} r_0(f)$ would have reduced to that of $\operatorname{tr} r(f)$ and $\operatorname{tr} r_E(f)$. But r and r_E are not admissible, so $\operatorname{tr} r(f)$ and $\operatorname{tr} r_E(f)$ make no sense.
Suppose f is right invariant under the open subgroup U of $\mathrm{GL}(2, O)$. Denote by A_0^U, A_c^U, A_E^U the spaces of U-invariant vectors in $A_{0,\alpha}$, $A_{c,\alpha}$, $A_{E,\alpha}$. Since $\operatorname{Im} r_0(f) \subset A_0^U$, we have $\operatorname{tr} r_0(f) = \operatorname{tr} r_0^U(f)$, where $r_0^U(f)$ is the restriction of $r_0(f)$ to A_0^U.
Denote by χ_m the characteristic function of the set

$$\{x \in \alpha^{\mathbb{Z}} \cdot \mathrm{GL}(2, F) \backslash \mathrm{GL}(2, \mathbb{A}); \operatorname{ht}(x) < m\}, \quad m > 0.$$

Denote by θ_m the operator of multiplication by χ_m on $A_{c,\alpha}$.

Proposition 5.1. (1) *For any $m > 0$, $\dim \theta_m(A_c^U) < \infty$.*
(2) *If $m >> 1$ then (a) θ_m acts as the identity on A_0^U, and (b) $\theta_m(A_E^U) \subset A_E^U$.*

Proof. (1) The support of χ_m is compact mod $Z(\mathbb{A})$, the quotient by the open U is then finite. (2a) A_0^U is finite dimensional, consisting of compactly supported forms. (2b) By (2a), $(1 - \theta_m)A_E^U = (1 - \theta_m)A_c^U$. This lies in A_E^U as U-invariant cusp forms are uniformly compactly supported. Hence $\theta_m(A_E^U) \subset A_E^U$. □

Denote by $r^U(f)$ and $r_E^U(f)$ the restrictions of $r(f)$ to A_c^U and A_E^U. For m such that $\theta_m(A_E^U) \subset A_E^U$, denote the restriction of θ_m to A_E^U again by θ_m. Then for $m >> 1$,

$$\operatorname{tr} r_0(f) = \operatorname{tr} r_0^U(f) = \operatorname{tr}(\theta_m r^U(f)) - \operatorname{tr}(\theta_m r_E^U(f)) = \operatorname{tr}(\theta_m r(f)) - \operatorname{tr}(\theta_m r_E^U(f)).$$

We then proceed to compute $\operatorname{tr}(\theta_m r(f))$ and $\operatorname{tr}(\theta_m r_E^U(f))$.

Proposition 5.2. *There exist $c_f \in E$ and $\alpha_m \in E$ with $\lim_{n\to\infty} \alpha_m = 0$, and*

$$\operatorname{tr}(\theta_m r(f)) = \sum_{1\le i\le 4} S_i(f) + c_f(m - \frac{1}{2}) + \alpha_m.$$

Proof. The map $\theta_m r(f) : A_{c,\alpha} \to A_{c,\alpha}$ is an integral operator with kernel $\chi_m(y)K_f(x,y)$, where $K_f(x,y) = \sum_{\gamma\in\alpha^{\mathbb{Z}}\cdot\mathrm{GL}(2,F)} f(x^{-1}\gamma y)$. Then

$$\operatorname{tr}(\theta_m r(f)) = \int_{\alpha^{\mathbb{Z}}\cdot\mathrm{GL}(2,F)\backslash\,\mathrm{GL}(2,\mathbb{A})} \chi_m(x)K_f(x,x)dx.$$

Lemma 5.3. *There exists $m_f > 0$ such that if $x \in \mathrm{GL}(2,\mathbb{A})$, $\gamma \in \alpha^{\mathbb{Z}}\,\mathrm{GL}(2,F)$, $\mathrm{ht}^+(x) > m_f$, $f(x^{-1}\gamma x) \neq 0$, then $\gamma \in \alpha^{\mathbb{Z}}A(F)N(F)$.*

Proof. We have $\gamma x = xy$, y in $\mathrm{supp}(f)$. Since $\mathrm{ht}^+(x) + \mathrm{ht}^+(\delta x) \le 0$ for $\delta \in \mathrm{GL}(2,F) - B(F)$, we have that $\mathrm{ht}^+(x) > 0$. If in addition we had $\mathrm{ht}^+(xy) > 0$, we would conclude that $\gamma \in \alpha^{\mathbb{Z}}B(F)$. The number $m_f = -\min\{\mathrm{ht}^+(z);\ z \in \mathrm{GL}(2,O)\cdot\mathrm{supp}(f)\}$ then has the property that $\mathrm{ht}^+(x) > m_f$, $y \in \mathrm{supp}(f)$, implies $\mathrm{ht}^+(xy) = \mathrm{ht}^+(x) + \mathrm{ht}^+(ky) > 0$, where $x = bk$ and $ky = b'k'$ so that $xy = bb'k$ (b, $b' \in B(\mathbb{A})$; k, $k' \in \mathrm{GL}(2,\mathbb{A})$). □

Denote by ξ_m the characteristic function of the set $\{x \in \mathrm{GL}(2,\mathbb{A});\ \mathrm{ht}^+(x) \ge m\}$, by $A'(F)$ the set of nonscalar diagonal matrices, and by Ell the set of elliptic matrices in $\mathrm{GL}(2,F)$, namely those whose eigenvalues are not in F. Put $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Lemma 5.4. *If m is big enough, then $\chi_m(y)K_f(x,x)$ is the sum of*

$$T_{1,m}(x) = \chi_m(x)\sum_{\gamma\in\alpha^{\mathbb{Z}}\cdot F^\times} f(\gamma), \qquad T_{2,m}(x) = \sum_{\gamma\in\alpha^{\mathbb{Z}}\cdot\mathrm{Ell}} f(x^{-1}\gamma x),$$

$$T_{3,m}(x) = \frac{1}{2}\sum_{\gamma\in\alpha^{\mathbb{Z}}\cdot A'(F)}\ \sum_{\delta\in A(F)\backslash\,\mathrm{GL}(2,F)} f(x^{-1}\delta^{-1}\gamma\delta x)\cdot(1 - \xi_m(\delta x) - \xi_m(w\delta x)),$$

$$T_{4,m}(x) = \sum_{a\in\alpha^{\mathbb{Z}}\cdot F^\times}\ \sum_{\delta\in F^\times N(F)\backslash\,\mathrm{GL}(2,F)} f(x^{-1}\delta^{-1}\left(\begin{smallmatrix} a & a \\ 0 & a \end{smallmatrix}\right)\delta x)\cdot(1 - \xi_m(\delta x)).$$

Proof. $T_{1,m}(x)$ is the contribution of the elements $\gamma \in \alpha^{\mathbb{Z}}\cdot F^\times$ in $\chi_m(x)K_f(x,x)$. We claim that the contribution of the elements $\gamma \in \alpha^{\mathbb{Z}}\cdot\mathrm{Ell}$ in $\chi_m(x)K_f(x,x)$ is $T_{2,m}(x)$. To show this, we need to see that if $x \in \mathrm{GL}(2,\mathbb{A})$, $\gamma \in \alpha^{\mathbb{Z}}\cdot\mathrm{Ell}$ and $\Phi(x^{-1}\gamma x) \neq 0$, then $\mathrm{ht}^+(x) < m$. Indeed, if $\mathrm{ht}(x) \ge m$ then there is some $\delta \in \mathrm{GL}(2,F)$ with $\mathrm{ht}^+(\delta x) \ge m$. Lemma 5.3 then implies that $\delta\gamma\delta^{-1} \in \alpha^{\mathbb{Z}}A(F)N(F)$, contradicting $\gamma \in \alpha^{\mathbb{Z}}\cdot\mathrm{Ell}$.
Denote by $T'_{3,m}(x)$ the contribution into $\chi_m(x)K_f(x,x)$ of the elements γ of the form $\alpha^j\gamma$, $j \in \mathbb{Z}$, $\gamma \in \mathrm{GL}(2,F)$ with distinct eigenvalues in F. By $T'_{4,m}(x)$

we denote the contribution of the elements $\alpha^j\gamma$, $j \in \mathbb{Z}$, $\gamma \in \mathrm{GL}(2,F)$, $\gamma \notin F^\times$ but the eigenvalues of γ are equal. We have

$$T'_{3,m}(x) = \frac{1}{2}\chi_m(x) \sum_{\gamma\in\alpha^{\mathbb{Z}}\cdot A'(F)} \sum_{\delta\in A(F)\backslash \mathrm{GL}(2,F)} f(x^{-1}\delta^{-1}\gamma\delta x).$$

The factor $\frac{1}{2}$ appears since $\mathrm{diag}(b,a)$ is conjugate to $\mathrm{diag}(a,b)$. To show that $T'_{3,m}(x) = T_{3,m}(x)$ it suffices to show that when $f(x^{-1}\delta^{-1}\gamma\delta x) \neq 0$,

$$\chi_m(x) = 1 - \xi_m(\delta x) - \xi_m(w\delta x),$$

namely if $\mathrm{ht}(x) \geq m$ then either $\mathrm{ht}^+(\delta x) \geq m$ or $\mathrm{ht}^+(w\delta x) \geq m$. So if $\mathrm{ht}(x) \geq m$, then there is some $\eta \in \mathrm{GL}(2,F)$ with $\mathrm{ht}^+(\eta x) \geq m$. By Lemma 5.3, $\eta\delta^{-1}\gamma\delta\eta^{-1} \in \alpha^{\mathbb{Z}}A(F)N(F)$, but this implies that $\eta\delta^{-1} \in A(F)N(F)$ or $\eta\delta^{-1}w \in A(F)N(F)$. Correspondingly, $\mathrm{ht}^+(\delta x) = \mathrm{ht}^+(\eta x) \geq m$ or $\mathrm{ht}^+(w\delta x) = \mathrm{ht}^+(\eta x) \geq m$, but both inequalities cannot hold simultaneously if $m > 0$.
Now

$$T'_{4,m}(x) = \chi_m(x) \sum_{a\in\alpha^{\mathbb{Z}}\cdot F^\times} \sum_{\delta\in F^\times N(F)\backslash \mathrm{GL}(2,F)} f(x^{-1}\delta^{-1} \left(\begin{smallmatrix} a & a \\ 0 & a \end{smallmatrix}\right) \delta x).$$

To show that this equals $T_{4,m}(x)$ we need to check that when

$$f(x^{-1}\delta^{-1} \left(\begin{smallmatrix} a & a \\ 0 & a \end{smallmatrix}\right) \delta x) \neq 0$$

and $\mathrm{ht}(x) \geq m$, then $\mathrm{ht}^+(\delta x) \geq m$. Suppose then that $\mathrm{ht}^+(\eta x) \geq m$ for $\eta \in \mathrm{GL}(2,F)$. Then by Lemma 5.3 we have

$$\eta\delta^{-1} \left(\begin{smallmatrix} a & a \\ 0 & a \end{smallmatrix}\right) \delta\eta^{-1} \in \alpha^{\mathbb{Z}}A(F)N(F).$$

Hence $\eta\delta^{-1} \in A(F)N(F)$, so that $\mathrm{ht}^+(\delta x) = \mathrm{ht}^+(\eta x) \geq m$. □

We conclude that $\mathrm{tr}\,\theta_m r(f) = \sum_{1\leq i\leq 4} t_{i,m}$ with

$$t_{i,m} = \int_{\alpha^{\mathbb{Z}}\cdot \mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A})} T_{i,m}(x)dx.$$

To prove the proposition it suffices to show that $t_{i,m} = S_i(f) + c_i(2m-1) + \beta_m$ for all i ($1 \leq i \leq 4$), where c_i does not depend on m and $\lim \beta_m = 0$. It is clear that $t_{1,m} \to S_1(f)$ as $m \to \infty$. As $T_{2,m}(x)$ is independent of m, $t_{2,m} = S_2(f)$. Now

$$\begin{aligned} t_{3,m} &= \frac{1}{2} \sum_{\gamma\in\alpha^{\mathbb{Z}}\cdot A'(F)} \int_{\alpha^{\mathbb{Z}}(A(F)\backslash \mathrm{GL}(2,\mathbb{A})} f(x^{-1}\gamma x)(1 - \xi_m(x) - \xi_m(wx))dx \\ &= \frac{1}{2} \sum_{\gamma\in\alpha^{\mathbb{Z}}\cdot A'(F)} \int_{A(\mathbb{A})\backslash \mathrm{GL}(2,\mathbb{A})} f(x^{-1}\gamma x)s(x)dx \end{aligned}$$

where

$$s(x) = \int_{\alpha^{\mathbb{Z}} A(F)\backslash A(\mathbb{A})} [1 - \xi_m(yx) - \xi_m(wyx)]dy$$

$$= \mathrm{vol}\{y \in \alpha^{\mathbb{Z}} A(F)\backslash A(\mathbb{A});\ \mathrm{ht}^+(yx) < n,\ \mathrm{ht}^+(wyx) < n\}.$$

Note that for $y \in A(\mathbb{A})$, $\mathrm{ht}^+(yx) = \mathrm{ht}^+(y) + \mathrm{ht}^+(x)$ and $\mathrm{ht}^+(wyx) = \mathrm{ht}^+(wx) - \mathrm{ht}^+(y)$. Hence

$$s(x) = |\{y \in A(\mathbb{A})/\alpha^{\mathbb{Z}} \cdot A(F);\ \mathrm{ht}^+(wx) - m < \mathrm{ht}^+(y) < m - \mathrm{ht}^+(x)\}|.$$

This is the number of integers between $\mathrm{ht}^+(wx) - m$ and $m - \mathrm{ht}^+(x)$. So $s(x) = 2m - 1 - \mathrm{ht}^+(x) - \mathrm{ht}^+(wx)$.

Lemma 5.5. *We have* $\mathrm{ht}^+(x) + \mathrm{ht}^+(wx) = -2r(x)$, *where if* $x = a\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right)k$, $a \in A(\mathbb{A})$, $k \in \mathrm{GL}(2, O)$ *and* $y \in \mathbb{A}$, *we put* $r(x) = \sum_v \max(0, \log_q |y_v|_v)$.

Proof. Note that y is determined up to a change $y \mapsto by + c$, $b \in O^\times$, $c \in O$, so $r(x)$ is well defined. The asserted relation does not change if x is replaced by axk, $a \in A(\mathbb{A})$, $k \in \mathrm{GL}(2, O)$, so we may assume $x = \left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) \in N(\mathbb{A})$. Then $\mathrm{ht}^+(x) = 0$, and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) = \begin{pmatrix} -\frac{1}{y} & 1 \\ 0 & y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{y} & 1 \end{pmatrix}$ implies that $\mathrm{ht}^+(wx) = -2r(x)$. □

Lemma 5.5 implies that

$$t_{3,m} = S_3(f) + (m - \frac{1}{2}) \sum_{\gamma \in \alpha^{\mathbb{Z}} \cdot A'(F)} \int_{A(\mathbb{A})\backslash \mathrm{GL}(2,\mathbb{A})} f(x^{-1}\gamma x)dx.$$

Next

$$t_{4,m} = \sum_{a \in \alpha^{\mathbb{Z}} \cdot F^\times} \int_{\alpha^{\mathbb{Z}} F^\times N(F)\backslash \mathrm{GL}(2,\mathbb{A})} f\left(x^{-1}\left(\begin{smallmatrix} a & a \\ 0 & a \end{smallmatrix}\right)x\right)(1 - \xi_m(x))dx$$

$$= \sum_{a \in \alpha^{\mathbb{Z}} \cdot F^\times} \int_{\{x \in \alpha^{\mathbb{Z}} F^\times N(F)\backslash \mathrm{GL}(2,\mathbb{A});\, \mathrm{ht}^+(x) < m\}} f\left(x^{-1}\left(\begin{smallmatrix} a & a \\ 0 & a \end{smallmatrix}\right)x\right)dx.$$

Recall that $\theta_{a,f}(t) = \int_{\alpha^{\mathbb{Z}} F^\times N(F)\backslash \mathrm{GL}(2,\mathbb{A})} f\left(x^{-1}\left(\begin{smallmatrix} a & a \\ 0 & a \end{smallmatrix}\right)x\right)t^{\mathrm{ht}^+(x)}dx$ is a Laurent series at $t = 0$ of a rational function of t with $\zeta_F(q^{-1}t)^{-1}\theta_{a,f}(t) \in \mathbb{C}[t, t^{-1}]$. Suppose $\theta_{a,f}(t) = \sum_k u_k(a)t^k$. Then $t_{4,m} = \sum_{a \in \alpha^{\mathbb{Z}} \cdot F^\times} \sum_{k<m} u_k(a)$. Since $\zeta_F(q^{-1}t)$ has a simple pole at $t = 1$, we have that $\theta_{a,f}(t) = \frac{\rho(a)}{1-t} + \overline{\theta}_{a,f}(t)$, with $\overline{\theta}_{a,f}(t)$ without poles on $0 < |t| \leq 1$. Then

$$\tilde{\theta}_{a,f}(t) = \frac{1}{2}(\theta_{a,f}(t) + \theta_{a,f}(t^{-1})) = \frac{1}{2}(\overline{\theta}_{a,f}(t) + \overline{\theta}_{a,f}(t^{-1})) + \frac{1}{2}\rho(a),$$

$$\tilde{\theta}_{a,f}(1) = \overline{\theta}_{a,f}(1) + \frac{1}{2}\rho(a) = \frac{1}{2}\rho(a) + \sum_k (u_k(a) - \rho(a))$$

$$= \lim_{m\to\infty}[\sum_{k<m} u_k(a) - (m-\frac{1}{2})\rho(a)].$$

Then

$$t_{4,m} = \sum_{a\in\alpha^{\mathbb{Z}}\cdot F^\times} \tilde{\theta}_{a,f}(1) + (m-\frac{1}{2})\rho(a) + \beta_m, \qquad \beta_m \to 0 \text{ as } m\to\infty,$$

and $S_4(f) = \sum_{a\in\alpha^{\mathbb{Z}}\cdot F^\times} \tilde{\theta}_{a,f}(1)$. Proposition 5.2 follows. □

Note that β_m is 0 for sufficiently large m, as will be seen below.

5.2 The Eisenstein contribution

Next we turn to computing $\mathrm{tr}(\theta_m r_E^U(f))$ for large m. Put $W_c^U = C_c^\infty(Y_\alpha)^U$, $W_M^U = (1+M)W_c^U$.

Proposition 5.6. *The operator E^* maps A_E^U isomorphically onto W_M^U.*

Proof. As $A_E^U = E(W_c^U)$ and $E^*E = 1+M$, it suffices to show that $\ker E^*E = \ker E$. For $\varphi \in \ker E^*E$ we have $(E\varphi, E\varphi) = (E^*E\varphi,\varphi) = 0$, hence $E\varphi = 0$. □

Definition 1. Denote by W_m^U the space of f in W_c^U with $f(x) = 0$ if $\mathrm{ht}^+(x) < m$. Denote by ξ_m also the operator $W_m^U \to W_m^U$ of multiplication by the characteristic function of the set $\{x \in Y_\alpha;\ \mathrm{ht}^+(x) \geq m\}$. [If $m > 0$ then ξ_m is a left inverse to the operator $1+M : W_m^U \to W_M^U$. Indeed, if f is in W_m^U, then $(Mf)(x) = 0$ already when $\mathrm{ht}^+(x) > -m$ since $\mathrm{ht}^+(wnx)+\mathrm{ht}^+(nx) < 0$ implies $\mathrm{ht}^+(wnx) < m$ and so $f(wnx) = 0$.] Hence $\pi^m = (1+M)\xi_m : W_M^U \to W_M^U$ satisfies $\pi^m\pi^m = \pi^m$, for $m > 0$. Put $\pi_m = 1-\pi^m$.

Proposition 5.7. *For sufficiently large m, E^* intertwines θ_m with π_m, thus $\pi_m E^* = E^*\theta_m$, namely the diagram*

$$\begin{array}{ccc} A_E^U & \xrightarrow{E^*} & W_M^U \\ \theta_m\downarrow & & \downarrow\pi_m \\ A_E^U & \xrightarrow{E^*} & W_M^U \end{array}$$

is commutative.

Proof. Suppose $f \in A_E^U$ and $(1-\theta_m)f = 0$. Then $f(x) = 0$ for x with $\mathrm{ht}(x) \geq m$. As $\xi_m(x) \neq 0$ only on x with $\mathrm{ht}^+(x) \geq m$, we have $0 = (1+M)\xi_m E^* f = (1-\pi_m)E^*f$, the last equality as $1-\pi_m = \pi^m = (1+M)\xi_m$. For such f we have $E^*\theta_m f = E^*f$ and $\pi_m E^* f = E^* f$.

If $f \in A_E^U$ and $\theta_m f = 0$, then by Proposition 4.34 there is $\varphi \in W_m^U$ with $f = E\varphi$. Then

$$\pi_m E^* f = \pi_m E^* E\varphi = \pi_m(1+M)\varphi = \pi_m(1+M)\xi_m\varphi = \pi_m\pi^m\varphi = 0,$$

hence $E^*\theta_m f = \pi_m E^* f$ for such f.

Any $f \in A_E^U$ can be written as $f = f_1 + f_2$, $f_1 = (1-\theta_m)f$, $f_2 = \theta_m f$, thus $\theta_m f_1 = 0$ and $(1-\theta_m)f_2 = 0$. □

Definition 2. Recall that $Y_\alpha = \alpha^{\mathbb{Z}} A(F)N(\mathbb{A})\backslash \mathrm{GL}(2,\mathbb{A})$. Denote by σ_c, σ_+, σ_M the representations of $\mathrm{GL}(2,\mathbb{A})$ in the spaces

$$W_c = C_c^\infty(Y_\alpha), \qquad W_+ = C_+^\infty(Y_\alpha), \qquad W_M = (1+M)C_c^\infty(Y_\alpha).$$

Consider $\sigma_c(f)$, $\sigma_+(f)$, $\sigma_M(f)$ as operators in the spaces W_c^U, W_+^U, W_M^U.

Corollary 5.8. *We have* $\mathrm{tr}(\theta_m \cdot r_E^U(f)) = \mathrm{tr}(\pi_m \cdot \sigma_M(f))$.

Proof. The operator E^* yields an isomorphism of $A_E^U = E(W_c^U)$ with W_M^U intertwining θ_m with π_m. □

In the proof of Proposition 4.35 we introduced a structure of $\mathbb{C}[z,z^{-1}]$-module on W_c^U and W_+^U, as well as isomorphisms $W_c^U \simeq W_0^U \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$ and $W_+^U \simeq W_0^U \otimes_{\mathbb{C}} \mathbb{C}((z))$, where $W_0^U = \{f \in W_c^U;\ f(x) = 0 \text{ if } \mathrm{ht}^+(x) \neq 0\}$. Under these isomorphisms, the operator $M : W_c^U \to W_+^U$ corresponds to the operator

$$M : W_0^U \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}] \to W_0^U \otimes_{\mathbb{C}} \mathbb{C}((z)),$$

which satisfies the conditions of Proposition 4.31, hence has the form

$$(Mu)(z) = P(z)u(z^{-1}) \qquad \text{for} \qquad u \in W_0^U \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$$

which is viewed as a function of z with values in W_0^U. Here $P(z)$ is a rational function in z with values in $\mathrm{Aut}\, W_0^U$, and $P(z^{-1}) = P(z)^{-1}$.
Now $\sigma_c(f)$ is an endomorphism of W_c^U as a $\mathbb{C}[z,z^{-1}]$-module. The corresponding endomorphism of the module $W_0^U \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$ is determined by a function $B(z)$ in $\mathrm{End}(W_0^U) \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$. The endomorphism of $W_0^U \otimes_{\mathbb{C}} \mathbb{C}((z))$ corresponding to the operator $\sigma_+(f)$ is determined by the same function $B(z)$. The relation

$$M\sigma_c(f) = \sigma_+(f)M \qquad \text{becomes} \qquad P(z)B(z^{-1})u(z^{-1}) = B(z)P(z)u(z^{-1})$$

for any $u \in W_0^U \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$, thus $B(z^{-1}) = P(z)^{-1}B(z)P(z)$.

Definition 3. Under the isomorphism $W_+^U \simeq W_0^U \otimes_{\mathbb{C}} \mathbb{C}((z))$, the subspace $W_M^U = (1+M)W_c^U$ is mapped onto the subspace L consisting of all rational functions of the form $u(z) + P(z)u(z^{-1})$, with $u \in W_0^U \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$. Put

$$L_m = L \cap (W_0^U \otimes_{\mathbb{C}} z^{-m+1}\mathbb{C}[[z]]).$$

Denote by L^m the set of rational functions of the form

$$u(z) + P(z)u(z^{-1}) \qquad \text{with} \qquad u \in W_0^U \otimes_{\mathbb{C}} z^{-m}\mathbb{C}[z^{-1}].$$

For sufficiently large m we have $L = L_m \oplus L^m$. Under the isomorphism $W_M^U \xrightarrow{\sim} L$, the operator $\pi_m : W_M^U \to W_M^U$ corresponds to the idempotent operator $L \to L$ with kernel L^m and image L_m. This projection will also be denoted by π_m. Thus $\mathrm{tr}(\pi_m\sigma_M(f)) = \mathrm{tr}(\pi_m B)$, where $B : L \to L$ is the operator of multiplication by $B(z)$. On the left, π_m is an operator on W_M^U, on the right, on L.

Fix Q_1, $Q_2 \in M(k, \mathbb{C}[z, z^{-1}])$, $k \geq 1$, such that $\det Q_i \neq 0$. Suppose the function $Q_2(z)^{-1}Q_1(z)$ is regular at $z = \infty$, thus $Q_1(z) \in Q_2(z)M(k, \mathbb{C}[[z^{-1}]])$, and the function $Q_1(z)^{-1}Q_2(z)$ is regular at $z = 0$, thus $Q_2(z) \in Q_1(z)M(k, \mathbb{C}[[z]])$. Put $R = \mathbb{C}[z, z^{-1}]^k$. For $m \geq 1$, put

$$R_m = R \cap z^{1-m}Q_1(z)\mathbb{C}[[z]]^k \cap z^{m-1}Q_2(z)\mathbb{C}[[z^{-1}]]^k.$$

Also put

$$R_-^m = z^{-m}Q_1(z)\mathbb{C}[z^{-1}]^k \qquad R_+^m = z^m Q_2(z)\mathbb{C}[z]^k.$$

Then $\dim R_m$ is finite.

Proposition 5.9. *We have* $R = R_-^m \oplus R_m \oplus R_+^m$,

$$R_m \oplus R_+^m = R \cap z^{1-m}Q_1(z)\mathbb{C}[[z]]^k$$

and

$$R_m \oplus R_-^m = R \cap z^{m-1}Q_2(z)\mathbb{C}[[z^{-1}]]^k.$$

Proof. The natural map $\varphi : R_-^m \to X_- = \mathbb{C}((z))^k / z^{1-m}Q_1(z)\mathbb{C}[[z]]^k$ is an isomorphism (note that $\mathbb{C}((z))/z^{1-m}\mathbb{C}[[z]] \simeq z^{-m}\mathbb{C}[z^{-1}]$ and $Q_1(z)$ is invertible in $\mathrm{GL}(k, \mathbb{C}((z)))$. The natural map

$$\psi : R_+^m \to X_+ = \mathbb{C}((z^{-1}))^k / z^{m-1}Q_2(z)\mathbb{C}[[z^{-1}]]^k$$

is then too. The natural map $f : R/R_m \to X_- \oplus X_+$ is injective (by definition of R_m as the intersection of R and the denominators of X_-, X_+) and the composition of the natural map $R_+^m \oplus R_-^m \to R/R_m$ with f is $\varphi \oplus \psi$. $\square$

Definition 4. (1) Denote by $\mathrm{pr}_m : R \to R$ the projection with kernel $R_+^m \oplus R_-^m$ and image R_m. (2) If $A(z)$ is a matrix in $M(k, \mathbb{C}[z, z^{-1}])$, denote by $A[z]$ also the corresponding automorphism of $R = \mathbb{C}[z, z^{-1}]^k$. Denote by A_0 the constant term of $A(z)$.

Proposition 5.10. *The trace* $\mathrm{tr}(\mathrm{pr}_m \cdot A[z])$ *is equal to*

$$(2m-1)\,\mathrm{tr}\, A_0 - \mathrm{res}_{z=0}\,\mathrm{tr}\, A(z)Q_1'(z)Q_1(z)^{-1}dz - \mathrm{res}_{z=\infty}\,\mathrm{tr}\, A(z)Q_2'(z)Q_2(z)^{-1}dz.$$

Proof. Define a projection $\mathrm{pr}_+^m : R \to R$ with image R_+^m and kernel $R_-^m + R_m$, and a projection $\mathrm{pr}_-^m : R \to R$ with image R_-^m and kernel $R_+^m + R_m$. Analogously to the decomposition $R = R_-^m \oplus R_m \oplus R_+^m$, consider the decomposition

$$R = z^{-m}\mathbb{C}[z^{-1}]^k \oplus (z^{1-m}\mathbb{C}[z]^k \cap z^{m-1}\mathbb{C}[z^{-1}]^k) \oplus z^m\mathbb{C}[z]^k,$$

namely the case where $Q_1 = 1 = Q_2$. Denote the associated projections by p_-^m, p_m, p_+^m. Since the space $z^{-m}\mathbb{C}[z^{-1}]^k / R_-^m \cap z^{-m}\mathbb{C}[z^{-1}]^k$ is finite dimensional, the operator $\mathrm{pr}_+^m - p_+^m$ has finite rank, and the operator $\mathrm{pr}_-^m - p_-^m$ has finite rank since $z^m\mathbb{C}[z]^k / R_+^m \cap z^m\mathbb{C}[z]^k$ is finite dimensional.

Lemma 5.11. *We have*

$$\operatorname{tr}(\mathrm{pr}_-^m \cdot A[z] - p_-^m \cdot A[z]) = \operatorname{res}_{z=0} \operatorname{tr} A(z) Q_1'(z) Q_1(z)^{-1} dz,$$

as well as

$$\operatorname{tr}(\mathrm{pr}_+^m \cdot A[z] - p_+^m \cdot A[z]) = \operatorname{res}_{z=\infty} \operatorname{tr} A(z) Q_2'(z) Q_2(z)^{-1} dz.$$

Proof. Denote by $\mathrm{Pr}_-^m : \mathbb{C}((z))^k \to \mathbb{C}((z))^k$ the projection with image $z^{-m}Q_1(z)\mathbb{C}[z^{-1}]^k$ and kernel $z^{1-m}Q_1(z)\mathbb{C}[[z]]^k$. Denote by $P_-^m : \mathbb{C}((z))^k \to \mathbb{C}((z))^k$ the projection with image $z^{-m}\mathbb{C}[z^{-1}]^k$ and kernel $z^{1-m}\mathbb{C}[[z]]^k$ (thus the case of $Q_1 = 1$). Denote by $A((z))$ the endomorphism of $\mathbb{C}((z))^k$ which is defined by multiplication by $A(z)$. Then $\mathrm{Pr}_-^m = Q_1((z)) \cdot P_-^m \cdot Q_1((z))^{-1}$. Now

$$\operatorname{Im}(\mathrm{Pr}_-^m \cdot A((z)) - P_-^m \cdot A((z))) \subset \mathbb{C}[z, z^{-1}]^k,$$

and the restriction of the operator

$$\mathrm{Pr}_-^m \cdot A((z)) - P_-^m \cdot A((z)) \qquad \text{to} \qquad \mathbb{C}[z, z^{-1}]^k \qquad (\subset \mathbb{C}((z))^k)$$

is equal to $\mathrm{pr}_-^m \cdot A[z] - p_-^m \cdot A[z]$. Hence

$$\operatorname{tr}(\mathrm{pr}_-^m \cdot A[z] - p_-^m \cdot A[z]) = \operatorname{tr}(\mathrm{Pr}_-^m \cdot A((z)) - P_-^m \cdot A((z)))$$

$$= \operatorname{tr}(Q_1((z)) \cdot P_-^m \cdot Q_1((z))^{-1} \cdot A((z)) - P_-^m \cdot A((z)))$$

$$= \operatorname{tr}(Q_1((z)) \cdot P_-^m \cdot C((z)) - P_-^m \cdot Q_1((z)) \cdot C((z))), \qquad C(z) = Q_1(z)^{-1}A(z).$$

As $\operatorname{tr} A(z)Q_1'(z)Q_1(z)^{-1} = \operatorname{tr} C(z)Q_1'(z)$, to prove the first claim of the lemma it suffices to show that

$$\operatorname{tr}(Q_1((z)) \cdot P_-^m \cdot C((z)) - P_-^m \cdot Q_1((z))C((z))) = \operatorname{res}_{z=0} \operatorname{tr} C(z)Q_1'(z)dz$$

for any $Q_1(z) \in M(k, \mathbb{C}[z, z^{-1}])$, $C(z) \in M(k, \mathbb{C}((z)))$. By linearity, it suffices to show this when the matrices $Q_1(z)$ and $C(z)$ have a single nonzero entry. Thus we may assume $k = 1$, and that $Q_1(z) = z^b$. Thus we need to verify that for any formal power series $c(z) = \sum_d c_d z^d$ in $\mathbb{C}((z))$, we have

$$\operatorname{tr}[(((z^b)) \cdot P_-^m - P_-^m \cdot ((z^b)))c((z))] = bc_{-b},$$

where the operations here are in $\mathbb{C}((z))$. The left side is equal to

$$\operatorname{tr}[(((z^b)) \cdot P_-^m \cdot ((z^{-b})) - P_-^m) \cdot ((z^b))c((z))] = \operatorname{tr}[(P_-^{m-b} - P_-^m) \cdot ((z^b))c((z))]$$

$$= \operatorname{tr} \begin{pmatrix} c_{-b} & c_{-b+1} & \cdots & c_{-1} \\ c_{-b-1} & c_{-b} & \cdots & c_{-2} \\ \vdots & \vdots & \ldots & \vdots \\ c_{1-2b} & c_{2-2b} & \cdots & c_{-b} \end{pmatrix} = bc_{-b}.$$

The second claim of the lemma is similarly proven. $\square$

As $\mathrm{pr}_m - p_m = (1 - \mathrm{pr}_-^m - \mathrm{pr}_+^m) - (1 - p_-^m - p_+^m) = (p_-^m - \mathrm{pr}_-^m) + (p_+^m - \mathrm{pr}_+^m)$, Lemma 5.11 implies that $\mathrm{tr}(\mathrm{pr}_m \cdot A[z] - p_m \cdot A[z])$

$$= -\operatorname{res}_{z=0} \mathrm{tr}[A(z)Q_1'(z)Q_1(z)^{-1}dz] - \operatorname{res}_{z=\infty} \mathrm{tr}[A(z)Q_2'(z)A_2(z)^{-1}dz].$$

Since $\mathrm{tr}(p_m \cdot A[z]) = (2m-1)\,\mathrm{tr}\,A_0$, the proposition follows. □

Proposition 5.12. *Let $\iota : \mathbb{C}[z, z^{-1}]^k \to \mathbb{C}[z, z^{-1}]^k$ be the involution $(\iota u)(z) = u(z^{-1})$. For sufficiently large m we have $2\,\mathrm{tr}(\iota \cdot \mathrm{pr}_m \cdot A[z]) = \mathrm{tr}\,A(1) + \mathrm{tr}\,A(-1)$.*

Proof. Write $A(z) = \sum_k A_k z^k$, $A_k \in M(k, \mathbb{C})$. Then $\mathrm{tr}(\iota \cdot p_m \cdot A[z]) = \sum_{|i|<m} \mathrm{tr}\,A_{2i}$. If m is big enough the right side here is equal to $\frac{1}{2}(\mathrm{tr}\,A(1) + \mathrm{tr}\,A(-1))$. It remains to show that $\mathrm{tr}(\iota \cdot \mathrm{pr}_m \cdot A[z]) = \mathrm{tr}(\iota \cdot p_m \cdot A[z])$ for large enough m. As $\mathrm{pr}_m - p_m = p_+^m - \mathrm{pr}_+^m + (p_-^m - \mathrm{pr}_-^m)$, it suffices to show that for large enough m

$$\mathrm{tr}(\iota \cdot (p_+^m - \mathrm{pr}_+^m) \cdot A[z]) = 0 = \mathrm{tr}(\iota \cdot (p_-^m - \mathrm{pr}_-^m) \cdot A[z]).$$

Note that $\mathrm{pr}_+^m = [z^m]\,\mathrm{pr}_+^0[z^{-m}]$ and $p_+^m = [z^m]p_+^0[z^{-m}]$, where as usual $[z^m]$ here means the operator of multiplication by z^m. The operators pr_+^m and p_+^m were defined only for $m > 0$, but the definition extends to $m = 0$ so that the two relations above hold. Now

$$\begin{gathered}\mathrm{tr}(\iota \cdot (p_+^m - \mathrm{pr}_+^m) \cdot A[z]) = \mathrm{tr}(\iota \cdot [z^m](p_+^0 - \mathrm{pr}_+^0)[z^{-m}] \cdot A[z]) \\ = \mathrm{tr}([z^{-m}]\iota \cdot (p_+^0 - \mathrm{pr}_+^0)[z^{-m}] \cdot A[z]) = \mathrm{tr}(\iota \cdot (p_+^0 - \mathrm{pr}_+^0)[z^{-m}] \cdot A[z][z^{-m}]) \\ = \mathrm{tr}(\iota \cdot (p_+^0 - \mathrm{pr}_+^0)[z^{-2m}] \cdot A[z]).\end{gathered}$$

Recall that $\dim V$ is finite, where $V = \mathrm{im}[\iota(p_+^0 - \mathrm{pr}_+^0)]$. If m is big enough then

$$[z^{-2m}] \cdot A[z]V \subset z^{-1}\mathbb{C}[z^{-1}]^k \cap z^{-1}Q_2(z)\mathbb{C}[[z^{-1}]]^k \subset \ker p_+^0 \cap \ker \mathrm{pr}_+^0\,.$$

Hence $\mathrm{tr}(\iota \cdot (p_+^0 - \mathrm{pr}_+^0)[z^{-2m}] \cdot A[z])$ is zero. Hence $\mathrm{tr}(\iota(p_+^m - \mathrm{pr}_+^m)A[z])$ is zero. The proof of $\mathrm{tr}(\iota(p_-^m - \mathrm{pr}_-^m)A[z]) = 0$ for large m is analogous. □

Definition 5. Fix $P \in \mathrm{GL}(k, \mathbb{C}(z))$ such that $P(z)$ is regular at $z = 0$ and $P(z)^{-1}$ is regular at $z = \infty$. Put $S = \mathbb{C}[z, z^{-1}]^k + P \cdot \mathbb{C}[z, z^{-1}]^k$,

$$S_m = S \cap z^{1-m}\mathbb{C}[[z]]^k \cap z^{m-1}P \cdot \mathbb{C}[[z]]^k, \qquad S^m = z^{-m}\mathbb{C}[z^{-1}]^k + z^m P \cdot \mathbb{C}[z]^k.$$

Fix B in $M(k, \mathbb{C}[z, z^{-1}])$ such that $P^{-1}BP$ lies in $M(k, \mathbb{C}[z, z^{-1}])$. Then $BS \subset S$. We denote by $[B]$ or $B[z]$ the operator $S \to S$ of multiplication by $B(z)$.

Proposition 5.13. *We have $S = S_m \oplus S^m$. Denote by $\mathrm{ps}_m : S \to S$ the projection with image S_m and kernel S^m. Then*

$$\begin{aligned}&\mathrm{tr}(\mathrm{ps}_m \cdot [B]) = (2m-1)\,\mathrm{tr}\,B_0 \\ &- \operatorname{res}_{z=\infty} \mathrm{tr}[B(z)P'(z)P(z)^{-1}]dz + \mathrm{tr}([B];\, S/\mathbb{C}[z, z^{-1}]^k).\end{aligned}$$

Here B_0 is the constant term of $B = B(z)$, and $\mathrm{tr}([B]; S/\mathbb{C}[z, z^{-1}]^k)$ denotes the trace of the endomorphism of $S/\mathbb{C}[z, z^{-1}]^k$ induced by multiplication by $B(z)$.

Proof. The space S is a k-dimensional free $\mathbb{C}[z,z^{-1}]$-submodule of $\mathbb{C}(z)^k$. Hence there exists a matrix D in $\mathrm{GL}(k,\mathbb{C}(z))$ such that $S = D{\cdot}\mathbb{C}[z,z^{-1}]^k$. Since S contains $\mathbb{C}[z,z^{-1}]^k$, D^{-1} lies in $M(k,\mathbb{C}[z,z^{-1}])$. Since S contains $P\cdot\mathbb{C}[z,z^{-1}]^k$ we deduce that $D^{-1}P \in M(k,\mathbb{C}[z,z^{-1}])$. Put $Q_1 = D^{-1}$, $Q_2 = D^{-1}P$. The function $Q_1(z)^{-1}Q_2(z) = P(z)$ is regular at $z=0$. The function $Q_2(z)^{-1}Q_1(z)$ is regular at $z=\infty$. Under the isomorphism $S\tilde{\to}\mathbb{C}[z,z^{-1}]^k$, $u\mapsto D^{-1}u$, the subspaces S_m and S^m correspond to the subspaces R_m and R^m of Proposition 5.9. The multiplication $[B] : S\to S$ corresponds to $[A] : \mathbb{C}[z,z^{-1}]^k \to \mathbb{C}[z,z^{-1}]^k$, $A = D^{-1}BD$. Then Proposition 5.10 implies the first part of the proposition, as well as the equality

$$\operatorname{tr}(\mathrm{ps}_m\cdot B[z]) = (2m-1)\operatorname{tr}A_0 - \operatorname{res}_{z=0}\operatorname{tr}A(z)Q_1'(z)Q_1(z)^{-1}dz$$
$$- \operatorname{res}_{z=\infty}\operatorname{tr}A(z)Q_2'(z)Q_2(z)^{-1}dz.$$

Here A_0 is the constant term of $A(z)$. We have

$$\operatorname{tr}(AQ_1'Q_1^{-1}) = -\operatorname{tr}(D^{-1}BD') = -\operatorname{tr}(BD'D^{-1}),$$

$$\operatorname{tr}(AQ_2'Q_2^{-1}) = -\operatorname{tr}(D^{-1}BP'P^{-1}D - D^{-1}BD') = \operatorname{tr}(BP'P^{-1}) - \operatorname{tr}(BD'D^{-1}).$$

As $A = D^{-1}BD$, we have $\operatorname{tr}A = \operatorname{tr}B$, and $\operatorname{tr}A_0 = \operatorname{tr}B_0$. Hence

$$\begin{aligned}\operatorname{tr}(\mathrm{ps}_m\cdot B[z]) = (2m-1)\operatorname{tr}B_0 - \operatorname{res}_{z=\infty}\operatorname{tr}B(z)P'(z)P(z)^{-1}dz\\ +\operatorname{res}_{z=0}\operatorname{tr}B(z)D'(z)D(z)^{-1}dz + \operatorname{res}_{z=\infty}\operatorname{tr}B(z)D'(z)D(z)^{-1}dz\\ +(2m-1)\operatorname{tr}B_0 - \operatorname{res}_{z=\infty}\operatorname{tr}B(z)P'(z)P(z)^{-1}dz\\ -\sum_{\zeta\in\mathbb{C}^\times}\operatorname{res}_{z=\zeta}\operatorname{tr}B(z)D'(z)D(z)^{-1}dz.\end{aligned}$$

Lemma 5.14. *Suppose $T\in\mathrm{GL}(k,\mathbb{C}((z)))$, $C\in M(k,\mathbb{C}[[z]])$ and $T^{-1}CT \in M(k,\mathbb{C}[[z]])$. Then $\operatorname{res}_{z=0}\operatorname{tr}C(z)T'(z)T(z)^{-1} = a-b$, where a denotes the trace of the operator multiplication by C in the space $(\mathbb{C}[[z]]^k + T\mathbb{C}[[z]]^k)/T\mathbb{C}[[z]]^k$, while b denotes the trace of multiplication by C in the space*

$$(\mathbb{C}[[z]]^k + T\mathbb{C}[[z]]^k)/\mathbb{C}[[z]]^k.$$

Proof. Both sides of the asserted equality do not change if (T,C) is replaced by (UTV, UCU^{-1}) where $U, V\in\mathrm{GL}(k,\mathbb{C}[[z]])$. We may then assume that T is a diagonal matrix, hence that $k=1$. When $k=1$ both sides of the asserted relation are simply $mC(0)$, where m is the multiplicity of zero of $T(z)$ at $z=0$. □

It follows from the lemma that $-\operatorname{res}_{z=\zeta}\operatorname{tr}(B(z)D'(z)D(z)^{-1})dz$ is just the trace of the operator of multiplication by $B(z)$ on the ζ component of the module $S/\mathbb{C}[z,z^{-1}]^k$. This, and the equality just before the lemma, implies the proposition. □

Suppose we have $P(z^{-1}) = P(z)^{-1}$. Replace the assumption $P(z)^{-1}B(z)P(z) \in M(k, \mathbb{C}[z, z^{-1}])$ in Proposition 5.13 with the stronger assumption

$$P(z)^{-1}B(z)P(z) = B(z^{-1}).$$

Recall that L is the space of all rational functions of the form $u(z)+P(z)u(z^{-1})$ with $u \in \mathbb{C}[z, z^{-1}]^m$. In view of the stronger assumption, L is invariant under multiplication by B.

Definition 6. Denote by B_L the operator of multiplication by B on L. Put $L_m = L \cap z^{1-m}\mathbb{C}[[z]]^k$. Denote by L^m the set of rational functions of the form $u(z) + P(z)u(z^{-1})$ with $u(z) \in z^{-m}\mathbb{C}[z^{-1}]^k$.

Proposition 5.15. *The space L_m is finite dimensional, and $L = L_m \oplus L^m$. Denote by $\pi_m : L \to L$ the projection with image L_m and kernel L^m. Suppose the function $P(z)$ is regular at $z = \pm 1$. Then for large enough m we have that* $\operatorname{tr}(\pi_m B_L)$ *equals*

$$(m - \frac{1}{2}) \operatorname{tr} B_0 - \frac{1}{2} \operatorname{res}_{z=\infty} \operatorname{tr}(B(z)P'(z)P(z)^{-1})dz$$

$$+\frac{c}{2} + \frac{1}{4}[\operatorname{tr}(B(1)P(1)) + \operatorname{tr}(B(-1)P(-1))].$$

Here B_0 is the constant term of $B(z)$, and c is the trace of the operator of multiplication by $B(z)$ in the space $(\mathbb{C}[z, z^{-1}]^k + P(z)\mathbb{C}[z, z^{-1}]^k)/\mathbb{C}[z, z^{-1}]^k$.

Proof. Let S, S_m, S^m, ps_m, B be as in Proposition 5.13. From $P(z^{-1}) = P(z)^{-1}$ it follows that if $u \in S$ then $\tilde{u}$, given by $\tilde{u}(z) = P(z)u(z^{-1})$, is also in S. Define $\tau : S \to S$ by $\tau(u) = \tilde{u}$. Then $\tau^2 = 1$, $L = \{u \in S;\, \tau(u) = u\}$, $L_m = S_m \cap L$, $L^m = S^m \cap L$, and

$$\operatorname{tr}(\pi_m B_L) = \frac{1}{2} \operatorname{tr}(\operatorname{ps}_m \cdot B[z]) + \frac{1}{2} \operatorname{tr}(\tau \cdot \operatorname{ps}_m \cdot B[z]).$$

The finite dimensionality of S_m and Proposition 5.13 then imply that the space L_m is finite dimensional, and $L = L_m \oplus L^m$. To deduce the last claim of the proposition from Proposition 5.13, it remains to show that

$$\operatorname{tr}(\tau \cdot \operatorname{ps}_m \cdot [B]) = \frac{1}{2}(\operatorname{tr}(B(1)P(1)) + \operatorname{tr}(B(-1)P(-1)))$$

for large enough m.

Let D, Q_1, Q_2 be as in Proposition 5.13. Then under the isomorphism $S \tilde{\to} \mathbb{C}[z, z^{-1}]^k$, $u \mapsto D^{-1}u$, the operator $\operatorname{ps}_m : S \to S$ translates into the operator pr_m (of Proposition 5.9), and multiplication by $B : S \to S$ translates into multiplication by

$$A = D^{-1}BD, \qquad \mathbb{C}[z, z^{-1}]^k \to \mathbb{C}[z, z^{-1}]^k.$$

The map $\tau : S \to S$ translates into

$[C]\iota : \mathbb{C}[z,z^{-1}]^k \to \mathbb{C}[z,z^{-1}]^k$, $(\iota u)(z) = u(z^{-1})$, $C(z) = D(z)^{-1}P(z)D(z^{-1})$.

Hence

$$\operatorname{tr}(\tau \cdot \mathrm{ps}_m \cdot B[z]) = \operatorname{tr}(C[z]\iota \cdot \mathrm{pr}_m \cdot A[z]) = \operatorname{tr}(\iota \,\mathrm{pr}_m \, A[z]C[z]),$$

which – by Proposition 5.9 – is

$$\frac{1}{2}(\operatorname{tr} A(1)C(1) + \operatorname{tr} A(-1)C(-1)) = \frac{1}{2}\operatorname{tr}(B(1)P(1) + \operatorname{tr} B(-1)P(-1));$$

note that $D(z)$ is regular at $z = \pm 1$, since so is $P(z)$. □

If $F \in M(k,\mathbb{C})$ and $Y \subset \mathbb{C}^k$ is an F-invariant subspace, write $\operatorname{tr}(F,Y)$ for the trace of F on Y.

Proposition 5.16. *Fix $P(z) \in \mathrm{GL}(k,\mathbb{C}(z))$ with $P(z^{-1}) = P(z)^{-1}$. Suppose that the function $P(z)$ is regular on $|z| = 1$ and at $z = 0$, and that it has order 1 at all its poles $\zeta_1,\dots,\zeta_s$ inside $\{z \in \mathbb{C};\ 0 < |z| < 1\}$. Denote by Y_i the image of the operator $\lim_{z\to\zeta_i}(z-\zeta_i)P(z)$ acting on $\mathbb{C}^k$. Fix $B(z) \in M(k,\mathbb{C}[z,z^{-1}])$ and suppose $B_1(z) = P(z)^{-1}B(z)P(z) \in M(k,\mathbb{C}[z,z^{-1}])$. Then*

$$\operatorname{tr}(\mathrm{ps}_m \cdot [B]) = (2m-1)\operatorname{tr} B_0 + \frac{1}{2\pi i}\int_{|z|=1} \operatorname{tr} B(z)P'(z)P(z)^{-1}dz$$

$$+ \sum_{1\le i\le s} \operatorname{tr}(B(\zeta_i) + B_1(\zeta_i^{-1}), Y_i),$$

with B_0 being the constant term of $B(z)$.
If in addition $B_1(z) = B(z^{-1})$ then

$$\operatorname{tr}(\pi_m B_L) = (m - \frac{1}{2})\operatorname{tr} B_0 + \frac{1}{4\pi i}\int_{|z|=1} \operatorname{tr} B(z)P'(z)P(z)^{-1}dz$$

$$+ \sum_{1\le i\le s} \operatorname{tr}(B(\zeta_i), Y_i) + \frac{1}{4}[\operatorname{tr}(B(1)P(1)) + \operatorname{tr}(B(-1)P(-1))].$$

Note that the subspace Y_i of $\mathbb{C}^k$ is invariant under $B(\zeta_i)$ and $B_1(\zeta_i^{-1})$.

Proof. In view of Propositions 5.13 and 5.15 it suffices to verify that

$$\frac{1}{2\pi i}\oint_{|z|=1} \operatorname{tr} B(z)P'(z)P(z)^{-1}dz + \sum_{1\le i\le s} \operatorname{tr}(B(\zeta_i) + B_1(\zeta_i^{-1}), Y_i)$$

$$= \operatorname{tr}([B], S/\mathbb{C}[z,z^{-1}]^k) - \mathrm{res}_{z=\infty} \operatorname{tr} B(z)P'(z)P(z)^{-1}dz,$$

where

$$S = \mathbb{C}[z,z^{-1}]^k + P(z)\mathbb{C}[z,z^{-1}]^k.$$

For any $\zeta \neq 0$ in $\mathbb{C}$ denote by M_ζ and N_ζ the ζ-components of the $\mathbb{C}[z,z^{-1}]$-modules $S/\mathbb{C}[z,z^{-1}]^k$ and $S/P(z)\mathbb{C}[z,z^{-1}]^k$, respectively. From Cauchy's formula and Lemma 5.14, it follows that

$$\frac{1}{2\pi i}\oint_{|z|=1} \operatorname{tr} B(z)P'(z)P(z)^{-1}dz = \sum_{1<|\zeta|<\infty} \operatorname{tr}([B], M_\zeta)$$

$$- \sum_{1<|\zeta|<\infty} \operatorname{tr}([B], N_\zeta) - \operatorname{res}_{z=\infty} \operatorname{tr}(B(z)P'(z)P(z)^{-1})dz.$$

On the other hand, $\operatorname{tr}([B], S/\mathbb{C}[z,z^{-1}]^k) = \sum_{\zeta\in\mathbb{C}^\times} \operatorname{tr}([B], M_\zeta)$. Hence the required identity follows from

$$\sum_{0<|\zeta|<1} \operatorname{tr}([B], M_\zeta) = \sum_{1\le i\le s} \operatorname{tr}(B(\zeta_i), Y_i),$$

$$\sum_{1<|\zeta|<\infty} \operatorname{tr}([B], N_\zeta) = \sum_{1\le i\le s} \operatorname{tr}(B_1(\zeta_i^{-1}), Y_i).$$

If $P(z)$ is regular at ζ then $M_\zeta = 0$. At each ζ_i, $P(z)$ has a pole of order one. Hence there exists isomorphisms $M_{\zeta_i}\tilde{\to}Y_i$ which translate the operator $[B] : M_{\zeta_i} \to M_{\zeta_i}$ to the operator of multiplication by $B(\zeta_i)$ on Y_i. This implies the first identity.
For the second identity, for any $\zeta \in \mathbb{C}^\times$, denote by $\overline{N}_\zeta$ the ζ-component of the module $(\mathbb{C}[z,z^{-1}]^k + P(z)^{-1}\mathbb{C}[z,z^{-1}]^k)/\mathbb{C}[z,z^{-1}]^k$. Multiplication by $P(z)^{-1}$ induces an isomorphism $N_\zeta\tilde{\to}\overline{N}_\zeta$. Under this isomorphism, multiplication by $B : N_\zeta \to N_\zeta$ translates into multiplication by $B_1 : \overline{N}_\zeta \to \overline{N}_\zeta$, hence $\operatorname{tr}([B], N_\zeta) = \operatorname{tr}([B_1], \overline{N}_\zeta)$. From $P(z)^{-1} = P(z^{-1})$ we deduce that $\overline{N}_\zeta = 0$ if $P(z)$ is regular at $z = \zeta^{-1}$, and that $\operatorname{tr}([B_1], \overline{N}_{\zeta_i^{-1}}) = \operatorname{tr}(B_1(\zeta_i^{-1}), Y_i)$. This implies the second identity, hence the proposition. □

5.3 Spectral terms

To deduce the trace formula from Proposition 5.16, we use properties of the function $M(\mu_1, \mu_2, t)$.
Recall that we have the projection $\pi_m : L \to L$ with kernel L^m and image L_m, and B_L denotes the operator of multiplication by $B(z)$ on L. The operator $P(z)$ is the restriction to the subspace of U-invariant vectors of the operator M on the space $I_0 = \oplus I_0(\mu_1, \mu_2)$ (μ_1, μ_2 range over the characters of $\mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}}$), which maps $I_0(\mu_1, \mu_2)$ to $I_0(\mu_2, \mu_1)$ via $M(\mu_1, \mu_2, z)$.

Proposition 5.17. *There exists $a_f \in \mathbb{C}$ such that for sufficiently large m,*

$$\operatorname{tr}(\pi_m B_L) = (m - \frac{1}{2})a_f - \sum_{5\le i\le 8} S_i(f).$$

Proof. By Proposition 4.29 the function $P(z)$ has two poles in the domain $|z| \leq 1$, namely at $z = \pm q^{-1/2}$, each of order 1. We have $P(z^{-1}) = P(z)^{-1}$ and $P(z)^{-1}B(z)P(z) = B(z^{-1})$. Hence the final claim of Proposition 5.16 applies and implies that for large enough m,

$$\operatorname{tr}(\pi_m[B]) = (m-\frac{1}{2}) \operatorname{tr} B_0 + \frac{1}{4\pi i}\oint_{|z|=1} \operatorname{tr} B(z)P'(z)P(z)^{-1}dz + \operatorname{tr}(B(q^{-1/2}), Y_+)$$

$$+ \operatorname{tr}(B(-q^{-1/2}), Y_-) + \frac{1}{4}[\operatorname{tr}(B(1)P(1)) + \operatorname{tr}(B(-1)P(-1))].$$

Here B_0 is the constant term of $B(z)$ and the image of the operator $\lim_{z\to\pm q^{-1/2}}(z \mp q^{-1/2})P(z)$ is denoted by $Y_\pm$. The proposition follows once we show that

$$\oint_{|z|=1} \operatorname{tr} B(z)P'(z)P(z)^{-1}dz = -4\pi i(S_5(f) + S_6(f)), \tag{1}$$

$$\operatorname{tr}(B(q^{-1/2}), Y_+) + \operatorname{tr}(B(-q^{-1/2}), Y_-) = -S_8(f), \tag{2}$$

$$\operatorname{tr}(B(1)P(1)) + \operatorname{tr}(B(-1)P(-1)) = -4S_7(f). \tag{3}$$

Denote by $r(z)$ the representation of $\mathrm{GL}(2,\mathbb{A})$ by right translation in $I(z) = \otimes_{\mu_1,\mu_2} I(\mu_1\nu_{z^{-1}}, \mu_2\nu_z)$. Here μ_1, μ_2 are characters of $\mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}}$. Let $r(z,f)$ be the convolution operator defined by $r(z)$ and the compactly supported function f in $C_c^\infty(\mathrm{GL}(2,\mathbb{A}))$. Identify, as usual, $I(z)$ to the space I_0, and consider $r(z,f)$ as an operator in I_0. From Proposition 4.36, $B(z)$ coincides with the restriction of $r(z,f)$ to I_0^U. Also, $P(z)$ coincides with the restriction of $M(z)$ to I_0^U. Hence the integral on the left of (1) equals

$$\oint_{|z|=1} \operatorname{tr} r(z,f)M'(z)M(z)^{-1}dz$$

$$= \sum_{\mu_1,\mu_2} \oint_{|z|=1} \operatorname{tr} I(\mu_2\nu_{z^{-1}}, \mu_1\nu_z, f)M'(\mu_1,\mu_2,z)M(\mu_1,\mu_2,z)^{-1}dz$$

$$= \sum_{\mu_1,\mu_2} \oint_{|z|=1} \operatorname{tr} M(\mu_1,\mu_2,z)^{-1}I(\mu_2\nu_{z^{-1}}, \mu_1\nu_z, f)M'(\mu_1,\mu_2,z)dz$$

$$= \sum_{\mu_1,\mu_2} \oint_{|z|=1} \operatorname{tr} I(\mu_1\nu_z, \mu_2\nu_{z^{-1}}, f)M(\mu_1,\mu_2,z)^{-1}M'(\mu_1,\mu_2,z)dz.$$

Then (1) follows from Proposition 4.9.

For (2), it follows from Proposition 4.29 that $Y_+ = L^U$, with $L = \oplus L_\mu$, $L_\mu \subset I(\mu,\mu)$ being generated by the function $x \mapsto \mu(x)$. The operator $r(q^{-1/2}, f)$ acts in L_μ as the operator of multiplication by $\int_{\mathrm{GL}(2,\mathbb{A})} f(x)\mu(\det x)dx$. Hence

$$\operatorname{tr}(B(q^{-1/2}), Y_+) = \operatorname{tr}(r(q^{-1/2}, f), L) = \sum_\mu \int_{\mathrm{GL}(2,\mathbb{A})} f(x)\mu(\det x)dx,$$

where μ ranges over the set of characters of $\mathbb{A}^\times/F^\times \cdot \alpha^{\mathbb{Z}}$. Similarly

$$\operatorname{tr}(B(q^{-1/2}), Y_-) = \operatorname{tr}(r(-q^{-1/2}, f), L)$$

$$= \sum_\mu \int_{\mathrm{GL}(2,\mathbb{A})} f(x)\mu(\det x)\nu_{-1}(\det x)dx.$$

Every character of $\mathbb{A}^\times$ which is trivial on $F^\times \cdot \alpha^{2\mathbb{Z}}$ is either trivial on $F^\times \cdot \alpha^{\mathbb{Z}}$ or its product with ν_{-1} is, so (2) follows.
For (3) note that

$$\operatorname{tr} B(1)P(1) = \operatorname{tr} r(1,f)M(1) = \sum_\mu \operatorname{tr} I(\mu,\mu,f)M(\mu,\mu,1) = -\sum_\mu \operatorname{tr} I(\mu,\mu,f)$$

by Proposition 4.30. Similarly $\operatorname{tr} B(-1)P(-1) = -\sum_\mu \operatorname{tr} I(\mu\nu_{-1}, \mu\nu_{-1}, f)$. □

This completes the proof of the trace formula.

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