Non-Supramenable Groups Acting on Locally Compact Spaces

Julian Kellerhals, Nicolas Monod, and Mikael Rørdam

Received: June 3, 2013
Revised: July 22, 2013

Communicated by Joachim Cuntz

Abstract. Supramenability of groups is characterised in terms of invariant measures on locally compact spaces. This opens the door to constructing interesting crossed product C^*-algebras for non-supramenable groups. In particular, stable Kirchberg algebras in the UCT class are constructed using crossed products for both amenable and non-amenable groups.

2010 Mathematics Subject Classification: 43A07, 46L55, 46L35
Keywords and Phrases: Supramenable groups, actions on locally compact spaces, purely infinite C^*-algebras and actions.

1. Introduction

A group Γ is called *amenable* if it carries an invariant finitely additive measure μ with $0 < \mu(\Gamma) < \infty$. This concept introduced by von Neumann [23] was originally motivated by the Hausdorff–Banach–Tarksi paradox [10, 4] and has become central in many aspects of group theory and beyond.

However, from the point of view of paradoxical decompositions, it is more natural to consider the following question of von Neumann [23, §4]. Let Γ be a group acting on a set X. Given a subset $A \subseteq X$, when is there an invariant finitely additive measure μ on X with $0 < \mu(A) < \infty$?

Following Rosenblatt [18], the group Γ is called *supramenable* if there is such a measure for every non-empty subset; it turns out that it is sufficient to consider the case $X = \Gamma$. Supramenability is much stronger than amenability: whilst it holds for commutative and subexponential groups, it fails already for metabelian groups (Example 2.8) and thus is not preserved under extensions. It passes to subgroups and quotients but is not known to be preserved under direct products.

One of the many equivalent characterisations of amenability is the existence of a non-zero invariant Radon measure on any compact Hausdorff Γ-space (see e.g. [16, 5.4]). It turns out that there is an analogous characterisation of supramenability using *locally compact* spaces. We shall establish this and a couple more characterisations:

Theorem 1.1. *The following conditions are equivalent for any group* Γ*.*

(i) Γ *is supramenable.*
(ii) *Any co-compact* Γ*-action on a locally compact Hausdorff space admits a non-zero invariant Radon measure.*
(iii) *The Roe algebra* $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ *contains no properly infinite projection.*
(iv) *There is no injective Lipschitz map from the free group* $\mathbb{F}_2$ *to* Γ*.*

(This also gives a partial answer to the questions raised after Theorem 6.3.1 in [19]. Regarding condition (iii), we recall that Γ is amenable if and only if the Roe algebra is itself not properly infinite [17]. Theorem 1.1 is proved below in Propositions 2.7, 3.4 and 5.3.)

We shall leverage the *failure* of condition (ii) in order to provide interesting C^*-algebras through the corresponding reduced crossed product construction. To this end, we need to establish that the locally compact Γ-space can be assumed to have several additional properties, as follows. Recall that a *Kirchberg algebra* is a C^*-algebra which is simple, purely infinite, nuclear and separable. Kirchberg algebras in the UCT class are completely classified by their K-theory, see [15] and [12].

Theorem 1.2. *Let* Γ *be a countable group. Then* Γ *admits a free, minimal, purely infinite action on the locally compact non-compact Cantor set* $\mathbf{K}^*$ *if and only if* Γ *is non-supramenable. The crossed product* C^**-algebra* $C_0(\mathbf{K}^*) \rtimes_{\mathrm{red}} \Gamma$ *associated with any such action will be a stable, simple, purely infinite* C^**-algebra.*

If the non-supramenable group Γ *is amenable, then the associated crossed product* C^**-algebra* $C_0(\mathbf{K}^*) \rtimes_{\mathrm{red}} \Gamma$ *will be a stable Kirchberg algebra in the UCT class.*

In the above, an action is called *purely infinite* if every compact-open subset is paradoxical in a sense made precise in Definition 4.4 below. It is a strengthening of the failure of condition (ii) in Theorem 1.1. As for the non-compact Cantor set $\mathbf{K}^*$, one can realise it e.g. as the usual (compact) Cantor set $\mathbf{K}$ with a point removed, or as $\mathbb{N} \times \mathbf{K}$.

In the second statement of Theorem 1.2, the amenability of Γ is only used to deduce that the action on $\mathbf{K}^*$ is amenable. But of course there are many amenable actions of non-amenable groups; therefore, we strive to produce more examples of C^*-algebras as above using crossed products for groups Γ that are not necessarily amenable. Here is a summary of some progress in that direction.

Theorem 1.3. *Let* Γ *be a countable group.*

(i) *If* Γ *contains an infinite exact subgroup, then* Γ *admits a free minimal amenable action on* $\mathbf{K}$ *or on* $\mathbf{K}^*$*; necessarily on* $\mathbf{K}^*$ *if* Γ *is non-exact.*

If Γ contains an infinite exact subgroup of infinite index, then Γ admits a free minimal amenable action on $\mathbf{K}^*$.

(ii) *If Γ contains an element of infinite order, or if Γ contains an infinite amenable subgroup of infinite index, then Γ admits a free minimal amenable action on $\mathbf{K}^*$ such that $\mathbf{K}^*$ admits an invariant non-zero Radon measure. The associated crossed product $C_0(\mathbf{K}^*) \rtimes_{\mathrm{red}} \Gamma$ is a stably finite simple separable nuclear C^*-algebra the UCT class.*

(iii) *If Γ contains an exact non-supramenable subgroup, then Γ admits a free minimal amenable purely infinite action on $\mathbf{K}^*$. The associated crossed product $C_0(\mathbf{K}^*) \rtimes_{\mathrm{red}} \Gamma$ is a stable Kirchberg algebra in the UCT class.*

Acknowledgements. Work supported in part by the Swiss National Science Foundation, the European Research Council, Danish National Research Foundation (DNRF) through the Centre for Symmetry and Deformation at University of Copenhagen, and The Danish Council for Independent Research, Natural Sciences.

2. Actions on locally compact spaces

It is well-known that a group is amenable if and only if whenever it acts on a compact Hausdorff space, then the space admits an invariant probability measure. We shall here characterise supramenable groups in a similar way by their actions on locally compact Hausdorff spaces.

Let us first recall the basic notions of comparison of subsets of a group Γ. If $A, B \subseteq \Gamma$, then write $A \sim_\Gamma B$ if there are finite partitions $\{A_j\}_{j=1}^n$ and $\{B_j\}_{j=1}^n$ of A and B, respectively, and elements $t_1, t_2, \dots, t_n \in \Gamma$ such that $A_j = t_j B_j$. Write $A \precsim_\Gamma B$ if $A \sim_\Gamma B_0$ for some $B_0 \subseteq B$. A set $A \subseteq \Gamma$ is said to be *paradoxical* if there are disjoint subsets A_0, A_1 of A such that $A \sim_\Gamma A_0 \sim_\Gamma A_1$. Tarski's theorem says that $A \subseteq \Gamma$ is non-paradoxical if and only if there is an invariant finitely additive measure μ defined on the entire power set, $P(\Gamma)$, of Γ, such that $\mu(A) = 1$. (Notice that the empty set is paradoxical.)

The following proposition is well-known. For the weaker case of functions with "A-bounded" support, see for example [9, Chap. 1].

Proposition 2.1. *Let Γ be a group and let μ be a finitely additive measure on Γ. Let V_μ be the subspace of $\ell^\infty(\Gamma)$ consisting of all $f \in \ell^\infty(\Gamma)$ such that $\mu(\mathrm{supp}(f)) < \infty$. It follows that there is a unique positive linear functional $I_\mu \colon V_\mu \to \mathbb{C}$ such that $I_\mu(1_E) = \mu(E)$ for all $E \subseteq \Gamma$ with $\mu(E) < \infty$. If μ is Γ-invariant, then so is I_μ.*

Proof. Let $\mathcal{F}$ be the collection of subsets of Γ of finite measure. For each $F \in \mathcal{F}$ let V_F be the subspace of $\ell^\infty(\Gamma)$ consisting of all functions with support in F. The family $\{V_F\}$ is upwards directed with union V_μ. Using that the set of bounded functions $F \to \mathbb{C}$ that take finitely many values is uniformly dense in V_F it is easy to see that there is a unique positive (necessarily bounded) linear functional $I_\mu^{(F)} \colon V_F \to \mathbb{C}$ such that $I_\mu^{(F)}(1_E) = \mu(E)$ for all $E \subseteq F$. It follows

that there exists a unique positive (unbounded) linear functional $I_\mu \colon V_\mu \to \mathbb{C}$ that extends all the $I_\mu^{(F)}$'s.
When μ is translation invariant, the invariance of I_μ follows from uniqueness of I_μ. □

Let Γ be a group which acts on a locally compact Hausdorff space X. We say that the action is *co-compact* if there is a compact subset K of X such that $\bigcup_{t\in\Gamma} t.K = X$. Any *minimal* action of any group on any locally compact Hausdorff space is automatically co-compact.
Observe that if K is as above, and if λ is a non-zero Radon measure on X, then $0 < \lambda(K) < \infty$.
If X is locally compact and Hausdorff and if Γ acts co-compactly on X, then there is a compact subset $K \subseteq X$ such that $\bigcup_{t\in\Gamma} t.K^{\mathrm{o}} = X$, where K^{o} denotes the interior of K. This follows easily from the fact that for each compact set $K' \subseteq X$ there is another compact set $K \subseteq X$ such that $K' \subseteq K^{\mathrm{o}}$.

Lemma 2.2. *Let Γ be a group acting co-compactly on a locally compact Hausdorff space X. For each compact subset K of X, such that $\bigcup_{t\in\Gamma} t.K^{\mathrm{o}} = X$, and for each $x_0 \in X$, put*

$$A(K, x_0) = \{t \in \Gamma \mid t.x_0 \in K\}.$$

If $A(K, x_0)$ is non-paradoxical in Γ for some K and x_0 as above, then there is a non-zero Γ-invariant Radon measure λ on X.

Proof. By Tarski's theorem there exists an invariant finitely additive measure μ on Γ such that $\mu(A(K, x_0)) = 1$.
Each compact set $L \subseteq X$ is contained in $\bigcup_{s\in F} s.K$ for some finite subset F of Γ. Hence

$$\mu\big(\{t \in \Gamma \mid t.x_0 \in L\}\big) \le \mu\Big(\bigcup_{s\in F} sA(K, x_0)\Big) \le |F|\mu(A(K, x_0)) < \infty. \tag{2.1}$$

For each $f \in C_c(X)$ let $\hat{f} \in \ell^\infty(\Gamma)$ be given by $\hat{f}(t) = f(t.x_0)$, $t \in \Gamma$. In the notation of Proposition 2.1, it follows from (2.1) that $\hat{f} \in V_\mu$. Let $I_\mu \colon V_\mu \to \mathbb{C}$ be the Γ-invariant functional associated with μ constructed in Proposition 2.1. Define $\Lambda \colon C_c(X) \to \mathbb{C}$ by $\Lambda(f) = I_\mu(\hat{f})$. As I_μ is Γ-invariant, so is Λ. By Riesz' representation theorem there is a Radon measure λ on X such that $\Lambda(f) = \int_X f\, d\lambda$. As Λ is Γ-invariant, so is λ.
If $f \in C_c(X)$ is such that $f \ge 1_K$, then $\hat{f} \ge 1_{A(K,x_0)}$, so $\Lambda(f) = I_\mu(\hat{f}) \ge \mu(A(K, x_0)) = 1$. This shows that $\lambda(K) \ge 1$, so λ is non-zero. □

The action of a group Γ on itself given by left-multiplication extends to an action of Γ on its beta-compactification, $\beta\Gamma$. We shall refer to this action as the *canonical action of* Γ *on* $\beta\Gamma$.

Definition 2.3. Fix a subset A of Γ. Denote by K_A its closure in $\beta\Gamma$, and put

$$X_A = \bigcup_{t\in\Gamma} t.K_A.$$

As $\beta\Gamma$ is Stonian (a.k.a. extremely disconnected), it follows that K_A is compact-open in $\beta\Gamma$. Moreover, X_A is an open Γ-invariant subset of $\beta\Gamma$. In particular, X_A is locally compact and Hausdorff, and Γ acts co-compactly on X_A. Note also that $\Gamma \subseteq X_A$ (if $A \neq \emptyset$), and that $\Gamma = X_A$ if and only if A is finite and non-empty.

We state below two easy lemmas that shall be used frequently in what follows.

Lemma 2.4. *Let Γ be a group.*

(i) *Let A and B be subsets of Γ. If $A \cap B = \emptyset$, then $K_A \cap K_B = \emptyset$.*
(ii) *For any subset A of Γ we have $A = K_A \cap \Gamma$.*
(iii) *If A and B are subsets of Γ, then $A \subseteq B$ if and only if $K_A \subseteq K_B$.*
(iv) *If K is a compact-open subset of $\beta\Gamma$ and if $A = K \cap \Gamma$, then $K = K_A$.*
(v) *If A is a subset of Γ and $t \in \Gamma$, then $t.K_A = K_{tA}$.*
(vi) *$\beta\Gamma$ and $\emptyset$ are the only Γ-invariant compact-open subset of $\beta\Gamma$.*

Proof. (i). As A is an open subset of $\beta\Gamma$ we have $K_B \subseteq \beta\Gamma \setminus A$. Hence $A \subseteq \beta\Gamma \setminus K_B$, which entails that $K_A \subseteq \beta\Gamma \setminus K_B$, because K_B is open.
(ii). It is clear that $A \subseteq K_A \cap \Gamma$. Conversely, if $t \in K_A \cap \Gamma$, then $\{t\}$ is an open subset of $\beta\Gamma$ that intersects K_A, so it also intersects A, i.e., $t \in A$.
(iii). The non-trivial implication follows from part (ii).
(iv). It is clear that $K_A \subseteq K$. If $x \in K$ and if V is an open neighbourhood of x, then $K \cap V$ is a non-empty open subset of $\beta\Gamma$. As Γ is an open and dense subset of $\beta\Gamma$ we deduce that $A \cap V = \Gamma \cap K \cap V$ is non-empty. As V was an arbitrary open neighbourhood of x, we conclude that $x \in K_A$.
(v). As $A \subseteq K_A$ we have $tA \subseteq t.K_A$, so $K_{tA} \subseteq t.K_A$. Applying this inclusion with t^{-1} in the place of t we get

$$t.K_A = t.K_{t^{-1}t.A} \subseteq t.\big(t^{-1}.K_{tA}\big) = K_{tA}.$$

(vi). This follows from (iv) and (v) and the fact that Γ and $\emptyset$ are the only Γ-invariant subsets of Γ. □

The lemma above says that there is a one-to-one correspondence between subsets of Γ and compact-open subsets of $\beta\Gamma$. This is not surprising. Indeed the function algebras $\ell^\infty(\Gamma)$ and $C(\beta\Gamma)$ are canonically isomorphic (by the definition of the beta-compactification!). The canonical isomorphism carries the projection $1_A \in \ell^\infty(\Gamma)$ onto $1_{K_A} \in C(\beta\Gamma)$. We shall work with both pictures in this paper.

If A and B are subsets of a group Γ, then write $A \propto_\Gamma B$ if A is *B-bounded* i.e., if $A \subseteq \bigcup_{t\in F} tB$ for some finite subset F of Γ.

Lemma 2.5. *Let A and B be subsets of a group Γ.*

(i) *$K_B \subseteq X_A$ if and only if $B \propto_\Gamma A$.*
(ii) *$X_A = X_B$ if and only if $A \propto_\Gamma B \propto_\Gamma A$.*
(iii) *If $A \propto_\Gamma B \propto_\Gamma A$, then A is paradoxical if and only if B is paradoxical.*

Proof. (i). If $B \subseteq \bigcup_{t\in F} tA$ for some finite subset F of Γ, then

$$K_B \subseteq K_{\bigcup_{t\in F} tA} = \bigcup_{t\in F} t.K_A \subseteq X_A$$

by Lemma 2.4(iii) and (v). Conversely, if K_B is a subset of $X_A = \bigcup_{t\in\Gamma} t.K_A$, then $K_B \subseteq \bigcup_{t\in F} t.K_A = K_{\bigcup_{t\in F} tA}$ for some finite subset F of Γ by compactness of K_B. Hence B is contained in $\bigcup_{t\in F} tA$ by Lemma 2.4(iii).
(ii). Note that $K_B \subseteq X_A$ if and only if $X_B \subseteq X_A$ because X_A is Γ-invariant. Thus (ii) follows from (i).
(iii). Suppose that $B \subseteq \bigcup_{t\in F} tA$ and $A \subseteq \bigcup_{t\in F'} tB$ for some finite subsets F and F' of Γ. Suppose that A is non-paradoxical. Then, by Tarski's theorem, we can find a finitely additive invariant measure λ on Γ such that $\lambda(A) = 1$. Since $\lambda(B) \le |F|\lambda(A)$ and $\lambda(A) \le |F'|\lambda(B)$, we conclude that $0 < \lambda(B) < \infty$, which in turns implies that B is non-paradoxical. □

It follows from Lemma 2.4(vi) and from Lemma 2.5 that if A is a non-empty subset of Γ, then X_A is compact if and only if $X_A = \beta\Gamma$ if and only if $\Gamma \propto A$.

Proposition 2.6. *Let A be a non-empty subset of Γ. There is a non-zero Γ-invariant Radon measure on X_A if and only if A is non-paradoxical in Γ.*

Proof. In the notation of Lemma 2.2 it follows from Lemma 2.4(ii) that $A = A(K_A, e)$, and hence that X_A admits a non-zero Γ-invariant Radon measure if A is non-paradoxical.
Suppose next that there is a non-zero Γ-invariant Radon measure λ on X_A. Then, necessarily, $0 < \lambda(K_A) < \infty$. Let Ω_A be the set of all A-bounded subsets of Γ, i.e., $E \in \Omega_A$ if and only if $E \propto_\Gamma A$ if and only if $K_E \subseteq X_A$, cf. Lemma 2.5(i). Let μ be the invariant finitely additive measure on Γ defined by

$$\mu(E) = \begin{cases} \lambda(K_E), & E \in \Omega_A, \\ \infty, & E \notin \Omega_A. \end{cases}$$

Use Lemma 2.4(i) to see that μ, indeed, is finitely additive. As $\mu(A) = \lambda(K_A)$ by definition we conclude that A is non-paradoxical. □

Proposition 2.7. *A group Γ is supramenable if and only if whenever it acts co-compactly on a locally compact Hausdorff space X, then X admits a non-zero Γ-invariant Radon measure.*

Proof. If Γ is not supramenable and if A is a non-empty paradoxical subset of Γ, then $\Gamma \curvearrowright X_A$ is an example of a co-compact action on a locally compact Hausdorff space that admits no non-zero invariant Radon measure, cf. Proposition 2.6.
The reverse implication follows from Lemma 2.2 (and the remark above that lemma). □

Proposition 2.7 says that every non-supramenable group admits a co-compact action on a locally compact Hausdorff space for which there is no non-zero invariant Radon measures. Towards proving Theorem 1.2, we shall later improve this result and show that one always can choose this space to be the locally compact non-compact Cantor set, and that one further can choose the action to be free, minimal and *purely infinite* (see Definition 4.4).

Example 2.8. Let Γ denote the $ax+b$ group with $a \in \mathbb{Q}^+$ and $b \in \mathbb{Q}$. It is well-known that Γ is non-supramenable. In fact, Γ contains a free semigroup of two generators. One can take the two generators to be $2x$ and $2x+1$.
It also follows from Proposition 2.7 that Γ is non-supramenable, as the canonical action of Γ on $\mathbb{R}$ admits no non-zero invariant Radon measure. We can use Lemma 2.2 to construct paradoxical subsets of Γ other than the ones coming from free sub-semigroups. Indeed, applying Lemma 2.2 to the compact set $K = [\alpha, \beta] \subseteq \mathbb{R}$ and to $x_0 = \gamma \in \mathbb{R}$ we find that

$$\{ax + b \mid \alpha \leq a\gamma + b \leq \beta\} \subseteq \Gamma$$

is paradoxical whenever $\alpha, \beta, \gamma \in \mathbb{R}$ and $\alpha < \beta$.

3. A geometric description of supramenable groups

We shall here describe supramenability in terms of the existence of injective quasi-isometric embeddings of free groups.

DEFINITION 3.1. Let Γ and Λ be groups. A map $f \colon \Gamma \to \Lambda$ is said to be *Lipschitz* if for every finite set $S \subseteq \Gamma$ there is a finite set $T \subseteq \Lambda$ such that

$$\forall x, y \in \Gamma : xy^{-1} \in S \implies f(x)f(y)^{-1} \in T. \tag{3.1}$$

A map $f \colon \Gamma \to \Lambda$ is said to be a *quasi-isometric embedding* if it is Lipschitz and satisfies that for every finite set $T \subseteq \Lambda$ there is a finite set $S \subseteq \Gamma$ such that

$$\forall x, y \in \Gamma : f(x)f(y)^{-1} \in T \implies xy^{-1} \in S. \tag{3.2}$$

This definition corresponds to the usual terminology if Γ and Λ are considered as metric spaces with a right-invariant word metric. If a map f has either of the two properties, the map $\widehat{f}$ defined by $\widehat{f}(x) = f(x^{-1})^{-1}$ satisfies its left-invariant analogue and vice versa.
If (3.1) holds for some $S \subseteq \Gamma$ and some $T \subseteq \Lambda$, then, for each integer $n \geq 1$, (3.1) also holds with S^n and T^n in the place of S and T, respectively. In particular, if Γ is finitely generated, then $f \colon \Gamma \to \Lambda$ is Lipschitz if (3.1) holds for some finite symmetric generating set S for Γ and some finite set $T \subseteq \Lambda$.
Every group homomorphism is Lipschitz, and every group homomorphism with finite kernel is a quasi-isometric embedding.
Let Γ be a group and let $A \subseteq \Gamma$. A map $\sigma \colon A \to \Gamma$ is said to be a *piecewise translation* if it is injective and if there is a finite set $S \subseteq \Gamma$ such that $\sigma(x)x^{-1} \in S$ for all $x \in A$. The composition of piecewise translations (when defined) is again a piecewise translation.
If $\sigma \colon A \to \Gamma$ is a piecewise translation, then $\sigma(A_0) \sim_\Gamma A_0$ for all $A_0 \subseteq A$. A non-empty subset $A \subseteq \Gamma$ is paradoxical if and only if there are piecewise translations $\sigma^{\pm} \colon A \to A$ with disjoint images. In this case, for each natural number n, there are piecewise translations $\sigma_j \colon A \to A$, $j = 1, 2, \dots, n$, with disjoint images.

LEMMA 3.2. *Let Γ and Λ be groups and let $f \colon \Gamma \to \Lambda$ be an injective Lipschitz map. Let A be a subset of Γ and let $\sigma \colon A \to \Gamma$ be a piecewise translation.*

It follows that the map $\tau\colon f(A) \to \Lambda$, given by $\tau \circ f = f \circ \sigma$, is a piecewise translation.

Proof. Injectivity (and well-definedness) of τ follows from injectivity of f and of σ.
There is a finite set $S \subseteq \Gamma$ such that $\sigma(x)x^{-1} \in S$ for all $x \in A$. As f is Lipschitz there is a finite set $T \subseteq \Lambda$ such that $f(x)f(y)^{-1} \in T$ whenever $xy^{-1} \in S$. Hence

$$\tau(f(x))f(x)^{-1} = f(\sigma(x))f(x)^{-1} \in T$$

for all $x \in A$, which proves that τ is a piecewise translation. □

Lemma 3.3. *Let Γ and Λ be groups and let $f\colon \Gamma \to \Lambda$ be an injective Lipschitz map. Then $f(A)$ is a paradoxical subset of Λ whenever A is a paradoxical subset of Γ.*

Proof. Assume that A is a non-empty paradoxical subset of Γ. Then there are piecewise translations $\sigma^\pm\colon A \to A$ with disjoint images. Put $B = f(A)$, and let $\tau^\pm\colon B \to B$ be given by $\tau^\pm \circ f = f \circ \sigma^\pm$. Then $\tau^\pm$ are piecewise translations (by Lemma 3.2) and $\tau^+(B) \cap \tau^-(B) = \emptyset$ (by injectivity of f). This shows that B is paradoxical. □

Proposition 3.4. *A group Γ is supramenable if and only if there is no injective Lipschitz map $f\colon \mathbb{F}_2 \to \Gamma$.*

Proof. The "only if" part follows from Lemma 3.3 and the fact that $\mathbb{F}_2$ is non-amenable and hence paradoxical.
Suppose that Γ is not supramenable and let A be a non-empty paradoxical subset of Γ. Then we can find four piecewise translations $\sigma^\pm, \tau^\pm\colon A \to A$ with disjoint images such that there exists $s_0 \in A$ not in the image of any of these four piecewise translations.
Let a, b denote the generators of $\mathbb{F}_2$ and define a map $f\colon \mathbb{F}_2 \to A$ by induction on the word length ℓ as follows. We first take care of length zero by setting $f(e) = s_0$. If $x \in \mathbb{F}_2$ has length $\ell(x) \geq 1$, and if $x = a^{\pm 1}x' \in \mathbb{F}_2$ is a reduced word, then put $f(x) = \sigma^\pm \circ f(x')$. Similarly, if $x = b^{\pm 1}x' \in \mathbb{F}_2$ is a reduced word, then let $f(x) = \tau^\pm \circ f(x')$.
We prove by contradiction that f is injective. Indeed, if not, we can choose two reduced words $x \neq y$ such that $f(x) = f(y)$ and with $\ell(x)$ minimal for these conditions. Because $\sigma^\pm$ and $\tau^\pm$ have disjoint images (not containing s_0), there exists $c \in \{a, a^{-1}, b, b^{-1}\}$ such that $x = cx'$ and $y = cy'$ are reduced words. But then $f(x') = f(y')$, contradicting the minimality of $\ell(x)$. Therefore f is injective.
Let us finally show that f is Lipschitz. By the remarks below Definition 3.1 we need only show that there is a finite set $T \subseteq \Gamma$ such that (3.1) holds with $S = \{a, a^{-1}, b, b^{-1}\}$. This is done by first choosing

$$T' = \left\{\sigma^\pm(x)x^{-1} \mid x \in A\right\} \cup \left\{\tau^\pm(x)x^{-1} \mid x \in A\right\}.$$

If $x, y \in \mathbb{F}_2$ and $xy^{-1} \in S$, then either $x = cy$ or $y = cx$ for some $c \in S$ and such that cy, respectively, cx, is reduced. In the former case $f(x) = \sigma^{\pm} \circ f(y)$ or $f(x) = \tau^{\pm} \circ f(y)$, hence $f(x)f(y)^{-1} \in T'$. In the latter case $f(y)f(x)^{-1} \in T'$. Finally we can choose $T = T' \cup T'^{-1}$. $\square$

Benjamini and Schramm proved in [5] that there is an injective quasi-isometric embedding from $\mathbb{F}_2$ into any non-amenable group Γ. This gives us the following sequence of inclusions:

$$\begin{aligned} & \{\text{groups of sub-exponential growth}\} \\ \subseteq\ & \{\text{supramenable groups}\} \\ =\ & \{\text{groups with no injective Lipschitz inclusions of } \mathbb{F}_2\} \\ \subseteq\ & \{\text{groups with no quasi-isometric embedding of } \mathbb{F}_2\} \\ \subset\ & \{\text{amenable groups}\} \end{aligned}$$

De Cornulier and Tessera showed in [7, 5.1] that any finitely generated solvable group with exponential growth contains a quasi-isometrically embedded free sub-semigroup on two generators. Solvable groups are amenable and the free group $\mathbb{F}_2$ embeds quasi-isometrically in the free semi-group on two generators. Therefore the last inclusion is strict and for solvable groups the first four sets are equal.

We conclude this section by using the theorem of Benjamini and Schramm mentioned above to show that each non-amenable group contains *small* non-empty paradoxical subsets.

DEFINITION 3.5. Let A be a subset of a group Γ.

(i) Write $A \ll \Gamma$ if $A \precsim_\Gamma \Gamma \setminus B$ for all A-bounded subsets B of Γ.
(ii) Say that A is *absorbing* if $\bigcap_{t \in S} tA \neq \emptyset$ for all finite sets $S \subseteq \Gamma$.
(iii) Write $A \ll^* \Gamma$ if all A-bounded subsets of Γ are non-absorbing.

For example, if A is a subgroup of Γ, then $A \ll \Gamma$ if and only if $|\Gamma : A| = \infty$. A subset of $\Gamma = \mathbb{Z}$ is absorbing if and only if it contains arbitrarily long consecutive sequences (compare (iii) below for the non-commutative case).

We list some elementary properties of the relations defined above.

LEMMA 3.6. *Let Γ be a group and let A, B be subsets of Γ.*

(i) *If $B \propto A$ and $A \ll \Gamma$, then $B \ll \Gamma$.*
(ii) *If $A \ll \Gamma$, then $A \ll^* \Gamma$.*
(iii) *A is absorbing if and only if for every finite subset $F \subseteq \Gamma$ there is $g \in \Gamma$ with $Fg \subseteq A$.*
(iv) *Suppose A absorbing. Then Γ is amenable if and only if there is a left-invariant mean on Γ supported on A.*

Proof. (i). As $B \propto A$, i.e., B is A-bounded, we have $B \subseteq \bigcup_{s \in S} sA$ for some finite $S \subseteq \Gamma$. Let n be the number of elements in S.

Let C be a B-bounded subset of Γ. Then C is also A-bounded. By the hypothesis on A there exists a piecewise translation $\sigma_1 \colon A \to \Gamma \setminus C$. Since

$(\sigma_1(A) \cup C) \propto A$ we can repeat the process and obtain piecewise translations $\sigma_1, \dots, \sigma_n \colon A \to \Gamma \setminus C$ with disjoint images. These piecewise translations can be assembled to one piecewise translation $\sigma \colon \bigcup_{s \in S} sA \to \Gamma \setminus C$. The restriction of σ to B will then witness the relation $B \precsim_\Gamma \Gamma \setminus C$.
(ii). Note first that if $E \subseteq \Gamma$ is such that $E \ll \Gamma$, then E cannot be absorbing. Indeed, there is a piecewise translation $\sigma \colon E \to \Gamma \setminus E$, and $\sigma(E) \subseteq \bigcup_{s \in S} sE$ for some finite $S \subseteq \Gamma$. Put $T = S \cup \{e\}$. Then $\bigcap_{t \in T} tE = \emptyset$.
Suppose that $A \ll^* \Gamma$ does not hold. Then there exists an absorbing set $B \subseteq \Gamma$ such that $B \propto A$. Hence $B \ll \Gamma$ does not hold, which by (i) implies that $A \ll \Gamma$ does not hold.
(iii). It suffices to observe that for any $F \subseteq \Gamma$ we have

$$\{g \in \Gamma : Fg \subseteq A\} \;=\; \bigcap_{t \in F^{-1}} tA.$$

(iv). This follows from the fact that left Følner sets can be arbitrarily translated by right multiplication. □

One can show that $A \ll^* \Gamma$ does not imply $A \ll \Gamma$ (there are counterexamples with $\Gamma = \mathbb{Z}$). We now turn to the existence of interesting small sets in groups.

Lemma 3.7. *Let Γ and Λ be groups and let $f \colon \Gamma \to \Lambda$ be an injective quasi-isometric embedding. Then $f(A) \ll \Lambda$ whenever $A \ll \Gamma$.*
In particular, each non-amenable group Γ contains a non-empty paradoxical subset A with $A \ll \Gamma$.

Proof. Suppose that $A \ll \Gamma$. Let T be a finite subset of Λ and find a finite set $S \subseteq \Gamma$ such that (3.2) holds. There is a piecewise translation $\sigma \colon A \to \Gamma \setminus \bigcup_{s \in S} sA$. This implies that $\sigma(x)y^{-1} \notin S$ for all $x, y \in A$. Let $\tau \colon f(A) \to \Lambda$ be defined by $\tau \circ f = f \circ \sigma$. Then τ is a piecewise translation by Lemma 3.2. Moreover,

$$\tau(f(x))f(y)^{-1} = f(\sigma(x))f(y)^{-1} \notin T$$

for all $x, y \in A$. Hence τ maps $f(A)$ into $\Lambda \setminus \bigcup_{t \in T} tf(A)$. This proves that $f(A) \ll \Lambda$.
If Γ is non-amenable then, as noted above, by the theorem of Benjamini and Schramm in [5] there is an injective quasi-isometric embedding $f \colon \mathbb{F}_2 \to \Gamma$. The free group $\mathbb{F}_2$ contains (several) non-amenable subgroups H of infinite index. For any such subgroup H we can put $A = f(H)$. Then $A \ll \Gamma$ by the first part of the lemma, and A is paradoxical by Lemma 3.3. □

In general, if we are not asking that A in the lemma above is paradoxical, we have the following:

Lemma 3.8. *Each infinite group Γ contains an infinite subset A such that $A \ll \Gamma$.*

Proof. Choose a sequence $x_1, x_2, x_3, \dots$ of elements in Γ such that

$$x_n \notin \{x_k x_\ell^{-1} x_m \mid 1 \le k, \ell, m < n\} \tag{3.3}$$

for all $n \geq 2$, and put $A = \{x_1, x_2, x_3, \dots\}$. We claim that $|sA \cap A| \leq 2$ for all $s \in \Gamma \setminus \{e\}$. Indeed, if $sA \cap A \neq \emptyset$ and $s \neq e$, then $sx_n = x_m$ for some $n \neq m$. Suppose that $k, \ell \in \mathbb{N}$ is another pair such that $sx_k = x_\ell$. Suppose also that $k > \ell$. Then $x_k = s^{-1}x_\ell = x_n x_m^{-1} x_\ell$, whence $k \leq \max\{n, m, \ell\}$ by (3.3), i.e. $k \leq \max\{n, m\}$. Similarly, if $\ell > k$, then $x_\ell = x_m x_n^{-1} x_k$, which entails that $\ell \leq \max\{n, m\}$. Now, interchanging the roles of the pairs (n, m) and (k, ℓ), we conclude that $\max\{n, m\} = \max\{k, \ell\}$ whenever $sx_n = x_m$ and $sx_k = x_\ell$. It follows easily from this observation that $|sA \cap A| \leq 2$.

Let $F \subset \Gamma$ be finite, choose $s \in \Gamma \setminus F$, and put

$$H = A \cap \bigcup_{t \in F} s^{-1}tA = \bigcup_{t \in F} (A \cap s^{-1}tA).$$

Then H is finite and

$$A \setminus H \sim_\Gamma s(A \setminus H) \subseteq \Gamma \setminus \bigcup_{t \in F} tA. \tag{3.4}$$

To show that $A \precsim_\Gamma \Gamma \setminus \bigcup_{t \in F} tA$ it now suffices to show that

$$H \precsim_\Gamma \Gamma \setminus \big(\bigcup_{t \in F} tA \cup sA \big) = \Gamma \setminus \bigcup_{t \in F'} tA,$$

where $F' = F \cup \{s\}$. As H is finite it suffices to show that $\big|\Gamma \setminus \bigcup_{t \in F'} tA\big| \geq |H|$. However, one can deduce from (3.4) that $\Gamma \setminus \bigcup_{t \in F} tA$ is infinite for *any* finite set $F' \subset \Gamma$; and this completes the proof. $\square$

4. Structure of crossed product C^*-algebras

We review here some, mostly well-known, results about the structure of crossed product C^*-algebras; and we introduce the notion of *purely infinite actions*. Recall first that a dynamical system $\Gamma \curvearrowright X$ is said to be *regular* if the full and the reduced crossed product C^*-algebras coincide, i.e., if the natural epimorphism

$$C_0(X) \rtimes_{\mathrm{full}} \Gamma \to C_0(X) \rtimes_{\mathrm{red}} \Gamma$$

is an isomorphism. Anantharaman-Delaroche proved in [1] that $\Gamma \curvearrowright X$ is regular if the action is *amenable* (in the sense of Anantharaman-Delaroche, [1, Definition 2.1]). Matsumura proved later that the converse also holds when X is compact and the group Γ is exact, [14].

Moreover, the action is amenable if and only if the crossed product C^*-algebra $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ is nuclear. Archbold and Spielberg proved in [3] that the full crossed product $C_0(X) \rtimes_{\mathrm{full}} \Gamma$ is simple if and only if the action $\Gamma \curvearrowright X$ is minimal, topologically free and regular. Combining this result of Archbold and Spielberg with the results of Anantharaman-Delaroche mentioned above, we get:

Proposition 4.1 (Anantharaman-Delaroche, Archbold–Spielberg). *Let Γ be a countable group acting on a locally compact Hausdorff space X. Then $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ is simple and nuclear if and only if $\Gamma \curvearrowright X$ is minimal, topologically free and amenable.*

Recall that a *Kirchberg algebra* is a C^*-algebra which is simple, purely infinite, nuclear and separable.

The proposition below is essentially contained in [13, Theorem 5 and its proof] by Laca and Spielberg. To deduce simplicity of $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ one can also use Archbold–Spielberg, [3]. Recall that a projection p in a C^*-algebra $\mathcal{A}$ is said to be *properly infinite* if $p \oplus p \precsim p$, i.e., $p \oplus p$ is Murray–von Neumann equivalent to a subprojection of p.

Proposition 4.2 (Archbold–Spielberg, Laca–Spielberg). *Let Γ be a countable group acting minimally and topologically freely on a metrizable totally disconnected locally compact Hausdorff space X. Suppose further that each non-zero projection in $C_0(X)$ is properly infinite in $C_0(X) \rtimes_{\mathrm{red}} \Gamma$. Then $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ is simple and purely infinite.*

J.-L. Tu proved in [22] (see also [21]) that the C^*-algebra associated with any amenable groupoid belongs to the UCT class. The crossed product $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ can be identified with the C^*-algebra of the groupoid $X \rtimes \Gamma$, which is amenable if Γ acts amenably on X. Hence $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ belongs to the UCT class whenever Γ acts amenably on X. Combining this deep theorem with Propositions 4.1 and 4.2 we get:

Corollary 4.3. *Let X be a metrizable totally disconnected locally compact Hausdorff space and let Γ be a group acting on X. Then $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ is a Kirchberg algebra in the UCT class if and only if the action of Γ on X is minimal, topologically free, amenable, and each non-zero projection in $C_0(X)$ is properly infinite in $C_0(X) \rtimes_{\mathrm{red}} \Gamma$.*

It would be very desirable to replace the last condition on the projections in $C_0(X)$ with purely dynamical conditions on the Γ-space X. This leads us to the notion of *purely infinite actions*, which we shall proceed to define.

Let Γ be a group acting on a locally compact totally disconnected Hausdorff space X. Let $\mathbb{K}(X)$ (or just $\mathbb{K}$ if X is understood) be the algebra of all compact-open subsets of X. A set $K \in \mathbb{K}$ is said to be $(X, \Gamma, \mathbb{K})$-*paradoxical* if there exist pairwise disjoint sets $K_1, K_2, \dots, K_{n+m} \in \mathbb{K}$ and elements $t_1, t_2, \dots, t_{n+m} \in \Gamma$ such that $K_j \subseteq K$ for all j and

$$K = \bigcup_{j=1}^{n} t_j.K_j = \bigcup_{j=n+1}^{n+m} t_j.K_j. \tag{4.1}$$

(We can think of this version of paradoxicality as a relative version of the classical notion of paradoxicality from the Hausdorff–Banach–Tarski paradox.)

Definition 4.4. An action of a group Γ on a totally disconnected Hausdorff space X is said to be *purely infinite* if every compact-open subset of X is $(X, \Gamma, \mathbb{K})$-paradoxical (in the sense defined above).

Let us for a moment turn to the C^*-algebra point of view. Let us first fix some (standard) notations for crossed product C^*-algebras. Let Γ be a group acting on a C^*-algebra $\mathcal{A}$ (where $\mathcal{A}$ for example could be $C_0(X)$ for some locally compact Hausdorff space X). Let the action of Γ on $\mathcal{A}$ be denoted by $t \mapsto \alpha_t \in \mathrm{Aut}(\mathcal{A})$, $t \in \Gamma$. If $\mathcal{A} = C_0(X)$ and Γ acts on X, then the action of Γ on $\mathcal{A}$ is given by $\alpha_t(f) = t.f$, where $(t.f)(x) = f(t^{-1}.x)$ for $f \in C_0(X)$ and $x \in X$. Let u_t, $t \in \Gamma$, denote the unitary elements in (the multiplier algebra of) $\mathcal{A} \rtimes_{\mathrm{red}} \Gamma$ that implement the action of Γ on $\mathcal{A}$, i.e., $\alpha_t(a) = u_t a u_t^*$ for $a \in \mathcal{A}$ and $t \in \Gamma$. The set of finite sums of the form $\sum_{t\in\Gamma} a_t u_t$, where $a_t \in \mathcal{A}$ and only finitely many a_t are non-zero, forms a uniformly dense *-subalgebra of $\mathcal{A} \rtimes_{\mathrm{red}} \Gamma$.

A projection $p \in \mathcal{A}$ is said to be $(\mathcal{A}, \Gamma)$-*paradoxical* if there are pairwise orthogonal subprojections $p_1, p_2, \dots, p_{n+m}$ of p in $\mathcal{A}$ and elements $t_1, t_2, \dots, t_{n+m}$ in Γ such that

$$p = \sum_{k=1}^{n} \alpha_{t_k}(p_k) = \sum_{k=n+1}^{n+m} \alpha_{t_k}(p_k). \tag{4.2}$$

We record some facts about paradoxical projections and sets:

(a) If p is a projection in a C^*-algebra $\mathcal{A}$ on which a group Γ acts, and if p is $(\mathcal{A}, \Gamma)$-paradoxical, then p is properly infinite in $\mathcal{A} \rtimes_{\mathrm{red}} \Gamma$.

(b) If K is a compact-open subset of a locally compact Hausdorff Γ-space X, then K is $(X, \Gamma, \mathbb{K})$-paradoxical if and only if the projection 1_K is $(C_0(X), \Gamma)$-paradoxical.

(c) If $A \subseteq \Gamma$, then A is paradoxical if and only if K_A is $(\beta\Gamma, \Gamma, \mathbb{K})$-paradoxical, which again happens if and only if 1_A is a properly infinite projection in $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ or, equivalently, if and only if 1_{K_A} is a properly infinite projection in $C(\beta\Gamma) \rtimes_{\mathrm{red}} \Gamma$.

Proof. (a). Let $p_1, p_2, \dots, p_{n+m} \in \mathcal{A}$ be pairwise orthogonal subprojections of p and $t_1, t_2, \dots, t_{n+m}$ in Γ be such that (4.2) holds. Let u_t be the unitaries in the multiplier algebra of $\mathcal{A}$ that implement the action α_t, $t \in \Gamma$, on $\mathcal{A}$. Define the following two elements in $\mathcal{A} \rtimes_{\mathrm{red}} \Gamma$:

$$v = \sum_{j=1}^{n} p_j u_{t_j}^*, \qquad w = \sum_{j=n+1}^{n+m} p_j u_{t_j}^*.$$

It then follows from (4.2) that

$$v^*v = p = w^*w, \qquad vv^* \perp ww^*, \qquad vv^* \le p \qquad ww^* \le p.$$

This shows that p is a properly infinite projection in $C_0(X) \rtimes_{\mathrm{red}} \Gamma$.

(b). This follows immediately from the fact that there is a bijective correspondence between compact-open subsets of X and projections in $C_0(X)$ (given by $K \leftrightarrow 1_K$).

(c). The first claim, that A is paradoxical if and only if K_A is $(\beta\Gamma, \Gamma, \mathbb{K})$-paradoxical, follows easily from Lemma 2.4. Indeed, A is paradoxical if and only if there exist pairwise disjoint subsets $A_1, A_2, \dots, A_{n+m} \subseteq A$ and elements

$t_1, t_2, \dots, t_{n+m}$ such that

$$A = \bigcup_{j=1}^{n} t_j A_j = \bigcup_{j=n+1}^{n+m} t_j A_j. \tag{4.3}$$

Taking closures (relatively to $\beta\Gamma$) we obtain that (4.1) holds with $K = K_A$ and $K_j = K_{A_j}$. Hence K_A is $(\beta\Gamma, \Gamma, \mathbb{K})$-paradoxical. For the reverse implication, if (4.1) holds with $K = K_A$, then intersect all sets in (4.1) with Γ and use Lemma 2.4 to conclude that (4.3) holds, so A is paradoxical.

It was shown in [17, Proposition 5.5] that if $A \subseteq \Gamma$, then 1_A is properly infinite in $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ if and only if A is paradoxical. The natural isomorphism from $\ell^\infty(\Gamma)$ to $C(\beta\Gamma)$ maps 1_A to 1_{K_A}, and this isomorphism preserves the property of being properly infinite in the crossed product by Γ. □

Lemma 4.5. *Let Γ be a group acting on a locally compact totally disconnected Hausdorff space X. Consider the following three conditions on a compact-open subset K of X:*

(i) *There is no Γ-invariant Radon measure μ on X such that $\mu(K) > 0$.*
(ii) *1_K is a properly infinite projection in $C_0(X) \rtimes_{\mathrm{red}} \Gamma$.*
(iii) *K is $(X, \Gamma, \mathbb{K})$-paradoxical.*

Then (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

Proof. (ii) $\Rightarrow$ (i). Assume (i) does not hold. Then there is an invariant Radon measure μ on X such that $\mu(K) = 1$. The measure μ extends to a densely defined (possibly unbounded) trace τ on $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ (cf. [17, Lemma 5.3]) such that $\tau(1_K) = \mu(K) = 1$. A properly infinite projection is either zero (or infinite) under any positive trace, so 1_K cannot be properly infinite in $C_0(X) \rtimes_{\mathrm{red}} \Gamma$.

(iii) $\Rightarrow$ (ii). This follows from (a) and (b) above. □

The proposition below follows immediately from Corollary 4.3 and from "(iii) $\Rightarrow$ (ii)" of the lemma above.

Proposition 4.6. *Let X be a metrizable totally disconnected locally compact Hausdorff space and let Γ be a group which acts on X in such a way that the action is topologically free, minimal, amenable and purely infinite. Then $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ is a Kirchberg algebra in the UCT class.*

Remark 4.7 (The type semigroup). It is an interesting question if the converse of Proposition 4.6 holds, or to what extend the reverse implications in Lemma 4.5 hold. In other words, if Γ acts freely, minimally and amenably on a metrizable totally disconnected locally compact Hausdorff space X such that $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ is a Kirchberg algebra, does it then follow that the action is purely infinite? As a Kirchberg algebra has no traces, the space X cannot have any non-zero Γ-invariant Radon measures if $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ is a Kirchberg algebra.

We can analyse these questions using the relative type semigroup $S(X, \Gamma, \mathbb{K})$ considered for example in [17, Section 5], see also [24]. This semigroup can be defined as being the universal ordered abelian semigroup generated by elements $[K]$, with $K \in \mathbb{K}$, subject to the relations

$$[t.K] = [K], \quad [K \cup K'] = [K] + [K'] \text{ if } K \cap K' = \emptyset, \quad [K] \le [K'] \text{ if } K \subseteq K',$$

where $K, K' \in \mathbb{K}$ and $t \in \Gamma$. As the set of compact-open sets is closed under forming set differences, we deduce that the ordering on $S(X, \Gamma, \mathbb{K})$ is the *algebraic order*: If $x, y \in S(X, \Gamma, \mathbb{K})$, then $x \le y$ if and only if there exists $z \in S(X, \Gamma, \mathbb{K})$ such that $y = x + z$.
In the language of the type semigroup, $K \in \mathbb{K}$ is $(X, \Gamma, \mathbb{K})$-paradoxical if and only if $2[K] \le [K]$.
Property (i) in Lemma 4.5 holds if and only if there exists a natural number n such that $(n+1)[K] \le n[K]$, which in turns implies that $m[K] \le n[K]$ for all $m \ge 1$. We conclude that the implication "(i) $\Rightarrow$ (iii)" holds for all compact-open subsets K of X if $S(X, \Gamma, \mathbb{E})$ is *almost unperforated*, that is, for all $x, y \in S(X, \Gamma, \mathbb{K})$ for all integers $n \ge 1$, $(n+1)x \le ny$ implies $x \le y$. In fact, "(i) $\Rightarrow$ (iii)" also holds under the much weaker comparability assumption that there exist integers $n, m \ge 2$ such that for all $x, y \in S(X, \Gamma, \mathbb{K})$, $nx \le my$ implies $x \le y$.
It has recently been shown by Ara and Exel in [2] that $S(X, \Gamma, \mathbb{K})$ need not be almost unperforated. They give counterexamples where Γ is a free group acting on the (locally compact, non-compact) Cantor set. It is not known if this phenomenon can occur also when the free group acts freely and minimally on the Cantor set. Nonetheless, the example by Ara and Exel indicates that the questions raised in this remark may all have negative answers.

We close this section with two lemmas that shall be used in Section 6. An open subset E of a Γ-space X is said to be Γ-*full* in X if $X = \bigcup_{t \in \Gamma} t.E$.

Lemma 4.8. *Let X be a totally disconnected locally compact Hausdorff space, let Γ be a group acting co-compactly on X, and let Y be a closed Γ-invariant subset of X.*

(i) *If K' is a compact-open Γ-full subset of Y, then there exists a compact-open Γ-full subset K of X such that $K' = K \cap Y$.*
(ii) *If K is a $(X, \Gamma, \mathbb{K})$-paradoxical compact-open subset of X, then $K \cap Y$ is a $(Y, \Gamma, \mathbb{K})$-paradoxical compact-open subset of Y.*

Proof. (i). There is a compact-open subset K_0 of X such that $K' = K_0 \cap Y$. It follows from the assumption that the action of Γ on X is co-compact that there exists a compact-open subset $L \subseteq X$ such that $X = \bigcup_{t \in \Gamma} t.L$. Since K' is Γ-full in Y and $L \cap Y$ is compact there is a finite subset $S \subseteq \Gamma$ such that $L \cap Y \subseteq \bigcup_{t \in S} t.K'$. Put

$$K = K_0 \cup \big(L \setminus \bigcup_{t \in S} t.K'\big).$$

Then K is compact-open, $K \cap Y = K_0 \cap Y = K'$, and $X = \bigcup_{t \in \Gamma} t.K$.
(ii). Follows from the definitions. □

Lemma 4.9. *Consider the canonical action of* Γ *on* $\beta\Gamma$. *Let* A *and* B *be subsets of* Γ. *Then* K_B *is a* Γ*-full subset of* X_A *if and only if* $A \propto_\Gamma B \propto_\Gamma A$.

Proof. By definition, K_B is Γ-full in X_A if and only if $X_B = X_A$. The lemma therefore follows from Lemma 2.5(ii). □

5. The Roe algebra

The Roe algebra associated with a group Γ is the reduced crossed product C^*-algebra $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$. We give below a characterisation of supramenability in terms of the Roe algebra.

Lemma 5.1. *Let* $p \in \ell^\infty(\Gamma)$ *and* $q \in \ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ *be projections that generate the same closed two-sided ideal in* $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$. *If* q *is properly infinite, then so is* p.

Proof. By the assumption that p and q generate the same closed two-sided ideal in the Roe algebra $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ it follows that $p \precsim q \otimes 1_n$ and $q \precsim p \otimes 1_m$ relatively to $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ for some integers $n, m \geq 1$. As q is properly infinite we have $q \otimes 1_k \precsim q$ for all $k \geq 1$. Hence

$$(p \otimes 1_m) \oplus (p \otimes 1_m) \precsim q \otimes 1_{2nm} \precsim q \precsim p \otimes 1_m,$$

so $p \otimes 1_m$ is properly infinite in the Roe algebra. It then follows from [17, Proposition 5.5] that p itself is properly infinite in the Roe algebra. □

Remark 5.2 (The Pedersen ideal). Gert K. Pedersen proved that every C^*-algebra $\mathcal{A}$ contains a smallest dense two-sided ideal, now called the *Pedersen ideal in* $\mathcal{A}$. The Pedersen ideal of $\mathcal{A}$ can be obtained as the intersection of all dense two-sided ideals in $\mathcal{A}$; and the content of Pedersen's result is that this ideal is again a dense ideal in $\mathcal{A}$.
For every positive element a in $\mathcal{A}$ and for every $\varepsilon > 0$ one can consider the "ε-cut-down", $(a-\varepsilon)_+ \in \mathcal{A}$, (which is the positive part of the self-adjoint element $a - \varepsilon \cdot 1_{\widetilde{\mathcal{A}}}$ in the unitization $\widetilde{\mathcal{A}}$ of $\mathcal{A}$). This element $(a-\varepsilon)_+$ belongs to the Pedersen ideal for every positive element a in $\mathcal{A}$ and for every $\varepsilon > 0$. If $p \in \mathcal{A}$ is a projection and if $0 < \varepsilon < 1$, then $(p-\varepsilon)_+ = (1-\varepsilon)p$. All projections in $\mathcal{A}$ therefore belong to the Pedersen ideal in $\mathcal{A}$.
The Pedersen ideal of a unital C^*-algebra is the algebra itself, i.e., unital C^*-algebras have no proper dense two-sided ideals. The Pedersen ideal of $\mathcal{K}(H)$, the compact operators on a Hilbert space H, is the algebra of all finite rank operators on H. The Pedersen ideal of $C_0(X)$, for some locally compact Hausdorff space X, is $C_c(X)$.
An element a in a C^*-algebra $\mathcal{A}$ is said to be *full*, if a is not contained in any proper closed two-sided ideal in $\mathcal{A}$. If $\mathcal{A}$ contains a full projection p, then the primitive ideal space of $\mathcal{A}$ is (quasi-)compact. This can be rephrased as follows: Whenever $\{\mathcal{I}_\alpha\}_{\alpha \in \mathbb{A}}$ is an increasing net of closed two-sided ideals in $\mathcal{A}$ such that $\mathcal{A} = \overline{\bigcup_{\alpha \in \mathbb{A}} \mathcal{I}_\alpha}$, then $\mathcal{A} = \mathcal{I}_\alpha$ for some $\alpha \in \mathbb{A}$. Indeed, $\bigcup_{\alpha \in \mathbb{A}} \mathcal{I}_\alpha$ is a dense two-sided ideal in $\mathcal{A}$, which therefore contains the Pedersen ideal, and hence contains all projections in $\mathcal{A}$. Thus $p \in \mathcal{I}_\alpha$ for some α, whence $\mathcal{A} = \mathcal{I}_\alpha$.

As mentioned in the previous section, it was shown in [17, Proposition 5.5] that if $A \subseteq \Gamma$, then 1_A is properly infinite in $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ if and only if A is paradoxical. We sharpen this result as follows:

Proposition 5.3. *The following two conditions are equivalent for every group* Γ*:*

(i) Γ *is supramenable.*
(ii) *The Roe algebra* $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ *contains no properly infinite projection.*

Proof. Suppose that (ii) does not hold. If Γ is non-exact, then Γ is non-amenable and in particular not supramenable, so (i) does not hold. Suppose that Γ is exact and that $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ contains a properly infinite projection p. Let $\mathcal{I}$ be the closed two-sided ideal in $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ generated by p. Then, by [20, Theorem 1.16], which applies because Γ is assumed to be exact and the action of Γ on $\ell^\infty(\Gamma)$ is free, $\mathcal{I}$ is the closed two-sided ideal generated by $\ell^\infty(\Gamma) \cap \mathcal{I}$. Let $\mathbb{A}$ be the directed net of all finite subsets of the set of all projections in $\ell^\infty(\Gamma) \cap \mathcal{I}$; and for each $\alpha \in \mathbb{A}$ let $\mathcal{I}_\alpha$ be the closed two-sided ideal in $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ generated by α. Then

$$\mathcal{I} = \overline{\bigcup_{\alpha \in \mathbb{A}} \mathcal{I}_\alpha}.$$

It therefore follows from Remark 5.2 that $\mathcal{I} = \mathcal{I}_\alpha$ for some $\alpha \in \mathbb{A}$. Let $q \in \ell^\infty(\Gamma)$ be the supremum of the projections belonging to α. Then $\mathcal{I}$ is equal to the closed two-sided ideal generated by q. Hence p and q generate the same closed two-sided ideal in the Roe algebra, whence q is properly infinite by Lemma 5.1.
If (ii) holds and if $A \subseteq \Gamma$, then 1_A is not a properly infinite projection in $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$, so A is non-paradoxical, cf. [17, Proposition 5.5] or claim (c) (below Definition 4.4). This shows that Γ is supramenable. □

Remark 5.4. A projection is said to be infinite if it is Murray–von Neumann equivalent to a proper subprojection of itself. A unital C^*-algebra is infinite if its unit is an infinite projection, or, equivalently, if it contains a non-unitary isometry. If a unital C^*-algebra contains an infinite projection, then it is infinite itself. The Roe algebra $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ is therefore infinite whenever Γ is non-supramenable.
The Roe algebra $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ is also infinite whenever Γ contains an element t of infinite order. Indeed, let $A = \{t^n \mid n \geq 0\}$. Then $tA \subset A$, so $u_t 1_A u_t^* = 1_{tA} < 1_A$, which shows that 1_A is an infinite projection in $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$. (As before we let $(u_t)_{t \in \Gamma}$ denote the unitaries in $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ implementing the action of Γ on $\ell^\infty(\Gamma)$.)
On the other hand, if Γ is locally finite (an increasing union of finite groups), then $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ is finite. To see this observe first that $C(X) \rtimes_{\mathrm{red}} \Gamma$ is finite whenever Γ is a finite group acting on a compact Hausdorff space X. Indeed, a unital C^*-algebra is finite if it admits a separating family of tracial states. A crossed product C^*-algebra $C(X) \rtimes_{\mathrm{red}} \Gamma$ is therefore finite if X admits a separating family of invariant probability measures. (By separating we mean

that every non-empty open set is non-zero on at least one of the probability measures in the family.) If Γ is finite then the family consisting of Γ-means of every probability measure on X will be such a separating family of invariant probability measures.

Suppose Γ is locally finite and write $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$, where $\Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_3 \subseteq \cdots$ is an increasing sequence of finite subgroups of Γ. Then

$$\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma = \varinjlim \ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma_n,$$

since $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma_n$ is isomorphic to $C^*\big(\ell^\infty(\Gamma) \cup \{u_t \mid t \in \Gamma_n\}\big)$. Any inductive limit of finite C^*-algebras is again finite, so $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ is finite because all $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma_n$ are finite.

It seems plausible that the Roe algebra $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ is finite if and *only if* Γ is locally finite.

In the proposition below we identify the Roe algebra with $C(\beta\Gamma) \rtimes_{\mathrm{red}} \Gamma$.

If $A \subseteq \Gamma$ and $\{U_i\}_{i \in I}$ is an increasing family of proper open Γ-invariant subsets of X_A, then $\bigcup_{i \in I} U_i$ is also a proper subset of X_A. (Otherwise K_A would be contained in one of the U_i's, which again would entail that $U_i = X_A$.) It follows that X_A contains a maximal proper open Γ-invariant subset, or, equivalently, that X_A contains a minimal (non-empty) closed Γ-invariant subset. This argument shows moreover that each non-emtpy closed invariant subset of X_A contains a minimal (non-empty) closed Γ-invariant subset. Recall from Definition 3.5 the definition of the notion $A \ll^* \Gamma$.

Proposition 5.5. *Let A be a non-empty subset of Γ and let Z be a minimal (non-empty) closed Γ-invariant subset of X_A. Then:*

(i) *Z is a locally compact totally disconnected Hausdorff space, and the action of Γ on Z is free and minimal.*
(ii) *The action of Γ on Z is purely infinite if A is paradoxical.*
(iii) *The action of Γ on Z is amenable if Γ is exact.*
(iv) *If $A \ll^* \Gamma$, then Z is necessarily non-compact; and if $A \ll^* \Gamma$ does not hold, then there exists a minimal (non-empty) closed Γ-invariant subset of X_A which is compact.*[1]

Proof. It is clear from its definition that Z is a closed Γ-invariant subspace of X_A, and hence itself a Γ-space. Being a closed subset of the totally disconnected locally compact Hausdorff space X_A, Z is also a totally disconnected locally compact Hausdorff space. By maximality of U, the action of Γ on Z is minimal. Freeness and amenability of an action of a group on a space pass to any Γ-invariant subspace. As Γ acts freely on $\beta\Gamma$ for all Γ we conclude that (i) holds. Moreover, if Γ is exact, then the action of Γ on $\beta\Gamma$ is amenable (see [6, Theorem 5.1.6]), so (iii) holds.

Let us prove (ii). Assume that A is paradoxical. Take a non-empty compact-open subset K of Z. Then K is Γ-full in Z by minimality of the action. We

[1]Added in proof: The existence of one compact minimal closed invariant subset of X_A does not in general imply that all minimal closed invariant subset of X_A are compact.

can therefore use Lemma 4.8(i) to find a compact-open subset K' of X_A which is Γ-full in X_A and satisfies $K = K' \cap Z$. Put $B = K' \cap \Gamma$. Then $K_B = K'$ by Lemma 2.4(iv), and $A \propto_\Gamma B \propto_\Gamma A$ by Lemma 4.9. Hence B is paradoxical by Lemma 2.5(ii). Claim (c) (below Definition 4.4) then states that $K' = K_B$ is $(X_A, \Gamma, \mathbb{K})$-paradoxical, and Lemma 4.8(ii) finally yields that K is $(Z, \Gamma, \mathbb{K})$-paradoxical.

(iv). Suppose that Z is compact. As $Z \subseteq X_A = \bigcup_{t\in\Gamma} t.K_A$, it follows that there exists a finite set $S \subseteq \Gamma$ such that $Z \subseteq \bigcup_{s\in S} s.K_A = K_B$, where $B = \bigcup_{s\in S} sA$. For each finite set $T \subseteq \Gamma$ we have

$$Z = \bigcap_{t\in T} t.Z \subseteq \bigcap_{t\in T} t.K_B = K_{\bigcap_{t\in T} tB}.$$

This shows that $\bigcap_{t\in T} tB$ is non-empty, so B is absorbing. Hence $A \ll^* \Gamma$ does not hold.

Suppose, conversely, that $A \ll^* \Gamma$ does not hold. Find an absorbing subset B of Γ so that $B \propto A$. Then $K_B \subseteq X_A$ by Lemma 2.5(i). Put $W = \bigcap_{t\in\Gamma} t.K_B$. Then W is compact and Γ-invariant. Moreover, since

$$\bigcap_{s\in S} t.K_B = K_{\bigcap_{s\in S} sB} \neq \emptyset$$

for all finite sets $S \subseteq \Gamma$, we conclude that W is non-empty. By the argument above the proposition we can find a minimal (non-empty) closed Γ-invariant subset Z of W. As W is compact, so is Z. $\square$

It should be remarked that the space Z from Proposition 5.5 may not be metrizable and that the crossed product $C_0(Z) \rtimes_{\text{red}} \Gamma$ accordingly may not be separable. It follows from Proposition 5.5 and from Proposition 4.1 that the crossed product $C_0(Z) \rtimes_{\text{red}} \Gamma$ is simple and nuclear if Γ is exact; and it follows from Proposition 5.5 and Proposition 4.6 that the crossed product is simple, nuclear and purely infinite (but not necessarily a Kirchberg algebra) if A moreover is paradoxical.

Recall from Lemma 2.5 that X_A itself is compact if and only if $X_A = \beta\Gamma$. Clearly, any minimal closed Γ-invariant subset of $\beta\Gamma$ is compact. However, a minimal closed invariant subset Z of X_A can be compact even when X_A is non-compact. This follows from Proposition 5.5(iv) if we can find an absorbing subset A of a group Γ such that $X_A \neq \beta\Gamma$ (or equivalently, such that Γ is not A-bounded). There are many such examples of $A \subseteq \Gamma$, eg. $\Gamma = \mathbb{Z}$ and $A = \mathbb{N}$.

Perhaps surprisingly it turns out to a subtle matter to decide when the space Z is non-discrete. Clearly, if A is non-empty and finite, then $Z = X_A = \Gamma$ and we are in the trivial situation of Γ acting on itself. More generally, if Z has an isolated point, then the Γ-space Z is conjugate to Γ. (Indeed, if $x_0 \in Z$ is an isolated point, then $Z = \Gamma.x_0$ by minimality of the action.)

The example below shows that Z (in Proposition 5.5) can be discrete (and hence trivial) even when A is infinite. More precisely, if $A \subseteq \Gamma$ is such that $sA \cap A$ is finite for all $e \neq s \in \Gamma$, then Z is discrete. On the other hand, if A is paradoxical, then Z cannot be discrete by Proposition 5.5(ii).

Example 5.6. Let Γ be a countably infinite group and suppose that $A \subseteq \Gamma$ is an infinite set such that $|sA \cap A| < \infty$ for all $s \in \Gamma \setminus \{e\}$. It was shown in Lemma 3.8 (and its proof) that each infinite group Γ contains such a subset A. Let $1_A \in \ell^\infty(\Gamma) \subseteq \ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ denote indicator function for A.

Inside the Roe algebra, $c_0(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ is the smallest non-zero ideal (every other non-zero ideal contains this ideal). If $A \subseteq \Gamma$ is as above, then the following holds in the quotient of the Roe algebra by $c_0(\Gamma) \rtimes_{\mathrm{red}} \Gamma$:

$$\forall x \in \ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma : 1_A\, x\, 1_A + c_0(\Gamma) \rtimes_{\mathrm{red}} \Gamma \;\in\; \ell^\infty(\Gamma) + c_0(\Gamma) \rtimes_{\mathrm{red}} \Gamma. \tag{5.1}$$

In other words, if $\pi \colon \ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma \to \big(\ell^\infty(\Gamma)/c_0(\Gamma)\big) \rtimes_{\mathrm{red}} \Gamma$ denotes the quotient mapping, then $\pi(1_A)$ is an abelian projection. In particular, the corner algebra $1_A(\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma)1_A$ has a character.

Let us see why (5.1) holds. It suffices to establish (5.1) for x in a dense subset of $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$, so we may assume that $x = \sum_{t \in F} f_t u_t$, where $F \subset \Gamma$ is finite, $f_t \in \ell^\infty(\Gamma)$, and $t \mapsto u_t$ is the canonical unitary representation of the action of Γ on $\ell^\infty(\Gamma)$. Now,

$$1_A x 1_A = \sum_{t \in F} 1_A f_t u_t 1_A = \sum_{t \in F} 1_A\, f_t\, 1_{tA} u_t = \sum_{t \in F} 1_{A \cap tA}\, f_t u_t.$$

But $1_{A \cap tA} \in c_0(\Gamma)$ whenever $t \neq e$, so $\pi(1_A x 1_A) = \pi(1_A f_e)$ and $1_A f_e \in \ell^\infty(\Gamma)$.

Let $A \subseteq \Gamma$ be as above, consider the open set X_A, and let U be a maximal proper open Γ-invariant subset of X_A, cf. Definition 2.3. Since A is infinite and Γ is dense in $\beta\Gamma$, we have $\Gamma \subseteq U \subset X_A$. We claim that the minimal Γ-space $Z = X_A \setminus U$ is, in fact, discrete. More precisely, the compact-open subset $K_A \cap Z$ of Z is a singleton, and hence an isolated point in Z. Indeed, the natural epimorphism $C_0(X_A) \rtimes_{\mathrm{red}} \Gamma \to C_0(Z) \rtimes_{\mathrm{red}} \Gamma$ maps 1_{K_A} onto $1_{K_A \cap Z}$ and its kernel contains $c_0(\Gamma) \rtimes_{\mathrm{red}} \Gamma$. We can therefore deduce from (5.1) that

$$\mathcal{A} := 1_{K_A \cap Z}\big(C_0(Z) \rtimes_{\mathrm{red}} \Gamma\big)1_{K_A \cap Z}$$

is abelian. As $C_0(Z) \rtimes_{\mathrm{red}} \Gamma$ is a simple C^*-algebra, so is $\mathcal{A}$, whence $\mathcal{A} \cong \mathbb{C}$. As $C(K_A \cap Z)$ is a sub-C^*-algebra of $\mathcal{A}$ this implies that $K_A \cap Z$ is a singleton.

6. Kirchberg algebras arising as crossed products by non-supramenable groups

We show here that every non-supramenable countable group admits a free, minimal, purely infinite[2] action on the locally compact, non-compact Cantor set; and that the action moreover can be chosen to be amenable if the group is exact. The (reduced) crossed product C^*-algebra associated with such an action will in the latter case be a stable Kirchberg algebra in the UCT class. Our construction is an adaption of the one from [17, Section 6], where it was shown that every exact non-amenable countable group admits a free, minimal, amenable action on the (compact) Cantor set such that the crossed product C^*-algebra is a unital Kirchberg algebra in the UCT class. (One can easily

[2]In the sense of Definition 4.4

modify the construction in [17] to make the action on the Cantor set purely infinite.)

Fix a countable group Γ and a non-empty subset A of Γ. Let $K_A \subseteq X_A \subseteq \beta\Gamma$ be as in Definition 2.3. Note that $C_0(X_A)$ is a Γ-invariant closed ideal in $C(\beta\Gamma)$, and the smallest such which contains the projection 1_{K_A}. Sometimes we prefer to work with $\ell^\infty(\Gamma)$ rather than $C(\beta\Gamma)$. To avoid confusion we denote by $\mathcal{C}_A$ the Γ-invariant ideal in $\ell^\infty(\Gamma)$ that corresponds to $C_0(X_A)$. Thus $\mathcal{C}_A$ is the smallest Γ-invariant closed ideal in $\ell^\infty(\Gamma)$ which contains 1_A.

Fix an increasing sequence $\{F_n\}_{n=1}^\infty$ of finite subsets of Γ with $\bigcup_{n=1}^\infty F_n = \Gamma$. Put $B_n = \bigcup_{t \in F_n} tA$, and put $p_n = 1_{B_n} \in \mathcal{C}_A$. Then $\{p_n\}_{n=1}^\infty$ is an increasing approximate unit for $\mathcal{C}_A$ consisting of projections. We shall use and refer to this approximate unit several times in the following.

It is well-known that the canonical action of Γ on $\beta\Gamma$ is free, and it was shown by Ozawa (see [6, Theorem 5.1.6]) that this action is amenable whenever Γ is exact. The two lemmas below tell us how these properties can be preserved after passing to a suitable separable sub-C^*-algebra of $\mathcal{C}_A$. If $\mathcal{A}$ is an abelian C^*-algebra, then let $\widehat{\mathcal{A}}$ denote its space of characters, so that $\mathcal{A} \cong C_0(\widehat{\mathcal{A}})$.

It was remarked in [17] that if a group Γ acts on a compact Hausdorff space X, and if there are projections $q_j^{(t)}$ in $C(X)$, for $t \in \Gamma \setminus \{e\}$ and for $j = 1,2,3$, such that

$$\alpha_t(q_j^{(t)}) \perp q_j^{(t)}, \qquad q_1^{(t)} + q_2^{(t)} + q_3^{(t)} = 1, \tag{6.1}$$

for all $e \neq t \in \Gamma$ and $j = 1,2,3$, then Γ acts freely on X. (As below Definition 4.4, $t \mapsto \alpha_t$ denotes the induced action on $C(X)$ given by $\alpha_t(f)(x) = f(t^{-1}.x)$.) Consider now the case where Γ acts on a *non-compact* locally compact Hausdorff space X and where $C_0(X)$ has an increasing approximate unit $\{p_n\}_{n=1}^\infty$ consisting of projections. It is then easy to see that Γ acts freely on X if there are projections $q_{j,n}^{(t)}$ in $C_0(X)$, for $t \in \Gamma \setminus \{e\}$, $j = 1,2,3$ and $n \in \mathbb{N}$, such that

$$\alpha_t(q_{j,n}^{(t)}) \perp q_{j,n}^{(t)}, \qquad q_{1,n}^{(t)} + q_{2,n}^{(t)} + q_{3,n}^{(t)} = p_n, \tag{6.2}$$

holds for all $e \neq t \in \Gamma$, $j = 1,2,3$ and $n \in \mathbb{N}$.

Lemma 6.1 (cf. [17, Lemma 6.3]). *There is a countable subset M' of $\mathcal{C}_A$ such that if $\mathcal{A}$ is any Γ-invariant sub-C^*-algebra of $\mathcal{C}_A$ which contains M', then $\{p_n\}_{n=1}^\infty$ (defined above) is an approximate unit for $\mathcal{A}$, and Γ acts freely on the character space $\widehat{\mathcal{A}}$ of A.*

Proof. By [17, Corollary 6.2] we can find projections $q_j^{(t)}$ in $\ell^\infty(\Gamma)$ for $e \neq t \in \Gamma$ and $j = 1,2,3$, such that (6.1) holds. The projections

$$q_{j,n}^{(t)} = q_j^{(t)} p_n, \qquad e \neq t \in \Gamma, \quad j = 1,2,3, \quad n \in \mathbb{N},$$

satisfy (6.2). As p_n belongs to $\mathcal{C}_A$ for each n, so does each $q_{j,n}^{(t)}$. Let

$$M' = \{q_{j,n}^{(t)} \mid t \in \Gamma,\ n \in \mathbb{N},\ j = 1,2,3\} \subseteq \mathcal{C}_A,$$

and let $\mathcal{A}$ be any Γ-invariant sub-C^*-algebra of $\mathcal{C}_A$ that contains M'. Then each p_n belongs to $\mathcal{A}$ by (6.2). Moreover, $\{p_n\}_{n=1}^\infty$ is an approximate unit for $\mathcal{A}$ because it is an approximate unit for $\mathcal{C}_A$. Finally, Γ acts freely on $\widehat{\mathcal{A}}$ because (6.2) holds. □

Lemma 6.2. *Suppose that Γ is an exact countable group. Then there is a countable subset M'' of $\mathcal{C}_A$ such that if $\mathcal{A}$ is any Γ-invariant sub-C^*-algebra of $\mathcal{C}_A$ which contains M'', then Γ acts amenably on $\widehat{\mathcal{A}}$.*

Proof. The action of Γ on $\beta\Gamma$ is amenable, cf. [6, Theorem 5.1.6]. Let $(m_i)_{i\in I}$ be a net of approximately invariant continuous means $m_i\colon \beta\Gamma \to \mathrm{Prob}(\Gamma)$ that witnesses the amenability of this action, cf. Definition 2.1 in [1]. Let $\widetilde{m}_i$ be the restriction of m_i to $X_A \subseteq \beta\Gamma$. Then $(\widetilde{m}_i)_{i\in I}$ is a net of approximate invariant continuous means $\widetilde{m}_i\colon X_A \to \mathrm{Prob}(\Gamma)$ that witnesses the amenability of the action of Γ on X_A. As Γ is countable and X_A is σ-compact, the approximate mean $(\widetilde{m}_i)_{i\in I}$ can be chosen to be countable.

The spectrum $Y := \widehat{\mathcal{A}}$ of a Γ-invariant sub-C^*-algebra $\mathcal{A}$ of $\mathcal{C}_A$ is a Γ-invariant quotient of the space $X_A = \widehat{\mathcal{C}_A}$. We wish to make sure that each of the functions $\widetilde{m}_i$ passes to this quotient to yield a continuous function $\widehat{m}_i\colon Y \to \mathrm{Prob}(\Gamma)$, in which case the net $(\widehat{m}_i)_{\in I}$ will witness that Γ acts amenably on Y. This will happen if for all $x, y \in X_A$, for which $\widetilde{m}_i^x \neq \widetilde{m}_i^y$ for some i, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

For each i and for each $t \in \Gamma$ let $f_{i,t}\colon X_A \to \mathbb{R}$ be the continuous function given by $f_{i,t}(x) = \widetilde{m}_i^x(\{t\})$, $x \in X_A$. For each $s \in \Gamma$ let $f_{i,t,s}$ be the restriction of $f_{i,t}$ to the compact-open subset $s.K_A$ of X_A. The countable set

$$M'' = \{f_{i,t,s} \mid i \in I,\ s,t \in \Gamma\},$$

will then separate any pair of points $x, y \in X_A$ that are separated by the $\widetilde{m}_i$'s. □

Recall the definition of paradoxicality of projections from Section 4.

Lemma 6.3 (cf. [17, Lemma 6.6]). *For each projection $p \in \mathcal{C}_A$, which is $(\ell^\infty(\Gamma), \Gamma)$-paradoxical, there is a finite set $M_p \subseteq \mathcal{C}_A$ such that whenever $\mathcal{A}$ is a Γ-invariant sub-C^*-algebra of $\mathcal{C}_A$ which contains $M_p \cup \{p\}$, then p is $(\mathcal{A}, \Gamma)$-paradoxical.*

Proof. Note first that if p in $\mathcal{C}_A$ is $(\ell^\infty(\Gamma), \Gamma)$-paradoxical, then it is also $(\mathcal{C}_A, \Gamma)$-paradoxical. Indeed, the projections $p_j \in \ell^\infty(\Gamma)$ that witness that p is $(\ell^\infty(\Gamma), \Gamma)$-paradoxical, cf. (4.2), will automatically belong to the closed ideal $\mathcal{C}_A$ in $\ell^\infty(\Gamma)$. Accordingly we can let M_p consist of the finitely many projections p_j from (4.2). □

Lemma 6.4. *Each Γ-full projection in $\mathcal{C}_A$ is $(\ell^\infty(\Gamma), \Gamma)$-paradoxical if $A \subseteq \Gamma$ is paradoxical.*

Proof. Take a Γ-full projection in $\mathcal{C}_A \subseteq \ell^\infty(\Gamma)$ and write it as 1_B where $B \subseteq \Gamma$. Identify 1_B with the Γ-full projection 1_{K_B} in $C_0(X_A)$. Then K_B is Γ-full in X_A, and so $A \propto_\Gamma B \propto_\Gamma A$ by Lemma 4.9. Hence B is paradoxical by Lemma 2.5(iii), so 1_B is a paradoxical projection, cf. claim (c) below Definition 4.4. □

Lemma 6.5 (cf. [17, Lemma 6.7]). *Let T be a countable subset of $\mathcal{C}_A$. Then there is a countable Γ-invariant set Q consisting of projections in $\mathcal{C}_A$ such that $T \subseteq C^*(Q)$.*

Proof. The proof is identical with the proof of [17, Lemma 6.7]. Use that $\mathcal{C}_A$ is of real rank zero (being a closed two-sided ideal in the real rank zero C^*-algebra $\ell^\infty(\Gamma)$). □

If the group Γ acts on a C^*-algebra $\mathcal{A}$ and if $p \in \mathcal{A}$ is a projection, then we say that p is Γ-*full* if p is not contained in any proper closed two-sided Γ-invariant ideal in $\mathcal{A}$.
In the rest of this section we let $t \mapsto \alpha_t$, $t \in \Gamma$, denote the canonical actions of Γ on $\ell^\infty(\Gamma)$, respectively, on $C(\beta\Gamma)$, as well as on all their Γ-invariant sub-C^*-algebras.

Proposition 6.6. *Let Γ be a non-supramenable group, and let A be a subset of Γ. As above, let $\mathcal{C}_A$ be the smallest closed Γ-invariant ideal in $\ell^\infty(\Gamma)$ that contains 1_A. It follows that there is a separable, Γ-invariant sub-C^*-algebra $\mathcal{A}$ of $\mathcal{C}_A$ such that*

(i) *$1_A \in \ell^\infty(\Gamma)$ is a Γ-full projection in $\mathcal{A}$.*
(ii) *$\mathcal{A}$ is generated as a C^*-algebra by its projections.*
(iii) *Every projection in $\mathcal{A}$, which is $(\ell^\infty(\Gamma), \Gamma)$-paradoxical, is also $(\mathcal{A}, \Gamma)$-paradoxical,*
(iv) *Γ acts freely on the character space $\widehat{\mathcal{A}}$ of A.*

Furthermore, if Γ is exact, then $\mathcal{A}$ can be chosen such that Γ acts amenably on $\widehat{\mathcal{A}}$.

Proof. If Γ is exact, then let $M = M' \cup M'' \cup \{1_A\}$, and let otherwise $M = M' \cup \{1_A\}$, where M' and M'' are as in Lemma 6.1 and Lemma 6.2, respectively. Suppose that $\mathcal{A}$ is a Γ-invariant (separable) sub-C^*-algebra of $\mathcal{C}_A$ which contains M. Then Γ acts freely on $\widehat{\mathcal{A}}$, the approximate unit $\{p_n\}_{n=1}^\infty$ defined above Lemma 6.1 is contained in and is an approximate unit for $\mathcal{A}$ by Lemma 6.1; and if Γ is exact, then Γ acts amenably on $\widehat{\mathcal{A}}$ by Lemma 6.2.
Let us show that 1_A is a Γ-full projection in any such C^*-algebra $\mathcal{A}$. Let $\mathcal{I}$ be the smallest closed Γ-invariant ideal in $\mathcal{A}$ which contains 1_A. Then $1_{tA} = \alpha_t(1_A) \in \mathcal{I}$ for all $t \in \Gamma$. Since $p_n = 1_{B_n}$ and $B_n = \bigcup_{t \in F_n} tA$ for some finite subset F_n of Γ, we see that $p_n \leq \sum_{t \in F_n} 1_{tA} \in \mathcal{I}$, so $p_n \in \mathcal{I}$ for all n. As $\{p_n\}_{n=1}^\infty$ is an approximate unit for $\mathcal{A}$ we conclude that $\mathcal{I} = \mathcal{A}$, so 1_A is Γ-full in $\mathcal{A}$.
Construct inductively countable Γ-invariant sets of projections

$$Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \cdots \subseteq \mathcal{C}_A, \qquad P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq \mathcal{C}_A,$$

such that $M \subseteq C^*(Q_0)$, P_n is the set of projections in $C^*(Q_n)$, and such that each projection in P_n, which is $(\ell^\infty(\Gamma), \Gamma)$-paradoxical, is $(C^*(Q_{n+1}), \Gamma)$-paradoxical. The set of projections in any separable abelian C^*-algebra is countable. Therefore P_n is countable if Q_n is countable.
The existence of Q_0 follows from Lemma 6.5. Suppose that $n \geq 0$ and $Q_n, P_n \subseteq \mathcal{C}_A$ have been found. The existence of Q_{n+1} then follows from Lemma 6.3 and Lemma 6.5.
Put

$$P_\infty = \bigcup_{n=1}^{\infty} P_n, \qquad \mathcal{A} = C^*(P_\infty).$$

Then $\mathcal{A}$ is a Γ-invariant separable sub-C^*-algebra of $\mathcal{C}_A$ which contains M. By definition, $\mathcal{A}$ is generated by its projections, so (ii) holds.
Observe that each projection in $\mathcal{A}$ actually belongs to P_∞. Indeed, if p is a projection in $\mathcal{A}$, then, for some n, there is a projection $p' \in C^*(P_n)$ such that $\|p - p'\| < 1$. Since $\mathcal{A}$ is commutative this entails that $p = p'$. Each projection in $C^*(P_n)$ belongs to P_{n+1}, so $p \in P_{n+1}$.
Let p be a projection in $\mathcal{A}$ which is $(\ell^\infty(\Gamma), \Gamma)$-paradoxical. Then $p \in P_n$ for some n. It follows that p is $(C^*(Q_{n+1}), \Gamma)$-paradoxical and hence also $(\mathcal{A}, \Gamma)$-paradoxical. Thus (iii) holds. □

Recall the definition of the relation $A \ll \Gamma$ from Definition 3.5.

Lemma 6.7. *Let Γ be a countable group and let A be a non-empty subset of Γ such that $A \ll \Gamma$. Let $\mathcal{A} \subseteq \mathcal{C}_A$ be as in Proposition 6.6. Then $\mathcal{A}$ is non-unital, and so is $\mathcal{A}/\mathcal{I}$ for every proper closed Γ-invariant ideal $\mathcal{I}$ in $\mathcal{A}$.*

Proof. As in the beginning of this section (and as in the proof of Proposition 6.6 above) let $\{p_n\}$ be the approximate unit for $\mathcal{C}_A$ and for $\mathcal{A}$, given by $p_n = 1_{B_n}$, where $B_n = \bigcup_{t \in F_n} tA$ for some increasing sequence, $\{F_n\}$, of finite subsets of Γ whose union is Γ.
Let $\mathcal{I}$ be a Γ-invariant closed two-sided ideal in $\mathcal{A}$ and suppose that $\mathcal{A}/\mathcal{I}$ is unital. Then $p_n + \mathcal{I}$ is a unit for $\mathcal{A}/\mathcal{I}$ for some n. Now, $A \precsim_\Gamma \Gamma \setminus B_n$ by the assumption that $A \ll \Gamma$. Hence there is a partition $A_1, \dots, A_k$ of A and elements $t_1, \dots, t_k$ in Γ such that $\{t_j A_j\}$ are pairwise disjoint subsets of $\Gamma \setminus B_n$. For each j,

$$0 = (1_{t_j A_j} + \mathcal{I})(1_{B_n} + \mathcal{I}) = (1_{t_j A_j} + \mathcal{I})(p_n + \mathcal{I}) = 1_{t_j A_j} + \mathcal{I}.$$

Hence $1_{t_j A_j}$ belongs to $\mathcal{I}$ for all j, so $1_A = \sum_{j=1}^n \alpha_{t_j^{-1}}(1_{t_j A_j})$ belongs to $\mathcal{I}$. Finally, because 1_A is a Γ-full projection in $\mathcal{A}$, we must have $\mathcal{I} = \mathcal{A}$. □

By the *locally compact non-compact Cantor set* $\mathbf{K}^*$ we shall mean the unique (up to homeomorphism) locally compact, non-compact, totally disconnected second countable Hausdorff space that has no isolated points. This set arises, for example, from the usual (compact) Cantor set $\mathbf{K}$ by removing one point. One can also realise it as the product $\mathbb{N} \times \mathbf{K}$, as the local field $\mathbb{Q}_p$, etc.

Proof of Theorem 1.2. The "only if" part follows from Proposition 2.7 (and Lemma 4.5). The conclusions about the crossed product $C_0(\mathbf{K}^*) \rtimes_{\mathrm{red}} \Gamma$ follow from Proposition 4.2 and Lemma 4.5, respectively, from Proposition 4.6.

Suppose that Γ is non-supramenable and let A be a non-empty paradoxical subset of Γ. If Γ is non-amenable, then choose A such that $A \ll \Gamma$, cf. Lemma 3.7. Let X_A and $\mathcal{C}_A$ be as defined in the beginning of this section, and let $\mathcal{A}$ be as in Proposition 6.6.

The C^*-algebra $\mathcal{A}$ contains the projection 1_A, and this projection is furthermore Γ-full in $\mathcal{A}$ by Proposition 6.6. It follows that $\mathcal{A}$ contains a maximal proper closed Γ-invariant ideal $\mathcal{I}$ (cf. Remark 5.2). Put $\mathcal{B} = \mathcal{A}/\mathcal{I}$. Let

$$X = \widehat{\mathcal{B}} \qquad \text{and} \qquad Y = \widehat{\mathcal{A}}.$$

Then $\mathcal{B} \cong C_0(X)$, $\mathcal{A} \cong C_0(Y)$, and X is a closed Γ-invariant subset of Y. We know from Proposition 6.6 that Γ acts freely on Y, and the action is moreover amenable if Γ is exact. These properties pass to the subset X, so the action of Γ on X is free, amenable (if Γ is exact), and minimal (by maximality of $\mathcal{I}$).

We know from Proposition 6.6 that $\mathcal{A}$ is generated by its projections. Hence $\mathcal{B}$ is also generated by its projections. Hence X and Y are totally disconnected. As $\mathcal{A}$ and $\mathcal{B}$ are separable we conclude that X and Y are second countable. We show below that the action of Γ on X is purely infinite. This clearly will imply that X has no isolated points (an isolated point is compact-open and of course never paradoxical). If Γ is non-amenable, then $A \ll \Gamma$ which by Lemma 6.7 entails that $\mathcal{B}$ is non-unital, whence X is non-compact. We arrive at the same conclusion (that X is non-compact) if Γ is amenable, because no action of an amenable group on a compact Hausdorff space can be purely infinite. We can therefore conclude that X is homeomorphic to $\mathbf{K}^*$.

We finally show that the action of Γ on X is purely infinite. Let K be a non-empty compact-open subset of X. Then K is Γ-full in X because X is a minimal Γ-space. Hence there exists a Γ-full compact-open subset K' of Y such that $K = K' \cap X$, cf. Lemma 4.8(i). Let $p \in \mathcal{A}$ be the projection that corresponds to the projection $1_{K'} \in C_0(Y)$. Then p is Γ-full in $\mathcal{A}$ and therefore also Γ-full in $\mathcal{C}_A$. Hence p is $(\ell^\infty(\Gamma), \Gamma)$-paradoxical because A is paradoxical, cf. Lemma 6.4. By the construction of $\mathcal{A}$, cf. Proposition 6.6, it follows that p is $(\mathcal{A}, \Gamma)$-paradoxical. Hence $1_{K'}$ is $(C_0(Y), \Gamma)$-paradoxical, so K' is $(Y, \Gamma, \mathbb{K})$-paradoxical, cf. claim (c) (below Definition 4.4). By Lemma 4.8(ii) this implies that K is $(X, \Gamma, \mathbb{K})$-paradoxical. □

7. AMENABLE ACTIONS OF NON-EXACT GROUPS ON LOCALLY COMPACT HAUSDORFF SPACES

It is well-known that only exact (discrete) group can act amenably on a compact Hausdorff space, cf. [6, Theorem 5.1.7]. We also know that any group (exact or not) admits an amenable action on some locally compact Hausdorff space, for example on the group itself. We shall here address the issue of when a non-exact group admits amenable actions on the locally compact non-compact

Cantor set $\mathbf{K}^*$, and whether the second part of Theorem 1.2 can be extended to all countable non-supramenable groups. The goal is to prove Theorem 1.3.

We first recall a notion of induced action. Let Γ_0 be a subgroup of a countable group Γ, and suppose that Γ_0 acts on a locally compact Hausdorff space X. Then Γ acts by left-multiplication on the first coordinate of $\Gamma \times X$, and Γ_0 acts on $\Gamma \times X$ by $s.(t,x) = (ts^{-1}, s.x)$, for $s \in \Gamma_0$, $(t,x) \in \Gamma \times X$. These two actions commute and thus Γ acts on $Y = (\Gamma \times X)/\Gamma_0$. This action is called the *induced* action. We record some facts about this construction:

Lemma 7.1. *Let $\Gamma_0 \curvearrowright X$ and $\Gamma \curvearrowright Y$ be as above. Then:*

(i) *Y is homeomorphic to $X \times (\Gamma/\Gamma_0)$.*
(ii) *If $X \cong \mathbf{K}$, then $Y \cong \mathbf{K}$ or $Y \cong \mathbf{K}^*$ according to whether $|\Gamma : \Gamma_0|$ is finite or infinite.*
(iii) *If $X \cong \mathbf{K}^*$, then $Y \cong \mathbf{K}^*$.*
(iv) *If $\Gamma_0 \curvearrowright X$ is free, then so is $\Gamma \curvearrowright Y$.*
(v) *If $\Gamma_0 \curvearrowright X$ is minimal, then so is $\Gamma \curvearrowright Y$.*
(vi) *If $\Gamma_0 \curvearrowright X$ is amenable, then so is $\Gamma \curvearrowright Y$.*
(vii) *If X admits a non-zero Γ_0-invariant Radon measure, then Y admits a non-zero Γ-invariant Radon measure.*
(viii) *If $\Gamma_0 \curvearrowright X$ is purely infinite, then so is $\Gamma \curvearrowright Y$.*

Proof. Let $\pi\colon \Gamma \times X \to Y$ denote the quotient mapping.
(i). Write $\Gamma = \bigcup_{\alpha \in I} t_\alpha \Gamma_0$ as a disjoint union of left cosets and put $A = \{t_\alpha \mid \alpha \in I\}$. Then $A \times X$ is a clopen subset of $\Gamma \times X$ which is a transversal for the action of Γ_0 on $\Gamma \times X$. The restriction of π to $A \times X$ therefore defines a homeomorphism $A \times X \to Y$.
(ii) and (iii) follow from (i).
(iv). Suppose that $t.\pi(s,x) = \pi(s,x)$ for some $t \in \Gamma$ and some $(s,x) \in \Gamma \times X$. Then there is $r \in \Gamma_0$ such that $(ts, x) = (sr^{-1}, r.x)$. As Γ_0 acts freely on X this implies that $r = e$, so $t = e$.
(v). Each $\Gamma \times \Gamma_0$ orbit on $\Gamma \times X$ is dense, if Γ_0 acts minimally on X. Hence each Γ orbit on Y is dense.
(vi). Let $m_i\colon X \to \mathrm{Prob}(\Gamma_0)$, $i \in I$, be approximate invariant continuous means that witness the amenability of the action of Γ_0 on X, cf. [1]. For each $m \in \mathrm{Prob}(\Gamma_0)$ and for each $t \in \Gamma$ define $m_t \in \mathrm{Prob}(\Gamma)$ by $m_t(E) = m(t^{-1}E \cap \Gamma_0)$, for $E \subseteq \Gamma$. Identify Y with $A \times X$ as in (i), and define

$$\widetilde{m}_i\colon A \times X \to \mathrm{Prob}(\Gamma) \quad \text{by} \quad \widetilde{m}_i^{(t_\alpha, x)} = (m_i^x)_{t_\alpha}, \quad x \in X,\ \alpha \in I.$$

Let $s \in \Gamma$ and $(t_\alpha, x) \in A \times X$ be given. Then $st_\alpha = t_\beta r$ for some (unique) $\beta \in I$ and $r \in \Gamma_0$, and $s.(t_\alpha, x) = (t_\beta, r.x)$. Let $E \subseteq \Gamma$. Then

$$\begin{aligned} \widetilde{m}_i^{s.(t_\alpha,x)}(E) &= \widetilde{m}_i^{(t_\beta, r.x)}(E) \;=\; m_i^{r.x}\big(t_\beta^{-1}E \cap \Gamma_0\big) \\ &= m_i^{r.x}\big(rt_\alpha^{-1}s^{-1}E \cap \Gamma_0\big). \end{aligned}$$

The latter expression is close to $m_i^x\big(t_\alpha^{-1}s^{-1}E\cap\Gamma_0\big)$ when i is large by approximate invariance of (m_i), and

$$s.\widetilde{m}_i^{(t_\alpha,x)}(E) \;=\; \widetilde{m}_i^{(t_\alpha,x)}(s^{-1}E) \;=\; m_i^x\big(t_\alpha^{-1}s^{-1}E\cap\Gamma_0\big).$$

This shows that $(\widetilde{m}_i)_i$ is an approximate invariant continuous mean.
(vii). Identify again Y with $A\times X$ as in (i). Let λ be a non-zero Γ_0-invariant Radon measure on X and let μ be counting measure on A. Then $\mu\otimes\lambda$ is a Γ-invariant non-zero Radon measure on $A\times X$.
(viii). If F and F' are disjoint compact-open subsets of Y both of which are (Y,Γ)-paradoxical, then $F\cup F'$ is also (Y,Γ)-paradoxical. We need therefore only show that $F=\pi(\{t\}\times E)$ is (Y,Γ)-paradoxical for each $t\in\Gamma$ and for each compact-open subset E of X. As E is (X,Γ_0)-paradoxical there are pairwise disjoint compact-open subsets $E_1,E_2,\dots,E_{n+m}$ of X and elements $t_1,t_2,\dots,t_{n+m}\in\Gamma_0$ such that (4.1) holds. Put $F_j=\pi(\{t\}\times E_j)$ and let $s_j\in\Gamma$ be such that $s_jt=tt_j$. Then $F_1,F_2,\dots,F_{n+m}$ are pairwise disjoint compact-open subsets of F. As

$$s_j.\pi(t,x)=\pi(s_jt,x)=\pi(tt_j,x)=\pi(t,t_j.x),$$

we see that $s_j.F_j=\pi(\{t\}\times t_j.E_j)$. This shows that

$$F=\bigcup_{j=1}^{n}s_j.F_j=\bigcup_{j=n+1}^{n+m}s_j.F_j,$$

and hence that F is (Y,Γ)-paradoxical. □

Proof of Theorem 1.3. In each case we have a subgroup Γ_0 of Γ which admits a free minimal amenable action on X, where $X=\mathbf{K}$ or $X=\mathbf{K}^*$. This action induces a free minimal amenable action of Γ on Y, where $Y=\mathbf{K}$ or $Y=\mathbf{K}^*$ (according to the conclusions of (ii) and (iii) in Lemma 7.1).
(i). Here Γ_0 acts on $X=\mathbf{K}$ in the prescribed way by [11] or by [17]. The former reference gives a free minimal action on $\mathbf{K}$ for every infinite Γ_0, and this action will be amenable if Γ_0 is amenable. The latter reference gives a free minimal amenable action on $\mathbf{K}$, whenever Γ_0 is exact and non-amenable. We conclude by inducing the action and recall that a non-exact group cannot act amenably on a compact set. If $|\Gamma:\Gamma_0|=\infty$, then $Y=\mathbf{K}^*$, see Lemma 7.1.
(ii). It is known that $\Gamma_0=\mathbb{Z}$ has many interesting minimal actions on $X=\mathbf{K}^*$, see [8]. These actions are necessarily free since every non-trivial subgroup of $\mathbb{Z}$ has finite index. Pick such an action; it is of course amenable since $\mathbb{Z}$ is amenable. Therefore it induces the desired action of Γ on $Y=\mathbf{K}^*$. If Γ_0 is infinite and amenable, then Γ_0 has a free minimal action on $X=\mathbf{K}$ by [11], and this action is necessarily amenable. Again, if $|\Gamma:\Gamma_0|=\infty$, then $Y=\mathbf{K}^*$.
(iii). Here Γ_0 is exact and non-supramenable. By Theorem 1.2 there is a free minimal amenable purely infinite action of Γ_0 on $X=\mathbf{K}^*$. This action induces a free minimal amenable purely infinite action of Γ on $Y=\mathbf{K}^*$. □

Questions 7.2. Let Γ be a countably infinite group.

(i) Does Γ admit a free minimal amenable action on $\mathbf{K}^*$?
In that case, $C_0(\mathbf{K}^*) \rtimes_{\rm red} \Gamma$ is a (non-unital) simple nuclear separable C^*-algebra in the UCT class.

(ii) Does Γ admit a free minimal amenable action on $\mathbf{K}^*$ which leaves invariant a non-zero Radon measure on $\mathbf{K}^*$?
In that case, $C_0(\mathbf{K}^*) \rtimes_{\rm red} \Gamma$ is a (non-unital) simple nuclear separable C^*-algebra in the UCT class with a densely defined trace (hence it is stably finite).

(iii) Does Γ admit a free minimal amenable purely infinite action on $\mathbf{K}^*$ if Γ is non-supramenable?
In that case, $C_0(\mathbf{K}^*) \rtimes_{\rm red} \Gamma$ is a stable Kirchberg algebra in the UCT class.

It follows from Theorem 1.3 that Question (i) has an affirmative answer if Γ contains an element of infinite order, or if Γ contains an infinite exact subgroup of infinite index, or if Γ contains an infinite exact subgroup and Γ itself is non-exact. Question (ii) has an affirmative answer if Γ contains an element of infinite order, or if Γ contains an infinite amenable subgroup of infinite index. Question (iii) has an affirmative answer if Γ contains an exact non-supramenable subgroup.

On the other hand, as for Question (i), we do not even know if every countable *amenable* group Γ admits a free minimal action on $\mathbf{K}^*$. If there is an (infinite) set $A \subseteq \Gamma$ such that $A \lll^* \Gamma$ and such that some minimal Γ-subspace of X_A is non-discrete, then the construction in Proposition 5.5 and Section 6 will give such an action. Example 5.6 shows that the existence of a non-discrete minimal Γ-subspace of X_A is not guaranteed, even if A is infinite.[3]

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[3]Added in proof: It will be shown in a subsequent paper by the third named author and H. Matui that each infinite countable group Γ indeed does contain a subset A with these properties.

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Julian Kellerhals
EPFL
SB-EGG
Station 8
1015 Lausanne
Switzerland
julian.kellerhals@epfl.ch

Nicolas Monod
EPFL
SB-EGG
Station 8
1015 Lausanne
Switzerland
nicolas.monod@epfl.ch

Mikael Rørdam
Department of Mathematics
University of Copenhagen
Universitetsparken 5
DK-2100
Copenhagen Ø
Denmark
rordam@math.ku.dk