Invariants of Upper Motives

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Abstract. Let H be a homology theory for algebraic varieties over a field k. To a complete k-variety X, one naturally attaches an ideal $H_X(k)$ of the coefficient ring $H(k)$. We show that, when X is regular, this ideal depends only on the upper Chow motive of X. This generalises the classical results asserting that this ideal is a birational invariant of smooth varieties for particular choices of H, such as the Chow group. When H is the Grothendieck group of coherent sheaves, we obtain a lower bound on the canonical dimension of varieties. When H is the algebraic cobordism, we give a new proof of a theorem of Levine and Morel. Finally we discuss some splitting properties of geometrically unirational field extensions of small transcendence degree.

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1. Introduction

The canonical dimension of a smooth complete algebraic variety measures to which extent it can be rationally compressed. In order to compute it, one usually studies the p-local version of this notion, called canonical p-dimension (p is a prime number). In this paper, we consider the relation of p-equivalence [KM13, §3] between complete varieties, constructed so that p-equivalent varieties have the same canonical p-dimension. This essentially corresponds to the relation of having the same upper motive with $\mathbb{F}_p$-coefficients (see Remark 2.2). For example two complete varieties X and Y are p-equivalent, for any p, as soon as there are rational maps $X \dashrightarrow Y$ and $Y \dashrightarrow X$. In order to obtain restrictions on the possible values of the canonical p-dimension of a variety, one is naturally led to study invariants of p-equivalence. We give a systematic way to produce such invariants (and in particular, birational invariants), starting from a homology theory. We provide examples related to K-theory and cycle

modules. We then describe the relation between two such invariants of a complete variety X: its index n_X, and the integer d_X defined as the g.c.d. of the Euler characteristics of the coherent sheaves of $\mathcal{O}_X$-modules. The latter invariant contains both arithmetic and geometric informations; this can be used to give bounds on the possible values of the index n_X (an arithmetic invariant) in terms of the geometry of X. For instance, a smooth, complete, geometrically rational (or merely geometrically rationally connected, when k has characteristic zero) variety of dimension $< p-1$ always has a closed point of degree prime to p. Some consequences of this statement are given in Corollary 7.9.
An immediate consequence of the Hirzebruch-Riemann-Roch theorem is that, for a smooth projective variety X and a prime number p, we have

$$\dim X \geq (p-1)(v_p(n_X) - v_p(\chi(X, \mathcal{O}_X)),$$

(we denote by $v_p(m)$ the p-adic valuation of the integer m). When the characteristic of the base field is different from p, a result of Zainoulline states that X is incompressible in case of equality (this follows by taking p-adic valuations in [Zai10, Corollary (A)]). Here we improve this result, and obtain in Proposition 5.4 the following bound on the canonical p-dimension of a regular complete variety X over a field characteristic not p (and also a weaker statement in characteristic p)

$$\operatorname{cdim}_p X \geq (p-1)(v_p(n_X) - v_p(d_X)).$$

We always obtain some bound on the canonical p-dimension, even when it is not equal to the dimension. By contrast, the result of [Zai10], or more generally any approach based on the degree formula (see [Mer03]), does not directly say anything about the canonical dimension of varieties which are not p-incompressible. One can sometimes circumvent this problem by exhibiting a smooth complete variety Y of dimension $\operatorname{cdim}_p(X)$ which is p-equivalent to X, and use the degree formula to prove that Y is p-incompressible (see [Mer03, §7.3] for the case of quadrics). But this requires to explicitly produce such a variety Y, and to find an appropriate characteristic number for Y.
Let us mention that the bound above tends to be sharp only when $\operatorname{cdim}_p(X)$ is not too large, compared with p (see Example 6.7).

Another aspect of the technique presented here concerns its application to the algebraic cobordism Ω of Levine and Morel. Their construction uses two distinct results known in characteristic zero:

(a) the resolution of singularities [Hir64], and
(b) the weak factorisation theorem [AKMW02, Wło03].

Some care is taken in the book [LM07] to keep track of which result uses merely (a), or the combination of (a) and (b); most deeper results actually use both. In [Hau12b], we proved that the existence of a weak form of Steenrod operations (which may be considered as a consequence of the existence of algebraic cobordism) only uses (a). Moreover, we proved that it suffices, for the purpose of the construction of these operations, to have a p-local version of resolution of singularities, which has been recently obtained in characteristic different from p by Gabber. By contrast, it is not clear what would be a p-local version of the weak factorisation theorem. In the present paper, we extend the list of results using only (a) by proving the theorem below, whose original proof in [LM07, Theorem 4.4.17] was based on (b) (and (a)). Our approach moreover allows us to give in Proposition 4.8 a p-local version of this statement, in terms of p-equivalence.

Theorem. *Assume that the base field k admits resolution of singularities. The ideal $M(X)$ of $\Omega(k)$ generated by classes of smooth projective varieties Y, of dimension $< \dim X$, and admitting a morphism $Y \to X$, is a birational invariant of a smooth projective variety X.*

The structure of the paper is as follows. In Section 2 we define the notion of p-equivalence, and describe its relation with canonical p-dimension. In Section 3 we introduce a (non-exhaustive) set of conditions on a pair of functors which are expected to be satisfied by a pair homology/cohomology of algebraic varieties. We verify these conditions for K-theory, cycles modules, and algebraic cobordism. In Section 4 we explain how to construct an invariant of p-equivalence classes starting from any pair satisfying the conditions of Section 3. We discuss each of the three situations mentioned above. In Section 5 we provide a relation between n_X and d_X, which are invariants produced by the method of Section 3. This yields the lower bound for canonical dimension. In Section 6, we provide examples, and compute the lower bound in some specific situations. In Section 7 we adopt the point of view of function fields, and consider splitting properties of geometrically unirational field extensions.

2. p-equivalence

We denote by k a fixed base field. A variety will be an integral, separated, finite type scheme over $\operatorname{Spec} k$. The function field of a variety X will be denoted by $k(X)$. When K is a field containing k, we also denote by K its spectrum, and for a variety X we write X_K for $X \times_k K$. The letter p will always denote a prime number.

A prime correspondence, or simply a *correspondence*, $Y \rightsquigarrow X$ is a diagram of varieties and proper morphisms $Y \leftarrow Z \to X$, where the map $Y \to Z$ is generically finite. The degree of this map is the *multiplicity* of the correspondence. Following [KM13, §3], we say that two varieties X and Y are *p-equivalent* if there are correspondences $Y \rightsquigarrow X$ and $X \rightsquigarrow Y$ of multiplicities prime to p. It is equivalent to require that each of the schemes $X_{k(Y)}$ and $Y_{k(X)}$ have a closed point of degree prime to p.
Note that a rational map $Y \dashrightarrow X$ between complete varieties gives rise to a correspondence $Y \rightsquigarrow X$ of multiplicity 1 (by taking for Z the closure in $X \times_k Y$ of the graph of the rational map). Thus two complete varieties X and Y are p-equivalent, for all p, as soon as there are rational maps $Y \dashrightarrow X$ and $X \dashrightarrow Y$. Therefore we can use p-equivalence to find birational invariants (as

in Corollary 4.10), but of course birational equivalence is a much finer relation (see for example Remark 4.6).
We will in general restrict our attention to complete regular varieties. In this case the relation of p-equivalence becomes transitive (this can be proved using [KM06, Lemma 3.2]). A consequence of a theorem of Gabber [ILO, X, Theorem 2.1] is that, when the characteristic of k is not p, any complete variety is p-equivalent to a projective regular variety (of the same dimension).

Let X be a complete regular variety. Its canonical p-dimension $\operatorname{cdim}_p(X)$ can be defined as the least dimension of a closed subvariety $Z \subset X$ admitting a correspondence $X \rightsquigarrow Z$ of multiplicity prime to p [KM06, Corollary 4.12].
It is proven in [KM13, Lemma 3.6] that smooth p-equivalent varieties have the same canonical p-dimension. A slight modification of the arguments used there, together with Gabber's theorem, yields the following statement.

PROPOSITION 2.1. *Assume that the characteristic of k is different from p. Let X be a complete regular variety. Then* $\operatorname{cdim}_p(X)$ *is the least dimension of a complete regular variety p-equivalent to X.*

REMARK 2.2 (Upper motives). Let X be a complete smooth variety. A summand of the Chow motive of X with $\mathbb{F}_p$-coefficients is called *upper* if it is defined by a projector of multiplicity $1 \in \mathbb{F}_p$ [Kar13, Definition 2.10].
If two smooth complete varieties have a common upper summand with $\mathbb{F}_p$-coefficients, then they are p-equivalent. The converse is true if the varieties are geometrically split and satisfy Rost nilpotence [Kar13, Corollary 2.15].

3. RING THEORIES AND MODULES

We denote by Ab the category of abelian groups. Let $\mathcal{V}$ be a full subcategory of the category of varieties and proper morphisms, and Reg a full subcategory of $\mathcal{V}$, which unless otherwise specified (in 3.2) will consist of the regular varieties in $\mathcal{V}$. Let R and H be two (covariant) functors $\mathcal{V} \to \mathsf{Ab}$. For a morphism f of $\mathcal{V}$, we denote by f_* either of the two corresponding morphisms in Ab. We consider the following conditions on R and H.

CONDITIONS 3.1. Let $X \in \mathcal{V}$ and $S \in \mathsf{Reg}$. Let $f\colon X \to S$ be a proper, dominant, generically finite morphism of degree d.

(R1) The group $R(S)$ has a structure of an associative ring with unit 1_S.
(R2) It is possible to find an element $u \in R(X)$ such that the element $f_*u - d \cdot 1_S$ is nilpotent in the ring $R(S)$.
(H1) There is a morphism of abelian groups
$$R(X) \otimes H(S) \to H(X) \quad ; \quad x \otimes s \mapsto x \cdot_f s.$$
(H2) The group $H(S)$ has a structure of a left $R(S)$-module such that
$$f_*(x \cdot_f s) = (f_*x) \cdot s.$$

We now describe some classical examples of such functors R and H. In all cases, R and H will correspond to cohomology theories, and satisfy additional properties which are logically irrelevant here. We tried to be provide minimal conditions making the proof of Lemma 4.1 below work.

3.1. Quillen K-theory. (See [Qui73, §7]) Let $\mathcal{V}$ be the category of varieties and proper morphisms, and let $X \in \mathcal{V}$. We let $R(X)$ be the Grothendieck group $K'_0(X)$ of the category of coherent sheaves of $\mathcal{O}_X$-modules, and $H(X)$ be the m-th K-group $K'_m(X)$ of this category.
We denote by $K_m(X)$ the m-th K-group of the category of locally free coherent sheaves on X. The tensor product induces a morphism

$$K_m(X) \otimes K'_0(X) \to K'_m(X) \quad ; \quad x \otimes y \mapsto x \cap y. \tag{1}$$

When S is regular, the map $- \cap [\mathcal{O}_S]$ induces an isomorphism

$$\varphi_S \colon K_m(S) \to K'_m(S). \tag{2}$$

With $m = 0$, the combination of (2) and (1) gives (R1) (here $1_S = [\mathcal{O}_S]$).
When $f \colon X \to S$ is a morphism, we have a morphism $f^* \colon K_m(S) \to K_m(X)$, and we can define

$$K'_0(X) \otimes K'_m(S) \to K'_m(X) \quad ; \quad x \otimes s \mapsto x \cdot_f s = f^* \circ \varphi_S^{-1}(s) \cap x,$$

proving (H1). Then (H2) follows from the projection formula.
We prove (R2) for $u = [\mathcal{O}_X]$. The element $x = f_* u - d \cdot [\mathcal{O}_S]$ belongs to the kernel of the restriction to the generic point morphism $K'_0(S) \to K'_0(k(S))$ (see e.g. [Hau13, Lemma 2.4]). This amounts to saying that its unique antecedent $y \in K_0(S)$ under (2) (with $m = 0$) has rank zero. Thus for any n, its n-th power y^n belongs to the n-th term of the gamma filtration. The image by (2) (with $m = 0$) of this term is contained in the n-th term of the topological filtration [SGA6, Exposé X, Corollaire 1.3.3]. The latter vanishes when $n > \dim S$, hence so does x^n.

3.2. Chow groups and cycle modules. Let us sketch how Chow groups and cycle modules can be made to fit into this framework, although the situation is somewhat degenerate. Let $\mathcal{V}$ be the category of varieties and proper morphisms, and R be the Chow group CH. The property (R2) is satisfied with $u = [X]$, since $f_*[X] = d \cdot [S]$.
Let M be a cycle module [Ros96, Definition 2.1]. We let $H(-) = A_*(-; M)$ be the Chow group with coefficients in M [Ros96, p.356].
When M is Quillen K-theory, the conditions (R1), (H1) and (H2) are verified in [Gil81, §8] using Bloch's formula.
Taking for Reg the subcategory of smooth varieties, (R1) is classical. When $X \to S$ is a morphism, with S a smooth variety, the pairing (H1)

$$\mathrm{CH}(X) \otimes A_*(S; M) \to A_*(X; M)$$

is defined by sending $x \otimes s$ to $g^*(x \times_k s)$. Here we use the cross product of [Ros96, §14], and g^* is the pull-back along the regular closed embedding

$g\colon X \to X \times_k S$ given by the graph of f, defined as the composite

$$A_*(X \times_k S; M) \xrightarrow{J(g)} A_*(N_g; M) \xrightarrow{(p^*)^{-1}} A_*(X; M),$$

where $J(g)$ is the deformation homomorphism [Ros96, §11] and p^* the isomorphism induced by the flat pull-back along the normal bundle $p\colon N_g = T_S \times_S X \to X$ to g [Ros96, Proposition 8.6]. Then (H2) is easily verified.

3.3. Algebraic cobordism. Assume that k admits resolution of singularities. In this paper, this will mean that k satisfies the conclusion of [LM07, Theorem A.1, p.233]. Note that it implies that k is perfect. Let $\mathcal{V}$ be the category of quasi-projective varieties and projective morphisms. Then Reg is the full subcategory of smooth quasi-projective varieties. We take for $R = H$ the algebraic cobordism Ω of [LM07]. We prove properties (R1), (H1), (H2), and, under the additional assumption that f is separable, we prove (R2).

Property (R1) is a classical property of Ω. We now prove (H1). We have by [LM07, Lemma 2.4.15]

$$\Omega(X) = \operatorname*{colim}_{\mathcal{C}} \Omega(Y),$$

where the colimit is taken over the category $\mathcal{C}$ of projective X-schemes Y which are smooth k-varieties; if $g\colon Y \to Z$ is a morphism in $\mathcal{C}$, the transition map is the push-forward $g_*\colon \Omega(Y) \to \Omega(Z)$. Then for $Y \in \mathcal{C}$, we can make $\Omega(S)$ act on $\Omega(Y)$ using the pull-back along $Y \to S$ and the ring structure on $\Omega(Y)$. The map g_* is easily seen to be $\Omega(S)$-linear using the projection formula. This gives an action on the colimit

$$\Omega(X) \otimes \Omega(S) \to \Omega(X), \tag{3}$$

proving (H1). Then property (H2) follows formally from the projection formula in the smooth case.

For any $T \in \mathcal{V}$, we consider the subgroup $\Omega(T)^{(n)}$ of $\Omega(T)$ generated by the images of g_*, where g runs over the projective morphisms $W \to T$ whose image has codimension $\geq n$ in T. When $T = S$ is smooth, one checks, using reduction to the diagonal and the moving lemma [LM07, Proposition 3.3.1], that the subgroups $\{\Omega(S)^{(n)}, n \geq 0\}$ define a ring filtration on $\Omega(S)$. Since $\Omega(S)^{(\dim X+1)} = 0$, any element of $\Omega(S)^{(1)}$ is nilpotent. Let u be the class in $\Omega(X)$ of any resolution of singularities of X. Since f separable, the element $f_*u - d \cdot 1_S$ vanishes when restricted to some non-empty open subvariety of S by [LM07, Lemma 4.4.5]. By the localisation sequence [LM07, Theorem 3.2.7], this means that this element belongs to $\Omega(S)^{(1)}$, proving (R2).

Let us mention that the pair $(R, H) = (\Omega(-), \Omega(-)_{(m)})$ satisfies Conditions 3.1, with f separable in (R2) (where $\Omega(T)_{(m)} = \Omega(T)^{(\dim T - m)}$). Indeed the main point is to see that the map (3) descends to a map

$$\Omega(X) \otimes \Omega^{(n)}(S) \to \Omega^{(n)}(X).$$

This can be seen using the fact that pull-backs along morphisms of smooth varieties respect the filtration by codimension of supports, a consequence of the moving lemma mentioned above. When $n = 1$, this can also be proved directly, using the fact that f is dominant and the localisation sequence.

4. The subgroup $H_X(k)$

In this section, (R, H) will be a pair of functors $\mathcal{V} \to \mathsf{Ab}$ satisfying Conditions 3.1. We assume that the base k belongs to $\mathcal{V}$, and therefore to Reg. If $X \in \mathcal{V}$ is complete, its structural morphism $x\colon X \to k$ is then in $\mathcal{V}$. We consider the subgroup

$$H_X(k) = \operatorname{im}\big(x_*\colon H(X) \to H(k)\big) \subset H(k).$$

In all examples considered in this paper, $H_X(k)$ is actually an $R(k)$-submodule of $H(k)$.

Lemma 4.1. *Let $S \in \mathsf{Reg}$ and $X \in \mathcal{V}$. Let $f\colon X \to S$ be a proper, dominant, generically finite morphism of degree d. Then the map*

$$f_*\colon H(X) \otimes \mathbb{Z}[1/d] \to H(S) \otimes \mathbb{Z}[1/d]$$

is surjective.

Proof. Using (R2), choose $u \in R(X)$ such that $f_*u - d \cdot 1_S$ is nilpotent. The element f_*u is then invertible in the ring $R(S) \otimes \mathbb{Z}[1/d]$. The lemma follows, since we have by (H2), for any $s \in H(S)$,

$$f_*\left(u \cdot_f \left((f_*u)^{-1} \cdot s\right)\right) = s. \qquad \square$$

We denote by $\mathbb{Z}_{(p)}$ the subgroup of $\mathbb{Q}$ consisting of those fractions whose denominator is prime to p.

Proposition 4.2. *Let X, Y be complete varieties, with $X \in \mathcal{V}$ and $Y \in \mathsf{Reg}$. Let $Y \rightsquigarrow X$ be a correspondence of multiplicity prime to p (resp. let $Y \dashrightarrow X$ be a rational map). Then, as subgroups of $H(k) \otimes \mathbb{Z}_{(p)}$ (resp. $H(k)$),*

$$H_Y(k) \otimes \mathbb{Z}_{(p)} \subset H_X(k) \otimes \mathbb{Z}_{(p)} \ \text{(resp. } H_Y(k) \subset H_X(k)\text{)}.$$

Proof. Let $Y \leftarrow Z \to X$ be a diagram giving the correspondence (resp. the correspondence of multiplicity 1 associated with the rational map). By Lemma 4.1, we have

$$H_Y(k) \otimes \mathbb{Z}_{(p)} \subset H_Z(k) \otimes \mathbb{Z}_{(p)} \ \text{(resp. } H_Y(k) \subset H_Z(k)\text{)}.$$

Since there is a proper morphism $Z \to X$, we have

$$H_Z(k) \subset H_X(k). \qquad \square$$

Corollary 4.3. *Assume that two complete varieties $Y, X \in \mathsf{Reg}$ are p-equivalent. Then, as subgroups of $H(k) \otimes \mathbb{Z}_{(p)}$, we have*

$$H_Y(k) \otimes \mathbb{Z}_{(p)} = H_X(k) \otimes \mathbb{Z}_{(p)}.$$

Corollary 4.4. *Let $Y, X \in \mathsf{Reg}$ be two complete varieties. Assume that there are rational maps $Y \dashrightarrow X$ and $X \dashrightarrow Y$. Then $H_Y(k) = H_X(k)$ as subgroups of $H(k)$.*

We now come back to the examples of theories given in Section 3.

4.1. CHOW GROUPS. We have $\mathrm{CH}(k) = \mathbb{Z}$, and for a complete variety X,

$$\mathrm{CH}_X(k) = n_X\mathbb{Z},$$

where n_X is the index of X, defined as the g.c.d. of the degrees of closed points of X. We denote the p-adic valuation of n_X by

$$n_p(X) = v_p(n_X).$$

We will need the following slightly more precise version of Proposition 4.2, when $R = H = \mathrm{CH}$.

PROPOSITION 4.5. *Let X and Y be complete varieties, with Y regular. Let $Y \rightsquigarrow X$ be a correspondence of multiplicity m. Then*

$$n_X \mid m \cdot n_Y.$$

Proof. Let $Y \xleftarrow{f} Z \xrightarrow{g} X$ be a diagram giving the correspondence. Let $y \in \mathrm{CH}_0(Y)$ be such that $\deg(y) = n_Y$. Using 3.2, we have

$$n_X \mid \deg \circ g_*([Z] \cdot_f y) = \deg \circ f_*([Z] \cdot_f y) = \deg((f_*[Z]) \cdot y) = \deg(m \cdot y) = m \cdot n_Y.$$

□

4.2. CYCLE MODULES. Let M be a cycle module, $H(-) = A_*(-; M)$, $R = \mathrm{CH}$ (see §3.2). We have $A_*(k; M) = M(k)$. For a complete variety X, we claim that the subgroup $H_X(k)$ of $M(k)$ is the image of the morphism

$$\bigoplus_{L/k \in \mathcal{F}_X} M(L) \to M(k), \tag{4}$$

where $\mathcal{F}_X$ is the class of finite field extensions L/k such that $X(L) \neq \emptyset$. Indeed the image of $A_n(X; M) \to A_n(k; M)$ vanishes when $n > 0$. The group $A_0(X; M)$ is by definition a quotient of the direct sum of the groups $M(k(x))$, over all closed points x of X. Moreover, for each such x, the extension $k(x)/k$ belongs to $\mathcal{F}_X$, and the map $M(k(x)) \to A_0(X; M) \to A_0(k; M) = M(k)$ is the transfer for the finite field extension $k(x)/k$. Conversely, if $L/k \in \mathcal{F}_X$, then there is a closed point x of X such that $k(x)/k$ is a subextension of L/k. Thus $M(L) \to M(k)$ factors through $M(k(x)) \to M(k)$. This proves the claim, and additionally shows that $\mathcal{F}_X$ may be replaced in (4) by the set of residue fields at closed points of X.
Corollary 4.3 asserts that, when X is smooth (or merely regular when M is Quillen K-theory), the subgroup $H_X(k) \otimes \mathbb{Z}_{(p)}$ of $M(k) \otimes \mathbb{Z}_{(p)}$ only depends on the p-equivalence class of X.

REMARK 4.6. The group $A_0(X; M)$ itself is known to be a birational invariant of a complete smooth variety X (see [Ros96, Corollary 12.10] and [KM13, Appendix RC]).

4.3. Grothendieck group. We have $K_0'(k) = \mathbb{Z}$, and when X is a complete variety,

$$(K_0')_X(k) = d_X\mathbb{Z},$$

for a uniquely determined positive integer d_X. The integer d_X is the g.c.d. of the integers

$$\chi(X, \mathcal{G}) = \sum_i (-1)^i \dim_k H^i(X, \mathcal{G}),$$

where $\mathcal{G}$ runs over the coherent sheaves of $\mathcal{O}_X$-modules. We denote the p-adic valuation of d_X by

$$d_p(X) = v_p(d_X).$$

Let us record for later reference a consequence of Proposition 4.2.

Proposition 4.7. *Let X and Y be complete varieties, with Y regular. Let $Y \rightsquigarrow X$ be a correspondence of multiplicity prime to p. Then*

$$d_p(X) \leq d_p(Y).$$

4.4. Algebraic cobordism. We say that two varieties X and Y are *separably p-equivalent* if there are diagrams of projective morphisms $Y \xleftarrow{f} Z \to X$ and $X \xleftarrow{g} Z' \to Y$ such that f and g are both separable and generically finite of degrees prime to p. For a projective variety X, and an integer $n \geq 0$, we write $\Omega_X(k)^{(n)}$ for the subgroup of $\Omega(k)$ generated by classes of smooth projective varieties Y admitting a morphism $Y \to X$ with image of codimension $\geq n$. In particular $\Omega_X(k)^{(0)} = \Omega_X(k)$.

Proposition 4.8. *Assume that k admits resolution of singularities. Let Y and X be smooth projective varieties of the same dimension, which are separably p-equivalent. Then, as subgroups of $\Omega(k) \otimes \mathbb{Z}_{(p)}$, for $n \geq 0$,*

$$\Omega_X(k)^{(n)} \otimes \mathbb{Z}_{(p)} = \Omega_Y(k)^{(n)} \otimes \mathbb{Z}_{(p)}.$$

Proof. Let d be the common dimension. The proposition follows by applying Corollary 4.3 (more precisely the analog statement for separably p-equivalent varieties) with $R = \Omega$ and $H = \Omega(-)_{(d-n)}$ (defined at the end of §3.3). □

For a projective variety X, let $M(X) \subset \Omega(k)$ be the ideal generated by the classes of smooth projective varieties Y of dimension $< \dim X$ admitting a morphism $Y \to X$. We have $M(X) \subset \Omega_X(k)^{(1)}$; when k admits weak factorisation, this inclusion is an equality [LM07, Theorem 4.4.16].

Proposition 4.9. *Assume that k admits resolution of singularities. Let Y and X be two smooth projective varieties of the same dimension. If there are rational maps $X \dashrightarrow Y$ and $Y \dashrightarrow X$, then $\Omega_X(k)^{(n)} = \Omega_Y(k)^{(n)}$ as subgroups of $\Omega(k)$, for any n. Moreover, $M(X) = M(Y)$ as ideals of $\Omega(k)$.*

Proof. The first statement follows by taking $R = \Omega$ and $H = \Omega(-)_{(\dim X - n)}$ in Corollary 4.4 (which only requires the validity of Conditions 3.1 for f birational, and in particular separable).

Let V be a smooth projective variety of dimension d. The subgroup $\Omega_{<d}(k) \subset \Omega(k)$ generated by the classes of smooth projective varieties of dimension $< d$ is a direct summand of $\Omega(k)$ (indeed $\Omega(k)$ is the quotient of the free group generated by the classes of smooth projective varieties, by relations respecting the dimensional grading). Let $\pi_{<d} \colon \Omega(k) \to \Omega_{<d}(k)$ be the projection. The subgroup $\pi_{<d}\Omega_V(k)^{(1)}$ is generated by the elements $\pi_{<d}[U]$, where U runs over the smooth projective varieties admitting a non-dominant morphism to V. The element $\pi_{<d}[U]$ is equal to $[U]$ if $\dim U < d$, and vanishes otherwise. So $\pi_{<d}\Omega_V(k)^{(1)}$ is the subgroup of $\Omega(k)$ generated by the classes of smooth projective varieties of dimension $< d$ admitting a morphism to V (necessarily non-dominant), while $M(V)$ is the ideal generated by the same elements. Thus $M(V)$ is the ideal of $\Omega(k)$ generated by $\pi_{<d}\Omega_V(k)^{(1)}$, and the second statement follows from the first, with $n = 1$. □

The corollary below was proved in [LM07, Theorem 4.4.17, 1.] under the additional assumption that k admits weak factorisation.

COROLLARY 4.10. *Assume that k admits resolution of singularities. The ideal $M(X)$ of $\Omega(k)$ is a birational invariant of a smooth projective variety X.*

5. RELATIONS BETWEEN n_X AND d_X

We have the obvious relation $d_X \mid n_X$. When $\dim X = 0$, we have $n_X = d_X$. A consequence of the next theorem is that $d_p(X) = n_p(X)$ when $\dim X < p - 1$.

THEOREM 5.1. *Let X be a complete variety. Assume that one of the following conditions holds.*

(i) The characteristic of k is not p.
(ii) We have $\dim X < p(p-1)$.
(iii) The variety X is regular and (quasi-)projective over k.

Then

$$n_p(X) \leq d_p(X) + \Big[\frac{\dim X}{p-1}\Big].$$

Proof. For (i) and (ii), we may assume that X is projective over k by Chow's lemma. Indeed if $X' \to X$ is an envelope [Ful98, Definition 18.3], then it follows from [Ful98, Lemma 18.3] that $n_X = n_{X'}$ and $d_X = d_{X'}$. The group $K_0'(X)$ is generated by the classes $[\mathcal{O}_Z]$, where Z runs over the closed subvarieties of X. Therefore d_X is the g.c.d. of the integers $\chi(Z, \mathcal{O}_Z)$, for Z as above. In particular, we can find a closed subvariety Z of X such that

$$d_p(X) = v_p(\chi(Z, \mathcal{O}_Z)).$$

We now claim that, under the assumption (i) or (ii), we have,

$$n_p(Z) \leq v_p(\chi(Z, \mathcal{O}_Z)) + \Big[\frac{\dim Z}{p-1}\Big]. \tag{5}$$

This will conclude the proof, since $n_p(X) \leq n_p(Z)$, and $\dim Z \leq \dim X$.

If we assume (ii), then $\dim Z < p(p-1)$, and (5) follows from [Hau12a, Proposition 9.1]. But the argument of loc. cit. can also be used in the situation (i). Namely, we let $\tau_n \colon K_0'(-) \to \mathrm{CH}_n(-) \otimes \mathbb{Q}$ be the map of [Ful98, Theorem 18.3] (it is the homological Chern character, denoted ch_n in [Hau12a]). Let $z \colon Z \to k$ be the structural morphism of Z. Then by [Ful98, Theorem 18.3 (1)]

$$\chi(Z, \mathcal{O}_Z) = \tau_0 \circ z_*[\mathcal{O}_Z] = z_* \circ \tau_0[\mathcal{O}_Z]. \tag{6}$$

If (i) holds, then [Hau12a, Theorem 4.2] says that the element $p^{[\dim Z/(p-1)]} \cdot \tau_0[\mathcal{O}_Z] \in \mathrm{CH}_0(Z) \otimes \mathbb{Q}$ belongs to the image of $\mathrm{CH}_0(Z) \otimes \mathbb{Z}_{(p)}$, hence can written as $b \otimes \lambda^{-1}$, with λ an integer prime to p, and $b \in \mathrm{CH}_0(Z)$. Thus:

$$\left[\frac{\dim Z}{p-1}\right] + v_p(z_* \circ \tau_0[\mathcal{O}_Z]) = v_p(z_* b) \geq n_p(Z).$$

Using (6), this gives (5).

Now we assume (iii). We identify the groups $K_0'(X)$ and $K_0(X)$ using (2), and take $a \in K_0(X)$. Let $x \colon X \to k$ be the structural morphism of X. Since x is quasi-projective and X regular, the morphism x is a local complete intersection (i.e. factors as a regular closed embedding followed by a smooth morphism); let $T_x \in K_0(X)$ be its virtual tangent bundle. We apply the Grothendieck-Riemann-Roch theorem [Ful98, Theorem 18.2], and get in $\mathbb{Q} = K_0'(k) \otimes \mathbb{Q} = \mathrm{CH}(k) \otimes \mathbb{Q}$ the equalities

$$x_* a = \mathrm{ch} \circ x_* a = x_* \circ \mathrm{Td}(T_x) \circ \mathrm{ch}\, a. \tag{7}$$

Here ch is the Chern character, with components $\mathrm{ch}^n \colon K_0(X) \to \mathrm{CH}^n(X) \otimes \mathbb{Q}$, and $\mathrm{Td} = \sum_n \mathrm{Td}^n$ is the Todd class. By [Hau12a, Lemma 6.3], the morphism

$$p^{[n/(p-1)]} \cdot \mathrm{Td}^n(T_x) \colon \ \mathrm{CH}^\bullet(X) \otimes \mathbb{Q} \to \mathrm{CH}^{\bullet+n}(X) \otimes \mathbb{Q}$$

sends the image of $\mathrm{CH}^\bullet(X) \otimes \mathbb{Z}_{(p)}$ to the image $\mathrm{CH}^{\bullet+n}(X) \otimes \mathbb{Z}_{(p)}$. The degree zero component of $p^{[\dim X/(p-1)]} \cdot \mathrm{Td}(T_X) \circ \mathrm{ch}\, a$ is

$$\sum_{n=0}^{\dim X} \left(p^{[(\dim X - n)/(p-1)]} \cdot \mathrm{Td}^{\dim X - n}(T_x)\right) \circ \left(p^{[n/(p-1)]} \cdot \mathrm{ch}^n a\right) \in \mathrm{CH}_0(X) \otimes \mathbb{Q}.$$

By Lemma 5.2 and the remark above, this element belongs to the image of $\mathrm{CH}_0(X) \otimes \mathbb{Z}_{(p)} \to \mathrm{CH}_0(X) \otimes \mathbb{Q}$. Using (7), we obtain,

$$\left[\frac{\dim X}{p-1}\right] + v_p(x_* a) \geq n_p(X).$$

The statement follows, since we can choose a such that $v_p(x_* a) = d_p(X)$. □

Lemma 5.2. *Let X be a variety, and $\mathrm{ch} \colon K_0(X) \to \mathrm{CH}(X) \otimes \mathbb{Q}$ the Chern character. Then for all integers n and elements $a \in K_0(X)$, we have*

$$p^{[n/(p-1)]} \cdot \mathrm{ch}^n a \in \mathrm{im}\left(\mathrm{CH}^n(X) \otimes \mathbb{Z}_{(p)} \to \mathrm{CH}^n(X) \otimes \mathbb{Q}\right).$$

Proof. This follows from the splitting principle. In more details, we proceed exactly as in [Hau12a, Lemma 6.3], using that ch factors through the operational Chow ring tensored with $\mathbb{Q}$, and that for a line bundle L,

$$\mathrm{ch}[L] = \sum_{n \geq 0} \frac{c_1(L)^n}{n!}. \qquad \square$$

Remark 5.3. From the proof, we see that the statement of Lemma 5.2 may be improved: the exponent $[n/(p-1)]$ can be replaced by $[(n-1)/(p-1)]$.

Proposition 5.4. *Let X be a complete regular variety. Then*

$$\mathrm{cdim}_p(X) \geq \begin{cases} (p-1)\cdot(n_p(X) - d_p(X)) & \text{if } k \text{ has characteristic} \neq p, \\ (p-1)\cdot \min(p, n_p(X) - d_p(X)) & \text{if } k \text{ has characteristic} = p. \end{cases}$$

Proof. Let $Z \subset X$ be a closed subvariety admitting a correspondence $X \rightsquigarrow Z$ of multiplicity prime to p, and such that $\dim Z = \mathrm{cdim}_p(X)$. By Proposition 4.7 we have $d_p(Z) \leq d_p(X)$. Since there is a morphism $Z \to X$, we have $n_p(X) \leq n_p(Z)$. We conclude by applying Theorem 5.1 (i), (ii) to the complete variety Z. $\square$

6. Examples

In view of Section 5, it may seem desirable to find conditions on a complete variety X that give upper bounds for $d_p(X)$.

Proposition 6.1. *Let X be a complete smooth variety. Assume that there is a field extension l/k, and a complete smooth l-variety Y such that $X_l \times_l Y$ is a rational l-variety. Then $d_X = 1$.*

Proof. It will be sufficient to prove that $\chi(X, \mathcal{O}_X) = \pm 1$. While doing so, we may extend scalars, and thus assume that $k = l$. The variety $Z = X \times_k Y$ is then rational. Since the coherent cohomology groups of the structure sheaf are birational invariants of a complete smooth variety [CR11, Theorem 3.2.8], so is its Euler characteristic. The structure sheaf of the projective space has Euler characteristic equal to 1, hence $\chi(Z, \mathcal{O}_Z) = 1$. Since $\chi(Z, \mathcal{O}_Z) = \chi(X, \mathcal{O}_X) \cdot \chi(Y, \mathcal{O}_Y)$ by [Ful98, Example 15.2.12], we are done. $\square$

Corollary 6.2. *Let X be a complete, smooth, geometrically rational variety. Then $d_X = 1$.*

Example 6.3 (Projective homogeneous varieties). Let X be a complete, smooth, geometrically connected variety, which is homogeneous under a semi-simple linear algebraic group. Then X is geometrically rational. Thus by Corollary 6.2, we have $d_X = 1$.

Proposition 6.4. *Assume that k has characteristic zero. Let X be a complete, smooth, geometrically rationally connected variety. Then $d_X = 1$.*

Proof. We proceed as in the proof of Proposition 6.1, and assume that X is rationally connected. Then the groups $H^i(X, \mathcal{O}_X)$ vanish for $i > 0$ by [Deb01, Corollary 4.18, a)], and therefore $\chi(X, \mathcal{O}_X) = 1$. $\square$

Remark 6.5 (Decomposition of the diagonal). The statement of Proposition 6.4 is more generally true (still in characteristic zero, for X smooth complete) under the assumption that the diagonal decomposes, i.e. that there is a zero-cycle z on X whose degree N is not zero, and a non-empty open subvariety U of X, such that the cycles $[U] \times_k z$ and $N \cdot [\Gamma_U]$ are rationally equivalent on $U \times_k X$, where $\Gamma_U \subset U \times_k X$ is the graph of $U \to X$. This is the case when $\mathrm{CH}_0(X_\Omega) = \mathbb{Z}$, where Ω is an algebraic closure of $k(X)$ (see e.g. the introduction of [Esn03]).

Example 6.6 (Complete intersections). Let H be a complete intersection of hypersurfaces in $\mathbb{P}^n$ of degrees $\delta_1, \cdots, \delta_m$, with $\delta_1 + \cdots + \delta_m \leq n$. When H is smooth, it is Fano, hence geometrically rationally chain connected. If, in addition, k has characteristic zero, the variety H is geometrically rationally connected, and Proposition 6.4 shows that $\chi(H, \mathcal{O}_H) = 1$. Alternatively, a direct computation shows that this is true in general (in any characteristic, for possibly singular H). This can be used to produce other sufficient conditions on a complete variety X for the equality $d_X = 1$, namely:
— X becomes isomorphic to such an H after extension of the base field,
— or X is smooth and becomes birational to such a smooth H after extension of the base field.

Example 6.7 (Hypersurfaces). Let H be a regular hypersurface of degree p and dimension $\geq p-1$. By Example 6.6, we have $d_p(H) = 0$. Assume that H has no closed point of degree prime to p. Then by Proposition 5.4, we have $\mathrm{cdim}_p(H) \geq p-1$.
In case $\dim H = p-1$, this bound is optimal, and H is p-incompressible (see also [Mer03, § 7.3] and [Zai10, Example 6.4]). A more general statement was proved in [Hau12a, Proposition 10.1].
When $p = 2$, we have $\mathrm{cdim}_2(H) \geq 1$. This bound is sharp when $\dim H = 2$, as can be seen by taking for H an anisotropic smooth projective Pfister quadric surface. In general this is far from being sharp: in characteristic not two, it is known that $\mathrm{cdim}_2(H) = 2^n - 1$, where n is such that $2^n - 1 \leq \dim H < 2^{n+1} - 1$ (see [Mer03, § 7]).

Example 6.8 (Separation). Let X and Y be complete varieties, with X regular. Assume that the degree of any closed point of Y is divisible by p, that $d_p(X) = 0$ (see in particular Example 6.6 and Proposition 6.4), and that

$$\dim Y < p - 1 \leq \dim X.$$

Then the degree of any closed point of $Y_{k(X)}$ is divisible by p. (Indeed, assuming the contrary, there would be a correspondence $X \rightsquigarrow Y$ of degree prime to p. Then $d_p(Y) = 0$ by Proposition 4.7, hence $n_p(Y) = 0$ by Theorem 5.1.)

Proposition 6.9. *Let $f \colon Y \dashrightarrow X$ be a rational map between complete regular varieties. Let F be the generic fiber of f, considered as a $k(X)$-variety. Then*

(i) If $d_p(F) = 0$ then $d_p(X) = d_p(Y)$.
(ii) We have $n_p(X) \leq n_p(Y) \leq n_p(X) + n_p(F)$.

Proof. Let us prove (i). The map f induces a correspondence $Y \xleftarrow{g} Z \xrightarrow{h} X$ with g birational. By Lemma 4.1 we have $d_Y = d_Z$.
We now prove the equality $d_p(X) = d_p(Z)$. We have a cartesian square

$$\begin{tikzcd} F \arrow[r, "\rho"] \arrow[d, "l"'] & Z \arrow[d, "h"] \\ k(X) \arrow[r, "\eta"'] & X \end{tikzcd}$$

Since $d_p(F) = 0$, there is $v \in K_0'(F) \otimes \mathbb{Z}_{(p)}$ such that

$$l_* v = 1 \in \mathbb{Z}_{(p)} = K_0'(k(X)) \otimes \mathbb{Z}_{(p)}.$$

Since $\rho^* \colon K_0'(Z) \to K_0'(F)$ is surjective (by the localisation sequence), we can find $u \in K_0'(Z) \otimes \mathbb{Z}_{(p)}$ such that $\rho^* u = v$. Then we have

$$\eta^* \circ h_* u = l_* \circ \rho^* u = l_* v = 1 = \eta^*[\mathcal{O}_X].$$

It follows that the element $h_* u - [\mathcal{O}_X]$ is in the kernel of η^*, hence is nilpotent; therefore $h_* u$ is invertible in $K_0'(X) \otimes \mathbb{Z}_{(p)}$. We can now conclude that $d_p(X) = d_p(Z)$, as in Lemma 4.1.
The first inequality in (ii) follows from Proposition 4.5. To prove the second, note that $F \subset Y_{k(X)}$ and use Lemma 7.5 below. □

Example 6.10. Let p be an odd prime. Let H be a regular hypersurface of degree p and dimension $p-1$, with $n_p(H) = 1$. Let X be a regular complete variety admitting a rational map $X \dashrightarrow H$, whose generic fiber F is also a hypersurface of degree p and dimension $p-1$.
We have $d_p(F) = d_p(H) = 0$ by Example 6.6, and therefore $d_p(X) = 0$ by Proposition 6.9 (i). Moreover $n_p(F) \leq 1$ by Theorem 5.1, hence by Proposition 6.9 (ii), we have $1 \leq n_p(X) \leq 2$.
We use Proposition 5.4. If $n_p(X) = 2$, then $\mathrm{cdim}_p(X) = \dim X = 2(p-1)$. If $n_p(X) = 1$, then $\mathrm{cdim}_p(X) \geq p-1$. This bound is sharp: if $n_p(F) = 0$, then X and H are p-equivalent, hence have the same canonical p-dimension; but $\mathrm{cdim}_p(H) = \dim H = p-1$.

7. Invariants of function fields

Definition 7.1. Let K/k be a finitely generated field extension. Let M be a regular projective variety such that $k(M)$ contains K as a k-subalgebra with $[k(M) : K]$ finite and prime to p. We define $d_p(K/k) := d_p(M)$ and $n_p(K/k) := n_p(M)$. By Lemma 7.4 below, and Corollary 4.3, these integers do not depend on the choice of M. If there is no such M, we set $d_p(K/k) = n_p(K/k) = \infty$ (by Gabber's theorem, this may only happen in characteristic p).

An alternative definition of $n_p(K/k)$ can be found in [Mer03, Remark 7.7]. Note that, when X is a complete variety, we have (by Proposition 4.7 and Proposition 4.5)

$$d_p(X) \leq d_p(k(X)/k) \qquad \text{and} \qquad n_p(X) \leq n_p(k(X)/k),$$

with equalities when X is regular (by Lemma 7.4).

Remark 7.2. This process can be used more generally to define invariants $H_{K/k}(k) \otimes \mathbb{Z}_{(p)}$, when the pair (R, H) satisfies Conditions 3.1.

Remark 7.3 (Geometrically unirational field extensions). When k has characteristic zero, it is possible to have some control on $d_p(K/k)$ without explicitly introducing a variety M as in Definition 7.1. Namely assume that there is a field extension l/k such that the ring $K \otimes_k l$ is contained in a purely transcendental field extension of l. Then $d_p(K/k) = 0$ for any p. (Indeed by Hironaka's resolution of singularities [Hir64], there is a smooth projective variety M with function field K. Then M is geometrically unirational, hence geometrically rationally connected, and $d_M = 1$ by Proposition 6.4.)

Lemma 7.4. *Let $X_1 \to X \leftarrow X_2$ be a diagram of varieties and proper morphisms, generically finite of degree prime to p. Then X_1 is p-equivalent to X_2.*

Proof. Consider the cartesian square

$$\begin{tikzcd} X' \arrow[r, "g_1"] \arrow[d, "g_2"'] & X_2 \arrow[d, "f_2"] \\ X_1 \arrow[r, "f_1"'] & X \end{tikzcd}$$

We claim that X' has an irreducible component which is generically finite of degree prime to p over X_1; this component then gives a correspondence $X_1 \rightsquigarrow X_2$ of multiplicity prime to p, and the lemma follows by symmetry.
Since f_i is generically finite and dominant, and X_i is integral, the generic fiber of f_i is the spectrum of $K_i := k(X_i)$ (for $i = 1, 2$). The ring $B = K_1 \otimes_{k(X)} K_2$ is artinian, and its spectrum is the generic fiber of g_2. Write $B = B_1 \times \cdots \times B_n$, with B_j an artinian local ring with residue field L_j (for $j = 1, \cdots, n$). We have

$$[K_2 : k(X)] = \dim_{K_1} B = \sum_{j=1}^{n} \dim_{K_1} B_j = \sum_{j=1}^{n} [L_j : K_1] \cdot \operatorname{length} B_j$$

(for the last equality, use e.g. [Ful98, Lemma A.1.3], replacing A, B, M with K_1, B_j, B_j). But the leftmost integer is prime to p by hypothesis. It follows that for some j the integer $[L_j : K_1]$ is prime to p. The closure of $\operatorname{Spec} L_j$ in X' then gives the required irreducible component. □

Lemma 7.5. *Let X and M be complete varieties, with M regular. Then*

$$n_p(X) \leq n_p(X_{k(M)}) + n_p(M).$$

Proof. Write $E = k(M)$. Let L/E be a finite field extension such that $X_E(L) \neq \emptyset$ and $v_p[L\colon E] = n_p(X_E)$. Then the closure of the image of the map induced by an L-point of X_E

$$\operatorname{Spec} L \hookrightarrow X_E = (\operatorname{Spec} E) \times_k X \to M \times_k X$$

gives a correspondence $M \rightsquigarrow X$, whose multiplicity is $[L\colon E]$. By Proposition 4.5, we have

$$n_p(X) \le v_p[L\colon E] + n_p(M) = n_p(X_E) + n_p(M). \qquad \square$$

PROPOSITION 7.6. *Let K/k be a finitely generated field extension. Let X be a complete variety. Then*

$$n_p(X) \le n_p(X_K) + n_p(K/k).$$

Proof. We may assume that there is a projective regular variety M with $[k(M) : K]$ finite and prime to p (otherwise the statement is empty). Since $n_p(X_{k(M)}) \le n_p(X_K)$, the statement follows from Lemma 7.5. $\square$

PROPOSITION 7.7. *Let K/k be a finitely generated field extension. Then*

$$n_p(K/k) \le d_p(K/k) + \left\lceil \frac{\operatorname{tr.deg}(K/k)}{p-1} \right\rceil.$$

Proof. As above, we may assume that there is a smooth projective variety M such that $[k(M) : K]$ is prime to p, and we apply Theorem 5.1 (iii) with $X = M$. $\square$

COROLLARY 7.8. *Let K/k be a field extension of transcendence degree $< p-1$, with $d_p(K/k) = 0$. Then $n_p(K/k) = 0$.*

COROLLARY 7.9. *Let K/k be a field extension of transcendence degree $< p-1$. Assume that $d_p(K/k) = 0$ (see in particular Remark 7.3). Then*

(i) The relative Brauer group $\ker(\operatorname{Br}(k) \to \operatorname{Br}(K))$ *has no p-primary torsion.*

(ii) Let α be a pure symbol in $K_*^M(k)/p$. Assume that k has characteristic zero. If $\alpha_K = 0$ then $\alpha = 0$.

Proof. We have $n_p(K/k) = 0$ by Corollary 7.8. To prove (i), let A be central simple k-algebra of p-primary exponent. Take for X the Severi-Brauer variety of A. Then for any field extension l/k, the class of $A \otimes_k l$ vanishes in the Brauer group of l if and only if $n_p(X_l) = 0$. By Proposition 7.6, we have $n_p(X_K) = n_p(X)$, and (i) follows.
To prove (ii), one can take for X a complete generic p-splitting variety [SJ06] for α, and argue as above. $\square$

EXAMPLE 7.10. Let D be a non-trivial p-primary central division k-algebra, and K/k a splitting field extension for D. Assume that K is the function field of a complete, smooth, geometrically rational variety (or more generally that $d_p(K/k) = 0$). Corollary 7.9, (i) says that $\operatorname{tr.deg}(K/k) \ge p-1$. One may ask whether we always have

$$\operatorname{tr.deg}(K/k) \ge \operatorname{ind}(D) - 1.$$

In other words, has the Severi-Brauer variety of D the smallest possible dimension among the complete, smooth, geometrically rational varieties whose function field splits D?

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