Algebraic Cycles and Fibrations

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Received: March 23, 2012
Revised: July 19, 2013

Communicated by Alexander Merkurjev

Abstract. Let $f : X \to B$ be a projective surjective morphism between quasi-projective varieties. The goal of this paper is the study of the Chow groups of X in terms of the Chow groups of B and of the fibres of f. One of the applications concerns quadric bundles. When X and B are smooth projective and when f is a flat quadric fibration, we show that the Chow motive of X is "built" from the motives of varieties of dimension less than the dimension of B.

2010 Mathematics Subject Classification: 14C15, 14C25, 14C05, 14D99
Keywords and Phrases: Algebraic cycles, Chow groups, quadric bundles, motives, Chow–Künneth decomposition.

Contents

For a scheme X over a field k, $\mathrm{CH}_i(X)$ denotes the rational Chow group of i-dimensional cycles on X modulo rational equivalence. Throughout, $f : X \to B$ will be a projective surjective morphism defined over k from a quasi-projective variety X of dimension d_X to an irreducible quasi-projective variety B of dimension d_B, with various extra assumptions which will be explicitly stated. Let h be the class of a hyperplane section in the Picard group of X. Intersecting with h induces an action $\mathrm{CH}_i(X) \to \mathrm{CH}_{i-1}(X)$ still denoted h. Our first observation is Proposition 1.6: when B is smooth, the map

$$(1) \qquad \bigoplus_{i=0}^{d_X-d_B} h^{d_X-d_B-i} \circ f^* \ : \ \bigoplus_{i=0}^{d_X-d_B} \mathrm{CH}_{l-i}(B) \longrightarrow \mathrm{CH}_l(X)$$

is injective for all l and a left-inverse can be expressed as a combination of the proper pushforward f_*, the refined pullback f^* and intersection with h. It is then not too surprising that, when both B and X are smooth projective, the morphism of Chow motives

$$\bigoplus_{i=0}^{d_X-d_B} h^{d_X-d_B-i} \circ f^* \ : \ \bigoplus_{i=0}^{d_X-d_B} \mathfrak{h}(B)(i) \longrightarrow \mathfrak{h}(X)$$

is split injective; see Theorem 1.4. By taking a cohomological realisation, for instance by taking Betti cohomology if $k \subseteq \mathbf{C}$, we thus obtain that the map

$$\bigoplus_{i=0}^{d_X-d_B} h^{d_X-d_B-i} \circ f^* \ : \ \bigoplus_{i=0}^{d_X-d_B} \mathrm{H}^{n-2i}(B, \mathbf{Q}) \longrightarrow \mathrm{H}^n(X, \mathbf{Q})$$

is split injective for all n and thus realises the left-hand side group as a sub-Hodge structure of the right-hand side group. This observation can be considered as a natural generalisation of the elementary fact that a smooth projective variety X of Picard rank 1 does not admit a non-constant dominant map to a smooth projective variety of smaller dimension.

Let now Ω be a universal domain containing k, that is an algebraically closed field containing k which has infinite transcendence degree over its prime subfield. Let us assume that there is an integer n such that the fibres X_b of f over Ω-points b of B satisfy $\mathrm{CH}_l(X_b) = \mathbf{Q}$ for all $l < n$. If f is flat, then Theorem 3.2 shows that (1) is surjective for all $l < n$. When X and B are both smooth projective, we deduce in Theorem 4.2 a direct sum decomposition of the Chow motive of X as

$$(2) \qquad \mathfrak{h}(X) \cong \bigoplus_{i=0}^{d_X-d_B} \mathfrak{h}(B)(i) \oplus M(n),$$

where M is isomorphic to a direct summand of the motive of some smooth projective variety Z of dimension $d_X - 2n$. This notably applies when X is a projective bundle over a smooth projective variety B to give the well-known isomorphism $\bigoplus_{i=0}^{d_X-d_B} \mathfrak{h}(B)(i) \xrightarrow{\simeq} \mathfrak{h}(X)$. Such a morphism is usually shown to be an isomorphism by an existence principle, namely Manin's identity principle. Here, we actually exhibit an explicit inverse to that isomorphism. The same arguments are used in Theorem 5.3 to provide an explicit inverse to the smooth blow-up formula for Chow groups. More interesting is the case when a smooth projective variety X is fibred by complete intersections of low degree. For instance, the decomposition (2) makes it possible in Corollary 4.4 to construct a Murre decomposition (see Definition 4.3) for smooth projective varieties fibred by quadrics over a surface, thereby generalising a result of del Angel and Müller–Stach [5] where a Murre decomposition was constructed for 3-folds fibred by conics over a surface, and also generalising a previous result [27] where, in particular, a Murre decomposition was constructed for 4-folds fibred by quadric surfaces over a surface. Another consequence of the decomposition (2) is that rational and numerical equivalence agree on smooth projective varieties X fibred by quadrics over a curve or a surface defined over a finite field; see Corollary 4.8. It should be mentioned that our approach bypasses the technique of Gordon–Hanamura–Murre [8], where Chow–Künneth decompositions are constructed from relative Chow–Künneth decompositions. In our case, we do not require the existence of a relative Chow–Künneth decomposition, nor do we require f to be smooth away from finitely many points as is the case in [8]. Finally, it should be noted, for instance if X is complex smooth projective fibred by quadrics, that (2) actually computes some of the Hodge numbers of X without going through a detailed analysis of the Leray–Serre spectral sequence.

More generally, we are interested in computing, in some sense, the Chow groups of X in terms of the Chow groups of B and of the fibres of f. Let us first clarify what is meant by "fibres". We observed in [27, Theorem 1.3] that if B is smooth and if a general fibre of f has trivial Chow group of zero-cycles (*i.e.* if it is spanned by the class of a point), then $f_* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(B)$ is an isomorphism with inverse a rational multiple of $h^{d_X - d_B} \circ f^*$. We thus see that, as far as zero-cycles on the fibres are concerned, it is enough to consider only the general fibre. For that matter, we show in Proposition 2.4 that, provided the ground field is a universal domain, it is actually enough that a very general fibre have trivial Chow group of zero-cycles. However, if one is willing to deal with positive-dimensional cycles, it is no longer possible to ignore the Chow groups of some of the fibres. For instance, if $\widetilde{X}_Y \to X$ is a smooth blow-up along a smooth center $Y \subseteq X$, then $\mathrm{CH}_1(\widetilde{X}_Y)$ is isomorphic to $\mathrm{CH}_1(X) \oplus \mathrm{CH}_0(Y)$, although a general fibre of $\widetilde{X}_Y \to X$ is reduced to a point and hence has trivial CH_1. One may argue that a smooth blow-up is not flat. Let us however consider as in [1] a complex flat conic fibration $f : X \to \mathbf{P}^2$, where X is smooth projective. All fibres F of f satisfy $\mathrm{CH}_0(F) = \mathbf{Q}$. A smooth fibre F of f is isomorphic to $\mathbf{P}^1$ and hence satisfies $\mathrm{CH}_1(F) = \mathbf{Q}$. A singular fibre F of f is either a double line, or the union of two lines meeting at a point. In the latter case $\mathrm{CH}_1(F) = \mathbf{Q} \oplus \mathbf{Q}$. This reflects in $\mathrm{CH}_1(X)$ and, as shown in [1], $\mathrm{CH}_1(X)_{\mathrm{hom}}$ is isomorphic to the Prym variety attached to the discriminant curve of f. This suggests that a careful analysis of the degenerations of f is required in order to derive some precise information on the Chow groups of X. On another perspective, the

following examples show what kind of limitation is to be expected when dealing with fibres with non-trivial Chow groups. For instance, consider a complex Enriques surface S and let $T \to S$ be the 2-covering by a $K3$-surface T. We know that $\mathrm{CH}_0(S) = \mathbf{Q}$, and the fibres T_s are disjoint union of two points and hence satisfy $\mathrm{CH}_0(T_s) = \mathbf{Q} \oplus \mathbf{Q}$. However, we cannot expect $\mathrm{CH}_0(T)$ to be correlated in some way to the Chow groups of S and of the fibres, because a theorem of Mumford [17] says that $\mathrm{CH}_0(T)$ is infinite-dimensional in a precise sense. Another example is given by taking a pencil of high-degree hypersurfaces in $\mathbf{P}^n$. Assume that the base locus Z is smooth. By blowing up Z, we get a morphism $\widetilde{\mathbf{P}}^n_Z \to \mathbf{P}^1$. In that case, $\mathrm{CH}_0(\mathbf{P}^1) = \mathbf{Q}$ and the CH_0 of the fibres is infinite-dimensional, but $\mathrm{CH}_0(\widetilde{\mathbf{P}}^n_Z) = \mathbf{Q}$.
Going back to the case where the general fibre of $f : X \to B$ has trivial Chow group of zero-cycles, we see that $\mathrm{CH}_0(X)$ is supported on a linear section of dimension d_B. We say that $\mathrm{CH}_0(X)$ has niveau d_B. More generally, Laterveer [15] defines a notion of niveau on Chow groups as follows. For a variety X, the group $\mathrm{CH}_i(X)$ is said to have *niveau* $\leq n$ if there exists a closed subscheme Z of X of dimension $\leq i + n$ such that $\mathrm{CH}_i(Z) \to \mathrm{CH}_i(X)$ is surjective, in other words if the i-cycles on X are supported in dimension $i + n$. It can be proved [27, Theorem 1.7] that if a general fibre F of $f : X \to B$ is such that $\mathrm{CH}_0(F)$ has niveau ≤ 1, then $\mathrm{CH}_0(X)$ has niveau $\leq d_B + 1$. In that context, a somewhat more precise question is: what can be said about the niveau of the Chow groups of X in terms of the niveau of the Chow groups of the fibres of $f : X \to S$? A statement one would hope for is the following: if the fibres X_b of $f : X \to B$ are such that $\mathrm{CH}_*(X_b)$ has niveau $\leq n$ for all Ω-points $b \in B$, then $\mathrm{CH}_*(X)$ has niveau $\leq n + d_B$. We cannot prove such a general statement but we prove it when some of the Chow groups of the fibres of f are either spanned by linear sections or have niveau 0, *i.e.* when they are finite-dimensional $\mathbf{Q}$-vector spaces. Precisely, if $f : X \to B$ is a complex projective surjective morphism onto a smooth quasi-projective variety B, we show that $\mathrm{CH}_l(X)$ has niveau $\leq d_B$ in the following cases:

- $\mathrm{CH}_i(X_b) = \mathbf{Q}$ for all $i \leq l$ and all $b \in B(\mathbf{C})$ (Theorem 6.10);
- $d_B = 1$ and $\mathrm{CH}_i(X_b)$ is finitely generated for all $i \leq l$ and all $b \in B(\mathbf{C})$ (Theorem 6.12);
- f is smooth away from finitely many points, $\mathrm{CH}_i(X_b) = \mathbf{Q}$ for all $i < l$ and $\mathrm{CH}_l(X_b)$ is finitely generated, for all $b \in B(\mathbf{C})$ (Theorem 6.13).

These results, which are presented in Section 6, complement the generalisation of the projective bundle formula of Theorem 3.2 by dropping the flatness condition on f and by requiring in some cases that the Chow groups of the fibres be finitely generated instead of one-dimensional. Their proofs use standard techniques such as localisation for Chow groups (for that matter, information on the Chow groups of the fibres of f is extracted from information on the Chow groups of the closed fibres of f in Section 2), relative Hilbert schemes and a Baire category argument. Let us mention that the assumption of Theorem 6.13 on the singular locus of f being finite is also required in [8] where the construction of relative Chow–Künneth decompositions is considered. Finally, Theorem 7.1 gathers known results about smooth projective varieties whose Chow groups have small niveau. Together with the results above, in Section 7, we prove some conjectures on algebraic cycles (such as Kimura's finite-dimensionality conjecture [12], Murre's conjectures [18], Grothendieck's standard conjectures [13], the Hodge conjecture) for some smooth projective varieties fibred by very low degree complete intersection, or by cellular varieties over surfaces. For instance, we show the existence of a Murre decomposition for smooth projective varieties fibred by cellular varieties over a curve (Proposition 7.7) and for 6-folds fibred by cubics over a curve (Proposition 7.4), and the standard conjectures for varieties fibred by smooth cellular varieties of dimension ≤ 4 (Proposition 7.7) or by quadrics (Proposition 7.3) over a surface.

Notations. We work over a field k and Ω denotes a universal domain that contains k. A *variety* over k is a reduced scheme of finite type over k. Throughout, $f : X \to B$ denotes a projective surjective morphism defined over k from a quasi-projective variety X of dimension d_X to an irreducible quasi-projective variety B of dimension d_B. Given a scheme X over k, the group $\mathrm{CH}_i(X)$ is the $\mathbf{Q}$-vector space with basis the i-dimensional irreducible reduced subschemes of X modulo rational equivalence. By definition, we set $\mathrm{CH}_j(X) = 0$ for $j < 0$ and we say that $\mathrm{CH}_i(X)$ is *finitely generated* if it is finitely generated as a $\mathbf{Q}$-vector space, *i.e.* if it is a finite-dimensional $\mathbf{Q}$-vector space. If Z is an irreducible closed subscheme of X, we write $[Z]$ for the class of Z in $\mathrm{CH}_*(X)$. If α is the class of a cycle in $\mathrm{CH}_*(X)$, we write $|\alpha|$ for the support in X of a cycle representing α. If Y is a scheme over k and if β is a cycle in $\mathrm{CH}_*(X \times Y)$, we define its transpose ${}^t\beta \in \mathrm{CH}_*(Y \times X)$ to be the proper pushforward of β under the obvious map $\tau : X \times Y \to Y \times X$. If X and Y are smooth projective, a cycle $\gamma \in \mathrm{CH}_*(X \times Y)$ is called a correspondence. The correspondence γ acts both on $\mathrm{CH}_*(X)$ and $\mathrm{CH}_*(Y)$ in the following way. Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the first and second projections, respectively. These are proper and flat and we may define, for $\alpha \in \mathrm{CH}_*(X)$, $\gamma_*\alpha := (p_Y)_*(\gamma \cdot p_X^*\alpha)$. Here "$\cdot$" is the intersection product on non-singular varieties as defined in [7, §8]. We then define, for $\beta \in \mathrm{CH}_*(Y)$, $\gamma^*\beta := ({}^t\gamma)_*\beta$. Given another smooth projective variety Z and a correspondence $\gamma' \in \mathrm{CH}_*(Y \times Z)$, the composite $\gamma' \circ \gamma \in \mathrm{CH}_*(X \times Z)$ is defined to be $(p_{XZ})_*(p_{XY}^*\gamma \cdot p_{YZ}^*\gamma')$, where $p_{XY} : X \times Y \times Z \to X \times Y$ is the projection and likewise for p_{XZ} and p_{YZ}. The composition of correspondence is compatible with the action of correspondences on Chow groups [7, §16].
Motives are defined in a covariant setting and the notations are those of [27]. Briefly, a Chow motive (or motive, for short) M is a triple (X, p, n) where X is a variety of pure dimension d_X, $p \in \mathrm{CH}_d(X \times X)$ is an idempotent ($p \circ p = p$) and n is an integer. The motive of X is denoted $\mathfrak{h}(X)$ and, by definition, is the motive $(X, \Delta_X, 0)$ where Δ_X is the class in $\mathrm{CH}_{d_X}(X \times X)$ of the diagonal in $X \times X$. We write $\mathbb{1}$ for the unit motive $(\mathrm{Spec}\, k, \Delta_{\mathrm{Spec}\, k}, 0) = \mathfrak{h}(\mathrm{Spec}\, k)$. With our covariant setting, we have $\mathfrak{h}(\mathbf{P}^1) = \mathbb{1} \oplus \mathbb{1}(1)$. A morphism between two

motives (X,p,n) and (Y,q,m) is a correspondence in $q\circ \mathrm{CH}_{d_X+n-m}(X\times Y)\circ p$. If $f:X\to Y$ is a morphism, Γ_f denotes the graph of f in $X\times Y$. By abuse, we also write $\Gamma_f \in \mathrm{CH}_{d_X}(X\times Y)$ for the class of the graph of f. It defines a morphism $\Gamma_f : \mathfrak{h}(X)\to\mathfrak{h}(Y)$. By definition we have $\mathrm{CH}_i(X,p,n)=p_*\,\mathrm{CH}_{i-n}(X)$ and $\mathrm{H}_i(X,p,n)=p_*\mathrm{H}_{i-2n}(X)$, where we write $\mathrm{H}_i(X):=\mathrm{H}^{2d-i}(X(\mathbf{C}),\mathbf{Q})$ for singular homology when $k\subseteq \mathbf{C}$, or $\mathrm{H}_i(X):=\mathrm{H}^{2d-i}(X_{\overline{k}},\mathbf{Q}_\ell)$ for ℓ-adic homology ($\ell\neq \mathrm{char}\,k$) otherwise.
Given an irreducible scheme Y over k, η_Y denotes the generic point of Y. If $f:X\to B$ and if Y is a closed irreducible subscheme of B, X_{η_Y} denotes the fibre of f over the generic point of Y and $X_{\overline{\eta}_Y}$ denotes the fibre of f over a geometric generic point of Y.

1. Surjective morphisms and motives

Let us start by recalling a few facts about intersection theory. Let $f:X\to Y$ be a morphism of schemes defined over k and let l be an integer. If f is proper, then there is a well-defined proper pushforward map $f_*:\mathrm{CH}_l(X)\to \mathrm{CH}_l(Y)$; see [7, §1.4]. If f is flat, then there is a well-defined flat pullback map $f^*:\mathrm{CH}^l(Y)\to\mathrm{CH}^l(X)$; see [7, §1.7]. Pullbacks can also be defined in the following two situations. On the one hand, if D is a Cartier divisor with support $\iota:|D|\hookrightarrow X$, there is a well-defined Gysin map $\iota^*:\mathrm{CH}_l(X)\to\mathrm{CH}_{l-1}(|D|)$ and the composite $\iota_*\circ\iota^*:\mathrm{CH}_l(X)\to\mathrm{CH}_{l-1}(X)$ does not depend on the linear equivalence class of D, that is, there is a well-defined action of the Picard group $\mathrm{Pic}(X)$ on $\mathrm{CH}_l(X)$; see [7, §2]. For instance, if X is a quasi-projective variety given with a fixed embedding $X\hookrightarrow\mathbf{P}^N$, then there is a well-defined map $h:\mathrm{CH}_l(X)\to\mathrm{CH}_{l-1}(X)$ given by intersecting with a hyperplane section of X. More generally, if $\tau:Y\hookrightarrow X$ is a locally complete intersection of codimension r, then there is a well-defined Gysin map $\tau^*:\mathrm{CH}_l(X)\to\mathrm{CH}_{l-r}(Y)$; see [7, §6]. For $n\geq 0$, we write $h^0=\mathrm{id}:\mathrm{CH}_l(X)\to\mathrm{CH}_l(X)$ and h^n for the n-fold composite $h\circ\ldots\circ h:\mathrm{CH}_l(X)\to\mathrm{CH}_{l-n}(X)$. By functoriality of Gysin maps [7, §6.5], if $\iota^n: H^n\hookrightarrow X$ denotes a linear section of codimension n, then the composite map $h\circ\ldots\circ h$ coincides with $\iota^n_*\circ(\iota^n)^*$. When X is smooth projective, we write Δ_{H^n} for the diagonal inside $H^n\times H^n$, and the correspondence $\Gamma_{\iota^n}\circ{}^t\Gamma_{\iota^n}=(\iota^n\times\iota^n)_*[\Delta_{H^n}]\in\mathrm{CH}_{d_X-n}(X\times X)$ induces a map $\mathrm{CH}_l(X)\to\mathrm{CH}_{l-n}(X)$ that coincides with the map h^n; see [7, §16]. By abuse, we also write $h^n=\Gamma_{\iota^n}\circ{}^t\Gamma_{\iota^n}$ for $n>0$ and $h^0:=[\Delta_X]$. On the other hand, if $f:X\to Y$ is a morphism to a non-singular variety Y and if $x\in\mathrm{CH}_*(X)$ and $y\in\mathrm{CH}_*(Y)$, then there is a well-defined refined intersection product $x\cdot_f y\in\mathrm{CH}_*(|x|\cap f^{-1}(|y|))$, where "$\cap$" denotes the scheme-theoretic intersection; see [7, §8]. The pullback f^*y is then defined to be the proper pushforward of $[X]\cdot_f y$ in $\mathrm{CH}_*(X)$. Let us denote $\gamma_f:X\to X\times Y$ the morphism $x\mapsto(x,f(x))$. Because Y is non-singular, this morphism is a locally complete intersection morphism and the pullback f^* is by definition $\gamma_f^*\circ p_Y^*$, where $p_Y:X\times Y\to Y$ is the projection and γ_f^* is the Gysin map; see [7, §8].

Finally, if f is flat, then this pullback map coincides with flat pullback [7, Prop. 8.1.2].

We have the following basic lemma.

Lemma 1.1. *Let $f : X \to B$ be a projective surjective morphism between two quasi-projective varieties. Let $X' \hookrightarrow X$ be a linear section of X of dimension $\geq d_B$. Then $f|_{X'} : X' \to B$ is surjective.*

Proof. Let $X \hookrightarrow \mathbf{P}^N$ be an embedding of X in projective space and let $H \hookrightarrow \mathbf{P}^N$ be a linear subspace such that X' is obtained as the pullback of X along $H \hookrightarrow \mathbf{P}^N$. The linear subvariety H has codimension at most $d_X - d_B$ in $\mathbf{P}^N$ while a geometric fibre of f has dimension at least $d_X - d_B$. Thus every geometric fibre of f meets H and hence X'. It follows that $f|_{X'}$ is surjective. □

Lemma 1.2. *Let $f : X \to B$ be a projective surjective morphism to a smooth quasi-projective variety B. Then there exists a positive integer n such that, for all i, $f_* \circ h^{d_X - d_B} \circ f^* : \mathrm{CH}_i(B) \to \mathrm{CH}_i(B)$ is multiplication by n. If moreover X is smooth and B is projective, then*

$$\Gamma_f \circ h^{d_X - d_B} \circ {}^t\Gamma_f = n \cdot \Delta_B \in \mathrm{CH}_{d_B}(B \times B).$$

Proof. Let $\iota' := \iota^{d_X - d_B} : H' \hookrightarrow X$ be a linear section of X of dimension d_B. We first check, for lack of reference, that $(f \circ \iota')^* = (\iota')^* \circ f^*$ on Chow groups. Here, $f \circ \iota'$ and f are morphisms to the non-singular variety B and as such the pullbacks $(f \circ \iota')^*$ and $(\iota')^*$ are the ones of [7, §8], while ι' is the inclusion of a locally complete intersection and as such the pullback $(\iota')^*$ is the Gysin pullback of [7, §6]. Let $\sigma \in \mathrm{CH}^*(B)$, then $(\iota')^* f^* \sigma = (\iota')^* \gamma_f^*([X] \times \sigma) = (\gamma_f \circ \iota')^*([X] \times \sigma)$, where the second equality follows from the functoriality of Gysin maps [7, §6.5]. Since $\gamma_f \circ \iota' = (\iota' \times \mathrm{id}_B) \circ \gamma_{f \circ \iota'}$, we get by using functoriality of Gysin maps once more that $(\iota')^* f^* \sigma = \gamma_{f\circ\iota'}^* (\iota' \times \mathrm{id}_B)^*([X] \times \sigma)$. Now, we have $(\iota' \times \mathrm{id}_B)^*([X] \times \sigma) = (\iota')^*[X] \times \sigma = [H'] \times \sigma$; see [7, Example 6.5.2]. We therefore obtain that $(\iota')^* f^* \sigma = \gamma_{f\circ\iota'}^*([H'] \times \sigma) := (f \circ \iota')^* \sigma$, as claimed.
Thus, since in addition both f and ι' are proper, we have by functoriality of proper pushforward $(f \circ \iota')_* (f \circ \iota')^* = f_* \iota'_* (\iota')^* f^* = f_* \circ h^{d_X - d_B} \circ f^*$. By Lemma 1.1, the composite morphism $g := f \circ \iota'$ is generically finite, of degree n say. It follows from the projection formula [7, Prop. 8.1.1(c)] and from the definition of proper pushforward that, for all $\gamma \in \mathrm{CH}_i(B)$,

$$f_* \circ h^{d_X - d_B} \circ f^* \gamma = g_*([H'] \cdot g^* \gamma) = g_*([H']) \cdot \gamma = n[B] \cdot \gamma = n\gamma.$$

Assume now that X and B are smooth projective. In that case, we have $\Gamma_f \circ h^{d_X - d_B} \circ {}^t\Gamma_f = \Gamma_g \circ {}^t\Gamma_g := (p_{1,3})_*(p_{1,2}^* {}^t\Gamma_g \cdot p_{2,3}^* \Gamma_g)$, where $p_{i,j}$ denotes projection from $B \times H' \times B$ to the (i,j)-th factor. By refined intersection, we see that $\Gamma_g \circ {}^t\Gamma_g$ is supported on $(p_{1,3})([{}^t\Gamma_g \times B] \cap [B \times \Gamma_g])$, which itself is supported on the diagonal of $B \times B$. Thus $\Gamma_f \circ h^{d_X - d_B} \circ {}^t\Gamma_f$ is a multiple of Δ_B. We have already showed that $f_* \circ h^{d_X - d_B} \circ f^* = (\Gamma_f \circ h^{d_X - d_B} \circ {}^t\Gamma_f)_*$ acts by multiplication by n on $\mathrm{CH}_i(B)$. Therefore, $\Gamma_f \circ h^{d_X - d_B} \circ {}^t\Gamma_f = n \cdot \Delta_B$. □

The following lemma is reminiscent of [7, Prop. 3.1.(a)].

LEMMA 1.3. *Let $f : X \to B$ be a projective morphism to a smooth quasi-projective variety B. Then, for all i, $f_* \circ h^l \circ f^* : \mathrm{CH}_i(B) \to \mathrm{CH}_{i+d_X-d_B-l}(B)$ is the zero map for all $l < d_X - d_B$. If moreover X is smooth and B is projective, then*

$$\Gamma_f \circ h^l \circ {}^t\Gamma_f = 0 \in \mathrm{CH}_{d_X-l}(B \times B) \text{ for all } l < d_X - d_B.$$

Proof. By refined intersection [7, §8], the pullback $f^*\alpha$ is represented by a well-defined class in $\mathrm{CH}_{i+d_X-d_B}(f^{-1}(|\alpha|))$ for any cycle $\alpha \in \mathrm{CH}_i(B)$. It follows that $h^l \circ f^*\alpha$ is represented by a well-defined class in $\mathrm{CH}_{i+d_X-d_B-l}(f^{-1}(|\alpha|))$. Since $f|_{f^{-1}(|\alpha|)} : f^{-1}(|\alpha|) \to |\alpha|$ is proper, we see by proper pushforward that $f_* \circ h^l \circ f^*\alpha$ is represented by a well-defined cycle $\beta \in \mathrm{CH}_{i+d_X-d_B-l}(|\alpha|)$. But then, $\dim|\alpha| = i$ so that if $l < d_X - d_B$, then $\mathrm{CH}_{i+d_X-d_B-l}(|\alpha|) = 0$.
Let us now assume that X and B are smooth projective. Let $\iota^l : H^l \hookrightarrow X$ be a linear section of X of codimension l, and let h^l be the class of $(\iota^l \times \iota^l)(\Delta_{H^l})$ in $\mathrm{CH}_{d_X-l}(X \times X)$. By definition we have $\Gamma_f \circ h^l \circ {}^t\Gamma_f = (p_{1,4})_*(p_{1,2}^*{}^t\Gamma_f \cdot p_{2,3}^*h^l \cdot p_{3,4}^*\Gamma_f)$, where $p_{i,j}$ denotes projection from $B \times X \times X \times B$ to the (i,j)-th factor. These projections are flat morphisms, therefore by flat pullback we have $p_{1,2}^*{}^t\Gamma_f = [{}^t\Gamma_f \times X \times B]$, $p_{2,3}^*h^l = [B \times \Delta_{H^l} \times B]$ and $p_{3,4}^*\Gamma_f = [B \times X \times \Gamma_f]$. By refined intersection, the intersection of the closed subschemes ${}^t\Gamma_f \times X \times B$, $B \times \Delta_{H^l} \times B$ and $B \times X \times \Gamma_f$ of $B \times X \times X \times B$ defines a $(d_X - l)$-dimensional class supported on their scheme-theoretic intersection $\{(f(h), h, h, f(h)) : h \in H^l\} \subset B \times X \times X \times B$. Since f is projective, this is a closed subset of dimension $d_X - l$. Also its image under the projection $p_{1,4}$ has dimension at most d_B, which is strictly less than $d_X - l$ by the assumption made on l. The projection $p_{1,4}$ is a proper map and hence, by proper pushforward, we get that $(p_{1,4})_*[\{(f(h), h, h, f(h)) \in B \times X \times X \times B : h \in H^l\}] = 0$. □

THEOREM 1.4. *Let $f : X \to B$ be a surjective morphism of smooth projective varieties over k. Consider the following two morphisms of motives*

$$\Phi := \bigoplus_{i=0}^{d_X-d_B} h^{d_X-d_B-i} \circ {}^t\Gamma_f \ : \ \bigoplus_{i=0}^{d_X-d_B} \mathfrak{h}(B)(i) \longrightarrow \mathfrak{h}(X)$$

and

$$\Psi := \bigoplus_{i=0}^{d_X-d_B} \Gamma_f \circ h^i \ : \ \mathfrak{h}(X) \longrightarrow \bigoplus_{i=0}^{d_X-d_B} \mathfrak{h}(B)(i).$$

Then $\Psi \circ \Phi$ is an automorphism.

Proof. The endomorphism $\Psi \circ \Phi : \bigoplus_{i=0}^{d_X-d_B} \mathfrak{h}(B)(i) \to \bigoplus_{i=0}^{d_X-d_B} \mathfrak{h}(B)(i)$ can be represented by the $(d_X - d_B + 1) \times (d_X - d_B + 1)$-matrix whose $(i,j)^{\text{th}}$-entries are the morphisms

$$(\Psi \circ \Phi)_{i,j} = \Gamma_f \circ h^{d_X-d_B-(j-i)} \circ {}^t\Gamma_f \ : \ \mathfrak{h}(B)(j-1) \to \mathfrak{h}(B)(i-1).$$

By Lemma 1.2, there is a non-zero integer n such that the diagonal entries satisfy $(\Psi \circ \Phi)_{i,i} = n \cdot \mathrm{id}_{\mathfrak{h}(B)(i-1)}$. By Lemma 1.3, $(\Psi \circ \Phi)_{i,j} = 0$ as soon as

$j > i$. Therefore

$$N := \mathrm{id} - \frac{1}{n} \cdot \Psi \circ \Phi$$

is a nilpotent endomorphism of $\bigoplus_{i=0}^{d_X - d_B} \mathfrak{h}(B)(i)$ with $N^{d_X - d_B + 1} = 0$. Let us define

$$\begin{aligned} \Xi &:= (n \cdot \mathrm{id} - n \cdot N)^{-1} \\ &= \frac{1}{n} \cdot \left(\mathrm{id} + N + N^2 + \cdot + N^{d_X - d_B}\right). \end{aligned}$$

It then follows that Ξ is the inverse of $\Psi \circ \Phi$. □

In the situation of Theorem 1.4, the morphism

$$\Theta := \Xi \circ \Psi$$

then defines a left-inverse to Φ and the endomorphism

$$p := \Phi \circ \Theta = \Phi \circ \Xi \circ \Psi \in \mathrm{End}(\mathfrak{h}(X))$$

is an idempotent.

Proposition 1.5. *With the notations above, the idempotent $p \in \mathrm{End}(\mathfrak{h}(X)) = \mathrm{CH}_{d_X}(X \times X)$ satisfies $p = {}^t p$. Moreover, the morphism $\Psi \circ p : (X, p) \to \bigoplus_{i=0}^{d_X - d_B} \mathfrak{h}(B)(i)$ is an isomorphism with inverse $p \circ \Phi \circ \Xi$.*

Proof. The second claim consists of the following identities: $\Psi \circ p \circ p \circ \Phi \circ \Xi = \Psi \circ p \circ \Phi \circ \Xi = \Psi \circ \Phi \circ \Xi \circ \Psi \circ \Phi \circ \Xi = \mathrm{id} \circ \mathrm{id} = \mathrm{id}$ and $p \circ \Phi \circ \Xi \circ \Psi \circ p = p \circ p \circ p = p$. As for the first claim, we have

$$p = \frac{1}{n} \cdot \Phi \circ \left(1 + N + \ldots + N^{d_X - d_B}\right) \circ \Psi.$$

Recall that $N = \mathrm{id} - \frac{1}{n} \cdot \Psi \circ \Phi$, so that it is enough to see that ${}^t(\Phi \circ \Psi) = \Phi \circ \Psi$. A straightforward computation gives

$$\Phi \circ \Psi = \sum_{i=0}^{d_X - d_B} h^{d_X - d_B - i} \circ {}^t\Gamma_f \circ \Gamma_f \circ h^i.$$

We may then conclude by noting that the correspondence $h \in \mathrm{CH}_{d_X - 1}(X \times X)$ satisfies $h = {}^t h$. □

Finally, let us conclude with the following counterpart of Theorem 1.4 that deals with the Chow groups of quasi-projective varieties.

Proposition 1.6. *Let $f : X \to B$ be a projective surjective morphism to a smooth quasi-projective variety B. Then the map*

$$\Phi_* = \bigoplus_{i=0}^{d_X - d_B} h^{d_X - d_B - i} \circ f^* \ : \ \bigoplus_{i=0}^{d_X - d_B} \mathrm{CH}_{l-i}(B) \longrightarrow \mathrm{CH}_l(X)$$

is split injective and its left-inverse is a polynomial function in f_*, f^* and h.

Proof. Thanks to Lemma 1.2 and to Lemma 1.3, there is a non-zero integer n such that

$$f_* \circ h^i \circ f^* : \mathrm{CH}_l(B) \to \mathrm{CH}_{l+d_X-d_B-i}(B)$$

is multiplication by n if $i = d_X - d_B$ and is zero if $i < d_X - d_B$.
Let us write Ψ_* for $\bigoplus_{j=0}^{d_X-d_B} f_* \circ h^j : \mathrm{CH}_l(X) \to \bigoplus_{j=0}^{d_X-d_B} \mathrm{CH}_{l-j}(B)$. In order to prove the injectivity of Φ_*, it suffices to show that the composite

$$\Psi_* \circ \Phi_* \ : \ \bigoplus_{i=0}^{d_X-d_B} \mathrm{CH}_{l-i}(B) \longrightarrow \mathrm{CH}_l(X) \longrightarrow \bigoplus_{j=0}^{d_X-d_B} \mathrm{CH}_{l-j}(B)$$

is an isomorphism. But then, as in the proof of Theorem 1.4, we see that $\Psi_* \circ \Phi_*$ can be represented by a lower triangular matrix whose diagonal entries' action on $\mathrm{CH}_{l-i}(B)$ is given by multiplication by n. □

Remark 1.7. Note that the conclusion of Proposition 1.6 also holds for a flat and projective surjective morphism $f : X \to B$ of quasi-projective varieties.

2. On the Chow groups of the fibres

In this section, we fix a universal domain Ω. The following statement was communicated to me by Burt Totaro.

Lemma 2.1. *Let $f : X \to B$ be a morphism of varieties over Ω and let F be a geometric generic fibre of f. Then there is a subset $U \subseteq B(\Omega)$ which is a countable intersection of nonempty Zariski open subsets such that for each point $b \in U$, there is an isomorphism from the field Ω to the field $\overline{\Omega(B)}$ such that this isomorphism turns the scheme X_b over Ω into the scheme F over $\overline{\Omega(B)}$. In other words, a very general fibre of f is isomorphic to F as an abstract scheme. Consequently, for each point $p \in U$, $\mathrm{CH}_i(X_b)$ is isomorphic to $\mathrm{CH}_i(F)$ for all integers i.*

Proof. There exist a countable subfield $K \subset \Omega$ and varieties X_0 and B_0 defined over K together with a K-morphism $f_0 : X_0 \to B_0$ such that $f = f_0 \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, \Omega$. Let us define $U \subseteq B(\Omega)$ to be $\bigcap_{Z_0} (B_0 \backslash Z_0)_\Omega(\Omega)$, where the intersection runs through all proper K-subschemes Z_0 of B_0. Note that there are only countably many such subschemes of B_0 and that U is the set of Ω-points of $B = B_0 \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, \Omega$ that do not lie above a proper Zariski-closed subset of B_0.
Let now $b : \mathrm{Spec}\, \Omega \to B$ be a Ω-point of B that lies in U, *i.e.* a point b such that the composite map $\beta : \mathrm{Spec}\, \Omega \xrightarrow{b} B \to B_0$ is dominant, or equivalently such that the composite map β factors as $\eta_{B_0} \circ \alpha$ for some morphism $\alpha : \mathrm{Spec}\, \Omega \to \mathrm{Spec}\, K(B_0)$, where $\eta_{B_0} : \mathrm{Spec}\, K(B_0) \to B_0$ is the generic point of B_0. Since X is pulled back from X_0 along $B \to B_0$, we see that X_b the fibre of f at b is the pull back of the generic fibre $(X_0)_{\eta_{B_0}}$ along α. Consider then $\overline{\eta_B} : \mathrm{Spec}\, \overline{\Omega(B)} \to B$ a geometric generic point of B such that $X_{\overline{\eta_B}} = F$. Since the composite map $\mathrm{Spec}\, \overline{\Omega(B)} \to B \to B_0$ factors through $\eta_{B_0} : \mathrm{Spec}\, K(B_0) \to B_0$, we see as before that F is the pull-back of the generic fibre $(X_0)_{\eta_{B_0}}$ along some morphism

$\alpha' : \operatorname{Spec} \overline{\Omega(B)} \to \operatorname{Spec} K(B_0)$. The fields $\overline{\Omega(B)}$ and Ω are algebraically closed fields of infinite transcendence degree over $K(B_0)$ and there thus exists an isomorphism $\overline{\Omega(B)} \cong \Omega$ fixing $K(B_0)$. Hence, the fibre X_b identifies with F after pullback by the isomorphism $\operatorname{Spec} \Omega \cong \operatorname{Spec} \overline{\Omega(B)}$ over $\operatorname{Spec} K(B_0)$.
The last statement follows from the fact that the Chow groups of a variety X over a field only depend on X as a scheme. Precisely, if one denotes $\psi_b : X_b \to F$ an isomorphism of schemes, then the proper pushforward map $(\psi_b)_* : \mathrm{CH}_i(X_b) \to \mathrm{CH}_i(F)$ is an isomorphism with inverse $(\psi_b^{-1})_* : \mathrm{CH}_i(F) \to \mathrm{CH}_i(X_b)$. □

The following lemma will be useful to refer to.

Lemma 2.2. *Let $f : X \to B$ be a projective surjective morphism defined over Ω onto a quasi-projective variety B. Assume that $\mathrm{CH}_l(X_b) = \mathbf{Q}$ (resp. $\mathrm{CH}_l(X_b)$ is finitely generated) for all $b \in B(\Omega)$. Then $\mathrm{CH}_l(X_{\eta_D}) = \mathbf{Q}$ (resp. $\mathrm{CH}_l(X_{\eta_D})$ is finitely generated) for all irreducible subvarieties D of X.*

Proof. Let D be an irreducible subvariety of B and let $\overline{\eta}_D \to D$ be a geometric generic point of D. By Lemma 2.1 applied to $X_D := X \times_B D$, there is a closed point $d \in D$ such that $\mathrm{CH}_l(X_{\overline{\eta}_D})$ is isomorphic to $\mathrm{CH}_l(X_d)$. By assumption $\mathrm{CH}_l(X_d) = \mathbf{Q}$ (resp. $\mathrm{CH}_l(X_d)$ is finitely generated). Therefore $\mathrm{CH}_l(X_{\overline{\eta}_D}) = \mathbf{Q}$ (resp. $\mathrm{CH}_l(X_{\overline{\eta}_D})$ is finitely generated), too. By a norm argument for Chow groups, the pullback map $\mathrm{CH}_l(X_{\eta_D}) \to \mathrm{CH}_l(X_{\overline{\eta}_D})$ is injective. Hence $\mathrm{CH}_l(X_{\eta_D}) = \mathbf{Q}$ (resp. $\mathrm{CH}_l(X_{\eta_D})$ is finitely generated). □

The following definition is taken from Laterveer [15].

Definition 2.3. Let X be a variety over k. The Chow group $\mathrm{CH}_i(X)$ is said to have *niveau* $\leq r$ if there exists a closed subscheme $Y \subset X$ of dimension $i+r$ such that the proper pushforward map $\mathrm{CH}_i(Y_\Omega) \to \mathrm{CH}_i(X_\Omega)$ is surjective.

Proposition 2.4. *Let $f : X \to B$ be a generically smooth, projective and dominant morphism onto a smooth quasi-projective variety B defined over Ω. Let n be a non-negative integer. The following statements are equivalent.*

(1) If F is a general fibre, then $\mathrm{CH}_0(F)$ has niveau $\leq n$;
(2) If F is a very general fibre, then $\mathrm{CH}_0(F)$ has niveau $\leq n$;
(3) If F is a geometric generic fibre, then $\mathrm{CH}_0(F)$ has niveau $\leq n$.

Proof. The implication (1) ⇒ (2) is obvious. Let us prove (2) ⇒ (3). Let X_b be a very general fibre and F a geometric generic fibre of f, and, by Lemma 2.1, let $\psi_b : X_b \to F$ be an isomorphism of schemes. Assume that there is a closed subscheme Z of dimension $\leq r$ in X_b, for some integer r, such that the proper pushforward $\mathrm{CH}_0(Z) \to \mathrm{CH}_0(X_b)$ is surjective. Then, denoting Z' the image of Z in F under ψ_b, functoriality of proper pushforwards implies that $\mathrm{CH}_0(Z') \to \mathrm{CH}_0(F)$ is surjective. We may then conclude by noting that the subscheme Z' has dimension $\leq r$ in F.
As for (3) ⇒ (1), let Y be a subvariety of F defined over $\overline{\eta}_B$ such that $\mathrm{CH}_0(Y) \to \mathrm{CH}_0(F)$ is surjective. The technique of decomposition of the diagonal of Bloch–Srinivas [2] gives $\Delta_F = \Gamma_1 + \Gamma_2 \in \mathrm{CH}_{\dim F}(F \times F)$, where Γ_1 is

supported on $F \times Y$ and Γ_2 is supported on $D \times F$ for some divisor D in F. Consider a Galois extension $K/\Omega(B)$ over which the above decomposition and the morphism $Y \to F$ are defined, and consider an étale morphism $U \to B$ with $\Omega(U) = K$ such that f restricted to U is smooth. Let u be a Ω-point of U and let X_u be the fibre of f over u. Then the decomposition $\Delta_F = \Gamma_1 + \Gamma_2$ specialises [7, §20.3] on $X_u \times X_u$ to a similar decomposition, where $\Gamma_1|_{X_u \times X_u}$ is supported on $X_u \times Y_u$ and $\Gamma_2|_{X_u \times X_u}$ is supported on $D_u \times X_u$. Letting it act on zero-cycles, we see that $\mathrm{CH}_0(X_u)$ is supported on Y_u. □

Remark 2.5. When $n = 0$ or $n = 1$, the statements of Proposition 2.4 are further equivalent to $\mathrm{CH}_0(F)$ having niveau $\leq n$ for F the generic fibre of f. Indeed, if X is a smooth projective variety such that $\mathrm{CH}_0(X)$ has niveau ≤ 1, then $\mathrm{CH}_0(X)$ is supported on a one-dimensional linear section [11, Proposition 1.6]. In particular, $\mathrm{CH}_0(X)$ is supported on a one-dimensional subvariety of X which is defined over a field of definition of X. Note that, for general n, it is a consequence of the Lefschetz hyperplane theorem and of the Bloch–Beilinson conjectures that if $\mathrm{CH}_0(X)$ has niveau $\leq n$, then $\mathrm{CH}_0(X)$ is supported on an n-dimensional linear section of X.

3. A generalisation of the projective bundle formula

We establish a formula that is analogous to the projective bundle formula for Chow groups. Our formula holds for flat morphisms, rather than Zariski locally trivial morphisms as is the case for the projective bundle formula. However, since a flat morphism does not have any local sections in general, it only holds with rational coefficients.

Proposition 3.1. *Let $f : X \to B$ be a flat projective surjective morphism of quasi-projective varieties. Let $l \geq 0$ be an integer. Assume that*

$$\mathrm{CH}_{l-i}(X_{\eta_{B_i}}) = \mathbf{Q}$$

for all $0 \leq i \leq \min(l, d_B)$ and for all closed irreducible subschemes B_i of B of dimension i, where η_{B_i} is the generic point of B_i.
Then the map

$$\Phi_* = \bigoplus_{i=0}^{d_X - d_B} h^{d_X - d_B - i} \circ f^* \ : \ \bigoplus_{i=0}^{d_X - d_B} \mathrm{CH}_{l-i}(B) \longrightarrow \mathrm{CH}_l(X)$$

is surjective.

Proof. The case when $d_B = 0$ is obvious. Let us proceed by induction on d_B. We have the localisation exact sequence

$$\bigoplus_{D \in B^1} \mathrm{CH}_l(X_D) \longrightarrow \mathrm{CH}_l(X) \longrightarrow \mathrm{CH}_{l-d_B}(X_{\eta_B}) \longrightarrow 0,$$

where the direct sum is taken over all irreducible divisors of B. If $l \geq d_B$, let Y be a linear section of X of dimension l. By Lemma 1.1, $f|_Y : Y \to B$ is surjective. The restriction map $\mathrm{CH}_l(X) \to \mathrm{CH}_{l-d_B}(X_{\eta_B})$ is the direct limit of the flat pullback maps $\mathrm{CH}_l(X) \to \mathrm{CH}_l(X_U)$ taken over all open subsets U of

B; see [3, Lemma 1A.1]. Therefore $\mathrm{CH}_l(X) \to \mathrm{CH}_{l-d_B}(X_{\eta_B})$ sends the class of Y to the class of Y_{η_B} inside $\mathrm{CH}_{l-d_B}(X_{\eta_B})$. But then this class is non-zero because the restriction to η_B of a linear section of Y of dimension d_B has positive degree. Furthermore, if $[B]$ denotes the class of B in $\mathrm{CH}_{d_B}(B)$, then the class of Y is equal to $h^{d_X-l} \circ f^*[B]$ in $\mathrm{CH}_l(X)$. Thus, since by assumption $\mathrm{CH}_{l-d_B}(X_{\eta_B}) = \mathbf{Q}$, the composite map

$$\mathrm{CH}_{d_B}(B) \overset{h^{d_X-l}\circ f^*}{\longrightarrow} \mathrm{CH}_l(X) \to \mathrm{CH}_{l-d_B}(X_{\eta_B})$$

is surjective.
Consider now the fibre square

$$\begin{tikzcd} X_D \arrow[r, "j'_D"] \arrow[d, "f_D"'] & X \arrow[d, "f"] \\ D \arrow[r, "j_D"] & B. \end{tikzcd}$$

Then $f_D : X_D \to D$ is flat and its fibres above points of D satisfy the assumptions of the theorem. Therefore, by the inductive assumption, we have a surjective map

$$\bigoplus_{i=0}^{d_X-d_B} h^{d_X-d_B-i} \circ f_D^* \ : \ \bigoplus_{i=0}^{d_X-d_B} \mathrm{CH}_{l-i}(D) \longrightarrow \mathrm{CH}_l(X_D).$$

Furthermore, since f is flat and j_D is proper, we have the formula [7, Prop. 1.7 & Th. 6.2]

$$j'_{D*} \circ h^{d_X-d_B-i} \circ f_D^* = h^{d_X-d_B-i} \circ f^* \circ j_{D*} \ : \ \mathrm{CH}_{l-i}(D) \to \mathrm{CH}_l(X).$$

Hence, the image of Φ_* contains the image of

$$\bigoplus_{D\in B^1} \bigoplus_{i=0}^{d_X-d_B} j'_{D*} \circ h^{d_X-d_B-i} \circ f_D^* \ : \ \bigoplus_{D\in B^1} \bigoplus_{i=0}^{d_X-d_B} \mathrm{CH}_{l-i}(D) \longrightarrow \mathrm{CH}_l(X).$$

Altogether, this implies that the map Φ_* is surjective. $\square$

We can now gather the statements and proofs of Propositions 1.6 and 3.1 into the following.

Theorem 3.2. *Let $f : X \to B$ be a flat and projective surjective morphism onto a quasi-projective variety B of dimension d_B. Let $l \geq 0$ be an integer. Assume that*

$$\mathrm{CH}_{l-i}(X_b) = \mathbf{Q} \textit{ for all } 0 \leq i \leq \min(l, d_B) \textit{ and for all points } b \textit{ in } B(\Omega).$$

Then the map

$$\Phi_* = \bigoplus_{i=0}^{d_X-d_B} h^{d_X-d_B-i} \circ f^* \ : \ \bigoplus_{i=0}^{d_X-d_B} \mathrm{CH}_{l-i}(B) \longrightarrow \mathrm{CH}_l(X)$$

is an isomorphism. Moreover the map

$$\Psi_* = \bigoplus_{i=0}^{d_X-d_B} f_* \circ h^i \ : \ \mathrm{CH}_l(X) \longrightarrow \bigoplus_{i=0}^{d_X-d_B} \mathrm{CH}_{l-i}(B)$$

is also an isomorphism. □

Proof. Let B_i be an irreducible closed subscheme of B of dimension i with $0 \leq i \leq \min(l, d_B)$. Since $\mathrm{CH}_{l-i}(X_b) = \mathbf{Q}$ for all points $b \in B(\Omega)$, Lemma 2.2 gives $\mathrm{CH}_{l-i}(X_{\eta_{B_i}}) = \mathbf{Q}$. Thus the theorem follows from a combination of Proposition 1.6 (and Remark 1.7) and Proposition 3.1. □

4. On the motive of quadric bundles

Let us first recall the following result.

Proposition 4.1 (Corollary 2.2 in [27]). *Let m and n be positive integers. Let (Y, q) be a motive over k such that $\mathrm{CH}_i(Y_\Omega, q_\Omega) = 0$ for all $i < n$ and $\mathrm{CH}_j(Y_\Omega, {}^tq_\Omega) = 0$ for all $j < m$. Then there exist a smooth projective variety Z over k of dimension $d_X - m - n$ and an idempotent $r \in \mathrm{End}(\mathfrak{h}(Z))$ such that (Y, q) is isomorphic to (Z, r, n).* □

The main result of this section is the following theorem.

Theorem 4.2. *Let $f : X \to B$ be a flat morphism of smooth projective varieties over k. Assume that there exists a positive integer n such that $\mathrm{CH}_l(X_b) = \mathbf{Q}$ for all $0 \leq l < n$ and for all points $b \in B(\Omega)$. Then there exists a smooth projective variety Z of dimension $d_X - 2n$ and an idempotent $r \in \mathrm{End}(\mathfrak{h}(Z))$ such that the motive of X admits a direct sum decomposition*

$$\mathfrak{h}(X) \cong \bigoplus_{i=0}^{d_X-d_B} \mathfrak{h}(B)(i) \oplus (Z, r, n).$$

Proof. With the notations of Theorem 1.4 and its proof the endomorphism $\Psi \circ \Phi \in \mathrm{End}\big(\bigoplus_{i=0}^{d_X-d_B} \mathfrak{h}(B)(i)\big)$ admits an inverse denoted Ξ. Proposition 1.5 then states that $p := \Phi \circ \Xi \circ \Psi \in \mathrm{End}(\mathfrak{h}(X))$ is a self-dual idempotent such that $(X, p) \cong \bigoplus_{i=0}^{d_X-d_B} \mathfrak{h}(B)(i)$. By Theorem 3.2, $(p_\Omega)_* : \mathrm{CH}_l(X_\Omega) \to \mathrm{CH}_l(X_\Omega)$ is an isomorphism for all $l < n$. It follows that $\mathrm{CH}_l(X_\Omega, p_\Omega) = \mathrm{CH}_l(X_\Omega)$ for all $l < n$ and thus that $\mathrm{CH}_l(X_\Omega, \mathrm{id}_\Omega - p_\Omega) = 0$ for all $l < n$. Because $p = {}^tp$, we also have $\mathrm{CH}_l(X_\Omega, \mathrm{id}_\Omega - {}^tp_\Omega) = 0$ for all $l < n$. Proposition 4.1 then yields the existence of a smooth projective variety Z of dimension $d_X - 2n$ such that $(X, \mathrm{id} - p)$ is isomorphic to a direct summand of $\mathfrak{h}(Z)(n)$. □

Our original motivation was to establish Murre's conjectures [18] for smooth projective varieties fibred by quadrics over a surface. The importance of Murre's conjectures was demonstrated by Jannsen who proved [11] that these hold true for all smooth projective varieties if and only if Bloch and Beilinson's conjecture holds true. In our covariant setting, Murre's conjectures can be stated as follows.

(A) There exist mutually orthogonal idempotents $\pi_0, \ldots, \pi_{2d} \in \mathrm{CH}_{d_X}(X \times X)$ adding to the identity such that $(\pi_i)_*\mathrm{H}_*(X) = \mathrm{H}_i(X)$ for all i. We say that X has a *Chow–Künneth decomposition*.
(B) $\pi_0, \ldots, \pi_{2l-1}, \pi_{d+l+1}, \ldots, \pi_{2d}$ act trivially on $\mathrm{CH}_l(X)$ for all l.
(C) $F^i \mathrm{CH}_l(X) := \mathrm{Ker}\,(\pi_{2l}) \cap \ldots \cap \mathrm{Ker}\,(\pi_{2l+i-1})$ doesn't depend on the choice of the π_j's. Here the π_j's are acting on $\mathrm{CH}_l(X)$.
(D) $F^1 \mathrm{CH}_l(X) = \mathrm{CH}_l(X)_{\mathrm{hom}} := \mathrm{Ker}\big(\mathrm{CH}_l(X) \to \mathrm{H}_{2l}(X)\big)$.

Definition 4.3. A variety X that satisfies conjectures (A), (B) and (D) is said to have a *Murre decomposition*.

In the particular case when f is a flat morphism whose geometric fibres are quadrics[1], Theorem 4.2 implies the following corollary. We write $\lfloor a \rfloor$ for the greatest integer which is smaller than or equal to the rational number a.

Corollary 4.4. *Let $f : X \to B$ be a flat morphism of smooth projective varieties over k. Assume that $\mathrm{CH}_l(X_b) = \mathbf{Q}$ for all $0 \leq l < \frac{d_X - d_B}{2}$ and for all points $b \in B(\Omega)$. For instance, the geometric fibres of f could either be quadrics or complete intersection of dimension 4 and bidegree $(2,2)$. Then*

- *If $d_B = 1$, then X is Kimura finite-dimensional* [12].
- *If $d_B \leq 2$, then X has a Murre decomposition.*
- *If $d_B = 3$, $d_X - d_B$ is odd and B has a Murre decomposition, then X has a Murre decomposition.*

Proof. By Theorem 4.2, there is a variety Z and an idempotent $r \in \mathrm{End}(\mathfrak{h}(Z))$ such that the motive of X admits a direct sum decomposition

$$\mathfrak{h}(X) \cong \bigoplus_{i=0}^{d_X - d_B} \mathfrak{h}(B)(i) \oplus (Z, r, \lfloor \frac{d_X - d_B + 1}{2} \rfloor),$$

where

$$d_Z = \begin{cases} d_B - 1 & \text{if } d_X - d_B \text{ is odd;} \\ d_B & \text{if } d_X - d_B \text{ is even.} \end{cases}$$

Thus, we only need to note that any direct summand of the motive of a curve is finite-dimensional [12] and that any direct summand of the motive of a surface has a Murre decomposition [27, Theorem 3.5]. Finally, let us mention that, when $d_B = 1$, it is not necessary to assume f to be flat to conclude that X is Kimura finite-dimensional; see Propositions 7.3 and 7.5 below. $\square$

Remark 4.5. Examples of 3-folds having a Murre decomposition include products of a curve with a surface [19], 3-folds rationally dominated by a product of curves [28] and uniruled 3-folds [5].

[1] Actually if f is flat and if its closed geometric fibres are quadrics, then all of its geometric fibres are quadrics. Conversely if the geometric fibres of f are quadrics of dimension $d_X - d_B$, then f is flat.

Remark 4.6 (The case of smooth families). Suppose $f : X \to B$ is a smooth morphism between smooth projective varieties with geometric fibres being quadric hypersurfaces. Iyer [10] showed that f is étale locally trivial and deduced that f has a relative Chow–Künneth decomposition. By using the technique of Gordon–Hanamura–Murre [8], it is then possible to prove that

$$\mathfrak{h}(X) \cong \begin{cases} \bigoplus_{l=0}^{d_X-d_B} \mathfrak{h}(B)(l) & \text{if } d_X - d_B \text{ is odd;} \\ \bigoplus_{l=0}^{d_X-d_B} \mathfrak{h}(B)(l) \oplus \mathfrak{h}(B)(\frac{d_X-d_B}{2}) & \text{if } d_X - d_B \text{ is even.} \end{cases}$$

Remark 4.7. Suppose $f : X \to S$ is a complex morphism from a smooth projective 3-fold X to a smooth projective surface S whose fibres are conics. In that case, Nagel and Saito [20] identify (up to direct summands isomorphic to $\mathbb{1}$ or $\mathbb{1}(1)$) the motive (Z, r) in the proof of Corollary 4.4 with the $\mathfrak{h}_1$ of the Prym variety P attached to a double-covering of the discriminant curve C of f. If now $f : X \to S$ is a flat complex morphism from a smooth projective variety X to a smooth projective surface S whose fibres are odd-dimensional quadrics, then, because the motive of a curve is Kimura finite-dimensional and by the Lefschetz $(1,1)$-theorem, one would deduce an identification of the $\mathfrak{h}_1$ of (Z, r) with $\mathfrak{h}_1(P)$ from an isomorphism of Hodge structures $\mathrm{H}^1(Z, r) \cong \mathrm{H}^1(P)$. Here, P again is the Prym variety attached to a double-covering of the discriminant curve C of f. Such an identification is currently being investigated by J. Bouali [4] by generalising the methods of [20].

COROLLARY 4.8. *Let $f : X \to B$ be a flat dominant morphism between smooth projective varieties defined over a finite field $\mathbf{F}$ whose geometric fibres are quadrics. If $d_B \leq 2$, then numerical and rational equivalence agree on X.*

Proof. As in the proof of Corollary 4.4, there is a direct sum decomposition

$$\mathfrak{h}(X) \cong \bigoplus_{i=0}^{d_X-2} \mathfrak{h}(B)(i) \oplus (Z, r, \lfloor \frac{d_X - d_B + 1}{2} \rfloor) \tag{3}$$

for some smooth projective variety Z, which is a curve if $d_X - d_B$ is odd and a surface if $d_X - d_B$ is even. Now the action of correspondences preserves numerical equivalence so that if α denotes the isomorphism from $\mathfrak{h}(X)$ to the right-hand side of (3) and if β denotes its inverse, then we have $\mathrm{CH}_l(X)_{\mathrm{num}} = \beta_*\alpha_* \mathrm{CH}_l(X)_{\mathrm{num}}$ for all l. In particular, $\mathrm{CH}_l(X)_{\mathrm{num}} = \beta_*\big(\bigoplus_{i=0}^{d_X-2} \mathrm{CH}_{l-i}(B)_{\mathrm{num}} \oplus r_* \mathrm{CH}_{l-m}(Z)_{\mathrm{num}}\big)$, where $m = \lfloor \frac{d_X-d_B+1}{2} \rfloor$. The corollary then follows from the fact that for any smooth projective variety Y defined over a finite field the groups $\mathrm{CH}_0(Y)_{\mathrm{num}}$, $\mathrm{CH}^1(Y)_{\mathrm{num}}$ and $\mathrm{CH}^0(Y)_{\mathrm{num}}$ are zero. □

5. On the motive of a smooth blow-up

Let X be a smooth projective variety over a field k and let $j : Y \hookrightarrow X$ be a smooth closed subvariety of codimension r. We write $\tau : \widetilde{X}_Y \to X$ for the blow-up of X along Y. Manin [16] showed by an existence principle that the natural map, which is denoted Φ below, $\mathfrak{h}(X) \oplus \bigoplus_{i=1}^{r-1} \mathfrak{h}(Y)(i) \longrightarrow \mathfrak{h}(\widetilde{X}_Y)$ is an isomorphism of Chow motives. Here, we make explicit the inverse to this isomorphism. An application is given by Proposition 5.4. The results of this section will not be used in the rest of the paper.

We have the following fibre square

$$\begin{tikzcd} D \arrow[r, "\widetilde{j}"] \arrow[d, "\tau_D"'] & \widetilde{X}_Y \arrow[d, "\tau"] \\ Y \arrow[r, "j"] & X \end{tikzcd}$$

where $\widetilde{j} : D \to \widetilde{X}_Y$ is the exceptional divisor and where $\tau_D : D \to Y$ is a $\mathbf{P}^{r-1}$-bundle over Y. Precisely $D = \mathbf{P}(\mathcal{N}_{Y/X})$ is the projective bundle over Y associated to the normal bundle $\mathcal{N}_{Y/X}$ of Y inside X. The tautological line bundle on $D = \mathbf{P}(\mathcal{N}_{Y/X})$ is $\mathcal{O}_{\mathbf{P}(\mathcal{N}_{Y/X})}(-1) = \mathcal{O}_{\widetilde{X}_Y}(D)|_D$. Let

$$H^{r-1} \subset \ldots \subset H^i \subset \ldots \subset H \subset D$$

be linear sections of D corresponding to the relatively ample line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{N}_{Y/X})}(1)$, where H^i has codimension i. Thus, if $D \hookrightarrow \mathbf{P}^M \times Y$ is an embedding over Y corresponding to $\mathcal{O}_{\mathbf{P}(\mathcal{N}_{Y/X})}(1)$, then H^i denotes the smooth intersection of D with $L^i \times Y$ for some linear subspace L^i of codimension i inside $\mathbf{P}^M$. Let us write $\iota^i : H^i \hookrightarrow D$ for the inclusion maps.
As we will be using repeatedly Manin's identity principle, let us mention that if Z is a smooth projective variety over k, then the blow-up of $X \times Z$ along $Y \times Z$ canonically identifies with $\tau \times \mathrm{id}_Z : \widetilde{X}_Y \times Z \to X \times Z$. We write H^i_Z for $H^i \times Z$.

Let us define the morphism of motives

$$\Phi := {}^t\Gamma_\tau \oplus \bigoplus_{i=1}^{r-1} \Gamma_{\widetilde{j}} \circ h^{r-1-i} \circ {}^t\Gamma_{\tau_D} \ : \ \mathfrak{h}(X) \oplus \bigoplus_{i=1}^{r-1} \mathfrak{h}(Y)(i) \longrightarrow \mathfrak{h}(\widetilde{X}_Y).$$

Here, h^l is the correspondence $\Gamma_{\iota^l} \circ {}^t\Gamma_{\iota^l}$; it coincides with the l-fold composite of $h := \Gamma_{\iota^1} \circ {}^t\Gamma_{\iota^1}$ with itself.
On the one hand, we have the following blow-up formula for Chow groups; see [16].

Proposition 5.1. *The induced map*

$$\Phi_* = \tau^* \oplus \bigoplus_{i=1}^{r-1} \widetilde{j}_* h^{r-1-i} {\tau_D}^* \ : \ \mathrm{CH}_l(X) \oplus \bigoplus_{i=1}^{r-1} \mathrm{CH}_{l-i}(Y) \longrightarrow \mathrm{CH}_l(\widetilde{X}_Y)$$

is an isomorphism. □

On the other hand, we define

$$\Psi := \Gamma_\tau \oplus \bigoplus_{i=1}^{r-1} (-1) \cdot \Gamma_{\tau_D} \circ h^{i-1} \circ {}^t\Gamma_{\widetilde{j}} \ : \ \mathfrak{h}(\widetilde{X}_Y) \longrightarrow \mathfrak{h}(X) \oplus \bigoplus_{i=1}^{r-1} \mathfrak{h}(Y)(i).$$

Let $(\Psi\circ\Phi)_{i,j}$ be the $(i,j)^{\text{th}}$ component of $\Psi\circ\Phi$, where $\mathfrak{h}(X)$ is by definition the 0^{th} coordinate of $\mathfrak{h}(X)\oplus\bigoplus_{i=1}^{r-1}\mathfrak{h}(Y)(i)$. Thus, if $i,j\neq 0$, then $(\Psi\circ\Phi)_{i,j}$ is a morphism $\mathfrak{h}(Y)(j)\to\mathfrak{h}(Y)(i)$; if $i\neq 0$, then $(\Psi\circ\Phi)_{i,0}$ is a morphism $\mathfrak{h}(X)\to\mathfrak{h}(Y)(i)$; if $j\neq 0$, then $(\Psi\circ\Phi)_{0,j}$ is a morphism $\mathfrak{h}(Y)(j)\to\mathfrak{h}(X)$; and $(\Psi\circ\Phi)_{0,0}$ is a morphism $\mathfrak{h}(X)\to\mathfrak{h}(X)$.
The following lemma shows that $\Psi\circ\Phi$ is a lower triangular matrix with invertible diagonal elements.

Lemma 5.2. *We have*

$$(\Psi\circ\Phi)_{i,j}=\begin{cases}0 & \text{if } i<j\\ 0 & \text{if } ij=0 \text{ unless } i=j=0\\ \Delta_X & \text{if } i=j=0\\ \Delta_Y & \text{if } i=j>0\end{cases}$$

Proof. The proposition consists of the following relations:

(1) $\Gamma_\tau\circ{}^t\Gamma_\tau=\Delta_X$.
(2) $\Gamma_{\tau_D}\circ h^{i-1}\circ{}^t\Gamma_{\widetilde{j}}\circ\Gamma_{\widetilde{j}}\circ h^{r-1-j}\circ{}^t\Gamma_{\tau_D}=0$ for all $1\leq i<j\leq r-1$.
(3) $\Gamma_{\tau_D}\circ h^{i-1}\circ{}^t\Gamma_{\widetilde{j}}\circ\Gamma_{\widetilde{j}}\circ h^{r-1-i}\circ{}^t\Gamma_{\tau_D}=-\Delta_Y$ for all $1\leq i\leq r-1$.
(4) $\Gamma_\tau\circ\Gamma_{\widetilde{j}}\circ h^{r-1-i}\circ{}^t\Gamma_{\tau_D}=0$ for all $1\leq i\leq r-1$.

Let us establish them. The morphism τ is a birational morphism so that the identity (1) follows from the projection formula as in the proof of Lemma 1.2. The proof of (4) is a combination of the fact that $\tau\circ\widetilde{j}=j\circ\tau_D$ and Lemma 1.3. As for (2) and (3), we claim that

$${}^t\Gamma_{\widetilde{j}}\circ\Gamma_{\widetilde{j}}=-\Gamma_\iota\circ{}^t\Gamma_\iota=-h\in\mathrm{CH}_{d_X-1}(D\times D).$$

Indeed, the action of h on $\mathrm{CH}_*(D)$ is given by intersecting with the class of H. Also, by [7, Prop. 2.6], the map $\widetilde{j}^*\widetilde{j}_*:\mathrm{CH}_*(D)\to\mathrm{CH}_{*-1}(D)$ is given by intersecting with the class of $D|_D$ which is precisely $-h$. The same arguments for the smooth blow-up of $X\times Z$ along $Y\times Z$ (whose exceptional divisor is $D\times Z\hookrightarrow\widetilde{X}_Y\times Z$), together with Manin's identity principle, yield the claim. In view of the above claim, (2) follows from Lemma 1.3 and (3) follows from Lemma 1.2. □

Thus the endomorphism

$$N:=\begin{pmatrix}\Delta_X & 0 & 0 & \cdots & 0\\ 0 & \Delta_Y & 0 & \cdots & 0\\ 0 & 0 & \ddots & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & \Delta_Y\end{pmatrix}-\Psi\circ\Phi$$

is a nilpotent endomorphism of $\mathfrak{h}(X)\oplus\bigoplus_{i=1}^{r-1}\mathfrak{h}(Y)(i)$ of index $\leq r-1$, *i.e.* $N^{r-1}=0$. The morphism

$$\Theta:=(\mathrm{id}+N+\ldots+N^{r-2})\circ\Psi$$

thus gives a left-inverse to Φ.

The main result of this section is then the following theorem.

Theorem 5.3. *The morphism Θ is the inverse of Φ.*

Proof. Let $p := \Phi \circ \Theta \in \mathrm{End}(\mathfrak{h}(\widetilde{X}_Y))$. Because Θ is a left-inverse to Φ, we see that p is an idempotent. The motive of $\widetilde{X}_Y$ thus splits as

$$\mathfrak{h}(\widetilde{X}_Y) = (\widetilde{X}_Y, p) \oplus (\widetilde{X}_Y, \mathrm{id} - p).$$

As a consequence of Proposition 5.1 and Lemma 5.2, we obtain that

$$\mathrm{CH}_*(\widetilde{X}_Y, p) = \mathrm{CH}_*(\widetilde{X}_Y).$$

Actually, since $(\widetilde{X \times Z})_{Y \times Z}$ canonically identifies with $\widetilde{X}_Y \times Z$, we get, thanks to [7, Prop. 16.1.1], that $\Phi \times \Delta_Z$ induces an isomorphism of Chow groups as in Proposition 5.1 and that $\Theta \times \Delta_Z$ is a left-inverse to $\Phi \times \Delta_Z$. Thus, we also have that

$$\mathrm{CH}_*(\widetilde{X}_Y \times Z, p \times \Delta_Z) = \mathrm{CH}_*(\widetilde{X}_Y \times Z).$$

Therefore

$$\mathrm{CH}_*(\widetilde{X}_Y \times Z, \Delta_{\widetilde{X}_Y} \times \Delta_Z - p \times \Delta_Z) = 0.$$

By Manin's identity principle it follows that

$$p = \Delta_{\widetilde{X}_Y}.$$

In other words, Θ is not only a left-inverse to Φ, it is also the inverse of Φ. $\square$

Let us now use Theorem 5.3 to study the birational invariance of some groups of algebraic cycles attached to smooth projective varieties. For a smooth projective variety X over k, we write $\mathrm{Griff}_l(X)$ for its Griffiths group $\mathrm{CH}_l(X)_{\mathrm{hom}}/\mathrm{CH}_l(X)_{\mathrm{alg}}$. We also write, when $k \subseteq \mathbf{C}$,

$$T^l(X) := \mathrm{Ker}\left(AJ^l : \mathrm{CH}^l(X)_{\mathrm{hom}} \to J^l(X)\right)$$

for the kernel of Griffiths' Abel–Jacobi map to the intermediate Jacobian $J^l(X)$ which is a quotient of $\mathrm{H}^{2l-1}(X, \mathbb{C})$.

If $\pi : \widetilde{X} \to X$ is a birational map, the projection formula implies that $\Gamma_\pi \circ {}^t\Gamma_\pi = \Delta_X$; see Lemma 1.2. Thus $\pi_*\pi^*$ acts as the identity on $\mathrm{CH}_l(X)$, $\mathrm{Griff}_l(X)$ and on $T^l(X)$. The following proposition shows that in some cases π_* and π^* are actually inverse to each other.

Proposition 5.4. *Let $\pi : \widetilde{X} \to X$ be a birational map between smooth projective varieties. Then $\pi^*\pi_*$ acts as the identity on $\mathrm{CH}_0(\widetilde{X})$, $\mathrm{Griff}_1(\widetilde{X})$, $\mathrm{Griff}^2(\widetilde{X})$, $T^2(\widetilde{X})$, $\mathrm{CH}^1(\widetilde{X})_{\mathrm{hom}}$ and $\mathrm{CH}^0(\widetilde{X})$.*

Proof. By resolution of singularities, there are morphisms $f : Y \to \widetilde{X}$ and $g : Y \to X$, which are composite of smooth blow-ups, such that $g = \pi \circ f$. The groups considered in the proposition behave functorially with respect to the action of correspondences. Therefore it is enough to prove the proposition when π is a smooth blow-up $\widetilde{X}_Y \to X$ as above. First note that $\Psi_i := \mathfrak{h}(\widetilde{X}_Y) \to \mathfrak{h}(Y)(i)$ acts as zero on the groups considered in the proposition when $1 \leq i \leq r-1$, for dimension reasons. Having in mind that the first column and the first row of N are zero, and expanding $\Phi \circ \Theta$, we see that $\Phi \circ \Theta$ acts like $\pi^*\pi_*$ on the groups of the proposition. By Theorem 5.3, the correspondence $\Phi \circ \Theta$ acts as the identity on $\mathrm{CH}_l(\widetilde{X})$. Thus $\pi^*\pi_*$ acts as the identity on the groups of the proposition. $\square$

Remark 5.5. As a consequence of Theorem 5.3, we obtain an explicit Chow–Künneth decomposition for a smooth blow-up $\widetilde{X}_Y$ in terms of Chow–Künneth decompositions of X and Y. Precisely, assume that X and Y are endowed with Chow–Künneth decompositions $\{\pi_X^i, 0 \leq i \leq 2\dim X\}$ and $\{\pi_Y^i, 0 \leq i \leq 2\dim Y\}$, respectively. Let us define

$$\pi^i_{\widetilde{X}_Y} := \Phi \circ \big(\pi^i_X \oplus \bigoplus_{j=1}^{r-1} \pi_Y^{i-2j}\big) \circ \Theta \in \mathrm{End}\big(\mathfrak{h}(\widetilde{X}_Y)\big).$$

Then $\{\pi^i_{\widetilde{X}_Y}, 0 \leq i \leq 2\dim \widetilde{X}_Y\}$ is a Chow–Künneth decomposition of $\widetilde{X}_Y$.

6. Chow groups of varieties fibred by varieties with small Chow groups

In this section, we consider a projective surjective morphism $f : X \to B$ onto a quasi-projective variety B. We are interested in obtaining information on the niveau of the Chow groups of X, under the assumption that the Chow groups of the fibres of f over generic points of closed subvarieties of B are either trivial ($= \mathbf{Q}$) or finitely generated. Contrary to Section 3, we do not assume that f is flat. Let l be a non-negative integer and let d_X and d_B be the dimensions of X and B, respectively. In §6.1, we assume that $\mathrm{CH}_{l-i}(X_{\eta_{D_i}}) = \mathbf{Q}$ for all $0 < i \leq d_B$ and all irreducible subvarieties $D_i \subset B$ of dimension i, and we deduce in Lemma 6.1 by a localisation sequence argument that $\mathrm{CH}_l(X)$ is spanned by $\mathrm{CH}_l(X_b)$ for all closed points b in B and by $\mathrm{CH}_l(H)$, where $H \hookrightarrow X$ is a linear section of X of dimension $d_X + l$. We then move on to study the subspace of $\mathrm{CH}_l(X)$ spanned by $\mathrm{CH}_l(X_b)$ for all closed points b in B in the case when $\mathrm{CH}_l(X_b) = \mathbf{Q}$ in §6.2, and in the case when $\mathrm{CH}_l(X_b)$ finitely generated in §6.3. The results of §§6.1 & 6.2 are combined into Proposition 6.5, while the results of §§6.1 & 6.3 are combined into Propositions 6.8 and 6.9 in §6.4. Finally, in §6.5, we use Lemma 2.1 to give statements that only involve the Chow groups of closed fibres when f is defined over the complex numbers; see Theorems 6.10, 6.12 and 6.13.

6.1. Some general statements.

Lemma 6.1. *Let $f : X \to B$ be a projective surjective morphism onto a quasi-projective variety B and let $H \hookrightarrow X$ be a linear section of dimension $\geq l + d_B$. Assume that $\mathrm{CH}_{l-i}(X_{\eta_{D_i}}) = \mathbf{Q}$ for all $0 < i \leq d_B$ and all irreducible subvarieties $D_i \subset B$ of dimension i. Then the natural map $\bigoplus_{b\in B} \mathrm{CH}_l(X_b) \oplus \mathrm{CH}_l(H) \to \mathrm{CH}_l(X)$ is surjective.*

Proof. We prove the proposition by induction on d_B. If $d_B = 0$ then the statement is obvious. Let us thus consider a morphism $f : X \to B$ and a linear section $\iota : H \hookrightarrow X$ as in the statement of the proposition with $d_B > 0$. By Lemma 1.1, f restricted to H is surjective. We have the localisation exact sequence

$$\bigoplus_{D \in B^1} \mathrm{CH}_l(X_D) \to \mathrm{CH}_l(X) \to \mathrm{CH}_{l-d_B}(X_{\eta_B}) \to 0.$$

Here, B^1 denotes the set of codimension-one closed irreducible subschemes of B. For any irreducible codimension-one subvariety $D \subset B$, the restriction of ι to $D \to B$ defines a linear section $\iota_D : H_D \hookrightarrow X_D$ of dimension $\geq l + d_B - 1$ of X_D. The restriction of $f : X \to B$ to $D \to B$ defines a surjective morphism $X_D \to D$ which together with the linear section ι_D satisfies the assumptions of the proposition. Therefore, by the induction hypothesis applied to $X_D \to D$, the map

$$\bigoplus_{d \in D} \mathrm{CH}_l(X_d) \oplus \mathrm{CH}_l(H_D) \to \mathrm{CH}_l(X_D)$$

is surjective. This yields an exact sequence

$$\bigoplus_{b \in B} \mathrm{CH}_l(X_b) \oplus \bigoplus_{D \in B^1} \mathrm{CH}_l(H_D) \to \mathrm{CH}_l(X) \to \mathrm{CH}_{l-d_B}(X_{\eta_B}) \to 0.$$

Since each of the proper inclusion maps $H_D \to X$ factors through $\iota : H \to X$, we see that the map $\bigoplus_{D \in B^1} \mathrm{CH}_l(H_D) \xrightarrow{\oplus(\iota_D)_*} \mathrm{CH}_l(X)$ factors through $\iota_* : \mathrm{CH}_l(H) \to \mathrm{CH}_l(X)$. In order to conclude, it is enough to prove that the composite map

$$\mathrm{CH}_l(H) \to \mathrm{CH}_l(X) \to \mathrm{CH}_{l-d_B}(X_{\eta_B})$$

is surjective. If $l < d_B$, then this is obvious. Let us then assume that $l \geq d_B$. Let Y be an irreducible subvariety of H of dimension l such that the composite $Y \hookrightarrow X \to B$ is dominant. Because $\mathrm{CH}_{l-d_B}(X_{\eta_B}) = \mathbf{Q}$ it is enough to see that the class of Y in $\mathrm{CH}_l(X)$ maps to a non-zero element in $\mathrm{CH}_{l-d_B}(X_{\eta_B})$. But, as in the proof of Proposition 3.1, $[Y]$ maps to $[Y_{\eta_B}] \neq 0 \in \mathrm{CH}_{l-d_B}(X_{\eta_B})$. $\square$

Here is an improvement of Lemma 6.1:

Lemma 6.2. *Let $f : X \to B$ be a projective surjective morphism onto a quasi-projective variety B and let $H \hookrightarrow X$ be a linear section of dimension $\geq l + d_B$. Assume that:*

- $\mathrm{CH}_{l-i}(X_{\eta_{D_i}}) = \mathbf{Q}$ *for all i such that $0 < i < d_B$ and all irreducible subvarieties $D_i \subset B$ of dimension i.*
- $\mathrm{CH}_{l-d_B}(X_{\eta_B})$ *is finitely generated.*

Then there exist finitely many closed subschemes $\mathcal{Z}_j$ of X of dimension l such that the natural map $\bigoplus_j \mathrm{CH}_l(\mathcal{Z}_j) \oplus \bigoplus_{b \in B} \mathrm{CH}_l(X_b) \oplus \mathrm{CH}_l(H) \to \mathrm{CH}_l(X)$ is surjective.

Proof. If $d_B = 0$ then the statement is obvious. Let us thus consider a morphism $f : X \to B$ and a linear section $\iota : H \hookrightarrow X$ as in the statement of the proposition with $d_B > 0$. The morphism f restricted to H is surjective; see Lemma 1.1. As in the proof of Lemma 6.1 we have the localisation exact sequence

$$\bigoplus_{D \in B^1} \mathrm{CH}_l(X_D) \to \mathrm{CH}_l(X) \to \mathrm{CH}_{l-d_B}(X_{\eta_B}) \to 0.$$

Each of the morphisms $X_D \to D$ satisfies the assumptions of Lemma 6.1 and by the same arguments as in the proof of Lemma 6.1 we get that the image of the map $\bigoplus_{b \in B} \mathrm{CH}_l(X_b) \oplus \mathrm{CH}_l(H) \to \mathrm{CH}_l(X)$ contains the image of the map $\bigoplus_{D \in B^1} \mathrm{CH}_l(X_D) \to \mathrm{CH}_l(X)$. Let now Z_j be finitely many closed subschemes of X_{η_B} whose classes $[Z_j] \in \mathrm{CH}_{l-d_B}(X_{\eta_B})$ generate $\mathrm{CH}_{l-d_B}(X_{\eta_B})$. By surjectivity of the map $\mathrm{CH}_l(X) \to \mathrm{CH}_{l-d_B}(X_{\eta_B})$ there are cycles $\alpha_j \in \mathrm{CH}_l(X)$ that map to $[Z_j]$. If $\mathcal{Z}_j$ is the support in X of any representative of α_j, we then have a surjective map $\bigoplus_j \mathrm{CH}_l(\mathcal{Z}_j) \to \mathrm{CH}_l(X) \to \mathrm{CH}_{l-d_B}(X_{\eta_B})$. It is then clear that the map $\bigoplus_j \mathrm{CH}_l(\mathcal{Z}_j) \oplus \bigoplus_{b \in B} \mathrm{CH}_l(X_b) \oplus \mathrm{CH}_l(H) \to \mathrm{CH}_l(X)$ is surjective. □

6.2. VARIETIES FIBRED BY VARIETIES WITH CHOW GROUPS GENERATED BY A LINEAR SECTION.

LEMMA 6.3. *Let $f : X \to B$ be a projective surjective morphism onto a quasi-projective variety B. Assume that $\mathrm{CH}_l(X_b) = \mathbf{Q}$ for all closed points $b \in B$. Then, if $H \hookrightarrow X$ is a linear section of dimension $\geq l + d_B$, we have*

$$\mathrm{Im}\big(\bigoplus_{b \in B} \mathrm{CH}_l(X_b) \to \mathrm{CH}_l(X)\big) \subseteq \mathrm{Im}\big(\mathrm{CH}_l(H) \to \mathrm{CH}_l(X)\big).$$

Proof. Let b be a closed point of B and fix $H \hookrightarrow X$ a linear section of dimension $\geq l + d_B$. The morphism f restricted to H is surjective; see Lemma 1.1. Let Z_l be an irreducible closed subscheme of X of dimension l which is supported on X_b. Since $f|_H : H \to B$ is a dominant projective morphism, its fibre H_b over b is non-empty and has dimension $\geq l$. By assumption $\mathrm{CH}_l(X_b) = \mathbf{Q}$, so that a rational multiple of $[Z_l]$ is rationally equivalent to an irreducible closed subscheme of H_b of dimension l. Therefore $[Z_l] \in \mathrm{CH}_l(X_b)$ belongs to the image of the natural map $\mathrm{CH}_l(H_b) \to \mathrm{CH}_l(X_b)$. Thus the image of $\mathrm{CH}_l(X_b) \to \mathrm{CH}_l(X)$ is contained in the image of $\mathrm{CH}_l(H) \to \mathrm{CH}_l(X)$. □

Remark 6.4. It is interesting to decide whether or not it is possible to parametrise such l-cycles by a variety of dimension d_B; see Proposition 6.7.

PROPOSITION 6.5. *Let $f : X \to B$ be a projective surjective morphism onto a quasi-projective variety B. Assume that $\mathrm{CH}_{l-i}(X_{\eta_{D_i}}) = \mathbf{Q}$ for all $0 \leq i \leq d_B$ and all irreducible subvarieties $D_i \subset B$ of dimension i. Then, if $H \hookrightarrow X$ is a linear section of dimension $\geq l + d_B$, the pushforward map $\mathrm{CH}_l(H) \to \mathrm{CH}_l(X)$ is surjective. In particular, $\mathrm{CH}_l(X)$ has niveau $\leq d_B$.*

Proof. This is a combination of Lemma 6.1 and Lemma 6.3. □

6.3. AN ARGUMENT INVOLVING RELATIVE HILBERT SCHEMES. Let $f : X \to B$ be a generically smooth, projective morphism defined over the field of complex numbers $\mathbf{C}$ onto a smooth quasi-projective variety B. Let $B^\circ \subseteq B$ be the smooth locus of f and let $f^\circ : X^\circ \to B^\circ$ be the pullback of $f : X \to B$ along the open inclusion $B^\circ \hookrightarrow B$ so that we have a cartesian square

$$\begin{tikzcd} X^\circ \arrow[r, hook] \arrow[d, "f^\circ"'] & X \arrow[d, "f"] \\ B^\circ \arrow[r, hook] & B. \end{tikzcd}$$

We assume that there is a non-negative integer l such that for all closed points $b \in B^\circ(\mathbf{C})$ the cycle class map $\mathrm{CH}_l(X_b) \to \mathrm{H}_{2l}(X_b)$ is an isomorphism.

Let $\pi_d : \mathrm{Hilb}^d_l(X/B) \to B$ be the relative Hilbert scheme whose fibres over the points b in B parametrise the closed subschemes of X_b of dimension l and degree d, and let $p_d : \mathcal{C}^d_l \to \mathrm{Hilb}^d_l(X/B)$ be the universal family over $\mathrm{Hilb}^d_l(X/B)$; see [14, Theorem 1.4]. We have the following commutative diagram, where all the morphisms involved are proper:

$$\begin{tikzcd} \mathcal{C}^d_l \arrow[r, "q_d"] \arrow[d, "p_d"'] & X. \arrow[ddl] \\ \mathrm{Hilb}^d_l(X/B) \arrow[d, "\pi_d"'] & \\ B & \end{tikzcd}$$

We then consider the disjoint unions $\mathrm{Hilb}_l(X/B) := \coprod_{d\geq 0} \mathrm{Hilb}^d_l(X/B)$ and $\mathcal{C}_l := \coprod_{d\geq 0} \mathcal{C}^d_l$, and denote $\pi : \mathrm{Hilb}_l(X/B) \to B$, $p : \mathcal{C}_l \to \mathrm{Hilb}_l(X/B)$ and $q : \mathcal{C}_l \to X$ the corresponding maps.
Let us then denote

$\mathrm{Irr}_l(X/B) := \{\mathcal{H} : \mathcal{H}$ is an irreducible component of $\mathrm{Hilb}^d_l(X/B)$ for some $d\}$.

For a subset $\mathcal{E} \subset \mathrm{Irr}_l(X/B)$, we define the following closed subscheme of B°:

$$Z_{\mathcal{E}} := B^\circ \cap \bigcap_{\mathcal{H}\in\mathcal{E}} \pi(\mathcal{H}).$$

We say that a finite subset $\mathcal{E}$ of $\mathrm{Irr}_l(X/B)$ is *spanning* at a point $t \in Z_{\mathcal{E}}(\mathbf{C})$ if $\mathrm{H}_{2l}(X_t)$ is spanned by the set $\{\, cl(q_*[p^{-1}(u)]) : u \in \mathcal{H}, \mathcal{H} \in \mathcal{E}, \pi(u) = t \,\}$. Note that, given $\mathcal{H} \in \mathrm{Irr}_l(X/B)$ and $u, u' \in \mathcal{H}$ such that $\pi(u) = \pi(u') = t$, $cl(q_*[p^{-1}(u)]) = cl(q_*[p^{-1}(u')]) \in \mathrm{H}_{2l}(X_t)$ if u and u' belong to the same connected component in $\pi^{-1}(t)$.

CLAIM. Let $\mathcal{E}$ be a finite subset of $\mathrm{Irr}_l(X/B)$ that is spanning at a closed point $t \in Z_{\mathcal{E}}(\mathbf{C})$. Then, for all points $s \in Z_{\mathcal{E}}(\mathbf{C})$ belonging to an irreducible component of $Z_{\mathcal{E}}$ that contains t, $\mathrm{H}_{2l}(X_s)$ is spanned by the set $\{\, cl(q_*[p^{-1}(v)]) : v \in \mathcal{H}, \mathcal{H} \in \mathcal{E}, \pi(v) = s \,\}$.

Indeed, consider any finite subset $\mathcal{E}$ of $\mathrm{Irr}_l(X/B)$. The local system of $\mathbf{Q}$-vector spaces $R_{2l}(f_0)_*\mathbf{Q}$ on B° restricts to a local system $(R_{2l}(f_0)_*\mathbf{Q})|_{Z_\mathcal{E}}$ on $Z_\mathcal{E}$. If t is a complex point on $Z_\mathcal{E}$, let r_t be the rank of the subspace of $\mathrm{H}_{2l}(X_t)$ spanned by $\{\, cl(q_*[p^{-1}(u)]) : u \in \mathcal{H}, \mathcal{H} \in \mathcal{E}, \pi(u) = t\,\}$. If we see this latter set as a set of sections at t of the local system $R_{2l}(f_0)_*\mathbf{Q}$, then these sections extend locally to constant sections of the local system $(R_{2l}(f_0)_*\mathbf{Q})|_{Z_\mathcal{E}}$ on $Z_\mathcal{E}$. This shows that the rank r_t is locally constant. If $\mathcal{E}$ is spanning at the point $t \in Z_\mathcal{E}(\mathbf{C})$, then r_t is maximal, equal to $\dim_\mathbf{Q} \mathrm{H}_{2l}(X_t)$. The subset of $Z_\mathcal{E}(\mathbf{C})$ consisting of points s in $Z_\mathcal{E}(\mathbf{C})$ for which $r_s = \dim_\mathbf{Q} \mathrm{H}_{2l}(X_s)$ is therefore both open and closed in $Z_\mathcal{E}$. It contains then the irreducible components of $Z_\mathcal{E}$ that contain t.

Lemma 6.6. *There exists a finite subset $\mathcal{E}$ of $\mathrm{Irr}_l(X/B)$ such that $B^\circ = Z_\mathcal{E}$ and such that $\mathcal{E}$ is spanning at every point $t \in B^\circ(\mathbf{C})$.*

Proof. By working component-wise, we may assume that B is irreducible. By assumption on $f : X \to B$, $\mathrm{H}_{2l}(X_t)$ is spanned by algebraic cycles on X_t for all points $t \in B^\circ(\mathbf{C})$. Thus, for all points $t \in B^\circ(\mathbf{C})$, there is a finite subset $\mathcal{E}_t$ of $\mathrm{Irr}_l(X/B)$ that is spanning at t. For each point t, choose an irreducible component $Y_{\mathcal{E}_t}$ of $Z_{\mathcal{E}_t}$ that contains t. According to the claim above, $\mathcal{E}_t$ is spanning at every point $s \in Y_{\mathcal{E}_t}(\mathbf{C})$. Now, we have $B^\circ(\mathbf{C}) = \coprod_{t \in B^\circ(\mathbf{C})} Y_{\mathcal{E}_t}(\mathbf{C})$. Since there are only countably many finite subsets of $\mathrm{Irr}_l(X/B)$ and since $Z_\mathcal{E}$ has only finitely many irreducible components, we see that the latter union is in fact a countable union. This yields that $B^\circ = Y_\mathcal{E}$ for some finite subset $\mathcal{E}$ of $\mathrm{Irr}_l(X/B)$ that is spanning at every point in $Y_\mathcal{E}(\mathbf{C})$. We then conclude that $B^\circ = Z_\mathcal{E}$ and that $\mathcal{E}$ is spanning at every point $t \in Y_\mathcal{E}(\mathbf{C}) = B^\circ(\mathbf{C})$. □

Proposition 6.7. *Let $f : X \to B$ be a generically smooth and projective morphism defined over $\mathbf{C}$ onto a smooth quasi-projective variety B. Let $B^\circ \subseteq B$ be the smooth locus of f. Assume that there is an integer $l \leq d_X - d_B$ such that for all closed points $b \in B^\circ(\mathbf{C})$ the cycle class map $\mathrm{CH}_l(X_b) \to \mathrm{H}_{2l}(X_b)$ is an isomorphism. Then $\mathrm{Im}\big(\bigoplus_{b \in B^\circ} \mathrm{CH}_l(X_b) \to \mathrm{CH}_l(X)\big)$ is supported on a closed subvariety of X of dimension $d_B + l$.*
If moreover X is smooth, then there exist a smooth quasi-projective variety $\widetilde{B}$ of dimension d_B and a correspondence $\Gamma \in \mathrm{CH}_{d_B+l}(\widetilde{B} \times X)$ such that $\Gamma_* : \mathrm{CH}_0(\widetilde{B}) \to \mathrm{CH}_l(X)$ is well-defined and

$$\mathrm{Im}\Big(\bigoplus_{b \in B^\circ} \mathrm{CH}_l(X_b) \to \mathrm{CH}_l(X)\Big) \;\subseteq\; \mathrm{Im}\big(\Gamma_* : \mathrm{CH}_0(\widetilde{B}) \to \mathrm{CH}_l(X)\big).$$

Proof. By Lemma 6.6, there exists a finite set $\mathcal{E}$ of irreducible components of $\mathrm{Hilb}_l(X/B)$ such that $B^\circ = Z_\mathcal{E}$ and such that for all points $t \in B^\circ(\mathbf{C})$ the set $\{\, cl(q_*[p^{-1}(u)]) : u \in \mathcal{H}, \mathcal{H} \in \mathcal{E}, \pi(u) = t\,\}$ spans $\mathrm{H}_{2l}(X_t)$. Denote $\mathcal{H}_i$ the irreducible components of $\mathrm{Hilb}_l(X/B)$ that belong to $\mathcal{E}$ and let $\widetilde{\mathcal{H}}_i \to \mathcal{H}_i$ be resolutions thereof. For all i, pick a smooth linear section $\widetilde{B}_i \to \widetilde{\mathcal{H}}_i$ of dimension d_B. Lemma 1.1 shows that $r_i : \widetilde{B}_i \to \widetilde{\mathcal{H}}_i \to \mathcal{H}_i \to B$ is surjective and a refinement of its proof shows that, for all points $b \in B(\mathbf{C})$, $r_i^{-1}(b)$ contains a point in every connected component of $\widetilde{\mathcal{H}}_{i,b}$. Consider then $p_i : (\mathcal{C}_l)|_{\widetilde{B}_i} \to \widetilde{B}_i$

the pullback of the universal family $p : \mathcal{C}_l \to \mathrm{Hilb}_l(X/B)$ along $\widetilde{B}_i \hookrightarrow \widetilde{\mathcal{H}}_i \to \mathcal{H}_i \hookrightarrow \mathrm{Hilb}_l(X/B)$. For each i, we have the following picture

$$\begin{tikzcd} (\mathcal{C}_l)|_{\widetilde{B}_i} \arrow[r, "q_i"] \arrow[d, "p_i"'] & X \\ \widetilde{B}_i & \end{tikzcd}$$

and we have

$$\mathrm{Im}\Big(\bigoplus_{b\in B^\circ} \mathrm{CH}_l(X_b) \to \mathrm{CH}_l(X)\Big) \ \subseteq\ \sum_i \mathrm{Im}\big((q_i)_* : \mathrm{CH}_l((\mathcal{C}_l)|_{\widetilde{B}_i}) \to \mathrm{CH}_l(X)\big)$$

so that the group on the left-hand side is supported on the union of the scheme-theoretic images of the morphisms q_i.

If X is smooth, we define $\Gamma_i \in \mathrm{CH}_{d_B+l}(\widetilde{B}_i \times X)$ to be the class of the image of $(\mathcal{C}_l)|_{\widetilde{B}_i}$ inside $\widetilde{B}_i \times X$. Because $q : \mathcal{C}_l^d \to X$ is proper for all $d \geq 0$, Γ_i has a representative which is proper over X. It is therefore possible [7, Remark 16.1] to define maps $(\Gamma_i)_* : \mathrm{CH}_0(\widetilde{B}_i) \to \mathrm{CH}_l(X)$ for all i. In fact, we have $(\Gamma_i)_* = (q_i)_* p_i^*$. Finally, we define $\widetilde{B}$ to be the disjoint union of the $\widetilde{B}_i$'s and $\Gamma \in \mathrm{CH}_{d_B+l}(\widetilde{B} \times X)$ to be the class of the disjoint union of the correspondences Γ_i. □

6.4. Complex varieties fibred by varieties with small Chow groups. From now on, the base field k is assumed to be the field of complex numbers $\mathbf{C}$.

Proposition 6.8. *Let $f : X \to C$ be a generically smooth, projective morphism defined over $\mathbf{C}$ to a smooth curve. Assume that*

- $\mathrm{CH}_l(X_c)$ *is finitely generated for all closed points* $c \in C$,
- $\mathrm{CH}_l(X_c) \to \mathrm{H}_{2l}(X_c)$ *is an isomorphism for a general closed point* $c \in C$,
- $\mathrm{CH}_{l-1}(X_\eta)$ *is finitely generated, where* η *is the generic point of* C.

Then $\mathrm{CH}_l(X)$ *has niveau* ≤ 1.

Proof. We have the localisation exact sequence

$$\bigoplus_{c\in C} \mathrm{CH}_l(X_c) \longrightarrow \mathrm{CH}_l(X) \longrightarrow \mathrm{CH}_{l-1}(X_\eta) \longrightarrow 0.$$

Let $Z_1, \ldots, Z_n$ be irreducible closed subschemes of X_η of dimension $l-1$ that span $\mathrm{CH}_{l-1}(X_\eta)$ and let $\mathcal{Z}_1, \ldots, \mathcal{Z}_n$ be closed subschemes of X of dimension l that restrict to $Z_1, \ldots, Z_n$ in X_η. Then by flat pullback the class of $\mathcal{Z}_j$ in $\mathrm{CH}_l(X)$ maps to the class of Z_j in $\mathrm{CH}_{l-1}(X_\eta)$ so that the composite map $\bigoplus_{j=1}^n \mathrm{CH}_l(\mathcal{Z}_j) \to \mathrm{CH}_l(X) \to \mathrm{CH}_{l-1}(X_\eta)$ is surjective.

Let $U \subseteq C$ be a Zariski-open subset of C such that for all closed points $c \in U$ the cycle class map $\mathrm{CH}_l(X_c) \to \mathrm{H}_{2l}(X_c)$ is an isomorphism. Up to shrinking U, we may assume that $f|_U : X|_U \to U$ is smooth. We may then apply

Proposition 6.7 to get a closed subscheme $\iota : D \hookrightarrow X$ of dimension $l+1$ such that $\iota_* \operatorname{CH}_l(D) \supseteq \operatorname{Im}\big(\bigoplus_{c\in U} \operatorname{CH}_l(X_c) \to \operatorname{CH}_l(X)\big)$.
As such, we have a surjective map

$$\bigoplus_{j=1}^{n} \operatorname{CH}_l(\mathcal{Z}_j) \oplus \bigoplus_{c\in C\setminus U} \operatorname{CH}_l(X_c) \oplus \operatorname{CH}_l(D) \longrightarrow \operatorname{CH}_l(X)$$

and it is straightforward to conclude. □

The next proposition is a generalisation of Proposition 6.8 to the case when the base variety B has dimension greater than 1.

PROPOSITION 6.9. *Let $f : X \to B$ be a generically smooth, projective morphism defined over $\mathbf{C}$ to a smooth quasi-projective variety B. Assume that the singular locus of f in B is finite and let U be the maximal Zariski-open subset of B over which f is smooth. Assume also that*

- $\operatorname{CH}_l(X_b)$ *is finitely generated for all closed points $b \in B$,*
- $\operatorname{CH}_l(X_b) \to \operatorname{H}_{2l}(X_b)$ *is an isomorphism for all closed points $b \in U$,*
- $\operatorname{CH}_{l-i}(X_{\eta_{D_i}}) = \mathbf{Q}$ *for all i such that $0 < i < d_B$ and all irreducible subvarieties $D_i \subset B$ of dimension i.*
- $\operatorname{CH}_{l-d_B}(X_{\eta_B})$ *is finitely generated.*

Then $\operatorname{CH}_l(X)$ has niveau $\leq d_B$.

Proof. Let $H \hookrightarrow X$ be a linear section of dimension $\geq l + d_B$. The restriction of f to H is surjective. Thanks to Lemma 6.2, there are finitely many closed subschemes $\mathcal{Z}_j$ of X of dimension l such that the natural map $\bigoplus_j \operatorname{CH}_l(\mathcal{Z}_j) \oplus \bigoplus_{b\in B} \operatorname{CH}_l(X_b) \oplus \operatorname{CH}_l(H) \to \operatorname{CH}_l(X)$ is surjective. By Proposition 6.7, there exists a closed subscheme $\iota : \widetilde{B} \hookrightarrow X$ of dimension $d_B + l$ such that the image of the map $\bigoplus_{b\in U} \operatorname{CH}_l(X_b) \to \operatorname{CH}_l(X)$ is contained in the image of the map $\iota_* : \operatorname{CH}_l(\widetilde{B}) \to \operatorname{CH}_l(X)$. Therefore the map

$$\bigoplus_{j} \operatorname{CH}_l(\mathcal{Z}_j) \oplus \operatorname{CH}_l(\widetilde{B}) \oplus \bigoplus_{b\in B\setminus U} \operatorname{CH}_l(X_b) \oplus \operatorname{CH}_l(H) \longrightarrow \operatorname{CH}_l(X)$$

is surjective. It is then straightforward to conclude. □

6.5. THE MAIN RESULTS. The field of complex numbers is a universal domain and in view of Section 2 we restate some of the results above in a more comprehensive way.

First, we deduce from Proposition 6.5 the following.

THEOREM 6.10. *Let $f : X \to B$ be a complex projective surjective morphism onto a quasi-projective variety B. Assume that $\operatorname{CH}_i(X_b) = \mathbf{Q}$ for all $i \leq l$ and all closed point $b \in B$. Then $\operatorname{CH}_i(X)$ has niveau $\leq d_B$ for all $i \leq l$.*

Proof. By Lemma 2.2, $\operatorname{CH}_i(X_{\eta_D}) = \mathbf{Q}$ for all $i \leq l$ and all irreducible subvarieties D of X. Proposition 6.5 implies that $\operatorname{CH}_i(X)$ has niveau $\leq d_B$. □

The following lemma gives a criterion for the second point in the assumptions of Propositions 6.8 and 6.9 to be satisfied.

LEMMA 6.11. *Let X be a smooth projective complex variety. Assume that* $\mathrm{CH}_i(X)$ *is finitely generated for all $i \leq l$. Then the cycle class map* $\mathrm{CH}_l(X) \to \mathrm{H}_{2l}(X)$ *is an isomorphism.*

Proof. The proof follows the same pattern as the proof of [24, Theorem 3.4] once it is noted that if $\mathrm{CH}_i(X)$ is finitely generated then the group of algebraically trivial cycles $\mathrm{CH}_i(X)_{\mathrm{alg}}$ is representable in the sense of [24, Definition 2.1]. Concretely, we get that the Chow motive of X is isomorphic to $\mathbb{1} \oplus \mathbb{1}(1)^{b_2} \oplus \ldots \oplus \mathbb{1}(l)^{b_{2l}} \oplus N(l+1)$ where b_i is the i-th Betti number of X and N is an effective motive. This yields that the cycle class map $\mathrm{CH}_i(X) \to \mathrm{H}_{2i}(X)$ is an isomorphism for all $i \leq l$. □

As a consequence of Proposition 6.8, we obtain

THEOREM 6.12. *Let $f : X \to C$ be a complex generically smooth projective surjective morphism onto a smooth complex curve. Assume that* $\mathrm{CH}_i(X_c)$ *is finitely generated for all closed points $c \in C$ and all $i \leq l$, then* $\mathrm{CH}_i(X)$ *has niveau ≤ 1 for all $i \leq l$.*

Proof. Let D be an irreducible component of C. Lemma 2.2 shows that $\mathrm{CH}_i(X_{\eta_D})$ is finitely generated for all $i \leq l$. Let $U \subseteq C$ be the smooth locus of f. It is an open dense subset of C. Then, for $c \in U$, the closed fibre X_c is smooth and the groups $\mathrm{CH}_i(X_c)$ are finitely generated for all $i \leq l$. By Lemma 6.11 the cycle class maps $\mathrm{CH}_i(X_c) \to \mathrm{H}_{2i}(X_c)$ are isomorphisms for all $c \in U$ and all $i \leq l$. Proposition 6.8 implies that $\mathrm{CH}_i(X)$ has niveau ≤ 1. □

And, as a consequence of Proposition 6.9, we obtain

THEOREM 6.13. *Let $f : X \to B$ be a complex projective surjective morphism onto a smooth quasi-projective variety B. Assume that the singular locus of f in B is finite, and assume also that*

- $\mathrm{CH}_i(X_b)$ *is finitely generated for all closed points $b \in B$ and all $i \leq l$,*
- $\mathrm{CH}_i(X_b) = \mathbf{Q}$ *for all but finitely many closed points $b \in B$ and all $i < l$.*

Then $\mathrm{CH}_i(X)$ *has niveau $\leq d_B$ for all $i \leq l$.*

Proof. Let D be an irreducible subvariety of X of positive dimension. By Lemma 2.2, we have $\mathrm{CH}_i(X_{\eta_D}) = \mathbf{Q}$ for all $i < l$. If we denote by U the smooth locus of f, then, by Lemma 6.11, the cycle class maps $\mathrm{CH}_i(X_b) \to \mathrm{H}_{2i}(X_b)$ are isomorphisms for all $b \in U$ and all $i \leq l$. Proposition 6.9 implies that $\mathrm{CH}_i(X)$ has niveau $\leq d_B$ for all $i \leq l$. □

7. APPLICATIONS

7.1. VARIETIES WITH SMALL CHOW GROUPS. In this section, we review the known results about varieties with Chow groups having small niveau; see Definition 2.3. Varieties are defined over an algebraically closed field k of characteristic zero and Ω denotes a universal domain over k. In that case, Grothendieck's standard conjectures for a smooth projective variety X over k reduce to the Lefschetz standard conjecture for X; see Kleiman [13].

Theorem 7.1. *Let X be a smooth projective variety of dimension d. Assume that the Chow groups $\mathrm{CH}_0(X_\Omega), \ldots, \mathrm{CH}_l(X_\Omega)$ have niveau $\leq n$.*

- *If $n = 3$ and $l = \lfloor \frac{d-4}{2} \rfloor$, then X satisfies the Hodge conjecture.*
- *If $n = 2$ and $l = \lfloor \frac{d-3}{2} \rfloor$, then X satisfies the Lefschetz standard conjecture.*
- *If $n = 1$ and $l = \lfloor \frac{d-3}{2} \rfloor$, then X has a Murre decomposition.*
- *If $n = 1$ and $l = \lfloor \frac{d-2}{2} \rfloor$, then X is Kimura finite-dimensional.*

Proof. The fourth item is proved in [26]. It is also proved there that if X is as in the third item, then X has a Chow–Künneth decomposition. That such a decomposition satisfies Murre's conjectures (B), (C) and (D) is proved in [25, §4.4.2]. The first item is proved in [15]. We couldn't find a reference for the proof of the second item so we include a proof here.

Since it is enough to prove the conclusion of the theorem for X_Ω, we may assume that X is defined over Ω. Laterveer used the assumptions on the niveau of the Chow groups to show [15, 1.7] that the diagonal Δ_X admits a decomposition as follows : there exist closed and reduced subschemes $V_j, W^j \subset X$ with $\dim V_j \leq j+2$ and $\dim W^j \leq n-j$, there exist correspondences $\Gamma_j \in \mathrm{CH}_n(X \times X)$ for $0 \leq j \leq \lfloor \frac{n-3}{2} \rfloor$ and $\Gamma' \in \mathrm{CH}_n(X \times X)$ such that each Γ_j is in the image of the pushforward map $\mathrm{CH}_n(V_j \times W^j)$, Γ' is in the image of the pushforward map $\mathrm{CH}_n(X \times W^{\lfloor \frac{n-1}{2} \rfloor})$, and

$$\Delta_X = \Gamma_0 + \ldots + \Gamma_{\lfloor \frac{n-3}{2} \rfloor} + \Gamma'.$$

Given j such that $0 \leq j \leq \lfloor \frac{n-3}{2} \rfloor$, let $\widetilde{V}_j$ and $\widetilde{W}^j$ denote desingularisations of V_j and W^j respectively. The action of Γ_j on $\mathrm{H}^k(X)$ then factors through $\mathrm{H}^k(\widetilde{V}_j)$ and through $\mathrm{H}_{2n-k}(\widetilde{W}^j)$. On the one hand, we have $\mathrm{H}_{2n-k}(\widetilde{W}^j) = \mathrm{H}^{k-2j}(\widetilde{W}^j)$ and hence if $k \leq 2j+1$ then the action of Γ_j on $\mathrm{H}^k(X)$ factors through the H^0 or the H^1 of a smooth projective variety. Since the Lefschetz standard conjecture is true in degrees ≤ 1, it follows that the action of Γ_j on $\mathrm{H}^k(X)$ factors through the H_0 or the H_1 of a smooth projective variety. On the other hand, we have $\mathrm{H}^k(\widetilde{V}_j) = \mathrm{H}_{4+2j-k}(\widetilde{V}_j)$ and hence if $k \geq 2j+2$ then Γ_j factors through the H_0, the H_1 or the H_2 of a smooth projective variety. Concerning the action of Γ' on $\mathrm{H}^k(X)$, it factors through $\mathrm{H}_{2n-k}(\widetilde{W}^{\lfloor \frac{n-1}{2} \rfloor})$ which vanishes for dimension reasons if $k < n$ when n is odd and if $k < n-1$ when n is even. When n is even and $k = n-1$, the action of Γ' on $\mathrm{H}^k(X)$ factors through the H_1 of a curve. Indeed this follows from a combination of the fact that it factors through $\mathrm{H}_{n+1}(\widetilde{W}^{\lfloor \frac{n-1}{2} \rfloor}) = \mathrm{H}^1(\widetilde{W}^{\frac{n-2}{2}})$ and of the validity of the Lefschetz standard conjecture in degree 1.

By the Lefschetz hyperplane theorem, we get that for $k < n$ the cohomology groups $\mathrm{H}^k(X)$ are generated algebraically (that is through the action of correspondences) by the H_0 of points, the H_1 of curves and the H_2 of surfaces. We may then conclude with [28, Proposition 3.19]. □

7.2. Varieties fibred by low-degree complete intersections. As explained by Esnault–Levine–Viehweg in the introduction of [6], it is expected from general conjectures on algebraic cycles, that if $Y \subset \mathbf{P}^n_k$ is a complete intersection of multidegree $d_1 \geq \ldots \geq d_r \geq 2$, then $\mathrm{CH}_l(Y) = \mathbf{Q}$ for all $l < \lfloor \frac{n-\sum_{i=2}^r d_i}{d_1} \rfloor$, see also Paranjape [22] and Schoen [23]. If there is no proof of the above for the moment, the following theorem however was proved.

Theorem 7.2 (Esnault–Levine–Viehweg [6]). *Let $Y \subset \mathbf{P}^n_k$ be a complete intersection of multidegree $d_1 \geq \ldots \geq d_r \geq 2$.*

- *If either $d_1 \geq 3$ or $r \geq l+1$, assume that $\sum_{i=1}^r \begin{pmatrix} l+d_i \\ l+1 \end{pmatrix} \leq n$.*
- *If $d_1 = \ldots = d_r = 2$ and $r \leq l$, assume that $\sum_{i=1}^r \begin{pmatrix} l+d_i \\ l+1 \end{pmatrix} = r(l+2) \leq n-l+r-1$.*

Then $\mathrm{CH}_{l'}(Y) = \mathbf{Q}$ for all $0 \leq l' \leq l$. □

Let us consider $f : X \to B$ a dominant morphism between smooth projective complex varieties whose closed fibres are complete intersections. Theorem 6.10, together with Theorem 7.2, shows that the niveau of the first Chow groups of X have niveau $\leq \dim B$. When X is fibred by very low-degree complete intersections, we can thus expect X to satisfy the assumptions of Theorem 7.1. In the remainder of this paragraph, we inspect various such cases.

7.2.1. *Varieties fibred by quadric hypersurfaces.* Let $Q \subset \mathbf{P}^n$ be a quadric hypersurface. Then $\mathrm{CH}_l(Q) = \mathbf{Q}$ for all $l < \frac{\dim Q}{2}$.

Proposition 7.3. *Let $f : X \to B$ be a dominant morphism between smooth projective complex varieties whose closed fibres are quadric hypersurfaces.*

- *If $\dim B \leq 1$, then X is Kimura finite-dimensional and satisfies Murre's conjectures.*
- *If $\dim B \leq 2$, then X satisfies Grothendieck's standard conjectures.*
- *If $\dim B \leq 3$, then X satisfies the Hodge conjecture.*

Proof. The fibres of f have dimension $\geq \dim X - \dim B$, so that X satisfies the assumptions of Theorem 6.10 with $l = \lfloor \frac{d_X - d_B - 1}{2} \rfloor$. Thus the Chow groups $\mathrm{CH}_0(X), \mathrm{CH}_1(X), \ldots, \mathrm{CH}_{\lfloor \frac{d_X - d_B - 1}{2} \rfloor}(X)$ have niveau $\leq d_B$. We can therefore conclude by Theorem 7.1. □

7.2.2. *Varieties fibred by cubic hypersurfaces.* Let $X \subset \mathbf{P}^n$ be a cubic hypersurface. Then

- $\mathrm{CH}_0(X) = \mathbf{Q}$ for $\dim X \geq 2$.
- $\mathrm{CH}_1(X) = \mathbf{Q}$ for $\dim X \geq 5$.
- $\mathrm{CH}_2(X) = \mathbf{Q}$ for $\dim X \geq 8$.

Note that Theorem 7.2 only gives $\mathrm{CH}_2(X) = \mathbf{Q}$ for $\dim X \geq 9$. The bound on the dimension of X was improved to $\dim X = 8$ by Otwinowska [21].

Proposition 7.4. *Let $f : X \to B$ be a dominant morphism between smooth projective complex varieties whose closed fibres are cubic hypersurfaces.*

- *If* $\dim X = 6$ *and* $\dim B = 1$*, then* X *satisfies Grothendieck's standard conjectures and has a Murre decomposition.*
- *If* $\dim X = 7$ *and if* $\dim B \leq 2$*, then* X *satisfies the Hodge conjecture.*
- *If* $\dim X = 9$ *and if* $\dim B \leq 1$*, then* X *satisfies the Hodge conjecture.*

Proof. We use Theorem 6.10 as in the proof of Proposition 7.3. In the first case, we get that $\mathrm{CH}_0(X)$ and $\mathrm{CH}_1(X)$ have niveau ≤ 1. In the second case we get that $\mathrm{CH}_0(X)$ and $\mathrm{CH}_1(X)$ have niveau ≤ 2 and in the third case we get that $\mathrm{CH}_0(X)$, $\mathrm{CH}_1(X)$ and $\mathrm{CH}_2(X)$ have niveau ≤ 1. We can then conclude in all three cases by Theorem 7.1. □

7.2.3. *Varieties fibred by complete intersections of bidegree* $(2,2)$. Let $X \subset \mathbf{P}^n$ be the complete intersection of two quadrics. By Theorem 7.2, $\mathrm{CH}_0(X) = \mathbf{Q}$; and if $\dim X \geq 4$, then $\mathrm{CH}_1(X) = \mathbf{Q}$.

Proposition 7.5. *Let* $f : X \to B$ *be a dominant morphism between smooth projective complex varieties whose closed fibres are complete intersections of bidegree* $(2,2)$.

- *If* $\dim B \leq 1$ *and* $\dim X \leq 5$*, then* X *is Kimura finite-dimensional.*
- *If* $\dim B \leq 1$ *and* $\dim X \leq 6$*, then* X *satisfies Murre's conjectures.*
- *If* $\dim B \leq 2$ *and* $\dim X \leq 6$*, then* X *satisfies Grothendieck's standard conjectures.*
- *If* $\dim B \leq 3$ *and* $\dim X \leq 7$*, then* X *satisfies the Hodge conjecture.*

Proof. The variety X satisfies the assumptions of Theorem 6.10 with $l = 1$ for $\dim X - \dim B \geq 4$ and with $l = 0$ in any case. Thus the Chow group $\mathrm{CH}_0(X)$ has niveau $\leq d_B$ and $\mathrm{CH}_1(X)$ has niveau $\leq d_B$ for $\dim X - \dim B \geq 4$. We can therefore conclude by Theorem 7.1. □

7.2.4. *Varieties fibred by complete intersections of bidegree* $(2,3)$. Let $X \subset \mathbf{P}^n$ be the complete intersection of a quadric and of a cubic. If $\dim X \geq 6$, then Hirschowitz and Iyer [9] showed $\mathrm{CH}_l(X) = \mathbf{Q}$ for $l \leq 1$. (The result of Esnault–Levine–Viehweg only says that $\mathrm{CH}_l(X) = \mathbf{Q}$ for $l \leq 1$ when $\dim X \geq 7$).

Proposition 7.6. *Let* $f : X \to C$ *be a dominant morphism from a smooth projective complex variety* X *to a smooth projective complex curve* C *whose closed fibres are complete intersections of bidegree* $(2,3)$ *of dimension* 6*. Then* X *satisfies the Hodge conjecture.*

Proof. By Theorem 6.10, we see that the Chow groups $\mathrm{CH}_0(X)$ and $\mathrm{CH}_1(X)$ have niveau ≤ 1. We can thus conclude by Theorem 7.1. □

7.3. Varieties fibred by cellular varieties. Let $f : X \to B$ be a complex dominant morphism from a smooth projective variety X to a smooth projective variety B whose closed fibres are cellular varieties (not necessarily smooth). In other words, X is a smooth projective complex variety fibred by cellular varieties over B. For example, if $\Sigma \subset B$ is the singular locus of f, then X could be such that $X|_{B\backslash\Sigma}$ is a rational homogeneous bundle over $B\backslash\Sigma$ (e.g.

a Grassmann bundle) and the closed fibres of f over Σ (the degenerate fibres) are toric. That kind of situation is reminiscent of the setting of [8].

Proposition 7.7. *Let $f : X \to B$ be a dominant morphism between smooth projective complex varieties whose closed fibres are cellular varieties.*

- *Assume B is a curve, then X is Kimura finite-dimensional and X satisfies Murre's conjectures.*
- *Assume $\dim B \leq 2$ and $\dim X \leq 6$. If f is connected and smooth away from finitely many points in B, then X satisfies Grothendieck's standard conjectures.*
- *Assume $\dim B \leq 3$ and $\dim X \leq 7$. If f is connected and smooth away from finitely many points in B, then X satisfies the Hodge conjecture.*

Proof. The Chow groups of cellular varieties are finitely generated. The first statement thus follows from Theorems 6.12 and 7.1. Let us now focus on the cases when $\dim B$ is either 2 or 3. It is a consequence of Mumford's theorem [17] that a connected smooth projective complex variety with finitely generated Chow group of zero-cycles actually has Chow group of zero-cycles generated by a point. Thus the second and third statements follow from Theorem 6.13 with $l = 1$, and from Theorem 7.1. $\square$

Acknowledgements

Thanks to Reza Akhtar and Roy Joshua for asking a question related to the Chow–Künneth decomposition of a smooth blow-up, to Sergey Gorchinskiy for suggesting parts of Proposition 2.4, to Mingmin Shen for a discussion related to the proof of Lemma 6.6, and to Burt Totaro for suggesting the statement of Lemma 2.1. Finally, many thanks to the referee for helpful comments and suggestions.

This work was supported by EPSRC Early Career Fellowship number EP/K005545/1.

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