Evidence for a Generalization
of Gieseker's Conjecture on Stratified Bundles
in Positive Characteristic

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Abstract. Let X be a smooth, connected, projective variety over an algebraically closed field of positive characteristic. In [Gie75], Gieseker conjectured that every stratified bundle (i.e. every $\mathcal{O}_X$-coherent $\mathscr{D}_{X/k}$-module) on X is trivial, if and only if $\pi_1^{\text{ét}}(X) = 0$. This was proven by Esnault-Mehta, [EM10]. Building on the classical situation over the complex numbers, we present and motivate a generalization of Gieseker's conjecture, using the notion of regular singular stratified bundles developed in the author's thesis and [Kin12a]. In the main part of this article we establish some important special cases of this generalization; most notably we prove that for not necessarily proper X, $\pi_1^{\text{tame}}(X) = 0$ implies that there are no nontrivial regular singular stratified bundles with abelian monodromy.

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1 Introduction and statement of the conjecture

Let X be a smooth, separated, connected scheme of finite type over an algebraically closed field k, and fix a base point $x \in X(k)$. For readability, we write $\Pi_X^{\text{ét}} := \pi_1^{\text{ét}}(X,x)$, and we denote by $\operatorname{Repf}_k^{\text{cont}} \Pi_X^{\text{ét}}$ the category of continuous representations of the profinite group $\Pi_X^{\text{ét}}$ on finite dimensional k-vector spaces equipped with the discrete topology.
If $k = \mathbb{C}$, then $\Pi_X^{\text{ét}}$ is the profinite completion of the abstract group $\Pi_X^{\text{top}} := \pi_1^{\text{top}}(X(\mathbb{C}),x)$, and if we write $\operatorname{Repf}_{\mathbb{C}} \Pi_X^{\text{top}}$ for the category of representations

of Π_X^{top} on finite dimensional $\mathbb{C}$-vector spaces, then $\mathrm{Repf}_{\mathbb{C}}^{\mathrm{cont}} \Pi_X^{\mathrm{\acute{e}t}}$ can be considered as the full subcategory of $\mathrm{Repf}_{\mathbb{C}} \Pi_X^{\mathrm{top}}$ having as objects precisely those representations which factor through a finite group. Since Π_X^{top} is finitely generated, the category $\mathrm{Repf}_{\mathbb{C}} \Pi_X^{\mathrm{top}}$ is controlled by $\mathrm{Repf}_{\mathbb{C}}^{\mathrm{cont}} \Pi_X^{\mathrm{\acute{e}t}}$, according to the following theorem:

THEOREM 1.1 (Grothendieck [Gro70], Malcev [Mal40]). *If $\phi : G \to H$ is a morphism of finitely generated groups, then the following statements are equivalent:*

(a) *The induced morphism $\widehat{G} \to \widehat{H}$ is an isomorphism, where $\widehat{(-)}$ denotes profinite completion.*

(b) *The induced $\otimes$-functor*

$$\mathrm{Repf}_{\mathbb{C}} H \to \mathrm{Repf}_{\mathbb{C}} G$$

is a $\otimes$-equivalence.

(c) *The induced $\otimes$-functor*

$$\mathrm{Repf}_{\mathbb{C}}^{\mathrm{cont}} \widehat{H} \to \mathrm{Repf}_{\mathbb{C}}^{\mathrm{cont}} \widehat{G}$$

is a $\otimes$-equivalence.

Accordingly, if $f : Y \to X$ is a morphism of smooth, connected, complex varieties, then the induced morphism $\Pi_Y^{\mathrm{\acute{e}t}} \to \Pi_X^{\mathrm{\acute{e}t}}$ (with respect to compatible base points) is an isomorphism if and only if $\mathrm{Repf}_{\mathbb{C}} \Pi_X^{\mathrm{top}} \to \mathrm{Repf}_{\mathbb{C}} \Pi_Y^{\mathrm{top}}$ is a $\otimes$-equivalence. This consequence was already noted in [Gro70].

To study the category $\mathrm{Repf}_{\mathbb{C}} \Pi_X^{\mathrm{top}}$, we invoke the Riemann-Hilbert correspondence as developed in [Del70]: It states that the choice of a base point $x \in X(\mathbb{C})$ gives a $\otimes$-equivalence $|_x$ of Tannakian categories between the category of regular singular flat connections on X and the category $\mathrm{Repf}_{\mathbb{C}} \Pi_X^{\mathrm{top}}$. Theorem 1.1 then translates into the following completely algebraic statement, in which we suppress the choice of base points from the notation:

COROLLARY 1.2. *If $f : Y \to X$ is a morphism of smooth, connected, separated, finite type $\mathbb{C}$-schemes, then the following are equivalent:*

(a) *The morphism $\Pi_Y^{\mathrm{\acute{e}t}} \to \Pi_X^{\mathrm{\acute{e}t}}$ induced by f is an isomorphism.*

(b) *Pull-back along f induces an equivalence on the categories of regular singular flat connections.*

(c) *Pull-back along f induces an equivalence on the categories of regular singular flat connections with* finite *monodromy.*

In particular: $\Pi_Y^{\mathrm{\acute{e}t}} = 0$ if and only if every regular singular flat connection on Y is trivial.

This article is devoted to the study of analogous statements in positive characteristics. Let k be an algebraically closed field of characteristic $p > 0$, and X a smooth, connected, separated, finite type k-scheme. In general, neither the category of vector bundles with flat connection, nor the category of coherent $\mathcal{O}_X$-modules with flat connection are Tannakian categories over k. Instead, we consider left-$\mathcal{D}_{X/k}$-modules which are coherent (and then automatically locally free) as $\mathcal{O}_X$-modules, where $\mathcal{D}_{X/k}$ is the ring of differential operators relative to k, as developed in [EGA4, §16]. Following [Gro68] and [SR72], we call such objects *stratified bundles*, and we write $\mathrm{Strat}(X)$ for the category of stratified bundles on X. Recall that in characteristic 0, giving a stratified bundle is equivalent to giving a vector bundle with flat connection. In positive characteristic, these notions are not equivalent. A stratified bundle is called trivial if it is isomorphic to $\mathcal{O}_X^n$ with the canonical diagonal left-$\mathcal{D}_{X/k}$-action. In [Kin12a] (see also [Kin12b]), a notion of regular singularity for stratified bundles is defined and studied, generalizing work of Gieseker ([Gie75]); for a summary see Section 2. We write $\mathrm{Strat}^{\mathrm{rs}}(X)$ for the full subcategory of $\mathrm{Strat}(X)$ with objects regular singular stratified bundles; after choosing a base point, it is a neutral Tannakian category over k.

Using the theory of Tannakian categories, there still is a procedure to attach to a stratified bundle a monodromy group and a monodromy representation at the base point $x \in X(k)$. The main result of [Kin12a] states that this procedure induces a $\otimes$-equivalence between the category of regular singular stratified bundles with finite monodromy and $\mathrm{Repf}_k^{\mathrm{cont}} \Pi_X^{\mathrm{tame}}$, where $\Pi_X^{\mathrm{tame}} := \pi_1^{\mathrm{tame}}(X, x)$ is the tame fundamental group as defined in [KS10]. This result suggests the following conjecture, completely analogous to Corollary 1.2:

CONJECTURE 1.3. *Let k be an algebraically closed field of characteristic $p > 0$. If $f : Y \to X$ is a morphism of smooth, connected, separated, finite type k-schemes, then the following are equivalent:*

(a) *The morphism $\Pi_Y^{\mathrm{tame}} \to \Pi_X^{\mathrm{tame}}$ induced by f is an isomorphism.*

(b) *Pull-back along f induces an equivalence on the categories of regular singular stratified bundles.*

(c) *Pull-back along f induces an equivalence on the categories of regular singular stratified bundles with* finite *monodromy.*

In particular: $\Pi_Y^{\mathrm{tame}} = 0$ *if and only if every regular singular stratified bundle on Y is trivial.*

For a slightly different exposition of this conjecture, see [Esn12]. Note that by the main result of [Kin12a] we always have (b)$\Rightarrow$(c)$\Leftrightarrow$(a).

The main part of this article is concerned with establishing special cases of and evidence for the validity of Conjecture 1.3, i.e. for the direction (a)$\Rightarrow$(b). We give a brief summary: Gieseker's original conjecture from [Gie75] corresponds to Conjecture 1.3 for a projective morphism $f : Y \to \mathrm{Spec}(k)$. His conjecture was proven by Esnault-Mehta, [EM10]. Thus Conjecture 1.3 generalizes

Gieseker's conjecture in two "directions": It allows non-projective varieties by using the notion of regular singularity, and it gives a relative formulation.
The main result of this article, proven in Section 4, is the following:

THEOREM 1.4 (see Theorem 4.2). *In the situation of Conjecture 1.3, assume that $X = \mathrm{Spec}(k)$, and that Y admits a good compactification. If $\Pi_Y^{\mathrm{ab},(p')} = 0$ and if Π_Y^{tame} does not have a quotient isomorphic to $\mathbb{Z}/p\mathbb{Z}$, then every regular singular stratified bundle with abelian monodromy is trivial.*
In other words: The abelianization of the pro-algebraic group associated with $\mathrm{Strat}^{\mathrm{rs}}(Y)$ (and the choice of a base point) is trivial.

Here a *good compactification* of Y is a smooth, proper k-scheme $\overline{Y}$ containing Y as a dense open subscheme, such that $\overline{Y} \smallsetminus Y$ is the support of a strict normal crossings divisor, and $(-)^{\mathrm{ab},(p')}$ denotes the maximal abelian pro-prime-to-p-quotient. This quotient is independent of the choice of the base point.
Theorem 1.4 is a consequence of the fact that under the given assumptions there are no nontrivial rank 1 stratified bundles, and no nontrivial regular singular extensions of two rank 1 stratified bundles. Hence we establish these facts first: in Section 3 we study stratified bundles of rank 1, and we prove a relative version of Conjecture 1.3 for stratified line bundles:

THEOREM 1.5 (see Theorem 3.10). *Let X and Y be smooth, connected, separated, finite type k-schemes, which admit good compactifications $\overline{X}$ and $\overline{Y}$ over k. If $f : Y \to X$ is a map extending to a morphism $\overline{f} : \overline{Y} \to \overline{X}$ such that f induces an isomorphism $\Pi_Y^{\mathrm{ab},(p')} \xrightarrow{\cong} \Pi_X^{\mathrm{ab},(p')}$ and such that the cokernel of the induced map $H^0(X, \mathcal{O}_X^\times) \to H^0(Y, \mathcal{O}_Y^\times)$ is a p-group, then pull-back along f induces an isomorphism*

$$\mathrm{Pic}^{\mathrm{Strat}}(X) \xrightarrow{\cong} \mathrm{Pic}^{\mathrm{Strat}}(Y)$$

where $\mathrm{Pic}^{\mathrm{Strat}}$ denotes the group of isomorphism classes of stratified line bundles.

Note that contrary to the situation over the complex numbers, a stratified line bundle in our context is always regular singular (Proposition 2.7). The assumption on the cokernel of $H^0(X, \mathcal{O}_X^\times) \to H^0(Y, \mathcal{O}_Y^\times)$ is trivially fulfilled if Y and X are proper. For further comments see Remark 3.11.
In particular we obtain:

COROLLARY 1.6. *Let Y, X be proper, smooth, connected k-schemes. If $f : Y \to X$ is a morphism such that f induces an isomorphism $\Pi_Y^{\mathrm{ab},(p')} \xrightarrow{\cong} \Pi_X^{\mathrm{ab},(p')}$, then f induces an isomorphism $\mathrm{Pic}^{\mathrm{Strat}}(X) \xrightarrow{\cong} \mathrm{Pic}^{\mathrm{Strat}}(Y)$.*

In the case that $X = \mathrm{Spec}\, k$, the assumption on the existence of a good compactification of Y is not necessary:

PROPOSITION 1.7 (see Proposition 3.16). *Let Y be a smooth, connected, separated, finite type k-scheme, such that $\Pi_Y^{\mathrm{ab},(p')} = 0$. Then $\mathrm{Pic}^{\mathrm{Strat}}(Y) = 0$.*

In Section 4 we study regular singular extensions of rank 1 stratified bundles, and prove Theorem 1.4.

For regular singular stratified bundles with not necessarily abelian monodromy, we establish the following two special cases of Conjecture 1.3:
It is well-known that $\pi_1^{\mathrm{tame}}(\mathbb{A}_k^n) = 0$ and in Section 5 we give a short proof of:

THEOREM 1.8 (see Theorem 5.1). *Every regular singular stratified bundle on $\mathbb{A}_k^n$ is trivial.*

This was already sketched in [Esn12] using a slightly different approach.

If $f : Y \to X$ is a universal homeomorphism, then f induces an isomorphism $\Pi_Y^{\mathrm{tame}} \xrightarrow{\cong} \Pi_X^{\mathrm{tame}}$ ([Vid01]), and in Section 6 we show:

THEOREM 1.9 (see Theorem 6.6). *If $f : Y \to X$ is a universal homeomorphism, then pull-back along f is an equivalence $\mathrm{Strat}^{\mathrm{rs}}(X) \to \mathrm{Strat}^{\mathrm{rs}}(X)$.*

In Section 2 we gather a few general facts about regular singular stratified bundles which will be needed in the course of the text.

2 Generalities on stratified bundles

Let k be an algebraically closed field of characteristic $p > 0$, and fix a smooth, separated, connected, finite type k-scheme X.

DEFINITION 2.1. A *stratified bundle* on X is a $\mathscr{D}_{X/k}$-module which is $\mathcal{O}_X$-coherent. A morphism of stratified bundles is a morphism of $\mathscr{D}_{X/k}$-modules. A stratified bundle E is called *trivial* if it is isomorphic to $\mathcal{O}_X^n$ with its canonical diagonal $\mathscr{D}_{X/k}$-action. Here $\mathscr{D}_{X/k}$ is the ring of differential operators on X, see [EGA4, §16].
We write $\mathrm{Strat}(X)$ for the category of stratified bundles.

REMARK 2.2. The name "stratified module" goes back to Grothendieck: In [Gro68] he defines the notion of a stratification relative to k on an $\mathcal{O}_X$-module E as an "infinitesimal descent datum", and since X is smooth over k, such a

datum is equivalent to the datum of a $\mathscr{D}_{X/k}$-action on E, compatible with its $\mathcal{O}_X$-structure, see e.g. [BO78, Ch. 2]. As in characteristic 0, one shows that a $\mathcal{O}_X$-coherent $\mathscr{D}_{X/k}$-module is automatically locally free, [BO78, 2.17]. Hence the name "stratified bundle".

The following result of Katz gives a different perspective on the notion of a stratified bundle:

THEOREM 2.3 (Katz, [Gie75, Thm. 1.3]). *Let $F : X \to X$ denote the absolute Frobenius. The category $\mathrm{Strat}(X)$ is equivalent to the following category:*

OBJECTS: *Sequences of pairs $E := (E_n, \sigma_n)_{n \geq 0}$ with E_n a vector bundle on X and $\sigma_n : E_n \xrightarrow{\cong} F^* E_{n+1}$ an $\mathcal{O}_X$-linear isomorphism.*

MORPHISMS: *A morphism $\phi : (E_n, \sigma_n) \to (E_n', \sigma_n')$ is a sequence of morphisms of vector bundles $\phi_n : E_n \to E_n'$, such that $(F^*\phi_{n+1})\sigma_n = \sigma_n' \phi_n$.*

The functor giving the equivalence assigns to a stratified bundle E the sequence $(E_n, \sigma_n)_{n \geq 0}$, with

$$E_n(U) = \left\{ e \in E(U) | D(e) = 0 \text{ for all } D \in \mathscr{D}_{X/k}^{\leq p^n}(U) \text{ with } D(1) = 0 \right\}$$

and $\sigma_n : E_n \to F^* E_{n+1}$ the isomorphism given by Cartier's theorem [Kat70, §5]. This functor is compatible with tensor products.

In the sequel we will freely switch between the two perspectives on stratified bundles. The description by the above theorem is especially nice when X is proper over k:

PROPOSITION 2.4 ([Gie75, Prop. 1.7]). *If X is proper over k, then the isomorphism class of a stratified bundle $E = (E_n, \sigma_n)_{n \geq 0}$ only depends on the isomorphism classes of the vector bundles $E_1, E_2, \ldots$.*

2.1 REGULAR SINGULAR STRATIFIED BUNDLES

We recall from [Kin12a] the definition of regular singular stratified bundles.

DEFINITION 2.5. Let X be a smooth, separated, finite type k-scheme.

(a) If $\overline{X}$ is a smooth, separated, finite type k-scheme containing X as as an open dense subscheme such that $\overline{X} \smallsetminus X$ is a strict normal crossings divisor, then the pair $(X, \overline{X})$ is called *good partial compactification of X*. If $\overline{X}$ is also proper, then $(X, \overline{X})$ is called *good compactification of X*.

(b) If $(X, \overline{X})$ is a good partial compactification, then $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \smallsetminus X)$ is by definition the sheaf of subalgebras of $\mathscr{D}_{\overline{X}/k}$ spanned over an open $\overline{U} \subset \overline{X}$ by those differential operators in $\mathscr{D}_{\overline{X}/k}(\overline{U})$, fixing all powers of the ideal of the divisor $(\overline{X} \smallsetminus X) \cap \overline{U}$. Lets make this explicit: If $x_1, \ldots, x_n \in H^0(\overline{U}, \mathcal{O}_{\overline{U}})$

are coordinates on $\overline{U}$, i.e. if they define an étale morphism $\overline{U} \to \mathbb{A}^n_k$, then $\mathscr{D}_{\overline{U}/k}$ is spanned by operators $\partial^{(m)}_{x_i}$, $i = 1, \ldots, r$, $m \geq 0$, which "behave" like $\frac{1}{m!}\partial^m/\partial x^m_i$. If $\overline{U} \cap (\overline{X} \smallsetminus X)$ is defined by $x_1 \cdot \ldots \cdot x_r$, then the subring $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \smallsetminus X)|_{\overline{U}}$ is spanned by $\delta^{(m)}_{x_i} := x^m_i \partial^{(m)}_{x_i}$, $i = 1, \ldots, r$, $\partial^{(m)}_{x_i}$, $i > r, m \geq 0$.

This ring can also be defined intrinsically as the ring of differential operators associated with a morphism of log-schemes, which, for example, is studied in [Mon02].

(c) If $(X, \overline{X})$ is a good partial compactification, then a stratified bundle $E \in \mathrm{Strat}(X)$ is called $(X, \overline{X})$-*regular singular* if E extends to an $\mathcal{O}_{\overline{X}}$-torsion free, $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \smallsetminus X)$-module.

We write $\mathrm{Strat}^{\mathrm{rs}}((X, \overline{X}))$ for the full subcategory of $\mathrm{Strat}(X)$ with objects the $(X, \overline{X})$-regular singular stratified bundles.

(d) A stratified bundle E is called *regular singular* if E is $(X, \overline{X})$-regular singular for every good partial compactification $(X, \overline{X})$ of X.

We write $\mathrm{Strat}^{\mathrm{rs}}(X)$ for the full subcategory of $\mathrm{Strat}(X)$ with objects the regular singular stratified bundles.

REMARK 2.6. The notion of $(X, \overline{X})$-regular singularity for stratified bundles first appeared (to the author's knowledge) in [Gie75] for good compactifications $(X, \overline{X})$. Gieseker attributes some of the ideas used in [loc. cit.] to Katz. In [Kin12a] and [Kin12b] this notion of regular singularity is extended to varieties for which resolution of singularities is not known to hold, and its connection with tame ramification is studied.

For the purpose of this article, the following facts are of importance:

PROPOSITION 2.7. *Let X be a smooth, separated, finite type k-scheme.*

(a) *If X admits a good compactification $(X, \overline{X})$, then a stratified bundle E on X is regular singular if and only if it is $(X, \overline{X})$-regular singular.*

(b) *If E is a stratified bundle of rank 1, then E is regular singular.*

(c) *If $\iota : \mathrm{Strat}^{\mathrm{rs}}(X) \to \mathrm{Strat}(X)$ denotes the inclusion functor, then for every object $E \in \mathrm{Strat}^{\mathrm{rs}}(X)$, ι induces an equivalence $\langle E \rangle_{\otimes} \xrightarrow{\cong} \langle \iota(E) \rangle_{\otimes}$.*

Proof. Statement (a) is [Kin12a, Prop. 7.5], (c) is [Kin12a, Prop. 4.5], and (b) is [Gie75, Lemma 3.12]. We recall some arguments for (b) from [loc. cit.] for convenience: Let A be a finite type k-algebra with coordinates $x_1, \ldots, x_n$, and M a free rank 1 module over $A[x^{-1}_1 \cdot \ldots \cdot x^{-1}_r]$, $1 \leq r \leq n$. Assume that M carries an action of $\mathscr{D}_{A[x^{-1}_1, \ldots, x^{-1}_r]/k}$, and that e is a basis of M. With the notation from Definition 2.5, (b), it suffices to show that $\delta^{(m)}_{x_i}(e) \subset eA$ for every $m > 0$

and every $1 \le i \le r$. By Theorem 2.3, for $N \in \mathbb{N}$ there exists a nonzero section $s \in A$ such that $\partial_{x_i}^{(m)}(se) = 0$ for all $m \le p^N$. But then in $\operatorname{Frac} A$ we have

$$\delta_{x_i}^{(m)}(e) = \delta_{x_i}^{(m)}(s^{-1}se) = \delta_{x_i}^{(m)}(s^{-1})se$$

and $\delta_{x_i}^{(m)}(s^{-1})s \in A$. □

REMARK 2.8. Recall that the analogous statement to (b) in characteristic 0 is false.

We recall the definition of the monodromy group of a stratified bundle:

DEFINITION 2.9. If $E \in \operatorname{Strat}(X)$, then $\langle E \rangle_\otimes$ is the Tannakian subcategory category of $\operatorname{Strat}(X)$ generated by E. If $\omega : \langle E \rangle_\otimes \to \operatorname{Vectf}_k$ is a fiber functor, then the associated k-group scheme is called the *monodromy group of E (with respect to ω)*. By [dS07], the monodromy group of a stratified bundle is always smooth.

Note that if E is regular singular, then by Proposition 2.7, (c), we get the same result whether we compute the monodromy group of E as an object of $\operatorname{Strat}^{rs}(X)$ or $\operatorname{Strat}(X)$.

Moreover, note that since k is algebraically closed, the isomorphism class of the monodromy group does not depend on the choice of the fiber functor.

To finish this subsection, we briefly recall the notion of exponents of regular singular stratified bundles. For more details see [Kin12a] or [Kin12b].

PROPOSITION 2.10. *Let $(X, \overline{X})$ be a good partial compactification of X with $D := \overline{X} \smallsetminus X$ is irreducible. Let $\overline{E}$ be a $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \smallsetminus X)$-module, which is $\mathcal{O}_{\overline{X}}$-locally free of finite rank. Then the following are true:*

(a) *([Gie75, Lemma 3.8]) There exists a decomposition*

$$\overline{E}|_D = \bigoplus_{\alpha \in \mathbb{Z}_p} F_\alpha,$$

with the property that if $x_1, \dots, x_n$ are local coordinates around the generic point of D, such that $D = (x_1)$, then $e + x_1\overline{E} \in F_\alpha$ if and only if

$$\delta_{x_1}^{(m)}(e) = \binom{\alpha}{m}e + x_1\overline{E}.$$

For the definition of $\delta_{x_1}^{(m)}$, see Definition 2.5, (b). Write $\operatorname{Exp}(\overline{E}) \subset \mathbb{Z}_p$ for the set of those $\alpha \in \mathbb{Z}_p$ for which $\operatorname{rank} F_\alpha \ne 0$.

(b) *([Kin12a, Prop. 4.12]) If $\overline{E}'$ is a second $\mathcal{O}_{\overline{X}}$-locally free, finite rank, $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \smallsetminus X)$-module, such that the stratified bundles $\overline{E}|_X$ and $\overline{E}'|_X$ are isomorphic, then the sets $\operatorname{Exp}(\overline{E})$ and $\operatorname{Exp}(\overline{E}')$ have the same image in $\mathbb{Z}_p/\mathbb{Z}$.*

DEFINITION 2.11 ([Kin12a, Def. 4.13]). If $(X, \overline{X})$ is a good partial compactification, E an $(X, \overline{X})$-regular singular stratified bundle and D an irreducible component of $\overline{X} \smallsetminus X$, then we define the *exponents of E along D* as follows: Let $\overline{U}$ be a sufficiently small open neighborhood of the generic point of D, such that there exists an $\mathcal{O}_{\overline{U}}$-locally free $\mathscr{D}_{\overline{U}/k}(\log \overline{U} \smallsetminus X)$-module extending $E|_{\overline{U} \cap X}$. Then the set of exponents of E along D is the image of the set $\mathrm{Exp}(\overline{E})$ in $\mathbb{Z}_p/\mathbb{Z}$ from Proposition 2.10. This construction is well-defined by Proposition 2.10.

Having all exponents equal to $0 \mod \mathbb{Z}$ is a stronger condition in characteristic $p > 0$ than in characteristic 0. In particular there are no regular singular stratified bundles with nontrivial "nilpotent residues":

PROPOSITION 2.12 ([Kin12a, Cor. 5.4]). *Let $(X, \overline{X})$ be a good partial compactification and E an $(X, \overline{X})$-regular singular stratified bundle. If the exponents of E along all components of $\overline{X} \smallsetminus X$ are 0 in $\mathbb{Z}_p/\mathbb{Z}$, then there exists a stratified bundle $\overline{E}$ on $\overline{X}$ extending E.*

3 SPECIAL CASE I: STRATIFIED LINE BUNDLES

We continue to denote by k an algebraically closed field of characteristic $p > 0$, and by X a smooth, separated, connected, finite type k-scheme.
We start with a group theoretic definition.

DEFINITION 3.1. If G is an abelian group, write $(G)_p$ for the projective system

$$G \xleftarrow{\cdot p} G \xleftarrow{\cdot p} \dots. \tag{1}$$

This defines an exact functor from the category of abelian groups into the category of pro-systems of abelian groups. We write $\varprojlim_p G := \varprojlim((G)_p)$, and $R^1\varprojlim_p G := R^1\varprojlim((G)_p)$. This construction defines a left exact functor $\varprojlim_p$ from the category of abelian groups to itself, and for a short exact sequence

$$0 \to A \to B \to C \to 0$$

of abelian groups, we have the long exact squence

$$0 \to \varprojlim_p A \to \varprojlim_p B \to \varprojlim_p C \to R^1\varprojlim_p A \to R^1\varprojlim_p B \to R^1\varprojlim_p C \to 0.$$

DEFINITION 3.2. We write $\mathrm{Pic}^{\mathrm{Strat}}(X)$ for the set of isomorphism classes of stratified bundles of rank 1. The tensor product of stratified bundles gives this set an abelian group structure.

By Theorem 2.3 we can associate with every stratified line bundle a sequence of elements $L_n \in \mathrm{Pic}\,X$, such that $L_{n+1}^p = L_n$. This gives a group homomorphism $\mathrm{Pic}^{\mathrm{Strat}}(X) \to \varprojlim_p \mathrm{Pic}\,X$.

DEFINITION 3.3. Denote by $\mathbb{I}(X)$ the subgroup of $\mathrm{Pic}^{\mathrm{Strat}}(X)$ of isomorphism classes of stratified line bundles (L_n, σ_n) with $L_n \cong \mathcal{O}_X$ for all n.

We clearly have the following:

PROPOSITION 3.4. *The morphisms described above fit in a functorial short exact sequence*

$$0 \to \mathbb{I}(X) \to \mathrm{Pic}^{\mathrm{Strat}}(X) \to \varprojlim_p \mathrm{Pic}(X) \to 0.$$

Moreover, it is not difficult to describe the group $\mathbb{I}(X)$ concretely:

PROPOSITION 3.5. *There is a functorial exact sequence*

$$0 \longrightarrow k^\times \longrightarrow H^0(X, \mathcal{O}_X^\times) \overset{\Delta}{\longrightarrow} \varprojlim_n H^0(X, \mathcal{O}_X^\times)/H^0(X, \mathcal{O}_X^\times)^{p^n} \longrightarrow \mathbb{I}(X) \longrightarrow 0$$

where the morphism $k^\times \to H^0(X, \mathcal{O}_X^\times)$ is the canonical inclusion, and Δ the diagonal map.

Proof. Since the base field k is algebraically closed of characteristic $p > 0$, the sequence is exact at $H^0(X, \mathcal{O}_X^\times)$. Indeed, $H^0(X, \mathcal{O}_X^\times)/k^\times$ is a finitely generated free abelian group, because of the exact sequence

$$0 \longrightarrow H^0(\overline{X}, \mathcal{O}_{\overline{X}}^\times) \longrightarrow H^0(X, \mathcal{O}_X^\times) \longrightarrow \bigoplus_i D_i \mathbb{Z} \longrightarrow \mathrm{Cl}(\overline{X}) \longrightarrow \mathrm{Cl}(X) \longrightarrow 0,$$

where $\overline{X}$ is any normal compactification of X, and D_i the irreducible components of $\overline{X} \smallsetminus X$. This shows that the kernel of Δ is precisely $k^\times$.
Thus we only need to construct the morphism

$$\phi : \mathrm{coker}(\Delta) \to \mathbb{I}(X),$$

and show that it is a natural isomorphism.
Given an element $\alpha := (\alpha_1, \alpha_2, \ldots) \in \varprojlim_n H^0(X, \mathcal{O}_X^\times)/H^0(X, \mathcal{O}_X^\times)^{p^n}$, take any sequence of lifts $a := (a_1, a_2, \ldots) \in H^0(X, \mathcal{O}_X^\times)^{\mathbb{N}}$, and define the stratified bundle $\Psi(a) := (\Psi_n(a), \sigma_n^a)_{n \geq 0}$ by setting $\Psi(a)_n = \mathcal{O}_X$, and by defining the transition isomorphisms $\sigma_n^a : \Psi(a)_n \to F^* \Psi(a)_{n+1}$ as follows: σ_0^a is multiplication by a_1, and $\sigma_n^a = (a_{n+1} a_n^{-1})^{1/p^n}$, $n > 0$. This works, since by definition $a_{n+1} \equiv a_n$ mod p^n. We have to check that this construction gives a well-defined map Ψ: If we pick a different sequence of lifts $a' := (a_1', a_2', \ldots)$ of $\alpha = (\alpha_1, \alpha_2, \ldots)$, then the resulting stratified line bundle $\Psi(a')$ is isomorphic to $\Psi(a)$. Indeed, if $b_n^{p^n} = a_n' a_n^{-1}$, then the sequence of isomorphisms $\phi_0 = \mathrm{id}$, $\phi_n : \Psi(a)_n \overset{\cdot b_n}{\longrightarrow} \Psi(a')_n$, $n > 1$, defines an isomorphism of stratified bundles. We write $\Psi(\alpha)$ for the isomorphism class of the stratified bundle $\Psi(a)$
It is readily checked that this map is surjective: If the stratified bundle $L = (L_n, \sigma_n)_{n \geq 0}$ is given by $L_n = \mathcal{O}_X$ and the transition morphisms σ_n, then $\sigma_n \in H^0(X, \mathcal{O}_X^\times)$, and $L = \Psi((\sigma_0, \sigma_0 \sigma_1^p, \sigma_0 \sigma_1^p \sigma_2^{p^2}, \ldots))$.

To compute the kernel, note that a stratification L on $\mathcal{O}_X$ which is given by transition morphisms σ_n, is trivial if and only if there exists a sequence $\phi_0, \phi_1, \ldots \in H^0(X, \mathcal{O}_X^\times)$ such that $\sigma_n \phi_{n+1}^p = \phi_n$, $n > 0$. By recursion, this means that

$$\phi_0 = \sigma_0 \sigma_1^p \cdot \ldots \cdot \sigma_n^{p^n} \phi_{n+1}^{p^{n+1}}$$

for every n. In other words: $L = \Psi(\Delta(\phi_0))$, so the kernel of Ψ is precisely $H^0(X, \mathcal{O}_X^\times)$. This completes the proof. $\qquad\square$

REMARK 3.6. (a) The description of $\mathbb{I}(X)$ in Proposition 3.5 was inspired by a similar description for stratified bundles line bundles on $k((t))$ in [MvdP03].

(b) Note that the abelian group $\varprojlim_n H^0(X, \mathcal{O}_X^\times)/H^0(X, \mathcal{O}_X^\times)^{p^n}$ is just the p-adic completion of the free finitely generated abelian group $H^0(X, \mathcal{O}_X^\times)/k^\times$, and the map Δ induces the canonical map from $H^0(X, \mathcal{O}_X^\times)/k^\times$ into its p-adic completion.

COROLLARY 3.7. *The group $\mathbb{I}(X)$ is trivial if and only if $H^0(X, \mathcal{O}_X^\times) = k^\times$.*

Proof. This follows from Proposition 3.5 and Remark 3.6, (b), because the map from $\mathbb{Z}^r$ into its p-adic completion is surjective if and only if $r = 0$. $\qquad\square$

Corollary 3.7 allows us to exhibit a class of examples of varieties X such that $\mathbb{I}(X)$ is trivial:

PROPOSITION 3.8. *If k is an algebraically closed field, and if X is a connected normal k-scheme of finite type, such that the maximal abelian pro-ℓ-quotient $\pi_1(X)^{\mathrm{ab},(\ell)}$ is trivial for some $\ell \neq \mathrm{char}(k)$, then $H^0(X, \mathcal{O}_X^\times) = k^\times$. In particular, if k has positive characteristic, then $\mathbb{I}(X) = 0$.*

Proof. This argument is due to Hélène Esnault. Assume $f \in H^0(X, \mathcal{O}_X^\times) \smallsetminus k^\times$. Then f induces a dominant morphism $f' : X \to \mathbb{G}_{m,k} \cong \mathbb{A}_k^1 \smallsetminus \{0\}$, as f' is given by the map $k[x^{\pm 1}] \to H^0(X, \mathcal{O}_X)$, $x \mapsto f$, which is injective if and only if f is transcendental over k. Thus f' induces an *open* morphism $\pi_1(X) \to \pi_1(\mathbb{G}_{m,k})$, see e.g. [Sti02, Lemma 4.2.10]. But under our assumption, the maximal abelian pro-ℓ-quotient of the image of this morphism is trivial, so the image of $\pi_1(X)$ cannot have finite index in the group $\pi_1(\mathbb{G}_{m,k})$, as in fact $\pi_1(\mathbb{G}_{m,k})^{(\ell)} \cong \widehat{\mathbb{Z}}^{(\ell)} = \mathbb{Z}_\ell$. $\qquad\square$

REMARK 3.9. We can modify the argument of Proposition 3.8 slightly to obtain: If k has positive characteristic p, and $\pi_1(X)^{\mathrm{ab},(p)} = 0$, then $H^0(X, \mathcal{O}_X) = k$. A consequence is a proof of the folklore fact that over a field k of positive characteristic, unlike in characteristic 0, no normal, connected, affine, finite type k-scheme X of positive dimension is simply-connected, or even has $\pi_1(X)^{\mathrm{ab},(p)} = 0$.

Indeed, if $f \in H^0(X, \mathcal{O}_X)$ is nonconstant, then f induces a dominant morphism $X \to \mathbb{A}_k^1$ and hence an open morphism $\pi_1^{\mathrm{ab},(p)}(X) \to \pi_1^{\mathrm{ab},(p)}(\mathbb{A}_k^1)$. But by [Kat86, 1.4.3, 1.4.4] we have $H^2(\pi_1(\mathbb{A}_k^1), \mathbb{F}_p) = 0$, so $\pi_1(\mathbb{A}_k^1)^{(p)}$ is free pro-p of rank $\dim_{\mathbb{F}_p} H^1(\mathbb{A}_k^1, \mathbb{F}_p) = \#k$. Thus the image of $\pi_1^{\mathrm{ab},(p)}(X)$ in this group can only have finite index if $\pi_1(X)^{\mathrm{ab},(p)} \neq 0$.

Recall that a *good compactification* of a smooth k-scheme X is a proper, smooth, finite type k-scheme $\overline{X}$, with a dominant open immersion $X \hookrightarrow \overline{X}$, such that $\overline{X} \smallsetminus X$ is the support of a strict normal crossings divisor.
The main goal of this section is to prove the following theorem:

THEOREM 3.10. *Let X, Y be smooth, separated, finite type k-schemes, and let $f : Y \to X$ be a morphism such that the following conditions are satisified:*

(a) *There exist good compactifications $\overline{X}$ and $\overline{Y}$ of X and Y, such that f extends to a morphism $\bar{f} : \overline{Y} \to \overline{X}$.*

(b) *f induces an isomorphism*

$$f_* : \pi_1^{\mathrm{ab},(p')}(Y) \to \pi_1^{\mathrm{ab},(p')} \pi_1(X), \tag{2}$$

(c) *The cokernel of the morphism*

$$H^0(X, \mathcal{O}_X^\times) \to H^0(Y, \mathcal{O}_Y^\times). \tag{3}$$

induced by f is a p-group.

Then pull-back along f induces an isomorphism

$$\mathrm{Pic}^{\mathrm{Strat}}(X) \xrightarrow{\cong} \mathrm{Pic}^{\mathrm{Strat}}(Y).$$

REMARK 3.11. (a) The morphism (3) is an isomorphism in the following two situations:

    (i) X, Y proper over k.

    (ii) $\pi_1^{\mathrm{ab},(p')}(Y) = \pi_1^{\mathrm{ab},(p')}(X) = 1$ (e.g. (2) is an isomorphism and $X = \mathrm{Spec}(k)$). See Proposition 3.8.

Thus in these two cases, the theorem establishes that f induces an isomorphism of abelian groups $\mathrm{Pic}^{\mathrm{Strat}}(X) \to \mathrm{Pic}^{\mathrm{Strat}}(Y)$, if (2) is an isomorphism.

(b) If (2) is an isomorphism, then (3) is always injective. Indeed, if $\alpha \in H^0(X, \mathcal{O}_X^\times)$ is a global unit, then α induces a morphism $X \to \mathbb{G}_m$, which is dominant if and only if α is non-constant. Hence the induced continuous morphism $\pi_1^{\mathrm{ab},(p')}(X) \to \widehat{\mathbb{Z}}^{(p')}$ is open if and only if α is non-constant. If (2) is an isomorphism, then $f^*\alpha$ is non-constant whenever α is. In particular, it follows that (3) is injective.

(c) Clearly (3) is not necessarily an isomorphism, even if (2) is: Just take the purely inseparable morphism $\mathbb{G}_m \to \mathbb{G}_m$ defined by taking the p-th root of a coordinate on $\mathbb{G}_m$.

(d) The question whether (3) has a p-group as cokernel whenever (2) is an isomorphism, belongs to the area of Grothendieck's anabelian geometry: The cokernel of (3) is finitely generated by [Kah06, Lemme 1], so we can split off the p-power torsion. If $\alpha \in H^0(Y, \mathcal{O}_Y^\times)$ is a global unit of order prime to p in the cokernel of (3), and if (2) is an isomorphism, then α comes from a global unit on X if and only if the induced morphism on fundamental groups

$$\pi_1^{\mathrm{ab},(p')}(X) \xrightarrow{\cong} \pi_1^{\mathrm{ab},(p')}(Y) \xrightarrow{\alpha_*} \pi_1^{\mathrm{ab},(p')}(\mathbb{G}_m) = \widehat{\mathbb{Z}}^{(p')}$$

is induced by a morphism of k-schemes $X \to \mathbb{G}_m$. The author does not know whether the condition that (2) is an isomorphism always implies that the cokernel of (3) is a p-group.

We need a few lemmas to prepare the proof of Theorem 3.10.

LEMMA 3.12. *If G is a finitely generated abelian group, then the functor $\varprojlim_p$ is naturally isomorphic to the functor which assigns to G its subgroup $G[p']$ of torsion elements of order prime to p.*
If G is a finite abelian group or an abelian group (not necessarily finitely generated) on which multiplication by p is surjective, then $R^1\varprojlim_p G = 0$.

Proof. In a finitely generated abelian group, an element is infinitely p-divisible if and only if it is torsion of order prime to p. Moreover, an element $x \in G[p']$ admits a unique p-th root in $G[p']$. It follows that the map $\varprojlim_p G \to G[p']$, $(x_1, x_2, \ldots) \mapsto x_1$ is an isomorphism.
For the second claim, if G is finite or if multiplication by p on G is surjective, then the projective system (1) satisfies the Mittag-Leffler condition, so $R^1\varprojlim_p G = 0$. $\qquad\square$

LEMMA 3.13. *Consider the following morphism of short exact sequences of abelian groups*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & D_1 & \longrightarrow & G_1 & \longrightarrow & F_1 & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & D_2 & \longrightarrow & G_2 & \longrightarrow & F_2 & \longrightarrow & 0
\end{array}
$$

with F_1, F_2 finitely generated, and such that multiplication by p on D_1, D_2 is surjective (e.g. D_1, D_2 divisible).
Assume that the following conditions are satisfied:

(a) *f is surjective with finite kernel.*

(b) $\ker(g)$ *contains no torsion elements of order prime to p.*

(c) *The restriction $h|_{F_1[p']} : F_1[p'] \to F_2[p']$ is surjective.*

Then the induced morphism $\varprojlim_p G_1 \to \varprojlim_p G_2$ is an isomorphism.

Proof. Since $R^1\varprojlim_p D_i = 0$ for $i = 1,2$ by Lemma 3.12, we get the following morphism of short exact sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \varprojlim_p D_1 & \longrightarrow & \varprojlim_p G_1 & \longrightarrow & \varprojlim_p F_1 & \longrightarrow & 0 \\
 & & \varprojlim_p f \downarrow & & \varprojlim_p g \downarrow & & \varprojlim_p h \downarrow & & \\
0 & \longrightarrow & \varprojlim_p D_2 & \longrightarrow & \varprojlim_p G_2 & \longrightarrow & \varprojlim F_2 & \longrightarrow & 0
\end{array}
$$

By the left-exactness of $\varprojlim_p$, we have $\varprojlim_p(\ker g) = \ker(\varprojlim_p g)$. But by assumption (a), $\ker g$ is an extension of two finitely generated groups, and hence finitely generated itself. It has no prime-to-p torsion by (b), so by Lemma 3.12, $\varprojlim_p(\ker g) = 0$. Thus $\varprojlim_p g$ is injective.

Next, by (a), f is surjective with finite kernel, which implies that $\varprojlim_p f$ is surjective, since $R^1\varprojlim_p(\ker(f)) = 0$ by Lemma 3.12.

Finally, by Lemma 3.12 we know that $\varprojlim_p h = h|_{F_1[p']} : F_1[p'] \to F_2[p']$. This morphism is surjective by (c). It follows that $\varprojlim_p g$ is also surjective, which completes the proof. $\square$

LEMMA 3.14. *If $f : Y \to X$ is a morphism of connected, smooth, separated, finite type k-schemes, such that f induces an isomorphism $\pi_1(Y)^{\mathrm{ab},(p')} \xrightarrow{\cong} \pi_1(X)^{\mathrm{ab},(p')}$, and if $\overline{X}, \overline{Y}$ are smooth compactifications of X, Y, such that f extends to $\bar{f} : \overline{Y} \to \overline{X}$, then the map*

$$
\pi_1(\overline{Y})^{\mathrm{ab},(p')} \to \pi_1(\overline{X})^{\mathrm{ab},(p')}
$$

induced by $\bar{f}$ is surjective with finite kernel.

Proof. The surjectivity is clear, as $\pi_1(X)^{\mathrm{ab},(p')}$ and $\pi_1(Y)^{\mathrm{ab},(p')}$ surject onto $\pi_1(\overline{X})^{\mathrm{ab},(p')}$ and $\pi_1(\overline{Y})^{\mathrm{ab},(p')}$, respectively. We use the theory of the Albanese variety: After choosing a base point $x \in X(k)$, there exists a unique semi-abelian variety Alb_X, together with a map $\mathrm{alb}_{X,x} : X \to \mathrm{Alb}_X$, such that $\mathrm{alb}_{X,x}(x) = 0$, and such that any map $g : X \to A$ from X to a semi-abelian variety A with $g(x) = 0$ factors uniquely through $\mathrm{alb}_{X,x}$. For more details, see e.g. [SS03]. The Albanese variety classifies abelian coverings of X in the following sense: For ℓ a prime different from p, there exists a canonical isomorphism

$$
\hom(H^1(X, \mathbb{Z}_\ell), \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong \mathrm{Alb}_X(k)\{\ell\},
$$

see [SS03, Cor. 4.3], where $\mathrm{Alb}_X(k)\{\ell\}$ is the the subgroup of $\mathrm{Alb}_X(k)$ of all ℓ-power torsion elements. As a semi-abelian variety, Alb_X can be written uniquely as an extension of an abelian variety by a torus. The unique abelian variety appearing in this description of Alb_X is canonically isomorphic to $\mathrm{Alb}_{\overline{X}}$. By Chevalley's structure theorem for algebraic groups, the connected component of the origin (with its reduced structure) K^0_{red} of the kernel of the morphism of group schemes $\mathrm{Alb}_Y \to \mathrm{Alb}_X$ is a semi-abelian variety, and since every map from an abelian variety into an affine variety is constant, it follows that the unique abelian variety quotient of K^0_{red} is precisely the connected component of the origin (with its reduced structure) of the kernel of $\mathrm{Alb}_{\overline{Y}} \to \mathrm{Alb}_{\overline{X}}$.

With that in mind, we see that the assumption that $\pi_1(Y)^{\mathrm{ab},(p')} \xrightarrow{\cong} \pi_1(X)^{\mathrm{ab},(p')}$ is an isomorphism implies that the map $\mathrm{Alb}_Y(k) \to \mathrm{Alb}_X(k)$ is an isomorphism on prime-to-p torsion, and hence K^0_{red} is the trivial semi-abelian variety. Then $\ker(\mathrm{Alb}_{\overline{Y}} \to \mathrm{Alb}_{\overline{Y}})^0_{\mathrm{red}}$ is the trivial abelian variety, which shows that the kernel of $\mathrm{Alb}_{\overline{Y}}(k) \to \mathrm{Alb}_{\overline{X}}(k)$ is a finite group.

Writing T_ℓ for the Tate module, we see that if ℓ is a prime different from p, then $T_\ell \mathrm{Alb}_{\overline{Y}} \to T_\ell \mathrm{Alb}_{\overline{X}}$ is injective. Finally, since $\overline{X}$ and $\overline{Y}$ are proper and smooth, we get a morphism of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \hom(\mathrm{NS}(\overline{Y})\{\ell\}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) & \longrightarrow & \pi_1(\overline{Y})^{\mathrm{ab},(\ell)} & \longrightarrow & T_\ell \mathrm{Alb}_{\overline{Y}} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \hom(\mathrm{NS}(\overline{X})\{\ell\}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) & \longrightarrow & \pi_1(\overline{X})^{\mathrm{ab},(\ell)} & \longrightarrow & T_\ell \mathrm{Alb}_{\overline{X}} & \longrightarrow & 0.
\end{array}
$$

The groups on the left are finite groups, so we deduce that $\pi_1(\overline{Y})^{\mathrm{ab},(\ell)} \to \pi_1(\overline{X})^{\mathrm{ab},(\ell)}$ always has a finite kernel, and is injective for all but finitely many ℓ. This implies that the kernel of $\pi_1(\overline{Y})^{\mathrm{ab},(p')} \to \pi_1(\overline{X})^{\mathrm{ab},(p')}$ is finite. $\qquad\square$

PROPOSITION 3.15. *With the notations and assumptions of Theorem 3.10, pullback along f induces an isomorphism*

$$
\varprojlim_p \mathrm{Pic}(X) \xrightarrow{\cong} \varprojlim_p \mathrm{Pic}(Y).
$$

Proof. By Lemma 3.14 the induced morphism $\pi_1(\overline{Y})^{\mathrm{ab},p'} \to \pi_1(\overline{X})^{\mathrm{ab},p'}$ is surjective with finite kernel. Define $K_X := \ker(\mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(X))$ and $K_Y := \ker(\mathrm{Pic}(\overline{Y}) \to \mathrm{Pic}(Y))$. We get a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K_X & \longrightarrow & \mathrm{Pic}(\overline{X}) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_Y & \longrightarrow & \mathrm{Pic}(\overline{Y}) & \longrightarrow & \mathrm{Pic}(Y) & \longrightarrow & 0
\end{array}
$$

with K_X, K_Y finitely generated abelian groups. We have a second exact sequence

$$
0 \to \mathrm{Pic}^0(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to \mathrm{NS}(\overline{X}) \to 0
$$

with $\mathrm{NS}(\overline{X})$ a finitely generated group, according to, e.g., [Kah06, Thm. 3]. Moreover, $\mathrm{Pic}^0(\overline{X})$ is the set of k-points of an abelian variety, so $\mathrm{Pic}^0(\overline{X})$ is a divisible abelian group.

Define $\mathrm{Pic}^0(X)$ as the image of $\mathrm{Pic}^0(\overline{X})$ in $\mathrm{Pic}(X)$, and $\mathrm{NS}(X)$ as $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$. Then $\mathrm{Pic}^0(X)$ still divisible, and $\mathrm{NS}(X)$ is still finitely generated. Pullback along f induces a morphism $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y)$, and hence a morphism $\mathrm{NS}(X) \to \mathrm{NS}(Y)$.

We obtain the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{NS}(X) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Pic}^0(Y) & \longrightarrow & \mathrm{Pic}(Y) & \longrightarrow & \mathrm{NS}(Y) & \longrightarrow & 0
\end{array}
\tag{4}
$$

We are now in the situation of Lemma 3.13, and check that the conditions (a), (b) and (c) from the lemma are satisfied for diagram (4).

For Lemma 3.13, (a) we have to show that $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y)$ is surjective with finite kernel.

For every n prime to p, Kummer theory shows that there a morphism of short exact sequences of abelian groups

$$
\begin{array}{ccccccccc}
0 & \to & H^0(X,\mathcal{O}_X^\times)/H^0(X,\mathcal{O}_X^\times)^n & \to & \mathrm{Hom}(\pi_1^{\mathrm{ab},(p')}(X),\mathbb{Z}/n\mathbb{Z}) & \to & \mathrm{Pic}(X)[n] & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(Y,\mathcal{O}_Y^\times)/H^0(Y,\mathcal{O}_Y^\times)^n & \to & \mathrm{Hom}(\pi_1^{\mathrm{ab},(p')}(Y),\mathbb{Z}/n\mathbb{Z}) & \to & \mathrm{Pic}(Y)[n] & \to & 0
\end{array}
\tag{5}
$$

where the two left vertical arrows are isomorphisms by the assumptions (b) and (c) of Theorem 3.10, and hence so is the third. Since by Lemma 3.14 the morphism $\pi_1(\overline{Y})^{\mathrm{ab},(p')} \to \pi_1(\overline{X})^{\mathrm{ab},(p')}$ is surjective with finite kernel, the same argument as above shows that for almost all n prime to p, $\mathrm{Pic}(\overline{X})[n] \xrightarrow{\cong} \mathrm{Pic}(\overline{Y})[n]$. Since $\mathrm{Pic}^0(\overline{X})$ is divisible, we have a short exact sequence

$$
0 \to \mathrm{Pic}^0(\overline{X})[n] \to \mathrm{Pic}(\overline{X})[n] \to \mathrm{NS}(\overline{X})[n] \to 0
$$

which shows that for almost all n prime to p, $\mathrm{Pic}^0(\overline{X})[n] \xrightarrow{\cong} \mathrm{Pic}^0(\overline{Y})[n]$. Since $\mathrm{Pic}^0(\overline{X})$ and $\mathrm{Pic}^0(\overline{Y})$ are sets of k-points of abelian varietes, and since pullback along f induces a morphism of the underlying abelian varieties, it follows that $\mathrm{Pic}^0(\overline{X}) \to \mathrm{Pic}^0(\overline{Y})$ is surjective with finite kernel. From the morphism of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K_X \cap \mathrm{Pic}^0(\overline{X}) & \longrightarrow & \mathrm{Pic}^0(\overline{X}) & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_Y \cap \mathrm{Pic}^0(\overline{Y}) & \longrightarrow & \mathrm{Pic}^0(\overline{Y}) & \longrightarrow & \mathrm{Pic}^0(Y) & \longrightarrow & 0
\end{array}
$$

we see that $\operatorname{Pic}^0(X) \to \operatorname{Pic}^0(Y)$ is surjective. To prove that its kernel is finite, it suffices to show that $\operatorname{coker}(K_X \cap \operatorname{Pic}^0(\overline{X}) \to K_Y \cap \operatorname{Pic}^0(\overline{Y}))$ is finite. For this, let ℓ be a prime different from p, and write $\overline{X} \smallsetminus X =: D_X = \bigcup_{i=1}^{r(X)} D_{X,i}$ with $D_{X,i}$ smooth divisors. Let $M_X := \bigoplus D_{X,i} \mathbb{Z}$ be the free $\mathbb{Z}$-module of rank $r(X)$, and similarly for Y. We have the associated Gysin exact sequence in étale cohomology [Mil80, Ch. VI, Cor. 5.3]:

$$
\begin{array}{ccccccc}
0 \longrightarrow H^1(\overline{X}, \mathbb{Z}_\ell(1)) \longrightarrow H^1(X, \mathbb{Z}_\ell(1)) \longrightarrow M_X \otimes \mathbb{Z}_\ell \xrightarrow{c_1^{\ell,X}} H^2(\overline{X}, \mathbb{Z}_\ell(1)) \\
\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \\
0 \longrightarrow H^1(\overline{Y}, \mathbb{Z}_\ell(1)) \longrightarrow H^1(Y, \mathbb{Z}_\ell(1)) \longrightarrow M_Y \otimes \mathbb{Z}_\ell \xrightarrow{c_1^{\ell,Y}} H^2(\overline{Y}, \mathbb{Z}_\ell(1))
\end{array}
$$

As we have seen, for almost all primes ℓ the two left vertical arrows are isomorphisms, so $\ker(c_1^{\ell,X}) \xrightarrow{\cong} \ker(c_1^{\ell,Y})$ for almost all ℓ. But by construction, we have a commutative diagram

$$
\begin{array}{ccc}
\ker(c_1^{\ell,X}) & \longrightarrow & (K_X \cap \operatorname{Pic}^0(\overline{X})) \otimes \mathbb{Z}_\ell \\
\downarrow & & \downarrow \\
\ker(c_1^{\ell,Y}) & \longrightarrow & (K_Y \cap \operatorname{Pic}^0(\overline{Y})) \otimes \mathbb{Z}_\ell
\end{array}
$$

This shows that the finitely generated group $\operatorname{coker}(K_X \cap \operatorname{Pic}^0(\overline{X}) \to K_Y \cap \operatorname{Pic}^0(\overline{Y}))$ is trivial after tensoring with $\mathbb{Z}_\ell$ for almost all ℓ and hence finite. This finishes the proof that condition (a) from Lemma 3.13, holds for diagram (4).

To show that Lemma 3.13, (b) holds for diagram (4) we note that we have seen below (5) that the kernel of $\operatorname{Pic}(X) \to \operatorname{Pic}(Y)$ contains no prime-to-p-torsion. Finally, lets check that Lemma 3.13, (c) holds. Since $\operatorname{Pic}^0(X)$ is divisible, for every n prime to p we get a commutative diagram

$$
\begin{array}{ccc}
\operatorname{Pic}(X)[n] & \longrightarrow\!\!\!\!\rightarrow & \operatorname{NS}(X)[n] \\
\downarrow{\scriptstyle \cong} & & \downarrow \\
\operatorname{Pic}(Y)[n] & \longrightarrow\!\!\!\!\rightarrow & \operatorname{NS}(Y)[n]
\end{array}
$$

with surjective horizontal arrows. This shows that $\operatorname{NS}(X)[p'] \to \operatorname{NS}(X)[p']$ is surjective, so Lemma 3.13, (c) holds for diagram (4). $\qquad\square$

With Proposition 3.15, the proof of Theorem 3.10 becomes simple:

Proof of Theorem 3.10. By Proposition 3.4, f induces a morphism of short

exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{I}(X) & \longrightarrow & \mathrm{Pic}^{\mathrm{Strat}}(X) & \longrightarrow & \varprojlim_p \mathrm{Pic}(X) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{I}(Y) & \longrightarrow & \mathrm{Pic}^{\mathrm{Strat}}(Y) & \longrightarrow & \varprojlim_p \mathrm{Pic}(Y) & \longrightarrow & 0
\end{array}
$$

By assumption there exist good compactifications $\overline{X}$ and $\overline{Y}$, such that f extends to $\overline{f} : \overline{Y} \to \overline{X}$. Then the right vertical morphism is an isomorphism by Proposition 3.15. We show that Proposition 3.5 implies that the left vertical arrow is an isomorphism under our assumptions: If $C := \mathrm{coker}(H^0(X, \mathcal{O}_X^\times) \to H^0(Y, \mathcal{O}_Y^\times))$, then by assumption C is a finitely generated abelian p-group, and hence a finite p-group. It follows that the diagonal map

$$
C \to \varprojlim_n C/p^n
$$

is an isomorphism. By the exactness of p-adic completion of finitely generated abelian groups, we have a canonical isomorphism

$$
\varprojlim_n C/p^n \cong \mathrm{coker}(\varprojlim_n H^0(X, \mathcal{O}_X^\times)/p^n \to \varprojlim_n H^0(Y, \mathcal{O}_Y^\times)/p^n),
$$

so looking at the long exact sequence attached to the morphism of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(X, \mathcal{O}_X^\times)/k^\times & \longrightarrow & \varprojlim_n H^0(X, \mathcal{O}_X^\times)/p^n & \longrightarrow & \mathbb{I}(X) & \longrightarrow & 0 \\
& & \downarrow & & {\scriptstyle \varprojlim f^*} \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(Y, \mathcal{O}_Y^\times)/k^\times & \longrightarrow & \varprojlim_n H^0(Y, \mathcal{O}_Y^\times)/p^n & \longrightarrow & \mathbb{I}(Y) & \longrightarrow & 0,
\end{array}
$$

we see that $\mathbb{I}(X) \to \mathbb{I}(Y)$ is surjective. But by Remark 3.11, (b) the two left vertical arrows are injective, so $\mathbb{I}(X) \cong \mathbb{I}(Y)$ as claimed. $\square$

To close this section, we show that in the case that $X = \mathrm{Spec}\, k$, the assumption that Y admits a good compactification is not necessary:

PROPOSITION 3.16. *Let Y be a smooth, connected, separated, finite type k-scheme, such that $\pi_1^{\mathrm{ab},(p')}(Y) = 0$. Then $\mathrm{Pic}^{\mathrm{Strat}}(Y) = 0$.*

In the proof of Proposition 3.16 we will use Nagata's theorem on compactifications (see [Lüt93]) to find a normal projective compactification $\overline{Y}$ of Y. In a preliminary version of this article we used de Jong's theorem on alterations ([dJ96]) to replace $\overline{Y}$ with a nice simplicial scheme, and then we studied the attached groups of simplicial line bundles. These arguments were long and technical.

Brian Conrad suggested working directly with the normal projective compactifications instead of using simplicial techniques: He pointed out a theorem by Lang stating that if $\overline{Y}$ is projective and normal, then the group $\mathrm{Cl}^{\mathrm{alg}}(\overline{Y})$ of classes of those Weil divisors modulo linear equivalence which are algebraically equivalent to 0 is the group of k-points of an abelian variety; for a modern treatment see [BGS11].
We are grateful to Brian Conrad for bringing this result to our attention.

Proof of Proposition 3.16. By Proposition 3.8, we know that $\mathbb{I}(Y) = 0$. Hence, according to Proposition 3.4, we only have to show that $\varprojlim_p \mathrm{Pic}(Y) = 0$. As in (5), Kummer theory shows that $\mathrm{Pic}(Y)[n] = 0$, whenever n is prime to p. Thus, Lemma 3.12 shows that it suffices to prove that $\mathrm{Pic}(Y)$ is a finitely generated group.
To this end, let $\overline{Y}$ be a normal projective compactification of Y, and $\mathrm{Cl}(\overline{Y})$ the group of Weil divisors on $\overline{Y}$ modulo linear equivalence. We then have a surjection $\mathrm{Cl}(\overline{Y}) \to \mathrm{Pic}(Y)$, and a short exact sequence

$$0 \longrightarrow \mathrm{Cl}^{\mathrm{alg}}(\overline{Y}) \longrightarrow \mathrm{Cl}(\overline{Y}) \longrightarrow \mathrm{NS}(\overline{Y}) \longrightarrow 0$$

with $\mathrm{NS}(\overline{Y})$ finitely generated by [Kah06, Thm. 3]. By the aforementionend result of Lang, $\mathrm{Cl}^{\mathrm{alg}}(\overline{Y})$ is the set of k-points of an abelian variety. More precisely, given a point $y \in \overline{Y}(k)$, there is an abelian variety A and a rational map $\alpha_y : \overline{Y} \dashrightarrow A$ defined around y, such that $\alpha_y(y) = 0$, and such that every rational map $\beta : Y \dashrightarrow B$ defined around y, with B an abelian variety and $\beta(y) = 0$, factors through α_y. The abelian variety A is the Albanese variety for rational maps, and its dual has the property that $A^\vee(k) = \mathrm{Cl}^{\mathrm{alg}}(\overline{Y})$. For a modern treatment see [BGS11].
Now to finish, note that since the kernel of the surjection $\mathrm{Cl}(\overline{Y}) \twoheadrightarrow \mathrm{Pic}\, Y$ is finitely generated, it follows that $\mathrm{Cl}(\overline{Y})[n] = 0$ for all but finitely many n. This implies that $\mathrm{Cl}^{\mathrm{alg}}(\overline{Y})[n] = 0$ for all but finitely many n, which shows that $\mathrm{Cl}^{\mathrm{alg}}(\overline{Y}) = 0$, since the underlying abelian variety must have dimension 0. It follows that $\mathrm{Cl}(\overline{Y}) = \mathrm{NS}(\overline{Y})$ is finitely generated and thus that $\mathrm{Pic}(Y)$ is finitely generated. $\square$

4 Special case II: Extensions of stratified bundles of rank 1

We continue to write k for an algebraically closed field of characteristic $p > 0$.

LEMMA 4.1. *Let X be a smooth, connected, separated k-scheme of finite type which admits a good compactification, and $\bar{x}$ a geometric point. If $\pi_1^{\mathrm{tame}}(X, \bar{x})$ does not have a quotient isomorphic to $\mathbb{Z}/p\mathbb{Z}$, then $\mathrm{Ext}^1_{\mathrm{Strat}^{\mathrm{rs}}(X)}(\mathcal{O}_X, \mathcal{O}_X) = 0$.*

Proof. Let $\overline{X}$ be a good compactification of X and E a regular singular stratified bundle on X, which is an extension of $\mathcal{O}_X$ by $\mathcal{O}_X$ in $\mathrm{Strat}^{\mathrm{rs}}(X)$. The aim is to show that E extends to a stratified bundle $\overline{E} \in \mathrm{Strat}(\overline{X})$; the assumption

on $\pi_1^{\mathrm{tame}}(X,\bar{x})$ implies that $\pi_1(\overline{X},\bar{x})$ does not have a quotient isomorphic to $\mathbb{Z}/p\mathbb{Z}$, so we can then apply [EM10, Prop. 2.4] to show that $\overline{E}$ is trivial.

To show that E extends to a stratified bundle $\overline{E}$ on $\overline{X}$, by Proposition 2.12 it suffices to show that the exponents of E along every component of the boundary divisor $\overline{X} \smallsetminus X$ are $0 \mod \mathbb{Z}$. Let $x_0 \in \overline{X}$ be a closed point lying on precisely one component of $\overline{X} \smallsetminus X$. Write $K_{x_0} := \mathrm{Frac}(\widehat{\mathcal{O}_{\overline{X},x_0}})$. Then, after choosing coordinates $x_1,\ldots,x_n$, K_{x_0} is isomorphic to the fraction field of the ring of formal power series $k[\![x_1,\ldots,x_n]\!]$. Write $\widehat{E} := E \otimes K_{x_0}$. The stratification on E gives $\widehat{E}$ the structure of a finite dimensional K_{x_0}-vector space with an action of the ring of differential operators $k[\partial_{x_i}^{(m)}|i=1,\ldots,n,m\geq 0]$, where the usual composition rules hold. Such an object is called an iterative differential module in [MvdP03]. The category $\mathrm{Strat}(K_{x_0})$ of such objects is "almost" a neutral Tannakian category, but there might not exist a k-valued fiber functor (for more details see [Kin12b, Ch. 1]). Fortunately, the sub-tensor category $\langle \widehat{E} \rangle_{\otimes} \subset \mathrm{Strat}(K_{x_0})$ spanned by $\widehat{E}$ admits a fiber functor ω by [Del90, Cor. 6.20], as k is algebraically closed. Composition of ω with the restriction functor $\langle E \rangle_{\otimes} \to \langle \widehat{E} \rangle_{\otimes}$ is a fiber functor for $\langle E \rangle_{\otimes}$, hence we get a morphism $G(\langle \widehat{E} \rangle_{\otimes}) \to G(\langle E \rangle_{\otimes})$ of the associated affine k-group schemes, and this morphism is a closed immersion by [DM82, Prop. 2.21].

Since E is an extension of $\mathcal{O}_X$ by $\mathcal{O}_X$, its monodromy group $G(\langle E \rangle_{\otimes})$ is a closed subgroup scheme of $\mathbb{G}_{a,k}$. But E is assumed to be regular singular, so [Gie75, Thm. 3.3] implies that $\widehat{E}$ is a direct sum of rank 1 objects of $\mathrm{Strat}(K_{x_0})$ and thus $G(\langle \widehat{E} \rangle_{\otimes})$ is a closed subgroup scheme of $\mathbb{G}_{m,k}^2$. We finally conclude that $\widehat{E} \cong K_{x_0}^2$ as an object of $\mathrm{Strat}(K_{x_0})$, since $\mathbb{G}_{a,k}$ does not have nontrivial diagonalizable subgroups, and then [Gie75, Thm. 3.3] implies that the exponents of E along the component on which x_0 lies are $0 \mod \mathbb{Z}$. We repeat the same argument for every component of the boundary $\overline{X} \smallsetminus X$, and hence complete the proof. $\qquad\square$

We can now prove one of the main results of this article:

THEOREM 4.2. *Let X be a smooth, connected, separated k-scheme of finite type which admits a good compactification, and $\bar{x}$ a geometric point. If $\pi_1^{\mathrm{tame}}(X,\bar{x})^{\mathrm{ab},(p')} = 0$, and if $\pi_1^{\mathrm{tame}}(X,\bar{x})$ does not have a quotient isomorphic to $\mathbb{Z}/p\mathbb{Z}$, then every regular singular stratified bundle on X which has abelian monodromy is trivial.*

Proof. Let E be a regular singular stratified bundle on X, and $\omega : \langle E \rangle_{\otimes} \to \mathrm{Vectf}_k$ a fiber functor. Assume that the associated k-group scheme $G(\langle E \rangle_{\otimes},\omega)$ is abelian. By [Wat79, 9.4] every irreducible object of $\langle E \rangle_{\otimes}$ has rank 1. By Proposition 3.16 every rank 1 object of $\mathrm{Strat}(X)$ is trivial, and by Lemma 4.1 there are no nontrivial extensions between trivial objects of rank 1. It follows that every object of $\langle E \rangle_{\otimes}$ is trivial. $\qquad\square$

5   Special case III: Affine spaces

We continue to denote by k an algebraically closed field of characteristic $p > 0$. It is well known that $\pi_1^{\mathrm{tame}}(\mathbb{A}_k^n) = 0$ for all $n \geq 0$.

Theorem 5.1. *Every regular singular stratified bundle on $\mathbb{A}_k^n$ is trivial.*

Remark 5.2.

- A slightly different approach to Theorem 5.1 was sketched in [Esn12, 4.4]

- Note that Theorem 5.1 follows directly from Proposition 3.16 and [Gie75, Thm. 5.3], which states that every regular singular stratified bundle on $\mathbb{P}_k^n \smallsetminus D$ is a direct sum of stratified line bundles, if D is a strict normal crossings divisor. Unfortunately [loc. cit.] is imprecisely stated (it is false for $n = 1$) and its proof is very complicated.

  Below we give a simple argument to prove Theorem 5.1, which is certainly implicitly contained in [Gie75].

Proof of Theorem 5.1. The case of $\mathbb{A}_k^1$ follows from [Gie75, Prop. 4.2] and Proposition 3.16.

We proceed by induction; let $n > 1$. Then the n-fold product $\mathbb{P}_k^1 \times_k \ldots \times_k \mathbb{P}_k^1$ is a good compactification of $\mathbb{A}_k^n$, and if E is a stratified bundle on $\mathbb{A}_k^n$, then E is regular singular if and only if it is $(\mathbb{A}_k^n, (\mathbb{P}_k^1)^n)$-regular singular by Proposition 2.7.

We compute the exponents of E along the divisor $\left(\mathbb{P}_k^1\right)^{n-1} \times_k \{\infty\} \subset (\mathbb{P}_k^1)^n$. To this end let $\overline{E}$ be a free $\mathcal{O}_{\mathbb{A}_k^{n-1} \times (\mathbb{P}_k^1 \smallsetminus \{0\})}$-module with $\mathscr{D}_{\mathbb{A}_k^{n-1} \times (\mathbb{P}_k^1 \smallsetminus \{0\})/k}(\log \mathbb{A}_k^{n-1} \times \{\infty\})$-action extending $E|_{\mathbb{A}_k^{n-1} \times \mathbb{G}_m}$. Choose coordinates $x_1, \ldots, x_n$ such that $\mathbb{A}_k^{n-1} \times_k (\mathbb{P}_k^1 \smallsetminus \{0\}) = \operatorname{Spec} k[x_1, \ldots, x_{n-1}, x_n^{-1}]$. By Proposition 2.10 there exists a basis $e_1, \ldots, e_r$ of the free module $\overline{E}$, such that

$$\delta_{x_n^{-1}}^{(m)}(e_i) = \binom{\alpha_i}{m} e_i + x_n^{-1} \overline{E}$$

with $\alpha_i \in \mathbb{Z}_p$ an exponent of $\overline{E}$ along $\mathbb{A}_k^{n-1} \times \{\infty\}$. But the same equation also holds modulo a prime ideal $(x_1 - a_1, \ldots, x_{n-1} - a_{n-1})$, $a_1, \ldots, a_{n-1} \in k$, so $\alpha_i \mod \mathbb{Z}$ is an exponent of the fiber $E|_{(a_1, \ldots, a_{n-1}) \times \mathbb{A}_k^1}$ along the divisor $(a_1, \ldots, a_{n-1}, \infty) \subset (a_1, \ldots, a_{n-1}) \times \mathbb{P}_k^1$.

The case $n = 1$ now shows that $\alpha_i \equiv 0 \mod \mathbb{Z}$, and hence the exponents of E along $\mathbb{A}_k^{n-1} \times_k \{\infty\}$ are $0 \mod \mathbb{Z}$. By Proposition 2.12 this means that E extends to an actual stratified bundle on $\mathbb{A}_k^{n-1} \times_k \mathbb{P}^1$.

But now we are done: The above argument shows that E extends to a stratified bundle on $(\mathbb{P}_k^1)^n$: First to $(\mathbb{P}_k^1)^n$ minus a closed subset of codimension ≥ 2, and then by [Kin12a, Lemma 2.5] to $(\mathbb{P}_k^1)^n$. But there are only trivial stratified bundles on $(\mathbb{P}_k^1)^n$, as it it is birational to $\mathbb{P}_k^n$, so E is trivial ([Gie75, Thm. 2.2]). $\square$

6 SPECIAL CASE IV: UNIVERSAL HOMEOMORPHISMS

We continue to denote by k an algebraically closed field of characteristic $p > 0$. Recall that by [EGA4, 18.12.11], a finite type morphism $f : Y \to X$ of finite type k-schemes is a universal homeomorphism if and only if it is finite, purely inseparable (i.e. universally injective) and surjective. It is proven in [SGA1, IX.4.10] that f induces an isomorphism $\pi_1^{\text{ét}}(Y) \xrightarrow{\cong} \pi_1^{\text{ét}}(X)$ (with appropriate choices of base points); it follows from [Vid01] that the same is true for π_1^{tame}. In this section we prove that pull-back along f is an equivalence $\text{Strat}^{\text{rs}}(X) \to \text{Strat}^{\text{rs}}(Y)$.

For a k-scheme X, we write $X^{(n)}$ for the base change of X along the n-th power of the absolute Frobenius of k, and by $F_{X/k}^{(n)} : X \to X^{(n)}$ the associated k-linear relative Frobenius. It follows from Theorem 2.3 that pull-back along $F_{X/k}^{(n)}$ induces an equivalence of categories $\text{Strat}(X^{(n)}) \to \text{Strat}(X)$. This remains true in the regular singular case: We first work with respect to one fixed good partial compactification $(X, \overline{X})$.

PROPOSITION 6.1 ("Frobenius descent"). *If $(X, \overline{X})$ is a good partial compactification, then the pair $(X^{(n)}, \overline{X}^{(n)})$ also is a good partial compactification, and $F_{X/k}^{(n)}$ induces an equivalence*

$$(F_{X/k}^{(n)})^* : \text{Strat}^{\text{rs}}((X^{(n)}, \overline{X}^{(n)})) \to \text{Strat}^{\text{rs}}((X, \overline{X})).$$

Proof. We may assume that $n = 1$, and write $F_{X/k} = F_{X/k}^{(1)}$.

Write $D := \overline{X} \smallsetminus X$ and $D^{(1)} := \overline{X}^{(1)} \smallsetminus X^{(1)}$. Since the functor $F_{X/k}^* : \text{Strat}(X^{(1)}) \to \text{Strat}(X)$ is an equivalence, it suffices to show that the essential image of $\text{Strat}^{\text{rs}}((X^{(1)}, \overline{X}^{(1)}))$ in $\text{Strat}(X)$ is $\text{Strat}^{\text{rs}}((X, \overline{X}))$, i.e. it suffices to show that a stratified bundle E on $X^{(1)}$ is $(X^{(1)}, \overline{X}^{(1)})$-regular singular if $F_{X/k}^* E$ is $(X, \overline{X})$-regular singular.

Let $j : X \hookrightarrow \overline{X}$ denote the inclusion. Assume that $F_{X/k}^* E$ is $(X, \overline{X})$-regular singular. Let E' be any torsion free coherent extension of E to $\overline{X}^{(1)}$ and $\overline{E}$ the $\mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)})$-module generated by E' in the $\mathscr{D}_{\overline{X}^{(1)}/k}(\log \mathscr{D}^{(1)})$-module $j_*^{(1)} E$. We need to show that $\overline{E}$ is $\mathcal{O}_{\overline{X}^{(1)}}$-coherent. Since $F_{\overline{X}/k}$ is faithfully flat, it suffices to show that $F_{\overline{X}/k}^* \overline{E}$ is $\mathcal{O}_{\overline{X}}$-coherent, [SGA1, Prop. VIII.1.10]. Define G to be the $\mathscr{D}_{\overline{X}/k}(\log D)$-module generated by $F_{\overline{X}/k}^* E'$ in $j_* F_{X/k}^* E = F_{\overline{X}/k}^* j_*^{(1)} E$. Then G is $\mathcal{O}_{\overline{X}}$-coherent by assumption, so the proof is complete if we can show that $G = F_{\overline{X}/k}^* \overline{E}$.

The $\mathcal{O}_{\overline{X}^{(1)}}$-module $j_*^{(1)} E$ naturally carries a $\mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)})$-action, and similarly for $j_* F_{X/k}^* E$. Then we can describe $F_{\overline{X}/k}^* \overline{E}$ as the image of the (pulled-

back) evaluation morphism

$$F^*_{\overline{X}/k}\left(\mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)}) \otimes_{\mathcal{O}_{\overline{X}^{(1)}}} E'\right) \to F^*_{\overline{X}/k} j^{(1)}_* E,$$

and G as the image of the evaluation morphism

$$\mathscr{D}_{\overline{X}/k}(\log D) \otimes_{\mathcal{O}_{\overline{X}}} F^*_{\overline{X}/k} E' \to j_* F^*_{X/k} E = F^*_{\overline{X}/k} j^{(1)}_* E.$$

These morphisms fit in a commutative diagram (writing $F = F_{\overline{X}/k}$ for legibility):

$$
\begin{array}{ccc}
F^*\left(\mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)}) \otimes_{\mathcal{O}_{\overline{X}^{(1)}}} E'\right) & \longrightarrow & F^* j^{(1)}_* E \\
\| & & \| \\
F^* \mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)}) \otimes_{F^{-1}\mathcal{O}_{\overline{X}^{(1)}}} F^{-1} E' & & \| \\
\gamma \otimes \mathrm{id} \uparrow & & \| \\
\mathscr{D}_{\overline{X}/k}(\log D) \otimes_{F^{-1}\mathcal{O}_{\overline{X}^{(1)}}} F^{-1} E' & & \| \\
\| & & \| \\
\mathscr{D}_{\overline{X}/k}(\log D) \otimes_{\mathcal{O}_{\overline{X}}} F^* E' & \longrightarrow & j_* F^* E,
\end{array}
$$

where

$$\gamma : \mathscr{D}_{\overline{X}/k}(\log D) \to F^*_{\overline{X}/k} \mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)}) = \mathcal{O}_{\overline{X}} \otimes_{F^{-1}\mathcal{O}_{\overline{X}^{(1)}}} F^{-1} \mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)})$$

is the canonical morphism coming via restriction from the morphism $\mathscr{D}_{\overline{X}/k} \to F^*_{\overline{X}/k} \mathscr{D}_{\overline{X}^{(1)}/k}$. It follows that $G = F^*_{\overline{X}/k} \overline{E}$, if γ is surjective.

This is a local question, so we may assume that $\overline{X} = \operatorname{Spec} A$ and $X = \operatorname{Spec} A[t_1^{-1}]$, with $t_1, \ldots, t_r$ local coordinates on A. Then $\overline{X}^{(1)} = \operatorname{Spec} A \otimes_{F_k} k$, and $t_1 \otimes 1, \ldots, t_r \otimes 1$ is a system of local coordinates for $A \otimes_{F_k} k$. The relative frobenius $F_{X/k}$ then maps $t_i \otimes 1 \mapsto t_i^p$, for $i = 1, \ldots, r$.

With the notation $\delta_{t_i}^{(p^m)}$ from Definition 2.5, (b), and recalling that $\delta_{t_i}^{(p^m)}$ "behaves like" $\frac{t_i^{p^m}}{p^m!} \partial^{p^m} / \partial t_i^{p^m}$, we claim that $\gamma(\delta_{t_i}^{(1)}) = 0$ and

$$\gamma(\delta_{t_i}^{(p^m)}) = \delta_{t_i \otimes 1}^{(p^{m-1})} \text{ for } m > 1, \tag{6}$$

which shows that the image of γ contains all the generators of the left-$\mathcal{O}_{\overline{X}}$-algebra $F^*_{X/k} \mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)})$, and thus that γ is surjective. As for the claim, it suffices to observe that for $s \geq 0$,

$$\delta_{t_i}^{(p^m)}((t_j^p)^s) = \begin{cases} 0 & i \neq j \\ \binom{sp}{p^m} & \text{otherwise,} \end{cases}$$

and finally that

$$\binom{sp}{p^m} \equiv \binom{s}{p^{m-1}} \mod p.$$

$\square$

REMARK 6.2. This proof does *not* show that all $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D)$-module descend to $\mathcal{O}_{\overline{X}^{(1)}}$-coherent $\mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)})$-modules. In fact, such a statement is false, due to the failure of Cartier's theorem [Kat70, §5.] for logarithmic connections. On the other hand, in [Lor00] a version of Cartier's theorem for log-schemes is developed, which is applied in [Mon02, Ch. 4] to construct a generalization of Frobenius descent to the logarithmic setting. For this however, the rings of coefficients have to be enlarged.

COROLLARY 6.3. *With the notations from Proposition 6.1, if $\overline{E}$ is a locally free $\mathcal{O}_{\overline{X}^{(1)}}$-coherent $\mathscr{D}_{\overline{X}^{(1)}/k}(\log D^{(1)})$-module with exponents $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}_p$, then $F_{\overline{X}/k}^* \overline{E}$ is a $\mathscr{D}_{\overline{X}/k}(\log D)$-module with exponents $p\alpha_1, \ldots, p\alpha_n$.*

Consequently, if $E \in \mathrm{Strat}^{\mathrm{rs}}((X^{(1)}, \overline{X}^{(1)}))$, with exponents $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}_p/\mathbb{Z}$, then $F_{X/k}^* E$ has exponents $p\alpha_1, \ldots, p\alpha_n \in \mathbb{Z}_p/\mathbb{Z}$.

Proof. The claim follows directly from the formula (6) and the fact that $\delta_{t_1}^{(1)}$ acts on $F_{\overline{X}/k}^* \overline{E} = \mathcal{O}_{\overline{X}} \otimes_{F^{-1}\mathcal{O}_{\overline{X}}^{(1)}} \overline{E}$ via $\delta_{t_1}^{(1)}(a \otimes e) = \delta_{t_1}^{(1)}(a) \otimes e$. $\square$

REMARK 6.4. Note that multiplication by p is an automorphism of the group $\mathbb{Z}_p/\mathbb{Z}$.

THEOREM 6.5. *Let $(X, \overline{X})$, $(Y, \overline{Y})$ be good partial compactifications and $\bar{f} : \overline{Y} \to \overline{X}$ a universal homeomorphism such that $\bar{f}(X) \subset \overline{Y}$. If we write $f := \bar{f}|_X$ then f induces an equivalence*

$$f^* : \mathrm{Strat}^{\mathrm{rs}}((X, \overline{X})) \xrightarrow{\cong} \mathrm{Strat}^{\mathrm{rs}}((Y, \overline{Y})).$$

Proof. The morphism $\bar{f}$ is finite of degree p^n for some n, and thus there is a morphism $\bar{g} : \overline{X} \to \overline{Y}^{(n)}$, such that $\bar{g}\bar{f} = F_{\overline{Y}/k}^{(n)}$, and such that $\bar{f}^{(n)}\bar{g} = F_{\overline{X}/k}^{(n)}$. Moreover, $\bar{g}(X) \subset Y^{(n)}$. Write $g := \bar{g}|_X$. Then $(gf)^*$ is an equivalence by Proposition 6.1, so f^* is essentially surjective. But $(f^{(n)}g)^*$ also is an equivalence, so $f^{(n),*}$ is full. This shows that f^* is full, and since f is faithfully flat, it follows that f is faithful as well. This finishes the proof. $\square$

THEOREM 6.6. *Let $f : Y \to X$ be a universal homeomorphism of smooth, separated, finite type k-schemes. Then f induces an equivalence*

$$f^* : \mathrm{Strat}^{\mathrm{rs}}(X) \xrightarrow{\cong} \mathrm{Strat}^{\mathrm{rs}}(Y).$$

Proof. Without loss of generality we may assume that X, Y are connected. The same argument as in the proof of Theorem 6.5 shows that the fact that the relative Frobenius induces an equivalence $\mathrm{Strat}(X^{(n)}) \to \mathrm{Strat}(X)$, implies that the functor $f^* : \mathrm{Strat}(X) \to \mathrm{Strat}(Y)$ is an equivalence.

If follows that we just need to check that f^*E is regular singular for $E \in \mathrm{Strat}(X)$, if and only if E is regular singular.

Assume that E is regular singular and let $(Y, \overline{Y})$ be a good partial compactification with $\overline{Y} \smallsetminus Y$ smooth with generic point η. Using that $k(X) \subset k(Y)$ is purely inseparable, one quickly checks that $R := \mathcal{O}_{\overline{Y}, \eta} \cap k(X)$ is a discrete valuation ring, that $\mathcal{O}_{Y, \eta}$ is the integral closure of R in $k(Y)$, and hence that the residue field of R has transcendence degree $\dim X - 1$ over k. This means that R is the local ring of a codimension 1 point on some model of $k(X)$. Hence there exists a good partial compactification $(X, \overline{X})$, such that f extends to a morphism $\bar{f} : \overline{Y} \to \overline{X}$ (after possibly removing a closed subset of codimension ≥ 2 from $\overline{Y}$). This shows that f^*E is $(Y, \overline{Y})$-regular singular. We repeat this for every good partial compactification $(Y, \overline{Y})$ to conclude that f^*E is regular singular.

Conversely, assume that f^*E is regular singular. To prove that E is regular singular we need to show that for every good partial compactification $(X, \overline{X})$, such that $\overline{X} \smallsetminus X$ is smooth with generic point ξ, there exists an open neighborhood $\overline{U}'$ of ξ, such that E is $(\overline{U}' \cap X, \overline{U}')$-regular singular.

Let $\overline{U} = \mathrm{Spec}\, \overline{A}$ be an affine open neighborhood of ξ. We may assume that $U := \overline{U} \cap X$ is also affine, say $U = \mathrm{Spec}\, A$. Because f is finite, $V := f^{-1}(U)$ is affine, say $V = \mathrm{Spec}\, B$, and $f|_V$ is a universal homeomorphism. Let $\overline{B}$ be the integral closure of $\overline{A}$ in B, and $\overline{V} := \mathrm{Spec}\, \overline{B}$. Then $\bar{g} : \overline{V} \to \overline{U}$ is a finite morphism. By construction $\overline{V}$ is normal, and $\bar{g}$ is a universal homeomorphism, because $k(X) \subset k(Y)$ is purely inseparable. We may shrink $\overline{U}$ around ξ to obtain an open neighborhood $\overline{U}' \subset \overline{U}$ of ξ, such that $\overline{V}' := \bar{g}^{-1}(\overline{U}')$ is smooth, and $\bar{g}' : \overline{V}' \to \overline{U}'$ is a universal homeomorphism. Moreover, writing $V' := \overline{V}' \cap Y$, $U' := \overline{U}' \cap X$, we see that $\bar{g}'|_{V'} = f|_{V'} : V' \to U'$. Since $(f|_{V'})^*(E|_{U'})$ is $(V', \overline{V}')$-regular singular by assumption, we can apply Theorem 6.5 to see that E is $(U', \overline{U}')$-regular singular. $\qquad\square$

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