The Virtual Haken Conjecture

Dedicated to Mike Freedman
on the occasion of his 60th birthday

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with an appendix by

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Abstract. We prove that cubulated hyperbolic groups are virtually special. The proof relies on results of Haglund and Wise which also imply that they are linear groups, and quasi-convex subgroups are separable. A consequence is that closed hyperbolic 3-manifolds have finite-sheeted Haken covers, which resolves the virtual Haken question of Waldhausen and Thurston's virtual fibering question. An appendix to this paper by Agol, Groves, and Manning proves a generalization of the main result of [1].

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1 Introduction

In this paper, we will be interested in word-hyperbolic fundamental groups of compact non-positively curved (NPC) cube complexes. These notions were introduced in the seminal paper of Gromov [20].

Theorem 1.1. *[24, Problem 11.7] [50, Conjecture 19.5] Let G be a word-hyperbolic group acting properly and cocompactly on a CAT(0) cube complex X. Then G has a finite index subgroup F acting faithfully and specially on X.*

Remark 1.2. The conclusion of the theorem means that the cube complex X/F satisfies the *special* conditions of Haglund-Wise, namely that the hyperplanes are embedded, and there are no self-osculating or inter-osculating hyperplanes [24, Definition 3.2]. We will not delve into the definition in this paper since we are interested in the implications of specialness, but will not be using the hyperplane conditions directly. Suffice it to say that fundamental groups of compact special cube complexes are precisely the convex subgroups of right-angled Artin groups, and therefore inherit many of the nice properties of these groups.

Corollary 1.3. *Let G be a non-elementary word-hyperbolic group acting properly and cocompactly on a CAT(0) cube complex X. Then G is linear, large, and quasi-convex subgroups are separable.*

Proof. Since G is virtually special by Theorem 1.1, linearity follows from [24, Theorem 4.4] and quasi-convex separability follows from [24, Theorem 1.3]. The implication of largeness is well-known, see e.g. [50, Theorem 14.10]. □

Remark 1.4. The condition here that G is word-hyperbolic is necessary, since there are examples of simple groups acting properly cocompactly on a product of trees [11]. By [24, Proposition 7.2], a quasi-convex subgroup $H < G$ is represented by a convex subcomplex $Y \subset X$ such that such that H acts cocompactly on Y. A group G is *large* if it has a finite-index subgroup $G' < G$ surjecting $\mathbb{Z} * \mathbb{Z}$.

Theorem 9.1. *Let M be a closed aspherical 3-manifold. Then there is a finite-sheeted cover $\tilde{M} \to M$ such that $\tilde{M}$ is Haken.*

Theorem 9.2. *Let M be a closed hyperbolic 3-manifold. Then there is a finite-sheeted cover $\tilde{M} \to M$ such that $\tilde{M}$ fibers over the circle. Moreover, $\pi_1(M)$ is LERF and large.*

Theorem 9.1 resolves a question of Waldhausen [49]. Moreover, Theorem 9.2 resolves [30, Problems 3.50-51] from Kirby's problem list, as well as [48, Questions 15-18].

There has been much work on the virtual Haken conjecture before for certain classes of manifolds. These include manifolds in the Snappea census [17], surgeries on various classes of cusped hyperbolic manifolds [2, 3, 4, 9, 13, 14, 15, 31, 38, 39], certain arithmetic hyperbolic 3-manifolds (see [46] and references therein), and manifolds satisfying various group-theoretic criteria [32, 33, 35]. A key breakthrough was made by Kahn and Markovic who proved that every closed hyperbolic 3-manifold contains an immersed quasi-fuchsian surface [28] (see also Lackenby [34] who had previously resolved the arithmetic case). The approach in this paper uses techniques from geometric group theory, and as such does not specifically rely on 3-manifold techniques, although some of the arguments (such as word-hyperbolic Dehn surgery and hierarchies) are inspired by 3-manifold techniques.

Here is a short summary of the approach to the proof of Theorem 1.1. In Section 4 we use a weak separability result (Theorem A.1) to find an infinite-sheeted regular cover $\mathcal{X}$ of X/G which has a collection of embedded compact 2-sided walls $\mathcal{W}$. This covering space has a finite hierarchy obtained by labeling the walls $\mathcal{W}$ with finitely many numbers (which we think of as colors), so that walls with the same color do not intersect, and cutting successively along the walls ordered by their labels to get an infinite collection of "cubical polyhedra", giving a "hierarchy" for $\mathcal{X}$. The goal is to construct a finite-sheeted cover which is "modeled" on this hierarchy for $\mathcal{X}$. We first construct a measure on the space of colorings of the wall graph of $\mathcal{X}$ in Section 5. We then refine the colors to reflect how each wall is cut up by previous stages of the hierarchy in Section 6. We use the measure to find a solution to certain gluing equations on the colored cubical polyhedra defined by the refined colorings, and use solutions to these equations to get the base case of the hierarchy in Section 7. We glue up successively each stage of the hierarchy, using a gluing theorem 3.1 to glue at each stage after passing to a finite-sheeted cover. The inductive hypotheses and inductive step of the proof of Theorem 1.1 are given in section 8.

Theorem A.1 generalizes the main result of [1], and is proved in the appendix which is joint work with Groves and Manning. The proof of Theorem A.1 relies on the Malnormal Special Quotient Theorem A.10 which is a result of Wise [50, Theorem 12.3].

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2 Definitions

We expect the reader to be familiar with non-positively curved (NPC) cube complexes [10], special cube complexes [24], and hyperbolic groups [20].

Definition 2.1. A flag simplicial complex is a complex determined by its 1-skeleton: for every clique (complete subgraph) of the 1-skeleton, there is a simplex with 1-skeleton equal to that subgraph. A non-positively curved (NPC) cube complex is a cube complex X such that for each vertex $v \in X$, the link $link_X(v)$ is a flag simplicial complex. If X is simply-connected, then X is CAT(0) [10]. More generally, an NPC cube *orbicomplex* or *orbihedron* is a pair (G, X), where each component of X is a CAT(0) cube complex and $G \to Aut(X)$ is a proper cocompact effective action. We will also call such pairs *cubulated groups* when X is connected. If G is torsion-free, then X/G is an NPC cube complex. When G has torsion, we may also think of the quotient X/G as an orbi-space in the sense of Haefliger [22, 23]. The orbihedra we will consider in this paper will have covering spaces which are cube complexes, so they are *developable*, in which case we can ignore subtleties arising in the theory of general orbi-spaces.

Gluing the cubes isometrically out of unit Euclidean cubes gives a canonical metric on an NPC cube complex.

Definition 2.2. Given an NPC cube complex X, the *wall* of X is an immersed NPC cube complex W (possibly disconnected). For each n-cube $[-1,1]^n \cong C \subset X$, take the n $n-1$-cubes obtained by cutting the cube in half (setting one coordinate $= 0$), called the *hyperplanes* of C. If a k-cube D is a face of an n-cube C, then there is a corresponding embedding of the hyperplanes of D as faces of the hyperplanes of C. Take the cube complex W with cubes given by hyperplanes of the cubes of X, and gluings given by inclusion of cube hyperplanes. This cube complex immerses into the cube complex X. We will call this immersed cube complex the *wall complex* of X. There is a natural line bundle over W obtained by piecing together the normal bundles of hyperplanes in each cube. If this line bundle is non-orientable, then the wall W is one-sided. Otherwise, it is 2-sided or co-orientable, and there are two possible co-orientations.

Definition 2.3. Let X be an NPC cube complex. A subcomplex $Y \subset X$ is *locally convex* if the embedding $Y \to X$ is a local isometry. Similarly, a combinatorial map $Y \looparrowright X$ between NPC cube complexes is called locally convex if it is a local isometry.

The condition of being a local isometry is equivalent to saying that Y is NPC, and for each vertex $v \in Y$, $link_Y(v) \subset link_X(v)$ is a very full subcomplex, which means that for any two vertices of $link_Y(v)$ which are joined by an edge in $link_X(v)$, they are also joined by an edge in $link_Y(v)$. For example, an embedded cube in an NPC cube complex is a locally convex subcomplex.

Definition 2.4 (Almost malnormal Collection). A collection of subgroups $H_1, \ldots, H_g$ of G is almost malnormal provided that $|H_i^g \cap H_j| < \infty$ unless $i = j$ and $g \in H_i$.

For example, finite collections of non-conjugate maximal elementary subgroups of a torsion-free hyperbolic group form an almost malnormal collection.

Definition 2.5. Let X be an NPC cube complex, $Y \subset X$ a locally convex subcomplex. We say that Y is *acylindrical* if any map $(S^1\times[0,1], S^1\times\{0,1\}) \to (X, Y)$ which is injective on π_1 is relatively homotopic to a map $(S^1\times[0,1], S^1\times\{0,1\}) \to (Y, Y)$.

In particular, if $Y \subset X$ is acylindrical, then the collection of subgroups of the fundamental groups of its components form a malnormal collection of subgroups of $\pi_1(X)$.

Definition 2.6. ([50, Definition 11.5]) Let $\mathcal{QVH}$ denote the smallest class of hyperbolic groups that is closed under the following operations.

1. $1 \in \mathcal{QVH}$
2. If $G = A *_B C$ and $A, C \in \mathcal{QVH}$ and B is finitely generated and embeds by a quasi-isometry in G, then G is in $\mathcal{QVH}$.
3. If $G = A*_B$ and $A \in \mathcal{QVH}$ and B is f.g. and embeds by a quasi-isometry, then G is in $\mathcal{QVH}$.
4. Let $H < G$ with $[G : H] < \infty$. If $H \in \mathcal{QVH}$ then $G \in \mathcal{QVH}$ (in particular with (1), any finite group $K \in \mathcal{QVH}$).

The notation $\mathcal{QVH}$ is meant to be an abbreviation for "quasi-convex virtual hierarchy". A *hierarchy* for a group is a sequence of decompositions as amalgamated products and HNN extensions, which would hold if item (4) were eliminated. In particular, items (2) and (3) may be replaced by G is a graph of groups, with vertex groups in $\mathcal{QVH}$ and edge groups quasi-convex and f.g. in G. A motivating example of a group in $\mathcal{QVH}$ is a 1-relator group with torsion [50, Theorem 18.1], or a closed-hyperbolic 3-manifold group which contains an embedded quasi-fuchsian surface [50, Section 14].

Remark 2.7. Let M be a connected closed oriented hyperbolic 3-manifold, such that there is an embedded surface $S \subset M$ which is quasi-fuchsian. Then $\pi_1 S < \pi_1 M$ is a quasi-convex subgroup. The subgroups of components of $\pi_1(M \backslash\backslash S)$ are geometrically finite Kleinian groups with non-trivial domain of discontinuity, and we may obtain $G = \pi_1 M$ as an amalgamated product or HNN extension from these groups. Then fundamental groups of 3-manifolds with boundary are Haken and therefore admit a finite-stage hierarchy. The rest of the surfaces in the hierarchy of M will be finitely generated subgroups of these groups, and therefore will also be geometrically finite and quasi-convex in $\pi_1 M$, by a theorem of Thurston [12, Theorem 2.1].

Definition 2.8. ([50, Definition 11.1]) Let $\mathcal{MQH}$ denote the smallest class of hyperbolic groups that is closed under the following operations.

1. If $|G| < \infty$, then $G \in \mathcal{MQH}$
2. If $G = A *_B C$ and $A, C \in \mathcal{MQH}$ and B is finitely generated, almost malnormal, and embeds by a quasi-isometry in G, then G is in $\mathcal{MQH}$.
3. If $G = A*_B$ and $A \in \mathcal{MQH}$ and B is f.g., almost malnormal, and embeds by a quasi-isometry, then G is in $\mathcal{MQH}$.

We will not define special cube complexes in this paper (see [24]). However, we note that a special cube complex with hyperbolic fundamental group has embedded components of the wall subcomplex, and therefore its fundamental group is in the class $\mathcal{QVH}$. Moreover, we have

THEOREM 2.9. *[50, Theorem 13.3] A torsion-free hyperbolic group is in $\mathcal{QVH}$ if and only if it is the fundamental group of a compact virtually special cube complex.*

The reader not familiar with virtually special cube complexes or their fundamental groups may therefore take this theorem as the defining property for a virtually special group which will be used in this paper. Also, we note that using subgroup separability of quasi-convex subgroups, Wise's theorem implies that a group is in $\mathcal{QVH}$ if an only if it is virtually in $\mathcal{MQH}$. We will make use of this form of the theorem, by proving that a cubulated hyperbolic group is virtually in $\mathcal{MQH}$ in the proof of Theorem 1.1.

We state here a lemma which will be used in the case that G has torsion.

LEMMA 2.10. *Let $G \in \mathcal{QVH}$, and suppose that G' is an extension of G by a finite group $K < G'$, so there is a homomorphism $\varphi : G' \to G$ such that $ker(\varphi) = K$. Then $G' \in \mathcal{QVH}$.*

Proof. First, we remark that we need only check this for central extensions of G, since the kernel of the homomorphism $G' \to Aut(K)$ given by the conjugacy action will be a finite central extension of a finite-index subgroup of G. However, this observation does not seem to simplify the argument.

We prove this by induction on the defining properties of $G \in \mathcal{QVH}$ (essentially, on the length of the quasiconvex hierarchy defining the quotient group G). Consider the set of all extensions of groups in $\mathcal{QVH}$ by the finite group K. We prove that this class lies in $\mathcal{QVH}$ by showing that these groups are closed under the four operations defining $\mathcal{QVH}$.

1. The extension of 1 by K is in $\mathcal{QVH}$ by property (4), so $K \in \mathcal{QVH}$.

2. Suppose $G = A *_B C$ and $A, C \in \mathcal{QVH}$ and B is finitely generated and embeds by a quasi-isometry in G.

   We see that for $A' = \varphi^{-1}(A), B' = \varphi^{-1}(B), C' = \varphi^{-1}(C)$, then φ is a quasi-isometry, so $B' < G'$ embeds quasi-isometrically in G', and is finitely generated since B is. Also, A', B', C' are finite extensions of A, B, C by K respectively. If $A', B', C' \in \mathcal{QVH}$, then $G' \in \mathcal{QVH}$ by condition (2).

3. Suppose $G = A*_B$ and $A \in \mathcal{QVH}$ and B is f.g. and embeds by a quasi-isometry.

   We see that for $A' = \varphi^{-1}(A), B' = \varphi^{-1}(B)$, then φ is a quasi-isometry, so $B' < G'$ embeds quasi-isometrically in G', and is finitely generated since B is. Also, A', B' are finite extensions of A, B by K respectively. If $A', B' \in \mathcal{QVH}$, then $G' \in \mathcal{QVH}$ by condition (3).

4. Suppose $H < G$ with $[G : H] < \infty$ and $H \in \mathcal{QVH}$.
   Then for $H' = \varphi^{-1}(H)$, we have $[G' : H'] = [G : H] < \infty$, and H' is a finite extension of H by K. So if $H' \in \mathcal{QVH}$, then $G' \in \mathcal{QVH}$ by condition (4).

Thus, we see that finite extensions of elements of $\mathcal{QVH}$ by K are in $\mathcal{QVH}$ by induction, so $G' \in \mathcal{QVH}$. $\square$

3 Virtual gluing

In this section, we introduce a technical theorem which will be used in the proof of Theorem 1.1. We call this a "virtual gluing theorem", because it will allow us to glue together certain subcomplexes of a cube complex which may agree only up to finite index, by first passing to a finite cover in which they match exactly by an isometry.

Theorem 3.1. *Let X be a compact cube complex which is virtually special and $\pi_1(X)$ hyperbolic (for each component of X if X is disconnected). Let $Y \subset X$ be an embedded locally convex acylindrical subcomplex such that there is an NPC cube orbi-complex Y_0 and a finite-sheeted cover $\pi : Y \to Y_0$. Then there exists a regular cover $\overline{X} \to X$ such that the preimage of $Y \leftarrow \overline{Y} \subset \overline{X}$ is a regular orbi-cover $\overline{Y} \to Y_0$. In other words, $\pi_1(\overline{Y}) \lhd \pi_1(Y_0)$ for each component, and for each component of Y_0, the components of $\overline{Y}$ covering this component are equivalent.*

Remark 3.2. Keep in mind that all of the complexes in the statement of this theorem may be disconnected.

Proof. We construct a bipartite graph of groups G with vertex groups given by the fundamental groups of components of X and Y_0, and edge groups given by the fundamental groups of components of Y. The inclusion $Y \subset X$ and the covering space $\pi : Y \to Y_0$ gives the inclusion maps of the edge groups into the vertex groups to give the graph of groups. By the Bestvina-Feighn combination theorem [6, Corollary 7], the fundamental group of this graph of groups is a hyperbolic group. Since the subcomplex Y is acylindrical in X and Y is a finite-sheeted cover of Y_0, the graph of groups G is 2-acylindrical (cylinders between edge groups have length at most 1). By [29, Theorem 1.1], each edge group is quasi-convex in G. Thus, G is in $\mathcal{QVH}$, and therefore is virtually special by Theorem 2.9. Let $Y' \to Y_0$ be a regular finite-sheeted covering space which factors through $Y \to Y_0$. By separability of quasi-convex subgroups [24, Theorem 1.3], there is a finite index subgroup $G' \lhd G$ such that the induced cover of Y_0 factors through Y'. The induced cover $\overline{X}$ of X has the desired property, since it induces a regular orbifold cover $\overline{Y} \to Y_0$. $\square$

Remark 3.3. It is possible to give a proof of this theorem using the techniques of [25, Theorem 6.1] rather than citing Theorem 2.9 which depends on [50].

4 Quotient complex with compact walls

In this section, we will be setting up for a proof of Theorem 1.1 by induction on dimension. As in the hypothesis of Theorem 1.1, let X be a CAT(0) cube complex, G a hyperbolic group acting properly and cocompactly on X. Recall that we say that (G, X) is a cubulated hyperbolic group. Since the action of G is proper and cocompact, the cube complex X is finite dimensional, locally finite, and quasi-isometric to G. To prove that G is virtually special, we may assume that G acts faithfully on X, since properness implies that the subgroup K acting trivially must be finite. If we can show that G/K is in $\mathcal{QVH}$, then by Lemma 2.10, we can conclude that G is also in $\mathcal{QVH}$. So we will assume that the action of G on X is faithful as well as proper and cocompact. The quotient X/G may be interpreted as an orbihedron [22, 23] if G has torsion. Moreover, there are finitely many orbits of walls $W \subset X$ since the action of G is cocompact. The stabilizer G_W of a wall W acts properly and cocompactly on W, and therefore is quasi-isometric to W. Therefore G_W is a quasi-convex subgroup of G since W is totally geodesic and therefore convex in X, and quasi-convexity is preserved by quasi-isometry of δ-hyperbolic metric spaces. Since the action of G_W on W is proper, the kernel of the action $G_W \to Aut(W)$ induced by restriction is finite. So there is a finite-index quotient group $G_W \twoheadrightarrow \overline{G}_W$ which acts faithfully on W.

Let $\{W_1, \ldots, W_m\}$ be orbit representatives for the walls of X under the action of G. By induction on the maximal dimension of a cube, we may assume that $\overline{G}_{W_i}$ is virtually special for $1 \leq i \leq m$. Therefore, by Lemma 2.10, G_{W_i} is also virtually special. In particular, for each i, there is a finite index torsion-free normal subgroup $G'_i \dot{\trianglelefteq} G_{W_i}$ such that W_i/G'_i is a special cube complex. There exists $R > 0$ such that if two walls $W, W' \subset X$ have the property that $d(W, W') > R$, then $|G_W \cap G_{W'}| < \infty$. This follows because the subgroups $G_W, G_{W'}$ are quasi-convex, so their intersection is also quasi-convex [10, Proposition 3.9], and therefore a hyperbolic subgroup [10, Proposition 3.7]. If $G_W \cap G_{W'}$ is infinite, then there exists an infinite order element $g \in G_W \cap G_{W'}$. Since g preserves both W and W', there is an axis for g in both totally geodesic subsets. But since the space X is δ-hyperbolic for some δ, this means that the axes must be distance $\leq 5\delta$ apart [10, Lemma 3.3(2)], and therefore $d(W, W') \leq 5\delta$.

For each $1 \leq i \leq m$, let

$$\mathcal{A}_i = \{G_{W_i} g G_{W_i} | d(g(W_i), W_i) \leq 2R\} - \{G_{W_i}\}.$$

Then $\mathcal{A}_i$ is finite for all i, as follows. Let $g \in G$ be an element such that $d(W_i, g(W_i)) \leq R$. Let γ be a geodesic of length $\leq R$ connecting W_i and $g(W_i)$. Let $D \subset W_i$ be a fundamental domain for the action of G_{W_i} on W_i. Let $w \in G_{W_i}$ be an element such that $w(\gamma) \cap D \neq \emptyset$. Then $d(wg(W_i), D) \leq R$. By local compactness, there are finitely many such translates of W_i of distance $\leq R$ from D. Any other $k \in G$ such that $k(W_i) = wg(W_i)$ has the property that $k \in wgG_{W_i}$. Thus, $k \in G_{W_i} g G_{W_i}$. So we see that there are finitely many double cosets $G_{W_i}gG_{W_i}$ such that $d(W_i, g(W_i)) \le R$.

The following lemma will give us a CAT(0) cube complex $\mathcal{X}$ with compact embedded wall components $\mathcal{W}$, which is a regular cover of the orbi-complex X/G. The construction of this cover will be crucial for the rest of the proof, in that we will model a hierarchy for a finite-sheeted cover of X/G on hierarchies for $\mathcal{X}$.

Lemma 4.1. *We may find a quotient group homomorphism $\phi : G \to \mathcal{G}$ such that for all $1 \le i \le m$ and for all $G_{W_i}gG_{W_i} \in \mathcal{A}_i$, $\phi(g) \notin \phi(G_{W_i})$ and $\phi(G_{W_j})$ is finite for all j. Moreover, we may assume that the action of $G_{W_i} \cap ker(\phi)$ does not exchange the sides of W_i (preserves the co-orientation), and that $ker(\phi)$ is torsion-free and $X^{(1)}/ker(\phi)$ contains no closed loops.*

Proof. For each W_i, the set of double cosets $\mathcal{A}_i = \{G_{W_i}gG_{W_i}, d(g(W_i), W_i) \le 2R\} - \{G_{W_i}\}$ is finite. Fix an element g such that $G_{W_i}gG_{W_i} \in \mathcal{A}_i$. Choose elements $g_1, \ldots, g_m$ such that $g_i = 1$ and $H = \langle G_{W_1}^{g_1}, \ldots, G_{W_i}, \ldots, G_{W_m}^{g_m} \rangle \cong G_{W_1}^{g_1} * \cdots * G_{W_i} * \cdots * G_{W_w}^{g_w}$ and $g \notin H$, and $H < G$ is quasiconvex. This may be arranged by standard ping-pong arguments. Then H is virtually special since it is a free product of virtually special groups by the induction hypothesis (see the discussion in the first paragraph of this section). By Theorem A.1, we may find a quotient $\phi_g : G \to \mathcal{G}_g$ such that $\phi_g(g) \notin \phi(H)$ and $\phi_g(H)$ is finite. Clearly then $\phi_g(G_{W_j})$ is finite for all j. Moreover, we may assume that $ker(\phi_g) \cap G_{W_i}$ is contained in the subgroup preserving the orientation on the normal bundle to W_i. Let $\mathcal{A}$ be the finitely many double coset representatives for $\cup_i \mathcal{A}_i$ we use in this construction.

Let $\mathcal{T} \subset G$ be a finite set of representatives for each conjugacy class of torsion elements of G, such that $\mathcal{T} \cap G_{W_j} = \emptyset$ for all j, and conjugacy class representatives of any group elements identifying endpoints of edges of $X^{(1)}$. We may also apply the same technique to find for each $g \in \mathcal{T}$ a homomorphism $\psi_g : G \to \mathcal{G}'$ such that $\psi_g(g) \neq 1$ and $\psi_g(G_{W_i})$ is finite for all i.

Define ϕ by $ker(\phi) = \cap_{g\in\mathcal{A}} ker(\phi_g) \cap_{g\in\mathcal{T}} ker(\psi_g)$, then $\phi : G \to \mathcal{G} = G/ker(\phi)$ has the desired properties. $\square$

Let $K = ker(\phi)$, where ϕ comes from the previous lemma, and let $\mathcal{X} = X/K$. Let $\mathcal{W}$ be the collection of walls of $\mathcal{X}$, so that each wall $W \in \mathcal{W}$ is a compact, embedded, 2-sided component of the walls of $\mathcal{X}$. Then $\mathcal{X}$ is an NPC cube complex, and for each i, the quotient $\mathcal{N}_R(W_i)/(G_{W_i} \cap K)$ embeds in $\mathcal{X}$ under the natural covering map, where $\mathcal{N}_R(W_i)$ is the neighborhood of radius R about W_i. Recall from the above discussion that R could be taken to be 10δ, where δ is the hyperbolicity constant of X.

Definition 4.2. Form a graph $\Gamma(\mathcal{X})$, with vertices $V(\Gamma(\mathcal{X}))$ consisting of the wall components of $\mathcal{W} \subset \mathcal{X}$, and edges $E(\Gamma(\mathcal{X}))$ consisting of pairs of walls (W_1, W_2) in $\mathcal{X}$ such that $d(W_1, W_2) \le R$. We have a natural action of $\mathcal{G}$ on $\Gamma(\mathcal{X})$.

5 Invariant coloring measures

Let Γ be a (simplicial) graph of bounded valence $\leq k$, and let G be a group acting cocompactly on Γ. Note that the quotient graph Γ/G may have loops and multi-edges, so in particular may not be simplicial. We will denote the vertices of Γ by $V(\Gamma)$, and the edges by $E(\Gamma) \subset V(\Gamma) \times V(\Gamma)$ consisting of the symmetric relation of pairs of adjacent vertices in Γ, so that Γ is defined by the pair $\Gamma = (V(\Gamma), E(\Gamma))$. Since Γ is simplicial, it has no loops, and therefore $E(\Gamma)$ does not meet the diagonal of $V(\Gamma) \times V(\Gamma)$.

Definition 5.1. An n-coloring of Γ is a map $c : V(\Gamma) \to \{1, \ldots, n\} = [n]$ such that for every edge $(u, v) \in E(\Gamma)$, we have $c(u) \neq c(v)$. Let $[n]^{V(\Gamma)}$ be the space of all n-colorings of the trivial graph $(V(\Gamma), \emptyset)$, and endow this with the product topology to make it a compact space (Cantor set). Then the space of n-colorings of Γ is naturally a closed G-invariant subspace of $[n]^{V(\Gamma)}$ which we will denote $C_n(\Gamma)$. The set $M(C_n(\Gamma))$ of probability measures on $C_n(\Gamma)$ endowed with the weak* topology is a convex compact metrizable set. Let $M_G(C_n(\Gamma)) \subset M(C_n(\Gamma)) \subset M([n]^{V(\Gamma)})$ denote the G-invariant measures.

Since we have assumed that the degree of every vertex of Γ is $\leq k$, then clearly $C_{k+1}(\Gamma)$ is non-trivial: order the vertices, and color each vertex inductively by one of $k+1$ colors not already used by one of its $\leq k$ neighbors.

Theorem 5.2. *The set $M_G(C_{k+1}(\Gamma))$ is non-empty, that is, there exists a G-invariant probability measure on the space of $k+1$-colorings of the graph Γ.*

Proof. For a G-invariant measure $\nu \in M_G([n]^{V(\Gamma)})$, we want to define a quantity which measures how far ν is from giving a G-invariant coloring measure in $M_G(C_n(\Gamma))$. For an edge $e = (u, v) \in E(\Gamma)$, let $B_e = \{f \in [n]^{V(\Gamma)} | f(u) = f(v)\}$. This is the subset of colorings of $V(\Gamma)$ which violate the coloring condition for Γ at the edge e, so that $C_n(\Gamma) = \cap_{e\in E(\Gamma)} B_e^c$. Let $\{e_1, \ldots, e_m\} \subset E(\Gamma)$ be a complete set of representatives of the orbits of the action of G on $E(\Gamma)$, which exists because we have assumed that the action of G on Γ is co-compact. For $\nu \in M_G([n]^{V(\Gamma)})$ define $weight(\nu) = \sum_{i=1}^{m} \nu(B_{e_i})$. If ν is a G-invariant coloring measure of Γ, then regarding $\nu \in M_G(C_n(\Gamma)) \subset M_G([n]^{V(\Gamma)})$, we have $weight(\nu) = 0$. Conversely, if $weight(\nu) = 0$ for $\nu \in M_G([n]^{V(\Gamma)})$, then $\nu \in M_G(C_n(\Gamma))$. To see this, let $supp(\nu) \subset [n]^{V(\Gamma)}$ be the support of ν, which is $\cap_{C \text{ compact}, \nu(C)=1} C$. Let $e \in E(\Gamma)$, then $\nu(B_e) = 0$, since there exists $e_i, g \in G$ such that $e = g(e_i)$, so $\nu(B_e) = \nu(B_{g(e_i)}) = \nu(B_{e_i}) = 0$ by G-invariance of ν and $weight(\nu) = 0$. Therefore $supp(\nu) \subset B_e^c$ for all $e \in E(\Gamma)$, and therefore $supp(\nu) \subset \cap_{e\in E(\Gamma)} B_e^c = C_n(\Gamma)$. So $\nu \in M_G(C_n(\Gamma))$.

Take the uniform measure μ_n on $[n]^{V(\Gamma)}$, which is the product of $V(\Gamma)$ copies of the uniform measure on $[n]$, so $\mu_n \in M([n]^{V(\Gamma)})$. Clearly μ_n is G-invariant under the action of G on $V(\Gamma)$, $\mu_n \in M_G([n]^{V(\Gamma)})$, since G permutes the uniform measures on $[n]$. We note that for the uniform measure μ_n, we have $\mu_n(B_e) = 1/n$. Then we see that $weight(\mu_n) = m/n$.

For $n > k+1$, we define a map $p_n : [n]^{V(\Gamma)} \to [n-1]^{V(\Gamma)}$ which depends on Γ and which is G-equivariant. For $c \in [n]^{V(\Gamma)}$ and $v \in V(\Gamma)$, define $p_n(c)(v) = c(v)$ if $c(v) < n$, and if $c(v) = n$, then $p_n(c)(v) = \min(\{1, \ldots, n-1\} - \{c(u) | (u, v) \in E(\Gamma)\})$. Since the degree of v is $\leq k$, this set is non-empty, and has a well-defined minimum which is $\leq k+1$. In other words, $p_n(c)$ assigns to each vertex colored n the smallest color not used by its neighbors, and otherwise does not change the color. Then $p_n(c)$ has the property that for any two vertices $u, v \in V(\Gamma)$ with $p_n(c)(u) = p_n(c)(v)$, then $c(u) = c(v)$. In particular, if c is an n-coloring of Γ, then $p_n(c)$ is an $n-1$-coloring of Γ. This implies that for all measures $\nu \in M_G([n]^{V(\Gamma)})$, $weight(p_{n*}(\nu)) \leq weight(\nu)$, where $p_{n*}(\nu)$ is the push-forward measure. Notice that the map p_n is continuous, since its definition is local, so that the push-forward is well-defined.

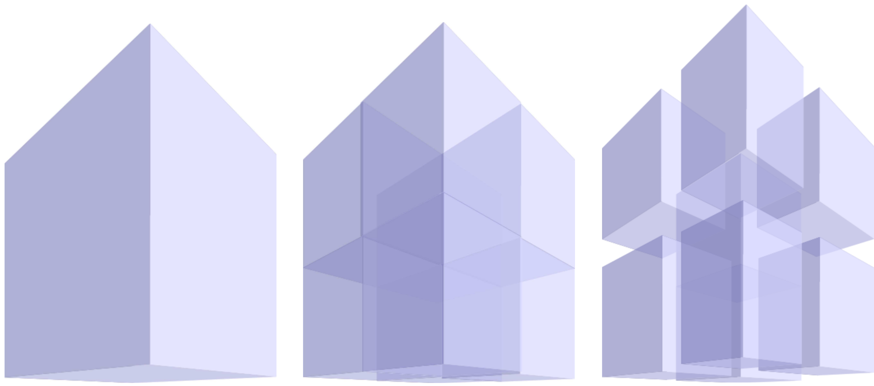
This gives a map $P_n : [n]^{V(\Gamma)} \to [k+1]^{V(\Gamma)}$ defined by $P_n = p_{k+1} \circ p_{k+2} \circ \cdots \circ p_n$. We get induced a map $P_{n*} : M([n]^{V(\Gamma)}) \to M([k+1]^{V(\Gamma)})$ by push-forward of measures, and induces a map by restriction $P_{n*} : M_G([n]^{V(\Gamma)}) \to M_G([k+1]^{V(\Gamma)})$ because the maps p_n are G-equivariant.

Finally, we have $weight(P_{n*}(\mu)) \leq weight(\mu)$ for any $\mu \in M_G([n]^{V(\Gamma)})$. In particular, $weight(P_{n*}(\mu_n)) \leq weight(\mu_n) = m/n$. Take a subsequence of $\{P_{n*}(\mu_n)\}$ converging to a G-invariant measure $\mu_\infty \in M_G([k+1]^{V(\Gamma)})$. Then $weight(\mu_\infty) = 0$, which implies that $\mu_\infty \in M_G(C_{k+1}(\Gamma))$. □

6 Cube complexes with boundary patterns

In this section, we introduce the cubical barycentric subdivision of a cube complex, cubical polyhedra, and cube complexes with boundary pattern. We then introduce equivalence classes of colored walls, faces, and polyhedra of the cube complex $\mathcal{X}$. These will be essential notions in the proof of Theorem 1.1.

Definition 6.1. Given a locally finite cube complex X, subdivide each n-cube into 2^n cubes of half the size to get a cube complex $\dot{X}$ (Figure 1(b)). This is called the *cubical barycentric subdivision*, and is analogous to the barycentric subdivision of a complex, in that one inserts new vertices in the barycenter of each cube, and connects each new barycenter vertex of each cube to the barycenter vertex of each cube of one higher dimension containing it, then filling in cubes using the flag condition (the difference with the usual barycentric subdivision is that one does not connect barycenter vertices of cubes to cubes of more than one dimension higher containing it). We may then regard the union of the hyperplanes $W \subset X$ as the union of the new topological codimension-one cubes of $\dot{X}$, which is the locally convex immersed complex $\dot{W} \looparrowright \dot{X}$ spanned by the barycenter vertices of $\dot{X}$. Consider splitting X along the hyperplanes W. By this, we mean remove each hyperplane, getting a disconnected complex, then put in 2^k copies of each codimension-k cube that is removed to get a complex $X \backslash\backslash W = \dot{X} \backslash\backslash \dot{W}$ (see Figure 1(c)). We will think of this as a cube complex "with boundary", where the boundary consists of the new cubes that were attached at the missing hyperplanes. What remains are stars of the vertices of X.



(a) Cube in X

(b) Cubical barycentric subdivision in $\dot{X}$

(c) Cube splitting $X\backslash\backslash W$

Figure 1: Subdividing and splitting a cube complex

Definition 6.2. A *cubical polyhedron* $\mathcal{P}$ is a CAT(0) cube complex with a distinguished vertex $v \in \mathcal{P}$ which is contained in every maximal cube. The star of v is combinatorially equivalent to $\mathcal{P}$, and $\mathcal{P}$ is determined by $link(v)$.

The stars of vertices in an NPC cube complex are cubical polyhedra, and if we split $\dot{X}$ along all of its hyperplanes, we get a union of stars of vertices and therefore cubical polyhedra.

Definition 6.3. A cube complex with boundary pattern is a cube complex X of bounded dimension together with locally convex subcomplexes $\{\partial_1 X, \ldots, \partial_n X\}$, $\partial_i X \subset X$ satisfying the following inductive definition (induct on the maximal dimension cube):

- For each i, there is an isometrically embedded open product neighborhood $\partial_i X = \partial_i X \times 0 \subset \partial_i X \times [0, 1) \subset X$. In particular, the dimension of each maximal cube of $\partial_i X$ is one less than the dimension of a cube of X containing it. For each cube $C \subset X$, the intersection $C \cap X_{i_1} \cap \cdots \cap X_{i_j}$ is a (possibly empty) face of C for all $\{i_1, \ldots, i_j\} \subset \{1, \ldots, n\}$.
- For each i, the subcomplex $\partial_i X$ forms a cube complex with boundary pattern $\{\partial_j X \cap \partial_i X | j \neq i\}$, with induced collar neighborhoods $(\partial_j X \cap \partial_i X) \times [0, 1) = (\partial_j X \times [0, 1)) \cap \partial_i X$.

What one may keep in mind for this definition is the analogy of a boundary pattern for a hierarchy of a 3-manifold, arising in the work of Haken [26].

If X is a cube complex with boundary pattern $\{\partial_1 X, \dots, \partial_n X\}$, then each $\partial_i X$ gets a co-orientation of the collar neighborhood $\partial_i X \times [0,1)$, pointing into X from 0 to 1 (into the cube complex).

EXAMPLES: Take a graph X, and let $\partial_0 X \subset X$ be the vertices of X which have degree 1, then X is a cube complex with boundary pattern $\partial_0 X$.

Take a cubical polyhedron $P(\Gamma)$ associated to a simplicial graph Γ. For each vertex $v \in \Gamma$, consider the subcomplex defined by $link(v) \subset \Gamma$, $P(link(v)) \subset P(\Gamma)$. Then $[0,1] \times P(link(v)) \subset P(\Gamma)$. The collection $\{\{1\} \times P(link(v)) | v \in V(\Gamma)\}$ forms a boundary pattern of $P(\Gamma)$. We will denote the union of this collection as $\partial P(\Gamma)$. We will call components $\{1\} \times P(link(v))$ the *facets* of $P(\Gamma)$. If a collection of facets of $P(\Gamma)$ have non-trivial intersection, then their intersection is a convex subcomplex we'll call a *face*. The minimal faces will be points, which we will call *vertices of* $P(\Gamma)$.

REMARK 6.4. The choice of terminology *cubical polyhedron* is meant to evoke a polyhedron. When X is PL equivalent to a manifold, then each component of $X \backslash\backslash W$ is homeomorphic to a ball, with the boundary pattern corresponding to the facets of the polyhedron.

DEFINITION 6.5. Let X be a cube complex with boundary pattern $\{\partial_1 X, \dots, \partial_n X\}$. Suppose there is an isometric involution $\tau : \partial_n X \to \partial_n X$ without fixed points, and with the property that $\tau(\partial_i X \cap \partial_n X) = \partial_i X \cap \partial_n X$ for $i < n$. Then we may form the quotient complex X/τ, where for each cube $c \subset \partial_n X$, $c \neq \tau(c)$, since τ is fixed-point free, we amalgamate the cubes $c \times [-1,0]$ (changing the collar parameter of c from $[0,1]$ to $[-1,0]$) and $\tau(c) \times [0,1]$ into a single cube isometric to $c \times [-1,1]$. We obtain an induced boundary pattern $\{\partial_i X/(\tau_{|\partial_i X \cap \partial_n X}) \ | i < n\}$. This operation on X is called *gluing a cube complex with boundary pattern*.

For the complex $\mathcal{X}$ with walls $\mathcal{W}$ constructed at the end of Section 4, let $\mathcal{P}(\mathcal{X})$ be the set of cubical polyhedra which are stars of vertices, and let $P_1, \dots, P_p$ be orbit representatives under the action of $\mathcal{G}$ of the cubical polyhedra of $\mathcal{X} \backslash\backslash \mathcal{W}$ which are vertex stars (we will think of these as the polyhedra obtained by splitting $\mathcal{X}$ along its walls). Similarly, let $W_1, \dots, W_w$ be orbit reps. of the walls $\mathcal{W}$ under the action of $\mathcal{G}$. Let $\mathcal{F}(\mathcal{X})$ denote the set of all cubical polyhedra of the walls $\mathcal{W}$. These are the stars of midpoints of edges of $\dot{\mathcal{X}}$ in $\dot{\mathcal{W}}$. Let $\mathcal{F} = \{F_1, \dots, F_f\}$ be orbit representatives of the action of $\mathcal{G}$ on $\mathcal{F}(\mathcal{X})$ (we will assume that each $F_i \subset W_j$ for some j). There is a canonical map $wall : \mathcal{F}(\mathcal{X}) \to V(\Gamma(\mathcal{X})) = \mathcal{W}$ defined by $wall(F) = W$ if $F \subset W \in V(\Gamma(\mathcal{X}))$. Notice that there is a one-to-one correspondence between P_i and the vertices of X/G, and between F_i and the edges of X/G since $X/G = \mathcal{X}/\mathcal{G}$.

Let $k = maxdegree(\Gamma(\mathcal{X}))$, then $C_{k+1}(\Gamma(\mathcal{X})) \neq \emptyset$. We define an equivalence relation $\simeq$ on $V(\Gamma(\mathcal{X})) \times C_{k+1}(\Gamma(\mathcal{X}))$.

DEFINITION 6.6. First, for $(v,c),(w,d) \in V(\Gamma(\mathcal{X})) \times C_{k+1}(\Gamma(\mathcal{X}))$, if $(v,c) \simeq (w,d)$, then we must have $v = w$ and $c(v) = d(w)$ (so the partition respects the colored vertex type). For each $v \in V(\Gamma(\mathcal{X}))$, we define the partition of $\{v\} \times C_{k+1}(\Gamma(\mathcal{X}))$ by induction on the color of v.

1. If $c(v) = d(v) = 1$, then $(v, c) \simeq (v, d) = (w, d)$.

2. If $c(v) = d(v) = 2$ and for all w such that $(w, v) \in E(\Gamma(\mathcal{X}))$, we have $c(w) = 1 \iff d(w) = 1$, then $(v, c) \simeq (v, d)$.

(j) The jth inductive step of the definition is given by: If $c(v) = d(v) = j$ with $2 \leq j \leq k+1$, and if for all w such that $(w, v) \in E(\Gamma(\mathcal{X}))$, we have $(w, c) \simeq (w, d)$ if $c(w) < j$ or $d(w) < j$, then $(v, c) \simeq (v, d)$.

REMARKS ON THE EQUIVALENCE RELATION $\simeq$: A coloring determines a hierarchy for $\mathcal{X}$, and an induced hierarchy on each wall of $\mathcal{X}$. The equivalence relation captures how each wall is cut up by previous stages of the hierarchy. This refinement is important for when we reconstruct the hierarchy to make sure after gluing up the jth level of the hierarchy that the $j-1$st levels and lower are still matching up to finite index.

Notice that the equivalence class of (v, c) where $c(v) = j$ depends only on c restricted to the ball of radius $j-1$ about v in $\Gamma(\mathcal{X})$. This implies that the equivalence classes are clopen sets as subsets of $\{v\} \times C_{k+1}(\Gamma(\mathcal{X}))$. In fact, if we think of the coloring c as a Morse function on the vertices $V(\Gamma(\mathcal{X}))$, then the equivalence class of (v, c) depends only on the "descending subgraph" of v, consisting of the union of all paths in $\Gamma(\mathcal{X})$ starting at v in which the values of c are decreasing.

We now want to define an equivalence relation $\simeq_{\mathcal{F}}$ on the set $\mathcal{F}(\mathcal{X}) \times C_{k+1}(\Gamma(\mathcal{X}))$.

DEFINITION 6.7. For $E \in \mathcal{F}(\mathcal{X})$, we decree $(E, c) \simeq_{\mathcal{F}} (E, d)$ if $(wall(E), c) \simeq (wall(E), d)$.

We define an equivalence relation $\simeq_{\mathcal{P}}$ on $\mathcal{P}(\mathcal{X}) \times C_{k+1}(\Gamma(\mathcal{X}))$.

DEFINITION 6.8. For $P \in \mathcal{P}(\mathcal{X})$, $(P, c) \simeq_{\mathcal{P}} (P, d)$ if for every facet $F \subset \partial P$, $(F, c) \simeq_{\mathcal{F}} (F, d)$. In particular, the colors $c(F)$ of the facets $F \subset \partial P$ depend only on the $\simeq_{\mathcal{P}}$ equivalence class of (P, c).

We have an action of $\mathcal{G}$ on each of these equivalence relations, by the action for $g \in \mathcal{G}$ given by $g \cdot (v, c) = (g \cdot v, c \circ g^{-1})$, for $(v, c) \in \mathcal{W} \times C_{k+1}(\Gamma(\mathcal{X}))$, and a similar formula for the action on faces and polyhedra. There are finitely many $\mathcal{G}$-orbits of equivalence classes under the action of $\mathcal{G}$, and we may find representatives among $\{W_1, \ldots, W_w\} \times C_{k+1}(\Gamma(\mathcal{X}))$, $\{F_1, \ldots, F_f\} \times C_{k+1}(\Gamma(\mathcal{X}))$, and $\{P_1, \ldots, P_p\} \times C_{k+1}(\Gamma(\mathcal{X}))$.

7 Gluing equations

We will consider weights on equivalence classes of polyhedra $\omega : \mathcal{P}(\mathcal{X}) \times C_{k+1}(\Gamma(\mathcal{X}))/\simeq_{\mathcal{P}} \to \mathbb{R}$ which are invariant under the action of $\mathcal{G}$, so that $\omega(g \cdot (P,c)) = \omega(P,c)$, for all $g \in \mathcal{G}$ and satisfying the *polyhedral gluing equations*. A weight ω will be determined by its values on a $\simeq_{\mathcal{P}}$-equivalence class $[(P_j,c)]_{\mathcal{P}}$, $1 \le j \le p, c \in C_{k+1}(\Gamma(\mathcal{X}))$, and therefore is determined by finitely many variables. Given polyhedra $P, P' \subset \dot{\mathcal{X}}$ sharing a facet $F \subset \partial P, F \subset \partial P'$, we get an equation on the weights for each equivalence class of $\{F\} \times C_{k+1}(\Gamma(\mathcal{X}))/\simeq_{\mathcal{F}}$. For each equivalence class $[(F,c)]_{\mathcal{F}} \in \{F\} \times C_{k+1}(\Gamma(\mathcal{X}))/\simeq_{\mathcal{F}}$, we have the equation

$$\sum_{[(P,d)]_{\mathcal{P}} | (F,d) \simeq_{\mathcal{F}} (F,c)} \omega([(P,d)]_{\mathcal{P}}) = \sum_{[(P',d)]_{\mathcal{P}} | (F,d) \simeq_{\mathcal{F}} (F,c)} \omega([(P',d)]_{\mathcal{P}}).$$

The polyhedral gluing equations on the polyhedra equivalence class weights are the equations obtained for each equivalence class $[(F,c)]_{\mathcal{F}}$. These equations are also $\mathcal{G}$-equivariant, so in particular are determined by the equations for equivalence classes $[(F_i,c)]_{\mathcal{F}}$, $1 \le i \le f, c \in C_{k+1}(\Gamma(\mathcal{X}))$. Thus, we have finitely many equations determined by equivalence classes $[(F_i,c)]_{\mathcal{F}}, 1 \le i \le f$ on finitely many variables $\omega([P_j,c)]_{\mathcal{P}}), 1 \le j \le p$, together with the equations determined by $\mathcal{G}$-invariance.

For a measure $\mu \in M_{\mathcal{G}}(C_{k+1}(\Gamma(\mathcal{X})))$, we get non-negative polyhedral weights $\mu([(P,c)]_{\mathcal{P}}) = \mu(\{d \in C_{k+1}(\Gamma(\mathcal{X}))| \ (P,c) \simeq_{\mathcal{P}} (P,d)\})$ (and $\mu([F,c]_{\mathcal{F}})$ is similarly defined for each facet F). These weights satisfy the polyhedral gluing equations. Consider a facet $F = \partial P \cap \partial P'$, and an equivalence class $[(F,c)]_{\mathcal{F}}$ which defines a gluing equation. Then using the additivity property of μ, we have

$$\sum_{[(P,d)]_{\mathcal{P}} | (F,d) \simeq_{\mathcal{F}} (F,c)} \mu([(P,d)]_{\mathcal{P}}) = \mu(\{d | (F,d) \simeq_{\mathcal{F}} (F,c)\}) =$$

$$\mu([(F,c)]_{\mathcal{F}}) = \sum_{[(P',d)]_{\mathcal{P}} | (F,d) \simeq_{\mathcal{F}} (F,c)} \mu([(P',d)]_{\mathcal{P}}).$$

So μ gives a non-negative real solution to the polyhedral gluing equations.

Since these equations are defined by finitely many linear equations with integral coefficients, there is a non-negative non-zero integral weight function satisfying the polyhedral gluing equations, $\Omega : \mathcal{P}(\mathcal{X}) \times C_{k+1}(\Gamma(\mathcal{X}))/\simeq_{\mathcal{P}} \to \mathbb{Z}_{\ge 0}$. In the next section we will use Ω to create a tower hierarchy which gives a finite-sheeted cover of X/G in $\mathcal{QVH}$.

8 Virtually gluing up the hierarchy

Let $(w,c) \in V(\Gamma(\mathcal{X})) \times C_{k+1}(\Gamma(\mathcal{X}))$. Let $\mathcal{W}^c_j = \cup\{w \in V(\Gamma(\mathcal{X}))|c(w) = j\} \subset \mathcal{W}$, $1 \le j \le k+1$, the union of walls colored j by c. Suppose that

$c(w) = j > 1$, then define $w_1^c = w\backslash\backslash(w \cap \mathcal{W}_1^c)$. We may think of w_1^c as being immersed in the wall $\dot{w}$. Then define inductively immersed complexes in $\dot{w}$ by $w_i^c = w_{i-1}^c\backslash\backslash(w_{i-1}^c \cap \mathcal{W}_i^c)$, for $2 \le i \le j-1$. We don't split w along $\mathcal{W}_j^c$ since $w \subset \mathcal{W}_j^c$. We will use the notation $w_{j-1}^c = w^c$, since the $j = c(w)$ is implicitly determined (if $j = 1$, then $w^c = w$). The complex w^c has a boundary pattern, given by $\partial_i(w^c) = w^c \cap \mathcal{W}_i^c$, $1 \le i \le j-1$.

Claim: If $(w, c) \simeq (w, d)$, then $w^c = w^d$ (as cube complexes with boundary pattern). In other words, w^c depends only on the $\simeq$-equivalence class $[(w, c)]$. This follows because w^c is determined by $w \cap \mathcal{W}_i^c$, $1 \le i \le j-1$, which depends only on the equivalence class of (w, c) since if v is a component of $\mathcal{W}_i^c$ with $w \cap v \neq \emptyset$, then $(w, v) \in E(\Gamma(\mathcal{X}))$.

Consider now the symmetries of w^c which preserve the equivalence class. That is, consider $Stab(w^c) \le \mathcal{G}$, given by $g \in \mathcal{G}$ such that $g(w^c) = w^c$ (in particular, $g(w) = w$) and $(w, c \circ g^{-1}) = (g(w), c \circ g^{-1}) \simeq (w, c)$. Now define $w_{\mathcal{G}}^c = w^c/Stab(w^c)$, with its corresponding boundary pattern $\partial_i(w_{\mathcal{G}}^c) = \partial_i(w^c)/Stab(w^c)$, $1 \le i \le j-1$. In general, $w_{\mathcal{G}}^c$ will be an orbihedron with boundary pattern.

For each j, $1 \le j \le k+1$, let $\mathcal{Y}_j = \sqcup_{[(w,c)],c(w)=j} w_{\mathcal{G}}^c$ (where we take precisely one $\mathcal{G}$-orbit representative of the equivalence relation $\simeq$ so that there are only finitely many equivalence classes $[(w, c)]$ up to the action of $\mathcal{G}$, and therefore $\mathcal{Y}_j$ is a compact orbi-complex). The orbi-complex $\mathcal{Y}_j$ has the property that for each $\mathcal{G}$-orbit of equivalence class $[(F, c)]_{\mathcal{F}}$ with $c(F) = j$, there is a unique representative of (F, c) in the complex $\mathcal{Y}_j$. At two extremes, we have $\mathcal{Y}_1 = \cup\{W_1/Stab(W_1), \dots, W_w/Stab(W_w)\}$, since the equivalence class depends only on the orbit of the walls under the action of $\mathcal{G}$. We have $\mathcal{Y}_{k+1} = \sqcup\{[(F, c)]_{\mathcal{F}}/Stab([F, c]_{\mathcal{F}}) | c(wall(F)) = k+1\}$, with boundary pattern $\partial_i F/Stab([F, c]_{\mathcal{F}}) = (F \cap \mathcal{W}_i^c)/Stab([F, c]_{\mathcal{F}})$, $1 \le i \le k$.

proof of Theorem 1.1. The idea of the proof is to reverse-engineer a (malnormal) quasi-convex hierarchy for a finite-sheeted cover of X/G, proving that G is in $\mathcal{MQH}$. This hierarchy will be realized as a sequence of graphs of acyclindrical spaces. This will imply that G is virtually special. The hierarchy is in some sense an approximation to hierarchies of $\mathcal{X}$, in that the hierarchies induced on the walls will be covers of hierarchies of walls associated to colorings of $\mathcal{X}$.

We will construct a sequence of (usually disconnected) finite cube complexes $\mathcal{V}_j$, $k+1 \ge j \ge 0$, with boundary pattern $\{\partial_1(\mathcal{V}_j), \dots, \partial_j(\mathcal{V}_j)\}$ which have the following properties:

1. there is a locally convex combinatorial immersion $\nu_j : \mathcal{V}_j \to \dot{X}/G = \dot{\mathcal{X}}/\mathcal{G}$
2. $\mathcal{V}_j$ is glued together from copies of $\mathcal{G}$-orbits of equivalence classes of polyhedra $\mathcal{P}(\mathcal{X}) \times C_{k+1}(\Gamma(\mathcal{X}))/\simeq_{\mathcal{P}}$ in such a way that if polyhedra $P, P' \subset \mathcal{V}_j$ share a facet F, then the induced equivalence class of F is the same. More formally, there is a decomposition of $\mathcal{V}_j$ into cubical polyhedra $\{P_h\}$, such

that there is a lift $P_h \to \dot{\mathcal{X}}$ to a polyhedron of $\dot{\mathcal{X}}$ (well-defined up to the action of $\mathcal{G}$) which projects to the map $\nu_{j|P_h} : P_h \to \dot{\mathcal{X}}/\mathcal{G}$. Moreover, there is a coloring $c_h \in C_{k+1}(V(\Gamma(\mathcal{X})))$, with a well-defined equivalence class associated to the lift $P_h \to \dot{\mathcal{X}}$. If P_g, P_h share a facet F, so that $F = \partial P_g \cap \partial P_h \subset \mathcal{V}_j$, then there is a lift $P_g \cup_F P_h \to \dot{\mathcal{X}}$ which projects to the map $\nu_{j|} : P_g \cup_F P_h \to \dot{\mathcal{X}}/\mathcal{G}$. We want the colorings to be compatible, in the sense that $(F, c_g) \simeq_{\mathcal{F}} (F, c_h)$. Thus, there is a well-defined map $c_j : \mathcal{F}(\mathcal{V}_j) \to [k+1]$.

3. The boundary of $\mathcal{V}_j$ is the union of all facets F contained in precisely one polyhedron $\partial P_g \subset \mathcal{V}_j$. Moreover, the boundary pattern $\partial_i \mathcal{V}_j = \cup_{F \in \mathcal{F}(\mathcal{V}_j), c_j(F)=i} F$, $1 \leq i \leq j$. Thus, a facet F is an interior facet (contained in the boundary of two polyhedra) if and only if $c_j(F) > j$.

4. The multiplicities of $\mathcal{G}$-orbits of equivalence classes of colored polyhedra making up $\mathcal{V}_j$ satisfy the polyhedral gluing equations. In particular, for each equivalence class $[(F,c)]_{\mathcal{F}}$, $F = \partial P \cap \partial P'$, the number of lifts $P_g \to P$ with coloring c_g which induce equivalent colorings $(F, c_g) \simeq_{\mathcal{F}} (F, c)$ on F is equal to the number of lifts $P_h \to P'$ which induce equivalent colorings $(F, c_h) \simeq_{\mathcal{F}} (F, c)$.

5. The complex $\mathcal{V}_j$ will admit a malnormal quasi-convex hierarchy, realized geometrically by cutting along the walls colored $j, \ldots, k+1$. So the components of $\mathcal{V}_j$ will have fundamental group in $\mathcal{MQH}$, and therefore will be virtually special by Theorem 2.9.

The base case $\mathcal{V}_{k+1}$ is the collection of equivalence classes of polyhedra given by the solution to the polyhedral gluing equations Ω found in the previous section. Recall we proved the existence of $\Omega : \mathcal{P}(\mathcal{X}) \times C_{k+1}(\Gamma(\mathcal{X}))/\simeq_{\mathcal{P}} \to \mathbb{Z}_{\geq 0}$ satisfying the polyhedral gluing equations. For each equivalence class $[(P_j, c)]_{\mathcal{P}}$, take $\Omega(P_j, c)$ copies of P_j, $1 \leq j \leq p$, keeping track of the coloring c associated to each copy of P_i, and take the disjoint union of these to get $\mathcal{V}_{k+1}$. Each polyhedron has a locally convex map to $\dot{X}/G$, so condition (1) holds. These have the empty gluing, each component of $\mathcal{V}_{k+1}$ has a lift to $\dot{\mathcal{X}}$ and coloring determined by the polyhedral equivalence class, so condition (2) holds. Every facet of $\mathcal{V}_{k+1}$ is a subset of $\partial \mathcal{V}_{k+1}$, so there are no restrictions on the facets and condition (3) holds. Property (4) holds trivially since Ω is a solution to the polyhedral gluing equations.

Now, suppose we have constructed $\mathcal{V}_j$ with these properties, for $1 \leq j \leq k+1$. Let's prove the existence of $\mathcal{V}_{j-1}$. The way that we will do this is to prove that $\partial_j \mathcal{V}_j$ covers components of $\mathcal{Y}_j$ with degree zero. By degree zero, we mean that for each facet of $\mathcal{Y}_j$, the number of facets of $\partial_j \mathcal{V}_j$ which cover the facet and have one co-orientation is equal to the number with the opposite co-orientation, where the co-orientation points into the adjacent polyhedron. Then we will appeal to Theorem 3.1 to take a cover $\tilde{\mathcal{V}}_j$ of $\mathcal{V}_j$ which may be glued along $\partial_j \tilde{\mathcal{V}}_j$ to form $\mathcal{V}_{j-1}$. We must further check that it satisfies the inductive hypotheses.

Claim: $\partial_j \mathcal{V}_j$ covers components of $\mathcal{Y}_j$ with degree zero.

First, note that condition (1) implies that each facet F of $\mathcal{V}_j$ is contained in at most two polyhedra of $\mathcal{V}_j$, because the map $\nu_j : \mathcal{V}_j \to \dot{X}/G$ is locally convex. In particular, the map is injective on links of vertices lifted to $\dot{\mathcal{X}}$, and therefore is also injective on links of facets lifted to $\dot{\mathcal{X}}$. Since facets of $\dot{\mathcal{X}}$ are contained in at most two polyhedra, the same holds for facets of $\mathcal{V}_j$. So the gluing given in condition (2) identifies facets of the polyhedra in pairs. As described in condition (3), the facets contained in exactly one polyhedron form the boundary of $\mathcal{V}_j$, and therefore a facet of $\mathcal{V}_j$ which is not in the boundary of $\mathcal{V}_j$ must be contained in precisely two polyhedra of $\mathcal{V}_j$. Also, because the map $\mathcal{V}_j \to \dot{X}/G$ is locally convex, the link of each polyhedron vertex of $\mathcal{V}_j$ is the link of a product of open intervals and half-open intervals. This implies that any path in $\partial_i \mathcal{V}_j$ may be deformed to lie in a sequence of adjacent facets of $\partial_i \mathcal{V}_j$, meeting in codimension-one facets of $\partial_i \mathcal{V}_j$. In fact, from the inductive construction, $\mathcal{V}_j$ will have a hierarchy of length $k+1-j$ that induces such a hierarchy on each boundary component as well.

Consider a polyhedral facet F involved in the boundary pattern $\partial_j \mathcal{V}_j$, which by hypothesis (2) has a lift $F \to \dot{\mathcal{X}}$ and an associated equivalence class $[(F,c)]_{\mathcal{F}}$, some $c \in C_{k+1}(\Gamma(\mathcal{X}))$. The adjacent polyhedron $\partial P \supset F$ has an equivalence class $[(P,d)]_{\mathcal{P}}$ that is a polyhedron of $\mathcal{V}_j$ by property (2) such that $(F,d) \simeq_{\mathcal{F}} (F,c)$. For a facet F' of ∂P adjacent to F with color $d(F') > j$, there must be an adjacent polyhedron $P' \subset \mathcal{V}_j$ containing $F' \subset \partial P'$, since this facet cannot occur as part of the boundary pattern of $\mathcal{V}_j$ by condition (3). Then there is a unique facet $F'' \subset \partial P'$ meeting F' such that $F' \cap F = F'' \cap F'$ and by condition (2) $F \cup_{F\cap F''} F'' \subset P \cup_{F'} P' \to \dot{\mathcal{X}}$ is some lift of the map $\nu_{j|}$ (the restriction of the map ν_j from condition (1)) such that $F \cup F'' \subset wall(F)$ (so $wall(F) = wall(F'')$, see Figure 2). Let $[(P',d')]_{\mathcal{P}}$ be the equivalence class associated to P' (which exists by condition (2)). Then $(F',d') \simeq_{\mathcal{F}} (F',d)$ by the condition (2). We have $(wall(F), wall(F')) \in E(\Gamma(\mathcal{X}))$. Also, $d(wall(F)) < d(wall(F')), d'(wall(F'')) < d'(wall(F'))$, by the inductive hypothesis on $\mathcal{V}_j$. Since $(F',d) \simeq_{\mathcal{F}} (F',d')$, and therefore $(wall(F'),d) \simeq (wall(F'),d')$, we have $(wall(F),d) = (wall(F''),d) \simeq (wall(F''),d')$ by one of the conditions of the equivalence relation $\simeq$. Also, the lift $F \cup_{F\cap F''} F'' \to wall(F)^d \looparrowright wall(F)$, since $d(F') > j$ by condition (j) of the equivalence relation 6.6.

Take a path $\alpha : I \to \partial_j \mathcal{V}_j$ starting at F, and going through a sequence of facets $F = F_0, F_1, F_2, \ldots, F_m$, such that F_i is associated to a coloring d_i. We may assume each of these facets intersects its neighbors in codimension-one facets of $\partial_j \mathcal{V}_j$, by the observation above. We see that once we choose a lift $F \to wall(F)^d \subset \mathcal{X}$, we get a lift $\tilde{\alpha} : I \to wall(F)^d$, and corresponding lifts $F_i \to wall(F)^d$. Moreover, $(wall(F), d_0) \simeq (wall(F), d_i)$. If α is a closed path so that $F_k = F$, then the lift $F_k \to wall(F)^d$ induces an equivalent coloring of $wall(F)$. Thus, we see that the lift $F \to wall(F)^d$ is well-defined up to the action of $Stab(wall(F)^d)$, so we get a well-defined lift of the component Z of

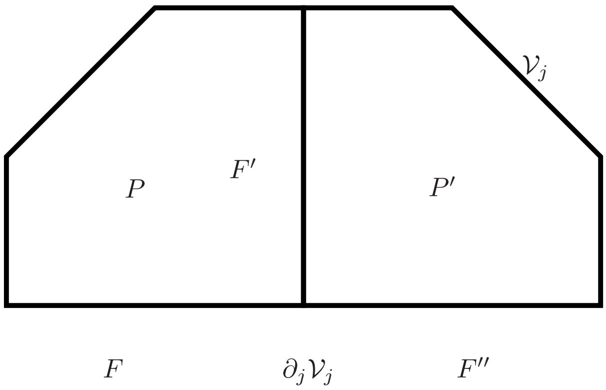


Figure 2: Developing $\partial_j\mathcal{V}_j$

$\partial_j\mathcal{V}_j$ containing F to a map $Z \to wall(F)^d/Stab(wall(F)^d) = wall(F)^d_{\mathcal{G}}$. Conversely, if a facet $F' \subset \partial P$ adjacent to F is colored $d(wall(F')) = i < j = d(wall(F))$, then F' must be part of the boundary pattern $\partial_i\mathcal{V}_j$ by condition (3). Then $F' \cap F \subset \partial_i(\partial_j\mathcal{V}_j)$. Thus, we have a map $Z \to wall(Z)^c_{\mathcal{G}}$ which is a covering projection onto the component of its image. The condition (4) ensures that the map $\partial_j\mathcal{V}_j \to \mathcal{Y}_j$ is degree zero, since for each facet equivalence class $[(F,c)]_{\mathcal{F}}$ with $c(F) = j$, there is a unique representative of the $\mathcal{G}$-orbit of (F,c) in the complex $\mathcal{Y}_j$. Thus, the number of representatives of $[(F,c)]_{\mathcal{F}}$ in $\partial_j\mathcal{V}_j$ with one co-orientation will cancel with the other co-orientation by the gluing equation for the class $[(F,c)]_{\mathcal{F}}$. This finishes the proof of the claim that $\partial_j\mathcal{V}_j$ covers components of $\mathcal{Y}_j$ with degree zero.

Next, we need to show that $\partial_j\mathcal{V}_j$ is acylindrical in $\mathcal{V}_j$ in order to apply Theorem 3.1. Suppose that there is an essential cylinder $(S^1 \times [0,1], S^1 \times \{0,1\}) \to (\mathcal{V}_j, \partial_j\mathcal{V}_j)$. By induction hypothesis (1), there is a map $\nu_j : \mathcal{V}_j \to \dot{X}/G$, inducing a map of the cylinder $S^1 \times [0,1] \to \dot{X}/G$. We may therefore choose a compatible elevation $(S^1 \times [0,1], S^1 \times \{0\}, S^1 \times \{1\}) \to (\dot{\mathcal{X}}, \dot{Y}_0, \dot{Y}_q)$, where $Y_0, Y_q \in V(\Gamma(\mathcal{X}))$, which must also be an essential cylinder. Moreover, since Y_0, Y_q are convex in $\mathcal{X}$, we may assume that the maps $S^1 \times \{0\} \to Y_0, S^1 \times \{1\} \to Y_1$ are geodesic. Since parallel geodesics are distance $\leq R = 2\delta$ apart in X (where δ is the hyperbolicity constant of X), we may assume that for some $z \in S^1$, $z \times [0,1]$ is a geodesic of length $\leq R$ connecting Y_0 and Y_q in $\mathcal{X}$. Therefore $(Y_0, Y_q) \in E(\Gamma(\mathcal{X}))$, so Y_0 and Y_q must have distinct colors in any coloring $c \in C_{k+1}(\Gamma(\mathcal{X}))$, so $c(Y_0) \neq c(Y_q)$. However, there must be a sequence of walls $Y_0, Y_1, \dots, Y_q$ such that the geodesic $z \times [0,1]$ intersects this sequence of walls. There will also be a sequence of facets $F_0, F_1, \dots, F_q$, $F_i \subset Y_i$ that the geodesic meets, and sequence of polyhedra $P_1, \dots, P_q$, with $F_{i-1} \cup F_i \subset \partial P_i, i = 1, \dots, q$ (see Figure 3). Moreover, $(Y_i, Y_j) \in E(\Gamma(\mathcal{X}))$ for all $0 \leq i < j \leq q$ since their distance is $\leq R$. Associated to each P_i is an equivalence class of colorings

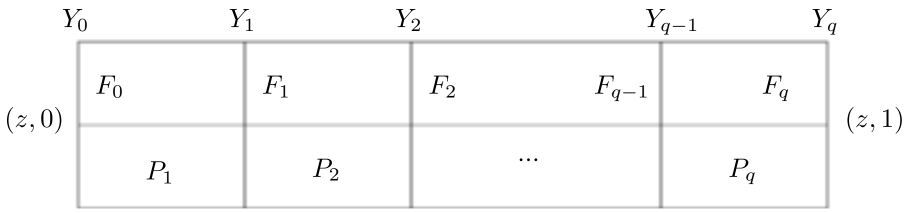


Figure 3: Sequence of walls meeting the geodesic

$[(P_i,d_i)]_{\mathcal{P}}$ by (2), and since the facets $F_1,\ldots,F_{q-1}$ are interior to $\mathcal{V}_j$, we must have $(F_i,d_{i-1})\simeq_{\mathcal{F}}(F_i,d_i)$ by (2). In particular, $d_{i-1}(Y_0)=d_i(Y_0), d_{i-1}(Y_q)=d_i(Y_q)$, $i=1,\ldots,q$, since $d_i(Y_i)>j$ since F_i is interior to $\mathcal{V}_j$ by induction hypothesis (3). But then $d_0(Y_0)=j, d_0(Y_q)=d_q(Y_q)=j$, which contradicts the fact that $d_0(Y_0)\neq d_0(Y_q)$ since $(Y_0,Y_q)\in E(\Gamma(\mathcal{X}))$. Thus, we conclude that the cylinder does not exist, and therefore $\partial_j\mathcal{V}_j$ is acylindrical in $\mathcal{V}_j$.

To recap, we have an acylindrical subcomplex $\partial_j\mathcal{V}_j\subset\mathcal{V}_j$. Moreover, the components Z of $\partial_j\mathcal{V}_j$ are partitioned into equivalence classes determined by the equivalence relation of the equivalence class of $wall(Z)$ together with coloring. Each component covers a component of $wall(Z)^c_{\mathcal{G}}$ for some $\simeq$ equivalence class $[(wall(Z),c)]$. Thus, there is a union of components $Z_j\subseteq\mathcal{Y}_j$ such that there is a cover $\partial_j\mathcal{V}_j\to Z_j$. Moreover, the cover is degree 0 with respect to the co-orientation. We split $\partial_j\mathcal{V}_j=\partial_j\mathcal{V}_j^{\uparrow}\sqcup\partial_j\mathcal{V}_j^{\downarrow}\sqcup\partial_j\mathcal{V}_j^{\circ}$, determined on each component by whether the cover of the corresponding component of Z_j preserves or reverses co-orientation, unless $Stab(wall(Z)^c)$ exchanges the sides of $wall(Z)$, in which case we may ignore the orientation and it lies in $\partial_j\mathcal{V}_j^{\circ}$.

By Theorem 3.1, there is a regular covering space of constant degree $\tilde{\mathcal{V}}_j\to\mathcal{V}_j$, with boundary pattern $\{\partial_1\tilde{\mathcal{V}}_j,\ldots,\partial_j\tilde{\mathcal{V}}_j\}$ given by the preimages of $\partial_i\mathcal{V}_j$, such that the induced covering space $\partial_j\tilde{\mathcal{V}}_j\to Z_j$ is regular. Since the degree of the cover is zero, we must have that the covers $\partial_j\tilde{\mathcal{V}}_j^{\uparrow}\to Z_j$ and $\partial_j\tilde{\mathcal{V}}_j^{\downarrow}\to Z_j$ are common covers. After gluing the co-oriented components of $\partial_j\tilde{\mathcal{V}}_j$, we may take two copies of the resulting complex, and glue the non-co-oriented components $\partial_j\tilde{\mathcal{V}}_j^{\circ}$ by co-orientation reversing isometries which exchange the sides in pairs (we'll rename the 2-fold cover $\tilde{\mathcal{V}}_j$ for simplicity). Thus, there is an isometric involution $\tau_j:\partial_j\tilde{\mathcal{V}}_j\leftrightarrow\partial_j\tilde{\mathcal{V}}_j$. We may form the quotient space $\mathcal{V}_{j-1}=\tilde{\mathcal{V}}_j/\tau_j$ by gluing the boundary pattern by τ_j. We need to check that the inductive hypotheses are satisfied for $\mathcal{V}_{j-1}$.

Since the involution τ_j reverses co-orientation, we can see that the combinatorial immersion $\tilde{\mathcal{V}}_j\to\mathcal{V}_j\to\dot{\mathcal{X}}/\mathcal{G}$ extends to an immersion $\mathcal{V}_{j-1}\to\dot{\mathcal{X}}/\mathcal{G}$. Moreover, since τ_j is an involution of the boundary pattern, we see that $\mathcal{V}_{j-1}$ has locally convex boundary since $\partial_j\tilde{\mathcal{V}}_j\subset\tilde{\mathcal{V}}_j$ has a collar neighborhood, and therefore the map to $\dot{\mathcal{X}}/\mathcal{G}$ is locally convex, so condition (1) is satisfied. The boundary pattern $\partial_i(\partial_j\mathcal{V}_j)$ is preserved by the involution τ_j, since the coloring of the boundary pattern is locally determined by the equivalence classes of walls being glued together for colors $i < j$. We define the boundary pattern of $\mathcal{V}_{j-1}$ by $\partial_i \mathcal{V}_{j-1} = \partial_i \tilde{\mathcal{V}}_j / \tau_{j|}$ (where $tau_{j|}$ is the restriction of τ_j to $\partial_i \tilde{\mathcal{V}}_j$). The interior facets will all have color $> j-1$, and the boundary facets will have color $\leq j-1$. So condition (3) is satisfied. Since we have glued $\mathcal{V}_{j-1}$ out of copies of colored polyhedra in a way consistent with the gluing equations, and taking regular covers preserves the gluing equations, conditions (2) and (4) are satisfied. Since by induction, each component of $\mathcal{V}_j$ has fundamental group in $\mathcal{MQH}$, so does $\tilde{\mathcal{V}}_j$. Each component of $\mathcal{V}_j$ is obtained from components of $\tilde{\mathcal{V}}_j$ by a graph of spaces with acylindrical edge spaces, which implies that the fundamental group is a graph of groups with vertex groups in $\mathcal{MQH}$ and edge groups malnormal. Thus, the fundamental group of each component of $\mathcal{V}_j$ is in $\mathcal{MQH}$, so property (5) holds.

So all of the inductive hypotheses are satisfied.

The complex $\mathcal{V}_0$ has trivial boundary pattern, and a locally convex map $\mathcal{V}_0 \to \dot{X}/G$. Therefore, this map is a finite-sheeted covering space. Moreover, by property (5) $\pi_1(\mathcal{V}_0) \in \mathcal{MQH}$. By [50, Theorem 13.3 or Theorem 11.2] (see also Theorem 2.9), $\mathcal{V}_0$ has a finite-sheeted special cover, and thus X/G does. This finishes the proof of Theorem 1.1. $\square$

9 Conclusion

Recall that a Haken 3-manifold is a compact irreducible orientable 3-manifold containing an embedded π_1-injective surface.

Theorem 9.1 (Virtual Haken conjecture [49]). *Let M be a closed aspherical 3-manifold. Then there is a finite-sheeted cover $\tilde{M} \to M$ such that $\tilde{M}$ is Haken.*

Theorem 9.2 (Virtual fibering conjecture, Question 18 [48]). *Let M be a closed hyperbolic 3-manifold. Then there is a finite-sheeted cover $\tilde{M} \to M$ such that $\tilde{M}$ fibers over the circle. Moreover, $\pi_1(M)$ is LERF and large.*

Proof of Theorems 9.1 and 9.2. From the geometrization theorem [44, 43, 40], it is well-known that the virtual Haken conjecture reduces to the case that M is a closed hyperbolic 3-manifold. For a closed hyperbolic 3-manifold, we have the following result of Bergeron-Wise based on work of Kahn-Markovic [28] (and making use of seminal results of Sageev on cubulating groups containing codimension-one subgroups [45]).

Theorem 9.3. *[5, Theorem 5.3] Let M be a closed hyperbolic* 3*-manifold. Then $\pi_1 M$ acts freely and cocompactly on a CAT(0) cube complex.*

Now, by Theorem 1.1, $\pi_1(M)$ is virtually special. This implies that $\pi_1 M$ is LERF and large following from the virtual specialness by Cor. 1.3. Therefore

M is virtually Haken, and in fact M is also virtually fibered by [50, Corollary 14.3]. □

We also have the following corollary, resolving a question of Thurston.

Corollary 9.4 (Question 15 [48]). Kleinian groups are LERF.

Proof. This follows combining 9.2 which proves that compact hyperbolic 3-manifold groups are LERF, together with the implication that therefore all finite-covolume Kleinian groups are LERF by [36, Proposition 5.3]. It is well known that any Kleinian group embeds in a finite covolume Kleinian group [41]. □

A Appendix: Filling Virtually Special Subgroups

by Ian Agol, Daniel Groves, and Jason Manning

This appendix will be devoted to proving the following theorem, which may be regarded as a generalization of the main theorem of [1], with the extra ingredient of the malnormal virtually special quotient Theorem A.10 [50, Theorem 12.3].

Theorem A.1. *Let G be a hyperbolic group, let $H \leq G$ be a quasi-convex virtually special subgroup. For any $g \in G - H$, there is a hyperbolic group $\mathcal{G}$ and a homomorphism $\phi : G \to \mathcal{G}$ such that $\phi(g) \notin \phi(H)$ and $\phi(H)$ is finite.*

Remark A.2. The conclusion of this theorem may be regarded as a weak version of subgroup separability. Under the hypotheses of the theorem, H is subgroup separable in G if one may also assume that the quotient group $\mathcal{G}$ is finite.

Remark A.3. It ought to be possible to prove this result using the techniques used by Wise to prove [50, Theorem 12.1]. However, we have decided to provide an alternative argument which gives a geometric perspective on the notion of height, and uses hyperbolic Dehn filling arguments from the literature instead of the small-cancellation theory developed in [50].

Notation A.4. In this appendix, we will sometimes use the notation $A \dot{\leq} B$ to indicate that A is a finite-index subgroup of B.

The height of a subgroup $H < G$ measures how far away H is from being (almost) malnormal.

Definition A.5. Let $H < G$. The *height* of H in G is the maximum number $n \geq 0$ so that there are distinct cosets $\{g_1H, \ldots, g_nH\}$ with $\bigcap_{i=1}^{n} g_iHg_i^{-1}$ infinite.

Thus a finite subgroup has height 0, an infinite almost malnormal subgroup has height 1, and so on. Let H be a quasiconvex subgroup of a hyperbolic group G. That H has finite height was proved in [19] (we give a new proof in Corollary A.40 below). Our proof of Theorem A.1 is by an induction on the height of H in G. In order to reduce the height of H, we first use H to produce an associated peripheral structure consisting of commensurators of infinite intersections of H with its conjugates. We then perform a Dehn filling on this peripheral structure.

We recall some more facts about quasiconvex subgroups of hyperbolic groups we'll need to define the peripheral structure:

Proposition A.6. Let G be a hyperbolic group.

1. [19, Lemma 2.7] Finite intersections of quasiconvex subgroups of G are quasiconvex.
2. [19, Lemma 2.9] Quasiconvex subgroups of G are finite index in their commensurators.

Definition A.7. We now define the malnormal core of H and peripheral system induced by H on G. Let h be the height of H in G. By [1, Corollary 3.5], there are finitely many H-conjugacy classes of minimal infinite subgroups of the form $H \cap H^{g_2} \cap H^{g_3} \cap \cdots \cap H^{g_j}$, where $1 \leq j \leq h$ and $\{g_1 = 1, g_2, \ldots, g_j\}$ are essentially distinct, in the sense that $g_i H = g_{i'} H$ if and only if $i = i'$. (Here minimality is with respect to inclusion.) These intersections are quasiconvex by Proposition A.6.(1).

Choose one H-conjugacy class of each such subgroup in H, and replace it with its commensurator in H to obtain a collection of quasi-convex subgroups $\mathcal{D}_0$ of H. Eliminating redundant entries which are H-conjugate, we obtain a collection $\mathcal{D}$, which we will call the *malnormal core* of H in G. The collection $\mathcal{D}$ gives rise to a peripheral system of subgroups $\mathcal{P}$ in G in two steps:

1. Change $\mathcal{D}$ to $\mathcal{D}'$ by replacing each $D \in D$ with $D' < G$ its commensurator in G.
2. Eliminate redundant entries of $\mathcal{D}'$ to obtain $\mathcal{P} \subset \mathcal{D}'$ which contains no two elements which are conjugate in G.

Call $\mathcal{P}$ the *peripheral structure on G induced by H*. This peripheral structure is only well-defined up to replacement of some elements of $\mathcal{P}$ by conjugates. On the other hand, replacing H by a commensurable subgroup of G does not affect the induced peripheral structure. We consider two peripheral structures on a group to be the same if the same group elements are parabolic (i.e. conjugate into $\bigcup \mathcal{P}$) in the two structures.

Remark A.8. From Proposition A.6.(2) it follows that each of the elements of $\mathcal{D}$ or $\mathcal{P}$ contains some such $H \cap H^{g_1} \cap \cdots \cap H^{g_j}$ as a finite index subgroup.

Example A.9. Suppose that $H < G$ has height 3. If $U = H \cap H^{g_1} \cap H^{g_2}$ is a minimal infinite intersection of G–conjugates of H, then U contributes at most 3 subgroups to the collection $\mathcal{D}$, but only one subgroup to $\mathcal{P}$. Suppose, for example, $G = \langle a, b\rangle$ is free, and $H = \langle a, bab^{-1}, b^2ab^{-2}\rangle$. Then $\mathcal{D}$ consists of three cyclic groups $\langle a\rangle$, $\langle bab^{-1}\rangle$, and $\langle b^2ab^{-2}\rangle$. These subgroups are not conjugate in H, but of course they are conjugate in G, so $\mathcal{P} = \{\langle a\rangle\}$.

For a second example with slightly different behavior, take $H = \langle a, ba^2b^{-1}\rangle$. Then $U = \langle a^2\rangle$, and we have to take commensurators in H to obtain $\mathcal{D} = \{\langle a\rangle, \langle ba^2b^{-1}\rangle\}$. Once again, $\mathcal{P} = \{\langle a\rangle\}$ consists of a single cyclic group.

Wise's malnormal virtually special quotient theorem [50, Theorem 12.3] is a key tool in our argument:

Theorem A.10. *Let G be a virtually special hyperbolic group. Let $\{H_1, \ldots, H_m\}$ be an almost malnormal collection of quasiconvex subgroups. Then there exists finite-index subgroups $\dot{H}_i \dot{\trianglelefteq} H_i$, $i = 1, \ldots, m$ such that for any further finite index subgroups $H_i' \dot{\le} \dot{H}_i$, the quotient $G/ \ll H_1', \ldots, H_m' \gg$ is virtually special.*

In the proof of Theorem A.1 (Section A.5), we will combine Theorem A.10 with the Dehn filling results A.43 and A.22 to find a hyperbolic quotient of G in which the image of H is virtually special, quasiconvex, and has smaller height than H.

A.1 Relative quasiconvexity and H-filling

In this subsection, we recall definitions of relative quasiconvexity and H–filling and state the main new Dehn filling result.

Our arguments use the *cusped space* $X(G, \mathcal{P}, S)$ associated to a group G, a system of subgroups $\mathcal{P}$ and a finite generating set S for G, assumed to satisfy the compatibility condition that $S \cap P$ generates P for each $P \in \mathcal{P}$. A definition can be found in [21], but we recall and slightly modify the definition here for convenience. The cusped space is built from a Cayley graph $\Gamma(G, S)$ for G by equivariantly attaching *combinatorial horoballs* to subgraphs of $\Gamma(G, S)$ corresponding to left cosets of elements of $\mathcal{P}$. It is convenient in this appendix to modify the definition of these combinatorial horoballs as follows:

Definition A.11. (Combinatorial horoball) Let $W \subset G$ be of the form tP for $t \in G$, $P \in \mathcal{P}$. Let $S_0 = S \cap P \setminus \{1\}$, and for $n > 0$, let $S_n = S_{n-1} \cup \{s_1s_2 \neq 1 \mid s_1, s_2 \in S_{n-1}\}$. Thus for each n, S_n is a generating set for P. Define $H(W)$ to be a 1–complex with vertex set $V = W \times \mathbb{Z}_{\geq 0}$, and edges as follows:

1. (vertical) For each $(w, n) \in V$, there is an edge from (w, n) to $(w, n+1)$.
2. (horizontal) For each $(w, n) \in V$, and each $s \in S_n$ there is an edge from (w, n) to (ws, n). In particular, if s has order 2, then there is a pair of edges between (w, n) and (ws, n).

Definition A.12. (Cusped space) Note that the full subgraph of $\Gamma(G, S)$ on W is the same as the full subgraph of $H(W)$ on $W \times \{0\}$, so there is a natural way to glue $H(W)$ to $W \subset \Gamma(G, S)$. The cusped space $X(G, \mathcal{P}, S)$ is obtained from $\Gamma(G, S)$ by gluing on $H(W)$ for all cosets tP with $t \in G$ and $P \in \mathcal{P}$.

Remark A.13. Using the modified definition A.11 to construct $X(G, \mathcal{P}, S)$ gives a cusped space with the same distance function on vertices as the one constructed in [21]. In particular, the coarse geometry is unchanged, and all the results we will apply from [21, 1, 36, 27] apply in the same way to the cusped space we are using here. In particular, the pair $(G, \mathcal{P})$ is relatively hyperbolic if and only if $X(G, \mathcal{P}, S)$ is Gromov hyperbolic. The advantage of the current construction is that the obvious G–action on $X(G, \mathcal{P}, S)$ is free, even if G contains torsion elements.

Definition A.14. By a *horoball* of a cusped space $X(G, \mathcal{P}, S)$, we will always mean either one of the glued-on graphs $H(W)$, or a full subgraph of such an $H(W)$ on the vertices $W \times \mathbb{Z}_{\geq R}$ for some positive integer R.

Definition A.15. (See [1, Section 3]) Let G be a hyperbolic group and H a quasi-convex subgroup, and let $\mathcal{P}$ and $\mathcal{D}$ be the induced peripheral structures on G and H described above. Let X be the cusped space of $(G, \mathcal{P})$ and Y the cusped space of $(H, \mathcal{D})$ (with respect to choices of generating sets). The inclusion $\phi\colon H \to G$ sends peripheral subgroups in $\mathcal{D}$ into (conjugates of) peripheral subgroups in $\mathcal{P}$, and so induces a proper H-equivariant Lipschitz map $\check{\phi}\colon Y \to X$. We say that $(H, \mathcal{D})$ is *C-relatively quasiconvex* in $(G, \mathcal{P})$ if $\check{\phi}$ is C-Lipschitz and has C-quasiconvex image in X.

In [36, Appendix A] it is explained that the above definition agrees with other notions of relative quasiconvexity, such as those in [27].

The following is proved in [1] under the assumption that G is torsion-free. It was extended to the general setting in [37].

Proposition A.16. [1, Proposition 3.12],[37, Corollary 1.9] The pairs $(H, \mathcal{D})$ and $(G, \mathcal{P})$ are both relatively hyperbolic and with these peripheral structures $(H, \mathcal{D})$ is a relatively quasi-convex subgroup of $(G, \mathcal{P})$.

Definition A.17. Let $(H, \mathcal{D})$ be a relatively quasi-convex subgroup of $(G, \mathcal{P})$, where $\mathcal{P} = \{P_1, \ldots, P_m\}$. Let $\{N_i \lhd P_i\}$ be given. The quotient

$$G(N_1, \ldots, N_m) := G/ \ll N_1 \cup \cdots \cup N_m \gg$$

is a *filling of* $(G, \mathcal{P})$. It is an *H–filling* if $N_i^g \subset P_i^g \cap H$ whenever $H \cap P_i^g$ is infinite.

Remark A.18. The current definition of H–filling agrees with the one in [1] only in case G is torsion-free. As explained in [36, Appendix B], Definition A.17 is the correct extension in case there is torsion.

Remark A.19. As explained in [1, Definition 3.2], an H-filling $G(N_1,\dots,N_m)$ induces a filling $H(K_1,\dots,K_n)$ of H: For each $D_i \in \mathcal{D}$, there is a $c_i \in G$ and $P_{j_i} \in \mathcal{P}$ so that $D_i \subseteq c_i P_{j_i} c_i^{-1}$. Then $K_i = c_i N_{j_i} c_i^{-1} \cap D_i$. The inclusion $H \hookrightarrow G$ induces a homomorphism $H(K_1,\dots,K_n) \to G(N_1,\dots,N_m)$.

Definition A.20. Let $(G,\mathcal{P})$ be a relatively hyperbolic group. We say that a statement S about fillings $G(N_1,\dots,N_m)$ holds *for all sufficiently long fillings* if there is a finite set $B \subset \bigcup \mathcal{P}$ so that whenever $G(N_1,\dots,N_m)$ is a filling so that $\bigcup_{i=1}^n N_i$ does not intersect B, then S holds.

Similarly, if $(H,\mathcal{D})$ is a relatively quasiconvex subgroup of $(G,\mathcal{P})$, a statement S holds *for all sufficiently long H–fillings* if there is a finite set $B \subset \bigcup \mathcal{P}$ so the statement S holds for all H–fillings $G(N_1,\dots,N_m)$ so that $\bigcup_{i=1}^n N_i$ does not intersect B.

Obviously if S holds for all sufficiently long fillings, then S holds for all sufficiently long H–fillings. The fundamental theorem of relatively hyperbolic Dehn filling can be stated:

Theorem A.21. *[42, 16] (cf. [21] in the torsion-free case) Let G be a group and $\mathcal{P} = \{P_1,\dots,P_m\}$ a collection of subgroups so that $(G,\mathcal{P})$ is relatively hyperbolic, and let $F \subset G$ be finite. Then for all sufficiently long fillings $\phi\colon G \to \bar{G} := G(N_1,\dots,N_m)$,*

1. *$\ker(\phi|_{P_i}) = N_i$ for each $P_i \in \mathcal{P}$;*
2. *$(\bar{G}, \{\phi(P_1),\dots,\phi(P_m)\})$ is relatively hyperbolic; and*
3. *$\phi|_F$ is injective.*

Our chief new Dehn filling result in this appendix is the following:

Theorem A.22. *Let G be hyperbolic, and let H be height $k \geq 1$ and quasi-convex in G. Suppose that $\mathcal{D}$ and $\mathcal{P} = \{P_1,\dots,P_m\}$ are as in Definition A.7. Then for all sufficiently long H–fillings*

$$\phi : G \to \bar{G} := G(N_1,\dots,N_m)$$

with $N_i \lhd P_i$ finite index for all i, the subgroup $\phi(H)$ is quasi-convex of height strictly less than k in the hyperbolic group $\bar{G}$.

Remark A.23. We proved Theorem A.22 in [1] under the assumption that G was torsion-free. Much of the proof from [1] still works without that assumption, but our argument that height is reduced in the quotient depended on the machinery of Part 2 of [21], in which torsion-freeness is assumed. Our main innovation in this appendix is a completely different proof that height decreases under Dehn filling.

A.2 Geometric finiteness

Geometric finiteness is a dynamical condition. We recall the relevant definitions, which are mostly standard. They originate in the study of Kleinian groups [18]; the fact that hyperbolic and relatively hyperbolic groups can be understood in this way is due to Bowditch [8] and Yaman [51]. The only nonstandard terminology in this section is that of *weak* geometric finiteness, in which finite parabolic subgroups are explicitly allowed.

We also make the following standing assumption (natural in our context): G *is finitely generated, and* $\mathcal{P}$ *is a finite collection of subgroups of* G*.*

Definition A.24. Let M be a compact metrizable space with at least 3 points, and let $\Theta(M)$ be the set of unordered distinct triples of points in M. Any action of G on M induces an action on $\Theta(M)$. The action of G on M is said to be a *convergence group action* if the induced action on $\Theta(M)$ is properly discontinuous.

Definition A.25. Suppose $G \curvearrowright M$ is a convergence group action. A point $p \in M$ is a *conical limit point* if there is a sequence $\{g_i\}_{i\in\mathbb{N}}$ and a pair of points a, b so that $g_i p \to b$ but for every $x \in M \setminus \{p\}$, we have $g_i x \to a$.

A point p is *parabolic* if $\mathrm{Stab}_G(p)$ is infinite but there is no infinite order $g \in G$ and $q \neq p \in M$ so that $\mathrm{Fix}(g) = \{p, q\}$.

A parabolic point p is called *bounded parabolic* if $\mathrm{Stab}_G(p)$ acts cocompactly on $M \setminus \{p\}$.

Definition A.26. The action $G \curvearrowright M$ is *geometrically finite* if every point in M is a conical limit point or a bounded parabolic point. Say that $(G, \mathcal{P})$ *acts geometrically finitely* on M if all of the following hold:

1. $G \curvearrowright M$ is a geometrically finite convergence action.
2. Each $P \in \mathcal{P}$ is equal to $\mathrm{Stab}_G(p)$ for some bounded parabolic point p.
3. For any bounded parabolic point p, the stabilizer $\mathrm{Stab}_G(p)$ is conjugate to exactly one element of $\mathcal{P}$.

Let X be a δ–hyperbolic G–space, so that $(G, \mathcal{P})$ acts geometrically finitely on ∂X. Then we say that $(G, \mathcal{P})$ acts *geometrically finitely on* X.

It is useful when talking about Dehn filling to allow parabolic subgroups to be finite. We will use the following definitions:

Definition A.27. Let $G \curvearrowright M$ be a convergence action, and say that $p \in M$ is a *finite parabolic point* if p is isolated and has finite stabilizer.

For $\mathcal{P}$ a finite collection of subgroups of G, write $\mathcal{P}_\infty$ for the subcollection of infinite subgroups, and $\mathcal{P}_f$ for the subcollection of finite subgroups. Suppose

G acts on the compact metrizable space M. Let M' be obtained from M by removing all isolated points. Say that $(G, \mathcal{P})$ acts *weakly geometrically finitely* on M (or that the action is *WGF*) if all of the following occur:

1. $(G, \mathcal{P}_\infty)$ acts geometrically finitely on M'.
2. Each $P \in \mathcal{P}_f$ is equal to $\mathrm{Stab}_G(p)$ for some $p \in M \setminus M'$.
3. Every $p \in M \setminus M'$ is a finite parabolic point, with stabilizer conjugate to exactly one element of $\mathcal{P}_f$.

Finally, if X is a δ–hyperbolic G–space, then we say that the action of $(G, \mathcal{P})$ on X is *WGF* whenever the action of $(G, \mathcal{P})$ on ∂X is WGF.

Proposition A.28. Let $(G, \mathcal{P})$ be relatively hyperbolic. Then the action of $(G, \mathcal{P})$ on its cusped space is WGF.

Conversely, if $(G, \mathcal{P})$ has a WGF action on a space M, then $(G, \mathcal{P})$ is relatively hyperbolic.

Proof. We first handle the case that $\mathcal{P} = \mathcal{P}_\infty$. In [21, Theorem 3.25] it is shown that the current definition of relative hyperbolicity is equivalent to several others, in particular to Gromov's original definition of relative hyperbolicity. The papers of Bowditch [7] and Yaman [51] combine to show that, for $\mathcal{P} = \mathcal{P}_\infty$, the pair $(G, \mathcal{P})$ is relatively hyperbolic according to Gromov's definition if and only if $(G, \mathcal{P})$ acts geometrically finitely on a compact perfect metrizable M.

The pair $(G, \mathcal{P})$ is relatively hyperbolic if and only if $(G, \mathcal{P}_\infty)$ is relatively hyperbolic. Indeed, for any generating set S of G, the cusped space $X_\infty = X(G, \mathcal{P}_\infty, S)$ quasi-isometrically embeds into $X = X(G, \mathcal{P}, S)$. The complement $X \setminus X_\infty$ is composed of combinatorial horoballs based on finite graphs. Thus X is quasi-isometric to X_∞ with rays attached coarsely equivariantly to the Cayley graph of G.

Suppose that $(G, \mathcal{P})$ is relatively hyperbolic, so that X is Gromov hyperbolic. Then X_∞ is also Gromov hyperbolic, and $(\partial X)'$ can be canonically identified with ∂X_∞. Thus G acts geometrically finitely on $(\partial X)'$. Moreover, the isolated points of ∂X are in one to one correspondence with the left cosets of elements of $\mathcal{P}_f$; the point corresponding to tP has (finite) stabilizer equal to tPt^{-1}. Thus $(G, \mathcal{P})$ acts weakly geometrically finitely on X.

Conversely, if $(G, \mathcal{P})$ has a WGF action on M, then $(G, \mathcal{P}_\infty)$ has a geometrically finite action on M', so $(G, \mathcal{P}_\infty)$ is relatively hyperbolic. Since $\mathcal{P} \setminus \mathcal{P}_\infty$ is composed of finite subgroups of G, the pair $(G, \mathcal{P})$ is also relatively hyperbolic. $\square$

Remark A.29. Given that $(G, \mathcal{P})$ is relatively hyperbolic if and only if $(G, \mathcal{P}_\infty)$ is relatively hyperbolic, it is often convenient to simply ignore the possibility of finite parabolics, as for example in [27]. In the present setting it is important to keep track of them, as otherwise we would not get uniform control of the geometry of cusped spaces of quotients, as in Theorem A.43 below.

Suppose that X and Y are δ-hyperbolic G-spaces. A G-equivariant quasi-isometry from X to Y induces a G-equivariant homeomorphism from ∂X to ∂Y. Since the property of being a (weakly) geometrically finite action on a δ-hyperbolic space is defined in terms of the boundary, we have the following result.

LEMMA A.30. *Suppose that $(G, \mathcal{P})$ admits a WGF action on a δ-hyperbolic space X (as in Definition A.26) and that $f\colon X \to Y$ is a G-equivariant quasi-isometry to another δ-hyperbolic G-space. Then the action of G on Y is WGF.*

A.3 HEIGHT FROM MULTIPLICITY

DEFINITION A.31. $(H, \mathcal{D}) < (G, \mathcal{P})$ is *fully quasiconvex* if it is relatively quasiconvex and whenever $gDg^{-1} \cap P$ is infinite, for $D \in \mathcal{D}, P \in \mathcal{P}$, then $[P : gDg^{-1}] < \infty$.

In this section, $(G, \mathcal{P})$ is relatively hyperbolic, and $(H, \mathcal{D})$ is a fully quasiconvex subgroup. We allow the possibility that $\mathcal{P}$ and $\mathcal{D}$ are empty.

If $\mathcal{P}$ (and therefore $\mathcal{D}$) is empty, we take Γ to be any graph on which G acts freely and cocompactly, and choose $\tilde{*}$ to be some arbitrary vertex. Otherwise, we take Γ to be the 1–skeleton of the cusped space $X(G, \mathcal{P}, S)$ (see Definition A.12).[4] In this case Γ contains a Cayley graph for G, and we take $\tilde{*} = 1 \in G \subset \Gamma$.

DEFINITION A.32. Let $R \geq 0$. An *R-hull for H acting on* Γ is a connected H-invariant sub-graph $\tilde{Z} \subset \Gamma$ so that all of the following hold.

1. $\tilde{*} \in \tilde{Z}$.
2. If γ is a geodesic in Γ with endpoints in the limit set of H, then the R–neighborhood of γ is contained in $\tilde{Z}$.
3. If $\mathcal{P}$ is nonempty and B is a horoball of Γ whose stabilizer in H is infinite, then $B' \subset \tilde{Z}$ where B' is some horoball nested in B.
4. The action of $(H, \mathcal{D})$ on $\tilde{Z}$ is WGF.

[4] Below, when applying Theorem A.43 to a quotient of Γ by G, we will consider a graph which is the cusped space with some extra loops attached (in an equivariant way) to some vertices. It is straightforward to check that our arguments work as written for this slightly different space. In fact, with only a little extra work, one can take Γ to be any graph with a free WGF G-action, but we decided to stick with the more restrictive setting in the interests of brevity.

Remark A.33. For Γ the Cayley graph of a hyperbolic group G, and H a λ–quasiconvex subgroup, it is easy to show that the $(R+\lambda+5\delta)$–neighborhood of H is an R–hull for H acting on Γ. We observe in any case that the R–neighborhood of a 0–hull is an R–hull. The existence of R–hulls for H acting on a cusped space is proved in Lemma A.41.

Example A.34. Let $G < PSL_2\mathbb{C}$ be a nonuniform lattice, for example a hyperbolic knot group. Let $\mathcal{P}$ be a collection of representatives of conjugacy classes of maximal parabolic subgroups. If Γ is a cusped space for $(G, \mathcal{P})$, then there is an equivariant quasi-isometry $\psi\colon \Gamma \to \mathbb{H}^3$. Let H be a geometrically finite subgroup of G which is fully quasiconvex, and let C_H be the convex hull of the limit set of H in $\mathbb{H}^3$. Then, for a sufficiently large neighborhood N of C_H, $\psi^{-1}(N)$ has exactly one unbounded H–equivariant component D. Depending on the quality of the quasi-isometry ψ, D may not even be a 0–hull for H, but it can be shown that any sufficiently large neighborhood of D is an R–hull for H. In general the behavior of the convex hull C_H in $\mathbb{H}^3$ is what we are trying to capture with the R–hull in Γ.

Let $\tilde{Z}$ be an R-hull for H acting on Γ, let $Z = \tilde{Z}/H$ be the quotient of $\tilde{Z}$ by the H–action, and let $Y = \Gamma/G$ be the quotient of Γ by the G–action. If we let $*_H \in Z$ and $* \in Y$ be the images of $\tilde{*}$, we obtain canonical surjections $s\colon \pi_1(Z, *_H) \to H$ and $s\colon \pi_1(Y, *) \to G$. Moreover the canonical map

$$i : Z \to Y$$

which is the composition of the inclusion $Z \hookrightarrow \Gamma/H$ with the quotient map $\Gamma/H \to Y = \Gamma/G$ induces the inclusion of H into G, in the sense that the diagram

$$\begin{tikzcd} \pi_1(Z, *_H) \arrow[r, "i_*"] \arrow[d, "s"] & \pi_1(Y, *) \arrow[d, "s"] \\ H \arrow[r] & G \end{tikzcd} \tag{1}$$

commutes.

Definition A.35. Let $n > 0$, and define the following subset of Z^n:

$$S_n = \{(z_1, \ldots, z_n) \mid i(z_1) = \cdots = i(z_n)\} \setminus \Delta \tag{2}$$

where $\Delta = \{(z_1, \ldots, z_n) \mid z_i = z_j \text{ for some } i \neq j\}$ is the "fat diagonal" of Z^n. Let $s : \pi_1(Z, *_H) \to H$ be the canonical surjection. Let $\varpi_1, \ldots, \varpi_n$ be the n projections of S_n to Z.

(In Stallings' language [47], S_n is that part of the *pullback* of n copies of $i\colon Z \to Y$ which lies outside Δ.)

Let C be a component of S_n, with a choice of basepoint $p = (p_1, \ldots, p_n)$. For $i \in \{1, \ldots, n\}$ define maps $\tau_{C,i}\colon \pi_1(C) \to H$ as follows:

Choose a maximal tree T in Z. For each vertex v of Z, the tree gives a canonical path σ_v from $*_H$ to v, allowing the fundamental groups of Z at different basepoints to be identified. To simplify notation define $\sigma_i = \sigma_{p_i}$. Now, the map $\varpi_i \colon C \to Z$ induces a well-defined map $(\varpi_i)_* \colon \pi_1(C,p) \to \pi_1(Z,*_H)$, taking a loop γ based at $p \in C$ to the loop $\sigma_i \varpi_i(\gamma)\bar{\sigma}_i$ (concatenating from left to right). We define $\tau_{C,i} = s \circ (\varpi_i)_* \colon \pi_1(C,p) \to H$. Since H acts on $\tilde{Z}$ by covering translations, the map s can be seen by lifting paths starting and finishing at the basepoint in the usual way. Once we've used the path in the maximal tree to make based loops in C map to paths in Z starting and finishing at the basepoint $*_H$, the same is true of the maps $\tau_{C,i}$.

Definition A.36. The *multiplicity* of $Z \to Y$ is the largest n so that S_n contains a component C so that for all $i \in 1,\dots,n$ the group

$$\tau_{C,i}\left(\pi_1(C)\right)$$

is an infinite subgroup of H.

Lemma A.37. *For a fixed component C of S_n, the groups*

$$A_i = \tau_{C,i}(\pi_1(C,p)) < H$$

are conjugate in G. Specifically, if σ_i are defined as above, and $g_{i,j}$ is represented by the loop $i \circ \sigma_i \cdot i \circ \bar{\sigma}_j$, then $g_{i,j}A_j g_{i,j}^{-1} = A_i$.

Proof. As in the above discussion, the basepoint of C is $p = (p_1,\dots,p_n)$, and for each i there is a canonical path σ_i in $T \subset Z$ connecting the basepoint $*_H$ of Z to p_i. We also recall the map $i \colon Z \to Y$ takes $*_H$ to $*$ and induces the inclusion $H < G$ in the sense that the diagram (1) above commutes.

Let $q = i(p_1) = \cdots = i(p_n)$. The paths $i \circ \sigma_i$ all begin at $*$ and end at q, so any concatenation of two of them gives a loop in Y representing an element $g_{i,j}$ of G conjugating one of the images of $\pi_1(C,p)$ to another. Precisely, for $i,j \in \{1,\dots,n\}$ we get an element $g_{i,j}$ represented by $i \circ \sigma_i \cdot i \circ \bar{\sigma}_j$ so that

$$g_{i,j}\tau_{C,j}(\alpha)g_{i,j}^{-1} = \tau_{C,i}(\alpha), \forall \alpha \in \pi_1(C,p). \tag{3}$$

□

We aim in this section for the following:

Theorem A.38. *Let R be bigger than the quasi-geodesic stability constant $D = D(\delta)$ specified in the proof below. With the above notation, the height of H in G is equal to the multiplicity of $Z \to Y$.*

Remark A.39. It is instructive to contemplate the proof of this theorem when G is a Kleinian group, and H a geometrically finite subgroup. Then it is not hard to verify that the multiplicity of a convex core of H is equal to the height of H. In fact, the arguments in this section are motivated by carrying this geometric argument over to the broader category of hyperbolic groups.

Before doing the proof, we state and prove a corollary.

Corollary A.40. [19] The height of a quasiconvex subgroup of a hyperbolic group is finite.

Proof. Suppose H is quasiconvex in G. Let Γ be a Cayley graph for G, so that $H \subset \Gamma$ is λ–quasiconvex. Let $\tilde{Z}$ be the $(R + \lambda + 5\delta)$–neighborhood of H in Γ. As noted in Remark A.33, $\tilde{Z}$ is an R–hull for H. Since G (resp. H) acts cocompactly on Γ (resp. $\tilde{Z}$), the complexes Z and Y are both finite. Thus S_n is empty for large n. □

Proof of Theorem A.38. We first bound multiplicity from below by height, and then conversely.

(multiplicity $\geq$ height): Suppose that H has height $\geq n$. There are then $(H, g_2H, \ldots, g_nH)$ all distinct so that $J = H \cap H^{g_2} \cap \cdots \cap H^{g_n}$ is infinite. Since $(G, \mathcal{P})$ is relatively hyperbolic, every infinite subgroup of G either contains a hyperbolic element or is conjugate into some $P \in \mathcal{P}$. This follows immediately from the classification of isometries of δ-hyperbolic spaces (see [20, Section 8.2, p. 211]) and from the definition of WGF action. The proof therefore breaks up naturally into these two cases.

Case 1. The intersection J contains a hyperbolic element a.

By replacing a by a power we may suppose that a has a (K, C)–quasi-geodesic axis $\tilde{\gamma}_a$, where K and C depend only on δ. Quasi-geodesic stability implies that $\tilde{\gamma}_a$ lies Hausdorff distance at most D from a geodesic, where D depends only on δ. So as long as $R > D$, the geodesic $\tilde{\gamma}_a$ lies in $\tilde{Z} \cap g_2\tilde{Z} \cap \cdots \cap g_n\tilde{Z}$. Let π_Z be the natural projection from $\tilde{Z}$ to Z. For $t \in \mathbb{R}$, define $\gamma_a : \mathbb{R} \to Z^n$ as follows:

$$\gamma_a(t) = \left(\pi_Z(\tilde{\gamma}_a(t)), \pi_Z(g_2^{-1}\tilde{\gamma}_a(t)), \ldots, \pi_Z(g_n^{-1}\tilde{\gamma}_a(t))\right)$$

Since G acts freely and the g_i are essentially distinct, γ_a misses the diagonal. Since its coordinates differ only by elements of G, γ_a has image in S_n. Moreover projection of γ_a to any component gives a loop of infinite order in H. Thus we've shown that a component of S_n has an element with infinite order projection to G, and therefore the multiplicity of H is $\geq n$.

Case 2. The intersection J is conjugate into $P \in \mathcal{P}$.

In this case J preserves some horoball B of Γ. By point (3) in the definition of R–hull, there is a horoball B' nested inside B so that $B' \subset \tilde{Z}$. By possibly replacing B' with a horoball nested further inside, we have

$$B' \subset \tilde{Z} \cap g_2\tilde{Z} \cap \cdots \cap g_n\tilde{Z}.$$

It follows that

$$A = \left\{ (\pi_Z(b), \pi_Z(g_2^{-1}(b)), \ldots, \pi_Z(g_n^{-1}(b))) \mid b \in B' \right\}$$

lies in some component C of S_n. Moreover, each $\tau_{C,i}(\pi_1(A)) < \tau_{C,i}(\pi_1(C))$ is conjugate to J, hence infinite.

(height $\geq$ multiplicity): Suppose the multiplicity of $Z \to Y$ is n. Let $C \subset S_n$ be a component with infinite fundamental group, and let $p = (p_1, \ldots, p_n) \in C$. We define the paths σ_i from $*_H$ to p_i as in the discussion before Definition A.36. Recall the homomorphisms

$$\tau_{C,i} \colon \pi_1(C, p) \to H < G$$

are defined by $\tau_{C,i}([\gamma]) = [i \circ (\sigma_i \cdot \gamma_i \cdot \bar{\sigma}_i)]$, for any loop $\gamma = (\gamma_1, \ldots, \gamma_n)$ based at p in C. According to Lemma A.37, if we let $A_i = \tau_{C,i}(\pi_1(C,p))$, and $g_{i,j} = [i \circ \sigma_i \cdot i \circ \bar{\sigma}_j]$, then

$$A_j^{g_{i,j}} = A_i.$$

In particular, writing $g_i = g_{1,i}$, we have

$$H \cap H^{g_2} \cdots \cap H^{g_n} \supseteq A_1$$

is infinite. To establish the height of H is at least n, we need to show that $(1, g_2, \ldots, g_n)$ are essentially distinct.

Let $\tilde{T}$ be the lift of T to Γ which includes the point $\tilde{*}$, and let $\gamma = (\gamma_1, \ldots, \gamma_i)$ be a loop in C based at p. For each i, the path σ_i has a unique lift to $\tilde{T}$. Let $\tilde{\gamma_i}$ be the unique lift of γ_i starting at the terminus of $\tilde{\sigma}_i$. Then $g_i(\gamma_i) = \gamma_1$.

Since $p \in C$ lies outside the fat diagonal of Z^n, the paths $\gamma_1, \ldots, \gamma_n$ are all distinct. In particular, the lifts $\tilde{\gamma_1}, \ldots, \tilde{\gamma_n}$ are also distinct.

Suppose that $(1, g_2, \ldots, g_n)$ are not essentially distinct. Then we would have (writing $g_1 = 1$) $g_j = g_i h$ for some $1 \leq i < j \leq n$ and some $h \in H$. But then $\tilde{\gamma}_j = h^{-1}\tilde{\gamma}_i$. Projecting back to Z we have $\gamma_j = \gamma_i$, contradicting the fact that γ misses the fat diagonal of Z^n. $\square$

A.4 Height decreases

Again, we have the setup: $(H, \mathcal{D})$ is fully quasiconvex in $(G, \mathcal{P})$. We now assume that $\mathcal{D}$ and $\mathcal{P}$ are nonempty, and that $\Gamma = X(G, \mathcal{P}, S)$ for some finite generating set S for G. The graph Γ is acted on weakly geometrically finitely by $(G, \mathcal{P})$. Moreover given any finite generating set T for H, there is a $\lambda > 0$ and an H–equivariant, proper, λ–Lipschitz map (defined in [1, Section 3])

$$\breve{\imath} : X(H, \mathcal{D}, T) \to \Gamma$$

$X(H, \mathcal{D}, T)$ with λ–quasiconvex image. We briefly recall the definition of this map on the vertex set of $X(H, \mathcal{D}, T)$. Vertices are labeled either by elements

$h \in H$ or by triples (hD, k, n) so that $D \in \mathcal{D}$, $k \in hD$, and $n \in \mathbb{N}$. For $h \in H \subset X(H, \mathcal{D}, T)$, $\breve{\iota}(h) = h$. For each $D \in \mathcal{D}$ choose $c_D \in G$ and $P_D \in \mathcal{P}$ so that $D < c_D P_D c_D^{-1}$. For a vertex of the form (hD, k, n), define $\breve{\iota}(hD, k, n)$ to be the vertex of $X(G, \mathcal{P}, S)$ labeled by $(hc_D P_D, hc_D, n)$.

The quasi-convexity of the image of $\breve{\iota}$ is the *definition* of relative quasi-convexity in [1]. This definition is shown to be equivalent to the usual ones in [36, Appendix A].

Recall that the cusped space is built by attaching combinatorial horoballs to a Cayley graph, so there are canonical inclusions $H \hookrightarrow X(H, \mathcal{D}, T)$ and $G \hookrightarrow \Gamma$. The map $\breve{\iota}$ extends the natural inclusion map of H into G.

Lemma A.41. *There is some N so that the N–neighborhood of the image of $\breve{\iota}$ is a 0–hull for the action of H on Γ.*

Proof. There are four conditions to check. We will prove that each of them hold for any large enough value of N, and then take the maximum of the four lower bounds.

For a number $M \geq 0$ and a subset $A \subset \Gamma$, let $\mathcal{N}_M(A)$ denote the closed M-neighborhood of A in Γ. Let $Y_M = \mathcal{N}_M\left(\breve{\iota}\left(X(H, \mathcal{D}, T)\right)\right)$.

Condition (1): Since $\tilde{*} = 1 \in H \subset G$, we have $\tilde{*} \in Y_M$ for any $M \geq 0$.

Condition (2): Suppose that $\xi_1, \xi_2 \in \Lambda H$, the limit set of H in $\partial\Gamma$, and suppose that l is a geodesic between ξ_1 and ξ_2. To satisfy the second condition of Definition A.32 (with $R = 0$), we need l to be contained in Y_M for large enough M. The points ξ_1 and ξ_2 are limits of elements of H. Since Y_0 is λ-quasi-convex, a geodesic between any two elements of H is contained in Y_λ. It is now straightforward to see that l is contained in $Y_{\lambda+2\delta}$.

Condition (3): Suppose that B is a horoball of Γ whose stabilizer in H is infinite. Algebraically, this gives peripheral subgroups D of H and P of G, and $g \in G$ so that $gDg^{-1} \cap P$ is infinite. The condition from Definition A.31 ensures that $\left[P \colon gDg^{-1}\right] < \infty$. This implies that there is some M_0 so that $B \subset Y_{M_0}$. Since there are only finitely many such D, P and g, up to the action of G, the number M_0 may be taken to work for all such horoballs B.

Condition (4): The final condition from Definition A.32 is that H acts weakly geometrically finitely on Y_N. This is true for any $N > 0$. Note that Y_N is quasi-convex and quasi-isometric to Y_0, the image of $\breve{\iota}$. Because the peripheral subgroups of H are finite index in maximal parabolic subgroups of G, the map $\breve{\iota}$ is a quasi-isometric embedding. (The proof is similar to the proof that a quasiconvex subgroup of a hyperbolic group is quasi-isometrically embedded.) The map $\breve{\iota}$ therefore gives an H-equivariant quasi-isometry between $X(H, \mathcal{D}, T)$ and Y_N. Since $(H, \mathcal{D})$ acts weakly geometrically finitely on $X(H, \mathcal{D}, T)$, it follows from Lemma A.30 that $(H, \mathcal{D})$ acts weakly geometrically finitely on Y_N. □

Definition A.42. Let $\tilde{Z}_0$ be the N–neighborhood of $\mathrm{Im}(\breve{\iota})$ for N sufficiently large that $\tilde{Z}_0$ is a 0–hull. For $R > 0$, let $\tilde{Z}_R$ be the R–neighborhood of $\tilde{Z}_0$. Clearly $\tilde{Z}_R$ is an R–hull. Let Z_R be the quotient of $\tilde{Z}_R$ by the H–action, and let Y be the quotient of Γ by the G–action, as in the previous section.

Theorem A.43. *Let G be hyperbolic, $H < G$ quasiconvex, and let $g \in G \setminus H$. Let $(G, \mathcal{P})$, $(H, \mathcal{D})$ be the relatively hyperbolic structures from Definition A.7. Let $\Gamma = X(G, \mathcal{P}, S)$ be a cusped space for G. Let A be a finite set in G.*

Then for all sufficiently long H–fillings $\phi\colon G \to \bar{G} = G(N_1, \ldots, N_m)$:

1. *If $K = \ker(\phi)$, then $\bar{\Gamma} := \Gamma/K$ is δ'–hyperbolic for some δ' independent of the filling, and is (except for trivial loops) equal to the cusped space for $(\bar{G}, \bar{\mathcal{P}})$. In particular $(\bar{G}, \bar{\mathcal{P}})$ is relatively hyperbolic.*
2. *Let $q\colon \Gamma \to \bar{\Gamma}$ be the quotient by the K–action. For some λ' independent of the filling,* $\mathrm{Image}(q \circ \check{\imath})$ *is λ'–quasiconvex. Thus the induced filling $(\bar{H}, \bar{\mathcal{D}})$ is relatively quasiconvex in $(\bar{G}, \bar{\mathcal{P}})$.*
3. *The induced map $H(K_1, \ldots, K_n) \to G(N_1, \ldots, N_m)$ (described in Remark A.19) is injective.*
4. *$\phi(g) \notin \phi(H)$.*
5. *$\phi|_A$ is injective.*

Proof. We argue the conclusions in order, beginning with (1). It is straightforward to see that Γ/K is almost the cusped space for $(\bar{G}, \bar{\mathcal{P}})$ as advertised, and we leave the details to the reader. The extra loops come from horizontal edges in horoballs between elements in the same K–orbit. Osin in [42] proves that, for all sufficiently long fillings, the constant in the linear relative isoperimetric inequality of $(\bar{G}, \bar{\mathcal{P}}, S)$ is at most three times that of $(G, \mathcal{P}, S)$. In [1, Proposition 2.3], it is explained how to use this bound to obtain a bound on the hyperbolicity constant of $X(\bar{G}, \bar{\mathcal{P}}, S)$. That proposition is stated with the hypothesis that G is torsion-free, but this hypothesis is not really necessary; we outline the argument. It is proved in [1, Lemma 2.2] that the linear isoperimetric constant of the coned-off Cayley complex can be estimated in terms of the linear relative isoperimetric constant for $(\bar{G}, \bar{\mathcal{P}}, S)$. This in turn gives an estimate for the linear isoperimetric constant of the cusped space, with respect to a natural system of 2–cells [21, Theorem 3.24]. From this estimate one obtains an estimate for the hyperbolicity constant δ' as in [10, Theorem III.H.2.9].

In the torsion-free setting, Claims (2)–(4) are the same as [1, Propositions 4.3–4.5]. As explained in [36, Appendix B], the proof of these claims reduces to the extension of certain technical lemmas [1, 4.1 and 4.2] to the case in which G may contain torsion. The necessary modifications to the proof are explained in [36, Appendix B]. These lemmas together imply the following. *Let K_H be the kernel of the filling of H induced by a sufficiently long H–filling of G. Let $h \in H$ and $\bar{h} = \phi(h)$. If $d(1, h) = d(1, K_H h)$, and γ is a geodesic from 1 to*

h, then there is a $10\delta'$–local geodesic (in particular a uniform quasi-geodesic) from 1 to $\bar{h}$ which is contained in a 2–neighborhood of $q(\gamma)$, and coincides with $q(\gamma)$ within $10\delta + 2d(1,g)$ of the Cayley graph of $\bar{G}$.

Given this statement, we sketch the proof of the statement (2) about uniform quasiconvexity; full details are in the proof of [1, Proposition 4.3]. It suffices to consider a geodesic σ from $1 \in \bar{G}$ to $\bar{h} = \phi(h)$ for $h \in H$, and show it lies in a uniform neighborhood of $\mathrm{Image}(q \circ \breve{\imath})$. We may assume that no element h' of $H \cap \phi^{-1}(\bar{h})$ is closer to 1 in Γ than h. Choose a geodesic γ from 1 to h in Γ. Using the italicized statement above, there is a $10\delta'$–local geodesic from 1 to $\bar{h}$ contained in a 2–neighborhood of $q(\gamma)$. Quasi-geodesic stability then tells us that σ is contained in a uniform neighborhood of the image of γ. Since H was relatively quasiconvex before filling, this implies that σ is contained in a uniform neighborhood of $\mathrm{Image}(q \circ \breve{\imath})$.

Next we sketch the proof of statement (3); full details are in the proof of [1, Proposition 4.4]. By way of contradiction, we let $h \in H$ be in the kernel of ϕ, but not in K_H, the kernel of the induced filling of H. We may suppose that $d(1,h) = d(1, K_H h)$. Let γ be a geodesic from 1 to h. The italicized statement implies that there is a $10\delta'$–local geodesic loop in a 2–neighborhood of $q(\gamma)$, and coinciding with $q(\gamma)$ in a $10\delta'$–neighborhood of the Cayley graph. There can be no such loop, and we obtain a contradiction.

Next we sketch statement (4); details can be found in the proof of [1, Proposition 4.5]. If $\phi(g) \in \phi(H)$, we choose $h \in H \cap \phi^{-1}(\phi(g))$ minimizing $d(1,h)$, and let γ be a geodesic from 1 to h. The italicized statement gives us a $10\delta'$–local geodesic from 1 to $\phi(g)$ whose length is at least $20\delta' + 4d(1,g)$. An easy computation gives a lower bound for $d(1,\phi(g)) > \frac{12}{7} d(1,g)$, which is impossible.

Finally, (5) is part of [42, Theorem 1.1]. □

Proposition A.44. For any $R' > 0$, there is an R so that $q(\tilde{Z}_R)$ is an R'–hull for the action of $\bar{H}$ on $\bar{\Gamma}$, for all sufficiently long fillings. (In particular, R does not depend on the choice of long filling.)

Proof. Let γ be a geodesic joining limit points of $\bar{H}$. The λ'–neighborhood of $W := q(\mathrm{Im}(\breve{\imath}))$ contains γ, by quasi-convexity. Thus the R–neighborhood of γ is contained in the $R + \lambda'$–neighborhood of W, hence in the image of $Z_{R+\lambda'}$.

The other conditions follow from the relative quasiconvexity of $\bar{H}$ in $\bar{G}$. □

Let $G \to \bar{G}$ be a sufficiently long filling to satisfy the conclusions of Theorem A.43, so that $\bar{\Gamma}$ is δ'–hyperbolic, $\mathrm{Im}(q \circ \breve{\imath})$ is λ'–quasiconvex, and so on.

Fix R' bigger than the constant $D(\delta')$ from Theorem A.38. Then $q(\tilde{Z}_R)$ detects height in any sufficiently large filling, in a sense which we will describe below.

Lemma A.45. *For all sufficiently long fillings $\phi\colon G \to G(N_1, \ldots, N_m)$, if $K = \ker(\phi)$, $K_H = K \cap H$ and $k \in K \setminus K_H$, then $k\tilde{Z}_R \cap \tilde{Z}_R = \emptyset$.*

Proof. The set $A = \{g \in G \mid g\tilde{Z}_R \cap \tilde{Z}_R \neq \emptyset\}$ is a finite union of left cosets of H,

$$A = \bigsqcup_{i=0}^{l} g_i H,\ g_0 = 1.$$

Applying Theorem A.43 for $g = g_1, \ldots, g = g_l$, we conclude that for all sufficiently long fillings, $\phi(g_i) \notin \phi(H)$ for $i > 0$. Equivalently $g_i h \notin K$ for any $h \in H$, and any $i > 0$. Thus for $k \in K \setminus K_H$, we have $k \notin A$, and so $k\tilde{Z}_R \cap \tilde{Z}_R = \emptyset$. $\square$

Let $\bar{\tilde{Z}}_R$ be the quotient of $\tilde{Z}_R$ by K_H, and let $\bar{Z}_R = Z_R$ be the quotient of $\bar{\tilde{Z}}_R$ by the action of H. By Lemma A.45, $\bar{\tilde{Z}}_R$ embeds in $\bar{\Gamma}$. Now we have a commutative diagram,

$$\begin{tikzcd}
\tilde{Z}_R \arrow[r] \arrow[d] \arrow[loop above, "H"] & \Gamma \arrow[d] \arrow[loop above, "G"] \\
\bar{\tilde{Z}}_R \arrow[r] \arrow[loop left, "H/K_H"] & \bar{\Gamma} \arrow[loop right, "G/K"]
\end{tikzcd}$$

where the horizontal maps are inclusions and the vertical maps are quotients by K_H and K respectively. After taking quotients by the relevant groups we get the diagram,

$$\begin{tikzcd}
Z_R \arrow[r, "i"] \arrow[d] & Y \arrow[d] \\
\bar{Z}_R \arrow[r, "\bar{\imath}"] & \bar{Y}
\end{tikzcd} \tag{4}$$

where the vertical maps are homeomorphisms, and the horizontal maps are immersions inducing the inclusions $H \to G$ and $\bar{H} \to \bar{G}$. Let

$$S_n = \{(z_1, \ldots, z_n) \in Z_R^n \mid i(z_1) = \cdots = i(z_n)\} \setminus \Delta$$

and

$$\bar{S}_n = \{(z_1, \ldots, z_n) \in \bar{Z}_R^n \mid \bar{\imath}(z_1) = \cdots = \bar{\imath}(z_n)\} \setminus \Delta.$$

The maps in diagram (4) induce a bijection of S_n with $\bar{S}_n$. For each $i \in \{1, \ldots, n\}$ and each component C of S_n the projections of Z_R^n to its factors induce maps

$$\tau_{C,i} \colon\ \pi_1(C) \to H,$$

and

$$\bar{\tau}_{C,i} \colon\ \pi_1(C) \to \bar{H}.$$

Since the quotient $Z_R = \tilde{Z}_R/H$ can also be thought of as $\tilde{\bar{Z}}_R/(H/K_H)$, we see that the homomorphisms $\bar{\tau}_{C,i}$ all factor as $\bar{\tau}_{C,i} = \phi|_H \circ \tau_{C,i}$, where ϕ is the filling map.

In particular, if γ is a loop in $\bar{S}_n$ so that $\bar{\tau}_{C,i}([\gamma])$ is infinite for each $i \in \{1, \dots, n\}$ then it must be that $\tau_{C,i}([\gamma])$ is already infinite for each i. Therefore we have the following result.

Corollary A.46. The height of $\bar{H}$ in $\bar{G}$ is at most the height of H in G.

We now specialize to the case that $(H, \mathcal{D}) < (G, \mathcal{P})$ comes from a quasiconvex subgroup H of a hyperbolic group G, so that $\mathcal{D}$ is the malnormal core of H and $\mathcal{P}$ the induced peripheral structure on G.

Theorem A.47. *Assume $\mathcal{P}$ is the peripheral structure induced on G by the quasiconvex subgroup H, and let $G \to G(N_1, \dots, N_m)$ be a sufficiently long H–filling. In case every filling kernel N_i has finite index in P_i, the height of $\bar{H}$ in $\bar{G}$ is strictly less than that of H in G.*

Proof. Suppose that H has height n in G and that, contrary to the conclusion, $\bar{H}$ has height n in $\bar{G}$.

Fix $R' > 0$. By Proposition A.44, there is an R so that for any long enough filling the set $q(\tilde{Z}_R)$ is an R'-hull for the action of $\bar{H}$ in $\bar{G}$. We choose R' large enough so that it satisfies the hypotheses of Theorem A.38. Specifically, we make sure $R' > D(\delta')$ for the universal constant of hyperbolicity δ' from Theorem A.43.

By Theorem A.38, the multiplicity of the map $\bar{\iota}\colon \bar{Z}_R \to \bar{Y}$ is n. Let $\bar{C}$ be a component of $\bar{S}_n$ (with basepoint $\bar{p} = (\bar{p}_1, \dots, \bar{p}_n)$) so that each of the subgroups $\bar{A}_i = \bar{\tau}_{\bar{C},i}\left(\pi_1\left(\bar{C}, \bar{p}\right)\right)$ are infinite.[5] By Lemma A.37, the groups $\bar{A}_i$ are all conjugate in $\bar{H}$, and so there are $\bar{g}_{i,j} \in \bar{G}$ so that $\bar{g}_{i,j}\bar{A}_j\bar{g}_{i,j}^{-1} = \bar{A}_i$. Since G is a hyperbolic group, any infinite subgroup contains an infinite order element, so let $\bar{a} \in \bar{A}_1$ be such an infinite order element, and suppose that $\gamma_{\bar{a}}$ is a loop in $\bar{C}$ based at $\bar{p}$ so that $\bar{\tau}_{C,1}([\gamma_{\bar{a}}]) = \bar{a}$.

Now, consider the diagram (4). The vertical maps are homeomorphisms, and so (as in the above discussion), induce a homeomorphism between $\bar{S}_n$ and S_n. Let C be the component of S_n corresponding to $\bar{C}$, let p be the associated basepoint, and let γ_a be the loop in C associated to $\gamma_{\bar{a}}$. As in the discussion above, the image of each $\tau_{C,i}([\gamma_a])$ is infinite in H. This shows that $a = \tau_{C,1}([\gamma_a])$ is an element of infinite order in the intersection in G of n essentially distinct conjugates of H. Thus a lies in a conjugate of an element of $\mathcal{D}$. But each element of $\mathcal{D}$ has finite image in $\bar{G}$, contradicting the assumption that $\bar{a}$ has infinite order. This completes the proof. □

We now prove Theorem A.22.

[5] We add a bar to our notation in the obvious way in the quotient.

Proof of Theorem A.22. We have G hyperbolic, $H < G$ quasiconvex and height $k \geq 1$. We then have $(H, \mathcal{D})$ relatively quasiconvex in $(G, \mathcal{P})$ where $\mathcal{D}$ is the malnormal core of H, and $\mathcal{P} = \{P_1, \ldots, P_m\}$ is the peripheral structure induced on G.

Let $\phi\colon G \to G(N_1, \ldots, N_m)$ be a sufficiently long H–filling, that the conclusions of Theorem A.43 and Theorem A.47 both hold, and suppose $N_i \dot{\trianglelefteq} P_i$ for each i.

By Theorem A.43, $\bar{G} = G(N_1, \ldots, N_m)$ is hyperbolic relative to $\bar{P} = \{P_1/N_1, \ldots, P_m/N_m\}$, and $(\bar{H}, \bar{\mathcal{D}})$ is relatively quasiconvex in $(\bar{G}, \bar{\mathcal{P}})$. Since all the peripheral subgroups are finite, $\bar{G}$ is hyperbolic, and $\bar{H}$ is a quasiconvex subgroup of $\bar{G}$. By Theorem A.47, the height of $\bar{H}$ in $\bar{G}$ is at most $k - 1$. $\square$

A.5 Proof of main result

In this subsection we prove the main result of this appendix, first recalling the statement:

Theorem A.1. Let G be a hyperbolic group, let $H \leq G$ be a quasi-convex virtually special subgroup. For any $g \in G - H$, there is a hyperbolic group $\mathcal{G}$ and a homomorphism $\phi : G \to \mathcal{G}$ such that $\phi(g) \notin \phi(H)$ and $\phi(H)$ is finite.

Proof. Let $H \leq G$ be quasiconvex and virtually special, and let $g \in G \setminus H$. Let h be the height of H in G. We will induct on the height, noting that the height zero (H finite) case holds trivially.

Let $\mathcal{P} = \{P_1, \ldots, P_m\}$ be the peripheral system associated to $H \leq G$, and $\mathcal{D}$ the peripheral system of H from Definition A.7. By Theorem A.10, there are finite-index subgroups $\dot{D}_j \dot{\trianglelefteq} D_j$ for each $D_j \in \mathcal{D}$ such that for any further finite-index subgroups $D'_j \dot{\leq} \dot{D}_j$, the quotient $H(D'_1, \ldots, D'_n) := H/ \ll \bigcup_j D'_j \gg$ is virtually special.

For each $D_j \in \mathcal{D}$, there is some unique P_{i_j} and some g_j so that

$$g_j^{-1} D_j g_j \dot{\leq} P_{i_j}.$$

The element g_j is not unique, but if g'_j is another such element, then $g_j^{-1} g'_j \in P_{i_j}$. In particular, different G–conjugates of D_j in P_{i_j} are actually P_{i_j}–conjugates, so there are only finitely many of them.

Let $P_i \in \mathcal{P}$, and let

$$\mathcal{S}_i = \{\dot{D}_j^g \mid D_j \in \mathcal{D}, g \in G, D_j^g \dot{\leq} P_i\}.$$

By the way $\mathcal{D}$ and $\mathcal{P}$ are defined, $\mathcal{S}_i$ is never empty. By the argument in the last paragraph, $\mathcal{S}_i$ is a finite collection, so $I_i := \bigcap \mathcal{S}_i \dot{\trianglelefteq} P_i$.

Theorems A.43 and A.22 imply that there is a finite subset $B \subset \bigcup \mathcal{P}$ so that whenever $\phi\colon G \to G(N_1, \ldots, N_m)$ is an H–filling (see Definition A.17) satisfying $(\bigcup N_i) \cap B = \emptyset$ and $N_i \dot{\triangleleft} P_i$, then:

1. The image $\phi(H)$ is quasiconvex of height $< h$ in the hyperbolic group $G(N_1, \ldots, N_m)$. (Theorem A.22)
2. $\phi(H) \cong H(K_1, \ldots, K_n)$, where $H(K_1, \ldots, K_n)$ is the induced filling of H, described in Remark A.19. (Theorem A.43.(3))
3. $\phi(g) \notin \phi(H)$. (Theorem A.43.(4))

Since H is virtually special, it is residually finite. Each P_i is a finite extension of a subgroup of H, so each P_i is residually finite. Hence there are normal subgroups $N_i \dot{\trianglelefteq} P_i$ so that $(\bigcup N_i) \cap B = \emptyset$. These normal subgroups need not define an H–filling, but we can instead consider the subgroups

$$N_i' = N_i \cap I_i.$$

Then $\phi\colon G \to G(N_1', \ldots, N_m')$ is an H–filling inducing a filling $H \to H(K_1, \ldots, K_n)$ satisfying the hypotheses of Theorem A.10. In particular, the image $\bar{H}$ of H in $\bar{G} := G(N_1', \ldots, N_m')$ is virtually special. By Theorem A.22, $\bar{G}$ is hyperbolic and $\bar{H} \leq \bar{G}$ is quasiconvex, of height $< h$. Moreover, Theorem A.43 implies $\phi(g) \notin \phi(H)$.

By induction, there is a quotient $\bar{\phi}\colon \bar{G} \to \mathcal{G}$ so that $\bar{\phi}(\phi(g)) \notin \bar{\phi}(\phi(H))$ and $\bar{\phi}(\phi(H))$ is finite. □

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