Nekovář Duality over p-Adic Lie Extensions of Global Fields

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Abstract. Tate duality is a Pontryagin duality between the ith Galois cohomology group of the absolute Galois group of a local field with coefficents in a finite module and the $(2-i)$th cohomology group of the Tate twist of the Pontryagin dual of the module. Poitou-Tate duality has a similar formulation, but the duality now takes place between Galois cohomology groups of a global field with restricted ramification and compactly-supported cohomology groups. Nekovář proved analogues of these in which the module in question is a finitely generated module T over a complete commutative local Noetherian ring R with a commuting Galois action, or a bounded complex thereof, and the Pontryagin dual is replaced with the Grothendieck dual T^*, which is a bounded complex of the same form. The cochain complexes computing the Galois cohomology groups of T and $T^*(1)$ are then Grothendieck dual to each other in the derived category of finitely generated R-modules. Given a p-adic Lie extension of the ground field, we extend these to dualities between Galois cochain complexes of induced modules of T and $T^*(1)$ in the derived category of finitely generated modules over the possibly noncommutative Iwasawa algebra with R-coefficients.

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1 Introduction

1.1 Duality

In [Nek], Nekovář gave formulations of analogues of Tate and Poitou-Tate duality for finitely generated modules over a complete commutative local Noetherian ring R with finite residue field of characteristic a fixed prime p. In the usual formulation of these dualities, one takes the Pontryagin dual, which does not in general preserve the property of finite generation. Nekovář takes the dual with respect to a dualizing complex of Grothendieck so as to have a duality between bounded complexes of R-modules with finitely generated cohomology groups. This paper is devoted to a generalization of this result to the setting of nonabelian p-adic Lie extensions.

Recall that a dualizing complex ω_R is a bounded complex of R-modules with cohomology finitely generated over R that has the property that for every complex M of finitely generated R-modules, the Grothendieck dual $\mathbf{R}\mathrm{Hom}_R(M, \omega_R)$ in the derived category of R-modules $\mathbf{D}(\mathrm{Mod}_R)$ has finitely generated cohomology, and moreover, the canonical morphism

$$M \longrightarrow \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, \omega_R), \omega_R)$$

is an isomorphism in $\mathbf{D}(\mathrm{Mod}_R)$. Such a complex exists and is unique up to quasi-isomorphism and translation (see [Har1]).

One can choose ω_R to be a bounded complex of injectives, in which case the derived homomorphism complexes are represented by the complexes of homomorphisms themselves. If R is regular, then R itself, as a complex concentrated in degree 0, is a dualizing complex, but R is not in general R-injective. If $R = \mathbb{Z}_p$, for instance, then the complex $[\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p]$ concentrated in degrees 0 and 1 provides a complex of injective $\mathbb{Z}_p$-modules quasi-isomorphic to $\mathbb{Z}_p$.

Let us explain Nekovář's theorem, and our generalization of it, in the setting of Poitou-Tate duality. Let F be a global field with characteristic not equal to p. Let S be a finite set of primes of F that, if F is a number field, contains all primes over p and any real places, and let $G_{F,S}$ denote the Galois group of the maximal unramified outside S extension of F. We remark that a finitely generated R-module has a canonical topology arising from its filtration by powers of the maximal ideal of R. Let T be a bounded complex in the category of finitely generated (topological) R-modules with R-linear continuous $G_{F,S}$-actions. We use $\mathbf{R}\Gamma(G_{F,S}, T)$ to denote the object in $\mathbf{D}(\mathrm{Mod}_R)$ corresponding to the complex of continuous $G_{F,S}$-cochains with coefficients in T, and we use $\mathbf{R}\Gamma_{(c)}(G_{F,S}, T)$ to denote the derived object attached to the complex of continuous compactly supported cochains (using Tate groups for real places in its definition as a cone: see Section 4.1).

There exists a bounded complex T^* of finitely generated R-modules with R-linear continuous $G_{F,S}$-actions that represents $\mathbf{R}\mathrm{Hom}_R(T, \omega_R)$ in the derived category of such modules. Nekovář's duality theorem is then as follows (see

[Nek, Proposition 5.4.3(ii)]).[1]

Theorem (Nekovář). *We have an isomorphism*

$$\mathbf{R}\Gamma(G_{F,S},T) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_R\Big(\mathbf{R}\Gamma_{(c)}(G_{F,S},T^*(1)),\omega_R\Big)[-3]$$

in the derived category of finitely generated R-modules.

In this paper, we consider a generalization of this to the setting of noncommutative Iwasawa theory. Suppose that F_∞ is a p-adic Lie extension of F contained in F_S. We denote by Γ the Galois group of the extension F_∞/F, and we let $\Lambda = R[\![\Gamma]\!]$ denote the resulting Iwasawa algebra over R. For a finitely generated R-module T with a continuous R-linear $G_{F,S}$-action, we define a finitely generated Λ-module $\mathscr{F}_\Gamma(T)$ with a continuous Λ-linear $G_{F,S}$-action by

$$\mathscr{F}_\Gamma(T) = \Lambda^\iota \otimes_R T,$$

where the superscript ι denotes that an element of Γ in Λ acts here on Λ by right multiplication by its inverse, and where $G_{F,S}$ acts on Λ through the quotient map $G_{F,S} \to \Gamma$ by left multiplication and then diagonally on the tensor product.

The $G_{F,S}$-cohomology of $\mathscr{F}_\Gamma(T)$ is interesting in that a version of Shapiro's lemma provides natural isomorphisms of continuous cohomology groups

$$H^n(G_{F,S},\mathscr{F}_\Gamma(T)) \cong \varprojlim_\alpha H^n(G_{F_\alpha,S},T)$$

for every $n \geq 0$, where the limit is taken over α indexing the finite Galois extensions F_α of F that are contained in F_∞ (cf. [Lim2, Lemma 5.3.1]). That is, the cohomology groups of $\mathscr{F}_\Gamma(T)$ are the Iwasawa cohomology groups of the module T itself for the extension F_∞/F, and this identification is one of Λ-modules. We also have the analogous statements for compactly supported cohomology, as seen in [Lim2, Proposition 5.3.3]. Therefore, we can reduce the question of finding dualities among Iwasawa cohomology groups of compact modules to that of obtaining dualities among cohomology groups for $G_{F,S}$ itself. Note that, however, it is not even a priori clear in this setting that $H^n(G_{F,S},\mathscr{F}_\Gamma(T))$ is a finitely generated Λ-module, let alone that we can find such a duality of Λ-modules.

The following is our main theorem (cf. Theorem 4.5.1).

Theorem. *We have isomorphisms*

$$\mathbf{R}\Gamma(G_{F,S},\mathscr{F}_\Gamma(T)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma_{(c)}(G_{F,S},\mathscr{F}_\Gamma(T^*)^\iota(1)),\Lambda\otimes^{\mathbf{L}}_R\omega_R\Big)[-3]$$

$$\mathbf{R}\Gamma_{(c)}(G_{F,S},\mathscr{F}_\Gamma(T)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_{F,S},\mathscr{F}_\Gamma(T^*)^\iota(1)),\Lambda\otimes^{\mathbf{L}}_R\omega_R\Big)[-3]$$

in the derived category of finitely generated Λ-modules.

[1] Actually, Nekovář's longer treatise is particularly concerned with a generalization of this duality that takes place between Selmer complexes, which we do not address in this article.

Nekovář proves the above theorem in the case that Γ is abelian [Nek, Theorem 8.5.6]. In fact, in that case, it is an almost immediate consequence of his above-mentioned theorem. That is, suppose for instance that $\Gamma \cong \mathbb{Z}_p^r$ for some $r \geq 1$. Then Λ is a complete commutative Noetherian local ring with finite residue field of characteristic p. Moreover, its dualizing complex is isomorphic to $\Lambda \otimes_R^{\mathbf{L}} \omega_R$ in the derived category of Λ-modules [Nek, Lemma 8.4.5.6]. Therefore, the commutative theory described above applies to Λ, and Nekovář is able to deduce the result from this. On the other hand, since we are working with nonabelian Γ and hence noncommutative Λ, we do not know that there exists a (nice enough) dualizing complex, and so the proof of our main theorem takes a different route. The idea is a simple one, though the proof is rather involved: after reducing to the case that R is regular and $\mathbb{Z}_p$-flat, we perform an inductive argument, using the grading on Λ arising from the powers of its augmentation ideal, to deduce our result from Nekovář's.

We remark that, in their manuscript on the noncommutative main conjecture, Fukaya and Kato stated an analogue of our main theorem, with Λ replacing $\Lambda \otimes_R^{\mathbf{L}} \omega_R$ [FK1, (1.6.12)], which in turn generalized a result of Burns and Flach [BF, Lemma 12(b)] for a narrower class of rings. The result of Fukaya-Kato applies to a more general class of (adic) rings Λ than ours and replaces $\mathscr{F}_\Gamma(T)$ and $\mathscr{F}_\Gamma(T^*)^\iota$ by a bounded complex of $\Lambda[G_{F,S}]$-modules X and its Λ-dual $\mathrm{Hom}_\Lambda(X, \Lambda)$. However, in order to be able to work with the Λ-dual, they assume that X consists of (finitely generated) projective Λ-modules. In the case of Iwasawa cohomology that we study, the complex T need not be quasi-isomorphic to a bounded complex of $R[G_{F,S}]$-modules that are projective and finitely generated over R. Moreover, if R is Gorenstein, then R serves as an R-dualizing complex, and our result reduces to a duality with respect to Λ itself, as in the result of Fukaya-Kato. We also note that Vauclair proved a noncommutative duality theorem for induced modules in the case that $R = \mathbb{Z}_p$ and T is $\mathbb{Z}_p$-free, via a rather different method [Vau, Theorem 6.4].

1.2 An application

Since applications of our main result are not discussed in the body of this paper, we end the introduction with an indication of one setting in which our results naturally apply. Fix $N \geq 1$ not divisible by p, and suppose that $p \geq 5$. Let $\mathbb{Z}_{p,N}$ denote the inverse limit of the rings $\mathbb{Z}/Np^r\mathbb{Z}$ over $r \geq 1$. Hida's ordinary cuspidal $\mathbb{Z}_p$-Hecke algebra $\mathfrak{h}$ of level Np^∞ is a direct product of local rings that is free of finite rank over the subalgebra $\Omega = \mathbb{Z}_p[\![1 + p\mathbb{Z}_p]\!]$ of the algebra $\mathbb{Z}_p[\![\mathbb{Z}_{p,N}^\times/\langle -1 \rangle]\!]$ of diamond operators in $\mathfrak{h}$ [Hid, Theorem 3.1]. Hida showed that the $\mathfrak{h}$-module $\mathcal{S}$ of Ω-adic cusp forms is Ω-dual to $\mathfrak{h}$ (see [Oht1, Theorem 2.5.3] for a proof), from which it follows that $\mathcal{S}$ is a dualizing complex for $\mathfrak{h}$.

The inverse limit

$$\mathscr{H} = \varprojlim_r H^1_{\text{ét}}(X_1(Np^r)_{/\bar{\mathbb{Q}}}, \mathbb{Z}_p(1))^{\mathrm{ord}}$$

of ordinary parts of étale cohomology groups of modular curves is an $\mathfrak{h}[G_{\mathbb{Q},S}]$-module for the dual action of $\mathfrak{h}$, where S is the set of primes dividing $Np\infty$. As an $\mathfrak{h}$-module, Ohta showed in [Oht2] that $\mathscr{H}$ is an extension of $\mathcal{S}$ by $\mathfrak{h}$. Since $\mathfrak{h}$ is not always Gorenstein (in Eisenstein components: see [Oht3, Corollary 4.2.13] for conditions), it is not at all clear that $\mathscr{H}$ is quasi-isomorphic to a bounded complex of $\mathfrak{h}[G_{\mathbb{Q},S}]$-modules that are finitely generated and projective over $\mathfrak{h}$, precluding the use of the duality result of Fukaya-Kato. On the other hand, there is a perfect, $\mathfrak{h}$-bilinear pairing $\mathscr{H} \times \mathscr{H} \to \mathcal{S}(1)$ which is $G_{\mathbb{Q},S}$-equivariant for the action of $\sigma \in G_{\mathbb{Q},S}$ on $\mathcal{S}$ via the diamond operator $\langle \chi(\sigma) \rangle$, where $\chi\colon G_{\mathbb{Q},S} \to \mathbb{Z}_{p,N}^{\times}$ is the cyclotomic character (see [FK2, Section 1.6], where it is quickly derived from a pairing of Ohta's [Oht1, Definition 4.1.17]). Now fix a p-adic Lie extension F_∞ of $\mathbb{Q}$ that is unramified outside S and contains $\mathbb{Q}(\mu_{Np^\infty})$, and set $\Gamma = \mathrm{Gal}(F_\infty/\mathbb{Q})$. The complex $\mathbf{R}\Gamma_{(c)}(G_{\mathbb{Q},S}, \mathscr{F}_\Gamma(\mathscr{H}))$ is the subject of the noncommutative Tamagawa number conjecture of Fukaya and Kato for $\mathscr{H}$ (though to be precise, said conjecture is only formulated in the case that $\mathscr{H}$ is $\mathfrak{h}$-projective), which is directly related to the noncommutative main conjecture for ordinary Ω-adic cusp forms (see [FK1, Sections 2.3 and 4.2]). It is also perhaps worth remarking that, in the commutative setting, the first Iwasawa cohomology group $H^1(G_{\mathbb{Q},S}, \mathscr{F}_\Gamma(\mathscr{H}))$ contains zeta elements constructed out of Kato's Euler system (see [Kat, Section 12] or [FK2, Section 3.2]).

For a finitely generated $\Lambda = \mathfrak{h}[\![\Gamma]\!]$-module A, let us use $A\langle \chi \rangle$ to denote the Λ-module that is A as an $\mathfrak{h}$-module but for which the original $\mathfrak{h}$-linear action of $\gamma \in \Gamma$ on A has been twisted by multiplication by the diamond operator $\langle \chi(\tilde{\gamma}) \rangle$, for any lift $\tilde{\gamma} \in G_{\mathbb{Q},S}$ of γ. Using the pairing on $\mathscr{H}$, our main result can be seen to yield two interesting isomorphisms in the derived category of finitely generated modules over, including

$$\mathbf{R}\Gamma_{(c)}(G_{\mathbb{Q},S}, \mathscr{F}_\Gamma(\mathscr{H})) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma(G_{\mathbb{Q},S}, \mathscr{F}_\Gamma(\mathscr{H})^\iota), \Lambda \otimes_{\mathfrak{h}}^{\mathbf{L}} \mathcal{S})\langle \chi \rangle[-3].$$

Perhaps more concretely, we have a spectral sequence

$$\mathrm{Ext}^i_{\Lambda^\circ}\left(H^{3-j}(G_{\mathbb{Q},S}, \mathscr{F}_\Gamma(\mathscr{H})^\iota), \Lambda \otimes_{\mathfrak{h}} \mathcal{S}\right)\langle \chi \rangle \Longrightarrow H^{i+j}_{(c)}(G_{\mathbb{Q},S}, \mathscr{F}_\Gamma(\mathscr{H})).$$

We remark that one has a similar result with $X_1(Np^r)$ replaced by $Y_1(Np^r)$: in this case, the pairing on $\mathscr{H}$ to $\mathcal{S}$ is replaced by a pairing between ordinary parts of cohomology and compactly supported cohomology groups to ordinary Ω-adic modular forms. We also note that Fouquet has constructed an analogue of Ohta's pairing for towers of Shimura curves attached to indefinite quaternion algebras over totally real fields [Fou, Proposition 2.8], providing a related setting for an application of our main result.

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2 Preliminaries

In this section, we lay out some facts regarding modules over profinite rings that we shall require later in the paper. Elsewhere in the paper, we use some of the standard terminology and conventions regarding complexes, cones, shifts, total complexes of homomorphisms and tensor products, and derived categories, as can be found in [Nek], or in [Lim2] off of which this paper builds. In particular, all sign conventions will be as in the latter two papers. For further background on derived categories, we suggest the excellent book [Wei], as well as the more advanced text [KS].

2.1 Tensor products and homomorphism groups

In this subsection, R is a commutative ring. By an R-algebra, we will mean a ring with a given homomorphism from R to its center. In this section, we construct derived bifunctors of homomorphism groups and tensor products for bimodules over R-algebras and study a few isomorphisms that result from these constructions.

We use Λ° to denote the opposite ring to an R-algebra Λ. Given two R-algebras Λ and Σ, we are interested in a subclass of the class of Λ-Σ-bimodules, namely, the class of Λ-Σ-bimodules with the extra property that the left R-action coincides with the right R-action. We can (and shall) identify the category of such Λ-Σ-bimodules with the category of $\Lambda \otimes_R \Sigma^\circ$-modules, and there are natural exact forgetful functors

$$\mathrm{res}_\Lambda \colon \mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ} \longrightarrow \mathrm{Mod}_\Lambda \quad \text{and} \quad \mathrm{res}_{\Sigma^\circ} \colon \mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ} \longrightarrow \mathrm{Mod}_{\Sigma^\circ},$$

which extend to exact functors on the derived categories. One observes that the categories $\mathrm{Mod}_{\Lambda\otimes_R R^\circ}$ and Mod_Λ are equivalent, as are $\mathrm{Mod}_{R\otimes_R\Sigma^\circ}$ and $\mathrm{Mod}_{\Sigma^\circ}$. We have following lemma, which extends [Yek, Lemma 2.1].

Lemma 2.1.1.

(a) *If Σ is a projective (resp., flat) R-algebra, then* res_Λ *preserves projective (resp., flat) modules. Similarly, if Λ is a projective (resp., flat) R-algebra, then* $\mathrm{res}_{\Sigma^\circ}$ *preserves projective (resp., flat) modules.*

(b) *If Σ is a flat R-algebra, then* res_Λ *preserves injectives. Similarly, if Λ is a flat R-algebra, then* $\mathrm{res}_{\Sigma^\circ}$ *preserves injectives.*

Proof. (a) We prove the statements of the first sentence, those of the second being a consequence. Suppose that Σ is a projective R-algebra. Since projective modules are the summands of free modules, it suffices to show that $\Lambda \otimes_R \Sigma^\circ$ is a projective Λ-module. Since Σ is a central R-algebra, we have $\Sigma \cong \Sigma^\circ$ as R-modules. Therefore, we have $\Sigma^\circ \oplus P \cong L$ for some projective R-module P and free R-module L. Then $\Lambda \otimes_R \Sigma^\circ$ is a direct summand of $\Lambda \otimes_R L$, which is a free Λ-module. Hence, $\Lambda \otimes_R \Sigma^\circ$ is a projective Λ-module.
Now suppose that Σ is flat over R. Since flat modules are direct limits of finitely generated free modules (see [Lam, Theorem 4.34]) and tensor products preserve direct limits, it suffices to show that $\Lambda \otimes_R \Sigma^\circ$ is a flat Λ-algebra. Since Σ is flat over R, we have that Σ is a direct limit of finitely generated free R-modules, which implies that $\Lambda \otimes_R \Sigma^\circ$ is a direct limit of finitely generated free Λ-modules.

(b) We shall prove this for res_Λ, the case of $\mathrm{res}_{\Sigma^\circ}$ being a consequence. The functor from Mod_Λ to $\mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ}$ sending M to $M \otimes_R \Sigma^\circ$ is exact by our assumption and is left adjoint to the functor res_Λ. The conclusion then follows from [Wei, Prop. 2.3.10]. $\square$

Let us briefly introduce the notions of q-projective and q-injective complexes of Λ-modules and several facts regarding them arising from the work of Spaltenstein, as can be found in [Spa], [Kel], [Lip, Section 2.3], and [KS, Chapter 14]. A complex P of Λ-modules is called q-projective if for every map $g\colon P \to B$ and quasi-isomorphism $s\colon A \to B$ of complexes of Λ-modules, there exists a map $f\colon P \to A$ of complexes of Λ-modules such that g and $s \circ f$ are homotopy equivalent. We remark that P is q-projective if and only if it is homotopy equivalent to a direct limit of bounded above complexes P_n of projective Λ-modules via maps $\iota_n\colon P_n \to P_{n+1}$ that are split injective in each degree with quotients $P_{n+1}/\iota_n(P_n)$ having zero differentials. In particular, bounded above complexes of projectives are q-projective. If A is a complex of Λ-modules, then there exists a quasi-isomorphism $P \to A$ with P q-projective.
We also have the dual notion of q-injective complexes. A complex I of Λ-modules is q-injective if for every map $f\colon A \to I$ and quasi-isomorphism $s\colon A \to B$ of complexes of Λ-modules, there exists a map $g\colon B \to I$ such that f and $g\circ s$ are homotopy equivalent. A complex I of Λ-modules is q-injective if and only if it is homotopy equivalent to an inverse limit of bounded below complexes I_n of injective Λ-modules via maps $\pi_n\colon I_{n+1} \to I_n$ that are split surjective in each degree with kernels $\ker \pi_n$ having zero differentials. In particular, bounded below complexes of injective Λ-modules are q-injective. If A is a complex of Λ-modules, then there exists a quasi-isomorphism $A \to I$ with I q-injective.
In addition to Λ and Σ, we now let Ω be an R-algebra. If A is a $\Lambda \otimes_R \Omega^\circ$-module and B is a $\Lambda \otimes_R \Sigma^\circ$-module, we give $\mathrm{Hom}_\Lambda(A, B)$ the structure of an $\Omega \otimes_R \Sigma^\circ$-module via the left Ω and right Σ-actions

$$(\omega \cdot f)(a) = f(a\omega) \quad \text{and} \quad (f \cdot \sigma)(a) = f(a)\sigma$$

for $f \in \operatorname{Hom}_\Lambda(A,B)$, $a \in A$, $\omega \in \Omega$, and $\sigma \in \Sigma$.

Moreover, if A is a complex of $\Lambda \otimes_R \Omega^\circ$-modules and B is a complex of $\Lambda \otimes_R \Sigma^\circ$-modules, we define a complex $\operatorname{Hom}_\Lambda(A,B)$ of $\Omega \otimes_R \Sigma^\circ$-modules by

$$\operatorname{Hom}^n_\Lambda(A,B) = \prod_{i \in \mathbb{Z}} \operatorname{Hom}_\Lambda(A^i, B^{i+n}),$$

with the usual differentials, as in [Lim2, Section 2].

Proposition 2.1.2. *Let A be a complex of $\Lambda \otimes_R \Omega^\circ$-modules, and let B be a complex of $\Lambda \otimes_R \Sigma^\circ$-modules. If Ω is a projective R-algebra or Σ is a flat R-algebra, then we have a derived bifunctor*

$$\mathbf{R}\operatorname{Hom}_\Lambda(-,-)\colon \mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Omega^\circ})^\circ \times \mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ}) \longrightarrow \mathbf{D}(\mathrm{Mod}_{\Omega\otimes_R\Sigma^\circ}),$$

and $\operatorname{Hom}_\Lambda(A,B)$ represents $\mathbf{R}\operatorname{Hom}_\Lambda(A,B)$ if A is q-projective as a complex of Λ-modules and Ω is R-projective or B is q-injective as a complex of Λ-modules and Σ is R-flat.

Proof. Let us first assume that Ω is a projective R-algebra. Since every complex of $\Lambda \otimes_R \Omega^\circ$-modules is quasi-isomorphic to a q-projective complex of $\Lambda \otimes_R \Omega^\circ$-modules, we have a derived functor

$$\mathbf{R}\operatorname{Hom}_\Lambda(-,B)\colon \mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Omega^\circ})^\circ \longrightarrow \mathbf{D}(\mathrm{Mod}_{\Omega\otimes_R\Sigma^\circ})$$

(see [Lip, Corollary 2.3.2.3]). Suppose that $f\colon B \to B'$ is a quasi-isomorphism of complexes of $\Lambda \otimes_R \Sigma^\circ$-modules. Let $\varepsilon\colon P \to A$ be a quasi-isomorphism of complexes of $\Lambda \otimes_R \Omega^\circ$-modules, where P is q-projective. Then $\mathbf{R}\operatorname{Hom}_\Lambda(P,B)$ (resp., $\mathbf{R}\operatorname{Hom}_\Lambda(P,B')$) is represented by $\operatorname{Hom}_\Lambda(P,B)$ (resp., $\operatorname{Hom}_\Lambda(P,B')$). Applying Lemma 2.1.1(a), we see that P is also a q-projective complex of Λ-modules, so by [Lip, Proposition 2.3.8], the induced map

$$f_*\colon \operatorname{Hom}_\Lambda(P,B) \longrightarrow \operatorname{Hom}_\Lambda(P,B')$$

is a quasi-isomorphism of complexes of abelian groups. Since f_* is a morphism of complexes of $\Omega \otimes_R \Sigma^\circ$-modules, it is a quasi-isomorphism of such complexes. Hence, f induces isomorphisms

$$\mathbf{R}\operatorname{Hom}_\Lambda(A,B) \longrightarrow \mathbf{R}\operatorname{Hom}_\Lambda(A,B'),$$

proving the existence of the derived bifunctor. Moreover, if A is q-projective as a complex of Λ-modules, then

$$\varepsilon_*\colon \operatorname{Hom}_\Lambda(A,B) \longrightarrow \operatorname{Hom}_\Lambda(P,B)$$

is a quasi-isomorphism of complexes of Σ°-modules, hence of $\Omega \otimes_R \Sigma^\circ$-modules as well. Thus, $\operatorname{Hom}_\Lambda(A,B)$ represents $\mathbf{R}\operatorname{Hom}_\Lambda(A,B)$, as desired.

If Σ is R-flat, the argument for the existence of the derived bifunctor and its computation by $\operatorname{Hom}_\Lambda(A,B)$ in the case that B is q-injective as a complex of

Λ-modules is the direct analogue of the above argument, employing Lemma 2.1.1(b). To see that the resulting derived functor agrees with that constructed above in the case that both Ω is R-projective and Σ is R-flat, consider a quasi-isomorphism $\varepsilon\colon P \to A$ as above and a quasi-isomorphism $\delta\colon B \to I$ with I a q-injective complex of $\Lambda \otimes_R \Sigma^\circ$-modules. As we have quasi-isomorphisms

$$\mathrm{Hom}_\Lambda(P,B) \xrightarrow{\delta_*} \mathrm{Hom}_\Lambda(P,I) \xleftarrow{\varepsilon_*} \mathrm{Hom}_\Lambda(A,I)$$

of complexes of $\Omega \otimes_R \Sigma^\circ$-modules, the derived functors coincide. □

Note that one always has a map $\mathrm{Hom}_\Lambda(A,B) \to \mathbf{R}\mathrm{Hom}_\Lambda(A,B)$ (canonical up to isomorphism in the derived category), induced either by a quasi-isomorphism $P \to A$ with P q-projective or a quasi-isomorphism $B \to I$ with I q-injective.

Corollary 2.1.3. *If Ω is a projective R-algebra, then we have a commutative diagram*

$$\begin{tikzcd}
\mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Omega^\circ})^\circ \times \mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ}) \arrow[r, "\mathbf{R}\mathrm{Hom}_\Lambda(-,-)"] \arrow[d, "\mathrm{res}_\Lambda \times \mathrm{id}"'] & \mathbf{D}(\mathrm{Mod}_{\Omega\otimes_R\Sigma^\circ}) \arrow[d, "\mathrm{res}_{\Sigma^\circ}"] \\
\mathbf{D}(\mathrm{Mod}_\Lambda)^\circ \times \mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ}) \arrow[r, "\mathbf{R}\mathrm{Hom}_\Lambda(-,-)"] & \mathbf{D}(\mathrm{Mod}_{\Sigma^\circ}).
\end{tikzcd}$$

In the case that Σ is a flat R-algebra, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Omega^\circ})^\circ \times \mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ}) \arrow[r, "\mathbf{R}\mathrm{Hom}_\Lambda(-,-)"] \arrow[d, "\mathrm{id}\times\mathrm{res}_\Lambda"'] & \mathbf{D}(\mathrm{Mod}_{\Omega\otimes_R\Sigma^\circ}) \arrow[d, "\mathrm{res}_{\Omega}"] \\
\mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Omega^\circ})^\circ \times \mathbf{D}(\mathrm{Mod}_\Lambda) \arrow[r, "\mathbf{R}\mathrm{Hom}_\Lambda(-,-)"] & \mathbf{D}(\mathrm{Mod}_{\Omega}).
\end{tikzcd}$$

If A is a complex of $\Omega \otimes_R \Lambda^\circ$-modules and B is a complex of $\Lambda \otimes_R \Sigma^\circ$-modules, we define a complex $A \otimes_\Lambda B$ of $\Omega \otimes_R \Sigma^\circ$-modules by

$$(A \otimes_\Lambda B)^n = \bigoplus_{i\in\mathbb{Z}} A^i \otimes_\Lambda B^{n-i},$$

again with the usual differentials, as in [Lim2, Section 2].

We also have a notion of a q-flat complex of Λ°-modules (see [Lip, Section 2.5] and [Spa, Section 5] in the case of commutative rings, the proofs being identical); that is, a complex of Λ°-modules A is said to be q-flat if for every quasi-isomorphism $B \to C$ of complexes of Λ-modules, the resulting map $A \otimes_\Lambda B \to A \otimes_\Lambda C$ is also a quasi-isomorphism. In particular, q-projective complexes of Λ°-modules are q-flat, any bounded above complex of flat Λ°-modules is q-flat, and any filtered direct limit of q-flat complexes is q-flat. As in the case of homomorphism complexes, the total tensor product induces derived bifunctors as follows. We omit the analogous proof.

Proposition 2.1.4. *Let A and B be complexes of $\Omega \otimes_R \Lambda^\circ$-modules and $\Lambda \otimes_R \Sigma^\circ$-modules, respectively. If either Ω or Σ is a flat R-algebra, then we have a derived bifunctor*

$$- \otimes_\Lambda^{\mathbf{L}} - \colon \mathbf{D}(\mathrm{Mod}_{\Omega \otimes_R \Lambda^\circ}) \times \mathbf{D}(\mathrm{Mod}_{\Lambda \otimes_R \Sigma^\circ}) \longrightarrow \mathbf{D}(\mathrm{Mod}_{\Omega \otimes_R \Sigma^\circ}),$$

and $A \otimes_\Lambda B$ represents $A \otimes_\Lambda^{\mathbf{L}} B$ if A is q-flat as a complex of Λ°-modules and Ω is R-flat or if B is q-flat as a complex of Λ-modules and Σ is R-flat.

Proposition 2.1.4 has the following direct corollary.

Corollary 2.1.5. *If Ω is a flat R-algebra, then we have a commutative diagram*

$$\begin{tikzcd}
\mathbf{D}(\mathrm{Mod}_{\Omega \otimes_R \Lambda^\circ}) \times \mathbf{D}(\mathrm{Mod}_{\Lambda \otimes_R \Sigma^\circ}) \arrow[r, "- \otimes_\Lambda^{\mathbf{L}} -"] \arrow[d, "\mathrm{res}_{\Lambda^\circ} \times \mathrm{id}"'] & \mathbf{D}(\mathrm{Mod}_{\Omega \otimes_R \Sigma^\circ}) \arrow[d, "\mathrm{res}_{\Sigma^\circ}"] \\
\mathbf{D}(\mathrm{Mod}_{\Lambda^\circ}) \times \mathbf{D}(\mathrm{Mod}_{\Lambda \otimes_R \Sigma^\circ}) \arrow[r, "- \otimes_\Lambda^{\mathbf{L}} -"] & \mathbf{D}(\mathrm{Mod}_{\Sigma^\circ}),
\end{tikzcd}$$

and if Σ is a flat R-algebra, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{D}(\mathrm{Mod}_{\Omega \otimes_R \Lambda^\circ}) \times \mathbf{D}(\mathrm{Mod}_{\Lambda \otimes_R \Sigma^\circ}) \arrow[r, "- \otimes_\Lambda^{\mathbf{L}} -"] \arrow[d, "\mathrm{id} \times \mathrm{res}_{\Lambda}"'] & \mathbf{D}(\mathrm{Mod}_{\Omega \otimes_R \Sigma^\circ}) \arrow[d, "\mathrm{res}_{\Omega}"] \\
\mathbf{D}(\mathrm{Mod}_{\Omega \otimes_R \Lambda^\circ}) \times \mathbf{D}(\mathrm{Mod}_{\Lambda}) \arrow[r, "- \otimes_\Lambda^{\mathbf{L}} -"] & \mathbf{D}(\mathrm{Mod}_{\Omega}).
\end{tikzcd}$$

We end this section with some general lemmas regarding the passing of tensor products through homomorphism groups and the resulting isomorphisms in the derived categories. Let us use $\bar{m}$ to denote the degree of an element m of a term of a complex M of modules over a ring. We fix a fourth R-algebra Ξ.

Lemma 2.1.6. *Suppose that Ξ is a flat R-algebra and Σ is a projective R-algebra. Let A be a complex of $\Omega \otimes_R \Lambda^\circ$-modules, let B be a complex of $\Lambda \otimes_R \Sigma^\circ$-modules, and let C be a complex of $\Omega \otimes_R \Xi^\circ$-modules. Fix a quasi-isomorphism $Q \to A$ of complexes of $\Omega \otimes_R \Lambda^\circ$-modules with Q q-flat over Λ°, a quasi-isomorphism $P \to B$ of complexes of $\Lambda \otimes_R \Sigma^\circ$-modules with P q-projective over Λ, and a quasi-isomorphism $C \to I$ of complexes of $\Omega \otimes_R \Xi^\circ$-modules with I q-injective over Ω. Then the adjunction isomorphism*

$$\begin{gathered}
\mathrm{Hom}_\Omega(A \otimes_\Lambda P, I) \longrightarrow \mathrm{Hom}_\Lambda(P, \mathrm{Hom}_\Omega(A, I)) \\
f \mapsto (p \mapsto (-1)^{\bar{a}\bar{p}} f(a \otimes p))
\end{gathered}$$

induces an isomorphism

$$\mathbf{R}\mathrm{Hom}_\Omega(A \otimes_\Lambda^{\mathbf{L}} B, C) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_\Lambda(B, \mathbf{R}\mathrm{Hom}_\Omega(A, C))$$

in $\mathbf{D}(\mathrm{Mod}_{\Sigma \otimes_R \Xi^\circ})$, as does the adjunction isomorphism

$$\mathrm{Hom}_\Omega(Q \otimes_\Lambda B, I) \longrightarrow \mathrm{Hom}_\Lambda(B, \mathrm{Hom}_\Omega(Q, I)).$$

Proof. By Propositions 2.1.2 and 2.1.4, the complex $\mathrm{Hom}_\Omega(A \otimes_\Lambda P, I)$ represents $\mathbf{R}\mathrm{Hom}_\Omega(A \otimes^{\mathbf{L}}_\Lambda B, C)$, and the complex $\mathrm{Hom}_\Lambda(P, \mathrm{Hom}_\Omega(A, I))$ represents $\mathbf{R}\mathrm{Hom}_\Lambda(B, \mathbf{R}\mathrm{Hom}_\Omega(A, C))$. Therefore, we are reduced for the first part to showing that the adjunction map is an isomorphism of complexes, and this is standard (see [Lim2, Lemma 2.2]). The second part follows easily from the first part and the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_\Omega(Q \otimes_\Lambda B, I) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}_\Lambda(B, \mathrm{Hom}_\Omega(Q, I)) \arrow[d] \\
\mathrm{Hom}_\Omega(Q \otimes_\Lambda P, I) \arrow[r, "\sim"] & \mathrm{Hom}_\Lambda(P, \mathrm{Hom}_\Omega(Q, I)),
\end{tikzcd}$$

in that the left-hand vertical map is a quasi-isomorphism. □

We consider the next isomorphism first on the level of complexes.

Lemma 2.1.7. *Let A be a bounded complex of $\Omega \otimes_R \Sigma^\circ$-modules that are flat as Σ°-modules, let B be a complex of $\Xi \otimes_R \Lambda^\circ$-modules that are finitely presented as Λ°-modules, and let C be a complex of $\Sigma \otimes_R \Lambda^\circ$-modules. Suppose also that either the terms of A are finitely presented as Σ°-modules or at least one of B and C is bounded above and at least one is bounded below. Then the map*

$$\begin{gathered}
A \otimes_\Sigma \mathrm{Hom}_{\Lambda^\circ}(B, C) \longrightarrow \mathrm{Hom}_{\Lambda^\circ}(B, A \otimes_\Sigma C) \\
a \otimes f \mapsto \big(b \mapsto a \otimes f(b)\big)
\end{gathered}$$

is an isomorphism of complexes of $\Omega \otimes_R \Xi^\circ$-modules.

Proof. That the stated map is a map of complexes is an easy check of the actions and signs. To see that it is an isomorphism, consider first the case that A, B, and C are concentrated in degree 0 and take a presentation

$$F_1 \longrightarrow F_2 \longrightarrow B \longrightarrow 0$$

with F_1 and F_2 finitely generated free Λ°-modules. Then the two rightmost vertical arrows in the commutative diagram with exact (noting the Σ°-flatness of A) rows

$$\begin{tikzcd}
0 \arrow[r] & A \otimes_\Sigma \mathrm{Hom}_{\Lambda^\circ}(B, C) \arrow[r] \arrow[d] & A \otimes_\Sigma \mathrm{Hom}_{\Lambda^\circ}(F_2, C) \arrow[r] \arrow[d] & A \otimes_\Sigma \mathrm{Hom}_{\Lambda^\circ}(F_1, C) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_{\Lambda^\circ}(B, A \otimes_\Sigma C) \arrow[r] & \mathrm{Hom}_{\Lambda^\circ}(F_2, A \otimes_\Sigma C) \arrow[r] & \mathrm{Hom}_{\Lambda^\circ}(F_1, A \otimes_\Sigma C)
\end{tikzcd}$$

are isomorphisms of Ω-modules by universal property of the direct sum and the commutativity of direct sums and tensor products, so the other is as well. Since the latter map is a morphism of complexes of $\Omega \otimes_R \Xi^\circ$-modules, we have the result in the case of modules.

In the general setting, we have that

$$\mathrm{Hom}^n_{\Lambda^\circ}(B, A \otimes_\Sigma C) = \prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\Lambda^\circ}\Big(B^j, \bigoplus_{i \in \mathbb{Z}} A^i \otimes_\Sigma C^{n-i+j}\Big).$$

Since B^j is finitely presented over Λ°, the argument given above implies that the latter group is naturally isomorphic to

$$\prod_{j \in \mathbb{Z}} \bigoplus_{i \in \mathbb{Z}} A^i \otimes_\Sigma \mathrm{Hom}_{\Lambda^\circ}(B^j, C^{n-i+j}).$$

On the other hand, we have

$$(A \otimes_\Sigma \mathrm{Hom}_{\Lambda^\circ}(B, C))^n = \bigoplus_{i \in \mathbb{Z}} A^i \otimes_\Sigma \Big(\prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\Lambda^\circ}(B^j, C^{n-i+j})\Big).$$

If the terms of A are finitely presented over Σ°, then we can use a finite presentation of A^i to see that the latter term is naturally isomorphic to

$$\bigoplus_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} A^i \otimes_\Sigma \mathrm{Hom}_{\Lambda^\circ}(B^j, C^{n-i+j}),$$

and the boundedness of A allows us to commute the direct product and direct sum. Finally, if at least one of B and C is bounded above and at least one is bounded below, then the products over $j \in \mathbb{Z}$ involve only finitely many nonzero terms (again using that A is bounded), and the result follows as direct sums commute with tensor products and each other. ☐

In the derived category, we have the following generalization of [Ven, Proposition 6.1].

Lemma 2.1.8. *Suppose that Ω is a flat R-algebra and Ξ is a projective R-algebra. Let A be a bounded above complex of $\Omega \otimes_R \Sigma^\circ$-modules, let B be a complex of $\Xi \otimes_R \Lambda^\circ$-modules, and let C be a bounded below complex of $\Sigma \otimes_R \Lambda^\circ$-modules. Suppose that there exists a quasi-isomorphism $Q \to A$ of $\Omega \otimes_R \Sigma^\circ$-modules, where Q is bounded and has terms that are flat as Σ°-modules. Suppose also that there is a quasi-isomorphism $P \to B$ of complexes of $\Xi \otimes_R \Lambda^\circ$-modules, where P is q-projective as a complex of Λ°-modules with terms that are finitely presented over Λ°. We assume also that either C is bounded, P is bounded above, or the terms of Q are finitely presented as Σ°-modules. Then the map*

$$Q \otimes_\Sigma \mathrm{Hom}_{\Lambda^\circ}(P, C) \longrightarrow \mathrm{Hom}_{\Lambda^\circ}(P, Q \otimes_\Sigma C)$$

gives rise to an isomorphism

$$A \otimes^{\mathbf{L}}_\Sigma \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(B, C) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(B, A \otimes^{\mathbf{L}}_\Sigma C)$$

in $\mathbf{D}(\mathrm{Mod}_{\Omega \otimes_R \Xi^\circ})$.

Proof. By Propositions 2.1.2 and 2.1.4, the complex $Q \otimes_\Sigma \mathrm{Hom}_{\Lambda^\circ}(P, C)$ represents $A \otimes_\Sigma^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(B, C)$ and the complex $\mathrm{Hom}_{\Lambda^\circ}(P, Q \otimes_\Sigma C)$ represents $\mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(B, A \otimes_\Sigma^{\mathbf{L}} C)$. That the stated map is an isomorphism follows from Lemma 2.1.7. □

Remark 2.1.9. In this article, we need only compute derived tensor products in the case that at least one of the complexes is bounded above and derived homomorphism complexes in the case that the target complex is bounded below. However, in Section 4.5, we will nevertheless be forced to represent derived homomorphism complexes using q-projective resolutions in an unbounded first variable. Hence, we have given a general treatment.

2.2 Modules over group rings

In this subsection, we preserve the notation of Section 2.1, and in addition, let G be a group. We begin by extending the notions of derived bifunctors of homorphism groups and tensor products to incorporate an additional action of G.

If A is a complex of $\Lambda[G]$-modules and B is a complex of $(\Lambda \otimes_R \Sigma^\circ)[G]$-modules, we give $\mathrm{Hom}_\Lambda(A, B)$ the structure of a complex of $\Sigma^\circ[G]$-modules via the standard G-action

$$(g \cdot f)(a) = g \cdot f(g^{-1}a)$$

for $g \in G$, $a \in A$, and $f \in \mathrm{Hom}_\Lambda(A, B)$. The following is then a slight weakening of the natural analogue of Proposition 2.1.2, written in condensed form.

Proposition 2.2.1. *There is a derived bifunctor*

$$\mathbf{R}\mathrm{Hom}_\Lambda(-, -) \colon \mathbf{D}(\mathrm{Mod}_{\Lambda[G]})^\circ \times \mathbf{D}(\mathrm{Mod}_{\Lambda \otimes_R \Sigma^\circ[G]}) \longrightarrow \mathbf{D}(\mathrm{Mod}_{\Sigma^\circ[G]}).$$

Moreover, $\mathbf{R}\mathrm{Hom}_\Lambda(A, B)$ *can be represented by* $\mathrm{Hom}_\Lambda(A, B)$ *if* A *is q-projective as a complex of* Λ*-modules or, if* Σ *is* R*-flat,* B *is q-injective as a complex of* Λ*-modules.*

Proof. Proposition 2.1.2 (with Ω replaced by $R[G]^\circ$ and Σ replaced by $\Sigma^\circ[G]$) implies the existence of derived bifunctors with the desired properties that take values in the category $\mathbf{D}(\mathrm{Mod}_{R[G]^\circ \otimes_R \Sigma^\circ[G]})$. There is a natural exact functor

$$\mathrm{Mod}_{R[G]^\circ \otimes_R \Sigma^\circ[G]} \longrightarrow \mathrm{Mod}_{\Sigma^\circ[G]}$$

taking M to the same Σ°-module with new G-action

$$g \cdot m = (g^{-1} \otimes g) \cdot m$$

for $g \in G$ and $m \in M$. This in turn induces a functor on derived categories, and composition of the above derived functor with this functor yields the result. □

Next, if A is a complex of $(\Omega \otimes_R \Lambda^\circ)[G]$-modules and B is a complex of $(\Lambda \otimes_R \Sigma^\circ)[G]$-modules, we give $A \otimes_\Lambda B$ the structure of a complex of $(\Omega \otimes_R \Sigma^\circ)[G]$-modules via the G-action defined by $g(a \otimes b) = ga \otimes gb$ for $a \in A$, $b \in B$, and $g \in G$. We state a very slight weakening (for brevity) of the analogue of Proposition 2.1.4.

Proposition 2.2.2. *If Ω or Σ is a flat R-algebra, then we have a derived bifunctor*

$$- \otimes^{\mathbf{L}}_\Lambda - \colon \mathbf{D}(\mathrm{Mod}_{\Omega\otimes_R\Lambda^\circ[G]}) \times \mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ[G]}) \longrightarrow \mathbf{D}(\mathrm{Mod}_{\Omega\otimes_R\Sigma^\circ[G]}).$$

Moreover, $A \otimes_\Lambda B$ represents $A \otimes^{\mathbf{L}}_\Lambda B$ if either Ω is R-flat and A is a q-flat as a complex of Λ°-modules or Σ is R-flat and B is q-flat as a complex of Λ-modules.

Proof. Proposition 2.1.4 with Ω replaced by $\Omega[G]$ and Σ° replaced by $\Sigma^\circ[G]$ yields the existence of a derived bifunctor

$$- \otimes^{\mathbf{L}}_\Lambda - \colon \mathbf{D}(\mathrm{Mod}_{\Omega\otimes_R\Lambda^\circ[G]}) \times \mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ[G]}) \longrightarrow \mathbf{D}(\mathrm{Mod}_{\Omega[G]\otimes_R\Sigma^\circ[G]})$$

which has the desired properties. There is a natural exact functor

$$\mathrm{Mod}_{\Omega[G]\otimes_R\Sigma^\circ[G]} \longrightarrow \mathrm{Mod}_{(\Omega\otimes_R\Sigma^\circ)[G]}$$

that takes an module M to the same $\Omega \otimes_R \Sigma^\circ$-module with new G-action

$$g \cdot m = (g \otimes g) \cdot m$$

for $g \in G$ and $m \in M$, and the desired bifunctor is the resulting composition of derived functors. □

Suppose now that Λ is an R-algebra and that we are given a homomorphism $\chi\colon G \to \mathrm{Aut}_{\Lambda^\circ}(\Lambda)$, allowing us to endow Λ with the structure of a $\Lambda^\circ[G]$-module. We denote the resulting module by ${}_\chi\Lambda$. (If χ is trivial, then we continue to write Λ for ${}_\chi\Lambda$.) We also have a map $\chi^{-1}\colon G \to \mathrm{Aut}_\Lambda(\Lambda)$ defined by $\chi^{-1}(g)(\lambda) = \lambda \cdot \chi(g^{-1})(1)$ for $g \in G$ and $\lambda \in \Lambda$, so the resulting object Λ_χ is a $\Lambda[G]$-module. The relationship between ${}_\chi\Lambda$ and Λ_χ can be expressed by the evaluation-at-1 maps

$$\mathrm{Hom}_\Lambda(\Lambda_\chi, \Lambda) \longrightarrow {}_\chi\Lambda \quad \text{and} \quad \mathrm{Hom}_{\Lambda^\circ}({}_\chi\Lambda, \Lambda) \longrightarrow \Lambda_\chi,$$

which are isomorphisms of $\Lambda^\circ[G]$ and $\Lambda[G]$-modules, respectively.

Lemma 2.2.3. *Suppose that Λ is a flat R-algebra, and let $\chi\colon G \to \mathrm{Aut}_{\Lambda^\circ}(\Lambda)$ be a homomorphism. For any bounded above complex A of $R[G]$-modules that are finitely presented over R and any bounded below complex B of R-modules, the two maps*

$$\theta\colon {}_\chi\Lambda \otimes_R \mathrm{Hom}_R(A, B) \longrightarrow \mathrm{Hom}_\Lambda(\Lambda_\chi \otimes_R A, \Lambda \otimes_R B)$$
$$\theta'\colon \Lambda_\chi \otimes_R \mathrm{Hom}_R(A, B) \longrightarrow \mathrm{Hom}_{\Lambda^\circ}({}_\chi\Lambda \otimes_R A, \Lambda \otimes_R B)$$

defined by $\theta(\lambda \otimes f)(\mu \otimes x) = \mu\lambda \otimes f(x)$ *and* $\theta'(\lambda \otimes f)(\mu \otimes x) = \lambda\mu \otimes f(x)$ *are isomorphisms of complexes of* $\Lambda^\circ[G]$*-modules and of complexes of* $\Lambda[G]$*-modules, respectively.*

Proof. We focus on the case of θ, as the other case follows from it. Since ${}_\chi\Lambda$ is R-flat, Lemma 2.1.7 (with $\Omega = \Lambda^\circ[G]$ and $\Xi = R[G]$) implies that the natural map

$$ {}_\chi\Lambda \otimes_R \mathrm{Hom}_R(A, B) \xrightarrow{\sim} \mathrm{Hom}_R(A, {}_\chi\Lambda \otimes_R B) $$

is an isomorphism of complexes of $\Lambda^\circ[G] \otimes_R R[G]$-modules if we take the actions of G independently, and hence of $\Lambda^\circ[G]$-modules if we take the G-actions prescribed earlier in this subsection. The inverse of the isomorphism provided by the evaluation map provides the first of the isomorphisms of complexes of $\Lambda^\circ[G]$-modules

$$ {}_\chi\Lambda \otimes_R B \xrightarrow{\sim} \mathrm{Hom}_\Lambda(\Lambda_\chi, \Lambda) \otimes_R B \xrightarrow{\sim} \mathrm{Hom}_\Lambda(\Lambda_\chi, \Lambda \otimes_R B), $$

the canonical second map being an isomorphism as Λ_χ is free of rank 1 over Λ. We therefore have an isomorphism

$$ \mathrm{Hom}_R(A, {}_\chi\Lambda \otimes_R B) \xrightarrow{\sim} \mathrm{Hom}_R(A, \mathrm{Hom}_\Lambda(\Lambda_\chi, \Lambda \otimes_R B)). $$

Finally, the inverse of the adjoint morphism is an isomorphism

$$ \mathrm{Hom}_R(A, \mathrm{Hom}_\Lambda(\Lambda_\chi, \Lambda \otimes_R B)) \xrightarrow{\sim} \mathrm{Hom}_\Lambda(\Lambda_\chi \otimes_R A, \Lambda \otimes_R B) $$

of complexes of $\Lambda^\circ[G]$-modules, and the resulting composite of three isomorphisms is easily computed to be θ. $\square$

2.3 Projective modules over a profinite ring

Let Λ be a profinite ring, and fix a directed fundamental system $\mathcal{I}$ of open neighborhoods of zero consisting of two-sided ideals of Λ. We say that a topological Λ-module M is endowed with the $\mathcal{I}$-adic topology if the collection $\{\mathfrak{A}M\}_{\mathfrak{A}\in\mathcal{I}}$ forms a fundamental system of neighborhoods of zero. It was shown in [Lim2, Section 3.1] that any finitely generated compact (Hausdorff) Λ-module necessarily has the $\mathcal{I}$-adic topology, and, moreover, any homomorphism between such modules is necessarily continuous.

In this subsection, we recall several facts about projective Λ-modules that will be of use to us. We denote the abelian category of compact Λ-modules by $\mathcal{C}_\Lambda$. The free profinite Λ-module on a set X is canonically isomorphic to the topological direct product of one copy of Λ for each element of X [Wil, Proposition 7.4.1], and a profinite Λ-module P is projective if and only if it is continuously isomorphic to a direct summand of the free profinite module on a set of generators of P [Wil, Proposition 7.4.7]. In particular, the category $\mathcal{C}_\Lambda$ has enough projectives. Any projective object in $\mathcal{C}_\Lambda$ that is finitely generated over Λ is a projective Λ-module, and conversely, any finitely generated projective

Λ-module endowed with the $\mathcal{I}$-adic topology is a projective object in $\mathcal{C}_\Lambda$ [Lim2, Proposition 3.1.8].

Recall that the projective dimension of an abstract Λ-module M is the minimum integer n (if it exists) such that there is a resolution of M by projective Λ-modules

$$0 \longrightarrow P^{-n} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow M \longrightarrow 0.$$

The topological projective dimension of a compact Λ-module is defined similarly, replacing projective Λ-modules by projective objects in $\mathcal{C}_\Lambda$.

For the remainder of the subsection, we suppose that Λ is left Noetherian. Note that any projective resolution P of a finitely generated Λ-module M is quasi-isomorphic to a projective resolution Q of M by finitely generated Λ-modules via a map $Q \to P$ compatible with the augmentations to M. In particular, the projective dimension of M is the length of its shortest resolution by projectives in the category of finitely generated Λ-modules. It follows that the notions of projective dimension and topological projective dimension coincide on finitely generated (compact) Λ-modules.

In general, if $\mathfrak{C}$ is a category with objects that are Λ-modules, we will use $\mathfrak{C}^{\Lambda-\mathrm{ft}}$ to denote the full subcategory of objects that are finitely generated (i.e., of finite type) over Λ. Since $\mathrm{Mod}_\Lambda^{\Lambda-\mathrm{ft}}$ has enough projectives, the equivalent category $\mathcal{C}_\Lambda^{\Lambda-\mathrm{ft}}$ also has enough projectives. Denote by $\mathbf{D}^-_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$ the full subcategory of the bounded above derived category $\mathbf{D}^-(\mathrm{Mod}_\Lambda)$ which has as its objects those bounded above complexes X of Λ-modules for which all of the $H^i(X)$ are finitely generated Λ-modules. The following standard lemma tells us that any such complex is quasi-isomorphic to a complex of finitely generated modules. (See [Nek, Proposition 3.2.6] for an analogous statement, which has a similar proof.)

Lemma 2.3.1. *Let Ω be a left Noetherian ring. Every bounded above complex X of Ω-modules for which every $H^i(X)$ is a finitely generated Ω-module has a quasi-isomorphic subcomplex of finitely generated Ω-modules.*

In sum, we have equivalences of categories

$$\mathbf{D}^-(\mathcal{C}_\Lambda^{\Lambda-\mathrm{ft}}) \xrightarrow{\sim} \mathbf{D}^-(\mathrm{Mod}_\Lambda^{\Lambda-\mathrm{ft}}) \xrightarrow{\sim} \mathbf{D}^-_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda),$$

the first being induced by the forgetful functor and the second by the inclusion of categories $\mathrm{Mod}_\Lambda^{\Lambda-\mathrm{ft}} \hookrightarrow \mathrm{Mod}_\Lambda$. We use these equivalences to identify these categories with each other.

2.4 Completed tensor products

In this subsection, we review some basic facts about completed tensor products and briefly study their derived functors. Let R be a commutative profinite ring, and let Λ, Ω, and Σ be profinite R-algebras, by which we shall mean that they are profinite and the maps from R to their centers are continuous.

Let $\mathcal{I}$ (resp., $\mathcal{J}$) denote a directed fundamental system of open neighborhoods of zero consisting of two-sided ideals of Λ (resp., Ω). We then define a completed tensor product algebra by

$$\Omega \,\hat{\otimes}_R\, \Lambda^\circ = \varprojlim_{\mathfrak{A}\in\mathcal{I},\ \mathfrak{B}\in\mathcal{J}} \Omega/\mathfrak{B} \otimes_R (\Lambda/\mathfrak{A})^\circ.$$

This is clearly a profinite R-algebra.

We shall denote the category of compact $\Omega \,\hat{\otimes}_R\, \Lambda^\circ$-modules by $\mathcal{C}_{\Omega-\Lambda}$. Let M be an object of $\mathcal{C}_{\Omega-\Lambda}$, and let N be an object of $\mathcal{C}_{\Lambda-\Sigma}$. By [RZ, Lemma 5.1.1(a)], we have that the set of open, finite index $\Omega \,\hat{\otimes}_R\, \Lambda^\circ$-submodules (resp., $\Lambda \,\hat{\otimes}_R\, \Sigma^\circ$-submodules) of M (resp., N) forms a basis of neighborhoods of zero on M (resp., N). We define the completed tensor product to be the compact $\Omega \,\hat{\otimes}_R\, \Sigma^\circ$-module

$$M \,\hat{\otimes}_\Lambda\, N = \varprojlim_{M',N'} M/M' \otimes_\Lambda N/N',$$

where M' and N' run through the respective bases for M and N, with the topology defined by the inverse limit. The completed tensor product is associative and commutative (i.e., $M \,\hat{\otimes}_\Lambda\, N \cong N \,\hat{\otimes}_{\Lambda^\circ}\, M$) in the same sense as the usual tensor product.

Note that the canonical Λ-balanced map

$$t\colon M \times N \longrightarrow M \,\hat{\otimes}_\Lambda\, N$$

induces a homomorphism

$$M \otimes_\Lambda N \longrightarrow M \,\hat{\otimes}_\Lambda\, N$$

of $\Omega \otimes_R \Sigma^\circ$-modules that has dense image. The completed tensor product of M and N then satisfies the following universal property (see [RZ, Section 5.5] in the case that $\Omega = \Sigma = R$).

Lemma 2.4.1. *For any compact $\Omega \,\hat{\otimes}_\Lambda\, \Sigma^\circ$-module L and any continuous, Λ-balanced, left Ω-linear and right Σ-linear map $f\colon M \times N \to L$, there is a unique continuous map $\hat{f}\colon M \,\hat{\otimes}_\Lambda\, N \to L$ of $\Omega \,\hat{\otimes}_\Lambda\, \Sigma^\circ$-modules such that $\hat{f} \circ t = f$.*

It follows that, in defining the completed tensor product, it suffices to run through a basis of neighborhoods of zero consisting of open $\Omega \,\hat{\otimes}_R\, \Lambda^\circ$-submodules of M and a basis of neighborhoods of zero consisting of open $\Lambda \,\hat{\otimes}_R\, \Sigma^\circ$-submodules of N.

The following is also standard.

Lemma 2.4.2. *Let M and N be objects of $\mathcal{C}_{\Omega-\Lambda}$ and $\mathcal{C}_{\Lambda-\Sigma}$, respectively.*

(*a*) *Suppose that $M = \varprojlim M_\alpha$ and $N = \varprojlim N_\beta$, where each M_α (resp., N_β) is a compact $\Omega \,\hat{\otimes}_R\, \Lambda^\circ$-module (resp., compact $\Lambda \,\hat{\otimes}_R\, \Sigma^\circ$-module). Then there is an isomorphism*

$$M \,\hat{\otimes}_\Lambda\, N \cong \varprojlim_{\alpha,\beta} M_\alpha \,\hat{\otimes}_\Lambda\, N_\beta$$

of compact $\Omega \,\hat{\otimes}_R\, \Sigma^\circ$-modules.

(*b*) *The canonical map* $M \otimes_\Lambda N \to M \hat{\otimes}_\Lambda N$ *is an isomorphism if either* M *is finitely generated as a* Λ°*-module or* N *is finitely generated as a* Λ*-module.*

(*c*) *The functor*

$$M \hat{\otimes}_\Lambda - \colon \mathcal{C}_{\Lambda-\Sigma} \longrightarrow \mathcal{C}_{\Omega-\Sigma}$$

is right exact.

The next lemma describes connections between projective objects in different categories of compact modules and flat objects in categories of abstract modules. A ring is said to be left (resp., right) coherent if all its finitely generated left (resp., right) ideals are finitely presented.

Lemma 2.4.3.

(*a*) *If* Λ *is left* (*resp., right*) *coherent, then every projective object in* $\mathcal{C}_{\Lambda^\circ}$ (*resp.,* $\mathcal{C}_\Lambda$) *is flat with respect to the usual left* (*resp., right*) *tensor product over* Λ.

(*b*) *If* Ω (*resp.,* Λ) *is a projective object of* $\mathcal{C}_R$*, then every projective object in* $\mathcal{C}_{\Omega-\Lambda}$ *is a projective object in* $\mathcal{C}_{\Lambda^\circ}$ (*resp.,* $\mathcal{C}_\Omega$).

Proof. (a) Since every projective object in $\mathcal{C}_{\Lambda^\circ}$ is continuously isomorphic to a direct summand of a direct product of copies of Λ°, it suffices to show that any direct product of copies of Λ is flat as an abstract Λ°-module. By a theorem of Chase (cf. [Lam, Theorem 4.47]), this is equivalent to the fact that Λ is left coherent.

(b) Suppose that Ω is projective in $\mathcal{C}_R$, so Ω is continuously isomorphic to a direct summand of a direct product of copies of R. Lemma 2.4.2(a) then tells us that $\Omega \hat{\otimes}_R \Lambda^\circ$ is topologically a direct summand of a direct product of copies of Λ°, hence is projective. □

Let G be a profinite group. We use $\mathcal{C}_{\Lambda,G}$ to denote the category of compact Λ-modules with a continuous commuting action of G, the morphisms being continuous homomorphisms of $\Lambda[G]$-modules. Note the following.

Remark 2.4.4. The category $\mathcal{C}_{\Lambda,G}$ is equivalent to the category $\mathcal{C}_{\Lambda[\![G]\!]}$, where $\Lambda[\![G]\!]$ is given the profinite topology defined by

$$\Lambda[\![G]\!] \cong \varprojlim_{\mathfrak{A} \in \mathcal{I}} \varprojlim_{N} (\Lambda/\mathfrak{A})[G/N],$$

where N runs over the open normal subgroups of G.

Consequently, $\mathcal{C}_{\Lambda,G}$ is an abelian category with enough projectives, and every element of $\mathcal{C}_{\Lambda,G}$ is an inverse limit of finite $\Lambda[G]$-quotients (see also [Lim2, Section 3.2]). Moreover, $\Lambda[\![G]\!]$ is a projective object of $\mathcal{C}_\Lambda$ by [RZ, Lemma

5.3.5(d)], so the forgetful functor $\mathcal{C}_{\Lambda,G} \to \mathcal{C}_\Lambda$ takes projective objects to projective objects. To shorten notation, we use $\mathcal{C}_{\Omega-\Lambda,G}$ to denote the category $\mathcal{C}_{\Omega \hat{\otimes}_R \Lambda^\circ,G}$.
We remark that if M and N are, respectively, objects of $\mathcal{C}_{\Omega-\Lambda,G}$ and $\mathcal{C}_{\Lambda-\Sigma,G}$, possibly with trivial G-actions, we may give $M \hat{\otimes}_\Lambda N$ the structure of an object of $\mathcal{C}_{\Omega-\Sigma,G}$ via the diagonal action of G. That is, the G-action is defined by choosing bases of open $(\Omega \hat{\otimes}_R \Lambda^\circ)[G]$-submodules of M (resp., $(\Lambda \hat{\otimes}_R \Sigma^\circ)[G]$-submodules of N) and taking the inverse limits of the tensor products of the finite quotients.

REMARK 2.4.5. The analogous result to Lemma 2.4.2 holds, as a consequence of said lemma, if we take M and N to be objects of $\mathcal{C}_{\Omega-\Lambda,G}$ and $\mathcal{C}_{\Lambda-\Sigma,G}$, respectively. (That is, in part (a), one must take M_α and N_β to be objects of these categories to attain an isomorphism in $\mathcal{C}_{\Omega-\Sigma,G}$, and in part (c), the functor is now a functor from $\mathcal{C}_{\Lambda-\Sigma,G}$ to $\mathcal{C}_{\Omega-\Sigma,G}$.)

Note that we may form completed tensor products of bounded above complexes, or of a bounded complex with any complex, as with the usual tensor products.

PROPOSITION 2.4.6. *If Ω is projective in $\mathcal{C}_R$, then the completed tensor product induces the following derived bifunctor*

$$- \hat{\otimes}^{\mathbf{L}}_\Lambda - \colon \mathbf{D}^-(\mathcal{C}_{\Omega-\Lambda,G}) \times \mathbf{D}^-(\mathcal{C}_{\Lambda-\Sigma,G}) \longrightarrow \mathbf{D}^-(\mathcal{C}_{\Omega-\Sigma,G}),$$

where $A \hat{\otimes}_\Lambda B$ represents $A \hat{\otimes}^{\mathbf{L}}_\Lambda B$ if the terms of A are projective as objects in $\mathcal{C}_{\Lambda^\circ}$. Furthermore, there is a commutative diagram

$$\begin{tikzcd}
\mathbf{D}^-(\mathcal{C}_{\Omega-\Lambda,G}) \times \mathbf{D}^-(\mathcal{C}_{\Lambda-\Sigma,G}) \arrow[r, "- \hat{\otimes}^{\mathbf{L}}_\Lambda -"] \arrow[d] & \mathbf{D}^-(\mathcal{C}_{\Omega-\Sigma,G}) \arrow[d] \\
\mathbf{D}^-(\mathcal{C}_{\Lambda^\circ,G}) \times \mathbf{D}^-(\mathcal{C}_{\Lambda-\Sigma,G}) \arrow[r, "- \hat{\otimes}^{\mathbf{L}}_\Lambda -"] & \mathbf{D}^-(\mathcal{C}_{\Sigma^\circ,G})
\end{tikzcd}$$

in which the vertical arrows are induced by forgetful functors, and if Λ is left Noetherian, there is a commutative diagram

$$\begin{tikzcd}
\mathbf{D}^-(\mathcal{C}_{\Omega-\Lambda,G}) \times \mathbf{D}^-(\mathcal{C}_{\Lambda-\Sigma,G}) \arrow[rr, "- \hat{\otimes}^{\mathbf{L}}_\Lambda -"] & & \mathbf{D}^-(\mathcal{C}_{\Omega-\Sigma,G}) \arrow[dd] \\
\mathbf{D}^-(\mathcal{C}_{\Omega-\Lambda,G}) \times \mathbf{D}^-(\mathcal{C}^{\Lambda-\mathrm{ft}}_{\Lambda-\Sigma,G}) \arrow[u] \arrow[d] & & \\
\mathbf{D}^-(\mathrm{Mod}_{(\Omega\otimes_R\Lambda^\circ)[G]}) \times \mathbf{D}^-(\mathrm{Mod}_{(\Lambda\otimes_R\Sigma^\circ)[G]}) \arrow[rr, "-\otimes^{\mathbf{L}}_\Lambda -"] & & \mathbf{D}^-(\mathrm{Mod}_{(\Omega\otimes_R\Sigma^\circ)[G]})
\end{tikzcd}$$

in which the vertical arrows are induced by forgetful functors and embeddings of categories.

Proof. The proof of the first part and the commutativity of the first set of diagrams follow by similar arguments to those of Propositions 2.1.2 and 2.2.2, making use of Lemma 2.4.3(b) (using bounded above complexes of projective objects in place of q-projective complexes). The commutativity of the final diagram then follows from Lemmas 2.4.2(b) and 2.4.3. □

Let $\mathrm{Ch}^b_{\Lambda^\circ-\mathrm{perf}}(\mathcal{C}_{\Omega-\Lambda,G})$ denote the category of bounded complexes in $\mathcal{C}_{\Omega-\Lambda,G}$ that are quasi-isomorphic to bounded complexes in $\mathcal{C}_{\Omega-\Lambda,G}$ of objects that are projective in $\mathcal{C}_{\Lambda^\circ}$, and let $\mathbf{D}^b_{\Lambda^\circ-\mathrm{perf}}(\mathcal{C}_{\Omega-\Lambda,G})$ denote its derived category.

PROPOSITION 2.4.7. *If Ω is projective in $\mathcal{C}_R$, then we have a derived bifunctor*

$$- \hat{\otimes}^{\mathbf{L}}_\Lambda - \colon \mathbf{D}^b_{\Lambda^\circ-\mathrm{perf}}(\mathcal{C}_{\Omega-\Lambda,G}) \times \mathbf{D}(\mathcal{C}_{\Lambda-\Sigma,G}) \longrightarrow \mathbf{D}(\mathcal{C}_{\Omega-\Sigma,G}),$$

where $A \hat{\otimes}_\Lambda B$ represents $A \hat{\otimes}^{\mathbf{L}}_\Lambda B$ if A is a bounded complex with terms that are projectives in $\mathcal{C}_{\Lambda^\circ}$.

Proof. Let B be a complex of objects of $\mathcal{C}_{\Omega-\Lambda,G}$. Since every object of $\mathrm{Ch}^b_{\Lambda^\circ-\mathrm{perf}}(\mathcal{C}_{\Omega-\Lambda,G})$ is by definition quasi-isomorphic to a bounded complex of objects that are acyclic for the functor $- \hat{\otimes}_\Lambda B$, we have a derived functor $- \hat{\otimes}^{\mathbf{L}}_\Lambda B$. Let A be a bounded complex in $\mathcal{C}_{\Omega-\Lambda,G}$ of projectives in $\mathcal{C}_{\Lambda^\circ}$ and $f\colon B \to B'$ be a quasi-isomorphism. Then f induces isomorphisms between the E_2-terms of the convergent spectral sequence

$$E_2^{i,j}(B) = H^i(A \hat{\otimes}_\Lambda H^j(B)) \Rightarrow H^{i+j}(A \hat{\otimes}_\Lambda B)$$

and its analogue for B', and therefore it induces isomorphisms on the abutments. □

2.5 IND-ADMISSIBLE MODULES

The notion of an ind-admissible $R[G]$-module was introduced in [Nek, Section 3.3] for a complete commutative Noetherian local ring R with finite residue field and a profinite group G. An $R[G]$-module is ind-admissible if it can be written as a union of $R[G]$-submodules that are finitely generated over R and on which G acts continuously with respect to the topology defined by the maximal ideal of R. In this section, we discuss an analogous construction of ind-admissible modules over noncommutative profinite rings.

As in Nekovář's treatment, we do not consider the seemingly delicate issue of placing topologies on ind-admissible modules, as it proves unnecessary. In particular, it is still possible to define the continuous cochain complex of an ind-admissible module as a direct limit.

We maintain the notation of Section 2.4. Moreover, we suppose that Ω is left Noetherian and Λ is right Noetherian. For an $(\Omega \otimes_R \Lambda^\circ)[G]$-module M, we denote by $\mathcal{S}(M)$ the set of $(\Omega \otimes_R \Lambda^\circ)[G]$-submodules of M that are finitely generated as Λ°-modules and on which Ω and G act continuously with respect to the $\mathcal{I}$-adic topology. The following is a straightforward generalization of [Nek, Lemma 3.3.2].

Lemma 2.5.1. *Let M be an $(\Omega \otimes_R \Lambda^\circ)[G]$-module.*

(*a*) *If $M' \in \mathcal{S}(M)$, then $N \in \mathcal{S}(M)$ for every $(\Omega \otimes_R \Lambda^\circ)[G]$-submodule N of M'.*

(*b*) *If $f\colon M \to N$ is a homomorphism of $(\Omega \otimes_R \Lambda^\circ)[G]$-modules and $M' \in \mathcal{S}(M)$, then $f(M') \in \mathcal{S}(N)$.*

(*c*) *If M', $M'' \in \mathcal{S}(M)$, then $M' + M'' \in \mathcal{S}(M)$.*

Proof. For part (a), Corollary 3.1.6 and Proposition 3.1.7 of [Lim2] imply that the subspace topology on N from the $\mathcal{I}$-adic topology on M' agrees with the $\mathcal{I}$-adic topology on N, which implies that Ω and G act continuously on N. In (b), the continuity of the Ω and G-actions on the finitely generated Λ°-module $f(M')$ is a consequence of the fact that the map $M' \to f(M')$ is a continuous quotient map with respect to the $\mathcal{I}$-adic topology (as follows from Corollary 3.1.5 and Proposition 3.1.7 of [Lim2]). Part (c) follows from (b), using the addition map $M \times M \to M$. □

Note that Lemma 2.5.1(c) implies that $\mathcal{S}(M)$ is a directed set with respect to inclusion. We say that an $(\Omega \otimes_R \Lambda^\circ)[G]$-module M is (right) ind-admissible if

$$M = \bigcup_{N \in \mathcal{S}(M)} N.$$

We list some basic properties of the full subcategory $\mathscr{I}_{\Omega-\Lambda,G}$ of $\mathrm{Mod}_{(\Omega\otimes_R\Lambda^\circ)[G]}$ with objects the ind-admissible $(\Omega \otimes_R \Lambda^\circ)[G]$-modules.

Lemma 2.5.2.

(*a*) *The category $\mathscr{I}_{\Omega-\Lambda,G}$ is abelian and stable under subobjects, quotients and colimits.*

(*b*) *The embedding functor*

$$i\colon \mathscr{I}_{\Omega-\Lambda,G} \hookrightarrow \mathrm{Mod}_{(\Omega\otimes_R\Lambda^\circ)[G]}$$

is exact and is left adjoint to the functor

$$j\colon \mathrm{Mod}_{(\Omega\otimes_R\Lambda^\circ)[G]} \to \mathscr{I}_{\Omega-\Lambda,G}$$

that takes a module M to the union of the elements of $\mathcal{S}(M)$.

(*c*) *The category $\mathscr{I}_{\Omega-\Lambda,G}$ has enough injectives.*

(*d*) *Let M be an ind-admissible $(\Omega \otimes_R \Lambda^\circ)[G]$-module, and let N be a finitely generated Λ°-submodule of M. Then $(\Omega\otimes_R\Lambda^\circ)[G]\cdot N$ is an ind-admissible $(\Omega \otimes_R \Lambda^\circ)[G]$-module which is a finitely generated Λ°-module.*

(*e*) *Let M be an $(\Omega \otimes_R \Lambda^\circ)[G]$-module. Then $M \in \mathcal{S}(M)$ if and only if M is an ind-admissible $(\Omega \otimes_R \Lambda^\circ)[G]$-module which is finitely generated as a Λ°-module.*

Proof. For parts (a), (b) and (c), similar arguments to those of [Nek, Proposition 3.3.5] apply. The "only if" direction of (e) is obvious. For (d), since N is Λ°-finitely generated, we can find a finite subset $\{M_1, \ldots, M_n\}$ of $\mathcal{S}(M)$ such that

$$N \subseteq M_1 + \cdots + M_n.$$

The assertion then follows from Lemma 2.5.1(c) and the "only if" direction of (e). The "if" direction of (e) follows from (d), since $M = (\Omega \otimes_R \Lambda^\circ)[G] \cdot M$. $\square$

If G is trivial, we write $\mathscr{I}_{\Omega-\Lambda}$ for $\mathscr{I}_{\Omega-\Lambda,G}$. We leave to the reader the proof of the following.

LEMMA 2.5.3.

(*a*) *Let M be an $(\Omega \otimes_R \Lambda^\circ)[G]$-module. If N is an $(\Omega \otimes_R \Lambda^\circ)[G]$-submodule of M, then $N \in \mathcal{S}(M)$ if and only if N is an object of $\mathcal{C}_{\Omega-\Lambda,G}^{\Lambda^\circ-\mathrm{ft}}$.*

(*b*) *The category $\mathscr{I}_{\Omega-\Lambda,G}$ is equivalent to the ind-category of $\mathcal{C}_{\Omega-\Lambda,G}^{\Lambda^\circ-\mathrm{ft}}$.*

(*c*) *The category $\mathscr{I}_{\Omega-\Lambda}$ is equivalent to the full subcategory of $\mathscr{I}_{\Omega-\Lambda,G}$ with objects the modules on which G acts trivially.*

REMARK 2.5.4. In the case that $\Omega = R$, we denote $\mathscr{I}_{R-\Lambda,G}$ by $\mathscr{I}_{\Lambda^\circ,G}$, and the latter category is equivalent to the ind-category of $\mathcal{C}_{\Lambda^\circ,G}^{\Lambda^\circ-\mathrm{ft}}$. The subcategory $\mathscr{I}_{\Lambda^\circ}$ is simply $\mathrm{Mod}_{\Lambda^\circ}$.

LEMMA 2.5.5. *Let A be an object of $\mathcal{C}_{\Omega,G}^{\Omega-\mathrm{ft}}$, and let B be an object of $\mathcal{C}_{\Omega-\Lambda,G}^{\Lambda^\circ-\mathrm{ft}}$. Then $\mathrm{Hom}_\Omega(A,B)$ with the $\mathcal{I}$-adic topology is an object of $\mathcal{C}_{\Lambda^\circ,G}^{\Lambda^\circ-\mathrm{ft}}$.*

Proof. Since A is finitely generated over Ω and B is a compact Ω-module, we have by [Lim2, Lemma 3.1.4(3)] that $\mathrm{Hom}_\Omega(A,B) = \mathrm{Hom}_{\Omega,\mathrm{cts}}(A,B)$, where the latter group is the group of continuous homomorphisms of Ω-modules. Note that Λ° acts continuously on $\mathrm{Hom}_\Omega(A,B)$ with respect to the compact-open topology by [Flo, Proposition 3(a)], and similarly G acts continuously on it as a consequence of [Flo, Lemma 2]. We have a continuous isomorphism of $\Lambda^\circ[G]$-modules

$$\mathrm{Hom}_\Omega(A,B) \xrightarrow{\sim} \varprojlim_\beta \mathrm{Hom}_\Omega(A,B_\beta),$$

where B_β runs over the finite Hausdorff $(\Omega \hat{\otimes}_R \Lambda^\circ)[G]$-quotients of B. As B_β is finite and therefore discrete, the group $\mathrm{Hom}_\Omega(A,B_\beta)$ is finite and discrete as well, so $\mathrm{Hom}_\Omega(A,B)$ is compact.

Note that $\mathrm{Hom}_\Omega(A,B)$ injects into $\mathrm{Hom}_\Omega(\Omega^r,B) \cong B^r$ for some $r \geq 0$, since A is Ω-finitely generated. We therefore have that $\mathrm{Hom}_\Omega(A,B)$ is Λ°-finitely generated. Finally, [Lim2, Proposition 3.1.7] then implies that the compact-open topology on $\mathrm{Hom}_\Omega(A,B)$ agrees with the $\mathcal{I}$-adic topology. $\square$

We define

$$\mathrm{Hom}_{\Omega,\mathrm{cts}}(-,-)\colon \mathcal{C}_{\Omega,G} \times \mathscr{I}_{\Omega-\Lambda,G} \longrightarrow \mathscr{I}_{\Lambda^\circ,G}.$$

by

$$\mathrm{Hom}_{\Omega,\mathrm{cts}}(A,B) = \varinjlim_{\alpha} \varinjlim_{\beta} \mathrm{Hom}_\Omega(A_\alpha, B_\beta),$$

upon making the identifications $A = \varprojlim A_\alpha$, where A_α runs over the quotients of A in $\mathcal{C}_{\Omega,G}^{\Omega-\mathrm{ft}}$, and $B = \varinjlim B_\beta$, where B_β runs over the elements of $\mathcal{S}(B)$, i.e., by Lemma 2.5.3, the subobjects of B that lie in $\mathcal{C}_{\Omega-\Lambda,G}^{\Lambda^\circ-\mathrm{ft}}$. Note that it actually suffices to let A_α and B_β run over cofinal subsets.

Remark 2.5.6. We note that

$$\mathrm{Hom}_{\Omega,\mathrm{cts}}(A,B) \cong \varinjlim_{\beta} \mathrm{Hom}_{\Omega,\mathrm{cts}}(A, B_\beta),$$

where $\mathrm{Hom}_{\Omega,\mathrm{cts}}(A,B_\beta)$ here is the $(\Omega \otimes_R \Lambda^\circ)[G]$-module of continuous Ω-module homomorphisms in the usual sense.

We also note the following.

Lemma 2.5.7. *Let A be an object of $\mathcal{C}_{\Omega,G}^{\Omega-\mathrm{ft}}$, and let B be an object of $\mathscr{I}_{\Omega-\Lambda,G}$. Then we have*

$$\mathrm{Hom}_{\Omega,\mathrm{cts}}(A,B) = \mathrm{Hom}_\Omega(A,B).$$

Proof. This follows from the computation

$$\mathrm{Hom}_{\Omega,\mathrm{cts}}(A,B) \cong \varinjlim_{\beta} \mathrm{Hom}_{\Omega,\mathrm{cts}}(A,B_\beta) = \varinjlim_{\beta} \mathrm{Hom}_\Omega(A,B_\beta) \cong \mathrm{Hom}_\Omega(A,B),$$

the latter isomorphism following from the case in which A is free of finite rank, since A is finitely presented over Ω. $\square$

As usual, we may extend our definition to consider the complex of continuous homomorphisms from a complex of compact modules to a complex of ind-admissible modules, supposing that at least one of these complexes is bounded above and at least one is bounded below.

Proposition 2.5.8. *There is a derived bifunctor*

$$\mathbf{R}\mathrm{Hom}_{\Omega,\mathrm{cts}}(-,-)\colon \mathbf{D}^-(\mathcal{C}_{\Omega,G})^\circ \times \mathbf{D}^+(\mathscr{I}_{\Omega-\Lambda,G}) \longrightarrow \mathbf{D}^+(\mathscr{I}_{\Lambda^\circ,G}),$$

and $\mathbf{R}\mathrm{Hom}_{\Omega,\mathrm{cts}}(A,B)$ can be represented by $\mathrm{Hom}_{\Omega,\mathrm{cts}}(A,B)$ if the terms of A are projective as objects of $\mathcal{C}_\Omega$. Moreover, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{D}^-(\mathcal{C}_{\Omega,G})^\circ \times \mathbf{D}^+(\mathscr{I}_{\Omega-\Lambda,G}) \arrow[rr, "\mathbf{R}\mathrm{Hom}_{\Omega,\mathrm{cts}}(-,-)"] & & \mathbf{D}^+(\mathscr{I}_{\Lambda^\circ,G}) \arrow[dd] \\
\mathbf{D}^-(\mathcal{C}_{\Omega,G}^{\Omega-\mathrm{ft}})^\circ \times \mathbf{D}^+(\mathscr{I}_{\Omega-\Lambda,G}) \arrow[u] \arrow[d] & & \\
\mathbf{D}^-(\mathrm{Mod}_{\Omega[G]})^\circ \times \mathbf{D}^+(\mathrm{Mod}_{(\Omega\otimes_R\Lambda^\circ)[G]}) \arrow[rr, "\mathbf{R}\mathrm{Hom}_\Omega(-,-)"] & & \mathbf{D}^+(\mathrm{Mod}_{\Lambda^\circ[G]})
\end{tikzcd}$$

in which the vertical arrows arise from forgetful functors and the natural embeddings of abelian categories.

Proof. The existence of the derived bifunctor in question follows from an argument similar to that of Proposition 2.1.2, and the second statement is a consequence of the fact that a projective object in $\mathcal{C}_{\Omega,G}$ is a projective object of $\mathcal{C}_\Omega$, noting Remark 2.5.6.

Let A be a bounded above complex of objects of $\mathcal{C}_{\Omega,G}^{\Omega-\mathrm{ft}}$, and let B be a bounded below complex of objects of $\mathscr{I}_{\Omega-\Lambda,G}$. Choose bounded above complexes L, P and Q, consisting of projective objects in $\mathcal{C}_{\Omega,G}$, $\mathrm{Mod}_{\Omega[G]}$ and $\mathrm{Mod}_\Omega^{\Omega-\mathrm{ft}}$ (i.e., $\mathcal{C}_\Omega^{\Omega-\mathrm{ft}}$) respectively, with quasi-isomorphisms to A of complexes in the respective categories. Projectivity yields maps $Q \to L$, $Q \to P$ and $P \to L$ of complexes in $\mathcal{C}_\Omega$, Mod_Ω and $\mathrm{Mod}_{\Omega[G]}$, respectively. We then have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\Omega,\mathrm{cts}}(L,B) \arrow[r] \arrow[d] & \mathrm{Hom}_{\Omega,\mathrm{cts}}(Q,B) \arrow[d] \\ \mathrm{Hom}_\Omega(P,B) \arrow[r] & \mathrm{Hom}_\Omega(Q,B). \end{tikzcd}$$

Here, the right-hand vertical arrow is the identity map by Lemma 2.5.7 (viewing Q as a complex of objects in $\mathcal{C}_{\Omega,G}^{\Omega-\mathrm{ft}}$ with trivial G-action), and the lower horizontal map is a quasi-isomorphism as a consequence of Proposition 2.2.1, as $\Omega[G]$ is Ω-free. We want to show that the left-hand vertical arrow is a quasi-isomorphism of complexes of $\Lambda^\circ[G]$-modules, and we will be done if we can show the upper horizontal arrow is a quasi-isomorphism of complexes of Λ°-modules.

In the case that B is a module, exactness of direct limit reduces us to the case that B is an object of $\mathcal{C}_{\Omega-\Lambda,G}^{\Lambda^\circ-\mathrm{ft}}$, hence of $\mathcal{C}_\Omega$. In this case, since both L and Q are complexes of projectives in $\mathcal{C}_\Omega$, we obtain that the map is a quasi-isomorphism.

In the general setting, since L and Q are bounded above and B is bounded below, we have

$$\mathrm{Hom}^n_{\Omega,\mathrm{cts}}(X,B) = \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\Omega,\mathrm{cts}}(X^j, B^{j+n})$$

for $X = L$ and $X = Q$, and the direct sum commutes with direct limits. As B is the direct limit of its truncations $\tau_{\le i}B$ (with ith terms $\ker d_B^i$), we may therefore assume that B is bounded with $B^j = 0$ for $j > i$. In this case, brutal truncations (i.e., $\sigma_{\le i}B$ having ith term B^i) provide an exact triangle

$$(\sigma_{\le i-1}B)[i-1] \longrightarrow B^i \longrightarrow (\sigma_{\le i}B)[i] \longrightarrow (\sigma_{\le i-1}B)[i],$$

and we are reduced recursively to the above-proven case of a module. □

The following analogue of Lemma 2.2.3 will be of later use.

Lemma 2.5.9. *Suppose that Λ is a flat, Noetherian R-algebra, and let $\chi\colon G \to \mathrm{Aut}_{\Lambda^\circ}(\Lambda)$ be a homomorphism. Let A be a bounded above complex of objects in $\mathcal{C}_{R,G}$, and let B be a bounded below complex of R-modules. Then there are isomorphisms*

$$\theta\colon {}_\chi\Lambda \otimes_R \mathrm{Hom}_{R,\mathrm{cts}}(A,B) \longrightarrow \mathrm{Hom}_{\Lambda,\mathrm{cts}}(\Lambda_\chi \hat{\otimes}_R A, \Lambda \otimes_R B)$$
$$\theta'\colon \Lambda_\chi \otimes_R \mathrm{Hom}_{R,\mathrm{cts}}(A,B) \longrightarrow \mathrm{Hom}_{\Lambda^\circ,\mathrm{cts}}({}_\chi\Lambda \hat{\otimes}_R A, \Lambda \otimes_R B)$$

that arise as direct limits of the maps in Lemma 2.2.3.

Proof. Write $A = \varprojlim A_\alpha$ with A_α an object of $\mathcal{C}_{R,G}^{R-\mathrm{ft}}$ and $B = \varinjlim B_\beta$ with B_β an object of $\mathcal{C}_R^{R-\mathrm{ft}}$. Note that

$$\Lambda_\chi \hat{\otimes}_R A \cong \varprojlim_\alpha (\Lambda_\chi \otimes_R A_\alpha),$$

so, applying Lemma 2.2.3, we have

$$\begin{aligned}\mathrm{Hom}_{\Lambda,\mathrm{cts}}(\Lambda_\chi \hat{\otimes}_R A, \Lambda \otimes_R B) &= \varinjlim_{\alpha,\beta} \mathrm{Hom}_\Lambda(\Lambda_\chi \otimes_R A_\alpha, \Lambda \otimes_R B_\beta)\\ &\cong \varinjlim_{\alpha,\beta} {}_\chi\Lambda \otimes_R \mathrm{Hom}_R(A_\alpha, B_\beta)\\ &\cong {}_\chi\Lambda \otimes_R \mathrm{Hom}_{R,\mathrm{cts}}(A,B).\end{aligned}$$

This yields the first isomorphism, and the argument for the second is similar. □

Let M be an ind-admissible $(\Omega \otimes_R \Lambda^\circ)[G]$-module. The continuous cochain complex of G with values in M is the complex of $\Omega \otimes_R \Lambda^\circ$-modules that is the direct limit of complexes

$$C(G,M) = \varinjlim_{N \in \mathcal{S}(M)} C(G,N),$$

where $C(G,N)$ is the usual complex of (inhomogeneous) continuous cochains, with N given the $\mathcal{I}$-adic topology.

We remark that, in this definition, it clearly suffices to take the direct limit over a cofinal subset of $\mathcal{S}(M)$. In particular, if M itself is finitely generated over Λ°, then by Lemma 2.5.2(e) the above definition of $C(G,M)$ as a direct limit agrees with its definition as continuous cochains, considering M as an object of $\mathcal{C}_{\Omega-\Lambda,G}$. Finally, we can extend the above definition to consider the total cochain complex of a complex M of ind-admissible modules, which has kth term

$$C^k(G,M) = \bigoplus_{i+j=k} C^i(G,M^j)$$

and differentials as in [Nek, (3.4.1.3)], and the cochain functor induces a functor

$$\mathbf{R}\Gamma(G,-)\colon \mathbf{D}^+(\mathscr{I}_{\Omega-\Lambda,G}) \longrightarrow \mathbf{D}^+(\mathrm{Mod}_{\Omega\otimes_R\Lambda^\circ})$$

between derived categories.

3 Cohomology groups of induced modules

In this paper, we are interested in the case that our profinite ring Λ is the Iwasawa algebra of a compact p-adic Lie group. With this in mind, we introduce some notation that will be used from this point forward. Fix a prime p. Let R be a commutative complete Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k, where k is finite of characteristic p. We let Γ denote a compact p-adic Lie group. We are interested in the completed group ring $\Lambda = R[\![\Gamma]\!]$.
We note that Λ is a profinite ring, endowed with the topology given by the canonical isomorphism

$$\Lambda \xrightarrow{\sim} \varprojlim_{U \in \mathscr{U}} \varprojlim_{n \geq 1} R/\mathfrak{m}^n[\Gamma/U],$$

where $\mathscr{U}$ denotes the set of open normal subgroups of Γ. In fact, it is a projective object in $\mathcal{C}_R$ (cf. [RZ, Lemma 5.3.5(d)]). Moreover, we have the following (cf. [Wil, Theorem 8.7.8]).

Proposition 3.0.1. *The ring Λ is Noetherian.*

Proof. As R is a complete local Noetherian ring with finite residue field k, the Cohen structure theorem [Coh, Theorem 12] implies that it is isomorphic to a quotient of a power series ring S in $n = \dim_k \mathfrak{m}/\mathfrak{m}^2$ variables over the ring of Witt vectors $\mathcal{O}$ of k. Therefore, $R[\![\Gamma]\!]$ is a quotient of $S[\![\Gamma]\!]$, so it suffices to prove that $S[\![\Gamma]\!]$ is Noetherian. Since $S[\![\Gamma]\!] \cong \mathcal{O}[\![\mathbb{Z}_p^n \times \Gamma]\!]$ and $\mathbb{Z}_p^n \times \Gamma$ is a compact p-adic Lie group, we have that $S[\![\Gamma]\!]$ is Noetherian by a mild extension of a classical theorem of Lazard's (cf. [Ven2, Corollary 2.4] and [Laz, Proposition V.2.2.4]). $\square$

Since Noetherian rings are necessarily coherent, we may apply Lemma 2.4.3(a) to conclude from Proposition 3.0.1 that Λ is a flat R-algebra.

3.1 Induced modules and descent

In this subsection, we will exhibit an interesting spectral sequence relating the cohomology of an induced module over an Iwasawa algebra to the cohomology of the module itself. We start work in a more general setting.
Let G be a profinite group. Let Σ be a left coherent and right Noetherian profinite R-algebra. For a complex M of objects in $\mathcal{C}_{\Sigma,G}$, we let $C(G,M)$ denote its total direct sum complex of continuous G-cochains (cf. the end of Section 2.5), we let $\mathbf{R}\Gamma(G,M)$ denote the corresponding object in $\mathbf{D}(\mathrm{Mod}_\Sigma)$, and we let $H^i(G,M)$ denote its ith continuous G-cohomology (or, more precisely, hypercohomology) group. As in [Lim2, Proposition 3.2.11], the functor

$$\mathbf{R}\Gamma(G,-)\colon \mathbf{D}^+(\mathcal{C}_{\Sigma,G}) \longrightarrow \mathbf{D}^+(\mathrm{Mod}_\Sigma)$$

is well-defined and exact. (While $\mathcal{C}_{\Sigma,G}$ may not have enough injectives, the treatment of [KS] asserts the existence of $\mathbf{D}^+(\mathcal{C}_{\Sigma,G})$ and $\mathbf{D}(\mathcal{C}_{\Sigma,G})$ under certain set-theoretic assumptions, which we make here.)

In the case that G has finite p-cohomological dimension, we can do better. For a double complex X (or, by abuse of notation, its total complex) and $n \in \mathbb{Z}$, we let $\tau^{II}_{\geq n}(X)$ (resp., $\tau^{II}_{\leq n}(X)$) be the total complex of the quotient complex (resp., subcomplex) of X with jth row equal to $\tau_{\geq n}(X^{\bullet j})$ (resp., $\tau_{\leq n}(X^{\bullet j})$). For lack of a sufficiently precise reference, we provide a short proof of the following.

Lemma 3.1.1. *Suppose that G has finite p-cohomological dimension. Then we have a convergent hypercohomology spectral sequence*

$$E_2^{i,j} = H^i(G, H^j(M)) \Rightarrow H^{i+j}(G, M)$$

for any complex M of objects in $\mathcal{C}_{\Sigma,G}$.

Proof. Since G has finite cohomological dimension, for sufficiently large k we have a convergent hypercohomology spectral sequence

$$E_2^{i,j} = H^i(G, H^j(M)) \Rightarrow H^{i+j}(\tau^{II}_{\leq k} C(G, M))$$

arising from the filtration on rows of the indicated truncation of the double G-cochain complex of M (cf. [NSW, §II.2]). As $C(G, M)$ is the direct limit of the complexes $\tau^{II}_{\leq k} C(G, M)$ under maps that are quasi-isomorphisms by the above convergence, the natural map

$$H^{i+j}(\tau^{II}_{\leq k} C(G, M)) \to H^{i+j}(G, M)$$

is an isomorphism by exactness of the direct limit. □

As quasi-isomorphisms of complexes induce isomorphisms on the truncated double complexes defining their hypercohomology spectral sequences, we have the following corollary.

Corollary 3.1.2. *Suppose that G has finite p-cohomological dimension. The functor $C(G, -)$ preserves quasi-isomorphisms of chain complexes and induces an exact functor*

$$\mathbf{R}\Gamma(G, -) \colon \mathbf{D}(\mathcal{C}_{\Sigma,G}) \to \mathbf{D}(\mathrm{Mod}_\Sigma).$$

Fix a profinite R-algebra Ω that is a projective object of $\mathcal{C}_R$ for the remainder of the section. We now derive the following spectral sequence.

Proposition 3.1.3. *Let Y be a complex of objects in $\mathcal{C}_{\Sigma,G}$, and let N be a bounded above complex of objects in $\mathcal{C}_{\Omega-\Sigma}^{\Sigma^\circ-\mathrm{ft}}$. Consider the conditions*

(1) *G has finite p-cohomological dimension,*

(2) *N is bounded with terms of finite projective dimension over Σ°.*

If (1) *holds and Y is bounded above,* (2) *holds and Y is bounded below, or both* (1) *and* (2) *hold, then we have an isomorphism*

$$N \otimes^{\mathbf{L}}_{\Sigma} \mathbf{R}\Gamma(G, Y) \xrightarrow{\sim} \mathbf{R}\Gamma(G, N \hat{\otimes}^{\mathbf{L}}_{\Sigma} Y)$$

in $\mathbf{D}(\mathrm{Mod}_\Omega)$.

Proof. Let L be a bounded above complex of projective objects in $\mathcal{C}_{\Omega-\Sigma}$ mapping quasi-isomorphically to N. If (1) holds and Y is bounded above, we set $P = L$, and we have by Proposition 2.4.6 that $P \mathbin{\hat{\otimes}_\Sigma} Y$ represents $N \mathbin{\hat{\otimes}^{\mathbf{L}}_\Sigma} Y$ in $\mathbf{D}^-(\mathcal{C}_{\Omega,G})$. Otherwise, there exists a quasi-isomorphic bounded quotient complex $P = \tau_{\ge n}L$ of L of objects in $\mathcal{C}_{\Omega-\Sigma}$, through which the quasi-isomorphism $L \to N$ factors, such that the terms of P are projective in $\mathcal{C}_{\Sigma^\circ}$. Proposition 2.4.7 then tells us that $P \mathbin{\hat{\otimes}_\Sigma} Y$ represents $N \mathbin{\hat{\otimes}^{\mathbf{L}}_\Sigma} Y$ in $\mathbf{D}(\mathcal{C}_{\Omega,G})$. In all cases, the terms of P are flat Σ°-modules by Lemma 2.4.3, so Proposition 2.1.4 tells us that $P \otimes_\Sigma C(G,Y)$ represents $N \otimes^{\mathbf{L}}_\Sigma \mathbf{R}\Gamma(G,Y)$ in $\mathbf{D}(\mathrm{Mod}_\Omega)$.
We have a map of complexes of Ω-modules

$$P \otimes_\Sigma C(G,Y) \longrightarrow C(G, P \mathbin{\hat{\otimes}_\Sigma} Y)$$

with sign conventions as in [Nek, Proposition 3.4.4]. (The continuity of the cochains in the image is insured, for instance, by the fact that any term of P is a topological direct summand of a direct product of copies of Σ.) It suffices to show that this map is a quasi-isomorphism.
Let Q be a bounded above complex of finitely generated projective Σ°-modules mapping quasi-isomorphically to N, which we take to be bounded if (2) holds. We then have a map of complexes of R-modules

$$Q \otimes_\Sigma C(G,Y) \xrightarrow{\sim} C(G, Q \otimes_\Sigma Y)$$

that, much as in [Nek, Proposition 3.4.4], is an isomorphism. (To see that it is an isomorphism, note that it is immediate if Q is a finitely generated free Σ°-module, and therefore also for direct summands of such, i.e., the finitely generated projective Σ°-modules. The case of a complex is then immediate.) By the projectivity of Q in both the categories of abstract and compact Σ°-modules, we have a commutative diagram

$$\begin{tikzcd}
Q \otimes_\Sigma C(G,Y) \arrow[r, "\sim"] \arrow[d] & C(G, Q \otimes_\Sigma Y) \arrow[d] \\
P \otimes_\Sigma C(G,Y) \arrow[r] & C(G, P \mathbin{\hat{\otimes}_\Sigma} Y)
\end{tikzcd}$$

in which the vertical arrows are quasi-isomorphisms of complexes of R-modules, noting Lemma 2.4.2(b) and, when (1) holds, Corollary 3.1.2. It follows that the lower horizontal arrow is a quasi-isomorphism as well. $\square$

We make a couple of remarks.

Remark 3.1.4. Suppose that G is a finite group. The proof of Proposition 3.1.3 under condition (2) then goes through word-for-word for bounded Y, with Tate cochains and the resulting derived complexes replacing the usual cochains of G and its derived complexes.

Remark 3.1.5. In [FK1, Proposition 1.6.5(3)], an identical isomorphism to that of Proposition 3.1.3 is proved for a very general class of "adic" rings Ω and Λ, with stronger conditions on G, N, and Y.

From now on, we focus on case that Σ is the Iwasawa algebra $\Lambda = R[\![\Gamma]\!]$. Fix a continuous homomorphism $\chi\colon G \to \Gamma$ of profinite groups. Since Γ may be viewed as a subgroup of $\mathrm{Aut}_{\Lambda^\circ}(\Lambda)$ by left multiplication, we may define the $\Lambda^\circ[G]$-module ${}_\chi\Lambda$ as in Section 2.2. If A is a $\Lambda \otimes_R \Lambda^\circ$-module, we then set ${}_\chi A = {}_\chi\Lambda \otimes_\Lambda A$ and $A_\chi = A \otimes_\Lambda \Lambda_\chi$, which are $\Lambda^\circ[G]$ and $\Lambda[G]$-modules, respectively.

Let M be a $R[G]$-module. As in [Lim2, Section 5.1], we define a $\Lambda[G]$-module $\mathscr{F}_\Gamma(M)$ by

$$\mathscr{F}_\Gamma(M) = \varprojlim_{U \in \mathscr{U}} (R[\Gamma/U]_\chi \otimes_R M)$$

with G acting diagonally and Λ acting on the left on the terms in the inverse limit. This construction applied to compact modules provides a functor

$$\mathscr{F}_\Gamma\colon \mathcal{C}_{R,G} \longrightarrow \mathcal{C}_{\Lambda,G},$$

which is exact as $R[\Gamma/U]$ is R-flat and the inverse limit is exact on inverse systems in $\mathcal{C}_{\Lambda,G}$. We extend this to an exact functor on complexes in the obvious fashion.

Given an complex T of objects in $\mathcal{C}_{R,G}$, we have, noting Lemma 2.4.2(a), a natural continuous isomorphism

$$\mathscr{F}_\Gamma(T) \cong \Lambda_\chi \hat{\otimes}_R T$$

of complexes in $\mathcal{C}_{\Lambda,G}$. We may therefore use Lemma 2.4.2(b) to note that $\mathscr{F}_\Gamma$ takes a complex T of objects in $\mathcal{C}_{R,G}^{R-\mathrm{ft}}$ to the complex $\Lambda_\chi \otimes_R T$ of objects in $\mathcal{C}_{\Lambda,G}^{\Lambda-\mathrm{ft}}$. We make the latter identification freely.

We now supply a key ingredient for descent.

Lemma 3.1.6. *Let T be a complex in $\mathcal{C}_{R,G}$. Let Γ' be a quotient of Γ by a closed normal subgroup, and set $\Lambda' = R[\![\Gamma']\!]$. Let N be a bounded above complex of objects of $\mathcal{C}_{\Omega-\Lambda'}$. Then N can be viewed as a complex of objects in $\mathcal{C}_{\Omega-\Lambda}$ via the quotient map $\pi\colon \Lambda \twoheadrightarrow \Lambda'$. Suppose either that T is bounded above or that N is bounded and its terms have finite projective dimension in $\mathcal{C}_{\Lambda^\circ}$ and in $\mathcal{C}_{(\Lambda')^\circ}$. Then π induces an isomorphism*

$$N \hat{\otimes}^{\mathbf{L}}_\Lambda \mathscr{F}_\Gamma(T) \xrightarrow{\sim} N \hat{\otimes}^{\mathbf{L}}_{\Lambda'} \mathscr{F}_{\Gamma'}(T)$$

in $\mathbf{D}^-(\mathcal{C}_{\Omega,G})$.

Proof. Let $Q \to N$ be a quasi-isomorphism, where Q is a bounded above complex of $\mathcal{C}_{\Omega-\Lambda'}$-projectives, and note that its terms are projective objects of $\mathcal{C}_{(\Lambda')^\circ}$ by Lemma 2.4.3(b). If N is bounded with terms of finite projective dimension in $\mathcal{C}_{(\Lambda')^\circ}$, then for sufficiently small n, the truncation $\tau_{\geq n}Q$ also consists of projectives in $\mathcal{C}_{(\Lambda')^\circ}$, so we may assume in that case that Q is bounded. Next, and similarly, let $P \to Q$ be a quasi-isomorphism in $\mathcal{C}_{\Omega-\Lambda}$ with P a bounded above complex in $\mathcal{C}_{\Omega-\Lambda}$ of projectives in $\mathcal{C}_{\Lambda^\circ}$, and suppose that P is bounded if N is bounded with terms having finite projective dimension in $\mathcal{C}_{\Lambda^\circ}$. The derived completed tensor products in question are then represented by $P \hat{\otimes}_\Lambda \Lambda_\chi \hat{\otimes}_R T$ and $Q \hat{\otimes}_{\Lambda'} \Lambda'_\chi \hat{\otimes}_R T$, respectively, by Propositions 2.4.6 and 2.4.7. Since the induced map $P \hat{\otimes}_\Lambda \Lambda_\chi \to Q \hat{\otimes}_{\Lambda'} \Lambda'_\chi$ is clearly a quasi-isomorphism and the terms of these complexes are projective in $\mathcal{C}_R$, we have by the same propositions the desired quasi-isomorphism between the two complexes. ◻

We are particularly interested in the case that the complex N above is the Iwasawa algebra of a quotient of Γ, for which the following lemma is crucial.

Lemma 3.1.7. *Let Γ_0 be a closed normal subgroup of Γ with no elements of order p. Set $\Gamma' = \Gamma/\Gamma_0$ and $\Lambda' = R[\![\Gamma']\!]$. Then Λ' has finite projective dimension over Λ.*

Proof. Let $\text{c-Tor}_i^{\Xi}(-,-)$ denote the ith derived bifunctor of the completed tensor product over a profinite R-algebra Ξ. A standard argument yields a convergent spectral sequence

$$H_i(\Gamma, \text{c-Tor}_j^R(\Lambda', Z)) \Rightarrow \text{c-Tor}_{i+j}^{\Lambda}(\Lambda', Z)$$

for any compact Λ-module Z, as in [NSW, §V.2, Exercise 3], where elements of Γ act inversely on Λ°-modules and diagonally on completed R-tensor products. Since Λ' is a projective object in $\mathcal{C}_R$, the spectral sequence degenerates to yield isomorphisms

$$H_n(\Gamma, \Lambda' \hat{\otimes}_R Z) \cong \text{c-Tor}_n^{\Lambda}(\Lambda', Z).$$

Moreover, by a standard extension of Shapiro's lemma [RZ, Theorem 6.10.9], we have

$$H_n(\Gamma_0, Z) \cong H_n(\Gamma, \Lambda' \hat{\otimes}_R Z).$$

Since Γ_0 has finite p-cohomological dimension by [Ser, Corollaire 1], there then exists $n_0 \geq 0$ independent of Z such that $\text{c-Tor}_n^{\Lambda}(\Lambda', Z) = 0$ for every $n > n_0$. It follows from [NSW, Proposition 5.2.11] that Λ' has finite topological projective dimension as a compact Λ-module, so it has finite projective dimension over Λ by the discussion of Section 2.3. ◻

The following descent spectral sequence is reduced by the above results to a special case of Proposition 3.1.3.

Theorem 3.1.8. *Let T be a complex in $\mathcal{C}_{R,G}$. Let Γ' be a quotient of Γ by a closed normal subgroup Γ_0, and set $\Lambda' = R[\![\Gamma']\!]$. Consider the conditions*

(1) *G has finite cohomological dimension,*

(2) *Γ_0 contains no elements of order p.*

Suppose that (1) *holds and T is bounded above,* (2) *holds and T is bounded below, or both* (1) *and* (2) *hold. Then we have an isomorphism*

$$\Lambda' \otimes^{\mathbf{L}}_{\Lambda} \mathbf{R}\Gamma(G, \mathscr{F}_{\Gamma}(T)) \xrightarrow{\sim} \mathbf{R}\Gamma(G, \mathscr{F}_{\Gamma'}(T))$$

in $\mathbf{D}(\mathrm{Mod}_{\Lambda'})$.

Proof. This is simply Proposition 3.1.3 for $N = \Lambda'$ and $Y = \mathscr{F}_{\Gamma}(T)$, noting Lemma 3.1.7 and applying the isomorphism of Lemma 3.1.6. □

3.2 Finite generation of cohomology groups

In this subsection, we shall show that the cohomology groups of induced modules are finitely generated under a certain assumption on the group. We maintain the notation of the previous subsection. In particular, Γ is a compact p-adic Lie group, R is a complete commutative local Noetherian ring with finite residue field of characteristic p, and $\Lambda = R[\![\Gamma]\!]$. Moreover, we are given a continuous homomorphism $\chi\colon G \to \Gamma$ from a profinite group G.

Lemma 3.2.1. *Suppose that Γ is pro-p. Let M be a compact Λ-module. Then M is finitely generated over Λ if and only if $R \otimes_{\Lambda} M$ is finitely generated over R.*

Proof. Since Γ is pro-p, the Jacobson radical $\mathfrak{M}$ of Λ is $\mathfrak{m}\Lambda + I$, where I is the augmentation ideal of Λ (see [NSW, Proposition 5.2.16(iii)]). This implies that

$$M/\mathfrak{M}M \cong M_{\Gamma}/\mathfrak{m}M_{\Gamma},$$

where $M_{\Gamma} = M/IM$. Therefore, Nakayama's lemma tells us that M_{Γ} is finitely generated over R if and only if $M/\mathfrak{M}M$ is finite. On the other hand, Nakayama's lemma for compact Λ-modules (cf. [NSW, Lemma 5.2.18(ii)]) tells us that M is finitely generated over Λ if and only if $M/\mathfrak{M}M$ is finite. □

Lemma 3.2.1 requires a compact Λ-module. Therefore, we give a sufficient condition for an abstract Λ-module to be a compact Λ-module under an appropriate topology.

Lemma 3.2.2. *Suppose that M is an abstract Λ-module which is the inverse limit of an inverse system of finite quotient modules. Then M is a compact Λ-module with respect to the resulting profinite topology.*

Proof. We need only to show that the Λ-action

$$\theta\colon \Lambda \times M \longrightarrow M$$

is continuous with respect to the topology given by the inverse limit. Suppose first that M is finite. Let $\mathfrak{M}$ denote the Jacobson radical of Λ. Then $\mathfrak{M}^n M$ stabilizes, and it follows from Nakayama's lemma that we have $\mathfrak{M}^n M = 0$ for large enough n. By [NSW, Corollary 5.2.19], there exist $r \geq 0$ and $U \in \mathscr{U}$

such that $\mathfrak{m}^r\Lambda + I(U) \subseteq \mathfrak{M}^n$, where $I(U)$ denotes the ideal of Λ generated by the augmentation ideal in $R[\![U]\!]$. Hence, θ is continuous with respect to the discrete topology on M.

In general, we can write $M \cong \varprojlim M/M_\alpha$, where $\{M_\alpha\}$ is a directed system of Λ-submodules of finite index. Let $(\lambda, x) \in \theta^{-1}(y + M_\alpha)$ for $\lambda \in \Lambda$ and $x, y \in M$. Since M/M_α is finite, it follows from the above discussion that there exist r and U such that

$$(\lambda + \mathfrak{m}^r\Lambda + I(U)) \cdot (x + M_\alpha) \subseteq y + M_\alpha,$$

as desired. □

We shall also require the following lemma.

Lemma 3.2.3. *If M is a finitely generated Λ-module, then $\mathrm{Tor}_i^\Lambda(R, M)$ is finitely generated over R for every $i \geq 0$.*

Proof. To see this, we first choose a resolution P of M consisting of finitely generated projective Λ-modules. Then $R \otimes_\Lambda P$ is a complex of finitely generated R-modules. Therefore, the homology groups $\mathrm{Tor}_i^\Lambda(R, M)$ of the latter complex are finitely generated over R. □

We recall from [Nek, Proposition 4.2.3] that if G is a profinite group such that $H^i(G, M)$ is finite for every finite G-module M of p-power order and $i \geq 0$, then $H^i(G, T)$ is a finitely generated R-module for every $T \in \mathcal{C}_{R,G}^{R-\mathrm{ft}}$ and $i \geq 0$. We will refer to a profinite group G as p-cohomologically finite if there exists an integer n such that, for every finite G-module M of p-power order, $H^i(G, M) = 0$ for all $i > n$ and $H^i(G, M)$ is finite for all i. We remark that if G is p-cohomologically finite, then so is every open subgroup of G, as follows directly from Shapiro's lemma.

Theorem 3.2.4. *Suppose that G is p-cohomologically finite. Let T be a complex of objects in $\mathcal{C}_{R,G}^{R-\mathrm{ft}}$. Then the cohomology groups $H^i(G, \mathscr{F}_\Gamma(T))$ are finitely generated over Λ for all i. That is, $\mathbf{R}\Gamma(G, \mathscr{F}_\Gamma(T))$ is an object of $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$.*

Proof. Suppose for now that T is concentrated in degree 0. Let Γ_0 be an open uniform normal pro-p subgroup of Γ, and set $H = \chi^{-1}(\Gamma_0)$. Fix a set of double coset representatives $\gamma_1, \ldots, \gamma_t$ of $\Gamma_0 \backslash \Gamma / \chi(G)$. For each i, define $\chi_i \colon G \to \Gamma$ by $\chi_i(g) = \gamma_i g \gamma_i^{-1}$. The reader may check that

$$\bigoplus_{i=1}^t \Big(\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} R[\![\Gamma_0]\!]_{\chi_i} \Big) \xrightarrow{\sim} \Lambda_\chi$$
$$(g \otimes \lambda_0)_i \mapsto \lambda_0 \gamma_i \chi(g)^{-1}$$

is an isomorphism of $R[\![\Gamma_0]\!][G]$-modules. Given this, Shapiro's lemma induces isomorphisms

$$H^j(G, \mathscr{F}_\Gamma(T)) \cong \bigoplus_{i=1}^t H^j(H, R[\![\Gamma_0]\!]_{\chi_i} \otimes_R T)$$

of $R[\![\Gamma_0]\!]$-modules. The finite generation of $H^j(G, \mathscr{F}_\Gamma(T))$ over Λ then reduces to the finite generation of each $H^j(H, R[\![\Gamma_0]\!]_{\chi_i} \otimes_R T)$ over $R[\![\Gamma_0]\!]$. Thus, we can and do assume that Γ has no elements of finite order.

Since R has finite projective dimension over Λ by Lemma 3.1.7, Theorem 3.1.8 provides the convergent spectral sequence

$$E_2^{r,s} = \mathrm{Tor}^{\Lambda}_{-r}(R, H^s(G, \mathscr{F}_\Gamma(T))) \Rightarrow H^{r+s}(G, T).$$

Since G has finite p-cohomological dimension, say n, we may suppose inductively that $H^i(G, \mathscr{F}_\Gamma(T))$ is a finitely generated Λ-module for all i greater than some $j \leq n$. Note that $E_\infty^{0,j}$ is a quotient of $E_2^{0,j}$ and that $H^j(G,T)$ is a finitely generated R-module by [Nek, Proposition 4.2.3]. Since $E_\infty^{0,j}$ is a subquotient of $H^j(G,T)$, it is also a finitely generated R-module. On the other hand, it follows from the definition of $E_\infty^{0,j}$ that the kernel of the surjective map $E_2^{0,j} \to E_\infty^{0,j}$ is isomorphic to a subquotient of the finite direct sum

$$\bigoplus_{i=j+1}^{n} E_2^{j-i-1,i} = \bigoplus_{i=j+1}^{n} \mathrm{Tor}^{\Lambda}_{i-j+1}(R, H^i(G, \mathscr{F}_\Gamma(T))).$$

By our induction hypothesis and Lemma 3.2.3, the above module is a finitely generated R-module. It follows that

$$E_2^{0,j} \cong R \otimes_\Lambda H^j(G, \mathscr{F}_\Gamma(T))$$

is a finitely generated R-module. As $H^j(G, \mathscr{F}_\Gamma(T))$ is an inverse limit of finite Λ-modules by [Lim2, Proposition 5.2.4], Lemma 3.2.2 gives it the structure of a compact Λ-module, and we may therefore apply Lemma 3.2.1 to conclude that $H^j(G, \mathscr{F}_\Gamma(T))$ is finitely generated over Λ.

Now let us work without condition on Γ or the complex T. By Lemma 3.1.1 and the exactness of $\mathscr{F}_\Gamma(-)$, we have the convergent spectral sequence of Λ-modules

$$H^r(G, \mathscr{F}_\Gamma(H^s(T))) \Rightarrow H^{r+s}(G, \mathscr{F}_\Gamma(T)).$$

Thus, $H^i(G, \mathscr{F}_\Gamma(T))$ has a filtration consisting of subquotients of the finitely generated Λ-modules $H^j(G, \mathscr{F}_\Gamma(H^{i-j}(T)))$ with $0 \leq j \leq n$. Consequently, $H^i(G, \mathscr{F}_\Gamma(T))$ is also a finitely generated Λ-module. □

Remark 3.2.5. In the case that Γ is abelian, Theorem 3.2.4 for bounded below T is essentially [Nek, Proposition 4.2.3], as $R[\![\Gamma_0]\!]$ for an open pro-p, torsion-free subgroup Γ_0 is itself a commutative complete local Noetherian ring with finite residue field. In the case that T is a bounded complex of R-projectives, it is a corollary of [FK1, Proposition 1.6.5(2)].

3.3 Duals of induced modules

In this subsection, we describe the pairings and the resulting cup products that will be used in proving a duality theorem for the cohomology of induced modules. Key to our discussion is the following lemma, which we record for convenience. We let Ω and Σ denote auxiliary Iwasawa algebras: e.g., $\Omega = R[\![\Phi]\!]$ for some compact p-adic Lie group Φ.

Lemma 3.3.1. *Suppose that M is a bounded above complex of objects of $\mathcal{C}_{\Omega-\Lambda,G}$, let N be a bounded above complex of objects of $\mathcal{C}_{\Lambda-\Sigma,G}$, and let L be a bounded above complex of objects of $\mathcal{C}_{\Omega-\Sigma,G}$. Then any map*

$$\phi\colon M \hat{\otimes}_\Lambda N \longrightarrow L$$

of complexes in $\mathcal{C}_{\Omega-\Sigma,G}$ gives rise to a cup product morphism

$$\cup\colon C(G,M) \otimes_\Lambda C(G,N) \longrightarrow C(G,L)$$

of complexes of $\Omega \otimes_\Lambda \Sigma^\circ$-modules.

Proof. Briefly, the point is that to give such a map ϕ is equivalent to producing a collection of continuous, Λ-balanced, G-equivariant, left Ω-linear, and right Σ-linear pairings

$$\langle -,- \rangle_{mn}\colon M^m \times N^n \longrightarrow L^{m+n}$$

that are compatible with coboundaries in the sense of [Lim2, Section 3.3]. The cup product $\cup$ then arises from the induced cup products

$$\cup_{mn}^{ij}\colon C^i(G,M^m) \otimes_\Lambda C^j(G,N^n) \longrightarrow C^{i+j}(G,L^{m+n})$$

by combining them to a map of the total complexes with appropriate signs as in [Nek, (3.4.5.2)]: $\cup = ((-1)^{in}\cup_{mn}^{ij})$. □

Let us fix some notation. Let $\iota\colon \Lambda \to \Lambda$ denote the unique continuous R-algebra homomophism that satisfies $\iota(\gamma) = \gamma^{-1}$ for all $\gamma \in \Gamma$. If A is a $\Lambda[G]$-module, we let A^ι denote the $\Lambda^\circ[G]$-module that is A as an $R[G]$-module but on which any $\lambda \in \Lambda$ now acts by left multiplication by $\iota(\lambda)$. We extend this to complexes in the obvious fashion. Of particular interest to us is $\mathscr{F}_\Gamma(T)^\iota$ for a complex T of objects in $\mathcal{C}_{R,G}$. Since ι induces a continuous isomorphism of $\Lambda^\circ[G]$-modules ${}_\chi\Lambda \xrightarrow{\sim} \Lambda^\iota_\chi$, we can and will make the identification

$$\mathscr{F}_\Gamma(T)^\iota = {}_\chi\Lambda \hat{\otimes}_R T,$$

of complexes in $\mathcal{C}_{\Lambda^\circ,G}$ from this point forward.

Lemma 3.3.2. *For objects M and N in $\mathcal{C}_{R,G}$, the pairing*

$$\langle -,- \rangle\colon \mathscr{F}_\Gamma(M) \times \mathscr{F}_\Gamma(N)^\iota \longrightarrow \Lambda \hat{\otimes}_R (M \hat{\otimes}_R N)$$
$$(\lambda \otimes m, \mu \otimes n) \mapsto \lambda\mu \otimes m \otimes n$$

is continuous, R-balanced, G-equivariant with respect to the trivial action on Λ in the target, and is left and right Λ-linear. Moreover, if M and N are bounded above complexes of objects in $\mathcal{C}_{R,G}$, we have a morphism

$$\phi\colon \mathscr{F}_\Gamma(M) \hat{\otimes}_R \mathscr{F}_\Gamma(N)^\iota \longrightarrow \Lambda \hat{\otimes}_R (M \hat{\otimes}_R N)$$

of complexes in $\mathcal{C}_{\Lambda-\Lambda,G}$.

Proof. We begin by remarking that by Lemma 2.4.1, it makes sense to define the pairing on pairs of tensors, identified with their images in the completed tensor product. The pairing is clearly R-balanced, and the reader can check the statements on the Λ, Λ°, and G-actions. The continuity of the pairing reduces immediately to the continuity of multiplication on Λ. The second statement follows easily. □

We next apply these pairings to study a duality between induced modules. As described in the introduction, the dualizing complex for R is an object ω_R of $\mathbf{D}^b_{R-\mathrm{ft}}(\mathrm{Mod}_R)$ with the property that for every object M of $\mathbf{D}(\mathrm{Mod}_R^{R-\mathrm{ft}})$, the object $\mathbf{R}\mathrm{Hom}_R(M, \omega_R)$ lies in $\mathbf{D}_{R-\mathrm{ft}}(\mathrm{Mod}_R)$ and the canonical morphism

$$M \longrightarrow \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, \omega_R), \omega_R)$$

is an isomorphism in $\mathbf{D}(\mathrm{Mod}_R)$. A dualizing complex exists for R and is unique up to translation and isomorphism in $\mathbf{D}^b_{R-\mathrm{ft}}(\mathrm{Mod}_R)$ (see [Har1, Ch. V]). We choose a bounded complex J_R of injective R-modules which represents a choice of the dualizing complex in $\mathbf{D}(\mathrm{Mod}_R)$. (See [Nek, Section 0.4], or Section 4.3 below, for the construction of such a complex.)

Let T be a bounded complex of objects in $\mathcal{C}^{R-\mathrm{ft}}_{R,G}$. Then $\mathrm{Hom}_R(T, J_R)$ is a bounded complex of "admissible" $R[G]$-modules with cohomology groups that are finitely generated over R (see [Nek, (4.3.2)]). By [Nek, Proposition 3.3.9], there is a subcomplex T^* of $\mathrm{Hom}_R(T, J_R)$ which is a complex of objects in $\mathcal{C}^{R-\mathrm{ft}}_{R,G}$ (giving said objects the $\mathfrak{m}$-adic topology) and is quasi-isomorphic to $\mathrm{Hom}_R(T, J_R)$ via the inclusion map.

We have a composite morphism

$$\pi\colon T \otimes_R T^* \longrightarrow T \otimes_R \mathrm{Hom}_R(T, J_R) \longrightarrow J_R$$

of complexes of $R[G]$-modules, the first morphism being induced by the inclusion and the second being the usual evaluation map. By Lemma 3.3.2, this in turn induces a composite map

$$\overline{\pi}\colon \mathscr{F}_\Gamma(T) \hat{\otimes}_R \mathscr{F}_\Gamma(T^*)^\iota \xrightarrow{\phi} \Lambda \otimes_R T \otimes_R T^* \xrightarrow{\mathrm{id}\otimes\pi} \Lambda \otimes_R J_R$$

of complexes of $(\Lambda \otimes_R \Lambda^\circ)[G]$-modules. Twisting $\overline{\pi}$ by the identity map on an auxilliary bounded above complex of objects of $\mathcal{C}^{\Lambda^\circ-\mathrm{ft}}_{\Omega-\Lambda}$, Lemma 3.3.1 now provides the cup product morphisms of the following lemma, which will be used in the next section. For a continuous character $\kappa\colon G \to R^\times$ and an $R[G]$-module M, we let $M(\kappa)$ denote the R-module M with the new commuting G-action given by the twist of the original by κ. (In addition, $M(\kappa)$ will be taken to maintain any topology and actions of profinite R-algebras with which M may be endowed.)

Lemma 3.3.3. *Let A be a bounded above complex of objects in $\mathcal{C}^{\Lambda^\circ-\mathrm{ft}}_{\Omega-\Lambda}$, and let $\kappa\colon G \to R^\times$ be a continuous homomorphism. The map $\overline{\pi}$ defined above induces a map*

$$\overline{\pi}_A\colon A \otimes_\Lambda \mathscr{F}_\Gamma(T) \hat{\otimes}_R \mathscr{F}_\Gamma(T^*)^\iota \longrightarrow A \otimes_R J_R$$

and, in turn, a well-defined cup product

$$C(G, A \otimes_\Lambda \mathscr{F}_\Gamma(T)) \otimes_R C(G, \mathscr{F}_\Gamma(T^*)^\iota(\kappa)) \longrightarrow C(G, A \otimes_R J_R(\kappa)),$$

which is a map of complexes of $\Omega \otimes_R \Lambda^\circ$*-modules.*

Note that, by definition, the adjoint maps (defined as in [Lim2, Lemma 2.2])

$$\mathrm{adj}(\pi)\colon T^* \longrightarrow \mathrm{Hom}_R(T, J_R) \quad \text{and} \quad \mathrm{adj}'(\pi)\colon T \longrightarrow \mathrm{Hom}_R(T^*, J_R)$$

of π are quasi-isomorphisms, the first being simply the inclusion of complexes already defined. Since the terms of J_R are injective R-modules, it follows from Proposition 2.2.1 that the derived adjoint maps

$$\mathbf{adj}(\pi)\colon T^* \longrightarrow \mathbf{R}\mathrm{Hom}_R(T, \omega_R) \quad \text{and} \quad \mathbf{adj}'(\pi)\colon T \longrightarrow \mathbf{R}\mathrm{Hom}_R(T^*, \omega_R)$$

are isomorphisms in $\mathbf{D}^b_{R-\mathrm{ft}}(\mathrm{Mod}_{R[G]})$, hence in $\mathbf{D}^b_{R-\mathrm{ft}}(\mathscr{I}_{R,G})$, considering the restriction of $\mathbf{R}\mathrm{Hom}_R(-,-)$ to a bifunctor

$$\mathbf{D}^-(\mathcal{C}_{R,G}^{R-\mathrm{ft}})^\circ \times \mathbf{D}^+(\mathrm{Mod}_R) \longrightarrow \mathbf{D}^+(\mathscr{I}_{R,G}),$$

(see [Nek, (3.5.9)]).

Though we shall not use it later, we feel it important to note that the derived adjoint maps of $\overline{\pi}$ are also isomorphisms. While this is easy enough to prove in the bounded below derived category $\mathbf{D}^+(\mathrm{Mod}_{\Lambda^\circ[G]})$ of abstract modules (in the case of $\mathbf{adj}(\overline{\pi})$), we are interested in G-cohomology groups, so we want such an isomorphism in $\mathbf{D}^+(\mathscr{I}_{\Lambda^\circ,G})$.

THEOREM 3.3.4. *For any bounded complexes T and T^* of objects in $\mathcal{C}_{R,G}^{R-\mathrm{ft}}$ such that T^* sits quasi-isomorphically as a subcomplex of the dual $\mathrm{Hom}_R(T, J_R)$, the derived adjoint maps*

$$\mathbf{adj}(\overline{\pi})\colon \mathscr{F}_\Gamma(T^*)^\iota \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda,\mathrm{cts}}(\mathscr{F}_\Gamma(T), \Lambda \otimes^{\mathbf{L}}_R \omega_R)$$
$$\big(\textit{resp., } \mathbf{adj}'(\overline{\pi})\colon \mathscr{F}_\Gamma(T) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ,\mathrm{cts}}(\mathscr{F}_\Gamma(T^*)^\iota, \Lambda \otimes^{\mathbf{L}}_R \omega_R)\big)$$

are isomorphisms in $\mathbf{D}^+(\mathscr{I}_{\Lambda^\circ,G})$ (resp., $\mathbf{D}^+(\mathscr{I}_{\Lambda,G})$). Moreover, the derived object $\mathbf{R}\mathrm{Hom}_{\Lambda,\mathrm{cts}}(\mathscr{F}_\Gamma(T), \Lambda \otimes^{\mathbf{L}}_R \omega_R)$ (resp., $\mathbf{R}\mathrm{Hom}_{\Lambda^\circ,\mathrm{cts}}(\mathscr{F}_\Gamma(T^*)^\iota, \Lambda \otimes^{\mathbf{L}}_R \omega_R)$) can be represented by $\mathrm{Hom}_\Lambda(\mathscr{F}_\Gamma(T), \Lambda \otimes_R J_R)$ (resp., $\mathrm{Hom}_{\Lambda^\circ}(\mathscr{F}_\Gamma(T^*)^\iota, \Lambda \otimes_R J_R)$).

Proof. Let $X = \mathbf{R}\mathrm{Hom}_{\Lambda,\mathrm{cts}}(\mathscr{F}_\Gamma(T), \Lambda \otimes^{\mathbf{L}}_R \omega_R)$. Since J_R is a complex of R-injectives and ${}_\chi\Lambda$ is flat over R, Lemma 2.5.9 implies that the functor F given by

$$\mathrm{Hom}_{\Lambda,\mathrm{cts}}(\Lambda_\chi \hat{\otimes}_R -, \Lambda \otimes_R J_R)\colon \mathrm{Ch}^-(\mathcal{C}_{R,G})^\circ \longrightarrow \mathrm{Ch}^+(\mathscr{I}_{\Lambda^\circ,G})$$

is exact. In particular, if $P \to T$ is a quasi-isomorphism with P a bounded above complex of projective objects in $\mathcal{C}_{R,G}$, then $F(T) \to F(P)$ is a quasi-isomorphism. As $\Lambda_\chi \hat{\otimes}_R P$ is a complex of objects in $\mathcal{C}_{\Lambda,G}$ that are projective in $\mathcal{C}_\Lambda$, we then have that $F(T)$ represents X by Proposition 2.5.8.

By Lemmas 2.4.2(b) and 2.5.7, the functor F equals $\mathrm{Hom}_\Lambda(\Lambda_\chi \otimes_R -, \Lambda \otimes_R J_R)$ on $\mathrm{Ch}^-(\mathcal{C}_{R,G}^{R-\mathrm{ft}})^\circ$. Therefore, X is represented by $\mathrm{Hom}_\Lambda(\mathscr{F}_\Gamma(T), \Lambda \otimes_R J_R)$ in $\mathbf{D}^+(\mathscr{I}_{\Lambda^\circ,G})$. Finally, since the map

$$T^* \longrightarrow \mathrm{Hom}_R(T, J_R)$$

is a quasi-isomorphism, the adjoint map of complexes

$${}_\chi\Lambda \otimes_R T^* \longrightarrow \mathrm{Hom}_\Lambda(\Lambda_\chi \otimes_R T, \Lambda \otimes_R J_R)$$

is a quasi-isomorphism of complexes of $\Lambda^\circ[G]$-modules by Lemma 2.2.3. Therefore, we have the result for $\mathbf{adj}(\overline{\pi})$, and the proof for $\mathbf{adj}'(\overline{\pi})$ is analogous. □

4 Duality over p-adic Lie extensions

We now turn to arithmetic. Here, we fix the notation that we shall use throughout this section. To start, let p be a prime. We let F be a global field of characteristic not equal to p. Let S be a finite set of primes of F that, in the case that F is a number field, contains all primes above p and all real places. Let S_f (resp., $S_\mathbb{R}$) denote the set of finite places (resp., real places) in S. Let $G_{F,S}$ denote the Galois group of the maximal unramified outside S extension of F. For a place v of F, let F_v denote the completion of F at v, and let G_v denote a fixed decomposition group for v in the absolute Galois group of F.
We fix a p-adic Lie extension F_∞ of F that is unramified outside S, and we let Γ denote its Galois group. We let R denote a complete commutative local Noetherian ring with finite residue field of characteristic p, and we set $\Lambda = R[\![\Gamma]\!]$. We take the homomorphism $\chi\colon G_{F,S} \to \Gamma$ of Section 3.1 to be restriction.

4.1 Iwasawa cohomology

We recall the following facts, all of which can all be found in [NSW, Chapters VII-VIII]. The $G_{F,S}$-cohomology groups of a finite $G_{F,S}$-module of p-power order are all finite. If p is odd or F has no real places, then $G_{F,S}$ has p-cohomological dimension at most 2 and so is p-cohomologically finite in the sense of Section 3.2. If v is a nonarchimedean place of F, then G_v has p-cohomological dimension equal to 2, and the G_v-cohomology groups of a finite G_v-module of p-power order are finite as well. Of course, the G_v-cohomology groups of a finite module are also finite for archimedean v, since G_v is of order dividing 2 for such places.
Recall (e.g., from [Nek, (5.7.2)]) that, for any profinite ring Ω, the (Tate) compactly supported $G_{F,S}$-cochain complex of a complex of objects M in $\mathcal{C}_{\Omega,G_{F,S}}$ is defined as

$$C_{(c)}(G_{F,S}, M) = \mathrm{Cone}\left(C(G_{F,S}, M) \to \bigoplus_{v \in S_f} C(G_v, M) \oplus \bigoplus_{v \in S_\mathbb{R}} \widehat{C}(G_v, M) \right)[-1],$$

where $\widehat{C}(G_v, M)$ is defined as in [Lim2, Section 3.4] and denotes the standard complete complex of Tate G_v-cochains for M.
We remark that, in Nekovář's notation, $C_{(c)}(G_{F,S}, M)$ is denoted $\widehat{C}_c(G_{F,S}, M)$. We use the notation of [FK1], where parentheses are used to distinguish the latter group from the compactly supported cochains $C_c(G_{F,S}, M)$, for which one uses the usual cohomology groups at real places. For archimedean v, we will abuse notation and use $\mathbf{R}\Gamma(G_v, M)$ to denote the derived object of the Tate cochains $\widehat{C}(G_v, M)$. These cochains may as well be taken to be zero for complex places or for real places if p is odd, $\mathbf{R}\Gamma(G_v, M)$ being a zero object in the derived category. We denote the derived object corresponding to $C_{(c)}(G_{F,S}, M)$ by $\mathbf{R}\Gamma_{(c)}(G_{F,S}, M)$ and its ith cohomology group by $H^i_{(c)}(G_{F,S}, M)$. By definition, we have an exact triangle[2]

$$\mathbf{R}\Gamma_{(c)}(G_{F,S}, M) \longrightarrow \mathbf{R}\Gamma(G_{F,S}, M) \longrightarrow \bigoplus_{v\in S} \mathbf{R}\Gamma(G_v, M)$$

in $\mathbf{D}(\mathrm{Mod}_\Omega)$. The results of and methods used in Section 3.1 allow us to prove the following descent result for such an exact triangle with induced coefficients.

PROPOSITION 4.1.1. *Let $\Gamma' = \mathrm{Gal}(F'_\infty/F)$ be a quotient of Γ by a closed normal subgroup, and set $\Lambda' = R[\![\Gamma']\!]$. Let T be a bounded above complex of objects in $\mathcal{C}_{R,G_{F,S}}$. Suppose that at least one of the following holds: (i) p is odd, (ii) F'_∞ has no real places, or (iii) T is bounded and F'_∞ has no real places that become complex in F_∞. Then we have an isomorphism of exact triangles*

$$\begin{tikzcd}
\Lambda' \otimes^{\mathbf{L}}_\Lambda \mathbf{R}\Gamma_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T)) \arrow[r, "\sim"] \arrow[d] & \mathbf{R}\Gamma_{(c)}(G_{F,S}, \mathscr{F}_{\Gamma'}(T)) \arrow[d] \\
\Lambda' \otimes^{\mathbf{L}}_\Lambda \mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_\Gamma(T)) \arrow[r, "\sim"] \arrow[d] & \mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_{\Gamma'}(T)) \arrow[d] \\
\Lambda' \otimes^{\mathbf{L}}_\Lambda \bigoplus_{v\in S} \mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T)) \arrow[r, "\sim"] & \bigoplus_{v\in S} \mathbf{R}\Gamma(G_v, \mathscr{F}_{\Gamma'}(T))
\end{tikzcd}$$

in $\mathbf{D}(\mathrm{Mod}_{\Lambda'})$.

Proof. We need only show that the lower and upper horizontal morphisms are isomorphisms. The upper morphism exists and is an isomorphism for T bounded above without additional assumption. That is, recall that the compactly supported cohomology groups of a module vanish in dimension greater than 3 [Nek, Lemma 5.7.3]. The hypercohomology spectral sequence for compactly supported cohomology therefore converges for bounded above complexes, yielding the existence of a functor

$$\mathbf{R}\Gamma_{(c)}(G_{F,S}, -) \colon \mathbf{D}^-(\mathcal{C}_{\Lambda,G_{F,S}}) \to \mathbf{D}^-(\mathrm{Mod}_\Lambda).$$

[2]We write an exact triangle $A \to B \to C \to A[1]$ more compactly as $A \to B \to C$ throughout.

The analogous result to Proposition 3.1.3 then holds by the original argument (in the case that (1) holds and Y is bounded above). That the upper morphism exists and is an isomorphism follows immediately as in Theorem 3.1.8.
The summands of the lower morphism corresponding to nonarchimedean v are isomorphisms, again for T bounded above without further assumption, by Theorem 3.1.8. For a real place v of F, the functor $\mathbf{R}\Gamma(G_v, -)$ is well-defined only on the bounded derived category. However, if v becomes complex in F'_∞, then the relevant hypercohomology spectral sequence implies that the composite functor $\mathbf{R}\Gamma(G_v, \mathscr{F}_{\Gamma'}(-))$ is both well-defined and zero on $\mathbf{D}(\mathcal{C}_{R,G_v})$. So, the v-summand of the lower horizontal morphism is trivially an isomorphism. If v splits completely in F_∞/F, then the modules Λ_χ and ${}_\chi\Lambda$ have trivial G_v-actions, so we have isomorphisms $\mathscr{F}_\Gamma(T) \cong \Lambda \otimes_R T$ and $\mathscr{F}_\Gamma(T^*)^\iota \cong \Lambda \otimes_R T^*$ of complexes in $\mathcal{C}_{\Lambda,G_v}$ and $\mathcal{C}_{\Lambda^\circ,G_v}$, respectively. Since G_v is finite, the terms of $\widehat{C}(G_v, M)$ for a G_v-module M are each naturally isomorphic to a finite direct sum of copies of M. With these identifications, the canonical map

$$\Lambda \otimes_R \widehat{C}(G_v, T) \xrightarrow{\sim} \widehat{C}(G_v, \Lambda \otimes_R T)$$

agrees with the identity map, so is an isomorphism. Of course, we have the corresponding result for Λ', and the isomorphism

$$\Lambda' \otimes^{\mathbf{L}}_\Lambda (\Lambda \otimes_R \widehat{C}(G_v, T)) \xrightarrow{\sim} \Lambda' \otimes^{\mathbf{L}}_R \widehat{C}(G_v, T)$$

in $\mathbf{D}(\mathrm{Mod}_{\Lambda'})$ provides the v-summand of the lower isomorphism. □

Remark 4.1.2. Proposition 4.1.1 holds with "bounded above" removed if we suppose that the kernel of $\Gamma \to \Gamma'$ has no elements of order p. This follows quickly if p is odd or F has no real places, but it requires some work in the remaining case that F'_∞ has no real places but F does. We omit this for purposes of brevity.

The next proposition shows in particular that if T is a bounded complex of objects of $\mathcal{C}^{R-\mathrm{ft}}_{R,G_{F,S}}$, then the global, local, and compactly supported cohomology groups of $\mathscr{F}_\Gamma(T)$ are finitely generated Λ-modules. (Here, we implicitly identify cochain complexes with their quasi-isomorphic truncations in the derived category.)

Proposition 4.1.3. *Let T be a complex of objects of $\mathcal{C}^{R-\mathrm{ft}}_{R,G_{F,S}}$.*

(a) *The complex $\mathbf{R}\Gamma_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T))$ lies in $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$ if F_∞ has no real places or p is odd and in $\mathbf{D}^-_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$ if T is bounded above.*

(b) *The complex $\mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_\Gamma(T))$ for $v \in S_f$ lies in $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$ if F_∞ has no real places or p is odd and in $\mathbf{D}^+_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$ if T is bounded below.*

(c) *If T is bounded and either F_∞ has no real places or p is odd, then $\mathbf{R}\Gamma_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T))$ and $\mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_\Gamma(T))$ lie in $\mathbf{D}^b_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$.*

(*d*) *For $v \in S_f$, the complex $\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T))$ lies in $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$ and in $\mathbf{D}^b_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$ if T is bounded.*

(*e*) *The complex $\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T))$ for $v \in S_\mathbb{R}$ lies in $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$ if T is bounded.*

Proof. For real v, any $i \in \mathbb{Z}$ and T concentrated in degree 0, the Tate cohomology group $\widehat{H}^i(G_v, \mathscr{F}_\Gamma(T))$ is a 2-torsion subquotient of $\mathscr{F}_\Gamma(T)$, as G_v has order 2. Hence, it is finitely generated over Λ as $\mathscr{F}_\Gamma(T)$ is. The hypercohomology spectral sequence allows one to pass to the case of a bounded complex T, and therefore $\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T))$ lies in $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$, hence (e). Recall also that $\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T))$ is a zero object for all complexes T if v extends to a complex place of F_∞.

If $v \in S$ is nonarchimedean, then G_v is p-cohomologically finite by our above remarks, and hence Theorem 3.2.4 implies that $\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T))$ lies in $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$. In the case that p is odd or F_∞ has no real places, Theorem 3.2.4 again implies that $\mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_\Gamma(T))$ lies in $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$. (For this, note that if $p = 2$ and F has a real place that becomes complex in F_∞, it will already have become complex in any extension with Galois group Γ/Γ_0, where Γ_0 is an open uniform pro-p subgroup of Γ.) In this case, it follows from the above-mentioned exact triangle that $\mathbf{R}\Gamma_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T))$ is in $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$. If T is bounded, then clearly all of the above complexes will additionally lie in the bounded derived category. In particular, we have (c) and (d) and part of each of (a) and (b).

The analogue of Theorem 3.2.4 for compactly supported cohomology holds for bounded above T since $H^i_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T))$ vanishes for sufficiently large i. The proof is essentially identical but uses the semilocal version of Shapiro's lemma [Nek, (8.5.3.2)] in the first step and the relevant hypercohomology spectral sequence in the last. Thus we have (a).

Finally, if T is concentrated in degree 0, the exact triangle tells us that $H^i(G_{F,S}, \mathscr{F}_\Gamma(T))$ is finitely generated over $\mathscr{F}_\Gamma(T)$ for sufficiently large i. Thus, we are still able to perform the inductive step in the proof of Theorem 3.2.4 to obtain that $\mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_\Gamma(T))$ sits in $\mathbf{D}^+_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$ for such T, and then for bounded below T via the hypercohomology spectral sequence. Thus, we have (b). $\square$

Remark 4.1.4. Part (d) of Proposition 4.1.3 makes sense and holds more generally for any complex T in $\mathcal{C}^{R-\mathrm{ft}}_{R,G_v}$, as Λ_χ has a G_v-action through the composite with the canonical map $G_v \to G_{F,S}$.

Remark 4.1.5. We may also consider the setting in which we take F itself to be a nonarchimedean local field of characteristic not equal to p. We then let G_F be its absolute Galois group, let Γ denote the Galois group of a p-adic Lie extension of F, and as before, set $\Lambda = R[\![\Gamma]\!]$. In this case, we obtain immediately from Theorem 3.2.4 that the cohomology groups $H^j(G_F, \mathscr{F}_\Gamma(T))$ are finitely generated Λ-modules for all j and any complex of objects of T of

$\mathcal{C}_{R,G_F}^{R-\mathrm{ft}}$, and consequently, that $\mathbf{R}\Gamma(G_F,\mathscr{F}_\Gamma(T))$ is an object of $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$ for any complex T of objects in $\mathcal{C}_{R,G_F}^{R-\mathrm{ft}}$.

4.2 Duality over local fields

In this subsection, we will state a version of Tate duality for Γ-induced modules with $\Lambda\otimes_R^{\mathbf{L}}\omega_R$ for a dualizing complex ω_R replacing $\mathbb{Q}_p/\mathbb{Z}_p$. Fix a nonarchimedean prime $v\in S$. We continue to suppose that Γ is a compact p-adic Lie group that is a quotient of $G_{F,S}$ defining an extension F_∞, though as we remark later, we could just as well assume it to be a quotient of G_v. Let T be a bounded complex of objects in $\mathcal{C}_{R,G_v}^{R-\mathrm{ft}}$. As before, we let J_R denote a complex of injective R-modules that represents ω_R, and we let T^* be a quasi-isomorphic subcomplex of $\mathrm{Hom}_R(T,J_R)$ consisting of objects of $\mathcal{C}_{R,G_v}^{R-\mathrm{ft}}$.
The cup product of Lemma 3.3.3 yields a map

$$C(G_v,\mathscr{F}_\Gamma(T))\otimes_R C(G_v,\mathscr{F}_\Gamma(T^*)^\iota(1))\longrightarrow \tau_{\geq 2}^{II}C(G_v,\Lambda\otimes_R J_R(1))$$

of $\Lambda\otimes_R\Lambda^\circ$-modules. Taking adjoints, we have the following maps of complexes of Λ-modules and Λ°-modules, respectively:

$$\begin{aligned}C(G_v,\mathscr{F}_\Gamma(T))&\longrightarrow \mathrm{Hom}_{\Lambda^\circ}\Big(C(G_v,\mathscr{F}_\Gamma(T^*)^\iota(1)),\tau_{\geq 2}^{II}C(G_v,\Lambda\otimes_R J_R(1))\Big),\\ C(G_v,\mathscr{F}_\Gamma(T^*)^\iota(1))&\longrightarrow \mathrm{Hom}_{\Lambda}\Big(C(G_v,\mathscr{F}_\Gamma(T)),\tau_{\geq 2}^{II}C(G_v,\Lambda\otimes_R J_R(1))\Big).\end{aligned}$$

From now on, we use $M^\vee$ to denote the Pontryagin dual of a locally compact abelian group M. We have the following analogue of [Nek, (5.2.1)].

Lemma 4.2.1. *Let A be a bounded complex in $\mathcal{C}_{\Lambda-\Lambda}^{\Lambda^\circ-\mathrm{ft}}$ with terms that are flat as R-modules. Then there exists a quasi-isomorphism*

$$q_A\colon (A\otimes_R J_R)[-2]\longrightarrow \tau_{\geq 2}^{II}C(G_v,A\otimes_R J_R(1)),$$

natural in A, of chain complexes of $\Lambda\otimes_R\Lambda^\circ$-modules.

Proof. We first note that if M is any finitely generated R-module with a trivial action of G_v, then Tate local duality as in [Lim2, Theorem 4.1.2] yields isomorphisms

$$H^2(G_v,A\otimes_R M(1))\cong H^0(G_v,(A\otimes_R M)^\vee)^\vee\cong A\otimes_R M,$$

of $\Lambda\otimes_R\Lambda^\circ$-modules. Now suppose that M is any (ind-admissible) $R[G_v]$-module with trivial G_v-action, and write $M=\bigcup M_\alpha$, where the M_α are finitely generated R-submodules of M. Since the terms of A are R-flat, $A^i\otimes_R M$ is the union of the $A^i\otimes_R M_\alpha$ for any $i\in\mathbb{Z}$, so

$$H^2(G_v,A\otimes_R M(1))\cong \varinjlim H^2(G_v,A\otimes_R M_\alpha(1))\cong \varinjlim A\otimes_R M_\alpha\cong A\otimes_R M.$$

As G_v is of p-cohomological dimension 2, we therefore have a quasi-isomorphism

$$A \otimes_R M[-2] \longrightarrow \tau_{\geq 2}^{II} C(G_v, A \otimes_R M(1)),$$

of complexes of $\Lambda \otimes_R \Lambda^\circ$-modules. The case of a bounded complex M of ind-admissible R-modules, e.g. $M = J_R$, then follows easily. Naturality is immediate from the construction. $\square$

Combining Lemma 4.2.1 for $A = \Lambda$ with the morphisms constructed above and passing to the derived category, we obtain morphisms as in the following theorem.

Theorem 4.2.2. *Let T be a bounded complex in $\mathcal{C}_{R,G_v}^{R-\mathrm{ft}}$. Then the morphisms*

$$\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R^{\mathbf{L}} \omega_R\Big)[-2]$$
$$\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T)), \Lambda \otimes_R^{\mathbf{L}} \omega_R\Big)[-2]$$

in $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$ and $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_{\Lambda^\circ})$, respectively, are isomorphisms.

Over the course of the next two subsections, we will show that the first of the morphisms of Theorem 4.2.2 is an isomorphism. The proof that the second is an isomorphism is completely analogous. By an argument similar to that in [Nek, Lemma 5.2.5], it suffices to prove the theorem for a particular translate of the dualizing complex. In view of this, we will assume that our choice of the dualizing complex satisfies [Nek, (2.5)(i)] throughout the next two subsections.

Remark 4.2.3. In the setting that F is a nonarchimedean local field of characteristic not equal to p, that F_∞ is any p-adic Lie extension of F, and that T is a bounded complex of objects in $\mathcal{C}_{R,G_F}^{R-\mathrm{ft}}$, the argument we are about to describe also carries over as in Remark 4.1.5 to prove the direct analogue of Theorem 4.2.2 with G_v replaced by G_F.

4.3 Change of rings

Recall that R is a complete commutative local Noetherian ring with maximal ideal $\mathfrak{m}$ and finite residue field k. In this subsection, we shall show that it suffices to prove Theorem 4.2.2 for any complete commutative local Noetherian ring S that has R as a quotient. (In this subsection only, S denotes such a ring, rather than the set of primes chosen above, which is used only implicitly.)
We suppose given a surjection $\psi \colon S \to R$, which necessarily induces an isomorphism on residue fields. We use $\mathfrak{m}_S$ to denote the maximal ideal of S, and we set

$$d = \dim_k \mathfrak{m}_S/\mathfrak{m}_S^2 - \dim_k \mathfrak{m}/\mathfrak{m}^2.$$

Note that if A is any complex of S-modules, then $\mathrm{Hom}_S(R, A)$ is isomorphic to the subcomplex of A with terms the torsion submodules under $\ker\psi$. In particular, if B is a complex of R-modules, then the resulting inclusion map induces an isomorphism

$$\mathrm{Hom}_R(B, \mathrm{Hom}_S(R, A)) \xrightarrow{\sim} \mathrm{Hom}_S(B, A),$$

as is easy enough to see by the usual adjointness principle. In particular, we have

$$\mathrm{Hom}_R(B, R^\vee) \xrightarrow{\sim} \mathrm{Hom}_S(B, S^\vee).$$

With this comparison of Matlis duals with respect to R and S in hand, we now compare Grothendieck duals with respect to these rings (see also [Har2]). For this, we choose particular dualizing complexes J_R and J_S of injectives as follows. Let $\bar{x}_1, \ldots, \bar{x}_r \in \mathfrak{m}$ with images forming a k-basis of $\mathfrak{m}/\mathfrak{m}^2$. Recall (cf. [Nek, Section 0.4]) that we may take J_R to be the Matlis dual of $C_R[r]$, where C_R is defined to be the complex

$$\left[R \longrightarrow \bigoplus_i R_{\bar{x}_i} \longrightarrow \bigoplus_{i<j} R_{\bar{x}_i \bar{x}_j} \longrightarrow \cdots \longrightarrow R_{\bar{x}_1 \cdots \bar{x}_r} \right]$$

in degrees $[0, r]$, with the usual Čech differentials, which is to say that

$$J_R = \mathrm{Hom}_R(C_R[r], R^\vee).$$

Note that J_R is a complex of R-injectives by the R-flatness of the terms of C_R and the R-injectivity of $R^\vee$.

We lift $\bar{x}_1, \ldots, \bar{x}_r$ to elements $x_1, \ldots, x_r$ of $\mathfrak{m}_S$ and extend to a sequence $x_1, \ldots, x_s$ such that the images of $x_1, \ldots, x_s$ form a basis of $\mathfrak{m}_S/\mathfrak{m}_S^2$ and the elements $x_{r+1}, \ldots, x_s$ map trivially to $\mathfrak{m}/\mathfrak{m}^2$. Modifying each x_i with $i > r$ by an element of $\mathfrak{m}_S^2$ if necessary, we may choose these elements so that they in fact map trivially to $\mathfrak{m}$. We use this sequence to define C_S, and we set $J_S = \mathrm{Hom}_S(C_S[s], S^\vee)$.

Lemma 4.3.1. *Let T be a bounded complex in $\mathcal{C}_{R,G_v}^{R-\mathrm{ft}}$, which we view also as a complex in $\mathcal{C}_{S,G_v}^{S-\mathrm{ft}}$ via ψ. Then ψ induces a natural map $J_R \to J_S[d]$ and in turn an isomorphism*

$$\mathrm{Hom}_R(T, J_R) \xrightarrow{\sim} \mathrm{Hom}_S(T, J_S)[d]$$

of complexes in $\mathscr{I}_{S,G_v}$.

Proof. The natural map $C_S \to C_R$ factors through an isomorphism $R \otimes_S C_S \xrightarrow{\sim} C_R$. This induces an isomorphism

$$J_R \xrightarrow{\sim} \mathrm{Hom}_S(R, J_S)[d],$$

which is given by the following composition of isomorphisms

$$\begin{aligned}\mathrm{Hom}_R(C_R[r], R^\vee) &\xrightarrow{\sim} \mathrm{Hom}_S(C_R[r], S^\vee) \xrightarrow{\sim} \mathrm{Hom}_S(R \otimes_S C_S[r], S^\vee) \\ &\xrightarrow{\sim} \mathrm{Hom}_S(R, \mathrm{Hom}_S(C_S[r], S^\vee)) \xrightarrow{\sim} \mathrm{Hom}_S(R, \mathrm{Hom}_S(C_S[s], S^\vee))[d],\end{aligned}$$

where the latter natural map is defined as in [Nek, (1.2.15)] (noting [Nek, (1.2.5)]). The result then follows from the isomorphisms

$$\mathrm{Hom}_R(T, J_R) \xrightarrow{\sim} \mathrm{Hom}_R(T, \mathrm{Hom}_S(R, J_S))[d] \xrightarrow{\sim} \mathrm{Hom}_S(T, J_S)[d].$$

□

We now proceed to the reductive step.

Proposition 4.3.2. *Let T be a bounded complex in $\mathcal{C}^{R\text{-ft}}_{R,G_v}$. Setting $\Omega = S[\![\Gamma]\!]$, there is a commutative diagram of natural morphisms*

$$\begin{tikzcd}
\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T)) \arrow[r] \arrow[d, "\wr"] & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-2] \arrow[d, "\wr"] \\
\mathbf{R}\Gamma(G_v, \Omega_\chi \otimes_S T) \arrow[r] & \mathbf{R}\mathrm{Hom}_{\Omega^\circ}\Big(\mathbf{R}\Gamma(G_v, {}_\chi\Omega \otimes_S T^*(1)), \Omega \otimes^{\mathbf{L}}_S \omega_S[d]\Big)[-2]
\end{tikzcd}$$

in which the horizontal morphisms are as in Theorem 4.2.2 and the vertical arrows are isomorphisms in $\mathbf{D}(\mathrm{Mod}_\Omega)$. In particular, Theorem 4.2.2 holds for R if it holds for S.

Proof. Note that we have

$$\Omega_\chi \otimes_S T \cong \Lambda_\chi \otimes_R T \quad (\text{resp., } {}_\chi\Omega \otimes_S T^* \cong {}_\chi\Lambda \otimes_R T^*)$$

so $\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T))$ (resp., $\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1))$) is the same complex for R and for S. By Lemma 4.3.1, and employing the S-flatness of Ω in applying Lemma 2.1.7, we have canonical isomorphisms

$$\Lambda \otimes_R J_R \xrightarrow{\sim} \Omega \otimes_S J_R \xrightarrow{\sim} \Omega \otimes_S \mathrm{Hom}_S(R, J_S)[d] \xrightarrow{\sim} \mathrm{Hom}_S(R, \Omega \otimes_S J_S)[d],$$

of complexes of $\Omega^\circ \otimes_S R \cong \Lambda^\circ$-modules, and the composition of the composite isomorphism with the natural inclusion

$$\mathrm{Hom}_S(R, \Omega \otimes_S J_S)[d] \longrightarrow \Omega \otimes_S J_S[d]$$

induces the right-hand vertical map in the proposition. That the diagram commutes then follows directly from the definition of the cup product in Lemma 3.3.3.

Let us fix a quasi-isomorphism $\iota\colon \Omega \otimes_S J_S \to K$, where K is a bounded below complex of injective Ω°-modules. We check that the map

$$\iota_R\colon \mathrm{Hom}_S(R, \Omega \otimes_S J_S) \longrightarrow \mathrm{Hom}_S(R, K)$$

induced by ι is a quasi-isomorphism, which implies that $\mathrm{Hom}_S(R, \Omega \otimes_S J_S)$ represents $\mathbf{R}\mathrm{Hom}_S(R, \Omega \otimes^{\mathbf{L}}_S \omega_S)$ in $\mathbf{D}(\mathrm{Mod}_{\Lambda^\circ})$. Let $\varepsilon\colon P \to R$ be a resolution of

R by a complex P of finitely generated free S-modules. We have a commutative diagram

$$\begin{tikzcd}
\Omega \otimes_S \mathrm{Hom}_S(R, J_S) \arrow[r, "\phi_R"] \arrow[d, "\mathrm{id}\otimes\varepsilon_J^*"] & \mathrm{Hom}_S(R, \Omega \otimes_S J_S) \arrow[r, "\iota_R"] \arrow[d, "\varepsilon_{\Omega\otimes J}^*"] & \mathrm{Hom}_S(R, K) \arrow[r, "\psi_R"] \arrow[d, "\varepsilon_K^*"] & \mathrm{Hom}_{\Omega^\circ}(\Omega \otimes_S R, K) \arrow[d, "(\mathrm{id}\otimes\varepsilon)_K^*"] \\
\Omega \otimes_S \mathrm{Hom}_S(P, J_S) \arrow[r, "\phi_P"] & \mathrm{Hom}_S(P, \Omega \otimes_S J_S) \arrow[r, "\iota_P"] & \mathrm{Hom}_S(P, K) \arrow[r, "\psi_P"] & \mathrm{Hom}_{\Omega^\circ}(\Omega \otimes_S P, K)
\end{tikzcd}$$

in which the vertical maps are all induced by ε. Moreover, note that the upper horizontal morphisms are of complexes of Λ°-modules and the others are of complexes of Ω°-modules. The maps ϕ_P and ϕ_R are isomorphisms, and the map $\mathrm{id} \otimes \varepsilon_J^*$ is a quasi-isomorphism by the S-injectivity of J_S and the S-flatness of Ω, so $\varepsilon_{\Omega\otimes J}^*$ is also a quasi-isomorphism. The maps ψ_R and ψ_P are isomorphisms by the usual adjointness of Hom and the tensor product, and the map $(\mathrm{id} \otimes \varepsilon)_K^*$ is a quasi-isomorphism by the Ω°-injectivity of K, so ε_K^* is a quasi-isomorphism as well. Finally, the map ι_P is a quasi-isomorphism by the S-projectivity of P, and it follows that ι_R is a quasi-isomorphism.
To finish the proof, we need only show that the morphism

$$\mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(X, \mathbf{R}\mathrm{Hom}_S(R, \Omega \otimes_S^{\mathbf{L}} \omega_S)\Big) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Omega^\circ}(X, \Omega \otimes_S^{\mathbf{L}} \omega_S)$$

is an isomorphism for $X = C(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1))$. It is easy to see that $\mathrm{Hom}_S(R, K)$ is a complex of injective Λ°-modules. Moreover, every Ω°-homomorphism from the complex X of Λ°-modules to K must factor through

$$\mathrm{Hom}_{\Omega^\circ}(\Lambda, K) \cong \mathrm{Hom}_{\Omega^\circ}(\Omega \otimes_S R, K) \cong \mathrm{Hom}_S(R, K)$$

of K, so the map

$$\mathrm{Hom}_{\Lambda^\circ}(X, \mathrm{Hom}_S(R, K)) \longrightarrow \mathrm{Hom}_{\Omega^\circ}(X, K)$$

induced by inclusion is an isomorphism, as desired. □

4.4 Duality over flat $\mathbb{Z}_p$-algebras

In this subsection, we will prove Theorem 4.2.2. As remarked in the proof of Proposition 3.0.1, every complete commutative Noetherian local ring with finite residue field of characteristic p is a quotient of a power series ring in finitely many variables over an unramified extension of $\mathbb{Z}_p$. Proposition 4.3.2 then allows us to reduce Theorem 4.2.2 to the case of such power series rings. We therefore can and do assume that R is $\mathbb{Z}_p$-flat throughout this subsection. We will require a bit more general of a derived adjoint map than the one we wish to prove is an isomorphism. The construction is found in the following easy lemma.

Lemma 4.4.1. *Let A be a bounded complex of R-flat objects in $\mathcal{C}_{\Lambda-\Lambda}^{\Lambda^\circ-\mathrm{ft}}$. Then we have a morphism*

$$\mathbf{R}\Gamma(G_v, A \otimes_\Lambda \mathscr{F}_\Gamma(T)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), A \otimes_R^{\mathbf{L}} \omega_R\Big)[-2]$$

in $\mathbf{D}(\mathrm{Mod}_\Lambda)$ *that is natural in the complex* A.

Proof. This is simply the derived adjoint of the composition of the cup product in Lemma 3.3.3 with the map to the truncation $\tau^{II}_{\geq 2} C(G_v, A \otimes_R J_R(1))$, which is quasi-isomorphic to $(A \otimes_R J_R)[-2]$ by Lemma 4.2.1. Naturality is immediate. □

We will mostly be interested in complexes of length 2, so we introduce the following notation. Suppose that B is an object in $\mathcal{C}^{\Lambda^\circ-\mathrm{ft}}_{\Lambda-\Lambda}$ and that A is a subobject of B, which is to say that it is a closed $\Lambda \hat{\otimes}_R \Lambda^\circ$-submodule with the subspace topology. Then for any bounded complex T of objects in $\mathcal{C}^{R-\mathrm{ft}}_{R,G_v}$, we define a complex

$$\mathscr{F}_{B/A}(T) = [A \to B] \otimes_\Lambda \mathscr{F}_\Gamma(T),$$

of objects in $\mathcal{C}_{\Lambda,G_v}$, where A and B are in degree -1 and 0 respectively.
Let $I = I(\Gamma)$ denote the augmentation ideal of Λ. We then have the following lemmas. Recall for the first that we assume that R is $\mathbb{Z}_p$-flat in this subsection.

LEMMA 4.4.2. *The following statements hold for any* $n \geq 0$.

(*a*) *The ideal* I^n *is a flat* R*-module.*

(*b*) *The module* I^n/I^{n+1} *is finitely generated and of finite projective dimension over* R.

Proof. (a) Suppose first that Γ is finite, and let $\mathcal{I}$ denote the augmentation ideal in $\mathbb{Z}_p[\Gamma]$. As $\mathbb{Z}_p[\Gamma]$ is finitely generated and free over $\mathbb{Z}_p$, so is $\mathcal{I}^n$. Since R is $\mathbb{Z}_p$-flat, the natural surjection $R \otimes_{\mathbb{Z}_p} \mathcal{I}^n \to I^n$ is an isomorphism, and therefore I^n is free over R.
In the general case, $\Gamma \cong \varprojlim \Gamma'$, where Γ' runs over the Galois groups of the finite Galois extensions of F in F_∞, and I^n is the inverse limit of the nth powers of the augmentation ideals $I(\Gamma')$ of the $R[\Gamma']$. As $I(\Gamma')^n \cong I^n \otimes_\Lambda R[\Gamma']$, an application of Lemma 2.4.2 yields an isomorphism

$$I^n \otimes_R B \xrightarrow{\sim} \varprojlim_{\Gamma'} (I(\Gamma')^n \otimes_R B)$$

for any ideal B of R. The left exactness of the inverse limit and the R-flatness of $I(\Gamma')^n$ then imply that the canonical map $I^n \otimes_R B \to I^n$ is an injection, proving the flatness of I^n.

(b) Let $\mathcal{I}$ denote the augmentation ideal in $\mathbb{Z}_p[\![\Gamma]\!]$. For each n, the composition

$$R \hat{\otimes}_{\mathbb{Z}_p} \mathcal{I}^n \longrightarrow R \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[\![\Gamma]\!] \xrightarrow{\sim} R[\![\Gamma]\!]$$

induces a surjection $R \hat{\otimes}_{\mathbb{Z}_p} \mathcal{I}^n \twoheadrightarrow I^n$ that fits into the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & R \hat{\otimes}_{\mathbb{Z}_p} \mathcal{I}^{n+1} \arrow[r] \arrow[d, two heads] & R \hat{\otimes}_{\mathbb{Z}_p} \mathcal{I}^n \arrow[r] \arrow[d, two heads] & R \hat{\otimes}_{\mathbb{Z}_p} (\mathcal{I}^n/\mathcal{I}^{n+1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^{n+1} \arrow[r] & I^n \arrow[r] & I^n/I^{n+1} \arrow[r] & 0.
\end{tikzcd}$$

Since the two vertical maps on the left are surjections, so is the one on the right. Note that $\mathcal{I}^n/\mathcal{I}^{n+1}$ is a quotient of nth tensor power of the maximal abelian pro-p quotient of Γ, a finitely generated $\mathbb{Z}_p$-module. Therefore, the existence of the above surjection implies that I^n/I^{n+1} is finitely generated over R. To see that the last assertion holds, we note that it follows from (a) that I^n/I^{n+1} has finite flat dimension. Since flat dimension coincides with projective dimension for every finitely generated module over a Noetherian ring (see [Wei, Proposition 4.1.5]), we have our assertion. $\square$

Lemma 4.4.3. *Let T be a bounded complex of objects in $\mathcal{C}_{R,G_v}^{R-\mathrm{ft}}$. Then we have a quasi-isomorphism*

$$\mathrm{Cone}\left(\mathscr{F}_{I^n/I^{n+1}}(T) \longrightarrow \mathscr{F}_{\Lambda/I^{n+1}}(T)\right) \xrightarrow{\sim} \mathscr{F}_{\Lambda/I^n}(T)$$

of complexes in $\mathcal{C}_{\Lambda,G_v}^{\Lambda-\mathrm{ft}}$. Moreover, we have an exact triangle

$$I^n/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R \longrightarrow \Lambda/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R \longrightarrow \Lambda/I^n \otimes_R^{\mathbf{L}} \omega_R$$

in $\mathbf{D}^b(\mathrm{Mod}_{\Lambda\otimes_R\Lambda^\circ})$.

Proof. Since the powers of I are R-flat by Lemma 4.4.2(a), it suffices in both cases to show that there is a quasi-isomorphism

$$\mathrm{Cone}\left([I^{n+1} \to I^n] \longrightarrow [I^{n+1} \to \Lambda]\right) \xrightarrow{\sim} [I^n \to \Lambda].$$

Note that the latter cone is precisely the complex

$$I^{n+1} \xrightarrow{f} I^{n+1} \oplus I^n \xrightarrow{g} \Lambda,$$

where $f(x) = (x, -x)$ and $g(x, y) = x + y$ for $x \in I^{n+1}$ and $y \in I^n$. One can now easily check that the diagram

$$\begin{tikzcd} I^{n+1} \arrow[r, "f"] \arrow[d] & I^{n+1} \oplus I^n \arrow[r, "g"] \arrow[d, "\nu"'] & \Lambda \arrow[d, equal] \\ 0 \arrow[r] & I^n \arrow[r] & \Lambda \end{tikzcd}$$

commutes, where ν is given by $\nu(x, y) = x + y$ for $x \in I^{n+1}$ and $y \in I^n$, and the vertical maps induce isomorphisms on cohomology. $\square$

We are now able to prove the following proposition, which is an important ingredient in the proof of Theorem 4.2.2.

Proposition 4.4.4. *Let T be a bounded complex of objects in $\mathcal{C}_{R,G_v}^{R-\mathrm{ft}}$. Then we have the following morphism of exact triangles*

$$\begin{CD}
\mathbf{R}\Gamma(G_v, \mathscr{F}_{I^n/I^{n+1}}(T)) @>>> \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), I^n/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R\Big)[-2] \\
@VVV @VVV \\
\mathbf{R}\Gamma(G_v, \mathscr{F}_{\Lambda/I^{n+1}}(T)) @>>> \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R\Big)[-2] \\
@VVV @VVV \\
\mathbf{R}\Gamma(G_v, \mathscr{F}_{\Lambda/I^n}(T)) @>>> \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda/I^n \otimes_R^{\mathbf{L}} \omega_R\Big)[-2]
\end{CD}$$

in $\mathbf{D}(\mathrm{Mod}_\Lambda)$.

Proof. By Lemma 4.4.2(a), we see that $I^n/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R$ and $\Lambda/I^n \otimes_R^{\mathbf{L}} \omega_R$ are represented by $[I^{n+1} \to I^n] \otimes_R J_R$ and $[I^n \to \Lambda] \otimes_R J_R$ respectively. Therefore, the commutativity of the diagram in the proposition follows from the naturality in Lemma 4.4.1. By Lemma 4.4.3, both columns are exact triangles. ☐

We now describe the idea of the proof of Theorem 4.2.2. We shall first prove that the morphism

$$\mathbf{R}\Gamma(G_v, \mathscr{F}_{\Lambda/I^n}(T)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda/I^n \otimes_R^{\mathbf{L}} \omega_R\Big)[-2]$$

is an isomorphism for all n. Then, Theorem 4.2.2 will follow from this by a limit argument. To show that above morphism is an isomorphism, we will utilize Proposition 4.4.4. Note that if any two of the horizontal morphisms in Proposition 4.4.4 is a quasi-isomorphism, so is the third one. Therefore, by an inductive argument, we are reduced to showing that the morphism

$$\mathbf{R}\Gamma(G_v, \mathscr{F}_{I^n/I^{n+1}}(T)) \to \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), I^n/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R\Big)[-2]$$

is an isomorphism for all $n \geq 0$.
Note that Γ acts trivially on I^n/I^{n+1}. Therefore, one may view I^n/I^{n+1} as a Λ°-module via the augmentation map $\Lambda \to R$. We now have the following lemma.

Lemma 4.4.5. *Let T be a bounded complex of objects in $\mathcal{C}_{R,G_v}^{R-\mathrm{ft}}$. Then we have the following isomorphisms*

$$\mathscr{F}_{I^n/I^{n+1}}(T) \xleftarrow{\sim} I^n/I^{n+1} \hat{\otimes}_\Lambda^{\mathbf{L}} \mathscr{F}_\Gamma(T) \xrightarrow{\sim} I^n/I^{n+1} \hat{\otimes}_R^{\mathbf{L}} T$$

in $\mathbf{D}^b(\mathcal{C}_{\Lambda,G_v})$. Therefore, we have an isomorphism

$$\mathbf{R}\Gamma(G_v, \mathscr{F}_{I^n/I^{n+1}}(T)) \xrightarrow{\sim} \mathbf{R}\Gamma(G_v, I^n/I^{n+1} \hat{\otimes}_R^{\mathbf{L}} T)$$

in $\mathbf{D}(\mathrm{Mod}_\Lambda)$.

Proof. Let P be a resolution of I^n/I^{n+1} consisting of projective objects in $\mathcal{C}_{\Lambda-\Lambda}$. Then there is a quasi-isomorphism $P \to [I^{n+1} \to I^n]$ of complexes in $\mathcal{C}_{\Lambda-\Lambda}$ which lifts the identity map on I^n/I^{n+1}. By Lemmas 2.4.3 and 4.4.2(a) this induces a quasi-isomorphism

$$P \otimes_\Lambda \Lambda_\chi \otimes_R T \longrightarrow [I^{n+1} \to I^n] \otimes_\Lambda \Lambda_\chi \otimes_R T$$

of bounded above complexes in $\mathcal{C}_{\Lambda,G_v}$, proving the first isomorphism. The second isomorphism is a special case of Lemma 3.1.6 with $N = I^n/I^{n+1}$ and Γ' trivial. Note that both quasi-isomorphisms of complexes factor through a quasi-isomorphic truncation of $P \otimes_\Lambda \Lambda_\chi \otimes_R T$, so they may be seen in the bounded derived category. □

LEMMA 4.4.6. *For each n, there is a commutative diagram*

$$\begin{tikzcd}
I^n/I^{n+1} \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma(G_v,T) \arrow[r] \arrow[d, "\wr"'] & I^n/I^{n+1} \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R\Big(\mathbf{R}\Gamma(G_v,T^*(1)),\omega_R\Big)[-2] \arrow[d, "\wr"] \\
\mathbf{R}\Gamma(G_v, I^n/I^{n+1} \hat{\otimes}^{\mathbf{L}}_R T) \arrow[r] \arrow[ddd, "\wr"'] & \mathbf{R}\mathrm{Hom}_R\Big(\mathbf{R}\Gamma(G_v,T^*(1)), I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R\Big)[-2] \arrow[d, "\wr"] \\
 & \mathbf{R}\mathrm{Hom}_R\Big(\mathbf{R}\Gamma(G_v,\mathscr{F}_\Gamma(T^*)^\iota(1)) \otimes^{\mathbf{L}}_\Lambda R, I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R\Big)[-2] \arrow[d, "\wr"] \\
 & \\
\mathbf{R}\Gamma(G_v, \mathscr{F}_{I^n/I^{n+1}}(T)) \arrow[r] & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v,\mathscr{F}_\Gamma(T^*)^\iota(1)), I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R\Big)[-2],
\end{tikzcd}$$

where the vertical morphisms are isomorphisms in $\mathbf{D}(\mathrm{Mod}_R)$.

Proof. By Lemma 4.4.2(b), we may choose a bounded resolution Q of I^n/I^{n+1} by finitely generated projective R-modules. By [Nek, Proposition 3.4.4], we have an isomorphism of complexes

$$\alpha\colon Q \otimes_R C(G_v,T) \xrightarrow{\sim} C(G_v, Q \otimes_R T)$$

that fits into the commutative diagram

$$\begin{tikzcd}
Q \otimes_R C(G_v,T) \otimes_R C(G_v,T^*(1)) \arrow[r] \arrow[d, "\alpha\otimes\mathrm{id}"'] & Q \otimes_R C(G_v, J_R(1)) \arrow[d, "\alpha'"] \\
C(G_v, Q \otimes_R T) \otimes_R C(G_v,T^*(1)) \arrow[r] & C(G_v, Q \otimes_R J_R(1)),
\end{tikzcd}$$

where α' is defined analogously to α and is also an isomorphism. Since J_R is a bounded complex of R-injectives, we may find homotopy inverses to the quasi-isomorphisms q_R and q_Q of Lemma 4.2.1 that fit in a commutative diagram

$$\begin{tikzcd}
Q \otimes_R C(G_v, J_R(1)) \arrow[r] \arrow[d, "\alpha'"] & Q \otimes_R \tau^{II}_{\geq 2} C(G_v, J_R(1)) \arrow[r] \arrow[d] & Q \otimes_R J_R[-2] \arrow[d, equal] \\
C(G_v, Q \otimes_R J_R(1)) \arrow[r] & \tau^{II}_{\geq 2} C(G_v, Q \otimes_R J_R(1)) \arrow[r] & Q \otimes_R J_R[-2].
\end{tikzcd}$$

We obtain the top commutative square in the lemma and the fact that the vertical morphisms therein are isomorphisms, the one on the right by Lemma 2.1.8 and [Nek, Proposition 4.2.3].

Let P be a resolution of I^n/I^{n+1} consisting of finitely generated projective Λ°-modules. We may view Q as a resolution of Λ°-modules via the augmentation map $\Lambda \to R$. Then, by Proposition 2.4.6, the map

$$P \otimes_\Lambda \mathscr{F}_\Gamma(T) = P \otimes_\Lambda \Lambda_\chi \otimes_R T \longrightarrow Q \otimes_\Lambda \Lambda_\chi \otimes_R T \cong Q \otimes_R T$$

is a quasi-isomorphism of complexes of objects in $\mathcal{C}_{R,G_v}$, and we let f denote its induced map on cochains. Let L be a resolution of R consisting of finitely generated projective Λ-modules. Then, by an opposite version of Theorem 3.1.8, we have a quasi-isomorphism

$$g \colon C(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)) \otimes_\Lambda L \longrightarrow C(G_v, T^*(1))$$

of complexes of R-modules. Noting Lemma 3.3.3, we have a commutative diagram

$$\begin{tikzcd}
C(G_v, P \otimes_\Lambda \mathscr{F}_\Gamma(T)) \otimes_R C(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)) \otimes_\Lambda L \arrow[r] \arrow[d, "f \otimes g"'] & C(G_v, P \otimes_R J_R(1)) \otimes_\Lambda L \arrow[d, "\varepsilon"] \\
C(G_v, Q \otimes_R T) \otimes_R C(G_v, T^*(1)) \arrow[r] & C(G_v, Q \otimes_R J_R(1)),
\end{tikzcd}$$

where ε is induced by the augmentation $L \to R$ and the map $P \to Q$.

Taking adjoints and applying the homotopy inverse to q_Q as above, we obtain the commutative diagram

$$\begin{tikzcd}
C(G_v, Q \otimes_R T) \arrow[r] & \mathrm{Hom}_R\Big(C(G_v, T^*(1)), Q \otimes_R J_R\Big)[-2] \arrow[d] \\
C(G_v, P \otimes_\Lambda \mathscr{F}_\Gamma(T)) \arrow[r] \arrow[u] \arrow[d, equal] & \mathrm{Hom}_R\Big(C(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)) \otimes_\Lambda L, Q \otimes_R J_R\Big)[-2] \arrow[d, "\wr"] \\
C(G_v, P \otimes_\Lambda \mathscr{F}_\Gamma(T)) \arrow[r] & \mathrm{Hom}_{\Lambda^\circ}\Big(C(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \mathrm{Hom}_R(L, Q \otimes_R J_R)\Big)[-2],
\end{tikzcd}$$

which yields upon passage to the derived category the lower part of the diagram in the statement of the lemma, the vertical morphisms therein being isomorphisms by Theorem 3.1.8, Lemma 4.4.5 and Lemma 2.1.6. □

Lemma 4.4.7. *The morphisms*

$$\mathbf{R}\Gamma(G_v, \mathscr{F}_{I^n/I^{n+1}}(T)) \to \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), I^n/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R\Big)[-2]$$

in $\mathbf{D}(\mathrm{Mod}_\Lambda)$ *are isomorphisms for every* $n \geq 0$.

Proof. By Corollary 2.1.3, it suffices to show that the above morphism is an isomorphism in $\mathbf{D}(\mathrm{Mod}_R)$. The morphism

$$\mathbf{R}\Gamma(G_v, T) \longrightarrow \mathbf{R}\mathrm{Hom}_R\Big(\mathbf{R}\Gamma(G_v, T^*(1)), \omega_R\Big)[-2]$$

in $\mathbf{D}(\mathrm{Mod}_R)$ is an isomorphism by [Nek, Proposition 5.2.4(ii)], and so the top morphism of the diagram in Lemma 4.4.6 is an isomorphism. Since all the vertical morphisms in the diagram are isomorphisms, it follows that the bottom morphism is also an isomorphism, as required. □

Proposition 4.4.8. *The morphisms*

$$\mathbf{R}\Gamma(G_v, \mathscr{F}_{\Lambda/I^n}(T)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda/I^n \otimes_R^{\mathbf{L}} \omega_R\Big)[-2]$$

in $\mathbf{D}(\mathrm{Mod}_\Lambda)$ *are isomorphisms for every* $n \geq 1$.

Proof. As seen in the above discussion, the preceding lemma allows us to perform an inductive argument using the morphism of exact triangles in Proposition 4.4.4 to obtain the required conclusion. □

We finish the proof of Theorem 4.2.2 by passing to the inverse limit.

Proof of Theorem 4.2.2. By Remark 4.1.4, there exists a quasi-isomorphism

$$W \longrightarrow C(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1))$$

with W a bounded above complex of finitely generated projective Λ°-modules. Since J_R has cohomology groups which are finitely generated over R, Lemma 2.3.1 implies the existence of a subcomplex C of J_R such that C is a complex of finitely generated R-modules and the inclusion $i\colon C \hookrightarrow J_R$ is a quasi-isomorphism. We fix such a C and write X_n for $[I^n \to \Lambda]$. The complex $\mathrm{Hom}_{\Lambda^\circ}(W, X_n \otimes_R C)$ represents

$$\mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda/I^n \otimes_R^{\mathbf{L}} \omega_R\Big),$$

and $\mathrm{Hom}_{\Lambda^\circ}(W, \Lambda \otimes_R C)$ represents

$$\mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R^{\mathbf{L}} \omega_R\Big),$$

since X_n is a complex of flat R-modules by Lemma 4.4.2(a). Now, for each n, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T)) \arrow[r] \arrow[d] & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-2] \arrow[d] \\
\mathbf{R}\Gamma(G_v, \mathscr{F}_{\Lambda/I^n}(T)) \arrow[r] & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda/I^n \otimes^{\mathbf{L}}_R \omega_R\Big)[-2]
\end{tikzcd}$$

which induces a commutative diagram

$$\begin{tikzcd}
H^i(G_v, \mathscr{F}_\Gamma(T)) \arrow[r] \arrow[d] & H^{2-i}(\mathrm{Hom}_{\Lambda^\circ}(W, \Lambda \otimes_R C)) \arrow[d] \\
H^i(G_v, \mathscr{F}_{\Lambda/I^n}(T)) \arrow[r] & H^{2-i}(\mathrm{Hom}_{\Lambda^\circ}(W, X_n \otimes_R C))
\end{tikzcd}$$

of cohomology groups. Since the maps in this diagram are compatible as we vary n, we obtain the commutative diagram

$$\begin{tikzcd}
H^i(G_v, \mathscr{F}_\Gamma(T)) \arrow[r] \arrow[d] & H^{2-i}(\mathrm{Hom}_{\Lambda^\circ}(W, \Lambda \otimes_R C)) \arrow[d] \\
\varprojlim_n H^i(G_v, \mathscr{F}_{\Lambda/I^n}(T)) \arrow[r] & \varprojlim_n H^{2-i}(\mathrm{Hom}_{\Lambda^\circ}(W, X_n \otimes_R C)).
\end{tikzcd}$$

It remains to show that the upper horizontal map in the latter diagram is an isomorphism. By Proposition 4.4.8, we know that the lower horizonal map is one. Noting that the maps $X_{n+1} \otimes_R C \to X_n \otimes_R C$ are injections of complexes with intersection $\Lambda \otimes_R C$ over all n, we have an isomorphism

$$\varprojlim_n \mathrm{Hom}_{\Lambda^\circ}(W, X_n \otimes_R C) \cong \mathrm{Hom}_{\Lambda^\circ}(W, \Lambda \otimes_R C)$$

of complexes of finitely generated Λ-modules. Since inverse limits are exact for finitely generated Λ-modules (being that they are compact), after taking cohomology groups, we have that the vertical map on the right is an isomorphism. On the other hand, we also have an isomorphism

$$\varprojlim_n (X_n \otimes_\Lambda \mathscr{F}_\Gamma(T)) \cong \mathscr{F}_\Gamma(T)$$

of complexes of objects in $\mathcal{C}^{\Lambda-\mathrm{ft}}_{\Lambda,G_v}$ and hence an isomorphism

$$\varprojlim_n C(G_v, X_n \otimes_\Lambda \mathscr{F}_\Gamma(T)) \cong C(G_v, \mathscr{F}_\Gamma(T)).$$

By [Lim2, Proposition 3.2.13], we then have

$$\varprojlim_n H^j(G_v, X_n \otimes_\Lambda \mathscr{F}_\Gamma(T)) \cong H^j(G_v, \mathscr{F}_\Gamma(T)).$$

Therefore, the vertical map on the left is also an isomorphism. Hence, the top map is an isomorphism, as required. □

4.5 Duality over global fields

We end this paper by describing the global analog of Theorem 4.2.2 that is our main result. Let T be a bounded complex of objects in $\mathcal{C}^{R-\mathrm{ft}}_{R,G_{F,S}}$, and we choose T^* in $\mathcal{C}^{R-\mathrm{ft}}_{R,G_{F,S}}$ as in Section 3.3. As in [Nek, (5.3.3)], we define two morphisms of complexes of $\Lambda \otimes_R \Lambda^\circ$-modules

$$\begin{aligned} {}_c\cup\colon C_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T)) \otimes_R C(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)) &\to C_{(c)}(G_{F,S}, \Lambda \otimes_R J_R(1)) \\ \cup_c\colon C(G_{F,S}, \mathscr{F}_\Gamma(T)) \otimes_R C_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)) &\to C_{(c)}(G_{F,S}, \Lambda \otimes_R J_R(1)) \end{aligned}$$

which are given by the formulas

$$\begin{aligned} (a, a_S)\,{}_c\cup b &= (a \cup b, a_S \cup_S \mathrm{res}_S(b)) \\ a \cup_c (b, b_S) &= (a \cup b, (-1)^{\bar{a}} \mathrm{res}_S(a) \cup_S b_S), \end{aligned}$$

where $\bar{a}$ denotes the degree of a, the direct sum of the restrictions to the primes in S is denoted res_S, the symbol $\cup$ is the total cup product

$$C(G_{F,S}, \mathscr{F}_\Gamma(T)) \otimes_R C(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)) \longrightarrow C(G_{F,S}, \Lambda \otimes_R J_R(1))$$

of Lemma 3.3.3, and $\cup_S$ is the direct sum of the corresponding local cup products. Much as in Lemma 4.2.1 (but now analogously to [Nek, Lemma 5.7.3]), we have a quasi-isomorphism

$$\Lambda \otimes_R J_R \longrightarrow \tau^{II}_{\geqslant 3} C_{(c)}(G_{F,S}, \Lambda \otimes_R J_R(1))$$

of complexes of $\Lambda \otimes_R \Lambda^\circ$-modules, which allows us to use adjoints to define the morphisms in the following theorem.

Theorem 4.5.1. *Let T be a bounded complex of objects in $\mathcal{C}^{R-\mathrm{ft}}_{R,G_{F,S}}$. Then we have an isomorphism*

$$\begin{tikzcd} \mathbf{R}\Gamma_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T)) \arrow[r, "\sim"] \arrow[d] & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-3] \arrow[d] \\ \mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_\Gamma(T)) \arrow[r, "\sim"] \arrow[d] & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-3] \arrow[d] \\ \bigoplus_{v \in S} \mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T)) \arrow[r, "\sim"] & \bigoplus_{v \in S} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-2] \end{tikzcd}$$

of exact triangles in $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$.

Proof. The morphism of exact triangles can be constructed as in [Lim2, Theorem 4.2.6]. We remark that it suffices to prove that any two of the three horizontal morphisms is an isomorphism, and we focus on the second and third. By an analogous result to Proposition 4.3.2, we may assume that R is regular and $\mathbb{Z}_p$-flat. That the second of the morphisms is an isomorphism follows as in proof of Theorem 4.2.2. In fact, aside from the obvious changes of notation for cochains and other objects, there are no significant changes to the proof. As for the third, note that for a nonarchimedean prime v in S, the map

$$\tau^{II}_{\geq 2} C(G_v, \Lambda \otimes_R J_R(1)) \longrightarrow \tau^{II}_{\geq 2}\Big(C_{(c)}(G_{F,S}, \Lambda \otimes_R J_R(1))[1]\Big)$$

is a quasi-isomorphism that induces the identity maps on the cohomology groups of each complex. Under this identification, the map on the v-summand in the lower horizontal morphism in the statement is exactly the first isomorphism of Theorem 4.2.2.

It remains to prove that, for a real place v, the morphisms

$$\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-2]$$

are isomorphisms in $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$. There are two cases: either v extends to a complex place or splits completely in F_∞. However, if v becomes complex, then both $\widehat{C}(G_v, \mathscr{F}_\Gamma(T))$ and $\widehat{C}(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1))$ are acyclic, and there is nothing to prove.

If v splits completely in F_∞, then the adjoint map of interest is identified (as in the proof of Proposition 4.1.1) with

$$\Lambda \otimes_R \widehat{C}(G_v, T) \longrightarrow \mathrm{Hom}_{\Lambda^\circ}\Big(\Lambda \otimes_R \widehat{C}(G_v, T^*(1)), \Lambda \otimes_R J_R\Big)[-2],$$
$$\lambda \otimes f \mapsto (\mu \otimes g \mapsto \lambda\mu \otimes f \cup g),$$

where

$$\cup \colon \widehat{C}(G_v, T) \otimes_R \widehat{C}(G_v, T^*(1)) \longrightarrow J_R[-2]$$

is given by cup product followed by the maps

$$\widehat{C}(G_v, J_R(1)) \to \tau^{II}_{\geq 2}\widehat{C}(G_v, J_R(1)) \to \tau^{II}_{\geq 2}\Big(C_{(c)}(G_{F,S}, J_R(1))[1]\Big) \to J_R[-2],$$

the latter being a homotopy inverse to the inclusion of complexes.

Suppose now that we have a quasi-isomorphism $D \to \widehat{C}(G_v, T^*(1))$ with D a q-projective complex of finitely generated R-modules. In the commutative diagram

$$
\begin{tikzcd}
\mathrm{Hom}_{\Lambda^\circ}\Big(\Lambda \otimes_R \widehat{C}(G_v, T^*(1)), \Lambda \otimes_R J_R\Big) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_{\Lambda^\circ}(\Lambda \otimes_R D, \Lambda \otimes_R J_R) \arrow[d, "\wr"] \\
\mathrm{Hom}_R\Big(\widehat{C}(G_v, T^*(1)), \Lambda \otimes_R J_R\Big) \arrow[r] \arrow[d] & \mathrm{Hom}_R(D, \Lambda \otimes_R J_R) \arrow[d, "\wr"] \\
\Lambda \otimes_R \mathrm{Hom}_R\Big(\widehat{C}(G_v, T^*(1)), J_R\Big) \arrow[r] & \Lambda \otimes_R \mathrm{Hom}_R(D, J_R),
\end{tikzcd}
$$

of canonical maps, the lower right vertical map is an isomorphism by Lemma 2.1.7 (the terms of D being finitely generated over R), and the lower horizontal morphism is a quasi-isomorphism by the R-injectivity of the terms of J_R and the R-flatness of Λ. It follows that the derived adjoint morphism is represented by a morphism

$$\Lambda \otimes_R \widehat{C}(G_v, T) \longrightarrow \Lambda \otimes_R \mathrm{Hom}_R\Big(\widehat{C}(G_v, T^*(1)), J_R\Big)[-2]$$

that by construction takes $\lambda \otimes f$ to $\lambda \otimes (g \mapsto f \cup g)$, which is the tensor product of the identity map with the derived adjoint that is already known to be a quasi-isomorphism by [Nek, (5.7.5)].

We are left to construct D, the existence of which is not so obvious as $\widehat{C}(G_v, T^*(1))$ need not have bounded cohomology. However, as is seen in [Spa, Lemma 3.3] (see also [Kel, Appendix]), such a complex may be constructed as a direct limit $\varinjlim D_n$ using maps $\alpha_n \colon D_n \to D_{n+1}$ that are split injective in each degree, and where

$$D_0 \longrightarrow \tau^{II}_{\leq 0} \widehat{C}(G_v, T^*(1))$$

is a quasi-isomorphism from a bounded above complex of finitely generated R-projectives and $E_n = D_n/\alpha_{n-1}(D_{n-1})$ for $n \geq 1$ is chosen to be quasi-isomorphic to

$$\mathrm{Cone}\Big(\tau^{II}_{\leq n-1}\widehat{C}(G_v, T^*(1)) \longrightarrow \tau^{II}_{\leq n}\widehat{C}(G_v, T^*(1))\Big),$$

which has bounded, R-finitely generated cohomology. Since R has finite global dimension, we may choose the E_n as bounded complexes of finitely generated projective R-modules such that, for each k, only finitely many E_n^k are nonzero, which is what was needed. □

We end with a straightforward remark.

Remark 4.5.2. There is also an analogous diagram and isomorphisms for the other adjoint, with morphisms as in the second map in Theorem 4.2.2, and this follows, for instance, by a nearly identical argument.

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