Projective Varieties with Bad Semi-stable Reduction at 3 Only

To I. R. Shafarevich, on the occasion of his 90th Birthday

Victor Abrashkin

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Abstract. Suppose $F=W(k)[1/p]$ where $W(k)$ is the ring of Witt vectors with coefficients in algebraically closed field k of characteristic $p\neq 2$. We construct integral theory of p-adic semi-stable representations of the absolute Galois group of F with Hodge-Tate weights from $[0,p)$. This modification of Breuil's theory results in the following application in the spirit of the Shafarevich Conjecture. If Y is a projective algebraic variety over $\mathbb{Q}$ with good reduction modulo all primes $l\neq 3$ and semi-stable reduction modulo 3 then for the Hodge numbers of $Y_{\mathbb{C}}=Y\otimes_{\mathbb{Q}}\mathbb{C}$, one has $h^2(Y_{\mathbb{C}})=h^{1,1}(Y_{\mathbb{C}})$.

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Introduction

Everywhere in the paper p is a fixed prime number, $p\neq 2$, k is algebraically closed field of charactersitic p, F is the fraction field of the ring of Witt vectors $W(k)$, $\bar F$ is a fixed algebraic closure of F and $\Gamma_F=\mathrm{Gal}(\bar F/F)$ is the absolute Galois group of F.
Suppose Y is a projective algebraic variety over $\mathbb{Q}$. Denote by $Y_{\mathbb{C}}$ the corresponding complex variety $Y\otimes_{\mathbb{Q}}\mathbb{C}$. For integers $n,m\geqslant 0$, set $h^n(Y_{\mathbb{C}})=\dim_{\mathbb{C}}H^n(Y_{\mathbb{C}},\mathbb{C})$ and $h^{n,m}(Y_{\mathbb{C}})=\dim_{\mathbb{C}}H^n(\Omega^m_{Y_{\mathbb{C}}})$.
The main result of this paper can be stated as follows.

Theorem 0.1. *If Y has semi-stable reduction modulo 3 and good reduction modulo all primes $l\neq 3$ then $h^2(Y_{\mathbb{C}})=h^{1,1}(Y_{\mathbb{C}})$.*

Remind that a generalization of the Shafarevich Conjecture about the non-existence of non-trivial abelian varieties over $\mathbb{Q}$ with everywhere good reduction was proved by Fontaine [16] and the author [2], and states that

$$h^1(Y_{\mathbb{C}}) = h^3(Y_{\mathbb{C}}) = 0, \quad h^2(Y_{\mathbb{C}}) = h^{1,1}(Y_{\mathbb{C}}) \tag{0.1}$$

if Y has everywhere good reduction. (The Shafarevich Conjecture appears then as the equality $h^1(Y_{\mathbb{C}}) = 0$.) This result became possible due to the following two important achievements of Fontaine's theory of p-adic crystalline representations:

— the Fontaine-Messing theorem relating etale and de Rham cohomology of smooth proper schemes over $W(k)$ in dimensions $[0,p)$, [15] (it was later proved by Faltings in full generality, [12]);
— the Fontaine-Laffaille integral theory of crystalline representations of Γ_F with Hodge-Tate weights from $[0,p-2]$, [13].
Note that the Fontaine-Laffaille theory works essentially for Hodge-Tate weights from $[0,p)$ but does not give all Galois invariant lattices in the corresponding crystalline representations. Nevertheless, this theory admits improvement developed by the author in [1]. As a result, there was obtained a suitable integral theory for the case of Hodge-Tate weights from $[0,p)$, which allowed us to prove some extras to statements (0.1), in particular, that modulo the Generalized Riemann Hypothesis one has $h^4(Y_{\mathbb{C}}) = h^{2,2}(Y_{\mathbb{C}})$.
Since that time there was a huge progress in the study of semi-stable p-adic representations. Tsuji [23] proved a semi-stable case of the relation between etale and crystalline cohomology and Breuil [5, 6] developed an analogue of the Fontaine-Laffaille theory in the context of semi-stable representations (even for ramified basic fields). The papers [4] and [21] studied the problem of the existence of abelian varieties over $\mathbb{Q}$ with only one prime of bad semi-stable reduction. Note that the progress in this direction is quite restrictive because our knowledge of algebraic number fields with prescribed ramification at a given prime number p (and unramified outside p) is very far from to be complete. Theorem 0.1 represents an exceptional situation where the standard tools: the Odlyzko estimates of the minimal discriminants of algebraic number fields and the modern computing facilities (SAGE) are sufficient to resolve upcoming problems. In addition, the proof of this theorem requires a modification of Breuil's theory to work with semi-stable representations of Γ_F with Hodge-Tate weights from $[0,p)$.
The structure of this paper can be described as follows.
In Section 1 we introduce the category $\underline{\mathcal{L}}^*$ of filtered (φ,N)-modules over $\mathcal{W}_1 := k[[u]]$. This is a special pre-abelian category, that is an additive category with kernels, cokernels and sufficiently nice behaving short exact sequences. Note that such categories play quite appreciable role in all our constructions. In Section 2 we construct the functor $\mathcal{V}^*$ from $\underline{\mathcal{L}}^*$ to the category of $\mathbb{F}_p[\Gamma_F]$-modules $\underline{\mathrm{M}\Gamma}_F$ by introducing a "truncated" version of Fontaine's ring of semi-stable periods $\hat{A}_{st}$. The functor $\mathcal{V}^*$ is not fully faithful but by taking into account the maximal etale subobjects of filtered modules from $\underline{\mathcal{L}}^*$ we define a

modification $\mathcal{CV}^*$ of $\mathcal{V}^*$. This functor gives already a fully faithful functor from $\underline{\mathcal{L}}^*$ to the category of cofiltered Γ_F-modules $\underline{\mathrm{CM}\Gamma}_F$. In Section 3 we give an interpretation of Breuil's theory in terms of $\mathcal{W} := W(k)[[u]]$-modules (Breuil worked with modules over the divided powers envelope of $\mathcal{W}$) by introducing the category of filtered (φ, N)-modules $\underline{\mathcal{L}}^{ft}$ over $\mathcal{W}$. The advantage of this construction is that the objects of this category appear as strict subquotients of p-divisible groups in suitable pre-abelian category. This allows us to use devissage despite that all involved categories are not abelian. We also introduce the subcategories $\underline{\mathcal{L}}^{u,ft}$ and, resp., $\underline{\mathcal{L}}^{m,ft}$ of unipotent and, resp., multiplicative objects in $\underline{\mathcal{L}}^{ft}$ and prove that any $\mathcal{L} \in \underline{\mathcal{L}}^{ft}$ is a canonical extension

$$0 \longrightarrow \mathcal{L}^u \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}^m \longrightarrow 0 \tag{0.2}$$

of a multiplicative object $\mathcal{L}^m$ by a unipotent object $\mathcal{L}^u$. In Section 4 we study Breuil's functor $\mathcal{V}^{ft} : \underline{\mathcal{L}}^{ft} \longrightarrow \underline{\mathrm{M}\Gamma}_F$ in the situation of Hodge-Tate weights from $[0, p)$. We show that on the subcategory $\underline{\mathcal{L}}^{u,ft}$ this functor is still fully faithful by proving that on the subcategory of killed by p unipotent objects the functors $\mathcal{V}^{ft}$ and $\mathcal{V}^*$ coincide. Then we show that for any killed by p object $\mathcal{L}$ of $\underline{\mathcal{L}}^{ft}$, the functor $\mathcal{V}^{ft}$ transforms the standard short exact sequence (0.2) into a short exact sequence in $\underline{\mathrm{M}\Gamma}_F$, which admits a functorial splitting. This splitting is used then to construct a modified version $\widetilde{\mathcal{CV}}^{ft} : \underline{\mathcal{L}}^{ft} \longrightarrow \underline{\mathrm{CM}\Gamma}_F$ of $\mathcal{V}^{ft}$, which is already fully faithful. This gives us an efficient control on all Galois invariant lattices of semi-stable representations with weights from $[0, p)$. Especially, we have an explicit description of all killed by p subquotients of such lattices and the corresponding ramification estimates. Finally, in Section 5 we give a proof of Theorem 0.1 following the strategy from [2].

Essentially, we obtain the following result: if V is a 3-adic representation of $\Gamma_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which is unramified outside 3 and is semi-stable at 3 then there is a $\Gamma_{\mathbb{Q}}$-equivariant filtration by $\mathbb{Q}_3$-subspaces $V = V_0 \supset V_1 \supset V_2 \supset V_3 = 0$ such that for $0 \leqslant i \leqslant 2$, the $\Gamma_{\mathbb{Q}}$-module V_i/V_{i+1} is isomorphic to the product of finitely many copies of the Tate twist $\mathbb{Q}_3(i)$. If $V = H^2_{et}(Y_F, \mathbb{Q}_3)$ then looking at the eigenvalues of the Frobenius morphisms of reductions modulo $l \neq 3$, we obtain that $V = V_1$ and $V_2 = 0$, and this implies that $h^2(Y_{\mathbb{C}}) = h^{1,1}(Y_{\mathbb{C}})$.

Note that our construction of the modification of Breuil's functor gives automatically the modification of the Fontaine-Laffaille functor, which essentially coincides with the modification constructed in [1]. It is worth mentioning that switching from Breuil's S-modules to $\mathcal{W}$-modules means moving in the direction of Kisin's approach [18] and recent approach to integral theory of p-adic representations by Liu [19, 20]. It would be also very interesting to study the opportunity to modify Breuil's functor over ramified base [8, 9] to the case of Hodge-Tate weights from $[0, p)$. Finally, mention quite surprising matching of the ramification estimates for semi-stable representations and the Leopoldt conjecture for the field $\mathbb{Q}(\sqrt[3]{3}, \zeta_9)$, cf. Section 5.

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1. The categories $\widetilde{\underline{\mathcal{L}}}^*$, $\widetilde{\underline{\mathcal{L}}}_0^*$, $\underline{\mathcal{L}}^*$, $\underline{\mathcal{L}}_0^*$

Remind that k is algebraically closed field of characteristic $p>2$. Let $\mathcal{W}=W(k)[[u]]$, where $W(k)$ is the ring of Witt vectors with coefficients in k and u is an indeterminate. Denote by σ the automorphism of $W(k)$ induced by the p-th power map on k and agree to use the same symbol for its continuous extension to $\mathcal{W}$ such that $\sigma(u)=u^p$. Denote by $N:\mathcal{W}\longrightarrow\mathcal{W}$ the continuous $W(k)$-linear derivation such that $N(u)=-u$.
We shall often use below the following statement.

Lemma 1.1. *Suppose L is a finitely generated $\mathcal{W}$-module and $\mathcal{A}$ is a σ-linear operator on L. Then the operator* $\mathrm{id}_L-\mathcal{A}$ *is epimorphic. If, in addition, $\mathcal{A}$ is nilpotent then* $\mathrm{id}_L-\mathcal{A}$ *is bijective.*

Proof. Part b) is obvious. In order to prove a) notice first that we can replace L by L/uL and, therefore, assume that L is a finitely generated $W(k)$-module. Clearly, it will be enough to consider the case $pL=0$. Then there is a decomposition of k-vector spaces $L=L_1\oplus L_2$, where $\mathcal{A}$ is invertible on L_1 and nilpotent on L_2. It remains to note that $L_1=L_0\otimes_{\mathbb{F}_p}k$, where L_0 is a finite dimensional $\mathbb{F}_p$-vector space such that $\mathcal{A}|_{L_0}=\mathrm{id}$. The existence of L_0 is a standard fact of σ-linear algebra: if $s=\dim_k L_1$ and $A\in M_s(k)$ is a matrix of $\mathcal{A}|_{L_1}$ in some k-basis of L_1 then $L_0=\{(x_1,\dots,x_s)\in k^s\mid (x_1^p,\dots,x_s^p)A=(x_1,\dots,x_s)\}$; the $\mathbb{F}_p$-linear space L_0 has dimension s because the corresponding equations determine an etale algebra of rank p^s over algebraically closed field k.

$\square$

Remark. In above Lemma and everywhere below we use the following agreement: $\mathcal{A}$ is nilpotent on L iff it is "topologically nilpotent", i.e. $\bigcap_n\mathcal{A}^n(L)=0$.

1.1. Definitions and general properties.

Let $\mathcal{W}_1=\mathcal{W}/p\mathcal{W}$ with induced σ, φ and N.

Definition. The objects of the category $\widetilde{\underline{\mathcal{L}}}_0^*$ are the triples $\mathcal{L}=(L,F(L),\varphi)$, where

- L and $F(L)$ are $\mathcal{W}_1$-modules such that $L\supset F(L)$;
- $\varphi:F(L)\longrightarrow L$ is a σ-linear morphism of $\mathcal{W}_1$-modules; (Note that $\varphi(F(L))$ is a $\sigma(\mathcal{W}_1)$-submodule in L.)

If $\mathcal{L}_1=(L_1,F(L_1),\varphi)$ is also an object of $\widetilde{\underline{\mathcal{L}}}_0^*$ then the morphisms $f\in\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0^*}(\mathcal{L}_1,\mathcal{L})$ are given by $\mathcal{W}_1$-linear maps $f:L_1\longrightarrow L$ such that $f(F(L_1))\subset F(L)$ and $f\varphi=\varphi f$.

Definition. The objects of the category $\widetilde{\underline{\mathcal{L}}}^*$ are the quadruples $\mathcal{L}=(L,F(L),\varphi,N)$, where

- $(L,F(L),\varphi)$ is an object of the category $\widetilde{\underline{\mathcal{L}}}_0^*$;
- $N:L\longrightarrow L/u^{2p}L$ is a $\mathcal{W}_1$-differentiation, i.e. for all $w\in\mathcal{W}_1$ and $l\in L$, $N(wl)=N(w)(l\bmod u^{2p}L)+wN(l)$;

• if $\mathcal{L}_1 = (L_1, F(L_1), \varphi, N)$ is another object of $\widetilde{\underline{\mathcal{L}}}^*$ then the morphisms $\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}^*}(\mathcal{L}_1, \mathcal{L})$ are given by $f : (L_1, F(L_1), \varphi) \longrightarrow (L, F(L), \varphi)$ from $\widetilde{\underline{\mathcal{L}}}_0^*$ such that $fN = Nf$. (We use the same notation f for the reduction of f modulo $u^{2p}L$.)

The categories $\widetilde{\underline{\mathcal{L}}}^*$ and $\widetilde{\underline{\mathcal{L}}}_0^*$ are additive.

Definition. The category $\underline{\mathcal{L}}_0^*$ is a full subcategory of $\widetilde{\underline{\mathcal{L}}}_0^*$ consisting of the objects $\mathcal{L} = (L, F(L), \varphi)$ such that
• L is a free $\mathcal{W}_1$-module of finite rank;
• $F(L) \supset u^{p-1}L$;
• the natural embedding $\varphi(F(L)) \subset L$ induces the identification $\varphi(F(L)) \otimes_{\sigma(\mathcal{W}_1)} \mathcal{W}_1 = L$.

Note that φ induces a map $F(L)/u^{2p}L \longrightarrow L/u^{2p}L$: use that $u^{2p}L \subset u^{p+1}F(L) \subset u^2F(L)$ and $\varphi(u^2F(L)) \subset u^{2p}L$. We shall denote this map by the same symbol φ.

Definition. The category $\underline{\mathcal{L}}^*$ is a full subcategory of $\widetilde{\underline{\mathcal{L}}}^*$ consisting of the objects $\mathcal{L} = (L, F(L), \varphi, N)$ such that
• $(L, F(L), \varphi) \in \underline{\mathcal{L}}_0^*$;
• for all $l \in F(L)$, $uN(l) \in F(L) \bmod u^{2p}L$ and $N(\varphi(l)) = \varphi(uN(l))$.

The categories $\underline{\mathcal{L}}_0^*$ and $\underline{\mathcal{L}}^*$ are additive.
In the case of objects $(L, F(L), \varphi, N)$ of $\underline{\mathcal{L}}^*$ the morphism N can be uniquely recovered from the $\mathcal{W}_1$-differentiation $N_1 = N \bmod u^pL$ due to the following property.

Proposition 1.2. *Suppose* $(L, F(L), \varphi) \in \underline{\mathcal{L}}_0^*$ *and* $N_1 : L \mapsto L/u^pL$ *is a* $\mathcal{W}_1$-*differentiation such that for any* $m \in F(L)$, $uN_1(m) \in F(L) \bmod u^pL$ *and* $N_1(\varphi(l)) = \varphi(uN_1(l))$. *Then there is a unique* $\mathcal{W}_1$-*differentiation* $N : L \longrightarrow L/u^{2p}L$ *such that* $N \bmod u^p = N_1$ *and* $(L, F(L), \varphi, N) \in \underline{\mathcal{L}}^*$.

Proof. Choose a $\mathcal{W}_1$-basis $m_1, \dots, m_s$ of $F(L)$. Then $l_1 = \varphi(m_1), \dots, l_s = \varphi(m_s)$ is a $\mathcal{W}_1$-basis of L and a $\sigma(\mathcal{W}_1)$-basis of $\varphi(F(L))$.
Let $N(l_i) := \varphi(uN_1(m_i)') \in L/u^{2p}L$, where $N_1(m_i)'$ are some lifts of $N_1(m_i)$ to $L/u^{2p}L$. Clearly, the elements $N(l_i) \in \varphi(F(L)) \subset L/u^{2p}L$ are well-defined (use that $\varphi(u^{p+1}L) \subset u^{2p}L$).
For any $l = \sum_i w_il_i \in L$, let $N(l) := \sum_i N(w_i)l_i + \sum_i w_iN(l_i)$. Then $N : L \longrightarrow L/u^{2p}$ is a $\mathcal{W}_1$-differentiation and $N \bmod u^p = N_1$. Clearly, N is the only candidate to satisfy the requirements of our Proposition.
Now suppose $m = \sum_i w_im_i \in F(L)$ with all $w_i \in \mathcal{W}_1$. Then $N(\varphi(m)) = \sum_i \sigma(w_i)l_i \bmod u^{2p}$. On the other hand, $\varphi(uN(m))$ equals

$$\sum_i u^p\sigma(N(w_i))l_i + \sum_i \varphi(w_iuN(m_i)) = \sum_i \sigma(w_i)l_i \bmod u^{2p}$$

because all $\sigma(N(w_i)) \in u^p\sigma(\mathcal{W}_1)$.
The proposition is proved. □

Remark. By above Proposition in the definition of objects of $\underline{\mathcal{L}}^*$ one can replace $N : L \longrightarrow L/u^{2p}L$ by $N_1 = N \bmod u^pL$ and use N as a unique extension of N_1 if neccessary. An example of the situation where we do need to extend N_1 is described in Proposition 1.3 below. Another situation is related to the definition of the truncated version R^0_{st} of $\hat{A}_{st}$ in Subsection 2. Here we need N to be defined modulo some smaller module than u^pL, e.g. $u^{p+1}L$. Our choice was done in favour of the module $u^{2p}L$ because it is the smallest possible module where the definition of N makes sense.

Proposition 1.3. *$\underline{\mathcal{L}}^*_0$ and $\underline{\mathcal{L}}^*$ are pre-abelian categories (cf. Appendix A for the concept of pre-abelian category).*

Proof. Suppose $\mathcal{S}$ is an additive category and $f \in \mathrm{Hom}_{\mathcal{S}}(A,B)$, $A,B \in \mathcal{S}$. Then $i \in \mathrm{Hom}_{\mathcal{S}}(K,A)$ is a kernel of f if for any $D \in \mathcal{S}$, the sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{S}}(D,K) \xrightarrow{i_*} \mathrm{Hom}_{\mathcal{S}}(D,A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{S}}(D,B)$$

is exact. Similarly, $j \in \mathrm{Hom}_{\mathcal{S}}(B,C)$, $B,C \in \mathcal{S}$, is a cokernel of f if for any $D \in \mathcal{S}$, the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{S}}(C,D) \xrightarrow{j^*} \mathrm{Hom}_{\mathcal{S}}(B,D) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{S}}(A,D)$$

is exact.

Let $FF_{\mathcal{W}_1}$ be the category of free $\mathcal{W}_1$-modules with filtration. This category is pre-abelian. More precisely, consider the objects $\mathcal{L} = (L, F(L))$ and $\mathcal{M} = (M, F(M))$ in $FF_{\mathcal{W}_1}$ and let $f \in \mathrm{Hom}_{FF_{\mathcal{W}_1}}(\mathcal{L},\mathcal{M})$.

Then $\mathrm{Ker}_{FF_{\mathcal{W}_1}} f$ is a natural embedding $i_{\mathcal{L}} : \mathcal{K} = (K, F(K)) \longrightarrow \mathcal{L}$, where $K = \mathrm{Ker}(f : L \longrightarrow M)$ and $F(K) = K \cap F(L)$. The coimage $\mathrm{Coim}_{FF_{\mathcal{W}_1}} f = \mathrm{Coker}_{FF_{\mathcal{W}_1}}(\mathrm{Ker}_{FF_{\mathcal{W}_1}} f)$ appears as a natural projection $j_{\mathcal{L}} : \mathcal{L} \longrightarrow \mathcal{L}' = (L', F(L'))$, where $L' = f(L)$ and $F(L') = f(F(L))$.

Similarly, $\mathrm{Coker} f$ is a natural projection $j_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{C} = (C, F(C))$, where $C = (M/L')/(M/L')_{tor}$ and $F(C) = j_{\mathcal{M}}(F(M))$. Then the image $\mathrm{Im}_{FF_{\mathcal{W}_1}} f = \mathrm{Ker}_{FF_{\mathcal{W}_1}}(\mathrm{Coker}_{FF_{\mathcal{W}_1}} f)$ is a natural embedding $\mathcal{M}' = (M', F(M')) \longrightarrow \mathcal{M}$, where M' is the kernel of $j_{\mathcal{M}}$ and $F(M') = F(M) \cap M'$.

As usually, there is a natural map $\mathcal{L}' \longrightarrow \mathcal{M}'$ induced by $L' \subset M'$. Note that $M/M' = C$ is free and M'/L' is torsion $\mathcal{W}_1$-modules and these properties completely characterize M' as a $\mathcal{W}_1$-submodule of M.

Now suppose $\mathcal{L} = (L, F(L), \varphi)$, $\mathcal{M} = (M, F(M), \varphi)$ are objects of $\underline{\mathcal{L}}^*_0$ and $f \in \mathrm{Hom}_{\underline{\mathcal{L}}^*_0}(\mathcal{L},\mathcal{M})$. Use the obvious forgetful functor $\underline{\mathcal{L}}^*_0 \longrightarrow FF_{\mathcal{W}_1}$ and the same notation for the corresponding images of $\mathcal{L}$, $\mathcal{M}$ and f. Show that $\mathcal{K} = \mathrm{Ker}_{FF_{\mathcal{W}_1}} f$ and $\mathcal{C} = \mathrm{Coker}_{FF_{\mathcal{W}_1}} f$ have the natural structures of objects of $\underline{\mathcal{L}}^*_0$ and with respect to this structure they become the kernel and, resp, cokernel of f in $\underline{\mathcal{L}}^*_0$. Indeed, $u^{p-1}K = u^{p-1}L \cap K \subset F(L) \cap K = F(K) = \mathrm{Ker}(f : F(L) \longrightarrow F(M))$. Therefore, $\varphi(F(K)) \subset K \cap \varphi(F(L))$ and there is a natural embedding $\iota : \varphi(F(K)) \otimes_{\sigma\mathcal{W}_1} \mathcal{W}_1 \subset K$. On the one hand,

$$\mathrm{rk}_{\sigma\mathcal{W}_1} \varphi(F(K)) = \mathrm{rk}_{\mathcal{W}_1} F(K) = \mathrm{rk}_{\mathcal{W}_1} K.$$

On the other hand, $F(L)/F(K) \subset L/K = L'$ have no $\mathcal{W}_1$-torsion. This implies that the quotient $\varphi(F(L))/\varphi(F(K))$ has no $\sigma\mathcal{W}_1$-torsion and the factor of $L = \varphi(F(L)) \otimes_{\sigma\mathcal{W}_1} \mathcal{W}_1$ by $\varphi(F(K)) \otimes_{\sigma\mathcal{W}_1} \mathcal{W}_1$ also has no $\mathcal{W}_1$-torsion. So, ι becomes the equality $\varphi(F(K)) \otimes_{\sigma\mathcal{W}_1} \mathcal{W}_1 = K$ and $\mathcal{K} = (K, F(K), \varphi) = \mathrm{Ker}_{\underline{\mathcal{L}}_0^*} f$. The above description of $\mathrm{Ker}_{\underline{\mathcal{L}}_0^*}$ implies that $u^{p-1}L' \subset F(L')$, $\varphi(F(L')) = \varphi(F(M))/\varphi(F(K))$ and $L' = \varphi(F(L')) \otimes_{\sigma\mathcal{W}_1} \mathcal{W}_1$. In other words, $\mathcal{L}' = (L', F(L'), \varphi) \in \underline{\mathcal{L}}_0^*$.

Now note that for $\mathcal{M}' = (M', F(M'))$, we have

$$u^{p-1}M' = (u^{p-1}M) \cap M' \subset F(M) \cap M' = F(M')$$

and, therefore, $F(M')/F(L')$ is torsion $\mathcal{W}_1$-module and

- $(\varphi(F(M')) \otimes_{\sigma\mathcal{W}_1} \mathcal{W}_1)/L'$ is torsion $\mathcal{W}_1$-module;

On the other hand, $F(M)/F(M') = F(C)$ is torsion free $\mathcal{W}_1$-module. This implies that $\varphi(F(M))/\varphi(F(M'))$ is torsion free $\sigma\mathcal{W}_1$-module and, therefore,

- $M/(\varphi(F(M')) \otimes_{\sigma\mathcal{W}_1} \mathcal{W}_1)$ is torsion free $\mathcal{W}_1$-module.

The above two conditions completely characterize M' as a submodule of M. Therefore, $\varphi(F(M')) \otimes_{\sigma\mathcal{W}_1} \mathcal{W}_1 = M'$, $\mathcal{M}' = (M', F(M'), \varphi) \in \underline{\mathcal{L}}_0^*$ and $(M/M', F(M)/F(M'), \varphi) = (C, F(C), \varphi) = \mathcal{C} \in \underline{\mathcal{L}}_0^*$. Now a formal verification shows that $\mathcal{C} = \mathrm{Coker}_{\underline{\mathcal{L}}_0^*} f$.

Again $\mathrm{Coim}_{\underline{\mathcal{L}}_0^*} f = (L', F(L'), \varphi)$ and $\mathrm{Im}_{\underline{\mathcal{L}}_0^*} f = (M', F(M'), \varphi)$ together with their natural embedding $\mathrm{Coim}_{\underline{\mathcal{L}}_0^*} f \longrightarrow \mathrm{Im}_{\underline{\mathcal{L}}_0^*} f$ in $\underline{\mathcal{L}}_0^*$. As a matter of fact, these two objects of the category $\underline{\mathcal{L}}_0^*$ do not differ very much due to the following Lemma.

Lemma 1.4. *$\varphi(F(L')) \supset u^p\varphi(F(M'))$ (and, therefore, $L' \supset u^pM'$).*

Proof of Lemma. Otherwise, there is an $l \in \varphi(F(L')) \setminus u^p\varphi(F(L'))$ such that $l \in u^{2p}\varphi(F(M'))$.

Form the sequence $l_n \in L'$ such that $l_1 = l$ and for all $n \geqslant 2$, $l_{n+1} = \varphi(u^{a_n}l_n)$, where $a_n \geqslant 0$ is such that $u^{a_n}l_n \in F(L') \setminus uF(L')$. Clearly, all $l_n \notin uF(L') \supset u^pL'$.

On the other hand, $l \in u^{2p}\varphi(F(M')) \subset u^{p+1}F(M')$ and, therefore, for all $n \geqslant 1$, $l_n \in \varphi^n(u^{2p}M') \subset u^{p^n+p}M'$. So, for $n \gg 0$, $l_n \in u^pL'$. The contradiction. $\square$

Now suppose $\mathcal{L} = (L, F(L), \varphi, N)$ and $\mathcal{M} = (M, F(M), \varphi, N)$ are objects of $\underline{\mathcal{L}}^*$ and $f \in \mathrm{Hom}_{\underline{\mathcal{L}}^*}(\mathcal{L}, \mathcal{M})$. Prove that the kernel $(K, F(K), \varphi)$ and the cokernel $(C, F(C), \varphi)$ of f in the category $\underline{\mathcal{L}}_0^*$ have a natural structure of objects of the category $\underline{\mathcal{L}}^*$.

Clearly, $N(K) \subset \mathrm{Ker}\left(f \bmod u^{2p} : L/u^{2p}L \longrightarrow M/u^{2p}M\right)$. The above Lemma 1.4 implies that $u^pL' \supset u^{2p}M'$ and we obtain the following natural maps

$$L'/u^pL' \xleftarrow{\alpha} L'/u^{2p}M' \xrightarrow{\beta} M'/u^{2p}M' \xrightarrow{\gamma} M/u^{2p}M.$$

Note that α is epimorphic but β and γ are monomorphic. This implies that $N(K) \subset \mathrm{Ker}(L/u^{2p}L \longrightarrow L'/u^pL')$ and

$$N(K) \bmod u^pL \subset \mathrm{Ker}(L/u^pL \longrightarrow L'/u^pL') = K/u^pK.$$

Therefore, by Proposition 1.2, N (as a unique lift of $N_1 = N \bmod u^p$) maps K to $K/u^{2p}K$ and $(K, F(K), \varphi, N) \in \underline{\mathcal{L}}^*$.
The above property of $\mathrm{Ker}_{\underline{\mathcal{L}}^*} f$ implies that $N(L') \subset L'/u^{2p}L'$. Now use that $u^pM' \subset L'$, $u^{2p}L' \subset u^{2p}M'$ and $N(u^pM') \subset u^pM/u^{2p}M$ to deduce that

$$N(u^pM') \subset (L'/u^{2p}M') \cap (u^pM/u^{2p}M) = u^pM'/u^{2p}M'.$$

So, $N \bmod u^p$ maps M' to M'/u^pM' and again by Proposition 1.2 $N(M') \subset M'/u^{2p}M'$. This means that the kernel of the above constructed $\mathrm{Coker} f$: $(M, F(M), \varphi) \longrightarrow (C, F(C), \varphi)$ is provided with the structure of object of the category $\underline{\mathcal{L}}^*$. Therefore, N induces the map $N : C \longrightarrow C/u^{2p}C$ and $(C, F(C), \varphi, N) \in \underline{\mathcal{L}}^*$. The proposition is proved. □

The above proof shows that the kernels and cokernels in the category $\underline{\mathcal{L}}^*$ appear on the level of filtered modules as the kernel and cokernel of the corresponding map of filtered modules $(L_1, F(L_1))$ to $(L, F(L))$ in the category of filtered $\mathcal{W}_1$-modules. Therefore, the category $\underline{\mathcal{L}}^*$ is special, cf. Appendix A, and we can apply the corresponding formalism of short exact sequences. In particular, if we take another object $\mathcal{L}_2 = (L_2, F(L_2), \varphi, N) \in \underline{\mathcal{L}}^*$ then

- $i \in \mathrm{Hom}_{\underline{\mathcal{L}}^*}(\mathcal{L}_1, \mathcal{L})$ is strict monomorphism iff $i : L_1 \longrightarrow L$ is injective and $i(L_1) \cap F(L) = i(F(L_1))$;
- $j \in \mathrm{Hom}_{\underline{\mathcal{L}}^*}(\mathcal{L}, \mathcal{L}_2)$ is strict epimorphism iff $j : L \longrightarrow L_2$ is epimorphic and $j(F(L)) = F(L_2)$.

As usually, cf. Appendix A, if a monomorphism i is strict then the monomorphism $j = \mathrm{Coker}\, i$ is strict, and if an epimorphism j is strict then the monomorphism $i = \mathrm{Ker}\, j$ is strict, and under these assumptions $0 \longrightarrow \mathcal{L}_1 \xrightarrow{i} \mathcal{L} \xrightarrow{j} \mathcal{L}_2 \longrightarrow 0$ is short exact sequence.

With relation to the above result that the categories $\underline{\mathcal{L}}_0^*$ and $\underline{\mathcal{L}}^*$ are pre-abelian, note that the situation with the categories $\widetilde{\underline{\mathcal{L}}}_0^*$ and $\widetilde{\underline{\mathcal{L}}}^*$ is different. Indeed, let $FM_{\mathcal{W}_1}$ be the category of filtered (not necessarily free) modules over $\mathcal{W}_1$. This category is pre-abelian: for $\mathcal{M}_i = (M_i, F(M_i))$, $i = 1, 2$, and $f \in \mathrm{Hom}_{FM_{\mathcal{W}_1}}(\mathcal{M}_1, \mathcal{M}_2)$, we have the following equalities $\mathrm{Ker}_{FM_{\mathcal{W}_1}} f = (\mathrm{Ker} f, \mathrm{Ker} f \cap F(M_1))$ (together with its natural embedding into $\mathcal{M}_1$) and $\mathrm{Coker}_{FM_{\mathcal{W}_1}} f = (\mathrm{Coker} f, F(M_2)/(f(M_1) \cap F(M_2))$ (together with the natural projection from $\mathcal{M}_2$).
Now suppose that $\mathcal{M}_i = (M_i, F(M_i), \varphi) \in \widetilde{\underline{\mathcal{L}}}_0^*$, $i = 1, 2$, and $f \in \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0^*}(\mathcal{M}_1, \mathcal{M}_2)$. Then $\mathrm{Ker}_{\widetilde{\underline{\mathcal{L}}}_0^*} f$ exists (and coincides on the level of filtered modules with $\mathrm{Ker}_{FM_{\mathcal{W}_1}} f$) but $\mathrm{Coker}_{\widetilde{\underline{\mathcal{L}}}_0^*} f$ exists (and coincides on the level of filtered modules with $\mathrm{Coker}_{FM_{\mathcal{W}_1}} f$) only if we have $f(F(M_1)) = f(M_1) \cap F(M_2)$. In particular, on the level of filtered modules the composition $\mathrm{Coker}_{FM_{\mathcal{W}_1}}(\mathrm{Ker}_{FM_{\mathcal{W}_1}} f)$ always makes sense and coincides with the natural projection $\mathcal{M}_1 \longrightarrow (f(M_1), f(F(M_1))$. Therefore, one can introduce the concept of strict epimorphism in $\widetilde{\underline{\mathcal{L}}}_0^*$: f is strict epimorphism iff $f(M_1) = M_2$ and $f(F(M_1)) = F(M_2)$.

The following situation will appear several times below.

Lemma 1.5. *Suppose* $\mathcal{M}_1, \mathcal{M}_2 \in \widetilde{\underline{\mathcal{L}}}_0^*$, $\iota \in \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0^*}(\mathcal{M}_1, \mathcal{M}_2)$ *is a strict epimorphism and* $\mathrm{Ker}_{\widetilde{\underline{\mathcal{L}}}_0^*}\iota = (K, K, \varphi)$. *Then for any* $\mathcal{L} \in \underline{\mathcal{L}}_0^*$,

$$\iota^* : \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0^*}(\mathcal{L}, \mathcal{M}_1) \longrightarrow \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0^*}(\mathcal{L}, \mathcal{M}_2)$$

is epimorphic. In addition, if $\varphi|_K$ *is nilpotent then* ι^* *is bijective.*

Proof. The structure of $\mathcal{L} = (L, F(L), \varphi)$ can be given by a vector $\bar{l} = (l_1, \dots, l_s)$ and a matrix $C \in M_s(\mathcal{W}_1)$ such that
— $l_1, \dots, l_s$ is a $\mathcal{W}_1$-basis of L;
— if $\bar{l}C = \bar{m} = (m_1, \dots, m_s)$ then $m_1, \dots, m_s$ is a $\mathcal{W}_1$-basis of $F(L)$;
— $\bar{l} = \varphi(\bar{m}) := (\varphi(m_1), \dots, \varphi(m_s))$.

Suppose $M_1 = (M_1, F(M_1), \varphi)$ and $M_2 = (M_2, F(M_2), \varphi)$.
Any $f \in \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0^*}(\mathcal{L}, \mathcal{M}_2)$ is given by $f(\bar{l}) \in M_2^s$ such that $f(\bar{l})C \in F(M_2)^s$ and $\varphi(f(\bar{l})C) = f(\bar{l})$.
Choose an $\hat{f}(\bar{l}) \in M_1^s$ such that $\hat{f}(\bar{l}) \bmod K = f(\bar{l})$. Then $\hat{f}(\bar{l})C$ modulo K belongs to $F(M_2)^s$ and, therefore, $\hat{f}(\bar{l})C \in F(M_1)^s$. Clearly, we have that $\bar{k}_0 := \varphi(\hat{f}(\bar{l})C) - \hat{f}(\bar{l}) \in K^s$. We must prove the existence of $\bar{k}_1 \in K^s$ such that $\varphi((\hat{f}(\bar{l}) + \bar{k}_1)C) = \hat{f}(\bar{l}) + \bar{k}_1$. This is equivalent to

$$\bar{k}_1 - \varphi(\bar{k}_1 C) = \bar{k}_0$$

and the existence of $\bar{k}_1$ follows from Lemma 1.1. This proves that ι^* is surjective. If $\varphi|_K$ is nilpotent then the bijectivity of ι^* follows in a similar way from part b) of Lemma 1.1. ☐

1.2. Standard exact sequences. Suppose $\mathcal{L} = (L, F(L), \varphi, N) \in \underline{\mathcal{L}}^*$. Introduce a σ-linear map $\phi : L \longrightarrow L$ by $\phi : l \mapsto \varphi(u^{p-1}l)$.

Definition. The object $\mathcal{L}$ is etale (resp., connected) if $\phi \bmod u$ is invertible (resp., nilpotent) on L/uL.

Let $\mathcal{L}(0) = (\mathcal{W}_1, \mathcal{W}_1 u^{p-1}, \varphi, N) \in \underline{\mathcal{L}}^*$, where $\varphi(u^{p-1}) = 1$ and $N(1) = u^p \bmod u^{2p}$. Then $\mathcal{L}(0)$ is etale. As a matter of fact, it is the simplest etale object of $\underline{\mathcal{L}}^*$ due to the following Lemma.

Lemma 1.6. *Suppose* $\mathcal{L} = (L, F(L), \varphi, N) \in \underline{\mathcal{L}}^*$ *is etale. Then* $\mathcal{L}$ *is a product of finitely many copies of* $\mathcal{L}(0)$.

Proof. If $\widetilde{L}_0 = \{l \in L/uL \mid \phi(l) = l\}$ then $L/uL = \widetilde{L}_0 \otimes_{\mathbb{F}_p} k$. Then there is a unique $\mathbb{F}_p$-submodule L_0 of L such that $\phi|_{L_0} = \mathrm{id}$ and $L = L_0 \otimes_{\mathbb{F}_p} \mathcal{W}_1$.
Suppose $l \in L_0$. Then $\varphi(u^{p-1}l) = l$ and $N(l) = N(\varphi(u^{p-1}l)) = \varphi(uN(u^{p-1}l)) = \varphi(u^p(l \bmod u^{2p}) + u^p N(l)) = u^p l \bmod u^{2p}L$. Therefore, if $e_1, \dots, e_s$ is an $\mathbb{F}_p$-basis of L_0 then all $(\mathcal{W}_1 e_i, \mathcal{W}_1 u^{p-1} e_i)$ determine the subobjects $\mathcal{L}_i \simeq \mathcal{L}(0)$ of $\mathcal{L}$ and $\mathcal{L} \simeq \mathcal{L}_1 \times \dots \times \mathcal{L}_s$. ☐

Proposition 1.7. *Any* $\mathcal{L}\in\underline{\mathcal{L}}^*$ *contains a unique maximal etale subobject* $(\mathcal{L}^{et},i^{et})$ *and a unique maximal connected quotient object* $(\mathcal{L}^c,j^c)$ *and the sequence* $0\longrightarrow\mathcal{L}^{et}\xrightarrow{i^{et}}\mathcal{L}\xrightarrow{j^c}\mathcal{L}^c\longrightarrow 0$ *is short exact.*

Proof. Let $\mathcal{L}=(L,F(L),\varphi,N)$ and, as earlier, let $\phi:L\longrightarrow L$ be such that for any $l\in L$, $\phi(l)=\varphi(u^{p-1}l)$. Then for $\widetilde{L}=L/uL$, we have the k-linear subspaces $\widetilde{L}^{et}$ and $\widetilde{L}^c$ in $\widetilde{L}$ such that $\tilde{\phi}:=\phi\operatorname{mod}u$ is invertible on $\widetilde{L}^{et}$ and nilpotent on $\widetilde{L}^c$ and $\widetilde{L}=\widetilde{L}^{et}\oplus\widetilde{L}^c$.
Then there is a unique $\mathcal{W}_1$-submodule L^{et} of L such that $\phi|_{L^{et}}$ is invertible and $L^{et}/uL^{et}=\widetilde{L}^{et}$. The filtered submodule $(L^{et},u^{p-1}L^{et})$ determines an etale subobject $\iota^{et}:\mathcal{L}^{et}\longrightarrow\mathcal{L}$. Clearly, $u^{p-1}L^{et}\subset L^{et}\cap F(L)$. If the inverse embedding does not take place then there is an $l\in L^{et}\setminus uL^{et}$ such that $u^{p-1}l\in uF(L)$. Therefore, $\phi(l)=\varphi(u^{p-1}l)\in u^pL$ but $\phi|_{L^{et}}$ is invertible. So, ι^{et} is strict monomorphism and we can consider $\operatorname{Coker}\iota^{et}=j^c:\mathcal{L}\longrightarrow\mathcal{L}^c$. Clearly, $\mathcal{L}^c$ is connected. The maximality properties of $\mathcal{L}^{et}$ and $\mathcal{L}^c$ are formally implied by the following easy statement:

if $\mathcal{L}_1\in\underline{\mathcal{L}}^*$ *is etale and* $\mathcal{L}_2\in\underline{\mathcal{L}}^*$ *is connected then* $\operatorname{Hom}_{\underline{\mathcal{L}}^*}(\mathcal{L}_1,\mathcal{L}_2)=0$. □

Suppose $\mathcal{L}=(L,F(L),\varphi,N)\in\underline{\mathcal{L}}^*$. Then $\varphi(F(L))$ is a $\sigma(\mathcal{W}_1)$-module and $L=\varphi(F(L))\otimes_{\sigma(\mathcal{W}_1)}\mathcal{W}_1$. If $l\in L$ and for $0\leqslant i<p$, $l^{(i)}\in F(L)$ are such that $l=\sum_{0\leqslant i<p}\varphi(l^{(i)})\otimes u^i$, set $V(l)=l^{(0)}$. Then $V\operatorname{mod}u$ is a σ^{-1}-linear endomorphism of the k-vector space L/uL.

Definition. The module $\mathcal{L}$ is multiplicative (resp., unipotent) if $\widetilde{V}:=V\operatorname{mod}u$ is invertible (resp., nilpotent) on $\widetilde{L}:=L/uL$.

Let $\mathcal{L}(1)=(\mathcal{W}_1,\mathcal{W}_1,\varphi,N)\in\underline{\mathcal{L}}^*$, where $\varphi(1)=1$ and $N(1)=0$. Then $\mathcal{L}(1)$ is multiplicative. As a matter of fact, it is the simplest multiplicative object of $\underline{\mathcal{L}}^*$ due to the following Lemma.

Lemma 1.8. *Suppose* $\mathcal{L}=(L,F(L),\varphi,N)\in\underline{\mathcal{L}}^*$ *is multiplicative, then* $\mathcal{L}$ *is the product of finitely many copies of* $\mathcal{L}(1)$.

Proof. Clearly, the embedding $F(L)\longrightarrow L$ induces the identification $F(L)/uF(L)=L/uL$ and, therefore, $F(L)=L$.
Let $\widetilde{L}_0\subset\widetilde{L}$ be such that $\widetilde{V}|_{\widetilde{L}_0}=\mathrm{id}$. If $l\in L$ is such that $l\operatorname{mod}uL\in\widetilde{L}_0$ then $\varphi(l)\equiv l\operatorname{mod}uL$. This implies the existence of a unique $l'\in L$ such that $l'\equiv l\operatorname{mod}uL$ and $\varphi(l')=l'$. In other words, there is an $\mathbb{F}_p$-submodule L_0 in L such that $L=L_0\otimes_{\mathbb{F}_p}\mathcal{W}_1$ and $\varphi|_{L_0}=\mathrm{id}$.
If $l\in L_0$ then $N(l)=N(\varphi(l))=\varphi(uN(l))=u^p\varphi(N(l))=0$. So, if $e_1,\dots,e_s$ is an $\mathbb{F}_p$-basis of L_0 then the filtered modules $(\mathcal{W}_ie_i,\mathcal{W}_1e_i)$ determine the subobjects $\mathcal{L}_i\simeq\mathcal{L}(1)$ of $\mathcal{L}$ and $\mathcal{L}\simeq\mathcal{L}_1\times\dots\times\mathcal{L}_s$. □

Proposition 1.9. *Any* $\mathcal{L}=(L,M,\varphi,N)\in\underline{\mathcal{L}}^*$ *contains a unique maximal multiplicative quotient object* $(\mathcal{L}^m,j^m)$ *and a unique maximal unipotent subobject*

$(\mathcal{L}^u, i^u)$ *and the sequence*

$$0 \longrightarrow \mathcal{L}^u \xrightarrow{i^u} \mathcal{L} \xrightarrow{j^m} \mathcal{L}^m \longrightarrow 0$$

is exact.

Proof. Let $\widetilde{L} = L/uL$, $\widetilde{M} = M/uM$ and $\widetilde{L} = \widetilde{L}^m \oplus \widetilde{L}^u$, where $\widetilde{V} := V \bmod u$ is invertible on $\widetilde{L}^m$ and nilpotent on $\widetilde{L}^u$.
Note that φ induces a σ-linear isomorphism $\tilde{\varphi} : \widetilde{M} \longrightarrow \widetilde{L}$. Denote by $\tilde{\imath}$: $\widetilde{M} \longrightarrow \widetilde{L}$ the k-linear morphism induced by the embedding $M \subset L$. With this notation, for any $l \in \widetilde{L}$, $\widetilde{V}(l) = \tilde{\imath}(\tilde{\varphi}^{-1}(l))$.
Consider the filtration $\widetilde{L}^u \supset \widetilde{V}\widetilde{L}^u \supset \cdots \supset \widetilde{V}^s\widetilde{L}^u = \{0\}$ and set for $1 \leqslant i \leqslant s+1$, $\widetilde{M}_i = \tilde{\varphi}^{-1}(\widetilde{V}^{i-1}\widetilde{L}^u)$. Then $\widetilde{M}_1 \supset \widetilde{M}_2 \supset \cdots \supset \widetilde{M}_s \supset \widetilde{M}_{s+1} = \{0\}$ and for $1 \leqslant i \leqslant s$,

$$\tilde{\imath}(\widetilde{M}_i) = \widetilde{V}^i\widetilde{L}^u = \tilde{\varphi}(\widetilde{M}_{i+1}). \tag{1.1}$$

For $1 \leqslant i \leqslant s+1$, introduce the $\mathcal{W}_1$-submodules $M_i^{(0)}$ of M such that $M_1^{(0)} \supset M_2^{(0)} \supset \cdots \supset M_s^{(0)} \supset M_{s+1}^{(0)} = 0$ and $M_i^{(0)}/uM_i^{(0)} = \widetilde{M}_i$ with respect to the natural projection $M \longrightarrow \widetilde{M}$. Then conditions (1.1) imply that for all i, $M_i^{(0)} \subset \varphi(M_{i+1}^{(0)}) \otimes_{\sigma\mathcal{W}_1} \mathcal{W}_1 + uL$.
Let $\widetilde{M}^m = \tilde{\varphi}^{-1}(\widetilde{L}^m)$ and let $M^m \subset M$ be a $\mathcal{W}_1$-submodule such that $M^m/uM^m = \widetilde{M}^m$ with respect to the natural projection $M \longrightarrow \widetilde{M}$. Then

$$M^m + uL = \varphi(M^m) \otimes_{\sigma\mathcal{W}_1} \mathcal{W}_1 + uL \tag{1.2}$$

and $M = M^m \oplus M_1^{(0)}$.
Prove the existence of "more precise" lifts $M_i^{(n)}$ of $\widetilde{M}_i$, where $0 \leqslant i \leqslant s+1$ and $n \geqslant 1$.

Lemma 1.10. *For all $n \geqslant 1$ and $0 \leqslant i \leqslant s+1$, there are $\mathcal{W}_1$-modules $M_i^{(n)}$ such that*

a) $M_1^{(n)} \supset M_2^{(n)} \supset \cdots \supset M_s^{(n)} \supset M_{s+1}^{(n)} = \{0\}$ *and* $M_i^{(n)}/uM_i^{(n)} = \widetilde{M}_i$ *with respect to the natural projection* $M \longrightarrow \widetilde{M}$;

b) $M_i^{(n)} \subset \varphi(M_{i+1}^{(n)}) \otimes_{\sigma\mathcal{W}_1} \mathcal{W}_1 + u\varphi(M_1^{(n)}) \otimes_{\sigma\mathcal{W}_1} \mathcal{W}_1 + u^{n+1}L$;

c) $M_i^{(n-1)} + u^nM = M_i^{(n)} + u^nM$.

Proof of Lemma. The modules $M_i^{(0)}$, $0 \leqslant i \leqslant s+1$, satisfy the requirements a) and b) of our Lemma. Therefore, we can assume that the modules $M_i^{(n)}$ satisfying the requirements a)-c) have been already constructed for $n = N-1$, where $N \geqslant 1$.
Note that $M = M^m \oplus M_1^{(N-1)}$ (it is known for $N = 1$ and follows from c) for $N > 1$). Therefore, (1.2) implies that

$$\begin{aligned} L = \varphi(M^m) \otimes_{\sigma\mathcal{W}_1} \mathcal{W}_1 + \varphi(M_1^{(N-1)}) \otimes_{\sigma\mathcal{W}_1} \mathcal{W}_1 \\ \subset M^m + \varphi(M_1^{(N-1)}) \otimes_{\sigma\mathcal{W}_1} \mathcal{W}_1 + uL. \end{aligned}$$

Therefore, for $1 \leqslant i \leqslant s$ (use b) for $n = N-1$),

$$M_i^{(N-1)} \subset \varphi(M_{i+1}^{(N-1)}) \otimes_{\sigma \mathcal{W}_1} \mathcal{W}_1 + u\varphi(M_1^{(N-1)}) \otimes_{\sigma \mathcal{W}_1} \mathcal{W}_1 + u^N M^m + u^{N+1} L$$

and we can define the submodules $M_i^{(N)}$ in such a way that the property c) holds for $n = N$

$$M_i^{(N)} + u^N M^m = M_i^{(N-1)} + u^N M^m \tag{1.3}$$

and

$$M_i^{(N)} \subset \varphi(M_{i+1}^{(N-1)}) \otimes_{\sigma \mathcal{W}_1} \mathcal{W}_1 + u\varphi(M_1^{(N-1)}) \otimes_{\sigma \mathcal{W}_1} \mathcal{W}_1 + u^{N+1} L. \tag{1.4}$$

Note that (1.3) implies that $\varphi(M_i^{(N)}) + u^{Np} L = \varphi(M_i^{(N-1)}) + u^{Np} L$ and, therefore, we can replace $\varphi(M_i^{(N-1)})$ and $\varphi(M_1^{(N-1)})$ by $\varphi(M_i^{(N)})$ and, resp. $\varphi(M_1^{(N)})$ in (1.4). The lemma is proved. $\square$

Let $M^u = \bigcap_{n \geqslant 0} (M_1^{(n)} + u^{n+1} M)$. Then $M^u / uM^u = \widetilde{M}^u$ with respect to the natural projection $M \longrightarrow \widetilde{M}$ and $M = M^m \oplus M^u$.
Let $L^u = \varphi(M^u) \otimes_{\sigma \mathcal{W}_1} \mathcal{W}_1$. Then $\mathrm{rk}_{\mathcal{W}_1} L^u = \mathrm{rk}_{\mathcal{W}_1} M^u$ and

$$L^u = \bigcap_{n \geqslant 0} \left(\varphi(M_1^{(n)}) \otimes_{\sigma \mathcal{W}_1} \mathcal{W}_1 + u^{(n+1)p} L \right) \supset M^u$$

(use Lemma 1.10b)). On the other hand,

$$L = \varphi(M^m \oplus M^u) \otimes_{\sigma \mathcal{W}_1} \mathcal{W}_1 = M^m \oplus L^u$$

implies that $M^u \supset u^{p-1} L^u$ and $L^u \cap M = M^u$. Therefore, the filtered module (L^u, M^u) defines a unipotent subobject $\mathcal{L}^u$ of $\mathcal{L}$ in the category $\underline{\mathcal{L}}_0^*$ and the natural embedding $\mathcal{L}^u \longrightarrow \mathcal{L}$ is strict.
Suppose $l \in M^u$ and $N(l) \equiv l_0 + l_1 \bmod u^p L$, where $l_0 \in M^m$ and $l_1 \in L^u$. Then $uN(l) \equiv ul_0 + ul_1 \in (M^m \oplus M^u) \bmod u^p L$ and $N(\varphi(l)) = \varphi(uN(l)) \equiv \varphi(ul_1) \bmod u^p L$ implies that $N(L^u) \subset L^u \bmod u^p L$. Then from Proposition 1.2 it follows that $\mathcal{L}^u$ is a subobject of $\mathcal{L}$ in the category $\underline{\mathcal{L}}^*$. Clearly, the quotient $\mathcal{L}/\mathcal{L}^u := \mathcal{L}^m$ is multiplicative.
The maximality of $\mathcal{L}^u$ and $\mathcal{L}^m$ are formally implied by the following easy property of objects $\mathcal{L}_1, \mathcal{L}_2 \in \underline{\mathcal{L}}^*$:
if $\mathcal{L}_1$ *is unipotent and* $\mathcal{L}_2$ *is multiplicative then* $\mathrm{Hom}_{\underline{\mathcal{L}}^*}(\mathcal{L}_1, \mathcal{L}_2) = 0$. $\square$

Using the above results we can introduce the subcategories $\underline{\mathcal{L}}^{*et}$, $\underline{\mathcal{L}}^{*c}$, $\underline{\mathcal{L}}^{*m}$, $\underline{\mathcal{L}}^{*u}$ in $\underline{\mathcal{L}}^*$. They consist of, resp., etale, connected, multiplicative and unipotent objects of the cattegory $\underline{\mathcal{L}}^*$. The correspondences $\mathcal{L} \mapsto \mathcal{L}^{et}$, $\mathcal{L} \mapsto \mathcal{L}^c$, $\mathcal{L} \mapsto \mathcal{L}^m$, $\mathcal{L} \mapsto \mathcal{L}^u$ determine the natural exact functors from $\underline{\mathcal{L}}^*$ to, resp., $\underline{\mathcal{L}}^{*et}$, $\underline{\mathcal{L}}^{*c}$, $\underline{\mathcal{L}}^{*m}$ and $\underline{\mathcal{L}}^{*u}$.

1.3. The category $\underline{\mathcal{L}}^*_{cr}$.

Proposition 1.11. *Suppose* $\mathcal{L} = (L, F(L), \varphi, N) \in \underline{\mathcal{L}}^*$. *Then the following conditions are equivalent:*
(a) $N(F(L)) \subset F(L) \bmod u^{2p}L$;
(b) $N(\varphi(F(L))) \subset u^pL \bmod u^{2p}L$.

Proof. $(a) \Rightarrow (b)$: if for any $l \in F(L)$, $N(l) \in F(L) \bmod u^{2p}L$ then $N(\varphi(l)) = \varphi(uN(l)) = u^p\varphi(N(l)) \in u^pL \bmod u^{2p}L$.
$(b) \Rightarrow (a)$: for any $l \in F(L)$, $\varphi(uN(l)) = N(\varphi(l)) \in u^pL \bmod u^{2p}L$; now use that φ induces embedding of $F(L)/uF(L)$ into L/u^pL to deduce that $uN(l) \in uF(L) \bmod u^{2p}L$, i.e. $N(l) \in F(L) \bmod u^{2p}L$ (use that $u^{p-1}L \subset F(L)$). □

Definition. The category $\underline{\mathcal{L}}^*_{cr}$ is a full subcategory of $\underline{\mathcal{L}}^*$ consisting of $(L, F(L), \varphi, N)$ such that $N : L \longrightarrow L$ satisfies the equivalent conditions from Proposition 1.11.

Remark. a) If $\mathcal{L} = (L, F(L), \varphi, N) \in \underline{\mathcal{L}}^*_{cr}$ then $N_1 = N \bmod u^p$ is a unique $\mathcal{W}_1$-differentiation $N_1 : L \longrightarrow L/u^p$ whose restriction to $\varphi(F(L))$ is the zero map. Therefore, any $\mathcal{L} \in \underline{\mathcal{L}}^*_0$ has at most one structure of object of the category $\underline{\mathcal{L}}^*$.
b) Any etale or multiplicative object from $\underline{\mathcal{L}}^*$ belongs to $\underline{\mathcal{L}}^*_{cr}$.
c) If f is a morphism in $\underline{\mathcal{L}}^*_{cr}$ then $\mathrm{Ker}_{\underline{\mathcal{L}}^*} f = \mathrm{Ker}_{\underline{\mathcal{L}}^*_{cr}} f$ and $\mathrm{Coker}_{\underline{\mathcal{L}}^*} f = \mathrm{Coker}_{\underline{\mathcal{L}}^*_{cr}} f$. In particular, we can introduce the full subcategories $\underline{\mathcal{L}}^{*et}_{cr}$, $\underline{\mathcal{L}}^{*c}_{cr}$, $\underline{\mathcal{L}}^{*m}_{cr}$, $\underline{\mathcal{L}}^{*u}_{cr}$ of, resp., etale, connected, multiplicative and unipotent objects of $\underline{\mathcal{L}}^*_{cr}$.

Proposition 1.12. *Suppose* $\mathcal{L} = (L, F(L), \varphi, N) \in \underline{\mathcal{L}}^*_{cr}$. *Then there is a* $\sigma(\mathcal{W}_1)$*-basis* $l_1, \dots, l_s$ *of* $\varphi(F(L))$ *and integers* $0 \leqslant c_i < p$, *where* $1 \leqslant i \leqslant s$, *such that* $u^{c_1}l_1, \dots, u^{c_s}l_s$ *is a* $\mathcal{W}_1$*-basis of* $F(L)$.

Proof. Choose a $\mathcal{W}_1$-basis $m_1, \dots, m_s$ of L such that for suitable integers $c_1, \dots, c_s$, the elements $u^{c_1}m_1, \dots, u^{c_s}m_s$ form a $\mathcal{W}_1$-basis of $F(L)$. Clearly all $0 \leqslant c_i < p$.
For $1 \leqslant i \leqslant s$ and $j \geqslant 0$, let $l_{ij} \in \varphi(F(L))$ be such that $m_i = \sum_{j \geqslant 0} u^j l_{ij}$. Note that $\{l_{i0} \mid 1 \leqslant i \leqslant s\}$ is a $\sigma(\mathcal{W}_1)$-basis of $\varphi(F(L))$ and it will be sufficient to prove that all $u^{c_i}l_{i0} \in F(L)$ because then the elements $l_i := l_{i0}$ will satisfy the requirements of our proposition.
For all $1 \leqslant i \leqslant s$, the element

$$N(u^{c_i}m_i) = -\sum_j (j + c_i)u^{j+c_i}(l_{ij} \bmod u^{2p}L) + \sum_j u^{j+c_i}N(l_{ij})$$

belongs to $F(L) \bmod u^{2p}L$ if and only if $\sum_j (j + c_i)u^{j+c_i}l_{ij} \in F(L)$. (Use that $u^pL \subset uF(L)$.) This implies that for all integers $k \geqslant 0$, $\sum_j (j + c_i)^k u^{j+c_i}l_{ij} \in F(L)$. Therefore, for any $\alpha \in \mathbb{Z}/p\mathbb{Z}$,

$$\sum_{(j+c_i) \bmod p = \alpha} u^{j+c_i}l_{ij} \in F(L).$$

In particular, taking $\alpha = c_i \bmod p$ and using that $u^p l_{ij} \in F(L)$, we obtain that $u^{c_i} l_{i0} \in F(L)$. $\square$

Remark. a) Suppose $\mathcal{L} = (L, F(L), \varphi) \in \underline{\mathcal{L}}_0^*$ and satisfies the conclusion of Proposition 1.12. Define the $\mathcal{W}_1$-differentiation $N_1 : L \longrightarrow L/u^p L$ by setting $N_1(l_1) = \cdots = N(l_s) = 0$. If $N : L \longrightarrow L/u^{2p} L$ is the extension of N_1 given by Propostion 1.2 then $(L, F(L), \varphi, N) \in \underline{\mathcal{L}}_{cr}^*$. In other words, Proposition 1.12 characterizes the objects of $\underline{\mathcal{L}}_0^*$ coming from $\underline{\mathcal{L}}_{cr}^*$.
b) For an object $(L, F(L), \varphi, N) \in \underline{\mathcal{L}}_{cr}^*$, Proposition 1.12 implies that if $\sum_{0 \leqslant i < p} u^i l_i \in F(L)$, where all $l_i \in \varphi(F(L))$, then all $u^i l_i \in F(L)$.

Consider the category of filtered Fontaine-Laffaille modules $\underline{\mathrm{MF}}_{p-1}$ from [13]. The objects of this category are finite dimensional k-vector spaces M with decreasing filtration of length p by subspaces $M = M^0 \supset M^1 \supset \cdots \supset M^{p-1} \supset M^p = 0$ and σ-linear maps $\varphi_i : M^i \longrightarrow M$ such that $\mathrm{Ker}\,\varphi_i \supset M^{i+1}$, where $0 \leqslant i < p$, and $\sum_i \mathrm{Im}\,\varphi_i = M$. The morphisms in $\underline{\mathrm{MF}}_{p-1}$ are the morphisms of filtered vector spaces which commute with the corresponding morphisms φ_i, $0 \leqslant i < p$.
The category $\underline{\mathrm{MF}}_{p-1}$ is abelian. The object M of $\underline{\mathrm{MF}}_{p-1}$ is:
— etale (resp., multiplicative) if $M^1 = 0$ (resp., $M = M^{p-1}$);
— connected (resp., unipotent) if M has no etale (resp., multiplicative) subquotient.
Introduce the full subcategories $\underline{\mathrm{MF}}_{p-1}^{et}$, $\underline{\mathrm{MF}}_{p-1}^{m}$, $\underline{\mathrm{MF}}_{p-1}^{c}$ and $\underline{\mathrm{MF}}_{p-1}^{u}$ of, resp., etale, multiplicative, connected and unippotent objects in $\underline{\mathrm{MF}}_{p-1}$. These subcategories are closed under the operations of taking subobjects and quotient objects and, therefore, are also abelian. For any $M \in \underline{\mathrm{MF}}_{p-1}$, there are standard exact sequences $0 \longrightarrow M^{et} \longrightarrow M \longrightarrow M^c \longrightarrow 0$ and $0 \longrightarrow M^u \longrightarrow M \longrightarrow M^m \longrightarrow 0$, where M^{et} (resp., M^u) is the maximal etale (resp., unipotent) subobject and M^c (resp., M^m) is the maximal connected (resp., multiplicative) quotient object.
The categories $\underline{\mathcal{L}}_{cr}^*$ and $\underline{\mathrm{MF}}_{p-1}$ do not differ very much.
Indeed, introduce the functor $\mathrm{Md} : \underline{\widetilde{\mathcal{L}}}^* \longrightarrow \underline{\widetilde{\mathcal{L}}}^*$ induced on the level of filtered modules by $(L, F(L)) \mapsto (L/u^p L, F(L)/u^p L)$. Denote by $\mathrm{Md}(\underline{\mathcal{L}}_{cr}^*)$ the full subcategory of $\underline{\widetilde{\mathcal{L}}}^*$ consisting of the objects $\mathrm{Md}(\mathcal{L})$, where $\mathcal{L} \in \underline{\mathcal{L}}_{cr}^*$.
Define the functor $\mathcal{F} : \underline{\mathrm{MF}}_{p-1} \longrightarrow \underline{\widetilde{\mathcal{L}}}^*$ as follows. Let $M \in \underline{\mathrm{MF}}_{p-1}$ with the corresponding filtration M^i and σ-linear morphisms φ_i, $0 \leqslant i < p$. Then on the level of objects, $\mathcal{F}(M) = (L, F(L), \varphi, N)$, where $L = M \otimes_k \mathcal{W}_1/u^p \mathcal{W}_1$, $F(L) = \sum_{0 \leqslant i < p} u^{p-1-i} \mathcal{W}_1 (M^i \otimes 1)$ and for any $m \in M_i$, $\varphi(u^{p-1-i} m_i) = \varphi_i(m_i)$. One can easily see that $\mathcal{F}$ is equivalence of the categories $\underline{\mathrm{MF}}_{p-1}$ and $\mathrm{Md}(\underline{\mathcal{L}}_{cr}^*)$.
Now the difference between the categories $\underline{\mathcal{L}}_{cr}^*$ and $\underline{\mathrm{MF}}_{p-1}$ is described by the following Proposition.

Proposition 1.13. *For $\mathcal{L}_1, \mathcal{L}_2 \in \underline{\mathcal{L}}_{cr}^*$, Md induces a surjection from $\mathrm{Hom}_{\underline{\mathcal{L}}_{cr}^*}(\mathcal{L}_1, \mathcal{L}_2)$ to $\mathrm{Hom}_{\underline{\widetilde{\mathcal{L}}}^*}(\mathrm{Md}(\mathcal{L}_1), \mathrm{Md}(\mathcal{L}_2))$ and its kernel coincides with*

$(i_{\mathcal{L}_2*}\circ j^*_{\mathcal{L}_1})\mathrm{Hom}_{\underline{\mathcal{L}}^*}(\mathcal{L}_1^m,\mathcal{L}_2^{et})$, *where* $i_{\mathcal{L}_2}:\mathcal{L}_2^{et}\longrightarrow\mathcal{L}_2$ *(resp.,* $j_{\mathcal{L}_1}:\mathcal{L}_1\longrightarrow\mathcal{L}_1^m$*) is the maximal etale subobject in* $\mathcal{L}_2$ *(resp., multiplicative quotient object for* $\mathcal{L}_1$*).*

Proof. For $\mathcal{L}_2=(L_2,F(L_2),\varphi,N)$, let $\phi:L_2\longrightarrow L_2$ be such that $\phi(l)=\varphi(u^{p-1}l)$ for any $l\in L_2$. Let $L_2^c=\{l\in L_2\mid \phi(l)^n\underset{n\to\infty}{\longrightarrow}0\}$ and let $\mathcal{L}_2'\in\widetilde{\underline{\mathcal{L}}}^*$ be the filtered module $(L_2/u^pL^c, F(L_2)/u^pL_2^c)$ with φ and N induced from $\mathcal{L}_2$. Then there are natural strict epimorphisms

$$\mathcal{L}_2\xrightarrow{\alpha}\mathcal{L}_2'\xrightarrow{\beta}\mathrm{Md}(\mathcal{L}_2),$$

where $\mathrm{Ker}\,\alpha$ is associated with the filtered module $(u^pL_2^c,u^pL_2^c)$ and $\mathrm{Ker}\,\beta$ — with $(u^pL_2/u^pL_2^c,u^pL_2/u^pL_2^c)$.
Clearly, $\varphi|_{u^pL_2^c}$ is nilpotent and then by Lemma 1.5,

$$\alpha_*:\mathrm{Hom}_{\underline{\mathcal{L}}^*}(\mathcal{L}_1,\mathcal{L}_2)\longrightarrow\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}^*}(\mathcal{L}_1,\mathcal{L}_2')$$

is bijective. Note that the natural embedding $L_2^{et}\longrightarrow L_2$ induces the identification $u^pL_2/u^pL_2^c=u^pL_2^{et}/u^{p+1}L_2^{et}$. Let $\mathcal{L}_2''=(u^pL_2,u^pL_2)\in\underline{\mathcal{L}}^*$ with induced φ and N. Then $\mathcal{L}_2''$ is multiplicative and there is a natural projection $\gamma:\mathcal{L}_2''\longrightarrow\mathrm{Ker}\,\beta$ such that $\mathrm{Ker}\,\gamma$ is associated with $(u^{p+1}L_2^{et},u^{p+1}L_2^{et})$. Note that φ is nilpotent on $u^{p+1}L_2^{et}$. Applying Lemma 1.5 we obtain that

$$\beta_*:\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}^*}(\mathcal{L}_1,\mathcal{L}_2')\longrightarrow\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}^*}(\mathcal{L}_1,\mathrm{Md}(\mathcal{L}_2))$$

is surjective and

$$\mathrm{Ker}\,\beta_*=\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}^*}(\mathcal{L}_1,\mathrm{Ker}\beta)=\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}^*}(\mathcal{L}_1,\mathcal{L}'')\simeq\mathrm{Hom}(\mathcal{L}_1^m,\mathcal{L}_2'').$$

It remains to note that $\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}^*}(\mathcal{L}_1^m,\mathcal{L}_2'')=\mathrm{Hom}_{\underline{\mathcal{L}}^*}(\mathcal{L}_1^m,\mathcal{L}_2^{et})$ via the natural embedding of $\mathcal{L}''$ into $\mathcal{L}_2^{et}$. ∎

Corollary 1.14. *The functor* $\mathrm{Md}\circ\mathcal{F}^{-1}$ *induces equivalence of the categories* $\underline{\mathcal{L}}_{cr}^{*c}$ *(resp.,* $\underline{\mathcal{L}}_{cr}^{*u}$*) and* $\underline{\mathrm{MF}}_{p-1}^c$ *(resp.,* $\underline{\mathrm{MF}}_{p-1}^u$*).*

1.4. Simple objects in $\underline{\mathcal{L}}^*$.

Definition. An object $\mathcal{L}$ of $\underline{\mathcal{L}}^*$ is simple if any strict monomorphism $i:\mathcal{L}_1\longrightarrow\mathcal{L}$ in $\underline{\mathcal{L}}^*$ is either isomorphism or the zero morphism. Equivalently, $\mathcal{L}$ is simple iff any strict epimorphism $j:\mathcal{L}\longrightarrow\mathcal{L}_2$ is either isomorphism or the zero morphism.

All simple objects in $\underline{\mathcal{L}}^*$ can be described as follows.
Let $[0,1]_p=\{r\in\mathbb{Q}\mid 0\leqslant r\leqslant 1, v_p(r)=0\}$, where v_p is a p-adic valuation. Then any $r\in[0,1]_p$ can be uniquely written as $r=\sum_{i\geqslant 1}a_ip^{-i}$, where the digits $0\leqslant a_i=a_i(r)<p$ form a periodic sequence. The minimal positive period of this sequence will be denoted by $s(r)$.
Let $\tilde r=1-r$. Then $\tilde r\in[0,1]_p$ and $\tilde r=\sum_{i\geqslant 1}\tilde a_ip^{-i}$, where for all $i\geqslant 1$, the digits $\tilde a_i=a_i(\tilde r)$ are such that $a_i+\tilde a_i=p-1$.

Definition. For $r\in[0,1]_p$, let $\mathcal{L}(r)=(L(r),F(L(r)),\varphi,N)$ be the following object of the category $\underline{\mathcal{L}}^*_{cr}$:
- $L(r)=\oplus_{i\in\mathbb{Z}/s(r)}\mathcal{W}_1 l_i$;
- $F(L(r))=\sum_{i\in\mathbb{Z}/s(r)}\mathcal{W}_1 u^{\tilde a_i}l_i$;
- for $i\in\mathbb{Z}/s(r)$, $\varphi(u^{\tilde a_i}l_i)=l_{i+1}$.
- N is uniquely recovered from the condition $N|_{\varphi(F(L))}=0\operatorname{mod}u^p$, cf. Proposition 1.2.

Remark. If $r=0$ or $r=1$ we obtain the objects $\mathcal{L}(0)$ and $\mathcal{L}(1)$ introduced in Subsection 1.2. Note also that $\mathcal{L}(r)$ is connected iff $r\neq 0$ and unipotent iff $r\neq 1$.

For $n\in\mathbb{N}$ and $r\in[0,1]_p$, set $r(n)=\sum_{i\geqslant 1}a_{i+n}(r)p^{-i}$. Extend this definition to any $n\in\mathbb{Z}$ by setting $r(n):=r(n+Ns(r))$ for a sufficiently large $N\in\mathbb{N}$.

Proposition 1.15. *a) If $r\in[0,1]_p$ then $\mathcal{L}(r)$ is simple;*
b) if $r_1,r_2\in[0,1]_p$ then $\mathcal{L}(r_1)\simeq\mathcal{L}(r_2)$ if and only if there is an $n\in\mathbb{Z}$ such that $r_1=r_2(n)$;
c) if $\mathcal{L}$ is a simple object of the category $\underline{\mathcal{L}}^*$ then there is an $r\in[0,1]_p$ such that $\mathcal{L}\simeq\mathcal{L}(r)$.

Proof. Lemma 1.16 below implies that the simple objects in the categories $\underline{\mathcal{L}}^*_{cr}$ and $\underline{\mathcal{L}}^*$ are the same. By Corollary 1.14, the functor $\mathrm{Md}\circ\mathcal{F}^{-1}$ transforms simple objects of $\underline{\mathcal{L}}^*$ to simple objects in $\underline{\mathrm{MF}}_{p-1}$. It remains to note that an analogue of our Proposition for the category $\underline{\mathrm{MF}}_{p-1}$ is proved in [13]. □

Lemma 1.16. *For any $\mathcal{L}\in\underline{\mathcal{L}}^*$, there is an $\mathcal{L}^{cr}\in\underline{\mathcal{L}}^*_{cr}$ and a strict monomorphism $\iota^{cr}\in\mathrm{Hom}_{\underline{\mathcal{L}}^*}(\mathcal{L}^{cr},\mathcal{L})$ such that if $\iota'\in\mathrm{Hom}_{\underline{\mathcal{L}}^*}(\mathcal{L}',\mathcal{L})$ is a strict monomorphism and $\mathcal{L}'\in\underline{\mathcal{L}}^*_{cr}$ then there is a strict monomorphism $\alpha:\mathcal{L}'\longrightarrow\mathcal{L}^{cr}$ such that $\iota'=\iota^{cr}\circ\alpha$.*

Proof of Lemma. Suppose $\mathcal{L}=(L,F(L),\varphi,N)$. Consider the k-linear space $M:=\varphi(F(L))/u^p\varphi(F(L))$. Let $\widetilde L=M\otimes_k(\mathcal{W}_1/u^p\mathcal{W}_1)=L/u^pL$, $\widetilde F=F(L)/u^pL$ and $\tilde\varphi:\widetilde F\longrightarrow M$ be the map induced by φ.
Proceed by induction to define for all $i\geqslant 1$, the subspaces $M_i\subset M$ and the $\mathcal{W}_1$-submodules $\widetilde F_i\subset\widetilde L$ as follows.
From the definition of $N:L\longrightarrow L/u^{2p}L$ it follows easily that N induces a k-linear map $\widetilde N_1:M\longrightarrow M$ and $\widetilde N_1^p=0$. Therefore, $M_1:=\operatorname{Ker}\widetilde N_1$ is a non-trivial subspace in M.
Suppose $i\geqslant 1$ and M_i has been already defined. Let $\widetilde F_i$ be the submodule of the elements of the form u^al in $M\otimes_k(\mathcal{W}_1/u^p\mathcal{W}_1)$, where $a\geqslant 0$, $l\in M_i$ and $u^al\in\widetilde F$. Then set $M_{i+1}=\tilde\varphi(\widetilde F_i)$.
Verify that for all indices i, $M_{i+1}\subset M_i$. If $i=1$ we must prove that $\widetilde N_1(M_2)=0$. Indeed, M_2 is spanned by $\varphi(u^al)$, where $l\in M_1$ and $u^al\in\widetilde F_1$. But $N(\varphi(u^al))=\varphi(uN(u^al))=\varphi(-u^{a+1}l+u^{a+1}N(l))\in u^pL$. If $i>1$ then we can assume by induction that $M_{i-1}\subset M_i$. This implies that $\widetilde F_{i-1}\subset\widetilde F_i$ and $M_i\subset M_{i+1}$.

We obtained a decreasing sequence of non-trivial finite dimensional k-linear spaces $\{M_i\mid i\geqslant 1\}$. For $i\gg 1$, these spaces become a non-trivial constant space $M^{cr}\subset M$ such that if $\widetilde{F}^{cr}=\{u^al\in\widetilde{F}\mid a\geqslant 0, l\in M^{cr}\}$ then $\tilde{\varphi}(\widetilde{F}^{cr})=M^{cr}$. This subspace M^{cr} has the maximality property: if $M'\subset M$ is such that for $\widetilde{F}'=\{u^al\in\widetilde{F}\mid a\geqslant 0, l\in M'\}$, $\tilde{\varphi}(\widetilde{F}')=M'$ then $M'\subset M^{cr}$. Indeed, show as earlier that $M'\subset M_1$ and then proceed by induction proving that $M'\subset M_i$ for all $i\geqslant 1$.

Now in notation from Subsection 1.3, there is an $\mathcal{L}^{cr}\in\underline{\mathcal{L}}^*_{cr}$ such that $\mathrm{Md}(\mathcal{L}^{cr})=(M^{cr}\otimes_k(\mathcal{W}_1/u^p\mathcal{W}_1),\widetilde{F}^{cr},\tilde{\varphi},\widetilde{N})$, where $\widetilde{N}|_{M^{cr}}=0$. Then from Proposition 1.13 it follows the existence of a strict monomorphism $\iota^{cr}:\mathcal{L}^{cr}\longrightarrow\mathcal{L}$. If $\iota':\mathcal{L}'=(L',F(L'),\varphi,N)\longrightarrow\mathcal{L}$ is strict monomorphism and $\mathcal{L}'\in\underline{\mathcal{L}}^*_{cr}$ then $\mathrm{Md}(\mathcal{L}')$ is associated with the filtered module $(M'\otimes_k(\mathcal{W}_1/u^p\mathcal{W}_1),\widetilde{F}')$ and by the above maximality property of M^{cr}, M' is a subspace in M^{cr} and $\mathrm{Md}(\mathcal{L}')$ is a strict subobject of $\mathrm{Md}(\mathcal{L}^{cr})$. This gives the required strict embedding α. The Lemma is proved. □

1.5. Extensions in $\underline{\mathcal{L}}^*$. Suppose $r_1,r_2\in[0,1]_p$. Choose an $s\in\mathbb{N}$ which is divisible by $s(r_1)$ and $s(r_2)$ and introduce the objects $\mathcal{L}_1=(L_1,F(L_1),\varphi,N)$ and $\mathcal{L}_2=(L_2,F(L_2),\varphi,N)$ of the category $\underline{\mathcal{L}}^*_{cr}$ as follows:

$L_1=\oplus_{i\in\mathbb{Z}/s}\mathcal{W}_1l_i^{(1)}$, $F(L_1)=\sum_{i\in\mathbb{Z}/s}\mathcal{W}_1u^{\tilde{a}_i}l_i^{(1)}$, where $r_1=\sum_{i\geqslant 1}a_ip^{-i}$ with the digits $0\leqslant a_i<p$, $\tilde{a}_i=(p-1)-a_i$ and for all $i\in\mathbb{Z}/s$, $\varphi(u^{\tilde{a}_i}l_i^{(1)})=l_{i+1}^{(1)}$;

$L_2=\oplus_{j\in\mathbb{Z}/s}\mathcal{W}_1l_j^{(2)}$, $F(L_2)=\sum_{j\in\mathbb{Z}/s}\mathcal{W}_1u^{\tilde{b}_j}l_j^{(2)}$, where $r_2=\sum_{j\geqslant 1}b_jp^{-j}$ with the digits $0\leqslant b_j<p$, $\tilde{b}_j=(p-1)-b_j$, and for all $j\in\mathbb{Z}/s$, $\varphi(u^{\tilde{b}_j}l_j^{(2)})=l_{j+1}^{(2)}$.

Lemma 1.17. *For $\kappa=1,2$, $\mathcal{L}_\kappa$ is isomorphic to the product of $s/s(r_\kappa)$ copies of the (simple) object $\mathcal{L}(r_\kappa)$.*

Proof. Take $\kappa=1$. For $\gamma\in\mathbb{F}_{p^s}$ and $\bar{\imath}\in\mathbb{Z}/s(r_1)$, let $m_{\bar{\imath}}(\gamma)=\sum_{i\bmod s(r_1)=\bar{\imath}}\sigma^i(\gamma)l_i^{(1)}$ and $M(\gamma)=\sum_{\bar{\imath}\in\mathbb{Z}/s(r_1)}\mathcal{W}_1m_{\bar{\imath}}(\gamma)\subset L_1$. Then all $\mathcal{M}(\gamma)=(M(\gamma),M(\gamma)\cap F(L_1),\varphi,N)$ with induced φ and N are subobjects of $\mathcal{L}_1$ isomorphic to $\mathcal{L}(r_1)$. If $\gamma_1,\dots,\gamma_d$ is an $\mathbb{F}_{p^{s(r_1)}}$-basis of $\mathbb{F}_{p^s}$ then $\mathcal{M}(\gamma_1)\times\dots\times\mathcal{M}(\gamma_d)$ is isomorphic to $\mathcal{L}_1$. (Use that $d=s/s(r_1)$ and $\det(\sigma^i(\gamma_j))\neq 0$, where for a given $\bar{\imath}$, i is such that $i\bmod s(r_1)=\bar{\imath}$ and $1\leqslant j\leqslant d$.) □

If $\mathcal{L}=(L,F(L),\varphi,N)\in\underline{\mathcal{L}}^*$ then we shall use the same notation $\mathcal{L}$ for the image $(L,F(L),\varphi)$ of $\mathcal{L}$ under the forgetful functor from $\underline{\mathcal{L}}^*$ to $\underline{\mathcal{L}}^*_0$. Clearly, this forgetful functor induces a group homomorphism $\mathrm{Ext}_{\underline{\mathcal{L}}^*}(\mathcal{L}_2,\mathcal{L}_1)\longrightarrow\mathrm{Ext}_{\underline{\mathcal{L}}^*_0}(\mathcal{L}_2,\mathcal{L}_1)$. Suppose $\mathcal{L}=(L,F(L),\varphi)\in\mathrm{Ext}_{\underline{\mathcal{L}}^*_0}(\mathcal{L}_2,\mathcal{L}_1)$. Consider a $\sigma(\mathcal{W}_1)$-linear section $S:l_j^{(2)}\mapsto l_j$, $j\in\mathbb{Z}/s$, of the corresponding epimorphic map $\varphi(F(L))\longrightarrow\varphi(F(L_2))$. Then

a) $L=L_1\oplus\left(\oplus_{j\in\mathbb{Z}/s}\mathcal{W}_1l_j\right)$;

b) for all indices $j\in\mathbb{Z}/s$, there are unique elements $v_j\in L_1$, such that $F(L)=F(L_1)+\sum_{j\in\mathbb{Z}/s}\mathcal{W}_1(u^{\tilde{b}_j}l_j+v_j)$ and $\varphi(u^{\tilde{b}_j}l_j+v_j)=l_{j+1}$;

c) $F(L)\supset u^{p-1}L$ if and only if for all $j\in\mathbb{Z}/s$, $u^{b_j}v_j\in F(L_1)$;

d) if $S':l_j^{(2)}\mapsto l'_j=l_j+\varphi(w_{j-1})$, where $j\in\mathbb{Z}/s$ and $w_{j-1}\in F(L_1)$, is another section of the epimorphism $\varphi(F(L))\longrightarrow\varphi(F(L_2))$ then for the corresponding elements $v'_j\in L_1$, $v'_j-v_j=w_j-u^{\tilde b_j}\varphi(w_{j-1})$.

The constructions from above items a)-d) can be summarized as follows.

Lemma 1.18. *Let $Z(\mathcal{L}_2,\mathcal{L}_1)=\{(v_j)_{j\in\mathbb{Z}/s}\in L_1^s\mid u^{b_j}v_j\in F(L_1)\}$ be a subgroup in L_1^s and let*

$$B(\mathcal{L}_2,\mathcal{L}_1)=\{(w_j-u^{\tilde b_j}\varphi(w_{j-1}))_{j\in\mathbb{Z}/s}\mid w_j\in F(L_1)\}$$

be a subgroup of $Z(\mathcal{L}_2,\mathcal{L}_1)$. Then there is a natural isomorphism of abelian groups $Z(\mathcal{L}_2,\mathcal{L}_1)/B(\mathcal{L}_2,\mathcal{L}_1)\simeq\mathrm{Ext}_{\underline{\mathcal{L}}_0^*}(\mathcal{L}_2,\mathcal{L}_1)$.

Proposition 1.19. *Any $\mathcal{L}\in\mathrm{Ext}_{\underline{\mathcal{L}}_0^*}(\mathcal{L}_2,\mathcal{L}_1)$ appears from a system of factors $(v_j)_{j\in\mathbb{Z}/s}\in Z(\mathcal{L}_2,\mathcal{L}_1)$ satisfying the following normalization condition*
(C1) *if $v_j=\sum_{i,t}\gamma_{ijt}u^tl_i^{(1)}$ with $\gamma_{ijt}\in k$, then $\gamma_{ij\tilde b_j}=0$.*

Proof. Choose a section S of the projection $\varphi(F(L))\longrightarrow\varphi(F(L_2))$ with the minimal set $\gamma(S)=\{(i,j,\tilde b_j)\mid\gamma_{ij\tilde b_j}\neq0\}$. Suppose $\gamma(S)\neq\emptyset$ (otherwise, the proposition is proved) and let $(v_j)_{j\in\mathbb{Z}/s}$ be the corresponding system of factors. Suppose $(i_0,j_0,\tilde b_{j_0})\in\gamma(S)$ and $\gamma=\gamma_{i_0j_0\tilde b_{j_0}}$. Replace $(v_j)_{j\in\mathbb{Z}/s}$ by an equivalent system $(v'_j)_{j\in\mathbb{Z}/s}$ via the elements $w_j\in F(L_1)$ such that $w_j=0$ if $j\neq j_0-1$ and $w_{j_0-1}=\sigma^{-1}(\gamma)u^{\tilde a_{i_0-1}}l_{i_0-1}^{(1)}$.
If $v'_j=\sum_{i,t}\gamma'_{ijt}u^tl_i^{(1)}$ then
— $\gamma'_{i_0j_0\tilde b_{j_0}}=0$;
— $\gamma'_{i_0-1,j_0-1,\tilde a_{i_0-1}}=\sigma^{-1}(\gamma)+\gamma_{i_0-1,j_0-1,\tilde a_{i_0-1}}$;
— for all remaining indices $\gamma'_{ijt}=\gamma_{ijt}$.

Then $\gamma(S')\subset\gamma(S)\setminus\{(i_0,j_0,\tilde b_{j_0})\}\cup\{(i_0-1,j_0-1,\tilde a_{i_0-1})\}$ and the minimality condition for S implies $(i_0-1,j_0-1,\tilde a_{i_0-1})\in\gamma(S')\setminus\gamma(S)$. Therefore, $\tilde a_{i_0-1}=\tilde b_{j_0-1}$, $\gamma_{i_0-1,j_0-1,\tilde a_{i_0-1}}=0$, $\gamma'_{i_0-1,j_0-1,\tilde b_{j_0-1}}=\sigma^{-1}(\gamma)$, and the new section S' again satisfies the minimality condition.
Repeating the above procedure we obtain for all $n\in\mathbb{Z}/s$, that $\tilde a_{i_0-n}=\tilde b_{j_0-n}$, that is $\tilde r_1(i_0)=\tilde r_2(j_0)$.
Choose $\beta\in k$ such that $\sigma^s(\beta)-\beta=\gamma$ and consider $w_j\in F(L_1)$ such that for all $0\leqslant n<s$, $w_{j_0+n}=\sigma^n(\beta)u^{\tilde b_{j_0+n}}l_{i_0+n}^{(1)}$. Then for the corresponding new system of factors $(v'_j)_{j\in\mathbb{Z}/s}$, where

$$v'_j=v_j+w_j-u^{\tilde b_j}\varphi(w_{j-1})=\sum_{i,t}\gamma'_{ijt}u^tl_i^{(1)},$$

one has $\gamma'_{i_0,j_0,\tilde b_{j_0}}=0$, and $\gamma_{ijt}=\gamma'_{ijt}$ if $(i,j,t)\neq(i_0,j_0,\tilde b_{j_0})$. This contradicts to the minimality condition for S. □

Proposition 1.20. *Any $\mathcal{L} \in \mathrm{Ext}_{\underline{\mathcal{L}}^*}(\mathcal{L}_2, \mathcal{L}_1)$ can be described via a system of factors $(v_j)_{j\in\mathbb{Z}/s}$, satisfying the above condition* (**C1**) *and the normalization condition*

(C2) *the coefficients $\gamma_{ijt} = 0$ if $t > \tilde{a}_i$.*

Proof. Suppose $v^{(0)} = (v_j)_{j\in\mathbb{Z}/s}$ is such that $v_{j_0} = \gamma u^{t_0} l_{i_0}^{(1)}$ with $\gamma \in k$, $t_0 > \tilde{a}_{i_0}$ and for $j \neq j_0$, $v_j = 0$. It will be sufficient to prove that any such system of factors is trivial.
Take the elements $w_j^{(0)}$, $j \in \mathbb{Z}/s$, such that $w_{j_0}^{(0)} = -\gamma u^{t_0} l_{i_0}^{(1)}$ and $w_j^{(0)} = 0$ if $j \neq j_0$. Then the corresponding equivalent system $(v_j^{(1)})_{j\in\mathbb{Z}/s}$ is such that $v_j^{(1)} = 0$ if $j \neq j_0 + 1$, and $v_{j_0+1}^{(1)} = \gamma^p u^{t_1} l_{i_0+1}^{(1)}$, where $t_1 = \tilde{b}_{j_0+1} + (t_0 - \tilde{a}_{i_0})p$. This implies that $t_1 \geqslant p > \tilde{a}_{i_0+1}$, $t_1 - \tilde{a}_{i_0+1} \geqslant t_0 - \tilde{a}_{i_0}$, and $t_1 - \tilde{a}_{i_0+1} > t_0 - \tilde{a}_{i_0}$ unless $\tilde{b}_{j_0+1} = 0$, $t_1 = p$ and $\tilde{a}_{i_0+1} = p - 1$.
Repeat this procedure by using for all $n \geqslant 0$, the appropriate elements $w_j^{(n)}$, $j \in \mathbb{Z}/s$, to obtain the equivalent systems of factors $(v_j^{(n)})_{j\in\mathbb{Z}/s}$ such that $v_j^{(n)} = 0$ if $j \neq j_0 + n$, and $v_{j_0+n}^{(n)} = \gamma^{p^n} u^{t_n} l_{i_0+n}^{(1)}$.
If $(\tilde{r}_2, \tilde{r}_1, t_0) \neq (0, 1, p)$ then $t_n \to \infty$ and we can use the elements $w_j = \sum_{n\geqslant 0} w_j^{(n)}$, $j \in \mathbb{Z}/s$, to trivialize the original system of factors $v^{(0)}$.
If $(\tilde{r}_2, \tilde{r}_1, t_0) = (0, 1, p)$, we can trivialize $v^{(0)}$ via the elements w_j, $j \in \mathbb{Z}/s$, where for $0 \leqslant n < s$, $w_{j_0+n} = \kappa^{p^n} u^p l_{i_0+n}^{(1)}$ and $\kappa \in k$ is such that $\sigma^s(\kappa) - \kappa = \gamma$. ∎

Proposition 1.21. *Suppose $\mathcal{L} = (L, F(L), \varphi) \in \mathrm{Ext}_{\underline{\mathcal{L}}_0^*}(\mathcal{L}_2, \mathcal{L}_1)$ is given via a system of factors $(v_j)_{j\in\mathbb{Z}/s}$ satisfying the normalization condition* (**C1**). *Then $\mathcal{L}$ comes from $\underline{\mathcal{L}}_{cr}^*$ if and only if all $v_j \in F(L_1)$.*

Proof. Let $N_1 : L \longrightarrow L/u^p L$ be a $\mathcal{W}_1$-differentiation such that for all $j \in \mathbb{Z}/s$, $N_1(l_j) = 0$ (and, of course, $N_1(l_j^{(1)}) = 0$). If all $v_j \in F(L_1)$, $F(L)$ is generated by the elements $u^{\tilde{b}_j} l_j$ and $u^{\tilde{a}_j} l_j^{(1)}$, $j \in \mathbb{Z}/s$. If m is any of these elements then the basic identity $N_1(\varphi(m)) = \varphi(uN_1(m))$ is, clearly, satisfied. By Proposition 1.2, N_1 can be extended to a unique $\mathcal{W}_1$-differentiation $N : L \longrightarrow L/u^{2p}$ and $\mathcal{L} = (L, F(L), \varphi, N) \in \underline{\mathcal{L}}_{cr}^*$.
Suppose now that $\mathcal{L} = (L, F(L), \varphi, N) \in \underline{\mathcal{L}}_{cr}^*$ and for all $j \in \mathbb{Z}/s$, $v_j = \sum_{i,t} \gamma_{ijt} u^t l_i^{(1)}$ with $\gamma_{ij\tilde{b}_j} = 0$. Consider the following congruence (use that $-u^{\tilde{b}_j} l_j \equiv v_j \bmod F(L)$)

$$N(u^{\tilde{b}_j} l_j + v_j) \equiv \sum_{i,t} \gamma_{ijt} (\tilde{b}_j - t) u^t l_i^{(1)} + u^{\tilde{b}_j} N(l_j) \mathrm{mod} F(L). \tag{1.5}$$

The condition $\mathcal{L} \in \underline{\mathcal{L}}_{cr}^*$ implies that $N(u^{\tilde{b}_j} l_j + v_j) \in F(L) \bmod u^{2p} L$ and $N(l_j) \in u^p L \bmod u^{2p} L \subset F(L) \bmod u^{2p} L$. This means that all $(\tilde{b}_j - t)\gamma_{ijt} u^t l_i^{(1)} \in F(L_1)$. Therefore, for $t \neq \tilde{b}_j$, $\gamma_{ijt} u^t l_i^{(1)} \in F(L_1)$, and $v_j \in F(L_1)$. The proposition is proved. ∎

Definition. A pair $(i_0, j_0) \in (\mathbb{Z}/s)^2$ is $(r_1, r_2)_{cr}$-admissible if $\tilde{a}_{i_0} \neq \tilde{b}_{j_0}$ and there is an $m_0 = m_{cr}(i_0, j_0) \in \mathbb{N}$ such that for $1 \leqslant m < m_0$, $\tilde{a}_{i_0+m} = \tilde{b}_{j_0+m}$ but $\tilde{a}_{i_0+m_0} > \tilde{b}_{j_0+m_0}$.

Remark. For any $(r_1, r_2)_{cr}$-admissible pair of indices (i_0, j_0), one has $\tilde{r}_1(i_0) > \tilde{r}_2(j_0)$ (or, equivalently, $r_1(i_0) < r_2(j_0)$).

Definition. For $(i_0, j_0) \in (\mathbb{Z}/s)^2$ and $\gamma \in k$, denote by $E_{cr}(i_0, j_0, \gamma)$ the extension $\mathcal{L} \in \mathrm{Ext}_{\underline{\mathcal{L}}^*_{cr}}(\mathcal{L}_2, \mathcal{L}_1)$ given by the system of factors $(v_j)_{j \in \mathbb{Z}/s}$ such that $v_{j_0} = \gamma u^{\tilde{a}_{i_0}} l_{i_0}^{(1)}$ and $v_j = 0$ if $j \neq j_0$.

Proposition 1.22. *Any element $\mathcal{L} \in \mathrm{Ext}_{\underline{\mathcal{L}}^*_{cr}}(\mathcal{L}_2, \mathcal{L}_1)$ can be obtained as a sum of $E_{cr}(i, j, \gamma_{ij})$, where $(i, j) \in (\mathbb{Z}/s)^2$ runs over the set of $(r_1, r_2)_{cr}$-admissible pairs and all coefficients $\gamma_{ij} \in k$.*

Proof. Propositions 1.19-1.21 imply that any $\mathcal{L} = (L, F(L), \varphi, N)$ from the group $\mathrm{Ext}_{\underline{\mathcal{L}}^*_{cr}}(\mathcal{L}_2, \mathcal{L}_1)$ can be presented as a sum of extensions $E_{cr}(i, j, \gamma_{ij})$, where $i, j \in \mathbb{Z}/s$ are such that $\tilde{a}_i \neq \tilde{b}_j$, and $\gamma_{ij} \in k$.
If $m_0 \in \mathbb{N}$ is such that for $1 \leqslant m < m_0$, one has $\tilde{a}_{i+m} = \tilde{b}_{j+m}$ but $\tilde{a}_{i+m_0} < \tilde{b}_{j+m_0}$, then the extension $E_{cr}(i, j, \gamma_{ij})$ is trivial, cf. the proof of Proposition 1.20. The proposition is proved. $\square$

The above proposition describes the subgroup $\mathrm{Ext}_{\underline{\mathcal{L}}^*_{cr}}(\mathcal{L}_2, \mathcal{L}_1)$ of $\mathrm{Ext}_{\underline{\mathcal{L}}^*}(\mathcal{L}_2, \mathcal{L}_1)$. In particular, working modulo this subgroup we can describe the extensions in the whole category $\underline{\mathcal{L}}^*$ via the systems of factors $(v_j)_{j \in \mathbb{Z}/s} \in Z(\mathcal{L}_2, \mathcal{L}_1)$ such that all $v_j = \sum_{i,t} \gamma_{ijt} u^t l_i^{(1)}$ satisfy the normalization conditions (C1) and

(C3) *if $t \geqslant \tilde{a}_i$ then $\gamma_{ijt} = 0$.*

Proposition 1.23. *Suppose the system of factors $(v_j)_{j \in \mathbb{Z}/s}$ satisfies the conditions* (**C1**) *and* (**C3**)*. If it determines $\mathcal{L} = (L, F(L), \varphi) \in \mathrm{Ext}_{\underline{\mathcal{L}}^*_0}(\mathcal{L}_2, \mathcal{L}_1)$ from the image of $\mathrm{Ext}_{\underline{\mathcal{L}}^*}(\mathcal{L}_2, \mathcal{L}_1)$ then:*

a) *$\gamma_{ijt} = 0$ if $t < \tilde{a}_i - 1$;*

b) *if $t = \tilde{a}_i - 1$ and there is an $m_0 \in \mathbb{N}$ such that for all $1 \leqslant m < m_0$, $\tilde{a}_{i+m} - 1 = \tilde{b}_{j+m}$ but $\tilde{a}_{i+m_0} - 1 > \tilde{b}_{j+m_0}$, then $\gamma_{ijt} = 0$;*

c) *if $t = \tilde{a}_i - 1$ and for all $m \in \mathbb{Z}/s$, $\tilde{a}_{i+m} - 1 = \tilde{b}_{j+m}$ then $\gamma_{ijt} = 0$.*

Proof. Suppose $\mathcal{L} = (L, F(L), \varphi, N) \in \mathrm{Ext}_{\underline{\mathcal{L}}^*}(\mathcal{L}_2, \mathcal{L}_1)$ and $(v_j)_{j \in \mathbb{Z}/s}$ describes the image of $\mathcal{L}$ in $\mathrm{Ext}_{\underline{\mathcal{L}}^*_0}(\mathcal{L}_2, \mathcal{L}_1)$. By the definition of N, $uN(u^{\tilde{b}_j} l_j + v_j) \in F(L)$, and this implies that $\gamma_{ijt} = 0$ if $t < \tilde{a}_i - 1$, $t \neq \tilde{b}_j$ (use congruence (1.5)). This proves a).
Now we can set for all indices i and j, $\gamma_{ij} := \gamma_{i,j,\tilde{a}_i - 1}$.
Let $\kappa_{ij} \in k$ be such that $N(l_j) \equiv \sum_i \kappa_{ij} l_i^{(1)} \mathrm{mod} u^p L$ and suppose $\gamma_{ij} \neq 0$ (this implies that $\tilde{b}_j \neq \tilde{a}_i - 1$). For $m \geqslant 0$, consider the relations

$$N(l_{j+m+1}) = \varphi(uN(u^{\tilde{b}_{j+m}} l_{j+m} + v_{j+m})). \tag{1.6}$$

If $m=0$ then (1.6) implies $\kappa_{i+1,j+1}=\gamma_{ij}^p(\tilde b_j-\tilde a_i+1)$. Suppose that there is an $m_0\geqslant 0$ such that for all $1\leqslant m<m_0$, $\tilde a_{i+m}-1=\tilde b_{j+m}$ but $\tilde a_{i+m_0}-1\neq\tilde b_{j+m_0}$. Then (1.6) together with (1.5) (where j is replaced by $j+m$) imply that for $1\leqslant m<m_0$,

$$\kappa_{i+m+1,j+m+1}=\kappa_{i+m,j+m}^p=\gamma_{ij}^{p^{m+1}}(\tilde b_j-\tilde a_i+1).$$

In particular, $N(l_{j+m_0})\operatorname{mod} u^pL$ contains $l_{i+m_0}^{(1)}$ with the coefficient $\gamma_{ij}^{p^{m_0}}(\tilde b_j-\tilde a_i+1)$. Therefore, $uN(u^{\tilde b_{j+m_0}}l_{j+m_0}+v_{j+m_0})\operatorname{mod} u^pL$ contains $l_{j+m_0}^{(1)}$ with the coefficient $u^{\tilde b_{j+m_0}+1}\gamma_{ij}^{p^{m_0}}(\tilde b_j-\tilde a_i+1)$. But this monomial must belong to $F(L_1)$. This proves that if $\gamma_{ij}\neq 0$ then $\tilde b_{j+m_0}+1>\tilde a_{i+m_0}$.
Finally, suppose that for all $m\geqslant 1$, $\tilde a_{i+m}-1=\tilde b_{j+m}$. Then $\tilde a_i-1=\tilde a_{i+s}-1=\tilde b_{j+s}=\tilde b_j$ and $\gamma_{ij}=\gamma_{i,j,\tilde a_i-1}=\gamma_{i,j,\tilde b_j}=0$. □

Remark. With notation from the proof of above proposition the elements $v_j=\sum_i\gamma_{ij}u^{\tilde a_i-1}l_i^{(1)}$ determine a system of factors from $Z(\mathcal L_2,\mathcal L_1)$ iff $\gamma_{ij}=0$ when either $\tilde a_i=0$ or $\tilde b_j=p-1$ (in this case v_j should belong to $F(L_1)$).

Definition. A pair $(i_0,j_0)\in(\mathbb Z/s)^2$ is $(r_1,r_2)_{st}$-admissible if:
- $\tilde b_{j_0}\neq p-1$ and $\tilde a_{i_0}\neq 0$, cf. above remark;
- $\tilde a_{i_0}-1\neq\tilde b_{j_0}$;
- there is an $m_0=m_{st}(i_0,j_0)\in\mathbb N$ such that for $1\leqslant m<m_0$, $\tilde a_{i_0+m}-1=\tilde b_{j_0+m}$ but $\tilde a_{i_0+m_0}-1<\tilde b_{j_0+m_0}$.

Definition. A pair $(i_0,j_0)\in(\mathbb Z/s)^2$ is $(r_1,r_2)_{sp}$-admissible if $i_0=0$ and for all $m\in\mathbb Z/s$, $\tilde a_m-1=\tilde b_{j_0+m}$.

Proposition 1.24. a) *If (i_0,j_0) is an $(r_1,r_2)_{st}$-admissible pair then $r_1(i_0)+1/(p-1)>r_2(j_0)$;*

b) *if $(0,j_0)$ is an $(r_1,r_2)_{sp}$-admissible pair then $r_1+1/(p-1)=r_2(j_0)$.*

Proof. a) Here for $1\leqslant m<m_0$, $a_{i_0+m}+1=b_{j_0+m}$ and $a_{i_0+m_0}\geqslant b_{j_0+m_0}$. Therefore,

$$r_1(i_0)+1/(p-1)>\sum_{1\leqslant m\leqslant m_0}(a_{i_0+m}+1)p^{-m}>$$
$$\sum_{1\leqslant m\leqslant m_0}b_{j_0+m}p^{-m}+\sum_{m>m_0}(p-1)p^{-m}\geqslant r_2(j_0).$$

The part b) can be obtained similarly. □

Using the calculations from the proof of Proposition 1.23 we obtain the following two statements.

Proposition 1.25. *Suppose $(i_0,j_0)\in(\mathbb Z/s)^2$ is $(r_1,r_2)_{st}$-admissible and $\gamma\in k$. Then there is a unique $E_{st}(i_0,j_0,\gamma)\in\operatorname{Ext}_{\underline{\mathcal L}^*}(\mathcal L_2,\mathcal L_1)$ given by the system of factors $(v_j)_{j\in\mathbb Z/s}$ such that $v_{j_0}=\gamma u^{\tilde a_{i_0}-1}l_{i_0}^{(1)}$ and $v_j=0$ if $j\neq j_0$, and the map N, which is uniquely determined by the condition:*

• *if* $j=j_0+m$ *with* $1\leqslant m\leqslant m_{st}(i_0,j_0)$ *then*

$$N(l_{j_0+m})\equiv\gamma^{p^m}(\tilde b_{j_0}-\tilde a_{i_0}+1)l^{(1)}_{i_0+m}\operatorname{mod}u^pL$$

and, otherwise, $N(l_j)\equiv 0\operatorname{mod}u^pL$.

Proposition 1.26. *Suppose* $(0,j_0)\in(\mathbb{Z}/s)^2$ *is* $(r_1,r_2)_{sp}$*-admissible and* $\gamma\in\mathbb{F}_q$, $q=p^s$. *Then there is a unique* $E_{sp}(j_0,\gamma)\in\operatorname{Ext}_{\underline{\mathcal{L}}^*}(\mathcal{L}_2,\mathcal{L}_1)$ *given by the zero system of factors and the map* N*, which is uniquely determined by the condition:*
• $N(l_{j_0+m})\equiv\gamma^{p^m}l^{(1)}_m\operatorname{mod}(u^pL)$, $m\in\mathbb{Z}/s$.

The following proposition gives the uniqueness property of the decomposition of elements of $\operatorname{Ext}_{\underline{\mathcal{L}}^*}(\mathcal{L}_2,\mathcal{L}_1)$ into a sum of standard extensions.

Proposition 1.27. *Any element* $\mathcal{L}\in\operatorname{Ext}_{\underline{\mathcal{L}}^*}(\mathcal{L}_2,\mathcal{L}_1)$ *appears as a unique sum of the extensions* $E_{cr}(i,j,\gamma^{cr}_{ij})$, $E_{st}(i,j,\gamma^{st}_{ij})$ *and* $E_{sp}(j,\gamma^{sp}_{0j})$*, where all* $\gamma^{cr}_{ij},\gamma^{st}_{ij}\in k$ *but* $\gamma^{sp}_{0j}\in\mathbb{F}_q$*, and* $\gamma^{cr}_{ij}=0$*, resp.* $\gamma^{st}_{ij}=0$, $\gamma^{sp}_{0j}=0$*, if the corresponding pair of lower indices is not* $(r_1,r_2)_{cr}$*-admissible, resp.* $(r_1,r_2)_{st}$*-admissible,* $(r_1,r_2)_{sp}$*-admissible.*

Proof. By Proposition 1.23, any $\mathcal{L}\in\operatorname{Ext}_{\underline{\mathcal{L}}^*}(\mathcal{L}_2,\mathcal{L}_1)$ can be decomposed as a sum of the above special extensions. To prove the uniqueness of such decomposition, assume that $\mathcal{L}$ represents a trivial element of $\operatorname{Ext}_{\underline{\mathcal{L}}^*}(\mathcal{L}_2,\mathcal{L}_1)$ and prove that all involved coefficients γ^{cr}_{ij}, γ^{st}_{ij} and γ^{sp}_{0j} are equal to 0.
The image of $\mathcal{L}$ in $\operatorname{Ext}_{\underline{\mathcal{L}}^*_0}(\mathcal{L}_2,\mathcal{L}_1)$ is given by the system of factors $(v^{cr}_j+v^{st}_j)_{j\in\mathbb{Z}/s}$ such that
— $v^{cr}_j=\sum_i\gamma^{cr}_{ij}u^{\tilde a_i}l^{(1)}_i$;

— $v^{st}_j=\sum_i\gamma^{st}_{ij}u^{\tilde a_i-1}l^{(1)}_i$.

Let $w_j\in F(L_1)$ be such that for all j, $v_j=w_j-u^{\tilde b_j}\varphi(w_{j-1})$.
If $w_j\equiv\sum_i\kappa_{ij}u^{\tilde a_i}l^{(1)}_i\operatorname{mod}uF(L)$ with $\kappa_{ij}\in k$, then for all i and j,

$$\gamma^{cr}_{ij}u^{\tilde a_i}+\gamma^{st}_{ij}u^{\tilde a_i-1}\equiv\kappa_{ij}u^{\tilde a_i}-\kappa^p_{i-1,j-1}u^{\tilde b_j}\operatorname{mod}u^{\tilde a_i+1}. \tag{1.7}$$

Suppose (i_0,j_0) is $(r_1,r_2)_{st}$-admissible. Then $\tilde a_{i_0}-1\neq\tilde b_{j_0}$ and comparing the coefficients for $u^{\tilde a_{i_0}-1}$ in (1.7) we deduce that $\gamma^{st}_{i_0j_0}=0$. Therefore, all $\gamma^{st}_{ij}=0$.
Suppose (i_0,j_0) is $(r_1,r_2)_{cr}$-admissible. Then for $m_0=m_{cr}(i_0,j_0)$, $\tilde a_{i_0}\neq\tilde b_{j_0}$, $\tilde a_{i_0+m}=\tilde b_{j_0+m}$ if $1\leqslant m<m_0$, and $\tilde a_{i_0+m_0}>\tilde b_{j_0+m_0}$. Then (1.7) implies that $\gamma^{cr}_{i_0j_0}=\kappa_{i_0j_0}$, $\kappa_{i_0+m,j_0+m}=\kappa^p_{i_0+m-1,j_0+m-1}$ for $1\leqslant m<m_0$, and $\kappa_{i_0+m_0-1,j_0+m_0-1}=0$. Therefore, $\gamma^{cr}_{i_0j_0}=0$.
Finally, $\mathcal{L}$ is the trivial element of the group $\operatorname{Ext}_{\underline{\mathcal{L}}^*}(\mathcal{L}_2,\mathcal{L}_1)$ and, therefore, for all j, $N(l_j)\in u^pL$. Then from the description of standard extensions $E_{sp}(j,\gamma^{sp}_{0j})$ in Proposition 1.26 it follows that all $\gamma^{sp}_{0j}=0$. ☐

2. The functor $\mathcal{CV}^* : \underline{\mathcal{L}}^* \longrightarrow \underline{\mathrm{CM}\Gamma}_F$

2.1. The object $\mathcal{R}^0_{st} \in \widetilde{\underline{\mathcal{L}}}^*$. Let $R = \varprojlim_n (\bar{O}/p)_n$ be Fontaine's ring; it has a natural structure of k-algebra via the map $k \longrightarrow R$ given by $\alpha \mapsto \varprojlim([\sigma^{-n}\alpha] \mod p)$, where for any $\gamma \in k$, $[\gamma] \in W(k) \subset \bar{O}$ is the Teichmüller representative of γ. Let m_R be the maximal ideal of R.

Choose $x_0 = (x_0^{(n)} \mod p)_{n\geqslant 0} \in R$ and $\varepsilon = (\varepsilon^{(n)} \mod p)_{n\geqslant 0}$ such that for all $n \geqslant 0$, $x_0^{(n+1)p} = x_0^{(n)}$ and $\varepsilon^{(n+1)p} = \varepsilon^{(n)}$ with $x_0^{(0)} = -p$, $\varepsilon^{(0)} = 1$ but $\varepsilon^{(1)} \neq 1$. We shall denote by v_R the valution on R such that $v_R(x_0) = 1$.

Let Y be an indeterminate.

Consider the divided power envelope $R\langle Y\rangle$ of $R[Y]$ with respect to the ideal (Y). If for $j \geqslant 0$, $\gamma_j(Y)$ is the j-th divided power of Y then $R\langle Y\rangle = \oplus_{j\geqslant 0} R\gamma_j(Y)$. Denote by R_{st} the completion $\prod_{j\geqslant 0} R\gamma_j(Y)$ of $R\langle Y\rangle$ and set, $\mathrm{Fil}^p R_{st} = \prod_{j\geqslant p} R\gamma_j(Y)$. Define the σ-linear morphism of the R-algebra R_{st} by the correspondence $Y \mapsto x_0^p Y$; it will be denoted below by the same symbol σ.

Introduce a $\mathcal{W}_1$-module structure on R_{st} by the k-algebra morphism $\mathcal{W}_1 \longrightarrow R_{st}$ such that $u \mapsto \iota(u) := x_0 \exp(-Y) = x_0 \sum_{j\geqslant 0}(-1)^j \gamma_j(Y)$. Set $F(R_{st}) = \sum_{0\leqslant i<p} x_0^{p-1-i} R\gamma_i(Y) + \mathrm{Fil}^p R_{st}$. Define the continuous σ-linear morphism of R-modules $\varphi : F(R_{st}) \longrightarrow R_{st}$ by setting for $0 \leqslant i < p$, $\varphi(x_0^{p-1-i}\gamma_i(Y)) = \gamma_i(Y)(1 - (i/2)x_0^p Y)$, and for $i \geqslant p$, $\varphi(\gamma_i(Y)) = 0$. Let N be a unique R-differentiation of R_{st} such that $N(Y) = 1$.

Proposition 2.1. a) *If $a \in R_{st}$ and $b \in F(R_{st})$ then*

$$\varphi(ab) = \sigma(a)\varphi(b) \mod x_0^{2p} R_{st};$$

b) *$\varphi \mod x_0^{2p} R_{st}$ is a σ-linear morphism of $\mathcal{W}_1$-modules;*
c) *for any $b \in R_{st}$ and $w \in \mathcal{W}_1$, $N(wb) = N(w)b + wN(b)$;*
d) *for any $l \in F(R_{st})$, $uN(l) \in F(R_{st})$ and*

$$N(\varphi(l)) = \varphi(uN(l)) \mod x_0^{2p} R_{st}.$$

Proof. a) It is sufficient to verify it for $a = Y$ and $b = x_0^{p-1-i}\gamma_i(Y)$, $0 \leqslant i < p$.
b) Use that the multiplication by $\sigma(u) = u^p$ comes as the multiplication by $\iota(u)^p = x_0^p \equiv x_0^p \exp(-x_0^p Y) = \sigma(\iota(u)) \mod x_0^{2p} R_{st}$.
c) Use that $N(\iota(u)) = -\iota(u)$.
d) It will be enough to check the identity for $l = x_0^{p-1-i}\gamma_i(Y)$ with $1 \leqslant i < p$. Then $N(\varphi(l)) = \gamma_{i-1}(Y)(1 - (1/2)(i+1)x_0^p Y)$. On the other hand, $uN(l) = x_0^{p-1-(i-1)}\gamma_{i-1}(Y)\exp(-Y)$ and $\varphi(uN(l))$ is equal to

$$\gamma_{i-1}(Y)\left(1 - \frac{i-1}{2}x_0^p Y\right)\exp(-x_0^p Y) \equiv \gamma_{i-1}(Y)\left(1 - \frac{i+1}{2}x_0^p Y\right) \mod x_0^{2p}$$

$\square$

Introduce a Γ_F-action on $R_{st} \mod x_0^{p^2/(p-1)} R_{st}$ as follows.

For any $\tau \in \Gamma_F$, let $k(\tau) \in \mathbb{Z}$ be such that $\tau(x_0) = \varepsilon^{k(\tau)} x_0$ and let $\widetilde{\log}(1+X) = X - X^2/2 + \cdots - X^{p-1}/(p-1)$ be the truncated logarithm. For any $\tau \in \Gamma_F$, define a linear map $\tau : R_{st} \longrightarrow R_{st}$ by extending the natural action of τ on R and setting for $\tau \in \Gamma_F$ and $j \geqslant 0$,

$$\tau(\gamma_j(Y)) := \sum_{0 \leqslant i \leqslant \min\{j, p-1\}} \gamma_{j-i}(Y) \gamma_i(\widetilde{\log} \varepsilon).$$

Note that the cocycle relation $\varepsilon^{k(\tau_1)} (\tau_1 \varepsilon)^{k(\tau)} = \varepsilon^{k(\tau_1 \tau)}$, where $\tau_1, \tau \in \Gamma_F$, implies the cocycle relation

$$k(\tau_1) \widetilde{\log} \varepsilon + k(\tau) \widetilde{\log}(\tau_1(\varepsilon)) \equiv k(\tau_1 \tau) \widetilde{\log} \varepsilon \bmod x_0^{p^2/(p-1)}.$$

(Use that $\widetilde{\log}(1+X)^k \equiv k \widetilde{\log}(1+X) \bmod (X^p)$ and $\varepsilon \equiv 1 \bmod x_0^{p/(p-1)}$.) In addition, for any $k \in \mathbb{Z}$, the obvious congruence

$$(1+X)^k = \exp(k \log(1+X)) \equiv \widetilde{\exp}(k \widetilde{\log}(1+X)) \bmod (X^p)$$

implies that for any $\tau \in \Gamma_F$, $\tau(x_0 \exp(-Y)) \equiv x_0 \exp(-Y) \bmod x_0^{p^2/(p-1)}$. Therefore, the correspondences $\gamma_j(Y) \mapsto \tau(\gamma_j(Y))$ induce a Γ_F-action on $\mathcal{W}_1$-algebra $R_{st} \bmod x_0^{p^2/(p-1)} R_{st}$, which extends the natural Γ_F-action on R.

Proposition 2.2. *For any* $\tau \in \Gamma_F$,
a) $\tau(F(R_{st})) = F(R_{st})$;

b) *for any* $a \in F(R_{st})$, $\tau(\varphi(a)) \equiv \varphi(\tau(a) \bmod x_0^{p+1/(p-1)} R_{st}$;

c) *for any* $b \in R_{st}$, $\tau(N(b)) = N(\tau(b))$.

Proof. The proof is straightforward in cases a) and c). Part b) follows by direct calculation from the following Lemma. □

Lemma 2.3. $\sigma(\widetilde{\log} \varepsilon)/x_0^p \equiv \widetilde{\log} \varepsilon \bmod x_0^{p+1/(p-1)} R$.

Proof. Consider Fontaine's element

$$t^+ = \log[\varepsilon] = \sum_{n \geqslant 1} (-1)^{n-1} \frac{([\varepsilon]-1)^n}{n} = \sum_{m \in \mathbb{Z}} p^m [\eta_m] \in A_{cr}$$

where all $\eta_m \in R$. Then $t^+ \in \mathrm{Fil}^1 A_{cr}$ and $\sigma t^+ = p t^+$. This implies for all $m \in \mathbb{Z}$, that $\eta_m = \sigma^{-m} \eta_0$.
Consider $\mathcal{H} \subset A_{cr}$ consisting of the elements of the form $\sum_{m \in \mathbb{Z}} p^m [r_m]$ such that for $m \leqslant 0$, $v_R(r_m) \geqslant p^2/(p-1)$ (this is automatic for $m \leqslant -2$), $v_R(r_1) \geqslant p^2/(p-1) - 1$ and $v_R(r_2) \geqslant p^2/(p-1) - 2$. Then $\mathcal{H}$ is an additive subgroup in A_{cr}.
Verify that

- *for all* $n \geqslant p$, $([\varepsilon]-1)^n/n \in \mathcal{H}$.

Indeed, the congruence $[\varepsilon] \equiv 1 + [a_0] \bmod pW(R)$ (where $a_0 = \varepsilon - 1$) implies that $[\varepsilon] = \lim_{m\to\infty}(1 + [\sigma^{-m}a_0])^{p^m}$. Therefore,

$$[\varepsilon] - 1 = \sum_{m\geqslant 0}[a_m]p^m = [a_0]\left(1 + \sum_{m\geqslant 1}p^m[b_m]\right),$$

where $v_R(a_m) = p^{1-m}/(p-1)$, $v_R(b_1) = -1$ and $v_R(b_2) = -1 - 1/p$.
If $n \not\equiv 0 \bmod p$ then $([\varepsilon]-1)^n \equiv [a_{0n}] + p[a_{1n}] + p^2[a_{2n}] \bmod p^3W(R)$ with $v_R(a_{0n}) = v_R(a_{1n}) + 1 = v_R(a_{2n}) + 2 = pn/(p-1)$. This proves that $([\varepsilon]-1)^n/n \in \mathcal{H}$ for all $n \not\equiv 0 \bmod p$, $n > p$.
As for all remaining $n \geqslant p$, just note that for all $M \geqslant 1$,

$$([\varepsilon]-1)^{p^M} \equiv [a_0]^{p^M}(1 + p^{M+1}[b_1] + p^{M+2}[b_{2M}])) \bmod p^{M+3}W(R),$$

where $v_R(b_{2M}) = -2$.

The above calculations mean that $t^+ \equiv \widetilde{\log}[\varepsilon] \bmod \mathcal{H}$. Therefore, if

$$\widetilde{\log}[\varepsilon] = [\omega_0] + p[\omega_1] + p^2[\omega_2] \bmod p^3W(R)$$

then $\omega_0 = \widetilde{\log}\varepsilon \equiv \eta_0 \bmod x_0^{p^2/(p-1)}R$,

$$\omega_1 \equiv \eta_1 \equiv \sigma^{-1}\eta_0 \equiv \sigma^{-1}\widetilde{\log}\varepsilon \bmod x_0^{p^2/(p-1)-1}R$$

and $\omega_2 \equiv \eta_2 \bmod x_0^{p^2/(p-1)-2}R$.
Now note that $\widetilde{\log}[\varepsilon] \in \mathrm{Fil}^1 A_{cr} \bigcap W(R)$, that is $\widetilde{\log}[\varepsilon]$ is divisible by $[x_0] + p$ in $W(R)$. The division algorithm gives $(\omega_1 - \omega_0/x_0)/x_0 \equiv \omega_2 \bmod x_0^{1/(p-1)}R$. Therefore, $\sigma(\omega_1) \equiv \sigma(\omega_0)/x_0^p \bmod x_0^p\sigma(\omega_2)R$. The lemma is proved. $\square$

By above results we can introduce $\mathcal{R}^0_{st} = (R^0_{st}, F(R^0_{st}), \varphi, N) \in \widetilde{\underline{\mathcal{L}}}^*$, where $R^0_{st} = R_{st} \bmod x_0^p\mathrm{m}_R$ and $F(R^0_{st}) = F(R_{st}) \bmod x_0^p\mathrm{m}_R$ with induced σ-linear map φ and $\mathcal{W}_1$-differentiation N. The above defined Γ_F-action on $R^0 \bmod x_0^p\mathrm{m}_R$ respects the structure of $\mathcal{R}^0_{st}$ as an object of the category $\widetilde{\underline{\mathcal{L}}}^*$. In our setting the filtered Galois module R^0_{st} plays a role of Fontaine's ring $\hat{A}_{st}$.

2.2. The functor $\mathcal{V}^*$. If $\mathcal{L} = (L, F(L), \varphi, N) \in \widetilde{\underline{\mathcal{L}}}^*$ then the triple $(L, F(L), \varphi)$ is an object of $\widetilde{\underline{\mathcal{L}}}_0^*$ which will be denoted below by the same symbol $\mathcal{L}$.

Definition. Let $\mathcal{R}^0 = (R^0, F(R^0), \varphi) \in \widetilde{\underline{\mathcal{L}}}_0^*$, where $R^0 = R/x_0^p\mathrm{m}_R$, $F(R^0) = x_0^{p-1}R^0$, the $\mathcal{W}_1$-module structure on R^0 is given via $u \mapsto x_0$ and ϕ is induced by the map $r \mapsto r/x_0^{p(p-1)}$, $r \in R$.

For any $\mathcal{L} = (L, F(L), \varphi, N) \in \underline{\mathcal{L}}^*$, consider the Γ_F-module $\mathcal{V}^*(\mathcal{L}) = \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}^*}(\mathcal{L}, \mathcal{R}^0_{st})$. If $f \in \mathcal{V}^*(\mathcal{L})$ and $i \geqslant 0$, introduce the k-linear morphisms $f_i : L \longrightarrow R^0$ such that for any $l \in L$, $f(l) = \sum_{i\geqslant 0} f_i(l)\gamma_i(Y)$. The correspondence $f \mapsto f_0$ gives the homomorphism of abelian groups $\mathrm{pr}_0 : \mathcal{V}^*(\mathcal{L}) \longrightarrow \mathcal{V}^*_0(\mathcal{L}) := \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0^*}(\mathcal{L}, \mathcal{R}^0)$.

Proposition 2.4. pr_0 *is isomorphism of abelian groups.*

Proof. Clearly, pr_0 is additive. Suppose $f\in \mathrm{Ker}\,\mathrm{pr}_0$. Then for all $i\geqslant 0$ and $l\in L$, $f_i(l)=f_0(N^i(l)))=0$, i.e. $f=0$.
Suppose $g\in \mathrm{Hom}_{\underline{\tilde{\mathcal{L}}}_0^*}(\mathcal{L},\mathcal{R}^0)$. This means that $g:L\longrightarrow R^0$ is a σ-linear morphism of $\mathcal{W}_1$-modules, $g(F(L))\subset F(R^0)$ and for any $l\in F(L)$, $g(\varphi(l))=(g(l)/x_0^{p-1})^p$.
Set for any $l\in L$, $f(l)=g(l)+g(Nl)\gamma_1(Y)+\dots+g(N^il)\gamma_i(Y)+\dots$. Then for any $l\in L$, $f(N(l))=N(f(l))$ and our Proposition is implied by the following Lemma. □

Lemma 2.5. a) *For any* $l\in L$, $f(ul)=x_0\exp(-Y)f(l)$*;*
b) *for any* $l\in F(L))$, $\varphi(f(l))=f(\varphi(l))$.

Proof of Lemma. a) For any $l\in L$, $f(ul)=\sum_{i\geqslant 0}g(N^i(ul))\gamma_i(Y)=$

$$x_0\sum_{i\geqslant 0}g((N-\mathrm{id})^il)\gamma_i(Y)=x_0\sum_{i,s}(-1)^{i-s}\binom{i}{s}g(N^sl)\gamma_s(Y)\gamma_j(Y)$$

$$=x_0\sum_{j,s}(-1)^jg(N^sl)\gamma_s(Y)\gamma_j(Y)=x_0\exp(-Y)f(l).$$

b) Let $l\in L$. Prove by induction on $i\geqslant 1$ that

$$N^i(\varphi(l))=\varphi((uN)^i(l))=-\frac{i(i-1)}{2}u^p\varphi(u^{i-1}N^{i-1}(l))+\varphi(u^iN^i(l)).$$

Then

$$g(N^i(\varphi(l)))=-\frac{i(i-1)}{2}x_0^p\left(\frac{g(u^{i-1}N^{i-1}(l))}{x_0^{p-1}}\right)^p+\left(\frac{g(u^iN^i(l))}{x_0^{p-1}}\right)^p$$

and $f(\varphi(l))$ is equal to $\sum_{i\geqslant 0}g(N^i(\varphi(l))\gamma_i(Y)=$

$$\sum_{i\geqslant 0}\left(\frac{g(N^il)}{x_0^{p-1-i}}\right)^p\left(\gamma_i(Y)-\frac{i(i+1)}{2}x_0^p\gamma_{i+1}(Y)\right)=\varphi(f(l)).$$

□

Corollary 2.6. a) *If* $\mathrm{rk}_{\mathcal{W}_1}L=s$ *then* $|\mathcal{V}^*(\mathcal{L})|=p^s$*;*
b) *the correspondence* $\mathcal{L}\mapsto\mathcal{V}^*(\mathcal{L})$ *induces an exact functor* $\mathcal{V}^*$ *from* $\underline{\mathcal{L}}^*$ *to the category of* $\mathbb{F}_p[\Gamma_F]$*-modules.*

Proof. a) Proceed as in [1, 3]. Suppose the structure of the filtered φ-module $\mathcal{L}$ is given by a choice of a $\mathcal{W}_1$-basis $m_1,\dots,m_s$ of $F(L)$ and a non-degenerate matrix $A\in M_s(\mathcal{W}_1)$ such that $(m_1,\dots,m_s)=(\varphi(m_1),\dots,\varphi(m_s))A$. Let $\bar{X}=(X_1,\dots,X_s)$ be a vector with s independent variables and let $R_0=\mathrm{Frac}\,R$. Consider the quotient $A_{\mathcal{L}}$ of the polynomial ring $R_0[\bar{X}]$ by the ideal generated by the coordinates of the vector $(\bar{X}A)^{(p)}-x_0^{p(p-1)}\bar{X}$. (For a matrix C the matrix $C^{(p)}$ is obtained by raising all elements of C to p-th power.) Then $A_{\mathcal{L}}$ is etale R_0-algebra of rank p^s (use that $(u^{p-1}I_s)A^{-1}\in M_s(\mathcal{W}_1)$)

and all its $\bar{R}_0$-points give rise to elements of the group $\mathrm{Hom}_{\underline{\widetilde{\mathcal{L}}}_0^*}(\mathcal{L},\mathcal{R})$, where $\mathcal{R}=(R,x_0^{p-1}R,\varphi)\in\underline{\widetilde{\mathcal{L}}}_0^*$ is such that for any $r\in R$, $\varphi(x_0^{p-1}r)=r^p$. It remains to note that $\varphi|_{x_0^p\mathrm{m}_R}$ is nilpotent, by Lemma 1.5, the natural projection $\mathcal{R}\longrightarrow\mathcal{R}^0$ induces bijection from $\mathrm{Hom}_{\underline{\widetilde{\mathcal{L}}}_0^*}(\mathcal{L},\mathcal{R})$ to $\mathrm{Hom}_{\underline{\widetilde{\mathcal{L}}}_0^*}(\mathcal{L},\mathcal{R}^0)=\mathcal{V}_0^*(\mathcal{L})$ and by Proposition 2.4, $|\mathcal{V}_0^*(\mathcal{L})|=|\mathcal{V}^*(\mathcal{L})|$.
b) This follows from a) because the functor $\mathcal{L}\mapsto\mathcal{V}_0^*(\mathcal{L})$ is left exact. □

Introduce the ideal $\widetilde{J}=\sum_{0\leqslant i<p}x_0^{p-i}\mathrm{m}_R\gamma_i(Y)+\mathrm{Fil}^pR_{st}^0$ of R_{st}^0. Then $F(R_{st}^0)\supset\widetilde{J}$ and $\varphi|_{\widetilde{J}}$ is nilpotent. For $\widetilde{\mathcal{R}}_{st}^0=(R_{st}^0/\widetilde{J},F(R_{st}^0)/\widetilde{J},\varphi\,\mathrm{mod}\,\widetilde{J})\in\underline{\widetilde{\mathcal{L}}}_0^*$, there is a natural projection $\mathcal{R}_{st}^0\longrightarrow\widetilde{\mathcal{R}}_{st}^0$ in $\underline{\widetilde{\mathcal{L}}}_0^*$ and for any $\mathcal{L}\in\underline{\mathcal{L}}_0^*$, $\mathrm{Hom}_{\underline{\widetilde{\mathcal{L}}}_0^*}(\mathcal{L},\mathcal{R}_{st}^0)=\mathrm{Hom}_{\underline{\widetilde{\mathcal{L}}}_0^*}(\mathcal{L},\widetilde{\mathcal{R}}_{st}^0)$. This implies the following description of the Γ_F-modules $\mathcal{V}^*(\mathcal{L})$ where $\mathcal{L}\in\underline{\mathcal{L}}^*$ (use the identification pr_0 of Proposition 2.4).

Corollary 2.7.

$$\mathcal{V}^*(\mathcal{L})=\left\{\sum_{0\leqslant i<p}N^{*i}(f_0)\gamma_i(Y)\,\mathrm{mod}\,\widetilde{J}\mid f_0\in\mathrm{Hom}_{\underline{\widetilde{\mathcal{L}}}_0^*}(\mathcal{L},\mathcal{R}^0)\right\}$$

Remark. a) In the above description of $\mathcal{V}^*(\mathcal{L})$, for any $l\in L$, $N^*(f_0)(l)=f_0(N(l))$. In addition, all $N^{*i}(f_0)\gamma_i(Y)$ depend just on $N_1=N\,\mathrm{mod}\,u^pL$.
b) If $\mathcal{L}\in\underline{\mathcal{L}}^{*u}$ then in the above Corollary we can replace $\mathcal{R}^0$ and $\widetilde{J}$ by, respectively, $\mathcal{R}^u=(R/x_0^pR,x_0^{p-1}R/x_0^pR,\varphi)\in\underline{\widetilde{\mathcal{L}}}_0^*$ and the ideal $\widetilde{J}^u=\sum_{0\leqslant i<p}Rx_0^{p-i}\gamma_i(Y)+\mathrm{Fil}^pR_{st}^0$. In particular, for unipotent modules the whole theory can be developed in the context of $k[u]/u^p$-modules.

2.3. The category $\underline{\mathrm{CM\Gamma}}_F$ and the functor $\mathcal{CV}^*$.

Definition. The objects of the category $\underline{\mathrm{CM\Gamma}}_F$ are the triples $\mathcal{H}=(H,H^0,j)$, where H,H^0 are finite $\mathbb{Z}_p[\Gamma_F]$-modules, Γ_F acts trivially on H^0 and $j:H\longrightarrow H^0$ is an epimorphic map of $\mathbb{Z}_p[\Gamma_F]$-modules. If $\mathcal{H}_1=(H_1,H_1^0,j_1)\in\underline{\mathrm{CM\Gamma}}_F$ then $\mathrm{Hom}_{\underline{\mathrm{CM\Gamma}}_F}(\mathcal{H}_1,\mathcal{H})$ consists of the couples (f,f^0), where $f:H_1\longrightarrow H$ and $f^0:H_1^0\longrightarrow H^0$ are morphisms of Γ_F-modules such that $jf=f^0j_1$.

The category $\underline{\mathrm{CM\Gamma}}_F$ is pre-abelian, cf. Appendix A, and its objects have a natural group structure. In particular, with above notation, $\mathrm{Ker}(f,f^0)=(\mathrm{Ker}f,j_1(\mathrm{Ker}f))$ together with the natural embedding to $\mathcal{H}_1$. Similarly, $\mathrm{Coker}(f,f^0)=(H/f(H_1),H^0/j(f(H_1)))$. For example, the map $(\mathrm{id},0):(H,H)\longrightarrow(H,0)$ has the trivial kernel and cokernel. In addition, the monomorphism $(f_1,f_1^0):\mathcal{H}_1\longrightarrow\mathcal{H}$ is strict if and only if $f_1(\mathrm{Ker}\,j_1)=f_1(H_1)\cap\mathrm{Ker}\,j$. Suppose $\mathcal{H}_2=(H_2,H_2^0,j_2)$ and $(f_2,f_2^0):\mathcal{H}\longrightarrow\mathcal{H}_2$ is an epimorphism. Then it is strict if and only if f_2^0 induces epimorphic map from $\mathrm{Ker}j$ to $\mathrm{Ker}\,j_2$. In $\underline{\mathrm{CM\Gamma}}_F$ we can use formalism of short exact sequenes and the corresponding 6-terms $\mathrm{Hom}_{\underline{\mathrm{CM\Gamma}}_F}-\mathrm{Ext}_{\underline{\mathrm{CM\Gamma}}_F}$ exact sequences, cf. Appendix A.

Definition. Suppose $\mathcal{L} \in \underline{\mathcal{L}}^*$ and $i^{et} : \mathcal{L}^{et} \longrightarrow \mathcal{L}$ is the maximal etale subobject. Then $\mathcal{CV}^* : \underline{\mathcal{L}}^* \longrightarrow \underline{\mathrm{CM}\Gamma}_F$ is the functor such that $\mathcal{CV}^*(\mathcal{L}) = (\mathcal{V}^*(\mathcal{L}), \mathcal{V}^*(\mathcal{L}^{et}), \mathcal{V}^*(i^{et}))$.

The simple objects in $\underline{\mathrm{CM}\Gamma}_F$ are of the form either $(H, 0, 0)$, where H is a simple $\mathbb{Z}_p[\Gamma_F]$-module, or $(\mathbb{F}_p, \mathbb{F}_p, \mathrm{id})$, where $\mathbb{F}_p$ is provided with the trivial Γ_F-action. In this context it will be very convenient to use the following formalism.
For $s \in \mathbb{N}$, consider Serre's fundamental characters $\chi_s : \Gamma_F \longrightarrow k^*$. Here for $\tau \in \Gamma_F$, $\chi_s(\tau) = (\tau x_s)/x_s \bmod x_0^p$, where $x_s \in R$ is such that $x_s^{p^s-1} = x_0$. If χ is any continuous (1-dimensional) character of Γ_F then there are $s, m \in \mathbb{N}$ such that $0 < m \leqslant p^s - 1$ and $\chi = \chi_s^m$. Set $r(\chi) = m/(p^s-1)$. Then $r(\chi)$ depends only on χ and the correspondence $\chi \mapsto r(\chi)$ gives a bijection from the set of all continuous (1-dimensional) characters of Γ_F with values in k^* to the set $[0,1]_p \setminus \{0\}$.
For $r \in [0,1]_p$, $r \neq 0$, introduce the Γ_F-module $\mathbb{F}(r)$ such that $\mathbb{F}(r) = \mathbb{F}_{p^{s(r)}}$, where $s(r)$ is the period of the p-digit expansion of r, cf. Subsection 1.2, with the Γ_F-action given by the character χ such that $r(\chi) = r$. We have:
— all $\mathbb{F}(r)$ are simple $\mathbb{Z}_p[\Gamma_F]$-modules;
— Γ_F-modules $\mathbb{F}(r_1)$ and $\mathbb{F}(r_2)$ are isomorphic if and only if there is an $n \in \mathbb{Z}$ such that $r_1 = r_2(n)$;
— any simple $\mathbb{Z}_p[\Gamma_F]$-module is isomorphic to some $\mathbb{F}(r)$.
It will be natural to set $\mathcal{F}(r) := (\mathbb{F}(r), 0, 0)$ for all $r \in (0,1]_p$, and to set separately $\mathcal{F}(0) := (\mathbb{F}_p, \mathbb{F}_p, \mathrm{id})$.
With above notation we have the following property, where the objects $\mathcal{L}(r)$ were introduced in Subsection 1.3.

Proposition 2.8. *For any $r \in [0,1]_p$, $\mathcal{CV}^*(\mathcal{L}(r)) = \mathcal{F}(r)$.*

Proof. The proof goes along the lines of Subsection 4.2 of [1], cf. also the beginning of Subsection 2.4 below. □

2.4. A criterion. Suppose $\mathcal{L}_1, \mathcal{L}_2$ are given in notation of Subsection 1.4 and $q = p^s$. Then for $i = 1, 2$, $\mathcal{V}^*(\mathcal{L}_i) = V_i$ are 1-dimensional vector spaces over $\mathbb{F}_q$ with Γ_F-action given by the character $\chi_i : \Gamma_F \longrightarrow k^*$ such that $r(\chi_i) = r_i$. (Note that $(q-1)r_i \in \mathbb{Z}$ and, therefore, $\chi_i(\Gamma_F) \subset \mathbb{F}_q^*$.) Choose $\pi_s \in \bar{F}$ such that $\pi_s^{q-1} = -p$. Then $F_s = F(\pi_s)$ is a tamely ramified extension of F of degree $q-1$ and all points of V_i are defined over F_s. We can identify V_i with the $\mathbb{F}_p[\Gamma_F]$-module $\mathbb{F}_q \bar{\pi}_s^{(q-1)r_i} \subset \bar{O}/p\bar{O}$, where $\bar{\pi}_s = \pi_s \bmod p$. These identifications allow us to fix the points $h_i^0 := \bar{\pi}_s^{(q-1)r_i} \in V_i$ and to identify V_i with the Γ_F-module $\{\alpha h_i^0 \mid \alpha \in \mathbb{F}_q\}$.
Suppose $h_1 \in V_1$. Define the homomorphism

$$F_{h_1} : \mathrm{Ext}_{\mathbb{F}_p[\Gamma_F]}(V_1, V_2) \longrightarrow Z^1(\Gamma_{F_s}, \mathbb{F}_q) = \mathrm{Hom}(\Gamma_{F_s}, \mathbb{F}_q),$$

where $\Gamma_{F_s} = \mathrm{Gal}(\bar{F}/F_s)$, as follows. If $V \in \mathrm{Ext}_{\mathbb{F}_q[\Gamma_F]}(V_1, V_2)$ and $h \in V$ is a lift of h_1 then for any $\tau \in \Gamma_F$, $F_{h_1}(V)(\tau) = a_\tau \in \mathbb{F}_q$, where $\tau h - h = a_\tau h_2^0$.

Clearly, $F_{h_1}(V)$ does not depend on a choice of h and it is the zero function if and only if the projection $V \longrightarrow V_1$ admits a Γ_F-equivariant section. In other words, we have the following criterion.

PROPOSITION 2.9. *V is the trivial extension if and only if for all $h_1 \in V_1$, one has $F_{h_1}(V) = 0$.*

2.5. GALOIS MODULES $\mathcal{V}^*(E_{cr}(i_0, j_0, \gamma))$. Suppose we have an object $\mathcal{L} = (L, F(L), \varphi, N)$ of the category $\underline{\mathcal{L}}^*_{cr}$. Then there is a special $\sigma(\mathcal{W}_1)$-basis $l_1, \dots, l_s$ of $\varphi(F(L))$ such that for some integers $0 \leqslant c_1, \dots, c_s < p$ and a matrix $A \in \mathrm{GL}_s(k)$, the elements $u^{c_1}l_1, \dots, u^{c_s}l_s$ form a $\mathcal{W}_1$-basis of $F(L)$ and $(\varphi(u^{c_1}l_1), \dots, \varphi(u^{c_s}l_s)) = (l_1, \dots, l_s)A$.
For $1 \leqslant i \leqslant s$, set $\tilde{c}_i = (p-1) - c_i$. The following Proposition is a special case of Corollary 2.7 (remind that $R^0 = R/x_0^p \mathrm{m}_R$).

PROPOSITION 2.10. *With above notation, $\mathcal{V}^*(\mathcal{L})$ is the $\mathbb{F}_p[\Gamma_F]$-module of all $(\theta_1, \dots, \theta_s) \bmod x_0^p \mathrm{m}_R \in (R^0)^s$ such that*

$$(\theta_1^p / x_0^{p\tilde{c}_1}, \dots, \theta_s^p / x_0^{p\tilde{c}_s}) = (\theta_1, \dots, \theta_s)A.$$

REMARK. In [1, 2] it was proved (in the context of the Fontaine-Laffaille theory) that the family of $\mathbb{F}_p[\Gamma_F]$-modules $\mathcal{V}^*(\mathcal{L})$, where $\mathcal{L} \in \underline{\mathcal{L}}^*_{cr}$, coincides with the family of all killed by p subquotients of crystalline representations of Γ_F with weights from $[0, p)$. This result can be also extracted from Subsection 4, where we establish that the family of $\mathbb{F}_p[\Gamma_F]$-modules $\mathcal{V}^*(\mathcal{L})$, where $\mathcal{L} \in \underline{\mathcal{L}}^*$, coincides with the family of all killed by p subquotients of semi-stable representations of Γ_F with weights from $[0, p)$.

For an $(r_1, r_2)_{cr}$-admissible pair $(i_0, j_0) \in (\mathbb{Z}/s)^2$ and $\gamma \in k$, use the description of $E_{cr}(i_0, j_0, \gamma)$ from Subsection 1.4. Then by Corollary 2.7, $V = \mathcal{V}^*(E_{cr}(i_0, j_0, \gamma))$ is identified with the additive group of all taken modulo $x_0^p \mathrm{m}_R$ solutions in R of the following system of equations

$$\begin{array}{rcll} X_i^{(1)p}/x_0^{pa_i} & = & X_{i+1}^{(1)}, & \text{for all } i \in \mathbb{Z}/s; \\ X_j^p / x_0^{pb_j} & = & X_{j+1} - \delta_{jj_0}\gamma^p X_{i_0+1}^{(1)}, & \text{for all } j \in \mathbb{Z}/s \end{array}$$

Note that the first group of equations describes $V_1 = \mathcal{V}^*(\mathcal{L}_1)$ and the correspondences $X_i^{(1)} \mapsto 0$ and $X_j \mapsto X_j^{(2)}$ with $i, j \in \mathbb{Z}/s$, define the map $V \longrightarrow V_2$, where $V_2 = \mathcal{V}^*(\mathcal{L}_2)$ is associated with all taken modulo $x_0^p \mathrm{m}_R$ solutions in R of the equations $X_j^{(2)p}/x_0^{pb_j} = X_{j+1}^{(2)}$, $j \in \mathbb{Z}/s$. As it was noted in Subsection 2.2, the corresponding Γ_F-action on V, V_1 and V_2 comes from the natural Γ_F-action on R^0.
Take $x_s \in R$ such that $x_s^{q-1} = x_0$ and $x_s \mapsto \pi_s \bmod p$ under the natural identification $R/x_0^p R \simeq \bar{O}/p\bar{O}$. (This identification is given by the correspondence $r = \varprojlim_n (r_n \bmod p) \mapsto r^{(1)} := \lim_{n \to \infty} r_{n+1}^{p^n}$.) For $i, j \in \mathbb{Z}/s$, set $x_0^{r_1(i)} := x_s^{(q-1)r_1(i)}$ and $x_0^{r_2(j)} := x_s^{(q-1)r_2(j)}$, and introduce the variables $Z_i^{(1)} = x_0^{-pr_1(i)} X_i^{(1)}$, $Z_j = x_0^{-pr_2(j)} X_j$, $Z_j^{(2)} = x_0^{-pr_2(j)} X_j^{(2)}$. Then the elements of V appear as

the taken modulo m_R solutions in $R_0 := \mathrm{Frac}(R)$ of the following system of equations

$$\begin{array}{ccll} Z_i^{(1)p} & = & Z_{i+1}^{(1)}, & \text{for all } i\in\mathbb{Z}/s; \\ Z_j^p & = & Z_{j+1}, & \text{for all } j\neq j_0+1; \\ Z_{j_0+1}-Z_{j_0+1}^q & = & \gamma^p Z_{i_0+1}^{(1)} x_0^{p(r_1(i_0)-r_2(j_0))} & \end{array}$$

Note that for the points $h_1^0\in V_1$ and $h_2^0\in V_2$ chosen in Subsection 2.4, one has $Z_i^{(1)}(h_1^0)=Z_i^{(2)}(h_2^0)=1$, where $i\in\mathbb{Z}/s$.
Suppose $\alpha\in\mathbb{F}_q$ and $h_1=\alpha h_1^0\in V_1$.
Let $\mathcal{F}_s=k((x_s))\subset R_0=\mathrm{Frac}\,R$. The field-of-norms functor gives a natural embedding of the absolute Galois group $\Gamma_{\mathcal{F}_s}$ of $\mathcal{F}_s$ into Γ_{F_s}, where $F_s=F(\pi_s)$. Then the restriction $F_{h_1}(V)|_{\Gamma_{\mathcal{F}_s}}$ of the cocycle

$$\{F_{h_1}(V)(\tau)=A_{\tau,\alpha}(i_0,j_0,\gamma)\in\mathbb{F}_q \mid \tau\in\Gamma_{F_s}\}$$

from Subsection 2.4 can be described as follows.
Let $U\in R_0$ be such that $U-U^q=\gamma x_0^{r_1(i_0)-r_2(j_0)}$. Then for any $\tau\in\Gamma_{\mathcal{F}_s}$, $\sigma^{j_0}(A_{\tau,\alpha}(i_0,j_0,\gamma))=\sigma^{i_0}(\alpha)(\tau(U)-U)$ and therefore

$$A_{\tau,\alpha}(i_0,j_0,\gamma)=\sigma^{i_0-j_0}(\alpha)\sigma^{-j_0}(\tau U-U).$$

The following Lemma is an immediate consequence of the definition of $(r_1,r_2)_{cr}$-admissible pairs.

Lemma 2.11. *With above notation let $C=-(q-1)(r_1(i_0)-r_2(j_0))$. Then C is a prime to p integer and $1\leqslant C\leqslant q-1$.*

2.6. Galois modules $\mathcal{V}^*(E_{st}(i_0,j_0,\gamma))$. For an $(r_1,r_2)_{st}$-admissible pair $(i_0,j_0)\in(\mathbb{Z}/s)^2$ and $\gamma\in k$, use the description of $E_{st}(i_0,j_0,\gamma)$ from Subsection 1.5.
By Subsection 2.2, $V=\mathcal{V}^*(E_{st}(i_0,j_0,\gamma))$ is identified (as an abelian group) with the solutions $\Big(\{X_i^{(1)} \mid i\in\mathbb{Z}/s\},\{X_j\mid j\in\mathbb{Z}/s\}\Big)\in R^{2s}$ of the following system of equations

$$\begin{array}{rcl} X_i^{(1)p}/x_0^{pa_i} & = & X_{i+1}^{(1)},\ \text{for all } i\in\mathbb{Z}/s; \\ X_j^p/x_0^{pb_j}+\delta_{jj_0}\gamma^pX_{i_0}^{(1)p}/x_0^{pa_{i_0}+p} & = & X_{j+1},\ \text{for all } j\in\mathbb{Z}/s \end{array} \tag{2.1}$$

The structure of V as an element of $\mathrm{Ext}_{\mathbb{F}_p[\Gamma_F]}(V_1,V_2)$ can be described along the lines of Subsection 2.5. The action of Γ_F on V comes from the natural Γ_F-action on $\widetilde{\mathcal{R}}_{st}^0$, and the embedding of V into $(R_{st}^0)^{2s}$ given by the following correspondences:
– if $i\in\mathbb{Z}/s$ then $X_i^{(1)}\mapsto X_i^{(1)}\bmod x_0^p\mathrm{m}_R$;
– if $j\notin\{j_0+1,\dots,j_0+m_0\}$ then $X_j\mapsto X_j\bmod x_0^p\mathrm{m}_R$;
– for $1\leqslant m\leqslant m_0$, $X_{j_0+m}\mapsto X_{j_0+m}+\gamma^{p^m}(\tilde b_{j_0}-\tilde a_{i_0}+1)X_{i_0+m}^{(1)}Y\bmod x_0^p\mathrm{m}_R$.

Similarly to Subsection 2.5, introduce new variables by the relations $Z_i^{(1)}=x_0^{-pr_1(i)}X_i^{(1)}$, $Z_i=x_0^{-pr_2(i)}X_i$ and $Z_i^{(2)}=x_0^{-pr_2(i)}X_i^{(2)}$, $i\in\mathbb{Z}/s$, and rewrite system of equations (2.1) in the following form:

$$\begin{array}{ccc} Z_i^{(1)p} & = & Z_{i+1}, \quad \text{for all } i \in \mathbb{Z}/s; \\ Z_j^p & = & Z_{j+1}, \quad \text{for all } j \neq j_0+1; \\ Z_{j_0+1} - Z_{j_0+1}^q & = & \gamma^p Z_{i_0+1}^{(1)} x_0^{p(r_1(i_0)-r_2(j_0)-1)} \end{array}$$

If $\alpha \in \mathbb{F}_q$ and $h_1 = \alpha h_1^0 \in V_1$, then the restriction to $\Gamma_{\mathcal{F}_s}$ of the cocycle $\{F_{h_1}(V)(\tau) = A_{\tau,\alpha}(i_0, j_0, \gamma) \mid \tau \in \Gamma_{F_s}\}$ can be described as follows. Let $U \in R_0$ be such that

$$U - U^q = \gamma x_0^{r_1(i_0)-r_2(j_0)-1}.$$

Then for any $\tau \in \Gamma_{\mathcal{F}_s}$, $\sigma^{j_0}(A_{\tau,\alpha}(i_0, j_0, \gamma)) = \sigma^{i_0}(\alpha)(\tau U - U)$. Thus

$$A_{\tau,\alpha}(i_0, j_0, \gamma) = \sigma^{i_0-j_0}(\alpha)\sigma^{-j_0}(\tau U - U).$$

The following Lemma is a direct consequence of the definition of $(r_1, r_2)_{st}$-admissible pairs, cf. also Proposition 1.24

Lemma 2.12. *Let $C = -(q-1)(r_1(i_0) - r_2(j_0) - 1)$. Then C is a prime to p integer such that $1 \leqslant C < (q-1)(1 + 1/(p-1))$.*

2.7. Galois modules $E_{sp}(j_0, \gamma)$. In this subsection $(0, j_0)$ is some $(r_1, r_2)_{sp}$-admissible pair (i.e. $r_1 + 1/(p-1) = r_0(j_0)$) and $\gamma \in \mathbb{F}_q$. Then $V = \mathcal{V}^*(E_{sp}(j_0, \gamma))$ is identified as an abelian group with the solutions

$$(\{X_i^{(1)} \mid i \in \mathbb{Z}/s\}, \{X_j^{(2)} \mid j \in \mathbb{Z}/s\}) \in R^{2s}$$

of the following system of equations

$$\begin{array}{ccc} X_i^{(1)p}/x_0^{pa_i} & = & X_{i+1}^{(1)}, \text{for all } i \in \mathbb{Z}/s, \\ X_j^{(2)p}/x_0^{pb_j} & = & X_{j+1}^{(2)}, \text{for all } j \in \mathbb{Z}/s. \end{array}$$

The corresponding Γ_F-action comes from the natural Γ_F-action on $\mathcal{R}_{st}^0$ and the embedding of V into $(R_{st}^0)^{2s}$ given by the following correspondences:

– if $i \in \mathbb{Z}/s$ then $X_i^{(1)} \mapsto X_i^{(1)} \bmod x_0^p \mathrm{m}_R$;

– if $m \in \mathbb{Z}/s$ then $X_{j_0+m}^{(2)} \mapsto X_{j_0+m}^{(2)} + \gamma^{p^m} X_m^{(1)} Y \bmod x_0^p \mathrm{m}_R$.

If $\alpha \in \mathbb{F}_q$ and $h_1 = \alpha h_1^0 \in V_1$ then the cocycle

$$\{F_{h_1}(V)(\tau) = A_{\tau,\alpha}^{sp}(j_0, \gamma) \mid \tau \in \Gamma_{F_s}\}$$

can be described as follows. Note that the point h_1 corresponds to the collection $(\{\sigma^i(\alpha) x_0^{pr_1(i)} | i \in \mathbb{Z}/s\}, \{\sigma^{i-j_0}(\alpha\gamma) x_0^{pr_1(i-j_0)} Y | i \in \mathbb{Z}/s\})$. Then for $\tau \in \Gamma_{F_s}$, $\tau(h_1)$ corresponds to the collection

$$(\{\sigma^i(\alpha) x_0^{pr_1(i)} \mid i \in \mathbb{Z}/s\}, \{\sigma^{i-j_0}(\alpha\gamma) x_0^{pr_1(i-j_0)} (Y + k(\tau)\widetilde{\log}\varepsilon) \mid i \in \mathbb{Z}/s\}).$$

Therefore, $\tau(h_1) - h_1$ corresponds to the collection

$$(\{\ 0 \mid i \in \mathbb{Z}/s\}, \{\sigma^{i-j_0}(\alpha\gamma) x_0^{pr_2(i)} k(\tau) \mid i \in \mathbb{Z}/s\}),$$

which corresponds to $\sigma^{-j_0}(\alpha\gamma) h_2^0$. Therefore, $A_{\tau,\alpha}^{sp}(j_0, \gamma) = \sigma^{-j_0}(\alpha\gamma) k(\tau)$. Notice that for any $\tau \in \Gamma_{\mathcal{F}_s} \subset \Gamma_{F_s}$, $A_{\tau,\alpha}^{sp}(j_0, \gamma) = 0$.

2.8. Fully faithfulness of $\mathcal{CV}^*$.
In this subsection we prove the following important property.

Proposition 2.13. *The functor $\mathcal{CV}^*$ is fully faithful.*

Proof. We must prove that for all $\mathcal{L}_1, \mathcal{L}_2 \in \underline{\mathcal{L}}^*$, the functor $\mathcal{CV}^*$ induces a bijective map

$$\Pi(\mathcal{L}_1, \mathcal{L}_2) : \mathrm{Hom}_{\underline{\mathcal{L}}^*}(\mathcal{L}_2, \mathcal{L}_1) \longrightarrow \mathrm{Hom}_{\underline{\mathrm{CM\Gamma}}_F}(\mathcal{CV}^*(\mathcal{L}_1), \mathcal{CV}^*(\mathcal{L}_2)).$$

By induction on lengths of composition series for $\mathcal{L}_1$ and $\mathcal{L}_2$ it will be sufficient to verify that for any two simple objects $\mathcal{L}_1$ and $\mathcal{L}_2$:

- $\Pi(\mathcal{L}_1, \mathcal{L}_2)$ is bijective;
- the functor $\mathcal{CV}^*$ induces injective map

$$\mathrm{E}\Pi\,(\mathcal{L}_1, \mathcal{L}_2) : \mathrm{Ext}_{\underline{\mathcal{L}}^*}(\mathcal{L}_2, \mathcal{L}_1) \longrightarrow \mathrm{Ext}_{\underline{\mathrm{CM\Gamma}}_F}(\mathcal{CV}^*(\mathcal{L}_1), \mathcal{CV}^*(\mathcal{L}_2)).$$

The first fact has been already checked in Subsection 2.3.
In order to verify the second property, notice that for any two objects $\mathcal{L}_1, \mathcal{L}_2 \in \underline{\mathcal{L}}^*$, the natural map

$$\mathrm{Ext}_{\underline{\mathrm{CM\Gamma}}_F}(\mathcal{CV}^*(\mathcal{L}_1), \mathcal{CV}^*(\mathcal{L}_2)) \longrightarrow \mathrm{Ext}_{\underline{\mathrm{M\Gamma}}_F}(\mathcal{V}^*(\mathcal{L}_1)), \mathcal{V}^*(\mathcal{L}_2))$$

is injective. Therefore, we can prove injectivity of $\mathrm{E}\Pi(\mathcal{L}_1, \mathcal{L}_2)$ on the level of functor $\mathcal{V}^*$. In addition, for $n_1, n_2 \in \mathbb{N}$, $\mathrm{Ext}_{\underline{\mathcal{L}}^*}(\mathcal{L}_2^{n_1}, \mathcal{L}_1^{n_2}) = \mathrm{Ext}_{\underline{\mathcal{L}}^*}(\mathcal{L}_2, \mathcal{L}_1)^{n_1 n_2}$ (the formation of Ext is compatible with direct sums). So, by Lemma 1.17, we can replace $\mathcal{L}_1$ and $\mathcal{L}_2$ by the objects introduced in Subsection 1.5 (where they are denoted also by $\mathcal{L}_1$ and $\mathcal{L}_2$).
By Proposition 1.27, any element of $\mathrm{Ext}_{\underline{\mathcal{L}}^*}(\mathcal{L}_2, \mathcal{L}_1)$ appears as a sum of standard extensions of the form $E_{cr}(i, j, \gamma_{ij})$, $E_{st}(i, j, \gamma_{ij})$ and $E_{sp}(j, \gamma_j^{sp})$. Here: a) $(i, j) \in (\mathbb{Z}/s)^2$ is either $(r_1, r_2)_{cr}$-admissible or $(r_1, r_2)_{st}$-admissible and all $\gamma_{ij} \in k$; b) $j \in \mathbb{Z}/s$ is such that $(0, j)$ is $(r_1, r_2)_{sp}$-admissible and $\gamma_j^{sp} \in \mathbb{F}_q$.

Remark. A couple (i, j) can't be both $(r_1, r_2)_{cr}$-admissible and $(r_1, r_2)_{st}$-admissible, but it can be $(r_1, r_2)_{cr}$-admissible and $(r_1, r_2)_{sp}$-admissible at the same time.

By Subsections 2.5-2.7, we can attach to these standard extensions the 1-cocycles $A_{\tau,\alpha}(i, j, \gamma_{ij})$ and $A^{sp}_{\tau,\alpha}(j, \gamma_j^{sp})$, where $\tau \in \Gamma_{F_s}$. It remains to prove that the sum of these cocycles is trivial only if all corresponding coefficients γ_{ij} and γ_j^{sp} are equal to 0.
First, we need the following lemma.

Lemma 2.14. *Suppose for all $(i, j) \in (\mathbb{Z}/s)^2$, the elements $U_{ij} \in R_0 = \mathrm{Frac}\, R$ are such that $U_{ij} - U_{ij}^q = \gamma_{ij} x_s^{-C_{ij}}$, where all $\gamma_{ij} \in k$ and all C_{ij} are prime to p natural numbers. For $\tau \in \Gamma_{\mathcal{F}_s}$, let $B_\tau(i, j, \gamma_{ij}) = \tau(U_{ij}) - U_{ij} \in \mathbb{F}_q$. If for all $\alpha \in \mathbb{F}_q$ and all $\tau \in \Gamma_{\mathcal{F}_s}$,*

$$\sum_{i,j,\in\mathbb{Z}/s} \sigma^{i-j}(\alpha)\sigma^{-j} B_\tau(i, j, \gamma_{ij}) = 0 \tag{2.2}$$

then all $\gamma_{ij}=0$.

Proof of Lemma. For different prime to p natural numbers C_{ij} the extensions $\mathcal{F}_s(U_{ij})$ behave independently. Therefore, we can assume that all $C_{ij}=C$ are the same.
Let $j_0=j_0(j)$ be such that $0\leqslant j_0<s$ and $j_0\equiv -j \bmod s$. Then (2.2) means that for any $\alpha\in\mathbb{F}_q$,

$$B_\alpha:=\sum_{i,j\in\mathbb{Z}/s}\sigma^{i-j}(\alpha)\sigma^{j_0}(U_{ij})\in\mathcal{F}_s.$$

Then

$$B_\alpha-B_\alpha^q=\sum_{j\in\mathbb{Z}/s}\left(\sum_{i\in\mathbb{Z}/s}\sigma^{i-j}(\alpha)\gamma_{ij}^{p^{-j}}\right)x_s^{-p^{j_0}C}.$$

Looking at the Laurent series of $B_\alpha\in\mathcal{F}_s$ we conclude that all $B_\alpha\in\mathbb{F}_q$. This means that for all $j\in\mathbb{Z}/s$ and $\alpha\in\mathbb{F}_q$, $\sum_{i\in\mathbb{Z}/s}\sigma^i(\alpha)\gamma_{ij}=0$ and, therefore, all $\gamma_{ij}=0$. The lemma is proved $\square$

Now suppose that for all $\alpha\in\mathbb{F}_q$ and $\tau\in\Gamma_{F_s}$, the sum of cocycles $A_{\tau,\alpha}(i,j,\gamma_{ij})$ and $A^{sp}_{\tau,\alpha}(j,\gamma_j^{sp})$ is zero. Restrict this sum to the subgroup $\Gamma_{\mathcal{F}_s}$. Then all *sp*-terms will disappear and by above Lemma 2.14 all $\gamma_{ij}=0$. So, for all $\tau\in\Gamma_{F_s}$ and $\alpha\in\mathbb{F}_q$, $\sum_{j\in\mathbb{Z}/s}\sigma^{-j}(\alpha\gamma_j^{sp})=0$, and this implies that all $\gamma_j^{sp}=0$. $\square$

COROLLARY 2.15. *The functor $\mathcal{V}^*$ is fully faithful on the subcategories of unipotent objects $\underline{\mathcal{L}}^{*u}$ and of connected objects $\underline{\mathcal{L}}^{*c}$.*

Proof. Indeed, on both categories the map $\Pi(\mathcal{L}_1,\mathcal{L}_2)$ is already bijective on the level of functor $\mathcal{V}^*$. $\square$

2.9. RAMIFICATION ESTIMATES. Suppose $\mathcal{L}\in\underline{\mathcal{L}}^*$ and $H=\mathcal{V}^*(\mathcal{L})$. For any rational number $v\geqslant 0$, denote by $\Gamma_F^{(v)}$ the ramification subgroup of Γ_F in upper numbering, [22].

PROPOSITION 2.16. *If $v>2-\frac{1}{p}$ then $\Gamma_F^{(v)}$ acts trivially on H.*

A proof can be obtained along the lines of the paper [17] (which adjusts Fontaine's approach from [14]). Alternatively, one can apply author's method from [3]: if $\tau\in\Gamma^{(v)}$ with $v>2-1/p$ then there is an automorphism ψ of R such that $\psi(x_0)=\tau(x_0)$ and ψ induces the trivial action on H; therefore we can assume that τ comes from the absolute Galois group of $k((x_0))$ and the characteristic p approach from [3] gives the ramification estimate which coincides with the required by the theory of field-of-norms.

COROLLARY 2.17. *If $\widetilde{F}$ is the common field-of-definition of points of $\mathbb{F}_p[\Gamma_F]$-modules $\mathcal{V}(\mathcal{L})$ for all $\mathcal{L}\in\underline{\mathcal{L}}^*$, then $v_p(\mathcal{D}(\widetilde{F}/F))<3-\frac{1}{p}$, where $\mathcal{D}(\widetilde{F}/F)$ is the different of the field extension $\widetilde{F}/F$.*

3. Semistable representations with weights from $[0, p)$ and filtered $\mathcal{W}$-modules

3.1. The ring S. Let $v = u + p \in \mathcal{W}$ and let S be the p-adic closure of the divided power envelope of $\mathcal{W}$ with respect to the ideal generated by v. Use the same symbols σ and N for natural continuous extensions of σ and N from $\mathcal{W}$ to S. For $i \geqslant 0$, denote by $\mathrm{Fil}^i S$ the i-th divided power of the ideal (v) in S. Then for $0 \leqslant i < p$, there are σ-linear morphisms $\phi_i = \sigma/p^i : \mathrm{Fil}^i S \longrightarrow S$. Note that $\phi_0 = \sigma$ and agree to use the notation φ for ϕ_{p-1}. One can see also that S is the p-adic closure of $W(k)[v_0, v_1, \dots, v_n, \dots]$, where $v_0 = v$ and for all $n \geqslant 0$, $v_{n+1}^p/p = v_n$.

Consider the ideals $\mathrm{m}_S = (p, v, v_1, \dots, v_n, \dots)$, $I = (p, v_1, v_2, \dots)$ and $J = (p, v_1 v, v_2, \dots, v_n, \dots)$ of S. Then

— m_S is the maximal ideal in S;

— $I = \mathrm{Fil}^p S + pS \supset J$;

— $\varphi(I) \subset S$ and $\varphi(J) \subset pS$;

— $\varphi(v^{p-1}) \equiv 1 - v_1 (\mathrm{mod}\, J)$ and $\varphi(v_1) \equiv 1 (\mathrm{mod}\, J)$.

3.2. The ring of semi-stable periods $\hat{A}_{st}$. Let R be Fontaine's ring and let $x_0, \varepsilon \in R$ be the elements chosen in Subsection 2.1.

Denote by A_{cr} the Fontaine crystalline ring. It is the p-adic closure of the divided power envelope of $W(R)$ with respect to the ideal $([x_0] + p)$ of $W(R)$, where $[x_0] \in W(R)$ is the Teichmüller representative of x_0. Then for $i \geqslant 0$, $\mathrm{Fil}^i A_{cr}$ is the i-th divided power of the ideal $([x_0] + p)$ in A_{cr}. Denote by $\sigma : A_{cr} \longrightarrow A_{cr}$ the natural morphism induced by the p-th power on R. Then for $0 \leqslant i < p$, there are σ-linear maps $\phi_i = \sigma/p^i : \mathrm{Fil}^i A_{cr} \longrightarrow A_{cr}$. We shall often use the simpler notation $\varphi = \phi_{p-1}$ and $F(A_{cr}) = \mathrm{Fil}^{p-1} A_{cr}$. Notice that A_{cr} is provided with the natural continuous Γ_F-action.

Let X be an indeterminate. Then $\hat{A}_{\mathrm{st}}$ is the p-adic closure of the ring $A_{cr}[\gamma_i(X) \mid i \geqslant 0] \subset A_{cr}[X] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, where for all $i \geqslant 0$, $\gamma_i(X) = X^i/i!$. The ring $\hat{A}_{st}$ has the following additional structures:

• the S-module structure given by the natural $W(k)$-algebra structure and the correspondence $u \mapsto [x_0]/(1 + X)$;

• the ring endomorphism σ, which is the extension of the above defined endomorphism σ of A_{cr} via the condition $\sigma(X) = (1 + X)^p - 1$;

• the continuous A_{cr}-derivation $N : \hat{A}_{\mathrm{st}} \longrightarrow \hat{A}_{\mathrm{st}}$ such that $N(X) = X + 1$;

• for any $i \geqslant 0$, the ideal $\mathrm{Fil}^i \hat{A}_{\mathrm{st}}$, which is the closure of the ideal $\sum_{i_1 + i_2 \geqslant i} \left(\mathrm{Fil}^{i_1} A_{cr}\right) \gamma_{i_2}(X)$;

• the action of Γ_F, which is the extension of the Γ_F-action on A_{cr} such that for all $\tau \in \Gamma_F$, $\tau(X) = [\varepsilon]^{k(\tau)}(X + 1) - 1$. Here all $k(\tau) \in \mathbb{Z}_p$ are such that $\tau(x_0) = \varepsilon^{k(\tau)} x_0$.

Note that for $0 \leqslant m < p$, $\sigma(\mathrm{Fil}^m \hat{A}_{st}) \subset p^m \hat{A}_{st}$ and, as earlier, we can set $\phi_m = p^{-m} \sigma|_{\mathrm{Fil}^m \hat{A}_{st}}$ and introduce the simpler notation $\varphi = \phi_{p-1}$ and $F(\hat{A}_{st}) = \mathrm{Fil}^{p-1} \hat{A}_{st}$.

3.3. CONSTRUCTION OF SEMI-STABLE REPRESENTATIONS OF Γ_F WITH WEIGHTS FROM $[0,p)$. For $0 \leqslant m < p$, consider the category $\widetilde{\mathcal{S}}_m$ of quadruples $\mathcal{M} = (M, \mathrm{Fil}^m M, \phi_m, N)$, where $\mathrm{Fil}^m M \subset M$ are S-modules, $\phi_m : \mathrm{Fil}^m M \longrightarrow M$ is a σ-linear map and $N : M \longrightarrow M$ is a $W(k)$-linear endomorphism such that for any $s \in S$ and $m \in M$, $N(sx) = N(s)x + sN(x)$ The morphisms of the category $\widetilde{\mathcal{S}}_m$ are S-linear morphisms of filtered modules commuting with the corresponding morphisms ϕ_m and N. Notice that for $0 \leqslant m < p$, $\hat{A}_{\mathrm{st}}$ has a natural structure of the object of the category $\widetilde{\mathcal{S}}_m$. As earlier, we shall use the simpler notation $\varphi = \phi_{p-1}$ and $F(M) = \mathrm{Fil}^{p-1}M$.
For $0 \leqslant m < p$, the Breuil category $\mathcal{S}_m$ of strongly divisible S-modules of weight $\leqslant m$ is a full subcategory of $\widetilde{\mathcal{S}}_m$ consisting of the objects $\mathcal{M} = (M, \mathrm{Fil}^m M, \phi_m, N)$ such that

(1) M is a free S-module of finite rank;
(2) $(\mathrm{Fil}^m S)M \subset \mathrm{Fil}^m M$;
(3) $(\mathrm{Fil}^m M) \cap pM = p\mathrm{Fil}^m M$;
(4) $\phi_m(\mathrm{Fil}^m M)$ spans M over S;
(5) $N\phi_m = p\phi_m N$;
(6) $(\mathrm{Fil}^1 S)N(\mathrm{Fil}^m M) \subset \mathrm{Fil}^m M$.

For $\mathcal{M} \in \mathcal{S}_m$, let $T^*_{st}(\mathcal{M})$ be the Γ_F-module of all S-linear and commuting with ϕ_m and N, maps $f : M \longrightarrow \hat{A}_{\mathrm{st}}$ such that $f(\mathrm{Fil}^m M) \subset \mathrm{Fil}^m \hat{A}_{\mathrm{st}}$. Then one has the following two basic facts:

- $T^*_{\mathrm{st}}(\mathcal{M})$ is a continuous $\mathbb{Z}_p[\Gamma_F]$-module without p-torsion, its $\mathbb{Z}_p$-rank equals $\mathrm{rk}_S M$, and $V^*_{\mathrm{st}}(\mathcal{M}) = T^*_{\mathrm{st}}(\mathcal{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable Γ_F-module with Hodge-Tate weights from $[0,m]$;
- any semi-stable representation of Γ_F with Hodge-Tate weights from $[0,m]$, $0 \leqslant m < p$, appears in the form $V^*_{\mathrm{st}}(\mathcal{M})$ for a suitable $\mathcal{M} \in \mathcal{S}_m$.

By Theorem 1.3 [6] these facts follow from the existence of strongly divisible lattices in $S \otimes_{\mathcal{W}} F$-modules associated with weakly admissible (ϕ_0, N)-modules with filtration of length m. Breuil proved this for all $m \leqslant p-2$ but his method can be easily extended to cover the case $m = p-1$ as well, cf. also [7].

3.4. THE CATEGORY $\underline{\mathcal{L}}^f$. In this section we introduce $\mathcal{W}$-analogues of Breuil's S-modules from the category $\mathcal{S}_{p-1}$ and prove that they can be also used to construct semi-stable representations of Γ_F with Hodge-Tate weights from $[0,p)$.

DEFINITION. Let $\widetilde{\underline{\mathcal{L}}}$ be the category of $\mathcal{L} = (L, F(L), \varphi, N_S)$, where $L \supset F(L)$ are $\mathcal{W}$-modules, $\varphi : F(L) \longrightarrow L$ is a σ-linear morphism of $\mathcal{W}$-modules and $N_S : L \longrightarrow L_S := L \otimes_{\mathcal{W}} S$ is such that for all $w \in \mathcal{W}$ and $l \in L$, $N_S(wl) = N(w)l + (w \otimes 1)N_S(l)$. For $\mathcal{L}_1 = (L_1, F(L_1), \varphi, N_S) \in \widetilde{\underline{\mathcal{L}}}$, the morphisms $\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}}(\mathcal{L}, \mathcal{L}_1)$ are $\mathcal{W}$-linear $f : L \longrightarrow L_1$ such that $f(F(L)) \subset F(L_1)$, $f\varphi = \varphi f$ and $fN_S = N_S(f \otimes 1)$.

Let $\mathcal{A}_{st} = (\hat{A}_{st}, F(\hat{A}_{st}), \varphi, N_S)$, where $N_S = N \otimes 1$. Then $\mathcal{A}_{st}$ is an object of the category $\widetilde{\underline{\mathcal{L}}}$.

Suppose $\mathcal{L} = (L, F(L), \varphi, N_S) \in \widetilde{\underline{\mathcal{L}}}$.
Set $L_S := L \otimes_{\mathcal{W}} S$, $F(L_S) = (F(L) \otimes 1)S + (L \otimes 1)\mathrm{Fil}^p S$, and $\varphi_S : F(L_S) \longrightarrow F(L_S)$ is a unique σ-linear map such that $\varphi_S|_{F(L)\otimes 1} = \varphi \otimes 1$ and for any $s \in \mathrm{Fil}^p S$ and $l \in L$, $\varphi_S(l \otimes s) = (\varphi(v^{p-1}l) \otimes 1)\varphi(s)/\varphi(v^{p-1})$.

Definition. Denote by $\underline{\mathcal{L}}^f$ the full subcategory in $\widetilde{\underline{\mathcal{L}}}$ consisting of the quadruples $\mathcal{L} = (L, F(L), \varphi, N_S)$ such that
• L is a free $\mathcal{W}$-module of finite rank;
• $v^{p-1}L \subset F(L)$, $F(L) \cap pL = pF(L)$ and $L = \varphi(F(L)) \otimes_{\sigma\mathcal{W}} \mathcal{W}$;
• for any $l \in F(L)$, $vN_S(l) \in F(L_S)$ and $\varphi_S(vN(l)) = cN_S(\varphi(l))$, where $c = 1 + u^p/p$.

It can be easily seen that for $\mathcal{L} = (L, F(L), \varphi, N_S) \in \underline{\mathcal{L}}^f$ and the map $N = N_S \otimes 1 : L_S \longrightarrow L_S$, the quadruple $\mathcal{L}_S = (L_S, F(L_S), \varphi_S, N)$ is the object of the category $\mathcal{S}_{p-1}$
The main result of this Subsection is the following statement.

Proposition 3.1. *For any $\mathcal{M} = (M, F(M), \varphi, N) \in \mathcal{S}_{p-1}$, there is an $\mathcal{L} = (L, F(L), \varphi, N_S) \in \underline{\mathcal{L}}^f$ such that $\mathcal{M} = \mathcal{L}_S$.*

Corollary 3.2. *a) If $\mathcal{L} \in \underline{\mathcal{L}}^f$ and $T^*_{st}(\mathcal{L}) = \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}}(\mathcal{L}, \hat{A}_{st})$ with the induced structure of $\mathbb{Z}_p[\Gamma_F]$-module then $V^*_{st}(\mathcal{L}) = T^*_{st}(\mathcal{L}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a semi-stable $\mathbb{Q}_p[\Gamma_F]$-module with Hodge-Tate weights from $[0, p)$ and $\dim_{\mathbb{Q}_p} V^*_{st}(\mathcal{L}) = \mathrm{rk}_{\mathcal{W}} L$.*
b) For any semi-stable $\mathbb{Q}_p[\Gamma_F]$-module V^*_{st} with Hodge-Tate weights from $[0, p)$, there is an $\mathcal{L} \in \underline{\mathcal{L}}^f$ such that $V^*_{st} \simeq V^*_{st}(\mathcal{L})$.

Proof of Proposition 3.1. Let d be a rank of M over S. If $L \subset M$ is a free $\mathcal{W}$-submodule of rank d and M is generated by the elements of L over S we say that L is $\mathcal{W}$-structural (with respect to M).
Let $F(L) = F(M) \cap L$.

Lemma 3.3. *If L is $\mathcal{W}$-structural for M then*

a) *$F(L) \supset v^{p-1}L$;*

b) *$F(L) \cap pL = pF(L)$;*

c) *$F(L)$ is a free $\mathcal{W}$-module of rank d.*

Proof. a) $v^{p-1}L \subset (\mathrm{Fil}^{p-1}S)M \cap L \subset F(M) \cap L = F(L)$.

b) $F(L) \cap pL = L \cap F(M) \cap pL = F(M) \cap pL = F(M) \cap pM \cap pL = pF(M) \cap pL = pF(L)$.

c) $F(L)$ has no p-torsion. Therefore, it will be sufficient to prove that $F(L)/pF(L)$ is a free $k[[u]]$-module of rank d. Consider the following natural embeddings of $k[[v]]$-modules

$$L/pL \supset F(L)/pF(L) \supset v^{p-1}L/pv^{p-1}L \simeq L/pL$$

(Use b) and that $pL \cap v^{p-1}L = pv^{p-1}L$.) It remains to note that L/pL is free of rank d over $k[[v]]$.
The Lemma is proved. □

Suppose L is $\mathcal{W}$-structural for M.

Lemma 3.4. *If L is $\mathcal{W}$-structural then $\varphi(F(L))$ spans M over S.*

Proof. The equality $S = \mathcal{W} + \mathrm{Fil}^p S$ implies that $M = L + (\mathrm{Fil}^p S)L = L + (\mathrm{Fil}^p S)M$. Therefore,

$$F(M) = F(M) \cap L + (\mathrm{Fil}^p S)M = F(L) + (\mathrm{Fil}^p S)L$$

(use that $F(M) \supset (\mathrm{Fil}^p S)M$) and in notation of Subsection 3.1 one has

$$F(M) = F(L) + v_1 L + JM.$$

This implies that $\varphi(F(L))$, $\varphi(v_1 L)$ and $\varphi(JM)$ span M over S. But for any $l \in L$, $\varphi(v_1 l) = \varphi(v_1)\varphi(v^{p-1}l)/\varphi(v^{p-1}) = (1-v_1)^{-1}\varphi(v^{p-1}l) \equiv \varphi(v^{p-1}l) \bmod \mathrm{m}_S M$. For similar reasons, $\varphi(JM) \subset pM \subset \mathrm{m}_S M$. This means that $\varphi(F(L))$ spans M modulo $\mathrm{m}_S M$. The lemma is proved. □

By above lemma it remains to prove the existence of a $\mathcal{W}$-structural L for M such that $\varphi(F(L)) \subset L$.
Let ϕ_0 be a σ-linear endomorphism of the S-module $M \in \mathcal{S}_{p-1}$ such that for all $m \in M$, $\phi_0(m) = \varphi(v^{p-1}m)/\varphi(v^{p-1})$. Clearly, $\phi_0(\mathrm{m}_S M) \subset \mathrm{m}_S M$ and, therefore, it induces a σ-linear endomorphism σ_0 of the k-vector space $M_k = M/\mathrm{m}_S M$.

Lemma 3.5. *Suppose $n \in \mathbb{Z}_{\geqslant 0}$, L is $\mathcal{W}$-structural and $\varphi(F(L)) \subset L + p^n M$. Then there is a $\mathcal{W}$-structural L' for M such that $\varphi(F(L')) \subset L' + p^n JM$.*

Proof. Denote by $F(L)_k$ the image of $F(L)$ in the k-vector space $M/\mathrm{m}_S M = L/(\mathrm{m}_S \cap \mathcal{W})L = L_k$. Let $s = \dim_k F(L)_k$, then $s \leqslant d = \dim_k L_k$. Choose a $\mathcal{W}$-basis $e^{(1)}, \ldots, e^{(d)}$ of L and a $\mathcal{W}$-basis $f^{(1)}, \ldots, f^{(d)}$ of $F(L)$ such that

- for $1 \leqslant i \leqslant s$, $f^{(i)} = e^{(i)}$ and for $s < i \leqslant d$, $f^{(i)} \in vL$.

It will be convenient to use the following vector notation: $\bar{e} = (\bar{e}_1, \bar{e}_2)$, where $\bar{e}_1 = (e^{(1)}, \ldots, e^{(s)})$ and $\bar{e}_2 = (e^{(s+1)}, \ldots, e^{(d)})$, and $\bar{f} = (\bar{f}_1, \bar{f}_2)$, where $\bar{f}_1 = \bar{e}_1$ and $\bar{f}_2 = (f^{(s+1)}, \ldots, f^{(d)})$.
Then in obvious notation one has $(\varphi(\bar{f}_1), \varphi(\bar{f}_2)) = (\bar{e}_1, \bar{e}_2)C$, where $C \in \mathrm{GL}_d(S)$. Clearly, $C \equiv C_0 + p^n v_1 C_1 \bmod p^n J$ with $C_0 \in \mathrm{GL}_d(\mathcal{W})$ and $C_1 \in \mathrm{M}_d(\mathcal{W})$. Clearly, $\varphi(F(L)) \subset L + p^n JM$ iff $C_1 \equiv 0 \bmod \mathrm{m}_S$. Choose $\bar{g} = (\bar{g}_1, \bar{g}_2) \in L^d$ and set

$$\bar{e}'_1 = (e'^{(1)}, \ldots, e'^{(s)}) = \bar{e}_1 + p^n(v_1 - v^{p-1})\bar{g}_1$$
$$\bar{e}'_2 = (e'^{(s+1)}, \ldots, e'^{(d)}) = \bar{e}_2 + p^n(v_1 - v^{p-1})\bar{g}_2$$

Clearly, the coordinates of $\bar{e}' = (\bar{e}'_1, \bar{e}'_2)$ give an S-basis of M and we can introduce the structural $\mathcal{W}$-module $L' = \sum_i \mathcal{W}e'^{(i)}$ for M.

Prove that the elements $e'^{(i)}$, $1\leqslant i\leqslant s$, and $f^{(i)}, s<i\leqslant d$, generate $F(L')\operatorname{mod} p^nJM$. Indeed, we have

$$L+p^nIM=L'+p^nIM \tag{3.1}$$

and this implies that the image $F(L)_k$ of $F(L)$ in L_k coincides with its analogue $F(L')_k$. In addition, for $1\leqslant i\leqslant s$,

$$e'^{(i)}\in L'\cap(F(L)+p^nIM)\subset L'\cap F(M)=F(L').$$

Therefore, it would be sufficient to prove that $(vL')\cap F(L')\operatorname{mod} p^nJM$ is generated by the images of $ve'^{(i)}$, $1\leqslant i\leqslant s$, and $f^{(s+1)},\dots,f^{(d)}$. But relation (3.1) implies that $vL+p^nJM=vL'+p^nJM$ and

$$(vL')\cap F(L')\operatorname{mod} p^nJM=(vL)\cap F(L)\operatorname{mod} p^nJM.$$

It remains to note that for $1\leqslant i\leqslant s$, $ve'^{(i)}\equiv ve^{(i)}\operatorname{mod} p^nJM$.
Therefore, we can define special bases for L' and $F(L')$ by the relations $\bar f_1'=\bar e_1'$ and $\bar f_2'=\bar f_2$ and obtain that

$$(\varphi(\bar f_1'),\varphi(\bar f_2'))=(\varphi(\bar f_1),\varphi(\bar f_2))+p^nv_1(\sigma_0\bar g_1,\bar 0)\operatorname{mod} p^nJM$$

and

$$\begin{aligned}(\varphi(\bar f_1'),\varphi(\bar f_2'))&\equiv(\bar e_1',\bar e_2')C_0+p^nv^{p-1}(\bar g_1,\bar g_2)C_0+\\&+p^nv_1((\bar e_1,\bar e_2)C_1-(\bar g_1,\bar g_2)C_0+(\sigma\bar g_1,\bar 0))\operatorname{mod} p^nJM\end{aligned}$$

So, $\varphi(F(L'))\subset L'+p^nJM$ if and only if there is an $\bar g=(\bar g_1,\bar g_2)\in L^d$ such that $(\sigma_0\bar g_1,\bar 0)\equiv(\bar g_1,\bar g_2)C_0+\bar h\operatorname{mod}(\mathrm{m}_S\cap\mathcal W)L$, where $\bar h=(\bar e_1,\bar e_2)C_1\in L$ and $C_0\operatorname{mod}\mathrm{m}_S\in\mathrm{GL}_d(k)$. The existence of such vector $\bar g$ is implied by Lemma 3.6 below. □

Lemma 3.6. *Suppose V is a d-dimensional vector space over k with a σ-linear endomorphism $\sigma_0:V\longrightarrow V$ and $\bar a=(\bar a_1,\bar a_2)\in V^d$, where $\bar a_1\in V^s$ and $\bar a_2\in V^{d-s}$. Then for any $C\in\mathrm{GL}_d(k)$ there is an $\bar g=(\bar g_1,\bar g_2)\in V^d$ with $\bar g_1\in V^s$ and $\bar g_2\in V^{d-s}$ such that*

$$(\sigma_0\bar g_1,\bar 0)=\bar gC+\bar a. \tag{3.2}$$

Proof. Let $C^{-1}=\begin{pmatrix}D_{11}&D_{12}\\D_{21}&D_{22}\end{pmatrix}$ with the block matrices of sizes $s\times s$, $(d-s)\times s$, $s\times(d-s)$ and $(d-s)\times(d-s)$. Then the equality (3.2) can be rewritten as

$$\begin{array}{rcl}(\sigma_0\bar g_1)D_{11}&=&\bar g_1+\bar a_1'\\(\sigma_0\bar g_1)D_{21}&=&\bar g_2+\bar a_2'\end{array}$$

where $(\bar a_1',\bar a_2')=\bar aC^{-1}$. Clearly, it will be sufficient to solve the first equation in $\bar g_1$, but this is a special case of Lemma 1.1. □

Lemma 3.7. *Suppose $n\geqslant 0$ and L is $\mathcal W$-structural for M such that $\varphi(F(L))\subset L+p^nJM$. Then there is a $\mathcal W$-structural L' for M such that $\varphi(F(L'))\subset L'+p^{n+1}M$.*

Proof. Suppose the coordinates of $\bar{e} \in M^d$ form a $\mathcal{W}$-basis of L and $D \in \mathcal{M}_d(\mathcal{W})$ is such that the coordinates of $\bar{f} = \bar{e}D$ form a $\mathcal{W}$-basis of $F(L)$. Then $\varphi(\bar{f}) = \bar{e} + p^n\bar{h}$, where $\bar{h} \equiv \bar{0} \bmod JM$. Let $\bar{e}' = \bar{e} + p^n\bar{h}$ and let L' be a $\mathcal{W}$-submodule in M spanned by the coordinates of $\bar{e}'$. Clearly, L' is $\mathcal{W}$-structural. Prove that $F(L')$ is spanned by the coordinates of $\bar{e}'D$. Indeed, suppose $\bar{e}$ and $\bar{e}'$ have the coordinates $e^{(i)}$ and, resp., $e'^{(i)}$, $1 \leqslant i \leqslant s$. Then for all i, $e'^{(i)} = e^{(i)} + p^n h^{(i)}$, where $h^{(i)} \in JM \subset (\mathrm{Fil}^p S)M$. This means that a $\mathcal{W}$-linear combination of $e^{(i)}$ belongs to $F(M)$ if and only if the same linear combination of $e'^{(i)}$ belongs to $F(M)$. This implies that $\bar{e}'D$ spans $F(L')$ over $\mathcal{W}$ because $\bar{e}D$ spans $F(L)$ over $\mathcal{W}$. Then $\varphi(F(L')) \subset L' + p^{n+1}M$ because $\varphi(\bar{h}) \in pM$ (use that $\varphi(J) \subset pS$) and

$$\varphi(\bar{e}'D) = \varphi(\bar{e}D + p^n\bar{h}D) = \bar{e} + p^n\bar{h} + p^n\varphi(\bar{h})\sigma(D) \equiv \bar{e}' \mathrm{mod} p^{n+1}M$$

□

It remains to notice that applying above Lemmas 3.6 and 3.7 one after another we shall obtain a sequence of $\mathcal{W}$-structural modules L_n such that for all $n \geqslant 0$, $L_n + p^{n+1}M = L_{n+1} + p^{n+1}M$, where $L_0 \otimes_{\mathcal{W}} S = M$. Therefore, $L = \varprojlim_n L_n/p^n$ is $\mathcal{W}$-structural and $\varphi(L) \subset L$.
The proposition is completely proved. □

3.5. The categories $\underline{\mathcal{L}}^t$ and $\underline{\mathcal{L}}^{ft}$.

Definition. $\mathcal{W}$-module L is p-strict if it is isomorphic to $\oplus_{1\leqslant i\leqslant s}\mathcal{W}/p^{n_i}$, where $n_1, \dots, n_s \in \mathbb{N}$.

In particular, if L is p-strict and $pL = 0$ then L is a free $\mathcal{W}_1$-module. The p-strict modules can be efficiently studied via devissage due to the following property.

Lemma 3.8. *L is p-strict if and only if pL and L/pL are p-strict.*

Proof. Specify Breuil's proof of a similar statement but for more complicated ring $S = \mathcal{W}^{DP}$ from [6]. □

Definition. Denote by $\underline{\mathcal{L}}^t$ the full subcategory in $\widetilde{\underline{\mathcal{L}}}$ consisting of the quadruples $\mathcal{L} = (L, F(L), \varphi, N_S)$ such that
- L is p-strict;
- $v^{p-1}L \subset F(L)$, $F(L) \cap pL = pF(L)$ and $L = \varphi(F(L)) \otimes_{\sigma\mathcal{W}} \mathcal{W}$;
- for any $l \in F(L)$, $vN_S(l) \in F(L_S)$ and $\varphi_S(vN_S(l)) = cN_S(\varphi(l))$, where $c = 1 + u^p/p$.

Definition. Denote by $\underline{\mathcal{L}}^t[1]$ the full subcategory in $\underline{\mathcal{L}}^t$, which consists of objects killed by p.

The category $\underline{\mathcal{L}}^t[1]$ is not very far from the category $\underline{\mathcal{L}}^*$ introduced in Section 1. Indeed, suppose $\mathcal{L} = (L, F(L), \varphi, N_S) \in \underline{\mathcal{L}}^t[1]$. Note that $N_S(L) \subset L_{S_1} := L \otimes_{\mathcal{W}_1} S_1 = L/u^pL \oplus (L \otimes 1)\mathrm{Fil}^p S_1$. (Remind that $S_1 = S/pS = \mathcal{W}_1/u^p\mathcal{W}_1 \oplus \mathrm{Fil}^p S_1$.) With this notation we have the following property.

Proposition 3.9. *There is a unique* $N : L \longrightarrow L/u^{2p}$ *such that*
a) for any $l \in L$, $N(l) \otimes 1 = cN_S(l)$ *in* L_{S_1}, *where* $c = 1 + u^p/p \in S^*$;
b) $(L, F(L), \varphi, N) \in \underline{\mathcal{L}}^*$.

Proof. Let $N_1 := cN_S : L \longrightarrow L_{S_1}$. Then for any $w \in \mathcal{W}_1$ and $l \in L$, one has $N_1(wl) = N(w)l + wN_1(l)$ (use that $N(c) = 0$ in S_1) and there is a commutative diagram (use that $\sigma(c) = 1$ in S_1)

$$\begin{tikzcd} F(L) \arrow[r, "\varphi"] \arrow[d, "uN_1"'] & L \arrow[d, "N_1"] \\ F(L)_S \arrow[r, "\varphi_S"] & L_{S_1} \end{tikzcd}$$

Prove that $N_1(\varphi(F(L)) \subset L/u^pL$ and, therefore, $N_1(L) \subset L/u^pL$.
Indeed, $(uN_1)(F(L)) \subset uN_1(L) \cap F(L)_S \subset (uL/u^pL \oplus (uL)\mathrm{Fil}^pS_1) \cap (F(L)/u^pL \oplus L\mathrm{Fil}^pS_1) \subset F(L)/u^pL \oplus (uL)\mathrm{Fil}^pS_1$. This implies that $N_1(\varphi(F(L)) \subset \varphi_S(uN_1(F(L))) \subset L/u^pL$ because $\varphi_S(u\mathrm{Fil}^pS_1) = 0$. So, by Proposition 1.3 there is a unique $N : L \longrightarrow L/u^{2p}$ such that $N \mathrm{mod}\, u^p = N_1$ and $(L, F(L), \varphi, N) \in \underline{\mathcal{L}}^*$. □

Corollary 3.10. *With above notation the correspondence*

$$(L, F(L), \varphi, N_S) \mapsto (L, F(L), \varphi, N)$$

induces the equivalence of categories $\Pi : \underline{\mathcal{L}}^t[1] \longrightarrow \underline{\mathcal{L}}^*$.

Proof. We must verify that our correspondence is surjective on objects and bijective on morphisms. The first holds because $N_S = c^{-1}N \bmod u^p$ and the second — because a $\mathcal{W}_1$-linear map f commutes with N iff it commutes with $N \bmod u^p$ (use Proposition 1.2) iff $f \otimes_{\mathcal{W}_1} S_1$ commutes with N_S. □

Corollary 3.11. *The category* $\underline{\mathcal{L}}^t$ *is preabelian.*

Proof. Corollary 3.10 and Proposition 1.3 imply that $\underline{\mathcal{L}}^t[1]$ is pre-abelian. This can be extended then to the whole category $\underline{\mathcal{L}}^t$ by Breuil's method from [6] via above Lemma 3.8. □

Note that if $\mathcal{L} = (L, F(L), \varphi, N_S)$ and $\mathcal{M} = (M, F(M), \varphi, N_S)$ are objects of $\underline{\mathcal{L}}^t$ and $f \in \mathrm{Hom}_{\underline{\mathcal{L}}}(\mathcal{L}, \mathcal{M})$ then:

- $\mathrm{Ker} f = (K, F(K), \varphi, N_S)$, where $K = \mathrm{Ker}(f : L \longrightarrow M)$ and $F(K) = F(L) \cap K$ with induced φ and N_S;

- $\mathrm{Coker} f = (C, F(C), \varphi, N_S)$, where $C = M/M'$, M' is equal to $(f(L) \otimes_{\mathcal{W}} \mathcal{W}[u^{-1}]) \cap M$ and $F(C) = F(M)/(M' \cap F(M))$ with induced φ and N_S;

- f is strict monomorphic means that $f : L \longrightarrow M$ is monomorphism of $\mathcal{W}$-modules, $\big(f(L) \otimes_{\mathcal{W}} \mathcal{W}[u^{-1}]\big) \cap M = f(L)$ (or, equivalently, $M/f(L)$ is p-strict) and $f(F(L)) = L \cap F(M)$;

- f is strict epimorphic means that f is epimorphism of p-strict modules and $f(F(L)) = F(M)$.

According to Appendix A, we can use the concept of p-divisible group $\{\mathcal{L}^{(n)}, i_n\}_{n\geqslant 0}$ in $\underline{\mathcal{L}}^t$. In this case $\mathcal{L}^{(n)} = (L_n, F(L_n), \varphi, N_S)$, where all L_n are free $\mathcal{W}/p^n$-modules of the same rank equal to the height of this p-divisible group. We have obvious equivalence of the category $\underline{\mathcal{L}}^f$ and the category of p-divisible groups of finite height in $\underline{\mathcal{L}}^t$.

Definition. Denote by $\underline{\mathcal{L}}^{ft}$ the full subcategory in $\underline{\mathcal{L}}^t$, which consists of strict subobjects of p-divisible groups in $\underline{\mathcal{L}}^t$. By $\underline{\mathcal{L}}^{ft}[1]$ we denote the full subcategory in $\underline{\mathcal{L}}^{ft}$ consisting of all objects killed by p.

It is easy to see that $\underline{\mathcal{L}}^{ft}$ contains all strict subquotients of the corresponding p-divisible groups. Contrary to the case of filtered modules coming from crystalline representations, the categories $\underline{\mathcal{L}}^{ft}$ and $\underline{\mathcal{L}}^t$ do not coincide but they have the same simple objects.
Note that the functor Π from Corollary 3.10 identifies simple objects of the categories $\underline{\mathcal{L}}^t$ and $\underline{\mathcal{L}}^*$ and for any two objects $\mathcal{L}_1, \mathcal{L}_2 \in \underline{\mathcal{L}}^t[1]$, we have a natural isomorphism $\mathrm{Ext}_{\underline{\mathcal{L}}^t[1]}(\mathcal{L}_1, \mathcal{L}_2) = \mathrm{Ext}_{\underline{\mathcal{L}}^*}(\Pi(\mathcal{L}_1), \Pi(\mathcal{L}_2))$. One can use the methods of Subsection 1.2 to extend the concepts of etale, connected, unipotent and multiplicative objects to the whole category $\underline{\mathcal{L}}^t$. The starting point for this extension is the case of $W(k)$-modules, which is well-known from the classical Dieudonne theory [10]. Then we obtain the following standard properties:

• for any $\mathcal{L} \in \underline{\mathcal{L}}^t$, there are a unique maximal etale subobject $(\mathcal{L}^{et}, i^{et})$ and a unique maximal connected quotient object $(\mathcal{L}^c, j^c)$ in $\underline{\mathcal{L}}^t$ such that the sequence $0 \longrightarrow \mathcal{L}^{et} \xrightarrow{i^{et}} \mathcal{L} \xrightarrow{j^c} \mathcal{L}^c \longrightarrow 0$ is exact and the correspondences $\mathcal{L} \mapsto \mathcal{L}^{et}$ and $\mathcal{L} \mapsto \mathcal{L}^c$ are functorial; if $\mathcal{L} \in \underline{\mathcal{L}}^{ft}$ then $\mathcal{L}^{et}$ and $\mathcal{L}^c$ are also objects of $\underline{\mathcal{L}}^{ft}$;

• for any $\mathcal{L} \in \underline{\mathcal{L}}^t$, there are a unique maximal unipotent subobject $(\mathcal{L}^u, i^u)$ and a unique maximal multiplicative quotient object $(\mathcal{L}^m, j^m)$ in $\underline{\mathcal{L}}^t$ such that the sequence $0 \longrightarrow \mathcal{L}^u \xrightarrow{i^u} \mathcal{L} \xrightarrow{j^m} \mathcal{L}^m \longrightarrow 0$ is exact and the correspondences $\mathcal{L} \mapsto \mathcal{L}^u$ and $\mathcal{L} \mapsto \mathcal{L}^m$ are functorial; if $\mathcal{L} \in \underline{\mathcal{L}}^{ft}$ then $\mathcal{L}^u$ and $\mathcal{L}^u$ are also objects of $\underline{\mathcal{L}}^{ft}$.

Denote by $\underline{\mathcal{L}}^{et,t}$, $\underline{\mathcal{L}}^{c,t}$, $\underline{\mathcal{L}}^{u,t}$ and $\underline{\mathcal{L}}^{m,t}$ the full subcategories in $\underline{\mathcal{L}}^t$ consisting of, resp., etale, connected, unipotent and multiplicative objects. We have also the corresponding full subcategories $\underline{\mathcal{L}}^{et,ft}$, $\underline{\mathcal{L}}^{c,ft}$, $\underline{\mathcal{L}}^{u,ft}$ and $\underline{\mathcal{L}}^{m,ft}$ in $\underline{\mathcal{L}}^{ft}$.
The results of Subsection 1.5 and Appendix A imply that in the category $\underline{\mathcal{L}}^{ft}$:

• there is a unique etale p-divisible group $\mathcal{L}^{\infty}(0) := \{\mathcal{L}^{(n)}(0), i_n\}_{n\geqslant 0}$ of height 1 such that $\mathcal{L}^{(1)}(0) = \mathcal{L}(0)$;

• there is a unique multiplicative p-divisible group of height 1, $\mathcal{L}^{\infty}(1) := \{\mathcal{L}^{(n)}(1), i_n\}_{n\geqslant 0}$ such that $\mathcal{L}^{(1)}(1) = \mathcal{L}(1)$;

• for any p-divisible group $\mathcal{L}^{\infty}$ there are functorial exact sequences of p-divisible groups

$$0 \longrightarrow \mathcal{L}^{\infty,et} \longrightarrow \mathcal{L}^{\infty} \longrightarrow \mathcal{L}^{\infty,c} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{L}^{\infty,u} \longrightarrow \mathcal{L}^{\infty} \longrightarrow \mathcal{L}^{\infty,m} \longrightarrow 0$$

Here $\mathcal{L}^{\infty,et}$ and $\mathcal{L}^{\infty,m}$ are products of several copies of $\mathcal{L}^{\infty}(0)$ and, resp., $\mathcal{L}^{\infty}(1)$, and $\mathcal{L}^{\infty,c}$ and $\mathcal{L}^{\infty,u}$ are p-divisible groups in the categories $\underline{\mathcal{L}}^{c,ft}$ and, resp., $\underline{\mathcal{L}}^{u,ft}$.

4. Semistable modular representations with weights $[0,p)$

In this section we prove that all killed by p subquotients of Galois invariant lattices of semistable $\mathbb{Q}_p[\Gamma_F]$-modules with Hodge-Tate weights $[0,p)$ can be obtained via the functor $\mathcal{V}^*$ from Section 2.

4.1. The functor $\mathcal{V}^t : \underline{\mathcal{L}}^t \longrightarrow \underline{\mathrm{M}\Gamma}_F$. For $n \geqslant 1$, introduce the objects $\mathcal{A}_{st,n} = (\hat{A}_{st,n}, F(\hat{A}_{st.n}), \varphi, N_S)$ of the category $\widetilde{\underline{\mathcal{L}}}$, with $\hat{A}_{st,n} = \hat{A}_{st}/p^n\hat{A}_{st}$, $F(\hat{A}_{st,n}) = F(\hat{A}_{st})/p^nF(\hat{A}_{st})$ and induced φ and N_S. Let $\mathcal{A}_{st,\infty} = (A_{st,\infty}, F(A_{st,\infty}), \varphi, N_S)$ be the inductive limit of all $\mathcal{A}_{st,n}$.
For $\mathcal{L} \in \underline{\mathcal{L}}^t$, set $\mathcal{V}^t(\mathcal{L}) = \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}}(\mathcal{L}, \mathcal{A}_{st,\infty})$ with the induced structure of Γ_F-module. This gives the functor $\mathcal{V}^t : \underline{\mathcal{L}}^t \longrightarrow \underline{\mathrm{M}\Gamma}_F$. We shall use the same notation for its restriction to the category $\underline{\mathcal{L}}^{ft}$.

Proposition 4.1. *Suppose $\mathcal{L} = (L, F(L), \varphi, N_S) \in \underline{\mathcal{L}}^t$. Then $N_S|_{\varphi(F(L))}$ is nilpotent.*

By devissage and Corollary 3.10 this is implied by the following statement for the objects of the category $\underline{\mathcal{L}}^*$.

Lemma 4.2. *If $\mathcal{L} = (L, F(L), \varphi, N) \in \underline{\mathcal{L}}^*$ then $N^p(\varphi(F(L)) \subset u^pL$.*

Proof. For any $l \in F(L)$, $N(\varphi(l)) = \varphi(uN(l))$. Use induction to prove that for $1 \leqslant m \leqslant p$, $N^m(\varphi(l)) \equiv \varphi(u^mN^m(l)) \bmod u^pL$ and use then that $\varphi(u^pN^p(l)) \in \varphi(uF(L)) \subset u^pL$. □

Proposition 4.3. *For $n \geqslant 1$, $\oplus_{j\geqslant 0}A_{cr,n}\gamma_j(\log(1+X))$ is the maximal $W(k)$-submodule of $\hat{A}_{st,n}$ where N is nilpotent.*

Proof. For any $j \geqslant 1$, one has $N(\gamma_j(\log(1+X)) = \gamma_{j-1}(\log(1+X))$ and N is nilpotent on $\oplus_{j\geqslant 0}A_{cr,n}\gamma_j(\log(1+X))$. Therefore, it will be sufficient to prove that

$$\mathrm{Ker}\left(N^p|_{\hat{A}_{st,1}}\right) = \underset{0\leqslant j<p}{\oplus} A_{cr,1}\gamma_j(\log(1+X)).$$

Let $C = \mathbb{F}_p\langle X\rangle$ be the divided power envelope of $\mathbb{F}_p[X]$ with respect to the ideal (X). Then $C = \mathbb{F}_p[X_0, X_1, \dots, X_n, \dots]_{<p}$ is the ring of polynomials in $X_i := \gamma_{p^i}(X)$, where for all $i \geqslant 0$, $X_i^p = 0$.
Let m_C be the maximal ideal of C and $Y = \log(1+X) \in C$. Then $Y \equiv X_0 - X_1 \bmod \mathrm{m}_C^2$ and for all $j \geqslant 0$, $\gamma_{p^j}(Y) \equiv X_j - X_{j+1} \bmod \mathrm{m}_C^2$. This implies that with $Y_j = \gamma_j(Y)$ for all $j \geqslant 0$,

$$C = \mathbb{F}_p[X_0, Y_0, \dots, Y_n, \dots]_{<p} = \mathbb{F}_p\langle Y\rangle[X]_{<p} = \underset{0\leqslant i<p}{\oplus} \mathbb{F}_p\langle Y\rangle\gamma_i(X).$$

So, $\hat{A}_{st,1} = \underset{\substack{j\geqslant 0\\ 0\leqslant i<p}}{\oplus} A_{cr,1}\gamma_i(X)\gamma_j(Y)$. Remind $N(X) = X+1$ and for $j\geqslant 1$, $N(\gamma_j(Y)) = \gamma_{j-1}(Y)$. Using that N^p is an $A_{cr,1}$-derivation, $N^p(X) = X+1$ and $N^p(\gamma_{j+p}(Y)) = \gamma_j(Y)$, we obtain that for any $P = \sum_{i,j}\alpha_{ij}X^i\gamma_j(Y) \in \mathbb{F}_p\langle Y\rangle[X]_{<p}$ with $\alpha_{ij}\in\mathbb{F}_p$,

$$N^p(P) = \sum_{i,j}\alpha_{ij}iX^i\gamma_j(Y) + \sum_{i,j}(i+1)\alpha_{i+1,j}\gamma_j(Y)X^i + \sum_{i,j}\alpha_{i,j+p}X^i\gamma_j(Y).$$

If $P\in \mathrm{Ker}N^p$ then for all involved indices i,j,

$$i\alpha_{ij} + (i+1)\alpha_{i+1,j} + \alpha_{i,j+p} = 0.$$

This implies that $\alpha_{ij}=0$ if either $i\neq 0$ or $j\geqslant p$.
Indeed, take $i=p-1$. Then $-\alpha_{p-1,j}+\alpha_{p-1,j+p}=0$. Because for $j\gg 0$, $\alpha_{p-1,j}=0$ it implies that all $\alpha_{p-1,j}=0$. Then proceed similarly with $i=p-2$ and so on. This proves that all $\alpha_{ij}=0$ if $i\neq 0$. It remains to note that for $i=0$, our relations give $\alpha_{0,j+p}=0$ for all $j\geqslant 0$. □

As earlier, consider the category $\widetilde{\underline{\mathcal{L}}}_0$. Remind that its objects are the triples $(L, F(L), \varphi)$, where $L\supset F(L)$ are $\mathcal{W}$-modules and $\varphi : F(L)\longrightarrow L$ is a σ-linear morphism. For any object $\mathcal{L} = (L, F(L), \varphi, N_S)\in\widetilde{\underline{\mathcal{L}}}$, agree to use the same notation $\mathcal{L}$ for the corresponding object $(L, F(L), \varphi)\in\widetilde{\underline{\mathcal{L}}}_0$.
For all $n\geqslant 0$, set $\mathcal{A}_{cr,n} = (A_{cr,n}, F(A_{cr,n}), \varphi)\in\widetilde{\underline{\mathcal{L}}}_0$ with $A_{cr,n} = A_{cr}/p^nA_{cr}$, $F(A_{cr,n}) = F(A_{cr})/p^nF(A_{cr})$ and induced φ. Here the $\mathcal{W}$-module structure on $A_{cr,n}$ is defined by the morphism of $W(k)$-algebras $\mathcal{W}\longrightarrow A_{cr,n}$ such that $u\mapsto [x_0]$. Denote by $\mathcal{A}_{cr,\infty}$ the inductive limit of all $\mathcal{A}_{cr,n}$.
Suppose $\mathcal{L}\in\underline{\mathcal{L}}^t$ and $f\in\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}}(\mathcal{L},\mathcal{A}_{st,n})$. Then by Propositions 4.1 and 4.3,

$$f(\varphi(F(L)))\subset \underset{j\geqslant 0}{\oplus} A_{cr,n}\gamma_j(\log(1+X)).$$

Consider the formal embedding of the algebra $A_{st,n}$ into the completion $\prod_{j\geqslant 0}A_{cr,n}\gamma_j(\log(1+X))$ of $\oplus_{j\geqslant 0}A_{cr,n}\gamma_j(\log(1+X))$ such that $X\mapsto \sum_{j\geqslant 1}\gamma_j(\log(1+X))$. Then any element of $A_{st,n}$ can be uniquely written in the form $\sum_{j\geqslant 0}a_j\gamma_j(\log(1+X))$, where all $a_j\in A_{cr,n}$. Note that the $\mathcal{W}$-module structure on $A_{st,n}$ is given via the map

$$u\mapsto [x_0]/(1+X) = [x_0]\sum_{j\geqslant 0}(-1)^j\gamma_j(\log(1+X)).$$

For $j\geqslant 0$, introduce the $W(k)$-linear maps $f_j\in\mathrm{Hom}(L, A_{cr,n})$ such that for any $l\in L$, one has $f(l) = \sum_{j\geqslant 0}f_j(l)\gamma_j(\log(1+X))$. Then using methods from [6] obtain the following property.

Proposition 4.4. *a) The correspondence $f\mapsto f_0$ induces isomorphism of abelian groups $\mathcal{V}^t(\mathcal{L}) = \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0}(\mathcal{L},\mathcal{A}_{cr,n})$;*
b) for any $j\geqslant 0$ and $l\in L$, $f_j(l) = f_0(N^j(l))$.

Corollary 4.5. *The functor $\mathcal{V}^t$ is exact.*

Proof. Let $\underline{\mathcal{L}}_0^t$ be the full subcategory of $\widetilde{\underline{\mathcal{L}}}_0$ consisting of the triples $(L, F(L), \varphi)$ coming from all $\mathcal{L} = (L, F(L), \varphi, N) \in \underline{\mathcal{L}}^t$. By Proposition 4.4 it will be sufficient to prove that the functor $\mathcal{V}_0^t : \underline{\mathcal{L}}_0^t \longrightarrow (Ab)$, such that $\mathcal{V}_0^t(\mathcal{L}) = \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0}(\mathcal{L}, \mathcal{A}_{cr,\infty})$, is exact. The verification can be done by devissage along the lines of paper [13]. $\square$

Remark. One can simplify the verification of above corollary by replacing $\mathcal{A}_{cr,1}$ by the corresponding object $\widetilde{\mathcal{A}}_{cr,1}$ related to the module $\widetilde{A}_{cr,1} = (R/x_0^p)T_1 \oplus (R/x_0^p)$ introduced in Subsection 4.2 below.

Corollary 4.6. *For $\mathcal{L} \in \underline{\mathcal{L}}^f$, let $\{\mathcal{L}^{(n)}, i_n\}_{n\geqslant 0}$ be the corresponding p-divisible group in the category $\underline{\mathcal{L}}^{ft}$. Then in notation of Corollary 3.2, $T_{st}^*(\mathcal{L}) = \varprojlim_n \mathcal{V}^t(\mathcal{L}^{(n)})$.*

4.2. The functor $\mathcal{V}[1]^*$. Note the following case of Proposition 4.4.

Proposition 4.7. *Suppose $\mathcal{L} = (L, F(L), \varphi, N) \in \underline{\mathcal{L}}^t[1]$. Then there is an isomorphism of abelian groups $\mathcal{V}^t(\mathcal{L}) \simeq \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0}(\mathcal{L}, \mathcal{A}_{cr,1})$. In addition, Γ_F acts on $\mathcal{V}^t(\mathcal{L})$ via its natural action on $A_{st,1}$ and the identification $\iota_{\mathcal{L}} : \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0}(\mathcal{L}, \mathcal{A}_{cr,1}) \longrightarrow \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}}(\mathcal{L}, \mathcal{A}_{st,1})$ such that if $f_0 \in \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0}(\mathcal{L}, \mathcal{A}_{cr,1})$ then for any $l \in L$,*

$$\iota_{\mathcal{L}}(f_0)(l) = \sum_{j\geqslant 0} f_0(N^j(l))\gamma_j(\log(1+X))$$

Introduce the functor $\mathcal{V}[1]^* := \mathcal{V}^t|_{\underline{\mathcal{L}}^t[1]} \circ \Pi^{-1} : \underline{\mathcal{L}}^* \longrightarrow \underline{\mathrm{M}\Gamma}_F$, where $\Pi : \underline{\mathcal{L}}^t[1] \longrightarrow \underline{\mathcal{L}}^*$ is the equivalence of categories from Corollary 3.10.

Proposition 4.8. *On the subcategory of unipotent objects $\underline{\mathcal{L}}^{*u}$ of $\underline{\mathcal{L}}^*$ the functors $\mathcal{V}[1]^*$ and $\mathcal{V}^*$ coincide.*

Proof. The definition of A_{cr} implies that $A_{cr,1} = (R/x_0^p)[T_1, T_2, \dots]_{<p}$, where for all indices $i \geqslant 1$, T_i comes from $\gamma_{p^i}([x_0]+p)$ and $T_i^p = 0$. Set $F(A_{cr,1}) = \mathrm{Fil}^{p-1}A_{cr,1} = (x_0^{p-1}R/x_0^pR) \oplus (R/x_0^p)I_1$, where the ideal I_1 is generated by all T_i. Then the corresponding map $\varphi : F(A_{cr,1}) \longrightarrow A_{cr,1}$ is uniquely determined by the conditions $\varphi(x_0^{p-1}) = 1 - T_1$, $\varphi(T_1) = 1$ and $\varphi(T_i) = 0$ if $i \geqslant 2$. In particular, $\varphi(A_{cr,1}) \subset (R/x_0^p)T_1 \oplus (R/x_0^p)$.
Let $\widetilde{A}_{cr,1} = A_{cr,1}/J_1$ with the induced structure of filtered φ-module $\widetilde{\mathcal{A}}_{cr,1}$, where the ideal J_1 of $A_{cr,1}$ is generated by the elements $T_1x_0^p$ and T_i with $i \geqslant 2$. Then the projection $A_{cr,1} \longrightarrow \widetilde{A}_{cr,1}$ induces for any object $\mathcal{L} = (L, F(L), \varphi, N)$ of the category $\underline{\mathcal{L}}^*$, the identification (use that $\varphi|_J = 0$)

$$\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0}(\mathcal{L}, \mathcal{A}_{cr,1}) = \mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0}(\mathcal{L}, \widetilde{\mathcal{A}}_{cr,1}).$$

Introduce $a_0, a_{-1} \in \mathrm{Hom}(L, R/x_0^p)$ such that for any $m \in L$, $f_0(m) = a_{-1}(m)T_1 + a_0(m)$. Note that a_0 and a_{-1} are $\mathcal{W}_1$-linear, where the multiplication by u on L correspondes to the multiplication by x_0 in R/x_0^p.
Then for any $m \in F(L)$, the requirement $f_0(\varphi(m)) = \varphi(f_0(m))$ is equivalent to the conditions

$$(4.1)\qquad \begin{aligned} a_0(\varphi(m)) &= a_{-1}(m)^p + \frac{a_0(m)^p}{x_0^{p(p-1)}} \\ a_{-1}(\varphi(m)) &= -\frac{a_0(m)^p}{x_0^{p(p-1)}} \end{aligned}$$

Note that these conditions depend only on $\bar{m} = m \bmod u^pL$.

Consider the operator $V : L \longrightarrow L$ from Subsection 1.5. Clearly, $V(u^pL) \subset uF(L)$ and for $\bar{L} := L/u^pL$, we obtain the induced operator $\bar{V} : \bar{L} \longrightarrow \bar{L}$ (use that $F(L)/uF(L) \subset L/u^pL$).
For any $m \in \bar{L}$, relations (4.1) can be rewritten as follows:

$$\begin{aligned} a_0(m) &= \frac{a_0(\bar{V}m)^p}{x_0^{p(p-1)}} + a_{-1}(\bar{V}m)^p \\ a_{-1}(m) &= -\frac{a_0(\bar{V}m)^p}{x_0^{p(p-1)}} \end{aligned}$$

Therefore, if $\mathcal{L}$ is unipotent then for any $m \in \bar{L}$,

$$a_{-1}(m) = -a_0(m) + a_{-1}(\bar{V}m)^p = -a_0(m) + a_{-1}(\bar{V}^2m)^{p^2} = \cdots = -a_0(m).$$

This implies that for any $m \in F(\bar{L})$, $a_0(\varphi(m)) = a_0(m)^p/x_0^{p(p-1)}$. In other words, we have a natural identification

$$\mathrm{Hom}_{\underline{\widetilde{\mathcal{L}}}_0}(\mathcal{L}, \widetilde{\mathcal{R}}^u) = \mathrm{Hom}_{\underline{\widetilde{\mathcal{L}}}_0}(\mathcal{L}, \widetilde{A}_{cr,1})$$

coming from the map of filtered φ-modules $\widetilde{\mathcal{R}}^u \longrightarrow \widetilde{\mathcal{A}}_{cr,1}$ given by the R-linear map $R/x_0^p \longrightarrow \widetilde{A}_{cr,1} = (R/x_0^p)T_1 \oplus (R/x_0^p)$ such that for any $r \in R/x_0^p$, $r \mapsto (-rT_1, r)$. (For the definition of $\widetilde{\mathcal{R}} \in \underline{\mathcal{L}}_0^*$ cf. Subsection 2.2.)
This implies that for all unipotent $\mathcal{L} \in \underline{\mathcal{L}}^{*u}$, there is a natural identification of Γ_F-modules $\mathcal{V}[1]^*(\mathcal{L}) = \mathcal{V}^*(\mathcal{L})$. Indeed, the above embedding $R/x_0^p \longrightarrow \widetilde{A}_{cr,1}$ can be extended to the embedding of $R_{st}/x_0^pR_{st}$ to

$$\widetilde{A}_{st,1} = \prod_{j\geqslant 0} \widetilde{A}_{cr,1}\gamma_j(\log(1+X)),$$

which induces the above identification. $\square$

4.3. Splittings Θ and $\widetilde{\Theta}$. Suppose $\mathcal{L} = (L, F(L), \varphi, N) \in \underline{\mathcal{L}}^*$. Then there is a standard short exact sequence

$$(4.2)\qquad 0 \longrightarrow \mathcal{L}^u \xrightarrow{i} \mathcal{L} \xrightarrow{j} \mathcal{L}^m \longrightarrow 0,$$

where $(\mathcal{L}^u, i)$ is the maximal unipotent subobject and $(\mathcal{L}^m, j)$ is the maximal multiplicative quotient of $\mathcal{L}$.
If $\mathcal{L}^m = (L^m, F(L^m), \varphi, N)$ then $F(L^m) = L^m = L_0 \otimes_{\mathbb{F}_p} \mathcal{W}_1$, where $L_0 = \{l \in L^m \mid \varphi(l) = l\}$. Suppose $S : L^m \longrightarrow F(L) \subset L$ is a $\mathcal{W}_1$-linear section. Then for any $l_0 \in L_0$, $S(l_0) = \varphi(S(l_0)) + g(l_0)$, where $g \in \mathrm{Hom}(L_0, L^u)$. If $S' : L^m \longrightarrow$

$F(L)$ is another $\mathcal{W}_1$-linear section then for any $l_0 \in L_0$, $S'(l_0) = \varphi(S'(l_0)) + g'(l_0)$. Here $g' \in \mathrm{Hom}(L_0, L^u)$ is such that for some $h \in \mathrm{Hom}(L_0, L^u)$, one has

$$(g' - g)(l_0) = h(l_0) - \varphi(h(l_0)).$$

Proposition 4.9. *a) There is a section S such that $g(L_0) \subset uL^u$.*
b) If $g(L_0), g'(L_0) \subset uL^u$ then $h(L_0) \subset uF(L^u)$.

Proof. a) It will be sufficient to prove that for any $l \in L^u$, there is an $h \in F(L^u)$ such that $l \equiv h - \varphi(h) \bmod uL^u$.
Suppose $n_0 \geqslant 1$ is such that $V^{n_0}(L^u) \subset uF(L^u)$. Then for all $n \geqslant n_0$, $V^n(L^u) \subset uF(L^u)$. Let $h = -(Vl + V^2 l + \dots + V^{n_0+1} l)$. By the definition of the operator V for all $1 \leqslant i \leqslant n_0 + 1$, $V^i l \in F(L^u)$ and $\varphi(V^i l) \equiv V^{i-1} l \bmod uL^u$. Therefore, $h \in F(L^u)$ and $\varphi(h) \equiv -(l + Vl + \dots + V^{n_0} l) \equiv -l + h \bmod uL^u$.
b) We must prove that if $h \in F(L^u)$ and $h - \varphi(h) \in uL^u$ then $h \in uF(L^u)$.
Indeed, we have $V(h) - h \in V(uL^u) \subset uF(L^u)$ and for all $n \geqslant 1$, $V^n(h) \equiv h \bmod uF(L^u)$ implies that $h \in uL^u$. Therefore, $\varphi(h) \in uL^u$ and $h \in uF(L^u)$. $\square$

Proposition 4.10. *With above notation the short exact sequence*

$$0 \longrightarrow \mathcal{V}[1]^*(\mathcal{L}^m) \longrightarrow \mathcal{V}[1]^*(\mathcal{L}) \longrightarrow \mathcal{V}[1]^*(\mathcal{L}^u) \longrightarrow 0$$

obtained from (4.2) *by applying $\mathcal{V}[1]^*$, has a canonical functorial splittings $\Theta : \mathcal{V}[1]^*(\mathcal{L}^u) \longrightarrow \mathcal{V}[1]^*(\mathcal{L})$ and $\widetilde{\Theta} : \mathcal{V}[1]^*(\mathcal{L}) \longrightarrow \mathcal{V}[1])^*(\mathcal{L}^m)$ in the category $\underline{\mathrm{M}\Gamma}_F$.*

Proof. It will be sufficient to prove the existence of a functorial splitting

$$\Theta : \mathrm{Hom}_{\underline{\widetilde{\mathcal{L}}}_0}(\mathcal{L}^u, \widetilde{\mathcal{A}}_{cr,1}) \longrightarrow \mathrm{Hom}_{\underline{\widetilde{\mathcal{L}}}_0}(\mathcal{L}, \widetilde{\mathcal{A}}_{cr,1})$$

of the epimorphism $\mathrm{Hom}_{\underline{\widetilde{\mathcal{L}}}_0}(\mathcal{L}, \widetilde{\mathcal{A}}_{cr,1}) \to \mathrm{Hom}_{\underline{\widetilde{\mathcal{L}}}_0}(\mathcal{L}^u, \widetilde{\mathcal{A}}_{cr,1})$, obtained from exact sequence (4.2).
Suppose $f_0 = (a_{-1}, a_0) : L^u \longrightarrow (R/x_0^p)T_1 \oplus (R/x_0^p)$ belongs to $\mathrm{Hom}_{\underline{\widetilde{\mathcal{L}}}_0}(\mathcal{L}^u, \widetilde{\mathcal{A}}_{cr,1})$. Here $a_{-1}, a_0 \in \mathrm{Hom}_{\mathcal{W}_1}(L^u, R/x_0^p)$ and for any $l \in L^u$, $a_{-1}(l) = -a_0(l)$, cf. Subsection 4.2.
Let $S : L^m \longrightarrow L$ be a $\mathcal{W}_1$-linear section such that for any $l \in L_0$, $S(l_0) = \varphi(S(l_0)) + g(l_0)$, where $g \in \mathrm{Hom}(L_0, uL^u)$.
Extend f_0 to $\Theta(f_0) = (a_{-1}, a_0) : L \longrightarrow (R/x_0^p)T_1 \oplus (R/x_0^p)$ by setting $a_0(S(l_0)) = -a_{-1}(S(l_0)) = \mathcal{X}$, where $\mathcal{X}$ is a unique element of R/x_0^p such that $\mathcal{X} - \mathcal{X}^p/x_0^{p(p-1)} = a_0(g(l_0))$. One can prove that $\Theta(f_0) \in \mathrm{Hom}_{\underline{\widetilde{\mathcal{L}}}_0}(\mathcal{L}, \widetilde{\mathcal{A}}_{cr,1})$ by verifying relations (4.1) with $m = S(l_0)$. $\square$

4.4. A modification of Breuil's functor. Remind that Breuil's functor $\mathcal{V}^t : \underline{\mathcal{L}}^t \longrightarrow \underline{\mathrm{M}\Gamma}_F$ attaches to any $\mathcal{L} \in \underline{\mathcal{L}}^t$, the Γ_F-module $\mathcal{V}(\mathcal{L}) = \mathrm{Hom}_{\underline{\widetilde{\mathcal{L}}}}(\mathcal{L}, \mathcal{A}_{st,\infty})$.

Proposition 4.11. *The functor $\mathcal{V}^t$ is fully faithful on the subcategory of unipotent objects $\underline{\mathcal{L}}^{t,u}$.*

Proof. Indeed, by Subsection 2.3, $\mathcal{V}[1]^*$ is fully faithful. Then the exactitude of $\mathcal{V}^t$ together with Proposition 4.8 implies that $\mathcal{V}^t|_{\underline{\mathcal{L}}^{u,t}}$ is fully faithful. $\square$

Proposition 4.10 implies that $\mathcal{V}^t$ is very far from to be fully faithful on the whole $\underline{\mathcal{L}}^t$: if $\mathcal{L}\in\underline{\mathcal{L}}^t[1]$ and $0\longrightarrow\mathcal{L}^u\longrightarrow\mathcal{L}\longrightarrow\mathcal{L}^m\longrightarrow 0$ is the standard exact sequence then the corresponding sequence of Γ_F-modules admits a functorial splitting.
Introduce a modification $\widetilde{\mathcal{V}}^{ft}:\underline{\mathcal{L}}^{ft}\longrightarrow\underline{\mathrm{M}\Gamma}_F$ of Breuil's functor.
Suppose $\mathcal{L}\in\underline{\mathcal{L}}^{ft}$. From the definition of the category $\underline{\mathcal{L}}^{ft}$ in Subsection 3 it follows the existence of $\mathcal{L}'\in\underline{\mathcal{L}}^{ft}$ such that $p\mathcal{L}'=\mathcal{L}$. More precisely, there are a strict monomorphism $i_{\mathcal{L}'}:\mathcal{L}\longrightarrow\mathcal{L}'$ and a strict epimorphism $j_{\mathcal{L}'}:\mathcal{L}'\longrightarrow\mathcal{L}$ such that $p\,\mathrm{id}_{\mathcal{L}'}=i_{\mathcal{L}'}\circ j_{\mathcal{L}'}$. (Note that $j_{\mathcal{L}'}\circ i_{\mathcal{L}'}=p\,\mathrm{id}_{\mathcal{L}}$.)
Consider the following short exact sequences

$$0\longrightarrow\mathcal{L}\xrightarrow{i_{\mathcal{L}'}}\mathcal{L}'\xrightarrow{C_p}{}_p\mathcal{L}'\longrightarrow 0 \tag{4.3}$$

$$0\longrightarrow\mathcal{L}'_p\xrightarrow{K_p}\mathcal{L}'\xrightarrow{j_{\mathcal{L}'}}\mathcal{L}\longrightarrow 0 \tag{4.4}$$

and consider the corresponding sequence of Γ_F-modules and their morphisms

$$\mathcal{V}^t({}_p\mathcal{L}'^u)\xrightarrow{\Theta}\mathcal{V}^t({}_p\mathcal{L}')\xrightarrow{\mathcal{V}^t(C_p)}\mathcal{V}^t(\mathcal{L}')\xrightarrow{\mathcal{V}^t(K_p)}\mathcal{V}^t(\mathcal{L}'_p)\xrightarrow{\widetilde{\Theta}}\mathcal{V}^t(\mathcal{L}'^m_p).$$

As earlier, for any $\mathcal{L}\in\underline{\mathcal{L}}^{ft}$, $\mathcal{L}^u$ is the maximal unipotent subobject and $\mathcal{L}^m$ is the maximal multiplicative quotient object for $\mathcal{L}$.

Lemma 4.12. $\mathrm{Ker}(\widetilde{\Theta}\circ\mathcal{V}^t(K_p))\supset\mathrm{Im}(\mathcal{V}^t(C_p)\circ\Theta)$.

Proof. The section Θ depends functorially on objects of the category $\underline{\mathcal{L}}^t[1]\supset\underline{\mathcal{L}}^{ft}[1]$. Therefore, we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{V}^t({}_p\mathcal{L}'^u) \arrow[r, "\mathcal{V}^t(C_p^u\circ K_p^u)"] \arrow[d, "\Theta"] & \mathcal{V}^t(\mathcal{L}'^u_p) \arrow[d, "\Theta"] \\
\mathcal{V}^t({}_p\mathcal{L}') \arrow[r, "\mathcal{V}^t(C_p\circ K_p)"] & \mathcal{V}^t(\mathcal{L}'_p)
\end{tikzcd}$$

and $\widetilde{\Theta}\circ\mathcal{V}^t(K_p)\circ\mathcal{V}^t(C_p)\circ\Theta=(\widetilde{\Theta}\circ\Theta)\circ\mathcal{V}^t(C_p^u\circ K_p^u)=0.$ $\square$

Definition. Set $\mathcal{V}^t_{\mathcal{L}'}(\mathcal{L})=\mathrm{Ker}(\widetilde{\Theta}\circ\mathcal{V}^t(K_p))/\mathrm{Im}\,(\mathcal{V}^t(C_p)\circ\Theta)$.

Proposition 4.13. *With above notation one has:*
a) $\mathcal{V}^t_{\mathcal{L}'}(\mathcal{L})=\mathrm{Coker}\,\mathcal{V}^t(C_p)=\mathcal{V}^t(\mathcal{L})$ *if* $\mathcal{L}\in\underline{\mathcal{L}}^{u,ft}$*;*
b) $\mathcal{V}^t_{\mathcal{L}'}(\mathcal{L})=\mathrm{Ker}\,\mathcal{V}^t(K_p)=\mathcal{V}^t(\mathcal{L})$ *if* $\mathcal{L}\in\underline{\mathcal{L}}^{m,ft}$*;*
c) for any $\mathcal{L}\in\underline{\mathcal{L}}^{ft}$*, we have the induced exact sequence of* Γ_F*-modules* $0\longrightarrow\mathcal{V}^t(\mathcal{L}^m)\longrightarrow\mathcal{V}^t_{\mathcal{L}'}(\mathcal{L})\longrightarrow\mathcal{V}^t(\mathcal{L}^u)\longrightarrow 0$. *This sequence depends functorially on the pair* $(\mathcal{L},\mathcal{L}')$.

Proof. The parts a) and b) are obtained directly from definitions. In order to prove c), note that $p\mathcal{L}'=\mathcal{L}$ implies that $p\mathcal{L}'^u=\mathcal{L}^u$ and $p\mathcal{L}'^m=\mathcal{L}^m$. This gives a functorial sequence

$$0\longrightarrow \mathcal{V}^t_{\mathcal{L}'^m}(\mathcal{L}^m)\longrightarrow \mathcal{V}^t_{\mathcal{L}'}(\mathcal{L})\longrightarrow \mathcal{V}^t_{\mathcal{L}'^u}(\mathcal{L}^u)\longrightarrow 0.$$

Then standard diagram chasing proves that this sequence is exact. □

Proposition 4.14. *Suppose for a given $\mathcal{L}\in\underline{\mathcal{L}}^{ft}$, the objects $\mathcal{L}',\mathcal{L}''\in\underline{\mathcal{L}}^{ft}$ are such that $p\mathcal{L}'=p\mathcal{L}''=\mathcal{L}$. Then there is a natural isomorphism $f(\mathcal{L}',\mathcal{L}'')$ of Γ_F-modules such that the following diagram is commutative*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{V}^t(\mathcal{L}^m) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathcal{V}^t_{\mathcal{L}''}(\mathcal{L}) \arrow[r] \arrow[d, "f(\mathcal{L}',\mathcal{L}'')"] & \mathcal{V}^t(\mathcal{L}^u) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathcal{V}^t(\mathcal{L}^m) \arrow[r] & \mathcal{V}^t_{\mathcal{L}'}(\mathcal{L}) \arrow[r] & \mathcal{V}^t(\mathcal{L}^u) \arrow[r] & 0
\end{tikzcd}$$

(The lines of this diagram are given by Prop 4.13)

Proof. By replacing $\mathcal{L}''$ by $\mathcal{L}'\prod_{\mathcal{L}}\mathcal{L}''$ with respect to strict epimorphisms $j_{\mathcal{L}'}$ and $j_{\mathcal{L}''}$, we can assume that there is a map $f:\mathcal{L}''\longrightarrow\mathcal{L}'$ which induces the identity map $p\mathcal{L}''=\mathcal{L}\longrightarrow p\mathcal{L}'=\mathcal{L}$. Then the existence of $f(\mathcal{L}',\mathcal{L}'')$ follows from functoriality and diagram chasing implies that it induces the identity maps on $\mathcal{V}^t(\mathcal{L}^u)$ and $\mathcal{V}^t(\mathcal{L}^m)$. □

Definition. For $\mathcal{L},\mathcal{L}'\in\underline{\mathcal{L}}^{ft}$ such that $p\mathcal{L}'=\mathcal{L}$, set $\widetilde{\mathcal{V}}^{ft}(\mathcal{L})=\mathcal{V}^t_{\mathcal{L}'}(\mathcal{L})$.

The correspondence $\mathcal{L}\longrightarrow\widetilde{\mathcal{V}}^{ft}(\mathcal{L})$ induces the additive exact functor $\widetilde{\mathcal{V}}^{ft}:\underline{\mathcal{L}}^{ft}\longrightarrow\underline{\mathrm{M}\Gamma}_F$.

4.5. φ-filtered module $\widetilde{\mathcal{A}}_{cr,2}\in\widetilde{\underline{\mathcal{L}}}_0$. Let $\xi=[x_0]+p\in W(R)\subset A_{cr}$, and for $n\in\mathbb{N}$, $\gamma_n(\xi)=\xi^n/n!$

Lemma 4.15. *If $n\geqslant 2p$ then $\varphi(\gamma_n(\xi))\in p^2A_{cr}$.*

Proof. We have $\varphi(\gamma_n(\xi))=(p^{n-p+1}/n!)([x_0]^p/p+1)^n$. Therefore, it will be sufficient to verify that for $n\geqslant 2p$, $v_p(n!)+p+1\leqslant n$. Using the estimate $v_p(n!)<n/(p-1)$ we obtain that the required inequality holds for $p\geqslant 5$ if $n\geqslant p+3$ and for $p=3$ if $n\geqslant 8$. It remains to check that our inequality holds for $p=3$ and $n\in\{6,7\}$. □

Let J_2 be the closed ideal in A_{cr} generated by $[x_0]^p\xi^p/p$ and all $\xi^n/n!$ with $n\geqslant 2p$. Then $J_2\subset F(A_{cr})$ and $\varphi(J_2)\subset p^2A_{cr}$. Introduce $\widetilde{A}_{cr,2}=A_{cr}/(J_2+p^2A_{cr})$ and consider the corresponding induced filtered φ-module $\widetilde{\mathcal{A}}_{cr,2}=(\widetilde{A}_{cr,2},F(\widetilde{A}_{cr,2}),\varphi)\in\widetilde{\underline{\mathcal{L}}}_0$. Clearly, for any $\mathcal{L}\in\underline{\mathcal{L}}_0^t$, the natural projection $\mathcal{A}_{cr,2}\to\widetilde{\mathcal{A}}_{cr,2}$ induces the identification $\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0}(\mathcal{L},\mathcal{A}_{cr,2})=\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0}(\mathcal{L},\widetilde{\mathcal{A}}_{cr,2})$.

Consider the structure of $\widetilde{\mathcal{A}}_{cr,2}$ more closely.

Let $T_1 = \xi^p/p$. With obvious notation the elements of $\widetilde{A}_{cr,2}$ can be written uniquely modulo the subgroup $[x_0^pR]T_1 + [x_0^{2p}R] + p[x_0^pR] + p^2W(R)$ in the form $[r_{-1}]T_1 + [r_0] + p[r_1]$, where $r_{-1}, r_0, r_1 \in R$. Informally, we shall use that $r_{-1}, r_1 \in R/x_0^p$ and $r_0 \in R/x_0^{2p}$. The $W(R)$-module structure on $\widetilde{A}_{cr,2}$ is induced by usual operations on Teichmuller's representatives and the relation $pT_1 \equiv [x_0]^p \bmod p^2W(R)$. (Use that $T_1 \equiv [x_0]^p/p + p[x_0]^{p-1} \bmod p^2W(R)$.)
The S-module structure on $\widetilde{A}_{cr,2}$ is induced by the $W(k)$-algebra morphism $S \longrightarrow W(R)$ such that $u \mapsto [x_0]$. Then $F(\widetilde{A}_{cr,2})$ is generated over $W(R)$ by the images of T_1 and ξ^{p-1}. Note that $\xi^{p-1} \equiv [x_0]^{p-1} - p[x_0]^{p-2} \bmod p^2W(R)$.
The map $\varphi : F(\widetilde{A}_{cr,2}) \longrightarrow \widetilde{A}_{cr,2}$ is uniquely determined by the knowledge of $\varphi(T_1)$ and $\varphi(\xi^{p-1})$. Note that

$$\varphi(T_1) = \left(\frac{1+[x_0]^p}{p}\right)^p \equiv 1 + [x_0]^p \bmod (J + p^2A_{cr,2} + p[\mathrm{m}_R])$$

$$\varphi(\xi^{p-1}) = \left(1 + \frac{[x_0]^p}{p}\right)^{p-1} \equiv 1 - T_1 \bmod (J + p^2A_{cr,2} + p[\mathrm{m}_R])$$

Suppose $\mathcal{L} \in \underline{\mathcal{L}}^{ft}[1]$ and $\mathcal{L}' \in \underline{\mathcal{L}}^{ft}$ is such that $p\mathcal{L}' = \mathcal{L}$. Consider short exact sequences (4.3) and (4.4). Then the points $f \in \mathcal{V}^t({}_p\mathcal{L}')$ and $\mathcal{V}^t(C_p)(f) \in \mathcal{V}^t(\mathcal{L}')$ are related via the commutative diagram

$$\begin{tikzcd}
\mathcal{L}' \arrow[r, "\mathcal{V}^t(C_p)(f)"] \arrow[d, "C_p"] & \widetilde{\mathcal{A}}_{cr,2} \arrow[d] \\
{}_p\mathcal{L}' \arrow[r, "f"] & \widetilde{\mathcal{A}}_{cr,1}
\end{tikzcd}$$

where the right vertical arrow is induced by the correspondence

$$[r_{-1}]T_1 + [r_0] + p[r_1] \mapsto [r_{-1}]T_1 + [r_0 \bmod x_0^p].$$

Similarly, the points $g \in \mathcal{V}^t(\mathcal{L}')$ and $\mathcal{V}^t(K_p)(g) \in \mathcal{V}^t(\mathcal{L}'_p)$ are related via the commutative diagram

$$\begin{tikzcd}
\mathcal{L}' \arrow[r, "g"] & \widetilde{\mathcal{A}}_{cr,2} \\
\mathcal{L}'_p \arrow[u, "K_p"] \arrow[r, "\mathcal{V}^t(K_p)(g)"] & \widetilde{\mathcal{A}}_{cr,1} \arrow[u]
\end{tikzcd}$$

where the right vertical arrow is induced by the correspondence

$$[r_{-1}]T_1 + [r_0] \mapsto [r_{-1}x_0^p] + p[r_0].$$

4.6. Filtered φ-modules $\mathcal{A}^0_{cr,1}$ and $\mathcal{A}^0_{cr,2}$. Let $A^0_{cr,2}$ be the $W(R)$-submodule of $\widetilde{A}_{cr,2}$ consisting of elements $[r_{-1}]T_1+[r_0]+p[r_1]$ such that $r_{-1}=-r_0 \bmod x_0^p$. Then $F(A^0_{cr,2})=F(\widetilde{A}_{cr,2})\cap A^0_{cr,2}$ is generated over $W(R)$ by $[x_0^{p-1}]T_1+\xi^{p-1}$ and the congruence

$$\varphi([x_0^{p-1}]T_1+\xi^{p-1})\equiv -T_1+1 \bmod (J_2+p^2A_{cr,2}+p[\mathrm{m}_R])$$

implies that $\varphi(F(A^0_{cr,2}))\subset A^0_{cr,2}$ and $A^0_{cr,2}=(A^0_{cr,2},F(A^0_{cr,2}),\varphi)\in\widetilde{\underline{\mathcal{L}}}_0$.
Note that $p\mathcal{A}^0_{cr,2}=(pA^0_{cr,2},pF(A^0_{cr,2}),\varphi)\in\widetilde{\underline{\mathcal{L}}}_0$. Then in notation from Subsection 4.4, one has:

- $\mathrm{Im}\,\Theta=\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0}({}_p\mathcal{L}',p\mathcal{A}^0_{cr,2})$;
- $\mathcal{V}^t(C_p)(\mathrm{Im}\,\Theta)=\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0}(\mathcal{L}',p\mathcal{A}^0_{cr,2})$;
- $\mathrm{Ker}\widetilde{\Theta}=\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0}(\mathcal{L}'_p,p\mathcal{A}^0_{cr,2})$;
- $\mathrm{Ker}(\widetilde{\Theta}\circ\mathcal{V}^t(K_p))=\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0}(\mathcal{L}',\mathcal{A}^0_{cr,2})$.

Therefore, $\widetilde{\mathcal{V}}^{ft}(\mathcal{L})=\mathcal{V}^t_{\mathcal{L}'}(\mathcal{L})=\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0}(\mathcal{L}',\mathcal{A}^0_{cr,2}/p\mathcal{A}^0_{cr,2})=\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}_0}(\mathcal{L},\mathcal{A}^0_{cr,2}/p\mathcal{A}^0_{cr,2})$.

4.7. The functor $\widetilde{\mathcal{CV}}^{ft}$. Let $\mathcal{L}\in\underline{\mathcal{L}}^{ft}$ and let $i^{et}:\mathcal{L}^{et}\longrightarrow\mathcal{L}$ be the maximal etale subobject of $\mathcal{L}$.

Definition. $\widetilde{\mathcal{CV}}^{ft}:\underline{\mathcal{L}}^{ft}\longrightarrow\underline{\mathrm{CM}\Gamma}_F$ is the functor induced by the correspondence $\mathcal{L}\mapsto\widetilde{\mathcal{CV}}^{ft}(\mathcal{L})=(\widetilde{\mathcal{V}}^{ft}(\mathcal{L}),\widetilde{\mathcal{V}}^{ft}(\mathcal{L}^{et}),\widetilde{\mathcal{V}}^{ft}(i^{et}))$.

The functor $\widetilde{\mathcal{CV}}^{ft}$ is not very far from Breuil's functor $\mathcal{V}^t$ but it satisfies the following important property.

Proposition 4.16. *The functor $\widetilde{\mathcal{CV}}^{ft}$ is fully faithful.*

Proof. By standard devissage it will be sufficient to verify this property for the restriction $\widetilde{\mathcal{CV}}^{ft}|_{\underline{\mathcal{L}}^{ft}[1]}$. Due to Proposition 2.13 it will be sufficient to verify that the functor $\widetilde{\mathcal{V}}^{ft}|_{\underline{\mathcal{L}}^{ft}[1]}\circ\Pi^{-1}$ coincides with the functor $\mathcal{V}^*$ from Subsection 2.2. This can be proved similarly to the proof of the corresponding fact for unipotent objects in Subsection 4.2 as follows.
Let

$$A^0_{st,2}=\prod_{j\geqslant 0}A^0_{cr,2}\gamma_j(\log(1+X))\subset\widetilde{A}_{st,2}=\prod_{j\geqslant 0}\widetilde{A}_{cr,2}\gamma_j(\log(1+X))$$

with induced structures of the objects $\mathcal{A}^0_{st,2}$ and $\widetilde{\mathcal{A}}_{st,2}$ of the category $\widetilde{\underline{\mathcal{L}}}$. Then from Subsection 4.6 it follows that

$$\mathcal{V}^t(\mathcal{L})=\mathrm{Hom}_{\widetilde{\underline{\mathcal{L}}}}(\mathcal{L},\mathcal{A}^0_{st,2}/p\mathcal{A}^0_{st,2}).$$

One can see easily that the correspondence

$$[r_0 \bmod x_0^p]T_1+[r_0]+p[r_1]\mapsto(r_0+x_0^pr_1)\bmod x_0^p\mathrm{m}_R$$

induces the morphism $\mathcal{A}^0_{cr,2}/p\mathcal{A}^0_{cr,2} \longrightarrow \mathcal{R}^0$ in the category $\widetilde{\underline{\mathcal{L}}}_0$. This morphism induces a unique identification of the abelian groups $\mathcal{V}^t(\mathcal{L})$ and $\mathrm{Hom}(\mathcal{L},\mathcal{R}^0) = \mathcal{V}^*(\mathcal{L})$. Now going to a suitable factor of the object $\mathcal{A}^0_{st,2}/p\mathcal{A}^0_{st,2}$ we obtain that this identification is compatible with the Γ_F-actions on both abelian groups. $\square$

Now we can describe all Galois invariant lattices of semi-stable $\mathbb{Q}_p[\Gamma_F]$-modules with weights from $[0,p)$.

COROLLARY 4.17. *Suppose V is a semi-stable representation of Γ_F with weights from $[0,p)$, $\dim_{\mathbb{Q}_p} V = s$ and T is a Γ_F-invariant lattice in V. Then there is a p-divisible group $\{\mathcal{L}^{(n)}, i_n\}_{n\geqslant 0}$ of height s in $\underline{\mathcal{L}}^{ft}$ such that* $\varprojlim_n \widetilde{\mathcal{CV}}^{ft}(\mathcal{L}^{(n)}) = (T, T^{et}, i^{et}) \in \underline{\mathrm{CM\Gamma}}_F$.

5. PROOF OF THEOREM 0.1.

As earlier, p is a fixed prime number, $p \neq 2$. Starting Subsection 5.2 we assume $p = 3$.

5.1. For all prime numbers l, choose embeddings of algebraic closures $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_l$ and use them to identify the inertia groups $I_l = \mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_{l,ur})$, where $\mathbb{Q}_{l,ur}$ is the maximal unramified extension of $\mathbb{Q}_l$, with the appropriate subgroups in $\Gamma_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.
Introduce the category $\underline{\mathrm{M\Gamma}}^t_{\mathbb{Q}}$. Its objects are the pairs $H_{\mathbb{Q}} = (H, \widetilde{H}_{st})$, where H is a finite $\mathbb{Z}_p[\Gamma_{\mathbb{Q}}]$-module unramified outside p and $\widetilde{H}_{st} = (H_{st}, H^0_{st}, i) \in \underline{\mathrm{CM\Gamma}}^{st}_F$, where $H|_{I_p} = H_{st}$, $F = W(\bar{\mathbb{F}}_p)[1/p]$ and $\underline{\mathrm{CM\Gamma}}^{st}_F$ is the image of the functor $\widetilde{\mathcal{CV}}^{ft}$ from Subsection 4.7. The morphisms in $\underline{\mathrm{M\Gamma}}^t_{\mathbb{Q}}$ are compatible morphisms of Galois modules. Clearly, the category $\underline{\mathrm{M\Gamma}}^t_{\mathbb{Q}}$ is special pre-abelian, cf. Appendix A.
Let $\underline{\mathrm{M\Gamma}}^t_{\mathbb{Q}}[1]$ be the full suibcategory of killed by p objects in $\underline{\mathrm{M\Gamma}}^t_{\mathbb{Q}}$. Denote by $\mathcal{K}(p)$ an algebraic extension of $\mathbb{Q}$ such that for any $H_{\mathbb{Q}} = (H, \widetilde{H}_{st}) \in \underline{\mathrm{M\Gamma}}^t_{\mathbb{Q}}[1]$, $\Gamma_{\mathcal{K}(p)} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathcal{K}(p))$ acts trivially on H. In other words, $\mathcal{K}(p)$ can be taken as a common field-of-definition of points of all such $\Gamma_{\mathbb{Q}}$-modules H.
Now assume that

(**C**) $\mathcal{K}(p)$ is totally ramified at p.

Under this assumption we have a natural identification $\mathrm{Gal}(\mathcal{K}(p)/\mathbb{Q}) = \mathrm{Gal}(\mathcal{K}(p)F/F)$, that is the Galois group of the global extension $\mathcal{K}(p)/\mathbb{Q}$ comes as the Galois group of its completion over F. Therefore, we can identify $\underline{\mathrm{M\Gamma}}^t_{\mathbb{Q}}[1]$ with the full subcategory of $\underline{\mathrm{CM\Gamma}}^{st}_F$, consisting of (H_{st}, H^0_{st}, i) such that $pH_{st} = 0$ and all points of H_{st} are defined over $\mathcal{K}(p)F$. In other words, the objects of $\underline{\mathrm{M\Gamma}}^t[1]$ can be described via our local results about killed by p subquotients of semistable representations of Γ_F.

Denote by $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{ft}[1]$ a full subcategory in $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{t}[1]$ which consists of killed by p subquotients of p-divisible groups in the category $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{t}$.
Let F' be the maximal tamely ramified extension of F in $\mathcal{K}(p)F$. Then $\mathrm{Gal}(F'/F)$ is abelian group of order prime to p (use that the residue field of F is algebraically closed) and $\mathrm{Gal}(\mathcal{K}(p)F/F')$ is a p-group. This gives an abelian extension $\mathcal{K}'$ of $\mathbb{Q}$ in $\mathcal{K}(p)$ of prime-to-p degree and such that $\mathcal{K}(p)/\mathcal{K}'$ is a p-extension. This extension is unramified outside p and, therefore, it coincides (use class field theory) with $\mathbb{Q}(\zeta_p)$. In particular, all simple objects in $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{t}[1]$ are of the form $\mathcal{F}(j)=(\mathbb{F}_p(j),0,0)$ if $1\leqslant j<p$ and $\mathcal{F}(0)=(\mathbb{F}_p(0),\mathbb{F}_p(0),\mathrm{id}\,)$ if $j=0$.
Let $\underline{\mathcal{L}}_{\mathbb{Q}}^{ft}[1]$ and $\underline{\mathcal{L}}_{\mathbb{Q}}^{t}[1]$ be the full subcategories of $\underline{\mathcal{L}}^{t}[1]$ mapped by the functor $\widetilde{\mathcal{CV}}^{ft}$ to the objects of $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{ft}[1]$ and, resp., $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{t}[1]$. Clearly, $\underline{\mathcal{L}}_{\mathbb{Q}}^{ft}[1]$ is a full subcategory in $\underline{\mathcal{L}}^{ft}[1]$ and the only simple objects in these categories are $\mathcal{L}(r)$, where $r\in\{j/(p-1)\mid j=0,1,\dots,p-1\}$.
Suppose $H^{\infty}=\{H_{\mathbb{Q}}^{(n)},i_n\}_{n\geqslant 0}$ is a p-divisible group in the category $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{t}$. Here all $H_{\mathbb{Q}}^{(n)}=(H^{(n)},\widetilde{H}_{st}^{(n)})$ are objects of the category $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{t}$. Let $\mathcal{L}\in\underline{\mathcal{L}}_{\mathbb{Q}}^{ft}[1]$ be such that $\widetilde{\mathcal{CV}}^{ft}(\mathcal{L})=\widetilde{H}_{st}^{(1)}$. Note that the maximal etale subobject $\mathcal{L}^{et}$ of $\mathcal{L}$ is isomorphic to $\mathcal{L}(0)^{n_{et}}$, where $n_{et}=n_{et}(\mathcal{L})\in\mathbb{Z}_{\geqslant 0}$, and $\mathcal{L}/\mathcal{L}^{et}$ has no simple subquotients isomorphic to $\mathcal{L}(0)$. Similarly, the corresponding maximal multiplicative quotient $\mathcal{L}^{m}$ is isomorphic to $\mathcal{L}(1)^{n_m}$, where $n_m=n_m(\mathcal{L})\in\mathbb{Z}_{\geqslant 0}$, and the kernel of the canonical projection $\mathcal{L}\longrightarrow\mathcal{L}^{m}$ has no simple subquotients isomorphic to $\mathcal{L}(1)$. Therefore, for any $\mathcal{M}\in\underline{\mathcal{L}}_{\mathbb{Q}}^{ft}[1]$,

$$\mathrm{Ext}_{\underline{\mathcal{L}}_{\mathbb{Q}}^{ft}[1]}(\mathcal{L}(0),\mathcal{M})=\mathrm{Ext}_{\underline{\mathcal{L}}_{\mathbb{Q}}^{ft}[1]}(\mathcal{M},\mathcal{L}(1))=0.$$

This implies that for any $H\in\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{ft}[1]$,

$$\mathrm{Ext}_{\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{ft}[1]}(H,\mathcal{F}(0))=\mathrm{Ext}_{\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{ft}[1]}(\mathcal{F}(1),H)=0.$$

Therefore, by Theorem A.5 of Appendix A there is an embedding of p-divisible groups $H^{\infty,m}\subset H^{\infty}$, where $H^{(1)m}=\mathcal{F}(1)^{n_m}$, and there is a projection of p-divisible groups $H^{\infty}\longrightarrow H^{\infty,et}$, where $H^{(1)et}=\mathcal{F}(0)^{n_{et}}$.
For similar reasons,

$$\mathrm{Ext}_{\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{ft}[1]}(\mathcal{F}(0),\mathcal{F}(0))=\mathrm{Ext}_{\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{ft}[1]}(\mathcal{F}(1),\mathcal{F}(1))=0$$

and by Theorem A.4 of Appendix A, the corresponding p-divisible groups $H_{\mathbb{Q}}^{\infty,m}$ and $H_{\mathbb{Q}}^{\infty,et}$ are unique. Therefore they coincide with the products of trivial p-divisible groups $(\mathbb{Q}_p/\mathbb{Z}_p)(p-1)$ and, resp., $(\mathbb{Q}_p/\mathbb{Z}_p)(0)$.
We state this result in the following form.

Proposition 5.1. *Under assumption* (C)*, for any p-divisible group H^{∞} in the category $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{t}$ there is a filtration of p-divisible groups*

$$H^{\infty}\supset H_1^{\infty}\supset H_0^{\infty}$$

such that $H_0^\infty = (\mathbb{Q}_p/\mathbb{Z}_p)(p-1)^{n_m}$, $H^\infty/H_1^\infty = (\mathbb{Q}_p/\mathbb{Z}_p)(0)^{n_{et}}$ *and all simple subquotients of* H_1^∞/H_0^∞ *belong to* $\{\mathbb{F}_p(j) | 1 \leqslant j \leqslant p-2\}$.

5.2. Assume that $p = 3$.

LEMMA 5.2. $\mathcal{K}(3) = \mathbb{Q}(\sqrt[3]{3}, \zeta_9)$, *where* ζ_9 *is 9-th primitive root of 1.*

This Lemma will be proved in Subsection 5.3 below.
In particular, $\mathcal{K}(3)$ satisfies the assumption (C).

PROPOSITION 5.3. *If* H^∞ *is a 3-divisible group in* $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^t$ *then in its filtration from Proposition 5.1 the 3-divisible group* $\hat{H}^\infty = H_1^\infty/H_0^\infty$ *is a product of finitely many trivial 3-divisible groups* $(\mathbb{Q}_3/\mathbb{Z}_3)(1)$.

Proof. Let $\widehat{\underline{\mathcal{L}}}_{\mathbb{Q}}$ be the full subcategory of $\underline{\mathcal{L}}_{\mathbb{Q}}^{ft}[1]$ consisting of objects $\mathcal{L}$ such that $\mathcal{L}^m = \mathcal{L}^{et} = 0$. This category has only one simple object $\mathcal{L}(1/2)$. Let $\widehat{\underline{\mathrm{M}\Gamma}}_{\mathbb{Q}}$ be the full subcategory in $\underline{\mathrm{M}\Gamma}_{\mathbb{Q}}^{ft}[1]$ consisting of the objects $\widetilde{\mathcal{CV}}^{ft}(\mathcal{L})$, where $\mathcal{L} \in \widehat{\underline{\mathcal{L}}}_{\mathbb{Q}}$. Then $\widehat{\underline{\mathcal{L}}}_{\mathbb{Q}}$ and $\widehat{\underline{\mathrm{M}\Gamma}}_{\mathbb{Q}}$ are antiequivalent categories and $\hat{H}^{(1)} \in \underline{\mathrm{M}\Gamma}_{\mathbb{Q}}$. By Theorems A.4 and A.5 our Proposition is implied by the following result. $\square$

PROPOSITION 5.4. $\mathrm{Ext}_{\widehat{\underline{\mathcal{L}}}_{\mathbb{Q}}}(\mathcal{L}(1/2), \mathcal{L}(1/2)) = 0$.

Proof. Consider the equivalence of the categories $\Pi : \underline{\mathcal{L}}^t \longrightarrow \underline{\mathcal{L}}^*$ from Corollary 3.10. This equivalence transforms the functor $\widetilde{\mathcal{CV}}^{ft}$ into the functor $\mathcal{CV}^*$ from Section 2, cf. the proof of Proposition 4.16. Therefore, the objects $\mathcal{L}$ of the category $\Pi(\underline{\mathcal{L}}_{\mathbb{Q}}^t) := \underline{\mathcal{L}}_{\mathbb{Q}}^*$ are characterised by the condition that all points of $\mathcal{V}^*(\mathcal{L})$ are defined over the field $\mathcal{K}(3)F$. The objects $\mathcal{L}$ of the category $\Pi(\widehat{\underline{\mathcal{L}}}_{\mathbb{Q}}) := \widehat{\underline{\mathcal{L}}}_{\mathbb{Q}}^*$ are characterised by the additional properties: they are all obtained by subsequent extensions via $\mathcal{L}(1/2)$ and $\mathcal{V}^*(\mathcal{L})$ appears as a subquotient of semi-stable representation of Γ_F with Hodge-Tate weights from [0,2].
Introduce the object $\mathcal{L}(1/2, 1/2) = (L, F(L), \varphi, N)$ of the category $\underline{\mathcal{L}}^*$ as follows:

- $L = \mathcal{W}_1 l \oplus \mathcal{W}_1 l_1$;
- $F(L)$ is spanned by ul_1 and $ul + l_1$;
- $\varphi(ul_1) = l_1$, $\varphi(ul + l_1) = l$;
- $N(l_1) \equiv 0 \bmod u^3 L$, $N(l) \equiv l_1 \bmod u^3 L$.

Clearly, $\mathcal{L}(1/2, 1/2)$ has a natural structure of an element of the group $\mathrm{Ext}_{\underline{\mathcal{L}}^*}(\mathcal{L}(1/2), \mathcal{L}(1/2))$.

LEMMA 5.5. a) $\mathcal{L}(1/2, 1/2) \in \underline{\mathcal{L}}_{\mathbb{Q}}^*$;
b) $\mathrm{Ext}_{\underline{\mathcal{L}}_{\mathbb{Q}}^*}(\mathcal{L}(1/2), \mathcal{L}(1/2)) \simeq \mathbb{Z}/3$ *and is generated by the class of* $\mathcal{L}(1/2, 1/2)$;
c) $\mathrm{Ext}_{\underline{\mathcal{L}}_{\mathbb{Q}}^*}(\mathcal{L}(1/2), \mathcal{L}(1/2, 1/2)) = \mathrm{Ext}_{\underline{\mathcal{L}}_{\mathbb{Q}}^*}(\mathcal{L}(1/2, 1/2), \mathcal{L}(1/2)) = 0$.

This Lemma will be proved in Subsection 5.4 below.
Lemma 5.5 implies that $\mathrm{Ext}_{\underline{\mathcal{L}}_{\mathbb{Q}}^*}(\mathcal{L}(1/2, 1/2), \mathcal{L}(1/2, 1/2)) = 0$ and, therefore, any object $\mathcal{L}$ of $\underline{\mathcal{L}}_{\mathbb{Q}}^*$ is the product of several copies of $\mathcal{L}(1/2)$ and $\mathcal{L}(1/2, 1/2)$.

Suppose $\mathcal{L} = \mathcal{L}_1 \times L(1/2,1/2) \in \widehat{\underline{\mathcal{L}}}^*_{\mathbb{Q}}$. Then there is a 3-divisible group $\widetilde{H}^{\infty}$ in $\underline{\mathrm{M}\Gamma}^t_{\mathbb{Q}}$ such that $\widetilde{H}^{(1)} = H' \times H(1/2,1/2)$, where H' and $H(1/2,1/2) = \mathcal{CV}^*(\mathcal{L}(1/2,1/2))$ belong to $\underline{\hat{\mathrm{M}\Gamma}}_{\mathbb{Q}}$. Clearly, we have $\mathrm{Ext}_{\underline{\mathrm{M}\Gamma}^{ft}_{\mathbb{Q}}[1]}(H', H(1/2,1/2)) = 0$ and applying Theorem A.5 we obtain a 3-divisible group H^{∞} in $\underline{\mathrm{M}\Gamma}^t_{\mathbb{Q}}$ such that $H^{(1)} = H(1/2,1/2)$. This implies the existence of 2-dimensional semi-stable (and non-crystalline) representation of Γ_F with the only simple subquotient $\mathbb{F}_3(1)$, that is for any Galois invariant lattice T of such representation, the Γ_F-module $T/3T$ has semi-simple envelope $\mathbb{F}_3(1) \times \mathbb{F}_3(1)$. This situation appears as a very special case of Breuil's description of 2-dimensional semi-stable (and non-crystalline) representations. According to Theorem 6.1.1.2 of [5] the corresponding semi-simple envelope is either $\mathbb{F}_3(0) \times \mathbb{F}_3(1)$ or $\mathbb{F}_3(1) \times \mathbb{F}_3(2)$. The proposition is proved. ◻

Now our main Theorem appears as the following Corollary.

Corollary 5.6. *If Y is a projective variety with semi-stable reduction modulo 3 and good reduction modulo all primes $l \neq 3$ then $h^2(Y_{\mathbb{C}}) = h^{1,1}(Y_{\mathbb{C}})$.*

Proof. Indeed, let V be the $\mathbb{Q}_3[\Gamma_F]$-module of 2-dimensional etale cohomology of Y. Then it is a semi-stable representation of F and its Γ_F-invariant lattice determines a 3-divisible group in the category $\underline{\mathrm{M}\Gamma}^t_{\mathbb{Q}}$. By Proposition 5.3 this 3-divisible group can be built from the Tate twists $(\mathbb{Q}_3/\mathbb{Z}_3)(i)$, $i = 0,1,2$. Equivalently, all Γ_F-equivariant subquotients of V are $\mathbb{Q}_3(i)$ with $i = 0,1,2$. Applying the Riemann Conjecture (proved by Deligne) to the reductions $Y \bmod l$ with $l \neq 3$, we obtain that $\mathbb{Q}(0)$ and $\mathbb{Q}(2)$ do not appear. Therefore, V is the product of finitely many $\mathbb{Q}_3(1)$ and $h^2(Y_{\mathbb{C}}) = h^{1,1}(Y_{\mathbb{C}})$. ◻

5.3. Proof of Lemma 5.2. Use the ramification estimate from Subsection 2.9 to deduce that the normalized discriminant of $\mathcal{K}(3)$ over $\mathbb{Q}$ satisfies the inequality $|D(\mathcal{K}(3)/\mathbb{Q})|^{[\mathcal{K}(3):\mathbb{Q}]^{-1}} < 3^{3-1/3} = 18.72075$. Then Odlyzko estimates imply that $[\mathcal{K}(3):\mathbb{Q}] < 230$ [11].

Let $K_0 = \mathbb{Q}(\zeta_9)$ and $K_1 = \mathbb{Q}(\sqrt[3]{3}, \zeta_9)$. Then K_0 is the maximal abelian extension of $\mathbb{Q}$ in $\mathcal{K}(3)$ and $K_1 \subset \mathcal{K}(3)$. We have also the inequality $[\mathcal{K}(3):K_1] < 60$ and, therefore, $\mathrm{Gal}(\mathcal{K}(3)/\mathbb{Q})$ is soluble.

Prove that $K_1 = \mathcal{K}(3)$.

Suppose the field K_2 is the maximal abelian extension of K_1 in $\mathcal{K}(3)$. One can apply the computer package SAGE to prove that the group of classes of K_1 is trivial. Therefore, K_2 is totally ramified at 3 and $\mathrm{Gal}(K_2/\mathbb{Q})$ coincides with the Galois group of the corresponding 3-completions. In particular, the maximal tamely ramified subextension of these completions comes from $\mathbb{Q}(\zeta_3)$ and, therefore, K_2/K_1 is 3-extension. Therefore, there is an $\eta \in O^*_{K_1}$ such that $K_1(\sqrt[3]{\eta}) \subset K_2$. Then a routine computation shows that the normalized discriminant for $K_1(\sqrt[3]{\eta})$ over $\mathbb{Q}$ is less than $3^{3-1/3}$ if and only if $\eta \equiv 1 \bmod O^{*3}_{K_1}(1+3O_{K_1})^{\times}$. The Lemma will be proved if we show that such $\eta \in O^{*3}_{K_1}$. (This is equivalent to the Leopoldt Conjecture for the field K_1.) This

was proved via a SAGE computer program written by R.Henderson (Summer-2009 Project at Durham University supported by Nuffield Foundation). This program, cf. Appendix B, constructed a basis $\varepsilon_i \bmod O_{K_1}^{*3}$, $1 \leqslant i \leqslant 9$, of $O_{K_1}^*/O_{K_1}^{*3}$ such that $18v_3(\varepsilon_i - 1)$ takes values in the set $\{1,2,4,5,7,8,10,13,16\}$. In other words, $v_3(\eta-1) \geqslant 1 > 16/18$ implies that $\eta \in O_{K_1}^{*3}$.
Lemma 5.2 is proved.

5.4. Proof of Lemma 5.5. a) Use the notation from the definition of the functor $\mathcal{V}^t$ in Subsection 4.
If $f_0 \in \mathcal{V}^t(\mathcal{L}(1/2,1/2))$ then the correspondence $f_0 \mapsto (f_0(l_1), f_0(l))$ identifies $\mathcal{V}^t(\mathcal{L}(1/2,1/2))$ with the $\mathbb{F}_3$-module of couples $(X_{10}, X_0) \in (R/x_0^6)^2$ such that $X_{10}^3/x_0^3 = X_{10}$ and $(X_0^3 + X_{10})/x_0^3 = X_0$. Then the $\mathbb{F}_3[\Gamma_F]$-module $\mathcal{V}^t(\mathcal{L}(1/2,1/2))$ is identified with the module formed by the images of all $(X_{10}, X_0 + X_{10}Y) \in (R_{st}^0)^2$ in $\widetilde{R}_{st}^0 = R_{st}^0/(x_0^3\mathrm{m}_R + x_0^2\mathrm{m}_R Y + x_0\mathrm{m}_R Y^2)$.
In particular, the corresponding Γ_F-action on $\mathcal{V}^t(\mathcal{L}(1/2,1/2))$ comes from the natural Γ_F-action on the residues of X_{10} and X_0 modulo $x_0^3\mathrm{m}_R$. Notice there is a natural Γ_F-equivariant identification

$$\iota : \mathrm{m}_R/(x_0^3\mathrm{m}_R) \longrightarrow \bar{\mathrm{m}}/3\bar{\mathrm{m}},$$

where $\bar{\mathrm{m}}$ is the maximal ideal of the valuation ring of $\bar{\mathbb{Q}}_3$. This isomorphism ι comes from the map $r \mapsto r^{(1)}$, where for $r = \varprojlim_n (r_n \bmod p)$, $r^{(1)} := \lim_{n\to\infty} r_{n+1}^{p^n}$.
Then Hensel's Lemma implies the existence of unique $Z_{10}, Z_0 \in \bar{\mathrm{m}}$ such that the following equalities hold $\iota(X_{10} \bmod x_0^3\mathrm{m}_R) = Z_{10} \bmod 3\bar{\mathrm{m}}$, $\iota(X_0 \bmod x_0^3\mathrm{m}_R) = Z_0 \bmod 3\bar{\mathrm{m}}$, $Z_{10}^3 + 3Z_{10} = 0$ and $Z_0^3 + 3Z_0 = -Z_{10}$.
Clearly, $F(Z_{10}, Z_0) = F(\zeta_9)$. Therefore, if $\tau \in \Gamma_F$ is such that $\tau(\zeta_9) = \zeta_9$ then $\tau(X_{10}) = X_{10}$ and $\tau(X_0) = X_0$.
Finally, it follows directly from definitions that if $\tau(\sqrt[3]{3}) = \sqrt[3]{3}$ then τ acts as identity on the image of Y in $\widetilde{R}_{st}^0$. The part a) of the Lemma is proved.
b) Suppose $\mathcal{L} = (L, F(L), \varphi, N) \in \mathrm{Ext}_{\underline{\mathcal{L}}_{\mathbb{Q}}^*}(\mathcal{L}(1), \mathcal{L}(1))$. Then $L = \mathcal{W}_1 l \oplus \mathcal{W}_1 l_1$, there is an $w \in \mathcal{W}_1$ such that $F(L)$ is spanned by ul_1 and $ul + wl_1$ over $\mathcal{W}_1$, and one has $\varphi(ul_1) = l_1$, $\varphi(ul + wl_1) = l$, $N(l_1) \in u^3L$ and $N(l_1) \equiv w^3 l_1 \bmod u^3 L$. Notice that $\mathcal{L}$ splits in $\underline{\mathcal{L}}^*$ iff $w \in u\mathcal{W}_1$. Therefore, we can assume that $w = \alpha \in k$.
Then the field-of-definition of all points of $\mathcal{V}^t(\mathcal{L})$ contains the field-of-definition of all solutions $(X_1, X) \bmod x_0^3\mathrm{m}_R \in (R/x_0^3\mathrm{m}_R)^2$ of the following congruences: $X_1^3/x_0^3 \equiv X_1 \bmod x_0^3\mathrm{m}_R$ and $(X^3 + \alpha^3 X_1)/x_0^3 \equiv X \bmod x_0^3\mathrm{m}_R$.
Let $x_1 \in R$ be such that $x_1^2 = x_0$. Then we can take $X_1 = x_1^3$ and for $T = X/x_1^3$ one has the following Artin-Schreier-type congruence:

$$T^3 - T \equiv -\alpha^3/x^6 \bmod \mathrm{m}_R.$$

Using calculations from above part a) we can conclude that $\mathcal{L} \in \underline{\mathcal{L}}_{\mathbb{Q}}^*$ if and only if the field-of-definition of $T \bmod \mathrm{m}_R$ over $k((x_1))$ belongs to the field-of-definition of $T_0 \bmod \mathrm{m}_R$ over $k((x_1))$, where $T_0^3 - T_0 \equiv -x_1^{-6} \bmod \mathrm{m}_R$. By Artin-Schreier theory this happens if and only if $\alpha \in \mathbb{F}_3$ and,therefore, $\mathcal{L} \simeq \mathcal{L}(1/2,1/2)$.

c) Suppose $\mathcal{L}=(L,F(L),\varphi,N)\in\mathrm{Ext}_{\underline{\mathcal{L}}^*_{\mathbb{Q}}}(\mathcal{L}(1/2),\mathcal{L}(1/2,1/2))$.
Then we can assume that:
— $L=\mathcal{W}_1l\oplus\mathcal{W}_1l_1\oplus\mathcal{W}_1m$;
— $F(L)$ is spanned over $\mathcal{W}_1$ by ul_1, $ul+l_1$ and $um+wl+w_1l_1$ with $w,w_1\in\mathcal{W}_1$;
— $\varphi(ul_1)=l_1$, $\varphi(ul+l_1)=l$ and $\varphi(um+wl+w_1l_1)=m$.
Then the condition $u^2m\in F(L)$ implies that $wl_1\in F(L)$, or $w\in u\mathcal{W}_1$ and we can assume that $w=0$. Then the submodule $\mathcal{W}_1m+\mathcal{W}_1l_1$ determines a subobject $\mathcal{L}'$ of $\mathcal{L}$, $\mathcal{L}'\in\underline{\mathcal{L}}^*_{\mathbb{Q}}$ and using calculations from b) we conclude that $w_1\in\mathbb{F}_3\bmod u\mathcal{W}_1$. Therefore, we can assume that $w_1=\alpha\in\mathbb{F}_3$ and for $m'=m-\alpha l$ we have $m'\in F(L)$ and $\varphi(um')=m'$, i.e. $\mathcal{L}$ is a trivial extension.
Now suppose $\mathcal{L}=(L,F(L),\varphi,N)\in\mathrm{Ext}_{\underline{\mathcal{L}}^*_{\mathbb{Q}}}(\mathcal{L}(1/2,1/2),\mathcal{L}(1/2))$.
Then we can assume that:
— $L=\mathcal{W}_1m\oplus\mathcal{W}_1m_1\oplus\mathcal{W}_1l$;
— $F(L)$ is spanned over $\mathcal{W}_1$ by ul, um_1+wl and $um+m_1+w_1l$ with $w,w_1\in\mathcal{W}_1$;
— $\varphi(ul)=l$, $\varphi(um_1+wl)=m_1$ and $\varphi(um+m_1+w_1l)=m$.
Again, the condition $u^2m\in F(L)$ implies that $w\in u\mathcal{W}_1$ and, therefore, we can assume that $w=0$. Then the quotient module $L/\mathcal{W}_1m_1$ is the quotient of $\mathcal{L}$ in the category $\underline{\mathcal{L}}^*$. This quotient must belong to the subcategory $\underline{\mathcal{L}}^*_{\mathbb{Q}}$. This implies that $w_1\in\mathbb{F}_3\bmod u\mathcal{W}_1$, and, as earlier, $\mathcal{L}$ becomes a trivial extension.
The Lemma is completely proved.

Appendix A. p-divisible groups in pre-abelian categories

A.1. Short exact sequences in pre-abelian categories.

A.1.1. *Pre-abelian categories.* Introduce the concept of special pre-abelian category following mainly [28], cf. also [25, 26, 29]. Remind that a category $\mathcal{S}$ is pre-abelian if it is additive and for any morphism $u\in\mathrm{Hom}_{\mathcal{S}}(A,B)$, there exist $\mathrm{Ker}\,u=(A_1,i)$ and $\mathrm{Coker}\,u=(B_1,j)$, where $i\in\mathrm{Hom}_{\mathcal{S}}(A_1,A)$ and $j\in\mathrm{Hom}_{\mathcal{S}}(B,B_1)$. For any objects $A,B\in\mathcal{S}$, let $A\prod B$ and $A\coprod B$ be their product and coproduct, respectively. There is a canonical isomorphism $A\prod B\simeq A\coprod B$ in S. More generally, for given morphisms:

- $\alpha\in\mathrm{Hom}_{\mathcal{S}}(C,A)$, $\beta\in\mathrm{Hom}_{\mathcal{S}}(C,B)$, there is a fibered coproduct $(A\coprod_C B,i_A,i_B)$, with $i_A\in\mathrm{Hom}_{\mathcal{S}}(A,A\coprod_C B)$, $i_B\in\mathrm{Hom}_{\mathcal{S}}(B,A\coprod_C B)$ which completes the diagram $A\xleftarrow{\alpha}C\xrightarrow{\beta}B$ to a cocartesian square;

- $f\in\mathrm{Hom}_{\mathcal{S}}(A,C)$ and $g\in\mathrm{Hom}_{\mathcal{S}}(B,C)$, there is a fibered product $(A\prod_C B,p_A,p_B)$, with $p_A\in\mathrm{Hom}_{\mathcal{S}}(A\prod_C B,A)$, $p_B\in\mathrm{Hom}_{\mathcal{S}}(A\prod_C B,B)$, which completes the diagram $A\xrightarrow{f}C\xleftarrow{g}B$ to a cartesian square.

Suppose $i\in\mathrm{Hom}_{\mathcal{S}}(A_1,A)$, $f\in\mathrm{Hom}_{\mathcal{S}}(A_1,B)$ and $(B\coprod_{A_1}A,i_A,i_B)$ is their fibered coproduct. If $(A_2,j)=\mathrm{Coker}\,i$ then there is a morphism j_B :

$B\coprod_{A_1} A \to A_2$ such that the following diagram

$$\begin{tikzcd} A_1 \arrow[r, "i"] \arrow[d, "f"] & A \arrow[r, "j"] \arrow[d, "i_A"] & A_2 \arrow[d, "id"] \\ B \arrow[r, "i_B"] & B\coprod_{A_1} A \arrow[r, "j_B"] & A_2 \end{tikzcd}$$

is commutative (use the zero morphism from B to A_2). A formal verification shows that $(A_2, j_B) = \operatorname{Coker} i_B$.
Suppose $j \in \operatorname{Hom}_{\mathcal{S}}(A, A_2)$, $g \in \operatorname{Hom}_{\mathcal{S}}(B, A_2)$ and $(B\prod_{A_2} A, p_B, p_A)$ is their fibered product. If $(A_1, i) = \operatorname{Ker} j$ then there is an $i_B : A_1 \to B\prod_{A_2} A$ (use the zero map from A_1 to B) such that the following diagram

$$\begin{tikzcd} A_1 \arrow[r, "i"] & A \arrow[r, "j"] & A_2 \\ A_1 \arrow[u, "id"] \arrow[r, "i_B"] & B\prod_{A_2} A \arrow[u, "p_A"] \arrow[r, "p_B"] & B \arrow[u, "g"] \end{tikzcd}$$

is commutative and $(A_1, i_B) = \operatorname{Ker} p_B$.

A.1.2. *Strict morphisms.* A morphism $u \in \operatorname{Hom}_{\mathcal{S}}(A, B)$ is strict if the canonical morphism $\operatorname{Coim} u = \operatorname{Coker}(\operatorname{Ker} u) \to \operatorname{Im} u = \operatorname{Ker}(\operatorname{Coker} u)$ is isomorphism. One can verify that always $\operatorname{Ker} u = (A_1, i)$ is a strict monomorphism and $\operatorname{Coker} u = (B_1, j)$ is a strict epimorphism. By definition, a sequence of objects and morphisms

$$0 \longrightarrow A_1 \xrightarrow{i} A \xrightarrow{j} A_2 \longrightarrow 0 \tag{A.1}$$

in $\mathcal{S}$ is short exact if $(A_1, i) = \operatorname{Ker} j$ and $(A_2, j) = \operatorname{Coker} i$. In particular, any strict monomorphism (resp. strict epimorphism) can be included in a short exact sequence.

Definition. A pre-abelian category is special if it satisfies the following two axioms:
SP1. if $\alpha : C \to A$ is strict monomorphism then $i_B : B \to A\coprod\limits_{C} B$ is also strict monomorphism;
SP2. if $f : A \to C$ is strict epimorphism then $p_B : A\prod\limits_{C} B \to B$ is also strict epimorphism.

Remark. A typical example of pre-abelian special category is the category of modules with filtration.

Consider short exact sequence (A.1) in $\mathcal{S}$. If $f \in \mathrm{Hom}_{\mathcal{S}}(A_1, B)$ then we have the following commutative diagram

(A.2)
$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "i"] \arrow[d, "f"] & A \arrow[r, "j"] \arrow[d, "i_A"] & A_2 \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & B \arrow[r, "i_B"] & A\coprod_{A_1} B \arrow[r, "j_B"] & A_2 \arrow[r] & 0
\end{tikzcd}$$

Then $j_B = \mathrm{Coker}\, i_B$ is strict epimorphism and by axiom SP1, i_B is strict monomorphism. Then $\mathrm{Ker}\, j_B = \mathrm{Ker}(\mathrm{Coker}\, i_B) = \mathrm{Im} i_B = (B, i_B)$ and, therefore, the lower row of the above diagram is exact.
Dually, for any $g \in \mathrm{Hom}_{\mathcal{S}}(B, A_2)$ there is a commutative diagram

(A.3)
$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "i"] & A \arrow[r, "j"] & A_2 \arrow[r] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "i_B"] \arrow[u, "id"] & A\prod_{A_2} B \arrow[r, "p_B"] \arrow[u, "p_A"] & B \arrow[r] \arrow[u, "g"] & 0
\end{tikzcd}$$

where $i_B = \mathrm{Ker}\, j_B$ is strict monomorphism, by Axiom SP2, p_B is strict epimorphism and the lower row of this diagram is exact.
With relation to above diagram (A.2) we have the following ptoperties.

Lemma A.1. a) *The natural map* $\delta : \mathrm{Ker}\, f \longrightarrow \mathrm{Ker}\, i_A$ *is isomorphism;*
b) *if f is strict epimorphism then* $\mathrm{Ker}\, i_A$ *is also strict epimorphism.*

Proof. a) Suppose that $\mathrm{Ker}\, f = (K_1, \alpha_1)$ and $\mathrm{Ker}\, i_A = (K, \alpha)$. Then $\delta : \mathrm{Ker}\, f \longrightarrow \mathrm{Ker}\, i_A$ appears from the universal property of (K, α) because $i_A \circ i \circ \alpha_1 = i_B \circ f \circ \alpha_1 = 0$. The relation $j \circ \alpha = j_B \circ i_A \circ \alpha = 0$ implies the existence of $\tilde{\alpha} : K \longrightarrow A_1$ such that $i \circ \tilde{\alpha} = \alpha$. Then $i_B \circ f \circ \tilde{\alpha} = i_A \circ \alpha = 0$ and $f \circ \tilde{\alpha} = 0$ (use that i_B is monomorphism). By the universal property of (K_1, α_1) this gives the map $\delta_1 : K \longrightarrow K_1$ such that $\alpha_1 \circ \delta_1 = \tilde{\alpha}$ and this map is inverse to δ.
b) Suppose f is a strict epimorphism, then $(B, f) = \mathrm{Coker}\, \alpha_1$. Let $(\widetilde{C}, \tilde{\jmath}) = \mathrm{Coker}\, \alpha$. By functoriality, there is $\varepsilon : B \longrightarrow \widetilde{C}$ such that $\varepsilon \circ f = \tilde{\jmath} \circ i$. Then $\tilde{\jmath}$ and ε define a unique $\omega : A \coprod_{A_1} B \longrightarrow \widetilde{C}$ such that $\omega \circ i_B = \varepsilon$ and $\omega \circ i_A = \tilde{\jmath}$. But $i_A \circ \alpha = 0$ implies by the universal property of $(\widetilde{C}, \tilde{\jmath})$ the map $\omega_1 : \widetilde{C} \longrightarrow A \coprod_{A_1} B$ and one can verify that it is inverse to ω. □

Remark. If f is strict monomorphism then i_A is also strict monomorphism by axiom **SP2**.

With relation to diagram (A.3) we have the following Lemma which is dual to above Lemma A.1.

Lemma A.2. a) *The natural map* $\mathrm{Coker}\, p_A \longrightarrow \mathrm{Coker}\, g$ *is isomorphism;* b) *if g is strict epimorphism then p_A is also strict epimorphism.*

Proof. The proof is dual to the proof of Lemma A.1. □

LEMMA A.3. *A composition of two strict monomorphisms (resp., epimorphisms) is again strict monomorphism (resp., epimorphism).*

Proof. It will be sufficient to consider only the case of monomorphisms. Suppose $i \in \mathrm{Hom}_{\mathcal{S}}(A_1, A)$ and $i_1 \in \mathrm{Hom}_{\mathcal{S}}(A, B)$ are two strict monomorphisms. Construct the following commutative diagram:

(A.4)
$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "i"] \arrow[d, "\beta"] & A \arrow[r, "j"] \arrow[d, "i_1"] & A_2 \arrow[r] \arrow[d, "i_{A_2}"'] & 0 \\
0 \arrow[r] & K \arrow[r, "\alpha"] & B \arrow[r, "i_B"] \arrow[d, "j_1"] & A_2 \coprod_A B \arrow[r] \arrow[d, "j"] & 0 \\
& & C_1 \arrow[r, "\gamma"] & C &
\end{tikzcd}$$

Here $j = \mathrm{Coker}\, i$ is strict epimorphism and the upper right square is cocartesian. Therefore, i_B is strict epimorphism and we obtain the second line which is short exact. The morphisms i_1 and i_{A_2} are strict monomorphisms and we can complete our diagram by $(C_1, j_1) = \mathrm{Coker}\, i_1$ and $(C, j) = \mathrm{Coker}\, i_{A_2}$. The maps β and γ are obtained by functoriality and we have proved that they are isomorphisms. Therefore, $i_1 \circ i = \beta \circ \alpha$ is strict (use that $\alpha = \mathrm{Ker}\, i_B$ is strict). $\square$

A.1.3. *Bifunctor* $\mathrm{Ext}_{\mathcal{S}}$. If $\mathcal{S}$ is pre-abelian category then in the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r] \arrow[d, "id"] & A \arrow[r] \arrow[d, "f"] & A_2 \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & A_1 \arrow[r] & A' \arrow[r] & A_2 \arrow[r] & 0
\end{tikzcd}$$

the morphism f is isomorphism. Therefore, we can introduce the set of equivalence classes of short exact sequences $\mathrm{Ext}_{\mathcal{S}}(A_2, A_1)$. This set is functorial in both arguments due to axioms SP1 and SP2.

Suppose the objects of $\mathcal{S}$ are provided with commutative group structure respected by morphisms of $\mathcal{S}$. Then for any $A, B \in \mathcal{S}$, $\mathrm{Ext}_{\mathcal{S}}(A, B)$ has a natural group structure, where the class of split short exact sequences plays a role of neutral element. Remind that the sum $\varepsilon_1 + \varepsilon_2$ of two extensions $\varepsilon_1 : 0 \longrightarrow A_1 \xrightarrow{i'} A' \xrightarrow{j'} A_2 \longrightarrow 0$ and $\varepsilon_2 : 0 \longrightarrow A_1 \xrightarrow{i''} A'' \xrightarrow{j''} A_2 \longrightarrow 0$ is the lower line of the following commutative diagram relating the rows

$l=\varepsilon_1\oplus\varepsilon_2$, $\nabla^*(l)$ and $(+)_*\nabla^*(l)$,

$$\begin{tikzcd}
l:0 \arrow[r] & A_1\prod A_1 \arrow[r, "i'\prod i''"] & A'\prod A'' \arrow[r, "j'\prod j''"] & A_2\prod A_2 \arrow[r] & 0\\
\nabla^*(l):0 \arrow[r] & A_1\prod A_1 \arrow[u, "id"] \arrow[r, "i'\prod i''"] \arrow[d, "+"] & A'\prod_{A_2} A'' \arrow[u] \arrow[r, "j'\prod j''"] \arrow[d] & A_2 \arrow[u, "\nabla"'] \arrow[r] \arrow[d, "id"] & 0\\
(+)_*\nabla^*(l):0 \arrow[r] & A_1 \arrow[r] & A'' \arrow[r] & A_2 \arrow[r] & 0
\end{tikzcd}$$

Here ∇ is the diagonal morphism, $+$ is the morphism of the group structure on $\mathcal{S}$. For any $f\in\mathrm{Hom}_{\mathcal{S}}(A_1,B)$ and $g\in\mathrm{Hom}_{\mathcal{S}}(B,A_2)$ the corresponding morphisms $f_*:\mathrm{Ext}_{\mathcal{S}}(A_2,A_1)\to\mathrm{Ext}_{\mathcal{S}}(A_2,B)$ and $g^*:\mathrm{Ext}_{\mathcal{S}}(A_2,A_1)\to\mathrm{Ext}_{\mathcal{S}}(B,A_1)$ are homomorphisms of abelian groups. The proof is completely formal and goes along the lines of [27].

Suppose $\varepsilon\in\mathrm{Ext}_{\mathcal{S}}(A_2,A_1)$, then the extension $\varepsilon+(-\mathrm{id})^*\varepsilon$ splits. We shall need below the following explicit description of this splitting.

Let $\varepsilon:0\longrightarrow A_1\xrightarrow{i}A\xrightarrow{j}A_2\longrightarrow 0$. Then $\varepsilon+(-\mathrm{id})^*\varepsilon$ is the lower row in the following diagram

$$\begin{tikzcd}
0 \arrow[r] & A_1\prod A_1 \arrow[r, "i\prod i"] \arrow[d] & A\prod_{A_2} A \arrow[r, "(j,j)"] \arrow[d] & A_2 \arrow[r] \arrow[d, "id"] & 0\\
0 \arrow[r] & A_1 \arrow[r] & A_0 \arrow[r] & A_2 \arrow[r] & 0
\end{tikzcd}$$

where the left vertical arrow is the cokernel of the diagonal embedding $\nabla: A_1\to A_1\prod A_1$. One can see that the epimorphic map $A_0\to A_1$, which splits the lower exact sequence, is induced by the morphism $p_1-p_2:A\prod_{A_2}A\to A$. Finally, one can apply Serre's arguments [30] to obtain for any short exact sequence $0\longrightarrow A_1\xrightarrow{i}A\xrightarrow{j}A_2\longrightarrow 0$ and any $B\in\mathcal{S}$, the following standard 6-terms exact sequences of abelian groups

$$0\longrightarrow\mathrm{Hom}_{\mathcal{S}}(B,A_1)\xrightarrow{i_*}\mathrm{Hom}_{\mathcal{S}}(B,A)\xrightarrow{j_*}\mathrm{Hom}_{\mathcal{S}}(B,A_2)$$

$$\xrightarrow{\delta}\mathrm{Ext}_{\mathcal{S}}(B,A_1)\xrightarrow{i_*}\mathrm{Ext}_{\mathcal{S}}(B,A)\xrightarrow{j_*}\mathrm{Ext}_{\mathcal{S}}(B,A_2)$$

$$0\longrightarrow\mathrm{Hom}_{\mathcal{S}}(A_2,B)\xrightarrow{j^*}\mathrm{Hom}_{\mathcal{S}}(A,B)\xrightarrow{i^*}\mathrm{Hom}_{\mathcal{S}}(A_1,B)$$

$$\xrightarrow{\delta}\mathrm{Ext}_{\mathcal{S}}(A_2,B)\xrightarrow{i^*}\mathrm{Ext}_{\mathcal{S}}(A,B)\xrightarrow{j^*}\mathrm{Ext}_{\mathcal{S}}(A_1,B)$$

A.2. p-divisible groups. In this section $\mathcal{S}$ is a special pre-abelian category consisting of group objects. Denote by $\mathcal{S}_1$ the full subcategory of objects killed by p in $\mathcal{S}$, where p is a fixed prime number. Clearly, $\mathcal{S}_1$ is again special pre-abelian.

A.2.1. *Basic definitions.* Consider an inductive system $(C^{(n)}, i^{(n)})_{n\geqslant 0}$ of objects of $\mathcal{S}$, where $C^{(0)} = 0$ and all $i^{(n)} : C^{(n)} \to C^{(n+1)}$ are strict monomorphisms. Let for all $n \geqslant m \geqslant 0$, $i_{mn} = i^{(n-1)} \circ \ldots \circ i^{(m+1)} \circ i^{(m)} \in \mathrm{Hom}_{\mathcal{S}}(C^{(m)}, C^{(n)})$. Then all i_{mn} are strict monomorphisms. Follow Tate's paper [Ta] to define a p-divisible group in $\mathcal{S}$ as an inductive system $(C^{(n)}, i^{(n)})_{n\geqslant 0}$ such that for all $0 \leqslant m \leqslant n$,

a) $\mathrm{Coker}\, i_{mn} = (C^{(n-m)}, j_{n,n-m})$, i.e. there are short exact sequences:

$$0 \longrightarrow C^{(m)} \xrightarrow{i_{mn}} C^{(n)} \xrightarrow{j_{n,n-m}} C^{(n-m)} \longrightarrow 0$$

b) there are commutative diagrams

$$\begin{tikzcd} C^{(n)} \arrow[rr, "p^m \mathrm{id}_{C^{(n)}}"] \arrow[dr, "j_{n,n-m}"'] & & C^{(n)} \\ & C^{(n-m)} \arrow[ur, "i_{n-m,n}"'] & \end{tikzcd}$$

The above definition implies the existence of the following commutative diagrams with exact rows (where $m \leqslant n \leqslant n_1$):

$$\begin{tikzcd} 0 \arrow[r] & C^{(m)} \arrow[r, "i_{mn}"] \arrow[d, "id"] & C^{(n)} \arrow[r, "j_{n,n-m}"] \arrow[d, "i_{nn_1}"] & C^{(n-m)} \arrow[r] \arrow[d, "i_{n-m,n_1-m}"] & 0 \\ 0 \arrow[r] & C^{(m)} \arrow[r, "i_{mn_1}"] & C^{(n_1)} \arrow[r, "j_{n_1,n_1-m}"] & C^{(n_1-m)} \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & C^{(n)} \arrow[r, "i_{nn_1}"] \arrow[d, "j_{nm}"] & C^{(n_1)} \arrow[r, "j_{n_1,n_1-n}"] \arrow[d, "j_{n_1,m+n_1-n}"] & C^{(n_1-n)} \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & C^{(m)} \arrow[r, "i_{m,m+n_1-n}"] & C^{(m+n_1-n)} \arrow[r, "j_{m+n_1-n,n_1-n}"] & C^{(n_1-n)} \arrow[r] & 0 \end{tikzcd}$$

Also, for all $n \geqslant m \geqslant 0$, one has

- $(C^{(m)}, i_{mn}) = \mathrm{Ker}(p^m id_{C^{(n)}})$, $(C^{(m)}, j_{nm}) = \mathrm{Coker}\,(p^{n-m} id_{C^{(n)}})$;
- $i_{mn} = i_{n-1,n} \circ \ldots \circ i_{m,m+1}$ and $j_{nm} = j_{m+1,m} \circ \ldots \circ j_{n,n-1}$.

The set of p-divisible groups in $\mathcal{S}$ has a natural structure of category. This category is pre-abelian. In particular,

$$0 \longrightarrow (C_1^{(n)}, i_1^{(n)})_{n\geqslant 0} \xrightarrow{(\gamma_n)} (C^{(n)}, i^{(n)})_{n\geqslant 0} \xrightarrow{(\delta_n)} (C_2^{(n)}, i_2^{(n)})_{n\geqslant 0} \longrightarrow 0$$

is a short exact sequence of p-divisible groups iff for all $n \geqslant 1$, there are the following commutative diagrams with short exact rows in $\mathcal{S}$

$$\begin{tikzcd} 0 \arrow[r] & C_1^{(n)} \arrow[r, "\gamma_n"] \arrow[d, "i_1^{(n)}"] & C^{(n)} \arrow[r, "\delta_n"] \arrow[d, "i^{(n)}"] & C_2^{(n)} \arrow[r] \arrow[d, "i_2^{(n)}"] & 0 \\ 0 \arrow[r] & C_1^{(n+1)} \arrow[r, "\gamma_{n+1}"] & C^{(n+1)} \arrow[r, "\delta_{n+1}"] & C_2^{(n+1)} \arrow[r] & 0 \end{tikzcd}$$

A.2.2. *A property of uniqueness of p-divisible groups.*

Theorem A.4. *Let D be an object of $\mathcal{S}_1$ such that $\mathrm{Ext}_{\mathcal{S}_1}(D,D)=0$. If $(C^{(n)},i^{(n)})_{n\geqslant 0}$ and $(C_1^{(n)},i_1^{(n)})_{n\geqslant 0}$ are p-divisible groups in $\mathcal{S}$ such that $C^{(1)}\simeq C_1^{(1)}\simeq D$ then these p-divisible groups are isomorphic.*

Proof. We must prove that for all $n\geqslant 1$, there are isomorphisms $f_n: C^{(n)}\to C_1^{(n)}$ such that $i_1^{(n)}\circ f_n=f_{n+1}\circ i^{(n)}$. Suppose $n_0\geqslant 1$ and all such isomorphisms have been constructed for $1\leqslant n\leqslant n_0$. Therefore, we can assume that $C^{(n)}=C_1^{(n)}$ for $1\leqslant n\leqslant n_0$. Consider the following commutative duagrams with exact rows:

(A.5)
$$\begin{tikzcd}
\varepsilon_{n_0+1}: & 0 \arrow[r] & C^{(1)} \arrow[r, "i_1"] & C^{(n_0+1)} \arrow[r, "j_1"] & C^{(n_0)} \arrow[r] & 0 \\
\varepsilon_{n_0}: & 0 \arrow[r] & C^{(1)} \arrow[r, "i"] \arrow[u, "\mathrm{id}"] & C^{(n_0)} \arrow[r, "j"] \arrow[u, "i^{(n_0)}"'] & C^{(n_0-1)} \arrow[r] \arrow[u, "i^{(n_0-1)}"'] & 0
\end{tikzcd}$$

(A.6)
$$\begin{tikzcd}
\varepsilon'_{n_0+1}: & 0 \arrow[r] & C^{(1)} \arrow[r, "i'_1"] & C_1^{(n_0+1)} \arrow[r, "j'_1"] & C^{(n_0)} \arrow[r] & 0 \\
\varepsilon_{n_0}: & 0 \arrow[r] & C^{(1)} \arrow[r, "i"] \arrow[u, "\mathrm{id}"] & C^{(n_0)} \arrow[r, "j"] \arrow[u, "i_1^{(n_0)}"'] & C^{(n_0-1)} \arrow[r] \arrow[u, "i^{(n_0-1)}"'] & 0
\end{tikzcd}$$

Here in standard notation of Subsection A.2.1, $i_1=i_{1,n_0+1}$, $i'_1=i'_{1,n_0+1}$, $i=i_{1n_0}$, $j=j_{n_0,n_0-1}$, $j_1=j_{n_0+1,n_0}$ and $j'_1=j'_{n_0+1,n_0}$ (the dash means that the corresponding morphism is related to the second p-divisible group). We must construct an isomorphism $f_{n_0+1}: C^{(n_0+1)}\to C_1^{(n_0+1)}$ such that $f_{n_0+1}\circ i^{(n_0)}=i_1^{(n_0)}$. Consider the following commutative diagram obtained from above two diagrams

(A.7)
$$\begin{tikzcd}
0 \to C^{(1)}\prod C^{(1)} \arrow[r, "i_1\prod i'_1"] & C^{(n_0+1)}\prod_{C^{(n_0)}} C_1^{(n_0+1)} \arrow[r, "(j_1,j'_1)"] & C^{(n_0)} \to 0 \\
0 \to C^{(1)}\prod C^{(1)} \arrow[r, "i\prod i"] \arrow[u, "\mathrm{id}"] & C^{(n_0)}\prod_{C^{(n_0-1)}} C^{(n_0)} \arrow[r, "(j,j)"] \arrow[u, "i^{(n_0)}\prod i_1^{(n_0)}"'] & C^{(n_0-1)} \to 0 \arrow[u, "i^{(n_0-1)}"']
\end{tikzcd}$$

Notice that the morphisms of multiplication by p in $C^{(n_0+1)}$ and $C_1^{(n_0+1)}$ can be factored as follows

$$\begin{tikzcd}
C^{(n_0+1)} \arrow[rr, "p"] \arrow[dr, "j_1"'] & & C^{(n_0+1)} \\
& C^{(n_0)} \arrow[ur, "i^{(n_0)}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1^{(n_0+1)} \arrow[rr, "p"] \arrow[dr, "j'_1"'] & & C_1^{(n_0+1)} \\
& C^{(n_0)} \arrow[ur, "i_1^{(n_0)}"'] &
\end{tikzcd}$$

Therefore, we obtain the following commutative diagram

(A.8)
$$\begin{tikzcd}
C_1^{(n_0+1)} \prod_{C^{(n_0)}} C^{(n_0+1)} \arrow[r, "p"] \arrow[d, "(j_1', j_1)"] & C_1^{(n_0+1)} \prod_{C^{(n_0)}} C^{(n_0+1)} \\
C^{(n_0)} \arrow[r, "\nabla"] & C^{(n_0)} \prod_{C^{(n_0-1)}} C^{(n_0)} \arrow[u, "i_1^{(n_0)} \prod i^{(n_0)}"']
\end{tikzcd}$$

(here ∇ is the diagonal morphism). Let $\alpha : C^{(1)} \prod C^{(1)} \to C^{(1)}$ be the cokernel of the diagonal morphism $\nabla : C^{(1)} \to C^{(1)} \prod C^{(1)}$. Clearly, ∇ and α are, resp., strict monomorphism and strict epimorphism. Set $(D_{n_0+1}, \alpha_1) = \mathrm{Coker}\,((i_1 \prod i_1') \circ \nabla)$ and $(D_{n_0}, \alpha_0) = \mathrm{Coker}\,((i \prod i) \circ \nabla)$. Applying α_* to diagram (A.7) obtain the two lower rows of the following diagram

(A.9)
$$\begin{tikzcd}
0 \arrow[r] & C^{(1)} \arrow[r] & D_0 \arrow[r] & C^{(1)} \arrow[r] & 0 \\
0 \arrow[r] & C^{(1)} \arrow[r] \arrow[u, "id"] & D_{n_0+1} \arrow[r] \arrow[u, "s"] & C^{(n_0)} \arrow[r] \arrow[u, "j_{n_0 1}"] & 0 \\
0 \arrow[r] & C^{(1)} \arrow[r] \arrow[u, "id"] & D_{n_0} \arrow[r] \arrow[u, "u"] & C^{(n_0-1)} \arrow[r] \arrow[u, "i_{n_0-1,n_0}"] & 0
\end{tikzcd}$$

Note that the middle line of this diagram equals $\varepsilon_{n_0+1} - \varepsilon'_{n_0+1} \in \mathrm{Ext}\,(C^{(n_0)}, C^{(1)})$, and at the third row we have a trivial extension. This implies the existence of the first row of our diagram. As it was pointed out earlier, a splitting of the third line can be done via the morphism f from the commutative diagram

(A.10)
$$\begin{tikzcd}
& C^{(n_0)} \prod_{C^{(n_0-1)}} C^{(n_0)} \arrow[dl, "p_1 - p_2"'] \arrow[dr, "\alpha_0"] & \\
C^{(1)} & & D_{n_0} \arrow[ll, "f"']
\end{tikzcd}$$

(Notice that the morphism $s : D_{n_0+1} \to D_0$ is the cokernel of the composition $\mathrm{Ker} f \to D_{n_0} \xrightarrow{u} D_{n_0+1}$.)

Above diagram (A.8) means that the morphism of multiplication by p on $C_1^{(n_0+1)} \prod_{C^{(n_0)}} C^{(n_0+1)}$ factors through the diagonal embedding of $C^{(n_0)}$ into $C^{(n_0)} \prod_{C^{(n_0-1)}} C^{(n_0)}$. From diagram (A.10) it follows that $p\,\mathrm{id}_{D_{n_0+1}}$ factors through the embedding $\mathrm{Ker} f \to D_{n_0} \xrightarrow{u} D_{n_0+1}$. Therefore, $pD_0 = 0$ i.e. the first line in diagram (A.9) is an element of the trivial group $\mathrm{Ext}_{\mathcal{S}_1}(C^{(1)}, C^{(1)}) = 0$. So, the second row in (A.9) is a trivial extension, i.e. the extensions ε_{n_0+1} and ε'_{n_0+1} from diagrams (A.5) and (A.6) are equivalent. This implies the existence of isomorphism f_{n_0+1}. □

A.2.3. *Splitting of extensions of p-divisible groups.*

Theorem A.5. *Suppose $(C^{(n)}, i^{(n)})_{n\geq 0}$ is a p-divisible group in the category $\mathcal{S}$ and there are $D_1, D_2 \in \mathcal{S}_1$ such that $C^{(1)} \in \mathrm{Ext}_{\mathcal{S}_1}(D_2, D_1)$ and $\mathrm{Ext}_{\mathcal{S}_1}(D_1, D_2) = 0$. Then there is an exact sequence of p-divisible groups*

$$0 \longrightarrow (C_1^{(n)}, i_1^{(n)})_{n\geqslant 0} \longrightarrow (C^{(n)}, i^{(n)})_{n\geqslant 0} \longrightarrow (C_2^{(n)}, i_2^{(n)})_{n\geqslant 0} \longrightarrow 0$$

in $\mathcal{S}$ such that $C_1^{(1)} = D_1$ and $C_2^{(1)} = D_2$.

Proof. We have the exact sequence $0 \longrightarrow D_1 \xrightarrow{i} C^{(1)} \xrightarrow{j} D_2 \longrightarrow 0$. We must show for all $n \geqslant 1$, the existence of objects $C_1^{(n)}$, strict monomorphisms $\gamma_n : C_1^{(n)} \to C^{(n)}$ and $i_1^{(n)} : C_1^{(n)} \to C_1^{(n+1)}$ such that $(C_1^{(n)}, i_1^{(n)})_{n\geqslant 0}$ is a p-divisible group, the system $(\gamma_n)_{n\geqslant 0}$ defines an embedding of this p-divisible group into the original p-divisible group $(C^{(n)}, i^{(n)})_{n\geqslant 0}$, $C_1^{(1)} = D_1$ and $\gamma_1 = i$. Agree to use for all $0 \leqslant m \leqslant n$, the notation i_{mn} and j_{nm} from Subsection A.2.1 for the original p-divisible group and set $C^{(n)} = C_{n0}$.
Illustrate the idea of proof by considering the case $n = 2$. Set $C_{11} = D_1$ and consider the following commutative diagram with exact rows ε_2 and $\varepsilon_2^{(1)} = i^*\varepsilon_2$:

$$\begin{tikzcd}
\varepsilon_2 : 0 \arrow[r] & C_{10} \arrow[r, "i_{12}"] & C_{20} \arrow[r, "j_{21}"] & C_{10} \arrow[r] & 0 \\
\varepsilon_2^{(1)} : 0 \arrow[r] & C_{10} \arrow[r, "i_{12}^{(1)}"] \arrow[u, "id"] & C_{21} \arrow[r, "j_{21}^{(1)}"] \arrow[u, "\gamma_2^{(1)}"] & C_{11} \arrow[r] \arrow[u, "i"] & 0
\end{tikzcd} \tag{A.11}$$

Then $\gamma_2^{(1)} \circ p\,\mathrm{id}_{C_{21}} = p\,\mathrm{id}_{C_{20}} \circ \gamma_2^{(1)} = i_{12} \circ j_{21} \circ \gamma_2^{(1)} = i_{12} \circ i \circ j_{21}^{(1)} = \gamma_2^{(1)} \circ i_{12}^{(1)} \circ i \circ j_{21}^{(1)}$. By Lemma A.2, $\gamma_2^{(1)}$ is (strict) monomorphism. Therefore, $p\,\mathrm{id}_{C_{21}} = i_{21}^{(1)} \circ i \circ j_{21}^{(1)}$. Then the morphism $j_* : \mathrm{Ext}_{\mathcal{S}}(C_{11}, C_{10}) \to \mathrm{Ext}_{\mathcal{S}}(C_{11}, D_2)$ induces the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C_{10} \arrow[r, "i_{12}^{(1)}"] \arrow[d, "j"] & C_{21} \arrow[r, "j_{21}^{(1)}"] \arrow[d, "f"] & C_{11} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & D_2 \arrow[r, "\alpha"] & D_{21} \arrow[r] & C_{11} \arrow[r] & 0
\end{tikzcd}$$

Here $p\,\mathrm{id}_{D_{21}} \circ f = f \circ p\,\mathrm{id}_{C_{21}} = f \circ i_{12}^{(1)} \circ i \circ j_{21}^{(1)} = \alpha \circ j \circ i \circ j_{21}^{(1)} = 0$. By Lemma A.1, f is (strictly) epimorphic. Therefore, $p\mathrm{id}_{D_{21}} = 0$, i.e. $D_{21} \in \mathrm{Ext}_{\mathcal{S}_1}(C_{11}, D_2) = 0$. Then the exact sequence $\mathrm{Hom}_{\mathcal{S}} - \mathrm{Ext}_{\mathcal{S}}$ implies the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C_{10} \arrow[r, "i_{12}^{(1)}"] & C_{21} \arrow[r, "j_{21}^{(1)}"] & C_{11} \arrow[r] & 0 \\
0 \arrow[r] & C_{11} \arrow[r, "i_{12}^{(2)}"] \arrow[u, "i"] & C_{22} \arrow[r, "j_{21}^{(2)}"] \arrow[u, "\gamma_2^{(2)}"] & C_{11} \arrow[r] \arrow[u, "id"] & 0
\end{tikzcd}$$

Verify that one can set $C_1^{(2)} = C_{22}$ and $i_1^{(1)} = i_{12}^{(2)}$. Indeed,

$$\gamma_2^{(2)} \circ p\,\mathrm{id}_{C_{22}} = p\,\mathrm{id}_{C_{21}} \circ \gamma_2^{(2)} = (i_{12}^{(1)} \circ i) \circ (j_{21}^{(1)} \circ \gamma_2^{(2)}) = \gamma_2^{(2)} \circ i_{12}^{(2)} \circ j_{21}^{(2)}$$

and because $\gamma_2^{(2)}$ is monomorphism (use axiom SP1), $p\,\mathrm{id}_{C_{22}} = i_{12}^{(2)} \circ j_{21}$. Thus, we constructed a segment of length 2 of the p-divisible group $(C_1^{(n)}, i_1^{(n)})_{n\geqslant 0}$. Consider the general case.

Lemma A.6. *In the category $\mathcal{S}$ there are the following commutative diagrams with exact lines:*

- *for $k \geqslant 1$,*

$$\begin{tikzcd}
E_k^1 : 0 \arrow[r] & C_{k-1,0} \arrow[r, "i_{k-1,k}^{(0)}"] & C_{k0} \arrow[r, "j_{k1}^{(0)}"] & C_{10} \arrow[r] & 0 \\
0 \arrow[r] & C_{k-1,0} \arrow[r, "i_{k-1,k}^{(1)}"] \arrow[u, "id"] & C_{k1} \arrow[r, "j_{k1}^{(1)}"] \arrow[u, "\gamma_k^{(1)}"] & C_{11} \arrow[r] \arrow[u, "i"] & 0
\end{tikzcd}$$

- *for $2 \leqslant t \leqslant k$,*

$$\begin{tikzcd}
E_k^t : 0 \arrow[r] & C_{k-1,t-2} \arrow[r, "i_{k-1,k}^{(t-1)}"] & C_{k,t-1} \arrow[r, "j_{k1}^{(t-1)}"] & C_{11} \arrow[r] & 0 \\
0 \arrow[r] & C_{k-1,t-1} \arrow[r, "i_{k-1,k}^{(t)}"] \arrow[u, "\gamma_{k-1}^{(t-1)}"] & C_{kt} \arrow[r, "j_{k1}^{(t)}"] \arrow[u, "\gamma_k^{(t)}"] & C_{11} \arrow[r] \arrow[u, "id"] & 0
\end{tikzcd}$$

- *for $1 \leqslant t < k$,*

$$\begin{tikzcd}
\Delta_k^t : 0 \arrow[r] & C_{k-1,t-1} \arrow[r, "i_{k-1,k}^{(t)}"] \arrow[d, "j_{k-1,k-2}^{(t-1)}"] & C_{kt} \arrow[r, "j_{k1}^{(t)}"] \arrow[d, "j_{k,k-1}^{(t)}"] & C_{11} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & C_{k-2,t-1} \arrow[r, "i_{k-2,k-1}^{(t)}"] & C_{k-1,t} \arrow[r, "j_{k-1,1}^{(t)}"] & C_{11} \arrow[r] & 0
\end{tikzcd}$$

- *for $1 \leqslant t < k$,*

$$\begin{tikzcd}
\Omega_k^t : 0 \arrow[r] & C_{kt} \arrow[r, "\gamma_k^{(t)}"] \arrow[d, "j_{k,k-1}^{(t)}"] & C_{k,t-1} \arrow[r, "f_k^{(t)}"] \arrow[d, "j_{k,k-1}^{(t-1)}"] & D_2 \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & C_{k-1,t} \arrow[r, "\gamma_{k-1}^{(t)}"] & C_{k-1,t-1} \arrow[r, "f_{k-1}^{(t)}"] & D_2 \arrow[r] & 0
\end{tikzcd}$$

where for all indices k, $C_{k0} = C^{(k)}$, $i_{k,k+1}^{(0)} = i^{(k)}$, $j_{k+1,1}^{(0)} = j_{k+1,1}$ and $j_{k+1,k}^{(0)} = j_{k+1,k}$.

Proof. Construct the diagram E_1^1 by setting $\gamma_1^{(1)} = i$, $j_{11}^{(0)} = \mathrm{id}_{C_{10}}$, $j_{11}^{(1)} = \mathrm{id}_{C_{11}}$. Then for any $k \geqslant 2$, the upper row of E_k^1 is the short exact sequence $\varepsilon_k \in \mathrm{Ext}_{\mathcal{S}}(C_{10}, C_{k-1,0})$ from the original p-divisible group $(C_{k0}, i^{(k)})_{k\geq 0}$. Then E_k^1 is just a standard diagram relating ε_k and $i^*\varepsilon_k$. For any $k \geq 2$, we have $(j_{k,k-1})_*\varepsilon_k = \varepsilon_{k-1}$, therefore, $(j_{k,k-1})_*(i^*\varepsilon_k) = i^*\varepsilon_{k-1}$ and we obtain Δ_k^1. The upper row of Ω_k^1 is obtained from the middle column of E_k^1 because $\mathrm{Coker}\,\gamma_k^{(1)} \simeq \mathrm{Coker}\, i = (D_2, j)$. Similarly, the lower row of Ω_k^1 is obtained from E_{k-1}^1. The left square of Ω_k^1 is commutative by the definition of $j_{k,k-1}^{(1)}$. The right square is commutative because Ω_k^1 relates diagrams E_k^1 and E_{k-1}^1.

Suppose now we are given integers $k_0 \geqslant 2$ and $t_0 < k_0$ such that the required diagrams E_k^t, Δ_k^t and Ω_k^t have been already constructed for all $k < k_0$ with all relevant t and for $k = k_0$ with $1 \leqslant t \leqslant t_0$. Clearly, all $i_{k,k-1}^{(t)}$ and $\gamma_k^{(t)}$ are strict monomorphisms and all $j_{k1}^{(t)}$, $j_{k,k-1}^{(t)}$ and $f_k^{(t)}$ are strict epimorphisms.

Constructing $E_{k_0}^{t_0+1}$. Apply $(f_{k_0-1}^{(t_0)})_*$ to $E_{k_0}^{t_0}$:

$$\begin{tikzcd}
\varepsilon_{k_0}^{(t_0)} : 0 \arrow[r] & C_{k_0-1,t_0-1} \arrow[r, "i_{k_0-1,k_0}^{(t_0)}"] \arrow[d, "f_{k_0-1}^{(t_0)}"'] & C_{k_0,t_0} \arrow[r, "j_{k_0 1}^{(t_0)}"] \arrow[d] & C_{11} \arrow[r] \arrow[d, "id"'] & 0 \\
0 \arrow[r] & D_2 \arrow[r] & D^* \arrow[r] & C_{11} \arrow[r] & 0
\end{tikzcd}$$

Then $\mathrm{Ker}\,(C_{k_0t_0} \to D^*) = (C_{k_0-1,t_0}, i_{k_0-1,k_0}^{(t_0)} \circ \gamma_{k_0-1}^{(t_0)})$. Consider the strict monomorphism $\gamma_{k_0t_0} := \gamma_{k_0}^{(1)} \circ \ldots \circ \gamma_{k_0}^{(t_0)} : C_{k_0t_0} \to C_{k_00}$ and its analogue $\gamma_{k_0-1,t_0-1} : C_{k_0-1,t_0-1} \to C_{k_0-1,0}$. Because $t_0 \neq k_0$, the diagrams $\Omega_{k_0}^{t_0}$ and $E_{k_0}^t$ give the commutative diagram

$$\begin{tikzcd}
C_{k_0t_0} \arrow[rr, "\gamma_{k_0t_0}"] \arrow[d, "j_{k_0,k_0-1}^{(t_0)}"'] & & C_{k_00} \arrow[d, "j_{k_0,k_0-1}^{(0)}"'] \\
C_{k_0-1,t_0} \arrow[r, "\gamma_{k_0-1}^{(t_0)}"] & C_{k_0-1,t_0-1} \arrow[r, "\gamma_{k_0-1,t_0-1}"] \arrow[d, "i_{k_0-1,k_0}^{(t_0)}"'] & C_{k_0-1,0} \arrow[d, "i_{k_0-1,k_0}^{(0)}"'] \\
& C_{k_0t_0} \arrow[r, "\gamma_{k_0t_0}"] & C_{k_00}
\end{tikzcd}$$

Then $p\mathrm{id}_{C_{k_00}} = i_{k_0-1,k_0}^{(0)} \circ j_{k_0,k_0-1}^{(0)}$ implies $p\mathrm{id}_{C_{k_0t_0}} = (i_{k_0-1,k_0}^{(t_0)} \circ \gamma_{k_0-1}^{(t_0)}) \circ j_{k_0,k_0-1}^{(0)}$, i.e. $p\mathrm{id}_{C_{k_0t_0}}$ factors through $\mathrm{Ker}(C_{k_0t_0} \to D^*)$ and $p\mathrm{id}_{D^*} = 0$. Then $\mathrm{Ext}_{\mathcal{S}_1}(C_{11}, D_2) = 0$ implies $(f_{k_0-1}^{(t_0)})_* \varepsilon_{k_0}^{(t_0)} = 0$, and 6-terms $\mathrm{Hom}_{\mathcal{S}} - \mathrm{Ext}_{\mathcal{S}}$ exact sequence gives $E_{k_0}^{t_0+1}$:

$$\begin{tikzcd}
0 \arrow[r] & C_{k_0-1,t_0-1} \arrow[r, "i_{k_0-1,k_0}^{(t_0)}"] & C_{k_0t_0} \arrow[r, "j_{k_0 1}^{(t_0)}"] & C_{11} \arrow[r] & 0 \\
0 \arrow[r] & C_{k_0-1,t_0} \arrow[r, "i_{k_0-1,k_0}^{(t_0+1)}"] \arrow[u, "\gamma_{k_0-1}^{(t_0)}"'] & C_{k_0,t_0+1} \arrow[r, "j_{k_0 1}^{(t_0+1)}"] \arrow[u, "\gamma_{k_0}^{(t_0+1)}"'] & C_{11} \arrow[r] \arrow[u, "id"] & 0
\end{tikzcd}$$

We shall denote the rows of this diagram by $\varepsilon_{k_0}^{(t_0)}$ and $\varepsilon_{k_0}^{(t_0+1)}$.

Constructing $\Delta_{k_0}^{t_0+1}$. Assume that $t_0+1<k_0$. The above extension $\varepsilon_{k_0}^{(t_0+1)}$ is not uniquely defined by $\varepsilon_{k_0}^{(t_0)}$. Show that its choice can be done in such a way that the diagram $\Delta_{k_0}^{t_0+1}$ commutes. Consider the short exact sequences from $\Omega_{k_0-1}^{t_0}$. They give rise to the following exact sequences of abelian groups, where $H := \mathrm{Hom}(C_{11}, D_2)$ and $E := \mathrm{Ext}(C_{11}, D_2)$

(A.12)

$$\begin{tikzcd}
H \arrow[r] \arrow[d, "id"] & \mathrm{Ext}(C_{11}, C_{k_0-1,t_0}) \arrow[r, "\gamma^{(t_0)}_{k_0-1*}"] \arrow[d, "j^{(t_0)}_{k_0-1,k_0-2*}"'] & \mathrm{Ext}(C_{11}, C_{k_0-1,t_0-1}) \arrow[r] \arrow[d, "j^{(t_0-1)}_{k_0-1,k_0-2*}"] & E \arrow[d, "id"] \\
H \arrow[r] & \mathrm{Ext}(C_{11}, C_{k_0-2,t_0}) \arrow[r, "\gamma^{(t_0)}_{k_0-2*}"] & \mathrm{Ext}(C_{11}, C_{k_0-2,t_0-1}) \arrow[r] & E
\end{tikzcd}$$

As we saw earlier, the commutativity of $E_{k_0}^{t_0+1}$ is equivalent to the following relation

$$(\gamma_{k_0-1}^{(t_0)})_* \varepsilon_{k_0}^{(t_0+1)} = \varepsilon_{k_0}^{(t_0)} \tag{A.13}$$

From $\Delta_{k_0}^{t_0}$ it follows that $\varepsilon_{k_0-1}^{(t_0)} = (j_{k_0-1,k_0-2}^{(t_0-1)})_* \varepsilon_{k_0}^{(t_0)}$, and from $E_{k_0-1}^{t_0+1}$ it follows that $(\gamma_{k_0-2}^{(t_0)})_* \varepsilon_{k_0-1}^{(t_0+1)} = \varepsilon_{k_0-1}^{(t_0)}$. Then (A.12) implies that $\varepsilon_{k_0}^{(t_0+1)}$ from (A.13) can be chosen in such a way that $(j_{k_0-1,k_0-2}^{(t_0)})_* \varepsilon_{k_0}^{(t_0+1)} = \varepsilon_{k_0-1}^{(t_0+1)}$. This gives the diagram $\Delta_{k_0}^{t_0+1}$.

Constructing $\Omega_{k_0}^{t_0+1}$. The above arguments imply that the left squares of diagrams $E_{k_0}^{t_0+1}$ and $E_{k_0-1}^{t_0}$ are related via the following commutative diagram

$$
\begin{tikzcd}
C_{k_0-1,t_0-1} \arrow[rrr, "i^{(t_0)}_{k_0-1,k_0}"] \arrow[dr, "j^{(t_0-1)}_{k_0-1,k_0-2}"] & & & C_{k_0t_0} \arrow[dl, "j^{(t_0)}_{k_0,k_0-1}"'] \\
& C_{k_0-1,t_0-2} \arrow[r, "i^{(t_0-1)}_{k_0-2,k_0-1}"] & C_{k_0-1,t_0-1} & \\
& C_{k_0-2,t_0-1} \arrow[u, "\gamma^{(t_0-1)}_{k_0-2}"] \arrow[r, "i^{(t_0)}_{k_0-2,k_0-1}"] & C_{k_0-1,t_0} \arrow[u, "\gamma^{(t_0)}_{k_0-1}"'] & \\
C_{k_0-1,t_0} \arrow[uuu, "\gamma^{(t_0-1)}_{k_0-1}"] \arrow[ur, "j^{(t_0)}_{k_0-1,k_0-2}"] \arrow[rrr, "i^{(t_0+1)}_{k_0-2,k_0-1}"] & & & C_{k_0,t_0+1} \arrow[uuu, "\gamma^{(t_0+1)}_{k_0}"'] \arrow[ul, "j^{(t_0+1)}_{k_0,k_0-1}"']
\end{tikzcd}
$$

From diagrams $\Omega^{t_0}_{k_0-1}$, $E^{t_0+1}_{k_0}$ and $E^{t_0}_{k_0-1}$ it follows that the induced map $\mathrm{Coker}\gamma^{(t_0+1)}_{k_0} \to \mathrm{Coker}\gamma^{(t_0)}_{k_0-1} \simeq D_2$ is isomorphism. This is equivalent to the existence of diagram $\Omega^{t_0+1}_{k_0}$. The lemma is proved. $\square$

For any $k \geqslant 1$, set $C_{kk} = C^{(k)}_1, i^{(k)}_{k-1,k} = i^{(k)}_1$. Then use diagrams E^k_k to define the inductive system $(C^{(k)}_1, i^{(k)}_1)_{k\geqslant 0}$. Denote by γ_k the strict monomorphism $\gamma^{(1)}_k \circ \ldots \circ \gamma^{(k)}_k : C^{(k)}_1 \to C^{(k)}$. From diagrams E^t_k, $1 \leq t \leq k$, obtain the following commutative diagrams:

$$
\begin{tikzcd}
0 \arrow[r] & C^{(k-1)} \arrow[r, "i^{(k)}"] & C^{(k)} \arrow[r, "j_{k1}"] & C^{(1)} \arrow[r] & 0 \\
0 \arrow[r] & C^{(k-1)}_1 \arrow[u, "\gamma_{k-1}"] \arrow[r, "i^{(k)}_1"] & C^{(k)}_1 \arrow[u, "\gamma_k"] \arrow[r, "j^{(k)}_{k1}"] & C^{(1)}_1 \arrow[u, "\gamma_1"] \arrow[r] & 0
\end{tikzcd}
\tag{A.14}
$$

It remains only to prove that the inductive system $(C^{(n)}_1, i^{(n)}_1)_{n\geq 0}$ is a p-divisible group in $\mathcal{S}$. From diagrams E^k_k and Δ^{k-1}_k obtain the following commutative diagrams with exact rows

(A.15)
$$\begin{tikzcd} 0 \arrow[r] & C_{k-1,k-1} \arrow[r, "i^{(k)}_{k-1,k}"] \arrow[d, "j^{(k-2)}_{k-1,k-2}\circ\gamma^{(k-1)}_{k-1}"'] & C_{kk} \arrow[r, "j^{(k)}_{k1}"] \arrow[d, "j^{(k-1)}_{k,k-1}\circ\gamma^{(k)}_{k}"] & C_{11} \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & C_{k-2,k-2} \arrow[r, "i^{(k-1)}_{k-2,k-1}"] & C_{k-1,k-1} \arrow[r, "j^{(k-1)}_{k-1,1}"] & C_{11} \arrow[r] & 0 \end{tikzcd}$$

If $k = 3$ then the left vertical morphism of this diagram is equal to $j^{(1)}_{21} \circ \gamma^{(2)}_2 = j^{(2)}_{21}$ and is a strict monomorphism. By induction all morphisms $j'_{k,k-1} := j^{(k-1)}_{k,k-1} \circ \gamma^{(k)}_k$ are strict epimorphisms and are included in the following commutative diagrams

(A.16)
$$\begin{tikzcd} C^{(k)} \arrow[r, "j_{k,k-1}"] & C^{(k-1)} \\ C^{(k)}_1 \arrow[u, "\gamma_k"] \arrow[r, "j'_{k,k-1}"] & C^{(k-1)}_1 \arrow[u, "\gamma_{k-1}"'] \end{tikzcd}$$

For $0 \leqslant m \leqslant n$, set $j'_{nm} = j'_{m+1,m} \circ \ldots \circ j'_{n,n-1}$ and $i'_{mn} = i'_{n-1,n} \circ \ldots \circ i'_{m,m+1}$. Composing diagrams (A.15) obtain the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & C^{(n-1)}_1 \arrow[r, "i'_{n-1,n}"] \arrow[d, "j'_{n-1,m-1}"] & C^{(n)}_1 \arrow[r, "j^{(n)}_{n1}"] \arrow[d, "j'_{nm}"] & C^{(1)}_1 \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & C^{(m-1)}_1 \arrow[r, "i'_{m-1,m}"] & C^{(m)}_1 \arrow[r, "j^{(m)}_{m1}"] & C^{(1)}_1 \arrow[r] & 0 \end{tikzcd}$$

Thus, $i'_{n-1,n}$ induces the isomorphism $\operatorname{Ker} j'_{n-1,m-1} \simeq \operatorname{Ker} j'_{nm}$. Therefore, $\operatorname{Ker} j'_{nm} = (C^{(n-m)}_1, i'_{n-m,n})$ if we prove that

(A.17)
$$\operatorname{Ker} j'_{k1} = (C^{k-1}_1, i'_{k-1,k}).$$

As we noticed earlier, $j'_{k1} = j_{21} \circ \ldots \circ j_{k,k-1}$. Therefore, diagrams (A.16) imply that $j_{k1} \circ \gamma_k = \gamma_1 \circ j'_{k1}$. Now diagram (A.14) implies that $\gamma_1 \circ j^{(k)}_{k1} = \gamma_1 \circ j'_{k1}$ and, therefore, $j^{(k)}_{k1} = j'_{k1}$ because γ_1 is monomorphism. Hence equality (A.17) folows from diagram (A.14) and $(C^{(n)}_1, i^{(n)}_1)_{n \geqslant 0}$ satisfies the part a) of the definition of p-divisible groups.

From diagrams (A.14) and (A.16) one can easily obtain for all indices $0 \leqslant m \leqslant n$, the commutativity of the following diagrams

$$\begin{tikzcd}
C^{(n)} \arrow[r, "j_{n,n-m}"] & C^{(n-m)} \arrow[r, "i_{n-m,n}"] & C^{(n)} \\
C_1^{(n)} \arrow[u, "\gamma_n"] \arrow[r, "j'_{n,n-m}"] & C_1^{(n-m)} \arrow[u, "\gamma_{n-m}"] \arrow[r, "i'_{n-m,n}"] & C_1^{(n)} \arrow[u, "\gamma_n"]
\end{tikzcd}$$

Because γ_n is monomorphism, the equality $i_{n-m,n} \circ j_{n,n-m} = p^m \mathrm{id}_{C^{(n)}}$ implies the equality $i'_{n-m,n} \circ j'_{n,n-m} = p^m \mathrm{id}_{C_1^{(n)}}$. This gives the part b) of the definition of p-divisible groups for $(C_1^{(n)}, i_1^{(n)})_{n \geqslant 0}$. The proposition is proved. $\square$

Appendix B. SAGE program

This program computes the class number of the field $\mathbb{Q}(\sqrt[3]{3}, \zeta_9)$ and finds the basis $f_1, f_2, \dots, f_9$ of the 3-subgroup of units in this field such that for the normalized 3-adic valuation v_3 and all $1 \leqslant i \leqslant 9$, the natural numbers $a_i = 18v_3(f_i \pm 1)$ are prime to 3 and $1 \leqslant a_1 < a_2 < \cdots < a_9$. The result appears as the vector $af = (a_1, a_2, \dots, a_9) = (1, 2, 4, 5, 7, 8, 10, 13, 16)$.

```
sage: L.<b>=NumberField(x^3-3);
sage: R.<t>=L[]
sage: M.<c>=L.extension(t^6+t^3+1);
sage: X.<d>=M.absolute_field();
sage: h=X.class_number();
sage: e=list(X.unit_group().gens())
sage: def p(x):
...       for i in range(1,3):
...          if valuation(norm(X(x+2*i-3)),3)!=0:
...               break
...       return valuation(norm(X(x+2*i-3)),3)
...
...
sage: a=[p(x) for x in e]
sage: f=[e.pop(a.index(min(a)))]
sage: while len(e)!=0:
...       a=[p(x) for x in e]
...       i0=a.index(min(a))
...
...       for j in range(len(f)):
...           for k in range(5):
...               s=0
...               if p(f[j]^(3^k))>p(e[i0]):
...                   break
```

```
...
...                 if p(e[i0])==p(f[j]^(3^k)):
...                     s=1
...                     break
...
...             if s==1:
...                 for i in range(1,3):
...                     if p(e[i0])<p(e[i0]/(f[j]^(i*3^k))):
...                         e[i0]=e[i0]/(f[j]^(i*3^k))
...                         break
...
...                 break
...             if j+1==len(f) and s==0:
...                       f.append(e.pop(i0))
...
sage: af=[p(x) for x in f];
sage: print h
sage: print af
1
[1, 2, 4, 5, 7, 8, 10, 13, 16]
```

Remark. First 4 lines introduce the field $X = \mathbb{Q}(\sqrt[3]{3}, \zeta_9)$; its elements appear as polynomials in variable d of degree $\leqslant 17$. Then we find the class number of X and form the array $e = (e[1], \dots, e[9])$ of minimal generators of the group U/U^3, where U is the group of units in X. Next block gives a standard procedure to determine for any $x \in U$ the maximal natural number $p(x)$ such that $x \pm 1$ is divisible precisely by $\pi^{p(x)}$, where $\pi \in X$, $(\pi^{18}) = (3)$. The remaining part of the program is based on a standard technique from Linear algebra to rearrange the given system of generators e into a new system f with required properties. As a matter of fact, we use that the class number of X is prime to 3 (it equals 1) by allowing $k < 5$ on line 21. (Any unit $x \equiv 1 \bmod \pi^{28}$ is a cube in the 3-completion of X by Hensel's Lemma and, therefore, is a cube in X.) The last two lines contain the values of the class number of X and the exponents $(a(f[1]), \dots, a(f[9]))$.

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Victor Abrashkin
Department of Mathematical Sciences
Durham University
Science Laboratories,
South Rd, Durham DH1 3LE
United Kingdom
victor.abrashkin@durham.ac.uk
and
Steklov Mathematical Institute
Gubkina str. 8,
119991 Moscow
Russia