

## STABLE MAPS AND CHOW GROUPS

D. HUYBRECHTS AND M. KEMENY

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ABSTRACT. According to the Bloch–Beilinson conjectures, an automorphism of a K3 surface  $X$  that acts as the identity on the transcendental lattice should act trivially on  $\mathrm{CH}^2(X)$ . We discuss this conjecture for symplectic involutions and prove it in one third of all cases. The main point is to use special elliptic K3 surfaces and stable maps to produce covering families of elliptic curves on the generic K3 surface that are invariant under the involution.

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0.1. Let  $X$  be a complex projective K3 surface with an automorphism  $f : X \xrightarrow{\sim} X$ . According to the general philosophy of the Bloch–Beilinson conjectures, the induced action of  $f$  on the kernel of the cycle map  $\mathrm{CH}^*(X) \rightarrow H^*(X, \mathbb{Z})$  should be determined by the action of  $f$  on the cokernel of the cycle map. More precisely, one expects the following to be true:

CONJECTURE 0.1.  $f^* = \mathrm{id}$  on  $\mathrm{CH}^2(X)_0$  if and only if  $f^* = \mathrm{id}$  on  $T(X)$ .

Here,  $\mathrm{CH}^2(X)_0 \subset \mathrm{CH}^2(X)$  is the degree zero part, i.e. the kernel of the cycle map  $\mathrm{CH}^2(X) \rightarrow H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ , and  $T(X) \subset H^2(X, \mathbb{Z})$  is the transcendental lattice which can be described as the orthogonal complement of the Néron–Severi group  $\mathrm{NS}(X) \subset H^2(X, \mathbb{Z})$ . Alternatively,  $T(X) \subset H^2(X, \mathbb{Z})$  is the smallest sub-Hodge structure such that  $H^{2,0}(X) \subset T(X) \otimes \mathbb{C}$ . Thus,  $f^* = \mathrm{id}$  on  $T(X)$  if and only if  $f$  acts trivially on  $H^{2,0}(X)$ . The latter is spanned by the unique (up to scaling) regular two-form  $\sigma \in H^0(X, \Omega_X^2)$ , which we think of as a holomorphic symplectic structure. For this reason, an automorphism  $f : X \xrightarrow{\sim} X$  with  $f^* = \mathrm{id}$  on  $T(X)$  is called a symplectomorphism.

It is well known that  $f^* = \mathrm{id}$  on  $\mathrm{CH}^2(X)_0$  implies that  $f$  acts trivially on  $T(X)$  (see e.g. [16, Ch. 23]). Appropriately rephrased, this holds for arbitrary

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smooth projective varieties and for arbitrary correspondences. It is the converse of the statement that is difficult and that shall be discussed here for symplectic involutions of K3 surfaces, i.e. automorphisms  $f$  of order two with  $f^*\sigma = \sigma$ .

0.2. K3 surfaces  $X$  endowed with a symplectic involution  $f : X \xrightarrow{\sim} X$  come in families. As shown by van Geemen and Sarti in [4], the moduli space of such  $(X, f)$  has one resp. two connected components in each degree  $2d > 0$  depending on the parity of  $d$ . To be more precise, let  $\Lambda_d$  be the lattice  $\mathbb{Z}\ell \oplus E_8(-2)$  with  $(\ell, \ell) = 2d$  and denote for  $d \equiv 0(2)$  by  $\tilde{\Lambda}_d$  the unique even lattice containing  $\Lambda_d$  with  $\tilde{\Lambda}_d/\Lambda_2 \cong \mathbb{Z}/2\mathbb{Z}$  and such that  $E_8(-2) \subset \tilde{\Lambda}_d$  is primitive (see [4, Prop. 2.2]).

Then for generic  $(X, f)$  one has  $\text{NS}(X) \cong \Lambda_d$  or  $\text{NS}(X) \cong \tilde{\Lambda}_d$ . The class  $\ell$  corresponds under this isomorphism to an ample line bundle  $L$  on  $X$  which spans the  $f$ -invariant part of  $\text{NS}(X)$ . If  $(X, f)$  is not generic, then one still finds  $\Lambda_d$  or  $\tilde{\Lambda}_d$  as a primitive sublattice in  $\text{NS}(X)$  with  $E_8(-2)$  as the orthogonal complement of the invariant part. Conversely, by the Global Torelli theorem any  $X$  parametrized by the (non-empty and in fact 11-dimensional) connected moduli spaces  $\mathfrak{M}_{\Lambda_d}$  or  $\mathfrak{M}_{\tilde{\Lambda}_d}$  of  $\Lambda_d$  resp.  $\tilde{\Lambda}_d$ -lattice polarized K3 surfaces comes with a symplectic involution  $f$  that is determined by its action  $= -\text{id}$  on  $E_8(-2)$  and  $= \text{id}$  on its orthogonal complement.

In other words, for each  $d \equiv 1(2)$  the moduli space of K3 surfaces  $X$  with a symplectic involution  $f$  and an invariant polarization of degree  $2d$  has one connected component  $\mathfrak{M}_{\Lambda_d}$ , whereas for  $d \equiv 0(2)$  it has two connected components,  $\mathfrak{M}_{\Lambda_d}$  and  $\mathfrak{M}_{\tilde{\Lambda}_d}$ . Thus the following theorem, the main result of the present paper, proves Conjecture 0.1 in one third of all possible cases.

**THEOREM 0.2.** *Let  $d \equiv 0(2)$  and  $(X, f) \in \mathfrak{M}_{\tilde{\Lambda}_d}$ . Then  $f^* = \text{id}$  on  $\text{CH}^2(X)$ .*

0.3. For  $d = 1$  (double covers of  $\mathbb{P}^2$ ) and  $d = 2$  (quartics in  $\mathbb{P}^3$ ) the conjecture is known to hold, see [2, 14, 15]. For  $d = 3$  (complete intersection of a cubic and a quadric in  $\mathbb{P}^4$ ) an interesting approach is outlined in [6]. Theorem 3.2 in [17] proves the conjecture for equivariant complete intersections in varieties with trivial Chow groups.

In [7] the conjecture has been proven for  $(X, f)$  in dense subsets of  $\mathfrak{M}_{\Lambda_d}$  and  $\mathfrak{M}_{\tilde{\Lambda}_d}$ . The proof there relies on Fourier–Mukai equivalences of the bounded derived category of coherent sheaves on  $X$  and it is not clear how to push the techniques further to cover generic and hence arbitrary  $(X, f)$ .

The techniques to prove Theorem 0.2 can be applied to symplectic automorphisms  $f : X \xrightarrow{\sim} X$  of order  $> 2$ . If the order of  $f$  is a prime  $p$ , then  $p = 2, 3, 5$ , or  $7$  (cf. [13]), and the results of [4] have in [3] been successfully generalized to cover also the cases  $p = 3, 5$ , and  $7$ . Our methods prove Conjecture 0.1 for many components of the moduli space of  $(X, f)$  in these cases too, see Section 5.

**THEOREM 0.3.** *For  $p = 3, 5$ , or  $7$  and  $d = ep$ , there exists one component of the moduli space of polarized K3 surfaces  $(X, L)$  with a symplectic automorphism  $f : X \xrightarrow{\sim} X$  of order  $p$  and  $L^2 = 2d$  such that Conjecture 0.1 holds true.  $\square$*

0.4. The proof of Theorem 0.2 neither uses derived categories as in [7] nor any deep cycle arguments as e.g. in [6]. As we shall explain in Section 1, it is enough to find a dominating family of integral genus one curves on  $X$  that are invariant under  $f$  and avoid the fixed points of  $f$ . The conjecture is then deduced from the absence of torsion in  $\mathrm{CH}^2(X)$ . It is not clear whether the existence of such a family should be expected in general, but it will be shown here for generic K3 surfaces parametrized by points in  $\mathfrak{M}_{\Lambda_d}$ . This is done in two steps. Firstly, we construct a family of genus one (reducible) curves on a particular elliptic K3 surface for which  $f$  is given by translation by a two-torsion section, see Section 3. Then, the theory of stable maps is applied to obtain the desired family for generic  $X$ .

The missing piece to prove Conjecture 0.1 in full generality, or at least for symplectic involutions, is the lack of special K3 surfaces in  $\mathfrak{M}_{\Lambda_d}$  for which appropriate families of genus one curves can be described explicitly.

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**ADDENDUM:** The results of this article have meanwhile been improved by C. Voisin and the first author. In [18] Voisin replaces singular elliptic curves in the quotient  $X/\langle f \rangle$  and their étale cover in  $X$ , as used in this paper, by smooth ample curves of high genus. A dimension estimate for the corresponding Prym variety yields finite dimensionality of the anti-invariant part of the Chow group and eventually  $f^* = \mathrm{id}$  on  $\mathrm{CH}^2(X)$  for symplectic involutions. In particular, neither degeneration to special elliptic K3 surfaces nor the deformation theory of stable maps, which we believe to be of independent interest, enter her arguments. In [8] we use instead arguments involving the derived category  $\mathrm{D}^b(\mathrm{Coh}(X))$  and lattice theory as developed in [7] to prove the result for symplectic automorphisms of prime order  $p \neq 2$ . Both results taken together prove  $f^* = \mathrm{id}$  on  $\mathrm{CH}^2(X)$  for symplectic automorphisms of arbitrary finite order.

## 1. COVERING FAMILIES OF ELLIPTIC CURVES

Let  $f : X \xrightarrow{\sim} X$  be a symplectic automorphism of finite order and denote its quotient by  $\bar{X} := X/\langle f \rangle$ , which is a singular K3 surface. If  $f$  has prime order  $p$ , then  $p = 2, 3, 5$ , or  $7$  (see [13]). In order to prove Conjecture 0.1 for symplectic automorphisms of finite order (and we do not have anything to say for automorphisms of infinite order), one can restrict to those. The number of fixed points of  $f$ , all isolated, can be determined by the Lefschetz fixed point formula. E.g. for a symplectic involution, i.e.  $p = 2$ , there are exactly eight fixed points.

In the following, a family  $\mathcal{C}_t \subset X$  of curves given by  $\mathcal{C} \subset S \times X$  is called dominating if the projection  $\mathcal{C} \rightarrow X$  is dominant, i.e. if the curves  $\mathcal{C}_t$  parametrized by the closed point  $t \in S$  cover a Zariski open subset of  $X$ .

PROPOSITION 1.1. *Let  $f : X \xrightarrow{\sim} X$  be a symplectic automorphism. Assume there exists a dominating family of integral  $f$ -invariant curves  $\mathcal{C}_t \subset X$  of geometric genus one with  $\mathcal{C}_t \cap \text{Fix}(f) = \emptyset$  for generic  $t$ . Then  $f^* = \text{id}$  on  $\text{CH}^2(X)$ .*

*Proof.* It suffices to prove that for generic  $x \in X$  the points  $x$  and  $y := f(x)$  are rationally equivalent, i.e.  $[x] = [y]$  in  $\text{CH}^2(X)$ . Since by Roitman's theorem  $\text{CH}^2(X)$  is torsion free (see e.g. [16, Ch. 22]), the latter is equivalent to  $[x] - [y]$  being torsion. For any morphism  $g : C \rightarrow X$  from a smooth irreducible curve  $C$  the induced  $g_* : \text{Pic}(C) = \text{CH}^1(C) \rightarrow \text{CH}^2(X)$  is a group homomorphism. Thus, if there exist lifts  $\tilde{x}, \tilde{y} \in C$  of  $x$  resp.  $y$  such that  $\mathcal{O}(\tilde{x} - \tilde{y}) \in \text{Pic}^0(C)$  is a torsion line bundle, then automatically  $[x] = [y]$  in  $\text{CH}^2(X)$ .

By assumption, a generic closed point  $x \in X$  lies on one of the curves  $\mathcal{C}_t$ . Since the curves  $\mathcal{C}_t$  are assumed to be  $f$  invariant,  $y = f(x)$  is contained in the same curve and  $f$  lifts to an automorphism  $\tilde{f}$  of the normalization  $C := \tilde{\mathcal{C}}_t$ , which is a smooth integral curve of genus one. As  $\mathcal{C}_t$  avoids the fixed points of  $f$ , the automorphism  $\tilde{f} : C \xrightarrow{\sim} C$  is fixed point free and hence  $D := C/\langle \tilde{f} \rangle$  is also smooth of genus one. After choosing origins for  $C$  and  $D$  appropriately,  $C \rightarrow D$  is a morphism of elliptic curves which can be viewed as a quotient of  $C$  by a finite subgroup  $\Gamma \subset C \cong \text{Pic}^0(C)$ . Hence points in the same fibre of  $C \rightarrow D$  differ by elements of  $\Gamma$ . In particular,  $\mathcal{O}_C(\tilde{x} - \tilde{y}) \in \Gamma \subset \text{Pic}^0(C)$  is a torsion line bundle.  $\square$

The problem now becomes to construct a family of genus one curves as required. We do not know how to do this directly. On the special elliptic surface considered in Section 3 a family of genus one curves is constructed, but the curves are not integral. They become integral only after deformations to the generic case.

## 2. STABLE MAPS TO K3 SURFACES

Let  $\mathcal{X} \rightarrow S$  be an irreducible family of K3 surfaces with a global line bundle  $\mathcal{L}$ . Consider the moduli stack  $\mathcal{M}_g(\mathcal{X}, \mathcal{L}) \rightarrow S$  of stable maps  $h : D \rightarrow \mathcal{X}_t$  to fibres of  $\mathcal{X} \rightarrow S$  such that  $D$  is of arithmetic genus  $g$  with  $h_*(D) \in |\mathcal{L}_t|$ . As the stack structure of  $\mathcal{M}_g(\mathcal{X}, \mathcal{L})$  is of no importance to us, we shall ignore it and treat  $\mathcal{M}_g(\mathcal{X}, \mathcal{L})$  as a moduli space. If we do not want to fix the linear equivalence class of the image curves, we will simply write  $\mathcal{M}_g(\mathcal{X})$ .

The following fact has been used in various contexts in the literature, but mostly for  $g = 0$  (see e.g. [1, 12]). We shall need the following statement for  $g = 1$ .

PROPOSITION 2.1. *Every irreducible component of  $\mathcal{M}_g(\mathcal{X}, \mathcal{L})$  is of dimension at least  $g + \dim(S)$ .*

*Proof.* The starting point is [10, Thm. 2.17]: For simplicity let  $\pi : \mathcal{X} \rightarrow S$  be a smooth projective family over an irreducible base  $S$  and let  $\mathcal{D} \rightarrow S$  be a flat and projective family of curves. Every irreducible component of  $\text{Mor}_S(\mathcal{D}, \mathcal{X})$

containing a morphism  $h : D := \mathcal{D}_0 \rightarrow X := \mathcal{X}_0$  is of dimension at least

$$(1) \quad \chi(D, h^* \mathcal{T}_X) + \dim(S).$$

The first term  $\chi(D, h^* \mathcal{T}_X) = h^0(D, h^* \mathcal{T}_X) - h^1(D, h^* \mathcal{T}_X)$  reflects the usual deformation-obstruction theory for the morphism  $h : D \rightarrow X$ . A priori, the obstructions to deforming the morphism  $h : D \rightarrow \mathcal{X}$  are contained in  $H^1(D, h^* \mathcal{T}_X)$ , which is part of an exact sequence

$$\dots \rightarrow H^1(D, h^* \mathcal{T}_X) \rightarrow H^1(D, h^* \mathcal{T}_X) \rightarrow H^1(D, h^* \pi^* \mathcal{T}_S) \rightarrow 0.$$

Since the morphism  $D \rightarrow X \subset \mathcal{X} \rightarrow S$  is constant, there are no obstructions to deforming it sideways at least when  $S$  is smooth. In other words, the obstructions to deforming  $h : D \rightarrow \mathcal{X}$  are contained in the image of  $H^1(D, h^* \mathcal{T}_X)$  which leads to the stronger bound in (1).

A similar argument allows one to treat the case of varying domain  $D$ . The usual obstruction theory for stable maps shows that  $\mathcal{M}_g(X, L)$  in  $[h : D \rightarrow X]$  is of dimension at least  $\chi(D, (h^* \Omega_X \rightarrow \Omega_D)^*)$ , where the two term complex  $h^* \Omega_X \rightarrow \Omega_D$  is concentrated in degree  $-1$  and  $0$ , see [5]. For  $X = \mathcal{X}_0$  in a family  $\mathcal{X} \rightarrow S$ , the analogue of (1) then says that  $\mathcal{M}_g(\mathcal{X})$  in a point corresponding to a stable map  $h : D \rightarrow X$  is of dimension at least

$$(2) \quad \chi(D, (h^* \Omega_X \rightarrow \Omega_D)^*) + \dim(S) = g - 1 + \dim(S)$$

The last equation follows from a standard Riemann–Roch calculation.

The remaining issue is to increase the bound by restricting to families  $\mathcal{X} \rightarrow S$  which come with a deformation  $\mathcal{L}$  of  $L := \mathcal{O}(h_*(D))$ . One can either invoke reduced deformation theory for K3 surfaces as developed recently in [11] in great detail or use the following trick.

Any given family  $(\mathcal{X}, \mathcal{L}) \rightarrow S$  with a polarization  $\mathcal{L}$  can be thickened to a family  $\tilde{\mathcal{X}} \rightarrow \tilde{S}$  with  $\dim \tilde{S} = \dim S + 1$  such that transversally to  $S \subset \tilde{S}$  the line bundle  $\mathcal{L}$  is obstructed (even to first order). More precisely, for  $t \in S$  the line bundle  $\mathcal{L}_t$  on  $\mathcal{X}_t$  deforms to first order in the direction of  $v \in T_{\tilde{S}, t}$  if and only if  $v \in T_{S, t} \subset T_{\tilde{S}, t}$ . If  $\mathcal{L}$  is fibrewise ample, then the thickening  $\tilde{\mathcal{X}} \rightarrow \tilde{S}$  can be explicitly described by using the twistor space construction for each fibre  $\mathcal{X}_t$  and the Kähler class given by  $c_1(\mathcal{L}_t)$ . (Note that in particular,  $\tilde{\mathcal{X}} \rightarrow \tilde{S}$  will in general not be projective.) Otherwise, one uses the standard deformation theory of K3 surfaces to produce such a family at least locally, which is enough for the following dimension count.

By the discussion above,  $\mathcal{M}_g(\tilde{\mathcal{X}})$  is in  $[h : D \rightarrow X]$  of dimension

$$g - 1 + \dim(\tilde{S}) = g + \dim(S).$$

On the other hand,  $h : D \rightarrow X$  cannot deform sideways in a tangent direction  $v \in T_{\tilde{S}, 0}$  that is not contained in  $T_{S, 0}$ , because  $\mathcal{O}(h_*(D)) = \mathcal{L}_0$ . This shows that the two moduli spaces  $\mathcal{M}_g(\tilde{\mathcal{X}})$  and  $\mathcal{M}_g(\mathcal{X})$  coincide near the point given by  $[h : D \rightarrow X]$ .  $\square$

COROLLARY 2.2. *Suppose the fibre  $\mathcal{M}_0$  of an irreducible component  $\mathcal{M} \subset \mathcal{M}_g(\mathcal{X}, \mathcal{L})$  is of dimension  $\leq g$  for some  $0 \in S$ . Then  $\mathcal{M}$  dominates  $S$ .  $\square$*

In other words, if the moduli space  $\mathcal{M}_g(\mathcal{X}_0, \mathcal{L}_0)$  of stable maps to one fibre  $\mathcal{X}_0$  has the expected dimension  $g$  in  $[h : D \rightarrow \mathcal{X}_0]$ , then  $h$  can be deformed to a stable map  $h_t : D_t \rightarrow \mathcal{X}_t$  to the generic fibre. To ensure that the condition is met, we shall later use the following criterion, c.f. [9, Cor. 1.2.5] and [12, Lem. 2.6].

PROPOSITION 2.3. *Suppose the stable map  $h : D \rightarrow X$  satisfies the following conditions:*

*i) If  $D_1, D_2, \dots, D_n$  are the components of  $D$ , then  $D_2, \dots, D_n$  are smooth and rational.*

*ii) The first component  $D_1$  is smooth of genus  $g$  and  $h|_{D_1} : D_1 \rightarrow X$  is an embedding.*

*iii) The morphism  $h$  is unramified.*

*iv) Two components  $D_i$  and  $D_j$  intersect transversally in one point if  $|i - j| = 1$  and not at all otherwise.*

*Then  $\mathcal{M}_g(X)$  is of dimension  $g$  in  $[h : D \rightarrow X]$ .*

*Proof.* We copy the argument from [1, Lem. 2.7]. First of all, since  $h$  is unramified, the complex  $h^*\Omega_X \rightarrow \Omega_D$  is a locally free sheaf of rank one concentrated in degree  $-1$ , the dual of which is denoted  $\mathcal{N}_h$ . Then, one proceeds by induction over  $n$  and uses the exact sequence

$$0 \rightarrow \mathcal{N}_h(-x)|_{D'} \rightarrow \mathcal{N}_h \rightarrow \mathcal{N}_h|_{D_n} \rightarrow 0,$$

where  $D' := D_1 \cup \dots \cup D_{n-1}$  and  $\{x\} = D_{n-1} \cap D_n$ . From the exact sequence

$$0 \rightarrow \mathcal{N}_h^*|_{D_n} \rightarrow h^*\Omega_X|_{D_n} \rightarrow \Omega_D|_{D_n} \rightarrow 0$$

and  $\Omega_D|_{D_n} \cong \mathcal{O}(-1)$ , one deduces  $\mathcal{N}_h|_{D_n} \cong \mathcal{O}(-1)$ . Thus,  $H^i(\mathcal{N}_h) \cong H^i(\mathcal{N}_h(-x)|_{D'})$ . On the other hand,  $\mathcal{N}_h(-x)|_{D'} = \mathcal{N}_{h'}$ , where  $h' := h|_{D'} : D' \rightarrow X$ . By induction this eventually yields  $H^i(\mathcal{N}_h) \cong H^i(\mathcal{N}_{D_1/X})$ . But clearly,  $h^0(\mathcal{N}_{D_1/X}) = h^0(D_1, \omega_{D_1}) = g$  and the deformations of  $D_1 \subset X$  are unobstructed.  $\square$

REMARK 2.4. Maybe more geometrically, the arguments show that deformations of  $h : D \rightarrow X$  are all given by deforming  $D_1 \subset X$ .

### 3. SPECIAL ELLIPTIC SURFACES

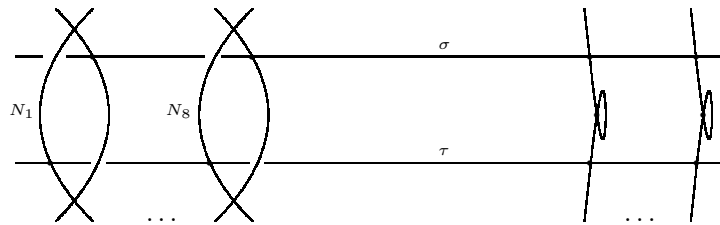
We follow [4, Sect. 4] for the construction of an elliptic K3 surface  $X \rightarrow \mathbb{P}^1$  with a symplectic involution given by a two-torsion section. Deformations of  $X$  will lead to K3 surfaces with Néron–Severi group  $\tilde{\Lambda}_{2d}$  with  $d = 2e > 2$ . The elliptic K3 surface  $X \rightarrow \mathbb{P}^1$  is described by an equation of the form

$$(3) \quad y^2 = x(x^2 + a(t)x + b(t))$$

with general  $a(t)$  and  $b(t)$  of degree 4 resp. 8. Then the fibration has two obvious sections: The section at infinity  $\sigma$  given by  $x = z = 0$ , which will serve

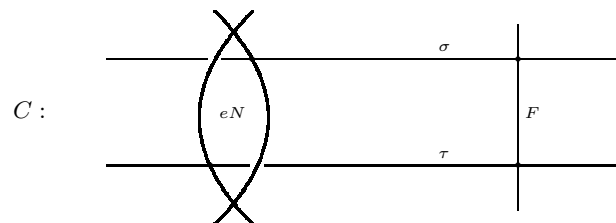
us as the zero section, and a disjoint section  $\tau$  given by  $x = y = 0$ . Using the explicit equation, one finds that  $\tau$  has order two. Thus, translation by  $\tau$  defines an involution  $f : X \xrightarrow{\sim} X$  which is symplectic.

Still following [4], one computes the singular fibres of  $X \rightarrow \mathbb{P}^1$ : There are eight fibres of type  $I_1$  (a rational curve with one node) and eight fibres of type  $I_2$  (the union of two copies of  $\mathbb{P}^1$  intersecting transversally in two points). They can be found over the zeroes of  $b \in H^0(\mathbb{P}^1, \mathcal{O}(8))$  resp.  $a^2 - 4b \in H^0(\mathbb{P}^1, \mathcal{O}(8))$ . The fixed points of  $f$  are the nodes of the eight  $I_1$ -fibres which are all avoided by  $\sigma$  and  $\tau$ . Moreover,  $f$  interchanges the two components of each  $I_2$ -fibre.



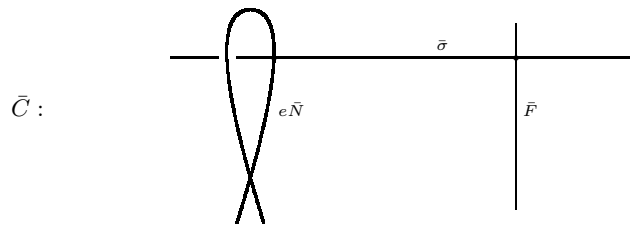
The components of the  $I_2$ -fibres not meeting  $\sigma$  are denoted  $N_1, \dots, N_8$ . Then  $\hat{N} = (1/2) \sum N_i \in \text{NS}(X)$ . Moreover, if  $F$  denotes the class of a generic fibre (and by abuse also a generic fibre itself), then  $\sigma$  and  $F$  span a hyperbolic plane and  $\tau = \sigma + 2F - \hat{N}$ . The Néron-Severi group of  $X$  (for general  $a$  and  $b$ ) is thus  $\langle \sigma, F \rangle \oplus \langle N_1, \dots, N_8, \hat{N} \rangle$ , which is of rank 10.

Next consider a curve of the form  $C = eN + F + \sigma + \tau$ , where  $N$  is one of the  $I_2$ -fibres, and let  $L := \mathcal{O}(C)$ . Then  $L$  is big and nef. Indeed,  $(L.L) = 4e > 0$  and  $C$  intersects all its irreducible components positively, e.g.  $(C.\sigma) = e - 1 > 0$ . In fact,  $L$  is ample as it clearly intersects all horizontal curves positively and has also positive intersection with all  $(-2)$ -curves (e.g. the two components of the  $I_2$ -fibres). Moreover,  $L$  is primitive, as  $(C.N_i) = 1$ . Since  $f$  respects the fibration and interchanges  $\sigma$  and  $\tau$ , the curve  $C$  is  $f$ -invariant and disjoint from  $\text{Fix}(f)$ .



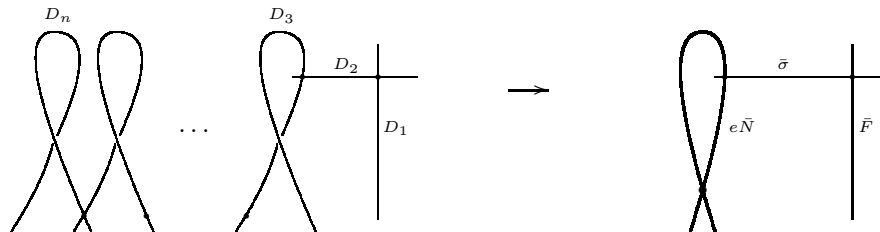
Let us now consider the quotient  $\bar{X} := X/\langle f \rangle$  which is a singular K3 surface with eight ordinary double points. Its minimal resolution  $Y \rightarrow \bar{X}$  comes with a natural elliptic fibration  $Y \rightarrow \mathbb{P}^1$ . Note that the fibres of type  $I_1$  and  $I_2$  are interchanged when passing from  $X$  to  $Y$ .

The quotient  $\bar{C} := C/\langle f \rangle \subset \bar{X}$  avoids the singular locus of  $\bar{X}$  and thus can also be viewed as a curve in  $Y$ . For the same reason, the line bundle  $L$  descends to an ample line bundle  $\bar{L}$  on  $\bar{Y}$ . Note that  $\bar{C}$  decomposes as  $\bar{C} = e\bar{N} + \bar{F} + \bar{\sigma}$ , where  $\bar{N}$  is an  $I_1$ -fibre of  $Y \rightarrow \mathbb{P}^1$ ,  $\bar{F}$  is a smooth fibre, and  $\bar{\sigma}$  is a section.



LEMMA 3.1. *There exists a stable map  $h : D \rightarrow \bar{X}$  of arithmetic genus one with image  $\bar{C}$  and such that  $\mathcal{M}_1(\bar{X}, \bar{L})$  is one-dimensional in  $h$ .*

*Proof.* Since  $\bar{C}$  avoids the singularities of  $\bar{X}$ , we can equally work with  $\bar{C} \subset Y$ . The curve  $D$  shall have components  $D_1, D_2, \dots, D_n$ ,  $n = e + 2$ , with  $D_1 \xrightarrow{\sim} \bar{F}$ ,  $D_2 \xrightarrow{\sim} \bar{\sigma}$ , and  $D_i \rightarrow \bar{N}$ ,  $i \geq 2$ , being the normalization. The gluing is defined according to the picture (cf. [1, 9, 12]):



Obviously,  $D$  is of arithmetic genus one and  $h_*(D) = \bar{C}$ . Moreover, the assumptions of Proposition 2.3 are satisfied and hence  $\mathcal{M}_1(Y)$  is of dimension one in  $[h : D \rightarrow Y]$ .  $\square$

Now consider a generic deformation

$$(4) \quad (\mathcal{X}, \mathcal{L}) \rightarrow S$$

of  $(X, L, f)$ , i.e.  $(\mathcal{X}_0, f_0) = (X, f)$  for a distinguished  $0 \in S$  and for generic  $t \in S$  the fibre  $\text{NS}(\mathcal{X}_t)$  has rank  $\rho = 9$  with  $f_t$ -invariant part spanned by  $\mathcal{L}_t$ . Taking quotients, one obtains a family of singular K3 surfaces  $\bar{\mathcal{X}} \rightarrow S$ . Clearly,  $L = \mathcal{O}(C)$  descends to the quotient  $\bar{X}$ , for  $C$  is  $f$ -invariant and avoids the fixed points. Hence the line bundle  $\mathcal{L}$  also descends to a relatively ample line bundle  $\bar{\mathcal{L}}$ . (The obstructions to deforming  $L$  resp.  $\bar{L}$  sideways are the same.) Note that for generic  $t \in S$  the line bundle  $\bar{\mathcal{L}}_t$  generates  $\text{Pic}(\bar{\mathcal{X}}_t)$ .

Let us apply the discussion of Section 2 to  $h : D \rightarrow \bar{\mathcal{X}}_0 = \bar{X}$ . So, consider the relative moduli space of stable maps of genus one  $\mathcal{M}_1(\bar{\mathcal{X}}, \bar{\mathcal{L}}) \rightarrow S$ .



COROLLARY 3.2. *The stable map  $h : D \rightarrow \bar{X}$  thus constructed deforms sideways to stable maps  $h_t : D_t \rightarrow \bar{\mathcal{X}}_t$ . Moreover, for generic  $t \in S$  the curve  $h_{t*}(D_t) \subset \bar{\mathcal{X}}_t$  and its preimage in  $\mathcal{X}_t$  are integral and disjoint from the singular locus resp. the fixed point set of  $f_t$ .*

*Proof.* The existence of the deformation to the nearby fibres follows directly from Proposition 2.3 and Corollary 2.2. Since  $\bar{C} = h_*(D)$  avoids the singularities of  $\bar{X}$ , this will hold for generic  $t$ . Clearly,  $h_{t*}(D_t) \in |\bar{\mathcal{L}}_t|$ . Therefore, since  $\mathcal{L}_t$  generates the invariant part of  $\text{NS}(\mathcal{X}_t)$  and hence  $\bar{\mathcal{L}}_t$  generates  $\text{NS}(\bar{\mathcal{X}}_t)$ , the curve  $h_{t*}(D_t)$  must be integral.

Suppose the preimage  $C_t$  of  $h_{t*}(D_t)$  were not integral for  $t$  generic, i.e.  $C_t = C'_t + C''_t$  with  $f_t(C'_t) = C''_t$ . (Use that  $f_t$  is an involution.) The two components would then specialize to  $C'$  resp.  $C''$  on  $X$  with  $C = C' + C''$  and  $f(C') = C''$ . We may assume that  $F \subset C'$ . But then also  $F = f(F) \subset f(C') = C''$  which eventually yields the contradiction that  $F$  appears with multiplicity at least two in  $C$ .  $\square$

REMARK 3.3. In fact, since the stable map  $h : D \rightarrow X$  deforms with the fibre component  $F$  in a one-dimensional family, also the deformations  $h_t : D_t \rightarrow \bar{\mathcal{X}}_t$  come in a family dominating  $\bar{\mathcal{X}}_t$ . Thus, one obtains a dominating family of integral genus one curves in the generic deformation  $\mathcal{X}_t$  that are  $f_t$ -invariant and avoid the fixed points of  $f_t$ .

4. PROOF OF THE MAIN THEOREM

The outcome of the above construction are generic K3 surfaces  $\mathcal{X}_t \in \mathfrak{M}_{\tilde{\Lambda}_d}$  with a symplectic involution  $f_t$  such that  $\bar{\mathcal{X}}_t = \mathcal{X}_t / \langle f_t \rangle$  contains a one-dimensional family of integral curves of geometric genus one that avoid the singular locus. This immediately leads to a proof of our main result.

THEOREM 4.1. *For all  $(X, f) \in \mathfrak{M}_{\tilde{\Lambda}}$ , the symplectic involution  $f : X \xrightarrow{\sim} X$  acts as id on  $\text{CH}^2(X)$ .*

*Proof.* The case  $d = 2$  follows from [14]. So we assume  $d = 2e > 2$ , i.e.  $e > 1$ . We first show that the above discussion combined with Proposition 1.1 proves the assertion for generic  $(X, f) \in \mathfrak{M}_{\tilde{\Lambda}_d}$ .

Consider a deformation (4) of the special elliptic K3 surface (3). Then for generic  $t \in S$  one has  $\text{NS}(\mathcal{X}_t) = \tilde{\Lambda}_d$ . Indeed, by [4, Prop. 2.7] only in this case do all  $f_t$ -invariant line bundles actually descend to the quotient  $\bar{\mathcal{X}}_t$ . Hence the elliptic K3 surfaces described by (3) can be connected to the generic K3 surface parametrized by  $\mathfrak{M}_{\tilde{\Lambda}_d}$ . Here we use that  $\mathfrak{M}_{\tilde{\Lambda}_d}$  is connected.

The generic fibre of the family (4) satisfies the assumption of Proposition 1.1. Indeed, by Corollary 3.2 and Remark 3.3 there exists a dominating family of integral curves of arithmetic genus one on the generic fibre  $\mathcal{X}_t$  that are invariant under the involution and avoid the fixed points.

Now consider an arbitrary  $(X, f) \in \mathfrak{M}_{\tilde{\Lambda}_d}$ . Then any  $x \in X$  can be viewed as a specialization of points  $x_t$  in generic deformations  $(\mathcal{X}_t, f_t) \in \mathfrak{M}_{\tilde{\Lambda}_d}$ . Clearly, the

points  $f_t(x_t)$  then specialize to  $f(x)$ . For generic  $\mathcal{X}_t$  we have already shown  $[x_t] = [f_t(x_t)]$  in  $\mathrm{CH}^2(\mathcal{X}_t)$  and thus specialization yields  $[x] = [f(x)]$  in  $\mathrm{CH}^2(X)$  for all  $x \in X$ .  $\square$

## 5. FURTHER COMMENTS

We briefly outline how to adapt our techniques to the case of symplectic automorphisms of prime order. For  $p = 3, 5$ , and  $7$ , Garbagnati and Sarti describe in [3, Thm. 4.1] lattices  $\Omega_p$  of rank  $12, 16$ , resp.  $18$  that are isomorphic to the anti-invariant part of  $f^*$  acting on  $H^2(X, \mathbb{Z})$ . Similar to the case  $p = 2$ , the generic polarized K3 surface  $(X, L)$  of degree  $2d$  with a symplectic automorphism  $f : X \xrightarrow{\sim} X$  of order  $p$  leaving  $L$  fixed has Picard group isomorphic to  $\Lambda_{p,d} := \mathbb{Z}L \oplus \Omega_p$  or possibly, if  $d \equiv 0(p)$ , isomorphic to a lattice  $\tilde{\Lambda}_{p,d}$  that contains  $\Lambda_{p,d}$  as a primitive sublattice of index  $p$ . In fact, the case  $\Lambda_{7,d}$  is not realized if  $d \equiv 0(7)$  (cf. [3, Prop. 5.2]), but unfortunately it is not known whether the lattices  $\tilde{\Lambda}_{p,d}$  are unique for given  $p$  and  $d \equiv 0(p)$  (see [3, Sec. 6]). The moduli spaces are of dimension  $7, 3$ , resp.  $1$ .

Examples of symplectic automorphisms of order  $3, 5$ , and  $7$  have been described in [3, Sec. 3.1]. They are again given by translation by a torsion section. The Picard numbers in these examples are  $14, 18$ , resp.  $20$  and in each case they correspond to points in (at least) one component of the moduli space of polarized K3 surfaces  $(X, L)$  with a symplectic automorphism  $f$  of degree  $L^2 = 2d$ . This leads to the following result:

**THEOREM 5.1.** *For  $p = 3, 5$ , or  $7$  and  $d = ep$ , there exists one component of the moduli space of polarized K3 surfaces  $(X, L)$  with a symplectic automorphism  $f : X \xrightarrow{\sim} X$  of order  $p$  and  $L^2 = 2d$  such that Conjecture 0.1 holds true.  $\square$*

It is very likely that for  $p = 7$  and  $d \equiv 0(7)$  the result can be strengthened to cover all K3 surfaces, as we would expect that  $\tilde{\Lambda}_{7,d}$  is in fact unique.

## REFERENCES

- [1] F. Bogomolov, B. Hassett, Y. Tschinkel *Constructing rational curves on K3 surfaces*, Duke Math. J. 157 (2011), 535–550.
- [2] A. Chatzistamatiou *First coniveau notch of the Dwork family and its mirror*, Math. Res. Lett. 16 (2009), 563–575.
- [3] A. Garbagnati, A. Sarti *Symplectic automorphisms of prime order*, J. Algebra 318 (2007), 323–350.
- [4] B. van Geemen, A. Sarti *Nikulin involutions on K3 surfaces*, Math. Z. 255 (2007), 731–753.
- [5] T. Graber, J. Harris, J. Starr *Families of rationally connected varieties*, J. Amer. Math. Soc. 16 (2003), 57–67.
- [6] V. Guletskii, A. Tikhomirov *Algebraic cycles on quadric sections of cubics in  $\mathbb{P}^4$  under the action of symplectomorphisms*, arXiv:1109.5725v1.

- [7] D. Huybrechts *Chow groups and derived categories of K3 surfaces*, Current developments in algebraic geometry, 177–195, Math. Sci. Res. Inst. Publ., 59, Cambridge Univ. Press, Cambridge, 2012.
- [8] D. Huybrechts *Symplectic automorphisms of K3 surfaces of arbitrary order*, Math. Res. Lett. 19 (2012), 947–951.
- [9] M. Kemeny *The universal Severi variety of rational curves on K3 surfaces*, Master thesis Bonn 2011. arXiv:1110.4266.
- [10] J. Kollár *Rational curves on algebraic varieties*, Ergebnisse 32 (3) Springer 1996.
- [11] M. Kool, R. Thomas *Reduced classes and curve counting on surfaces I: theory*, arXiv:1112.3069.
- [12] J. Li, Ch. Liedtke *Rational Curves on K3 Surfaces*, Invent. Math. 188, (2012), 713–727.
- [13] V. Nikulin *Finite groups of automorphisms of Kähler K3 surfaces*, Proc. Moscow Math. Society 38 (1980), 71–135.
- [14] C. Pedrini *On the finite dimensionality of a K3-surface*, Manuscripta Math. 138 (2012), 59–72.
- [15] C. Voisin *Sur les zéro-cycles de certaines hypersurfaces munies d'un automorphisme*, Ann. Scuola Norm. Sup. Pisa Cl. Si. (4) 19, (1992) 473–492.
- [16] C. Voisin *Théorie de Hodge et géométrie algébrique complexe*, Cours spécialisés 10. SMF (2002).
- [17] C. Voisin *The generalized Hodge and Bloch conjectures are equivalent for general complete intersections*, to appear in Annales Scientifiques de l'ENS. arXiv:1107.2600.
- [18] C. Voisin *Symplectic involutions of K3 surfaces act trivially on  $CH_0$* , Doc. Math. 17 (2012), 851–860.

Daniel Huybrechts  
Mathematisches Institut  
Universität Bonn  
Endenicher Allee 60  
53115 Bonn  
Germany  
huybrech@math.uni-bonn.de

Michael Kemeny  
Mathematisches Institut  
Universität Bonn  
Endenicher Allee 60  
53115 Bonn  
Germany  
michael.kemeny@gmail.com

