On Zagier's Conjecture for Base Changes of Elliptic Curves

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Abstract. Let E be an elliptic curve over $\mathbf{Q}$, and let F be a finite abelian extension of $\mathbf{Q}$. Using Beilinson's theorem on a suitable modular curve, we prove a weak version of Zagier's conjecture for $L(E_F, 2)$, where E_F is the base change of E to F.

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Introduction

Zagier conjectured in [19] very deep relations between special values of zeta functions at integers, special values of polylogarithms at algebraic arguments and K-theory. While the original conjectures concerned the Dedekind zeta function of a number field and Artin L-functions, theoretical and numerical results by many authors suggested an extension of these conjectures to elliptic curves (see [20] for an historical account). A precise formulation for elliptic curves over number fields was given by Wildeshaus in [17]. The conjecture on $L(E, 2)$, where E is an elliptic curve over $\mathbf{Q}$, was proved by Goncharov and Levin in [11]. In this article, we prove an analogue of Goncharov and Levin's result for the base change of E to an arbitrary abelian number field.

Let E be an elliptic curve defined over $\mathbf{Q}$. Let $F \subset \overline{\mathbf{Q}}$ be a finite abelian extension of $\mathbf{Q}$, and let E_F be the base change of E to F. The L-function $L(E_F, s)$ admits a factorization $\prod_{\chi \in \widehat{G}} L(E \otimes \chi, s)$, where $\widehat{G}$ is the group of $\overline{\mathbf{Q}}^{\times}$-valued characters of $G = \mathrm{Gal}(F/\mathbf{Q})$. Each factor $L(E \otimes \chi, s)$ has an analytic continuation to $\mathbf{C}$ with a simple zero at $s = 0$. The functional equation relates $L(E_F, 2)$ with the leading term of $L(E_F, s)$ at $s = 0$.

Let D_E (resp. J_E) be the Bloch elliptic dilogarithm (resp. its "imaginary" cousin) on $E(\mathbf{C}) \cong \mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$ (see §2-3 for the definitions). These functions induce linear maps on the free abelian group $\mathbf{Z}[E(\mathbf{C})]$, and thus on $\mathbf{Z}[E(\overline{\mathbf{Q}})]$ after fixing an embedding $\iota : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$. Let $\mathbf{Z}[E(\overline{\mathbf{Q}})]^{G_F}$ denote the subgroup of divisors fixed by $G_F := \mathrm{Gal}(\overline{\mathbf{Q}}/F)$. It carries a natural action of G. The main theorem of this article can be stated as follows.

Theorem 1. *There exists a divisor $\ell \in \mathbf{Z}[E(\overline{\mathbf{Q}})]^{G_F}$ such that for every character $\chi \in \widehat{G}$, the following identity holds*

$$L'(E\otimes\chi,0) \sim_{\mathbf{Q}^\times} \begin{cases} \frac{1}{\pi}\sum_{\sigma\in G}\chi(\sigma)D_E(\ell^\sigma) & \text{if } \chi \text{ is even,} \\ \frac{1}{\pi\Im(\tau)}\sum_{\sigma\in G}\chi(\sigma)J_E(\ell^\sigma) & \text{if } \chi \text{ is odd.}\end{cases} \tag{1}$$

Using the Dedekind-Frobenius formula for group determinants, we deduce from Theorem 1 the following result. Let us denote the elements of G by $\sigma_1,\ldots,\sigma_d$ (resp. $\sigma_1,\overline{\sigma_1},\ldots,\sigma_{d/2},\overline{\sigma_{d/2}}$) if F is real (resp. complex), with $d=[F:\mathbf{Q}]$.

Corollary (Weak version of Zagier's conjecture for $L(E_F,2)$).
Let $\ell \in \mathbf{Z}[E(\overline{\mathbf{Q}})]^{G_F}$ be a divisor satisfying the identities (1) of Theorem 1. For any i, define $\ell_i = \ell^{\sigma_i^{-1}}$. If F is real, then we have

$$L(E_F,2) \sim_{\mathbf{Q}^\times} \pi^d \cdot \det\big(D_E(\ell_i^{\sigma_j})\big)_{1\le i,j\le d}. \tag{2}$$

If F is complex, then we have

$$L(E_F,2) \sim_{\mathbf{Q}^\times} \frac{\pi^d}{\Im(\tau)^{d/2}} \cdot \det\big(D_E(\ell_i^{\sigma_j})\big)_{1\le i,j\le d/2} \cdot \det\big(J_E(\ell_i^{\sigma_j})\big)_{1\le i,j\le d/2}. \tag{3}$$

Remarks. 1. Wildeshaus's formulation of the conjecture [17, Conjecture, Part 2, p. 366] uses Kronecker double series instead of D_E and J_E. The link between these objects is classical (see the proof of Prop. 6). We have chosen to formulate our results in terms of D_E and J_E because these functions are easier to compute numerically and make apparent the distinction according to the parity of χ.

2. Because of the definition of ℓ_i, the determinant appearing in (2) is a group determinant, indexed by G. In fact, the eigenvalues of the matrix $\big(D_E(\ell_i^{\sigma_j})\big)$ are precisely the sums $\sum_{\sigma\in G}\chi(\sigma)D_E(\ell^\sigma)$ appearing in Theorem 1. This is an algebraic counterpart of the factorization of the L-value of E_F as a product of twisted L-values.

3. The divisor ℓ produced by Theorem 1 satisfies Goncharov and Levin's conditions [11, (2)-(4)]. Following [20], let $\mathcal{A}_{E/F} \subset \mathbf{Z}[E(\overline{\mathbf{Q}})]^{G_F}$ be the group of divisors satisfying these conditions. The strong version of Zagier's conjecture predicts that if F is real (resp. complex), then *for any* divisors $\ell_1,\ldots,\ell_d \in \mathcal{A}_{E/F}$ (resp. $\ell_1,\ldots,\ell_{d/2}\in\mathcal{A}_{E/F}$), the right-hand side of (2) (resp. (3)) is a rational multiple of $L(E_F,2)$ (possibly equal to zero). As in the case $F=\mathbf{Q}$, this strong conjecture is beyond the reach of current technology.

In order to prove Theorem 1, we prove a weak version of Beilinson's conjecture for the special value $L^{(d)}(E_F, 0)$ (see §3 for the definition of the objects involved in the following theorem).

Theorem 2. *There exists a subspace $\mathcal{P}_{E/F} \subset H^2_{\mathcal{M}/\mathbf{Z}}(E_F, \mathbf{Q}(2))$ such that $R_{E/F} \coloneqq \mathrm{reg}_{E/F}(\mathcal{P}_{E/F})$ is a $\mathbf{Q}$-structure of $H^1(E_F(\mathbf{C}), \mathbf{R})^-$ and*

$$\det(R_{E/F}) = L^{(d)}(E_F, 0) \cdot \det(H^1(E_F(\mathbf{C}), \mathbf{Q})^-). \tag{4}$$

We prove Theorem 2 by using Beilinson's theorem on a suitable modular curve. More precisely, we make use of a result of Schappacher and Scholl [15] on the (non geometrically connected) modular curve $X_1(N)_F$, where N is the conductor of E. We therefore need to work in the adelic setting. We establish a divisibility statement in the Hecke algebra of $X_1(N)_F$ in order to get the desired result for E_F.

The methods used in this article are of inexplicit nature and do not give rise, in general, to explicit divisors. However, Theorem 1 and its corollary can be made explicit in the particular case of the elliptic curve $E = X_1(11)$ and the maximal real subfield $F = \mathbf{Q}(\zeta_{11})^+$ inside the cyclotomic field $\mathbf{Q}(\zeta_{11})$. In fact, we may take for ℓ a divisor on the cuspidal subgroup of E. The tools for proving this are Kato's explicit version of Beilinson's theorem for the modular curve $X_1(N)_{\mathbf{Q}(\zeta_m)}$, the work of the author [3], as well as a technique used by Mellit [13] to get new relations between values of the elliptic dilogarithm. We hope to give soon an expanded account of this example.

The organization of the article is as follows. In §1, we recall well-known facts about $L(E_F, s)$. In §2 and §3, we recall the definition of the regulator map and we compute it for E_F (Prop. 9). In §4, we explain the adelic setting for modular curves. In §5, we prove the divisibility we need in the Hecke algebra (Prop. 16). Finally, we give in §6 the proofs of the main results. We conclude with some remarks and a conjecture in the case $F/\mathbf{Q}$ is not abelian.

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1 The L-function of the base change

By the Kronecker-Weber theorem, we have $F \subset \mathbf{Q}(\zeta_m)$ for some $m \geq 1$, so that G is a quotient of $(\mathbf{Z}/m\mathbf{Z})^\times$ and $\widehat{G}$ can be identified with a subgroup of the Dirichlet characters modulo m.

Let $f = \sum_{n \geq 1} a_n q^n \in S_2(\Gamma_0(N))$ be the newform associated to E. For any $\chi \in \widehat{G}$, define $L(E \otimes \chi, s) \coloneqq L(f \otimes \chi, s)$, where $f \otimes \chi$ is the unique newform of weight 2 whose p-th Fourier coefficient is $a_p \chi(p)$ for every prime $p \nmid Nm$. The L-function of E_F has the following description.

Proposition 3. *The following identity holds :*

$$L(E_F, s) = \prod_{\chi \in \widehat{G}} L(f \otimes \chi, s). \tag{5}$$

Proof. Let $\rho = (\rho_\ell)_\ell$ be the compatible system of 2-dimensional ℓ-adic representations of $G_{\mathbf{Q}}$ attached to f by Deligne [5]. By modularity $L(E_F, s) = L(\rho|_{G_F}, s)$. Using Artin's formalism for L-functions, we have

$$L(\rho|_{G_F}, s) = L(\mathrm{Ind}_{G_F}^{G_{\mathbf{Q}}}(\rho|_{G_F}), s). \tag{6}$$

If $\mathbf{1}_{G_F}$ denotes the trivial representation of G_F, we have

$$\begin{aligned} \mathrm{Ind}_{G_F}^{G_{\mathbf{Q}}}(\rho|_{G_F}) &= \mathrm{Ind}_{G_F}^{G_{\mathbf{Q}}}(\mathbf{1}_{G_F} \otimes \mathrm{Res}_{G_{\mathbf{Q}}}^{G_F} \rho) \\ &\cong \mathrm{Ind}_{G_F}^{G_{\mathbf{Q}}}(\mathbf{1}_{G_F}) \otimes \rho \\ &\cong \bigoplus_{\chi \in \widehat{G}} \rho \otimes \chi. \end{aligned} \tag{7}$$

(Here we chose embeddings $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}_\ell}$.) Finally, since an irreducible ℓ-adic representation of $G_{\mathbf{Q}}$ is determined by the traces of all but finitely many Frobenius elements, the compatible system associated to $f \otimes \chi$ is $\rho \otimes \chi$, so that $L(\rho \otimes \chi, s) = L(f \otimes \chi, s)$ for any $\chi \in \widehat{G}$. □

Proposition 4. *We have* $L(E_F, 2) \sim_{\mathbf{Q}^\times} \pi^{2d} L^{(d)}(E_F, 0)$, *where* $L^{(d)}(E_F, 0)$ *denotes the* d*-th derivative at* $s = 0$.

Proof. Since each $L(f \otimes \chi, s)$ has a simple zero at $s = 0$, we get

$$\frac{L^{(d)}(E_F, 0)}{d!} = \prod_{\chi \in \widehat{G}} L'(f \otimes \chi, 0), \tag{8}$$

Let $N_{f\otimes\chi}$ be the level of the newform $f \otimes \chi$. Putting $\Lambda(f \otimes \chi, s) = N_{f\otimes\chi}^{s/2} (2\pi)^{-s} \Gamma(s) L(f \otimes \chi, s)$, we have [7, §5]

$$\Lambda(f \otimes \chi, s) = -w_{f\otimes\chi} \Lambda(f \otimes \overline{\chi}, 2 - s) \qquad (s \in \mathbf{C}) \tag{9}$$

where $w_{f\otimes\chi}$ is the pseudo-eigenvalue of $f \otimes \chi$ with respect to the Atkin-Lehner involution of level $N_{f\otimes\chi}$. Note that (9) implies $w_{f\otimes\chi} w_{f\otimes\overline{\chi}} = 1$. Letting $w = \prod_{\chi \in \widehat{G}} w_{f\otimes\chi}$, we have

$$w^2 = \prod_{\chi \in \widehat{G}} w_{f\otimes\chi} w_{f\otimes\overline{\chi}} = 1 \tag{10}$$

so that $w = \pm 1$. Moreover $\Lambda(f \otimes \chi, 0) = L'(f \otimes \chi, 0)$ and $\Lambda(f \otimes \overline{\chi}, 2) = (N_{f\otimes\overline{\chi}}/4\pi^2) L(f \otimes \overline{\chi}, 2)$. Taking the product over $\chi \in \widehat{G}$ yields the result. □

2 The regulator map on Riemann surfaces

In this section, we recall the definition of the regulator map on compact Riemann surfaces [8, §1], and its computation in the case of elliptic curves.
Let X be a compact connected Riemann surface, and $\mathcal{M}(X)$ be its field of meromorphic functions. For any $f,g \in \mathcal{M}(X)^\times$, consider the 1-form

$$\eta(f,g) := \log|f| \cdot \operatorname{darg}(g) - \log|g| \cdot \operatorname{darg}(f). \tag{11}$$

For any $f \in \mathcal{M}(X)\backslash\{0,1\}$, the differential form $\eta(f,1-f)$ is exact on $X\backslash f^{-1}(\{0,1,\infty\})$. More precisely $\eta(f,1-f) = d(D\circ f)$, where D is the Bloch-Wigner dilogarithm function [18]. Let $K_2(\mathcal{M}(X))$ be the Milnor K_2-group associated to $\mathcal{M}(X)$. The *regulator map* on X is the unique linear map

$$\operatorname{reg}_X : K_2(\mathcal{M}(X)) \to H^1(X,\mathbf{R}) \tag{12}$$

such that for any $f,g \in \mathcal{M}(X)^\times$ and any holomorphic 1-form ω on X, we have

$$\int_X \operatorname{reg}_X\{f,g\} \wedge \omega = \frac{1}{2\pi} \int_X \eta(f,g) \wedge \omega. \tag{13}$$

The map reg_X is well-defined by exactness of $\eta(f,1-f)$ and Stokes' theorem. The construction of reg_X easily extends to the case where X is compact but not connected. Indeed, put $\mathcal{M}(X) := \prod_{i=1}^r \mathcal{M}(X_i)$ where $X_1,\ldots,X_r$ are the connected components of X. Then $K_2(\mathcal{M}(X)) \cong \bigoplus_i K_2(\mathcal{M}(X_i))$ as well as $H^1(X,\mathbf{R}) \cong \bigoplus_i H^1(X_i,\mathbf{R})$, and we define reg_X to be the direct sum of the maps reg_{X_i} for $1 \le i \le r$.
Let us recall the classical computation of the regulator map on a complex torus [1, §4]. Let $E_\tau := \mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$ with $\tau \in \mathbf{C}$, $\Im(\tau) > 0$. The map $z \mapsto \exp(2i\pi z)$ induces an isomorphism $E_\tau \cong \mathbf{C}^\times/q^{\mathbf{Z}}$, where $q := \exp(2i\pi\tau)$. Let $D_q : E_\tau \to \mathbf{R}$ be the Bloch elliptic dilogarithm, defined by $D_q([x]) = \sum_{n=-\infty}^{\infty} D(xq^n)$ for any $x \in \mathbf{C}^\times$. We will also use the function $J_q : E_\tau \to \mathbf{R}$, which is defined as follows. Let $J : \mathbf{C}^\times \to \mathbf{R}$ be the function defined by $J(x) = \log|x| \cdot \log|1-x|$ if $x \neq 1$, and $J(1) = 0$. Following [18], we put

$$J_q([x]) = \sum_{n=0}^{\infty} J(xq^n) - \sum_{n=1}^{\infty} J(x^{-1}q^n) + \frac{1}{3}\log^2|q| \cdot B_3\Big(\frac{\log|x|}{\log|q|}\Big) \qquad (x \in \mathbf{C}^\times) \tag{14}$$

where $B_3 = X^3 - \frac{3}{2}X^2 + \frac{X}{2}$ is the third Bernoulli polynomial. The function J_q is well-defined since $J(x) + J(\frac{1}{x}) = \log^2|x|$ and $B_3(X+1) - B_3(X) = 3X^2$. Both functions D_q and J_q extend to linear maps $\mathbf{Z}[E_\tau] \to \mathbf{R}$, by setting $D_q(\sum_i n_i[P_i]) := \sum_i n_i D_q(P_i)$ and similarly for J_q.

Definition 5. For any $f,g \in \mathcal{M}(E_\tau)^\times$ with divisors $\operatorname{div}(f) = \sum_i m_i[P_i]$ and $\operatorname{div}(g) = \sum_j n_j[Q_j]$, the divisor $\beta(f,g) \in \mathbf{Z}[E_\tau]$ is defined by

$$\beta(f,g) = \sum_{i,j} m_i n_j [P_i - Q_j]. \tag{15}$$

The following classical result expresses the regulator map on E_τ in terms of D_q and J_q.

Proposition 6. *For any* $f, g \in \mathcal{M}(E_\tau)^\times$, *we have*

$$\int_{E_\tau} \eta(f,g) \wedge dz = (D_q - iJ_q)(\beta(f,g)). \tag{16}$$

Proof. We have $\int_{E_\tau} \eta(f,g) \wedge dz = -\frac{\Im(\tau)^2}{\pi} K_{2,1,\tau}(\beta(f,g))$ by [1, §4.3] and [6, (6.2)], where $K_{2,1,\tau}$ is the linear extension of the following Eisenstein-Kronecker series on E_τ :

$$K_{2,1,\tau}(z) := \sum_{\substack{\lambda \in \mathbf{Z}+\tau\mathbf{Z} \\ \lambda \neq 0}} \frac{\exp\left(\frac{2i\pi}{\tau-\overline{\tau}}(z\overline{\lambda} - \overline{z}\lambda)\right)}{\lambda^2\overline{\lambda}} \qquad (z \in \mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})). \tag{17}$$

The result now follows from the formula $-\frac{\Im(\tau)^2}{\pi} K_{2,1,\tau} = D_q - iJ_q$, for which we refer to [2, Thm 10.2.1] and [18, §2, p. 616]. □

3 The regulator map on the base change

Let X be a connected (but not necessarily geometrically connected) smooth projective curve over $\mathbf{Q}$. Its function field $\mathbf{Q}(X)$ embeds into $\mathcal{M}(X(\mathbf{C}))$, so we get a natural map $K_2(\mathbf{Q}(X)) \to K_2(\mathcal{M}(X(\mathbf{C})))$. Let c denote the complex conjugation on $X(\mathbf{C})$. For any $f, g \in \mathbf{Q}(X)^\times$, we have $c^*\eta(f,g) = -\eta(f,g)$, so that (12) induces a map

$$K_2(\mathbf{Q}(X)) \to H^1(X(\mathbf{C}), \mathbf{R})^-, \tag{18}$$

where $(\cdot)^-$ denotes the (-1)-eigenspace of c^*.

Let $K_2(X)$ be the Quillen algebraic K_2-group associated to X. Recall that the motivic cohomology group $H^2_{\mathcal{M}}(X, \mathbf{Q}(2)) := K_2^{(2)}(X)$ is defined as the second Adams eigenspace of $K_2(X) \otimes \mathbf{Q}$. The exact localization sequence in K-theory yields a canonical injective map $K_2(X) \otimes \mathbf{Q} \hookrightarrow K_2(\mathbf{Q}(X)) \otimes \mathbf{Q}$ which is compatible with the Adams operations, so that in fact $K_2^{(2)}(X) = K_2(X) \otimes \mathbf{Q}$. The *integral subspace* $H^2_{\mathcal{M}/\mathbf{Z}}(X, \mathbf{Q}(2)) \subset H^2_{\mathcal{M}}(X, \mathbf{Q}(2))$ is the image of the map $K_2(\mathcal{X}) \otimes \mathbf{Q} \to K_2(X) \otimes \mathbf{Q}$ for any proper regular model $\mathcal{X}/\mathbf{Z}$ of X (see [16] for a definition in a more general setting). Tensoring (18) with $\mathbf{Q}$ and restricting to the integral subspace gives the *Beilinson regulator map* on X :

$$\mathrm{reg}_X : H^2_{\mathcal{M}/\mathbf{Z}}(X, \mathbf{Q}(2)) \to H^1(X(\mathbf{C}), \mathbf{R})^-. \tag{19}$$

Note that the real vector space $H^1(X(\mathbf{C}), \mathbf{R})^-$ admits the natural $\mathbf{Q}$-structure $H_X := H^1(X(\mathbf{C}), \mathbf{Q})^-$.

Any finite morphism $\varphi : X \to Y$ between smooth projective curves over $\mathbf{Q}$ induces maps $\varphi^* : K_2(Y) \to K_2(X)$ and $\varphi_* : K_2(X) \to K_2(Y)$, the latter being defined by $K_2(X) \xrightarrow{\cong} K_2'(X) \xrightarrow{\varphi_*} K_2'(Y) \xleftarrow{\cong} K_2(Y)$. It is known that $\varphi^* \otimes \mathbf{Q}$ and $\varphi_* \otimes \mathbf{Q}$ preserve the integral subspaces [16, Thm 1.1.6(i)]. Moreover, the Beilinson regulator maps associated to X and Y are compatible with φ^* and φ_* (this can be seen at the level of Riemann surfaces).

Let us return to our elliptic curve E. Fix an isomorphism $E(\mathbf{C}) \cong E_\tau$ which is compatible with complex conjugation, and let $q = \exp(2i\pi\tau)$. Let D_E and J_E be the real-valued functions on $E(\mathbf{C})$ induced by D_q and J_q respectively[1]. The space $H^1(E(\mathbf{C}), \mathbf{Q})^\pm$ is generated by the 1-form $\eta^\pm$, with

$$\eta^+ = dz + d\overline{z} \qquad \text{and} \qquad \eta^- = \frac{dz - d\overline{z}}{\tau - \overline{\tau}}. \tag{20}$$

Lemma 7. *Let $f, g \in \mathbf{C}(E)^\times$ and $\ell = \beta(f,g)$. We have*

$$\mathrm{reg}_{E(\mathbf{C})}\{f,g\} = -\frac{1}{2\pi}\Big(D_E(\ell)\cdot\eta^- + \frac{J_E(\ell)}{2\Im(\tau)}\cdot\eta^+\Big). \tag{21}$$

Proof. Put $\mathrm{reg}_{E(\mathbf{C})}\{f,g\} = a^+\eta^+ + a^-\eta^-$ with $a^+, a^- \in \mathbf{R}$. Taking the wedge product with dz and integrating over $E(\mathbf{C})$ yields

$$\int_{E(\mathbf{C})} \mathrm{reg}_{E(\mathbf{C})}\{f,g\} \wedge dz = -a^- + 2i\Im(\tau)a^+. \tag{22}$$

Using (13) with Prop. 6 and identifying the real and imaginary parts gives the lemma. ☐

Let Σ be the set of embedding of F into $\mathbf{C}$. We consider $E_F = E \times_{\mathrm{Spec}\,\mathbf{Q}} \mathrm{Spec}\, F$ as a scheme over $\mathrm{Spec}\,\mathbf{Q}$, so that $E_F(\mathbf{C})$ is the disjoint union of d copies of $E(\mathbf{C})$. In particular

$$H^1(E_F(\mathbf{C}), \mathbf{R}) \cong \bigoplus_{\psi\in\Sigma} H^1(E(\mathbf{C}), \mathbf{R}) \tag{23}$$

and $H^1(E_F(\mathbf{C}), \mathbf{Q})$ decomposes accordingly. The group G acts from the right on E_F. This induces a left action of G on $H^1(E_F(\mathbf{C}), \mathbf{Q})$. For any character $\chi \in \widehat{G}$, consider the idempotent

$$e_\chi := \frac{1}{|G|}\sum_{\sigma\in G} \chi(\sigma)\cdot[\sigma] \in \overline{\mathbf{Q}}[G]. \tag{24}$$

It acts on $H^1(E_F(\mathbf{C}), \overline{\mathbf{Q}} \otimes \mathbf{R})$. For any $\psi \in \Sigma$, let $\eta^\pm(\psi)$ be the 1-form $\eta^\pm$ sitting in the ψ-component of (23). Note that the embedding $\iota : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ induces a distinguished element $\iota \in \Sigma$. Define

$$\eta_\chi = \begin{cases} e_\chi(\eta^-(\iota)) & \text{if } \chi \text{ is even,} \\ e_\chi(\eta^+(\iota)) & \text{if } \chi \text{ is odd.} \end{cases} \tag{25}$$

[1] The lattice $\mathbf{Z} + \tau\mathbf{Z}$ is uniquely determined by E, and q is a well-defined real number such that $0 < |q| < 1$. But the pair (D_E, J_E) is defined only up to sign (choosing an isomorphism $E(\mathbf{C}) \cong E_\tau$ amounts to specifying an orientation of $E(\mathbf{R})$).

Lemma 8. *If $c : E_F(\mathbf{C}) \to E_F(\mathbf{C})$ is the map induced by complex conjugation on* $\operatorname{Spec}\mathbf{C}$*, then $c^*\eta_\chi = -\eta_\chi$.*

Proof. For any $\psi \in \Sigma$, we have $c^*\eta^\pm(\psi) = \pm\eta^\pm(\overline{\psi})$. It follows that

$$\begin{aligned} c^*\eta_\chi &= \frac{1}{|G|}\sum_{\sigma\in G}\chi(\sigma)c^*(\sigma\cdot\eta^{-\chi(-1)}(\iota)) \\ &= \frac{-\chi(-1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)(\overline{\sigma}\cdot\eta^{-\chi(-1)}(\iota)). \end{aligned}$$

Since $\chi(-1)\chi(\sigma) = \chi(\overline{\sigma})$, we get the result. □

The map β induces a linear map $F(E)^\times \otimes F(E)^\times \to \mathbf{Z}[E(\overline{\mathbf{Q}})]^{G_F}$, which we still denote by β. The following proposition computes explicitly the regulator map associated to E_F.

Proposition 9. *Let $\gamma \in F(E)^\times \otimes F(E)^\times$ and $\ell = \beta(\gamma)$. For any $\chi \in \widehat{G}$, we have $e_\chi \operatorname{reg}_{E/F}([\gamma]) = \mu_\chi(\ell)\cdot\eta_\chi$, where $\mu_\chi(\ell) \in \overline{\mathbf{Q}}\otimes\mathbf{R}$ is given by*

$$\mu_\chi(\ell) = \begin{cases} -\frac{1}{2\pi}\sum_{\sigma\in G}\chi(\sigma)\otimes D_E(\ell^\sigma) & \text{if } \chi \text{ is even,} \\ -\frac{1}{4\pi\Im(\tau)}\sum_{\sigma\in G}\chi(\sigma)\otimes J_E(\ell^\sigma) & \text{if } \chi \text{ is odd.} \end{cases} \tag{26}$$

Proof. Put $r = \operatorname{reg}_{E/F}([\gamma])$. By Lemma 7, the ψ-component of r is

$$r_\psi = -\frac{1}{2\pi}\Big(D_E(\psi(\ell))\cdot\eta^-(\psi) + \frac{J_E(\psi(\ell))}{2\Im(\tau)}\cdot\eta^+(\psi)\Big). \tag{27}$$

Since $e_\chi(r)$ and η_χ belong to the same G-eigenspace, it suffices to compare their ι-components. By definition, we have $(\eta_\chi)_\iota = \frac{1}{|G|}\eta^{\chi(-1)}$. Moreover

$$e_\chi(r)_\iota = \frac{1}{|G|}\sum_{\sigma\in G}\chi(\sigma)\otimes(\sigma\cdot r)_\iota = \frac{1}{|G|}\sum_{\sigma\in G}\chi(\sigma)\otimes r_{\iota\circ\sigma} \tag{28}$$

$$= -\frac{1}{2\pi|G|}\sum_{\sigma\in G}\chi(\sigma)\otimes\big(D_E(\ell^\sigma)\cdot\eta^- + \frac{J_E(\ell^\sigma)}{2\Im(\tau)}\cdot\eta^+\big). \tag{29}$$

But $D_E(\overline{P}) = D_E(P)$ and $J_E(\overline{P}) = -J_E(P)$ for any $P \in E(\mathbf{C})$, so that the terms involving J_E (resp. D_E) cancel out if χ is even (resp. odd). □

4 Modular curves in the adelic setting

Let $\mathbf{A}_f$ be the ring of finite adèles of $\mathbf{Q}$. For any compact open subgroup $K \subset \mathrm{GL}_2(\mathbf{A}_f)$, there is an associated smooth projective modular curve $\overline{M}_K$ over $\mathbf{Q}$. For example $X(N) = \overline{M}_{K(N)}$ and $X_1(N) = \overline{M}_{K_1(N)}$, where

$$K(N) = \ker(\mathrm{GL}_2(\widehat{\mathbf{Z}}) \to \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})) \tag{30}$$

$$K_1(N) = \Big\{g \in \mathrm{GL}_2(\widehat{\mathbf{Z}}); g \equiv \begin{pmatrix} * & * \\ 0 & 1\end{pmatrix} \pmod N\Big\}. \tag{31}$$

The Riemann surface $\overline{M}_K(\mathbf{C})$ can be identified with the compactification of $\mathrm{GL}_2(\mathbf{Q})\backslash(\mathfrak{h}^{\pm} \times \mathrm{GL}_2(\mathbf{A}_f))/K$. The set of connected components of $\overline{M}_K(\mathbf{C})$ is in bijection with $\widehat{\mathbf{Z}}^{\times}/\det(K)$. For any $g \in \mathrm{GL}_2(\mathbf{A}_f)$, we have an isomorphism $g : \overline{M}_K \xrightarrow{\cong} \overline{M}_{g^{-1}Kg}$ over $\mathbf{Q}$, which is given on the complex points by $(\tau, h) \mapsto (\tau, hg)$. For any compact open subgroups $K' \subset K$ of $\mathrm{GL}_2(\mathbf{A}_f)$, we have a finite morphism $\pi_{K',K} : \overline{M}_{K'} \to \overline{M}_K$.

The Hecke algebra $\mathcal{H}_K$ is the space of functions $K\backslash \mathrm{GL}_2(\mathbf{A}_f)/K \to \overline{\mathbf{Q}}$ with finite support, equipped with the convolution product [4]. It acts on $H^1(\overline{M}_K(\mathbf{C}), \overline{\mathbf{Q}})$ and $\Omega^1(\overline{M}_K) \otimes \overline{\mathbf{Q}}$. Let $\mathbf{T}_K = \mathbf{T}_{\overline{M}_K}$ be the image of $\mathcal{H}_K$ in $\mathrm{End}_{\overline{\mathbf{Q}}}(\Omega^1(\overline{M}_K) \otimes \overline{\mathbf{Q}})$. Let

$$\langle \cdot, \cdot \rangle : H^1(\overline{M}_K(\mathbf{C}), \mathbf{R})^- \times \big(\Omega^1(\overline{M}_K) \otimes \mathbf{R}\big) \to \mathbf{R} \tag{32}$$

be the perfect pairing induced by Poincaré duality. For any $T \in \mathcal{H}_K$, we have $\langle T\eta, \omega \rangle = \langle \eta, T'\omega \rangle$, where $T' \in \mathcal{H}_K$ is defined by $T'(g) = T(g^{-1})$, so that the action of $\mathcal{H}_K$ on $H^1(\overline{M}_K(\mathbf{C}), \overline{\mathbf{Q}} \otimes \mathbf{R})^-$ factors through $\mathbf{T}_K$.

Following [15, 1.1.1], let $\mathcal{Q}_K \subset K_2(\overline{M}_K) \otimes \mathbf{Q}$ be the subspace of Beilinson elements, and let

$$\mathcal{P}_K = \bigcup_{K' \subset K} (\pi_{K',K})_* \mathcal{Q}_{K'} \subset K_2(\overline{M}_K) \otimes \mathbf{Q}. \tag{33}$$

Schappacher and Scholl [15, 1.1.2] proved that $\mathcal{P}_K \subset H^2_{\mathcal{M}/\mathbf{Z}}(\overline{M}_K, \mathbf{Q}(2))$ and that $\mathrm{reg}_{\overline{M}_K}(\mathcal{P}_K)$ is a $\mathbf{Q}$-structure of $H^1(\overline{M}_K(\mathbf{C}), \mathbf{R})^-$ whose determinant with respect to the natural $\mathbf{Q}$-structure $H_{\overline{M}_K}$ is given by the leading term of $L(h^1(\overline{M}_K), s)$ at $s = 0$.

In the following, we assume $K = \prod_p K_p$, where K_p a compact open subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$. The Hecke algebra then decomposes as a restricted tensor product $\mathcal{H}_K = \otimes'_p \mathcal{H}_{K_p}$. For any prime p, let $\widetilde{T}(p) \in \mathcal{H}_K$ (resp. $\widetilde{T}(p,p) \in \mathcal{H}_K$) be the characteristic function of $K\begin{pmatrix} \varpi_p & 0 \\ 0 & 1 \end{pmatrix}K$ (resp. $K\begin{pmatrix} \varpi_p & 0 \\ 0 & \varpi_p \end{pmatrix}$), where $\varpi_p \in \mathbf{A}_f^{\times}$ has component p at the place p, and 1 elsewhere. Let $T(p)$ (resp. $T(p,p)$) be the image of $\widetilde{T}(p)$ (resp. $\widetilde{T}(p,p)$) in $\mathbf{T}_K$. When K needs to be specified, we write $T(p)_K$ or $T(p)_{\overline{M}_K}$.

For any integer $M \geq 1$, we let $\mathcal{H}_K^{(M)} \subset \mathcal{H}_K$ be the subalgebra generated by the $\mathcal{H}_{K_p}$ for $p \nmid M$. We use the notation $\mathbf{T}_K^{(M)}$ for the corresponding subalgebra of $\mathbf{T}_K$.

Lemma 10. *If $K(M) \subset K$ then $\mathbf{T}_K^{(M)}$ is in the center of $\mathbf{T}_K$.*

Proof. For any prime $p \nmid M$, we have $K_p = \mathrm{GL}_2(\mathbf{Z}_p)$ and by Satake the map $\overline{\mathbf{Q}}[T, S, S^{-1}] \to \mathcal{H}_{K_p}$ given by $T \mapsto \widetilde{T}(p)$ and $S \mapsto \widetilde{T}(p,p)$ is an isomorphism. In particular $\mathcal{H}_{K_p}$ is contained in the center of $\mathcal{H}_K$, whence the result. □

Let $U_F \subset \widehat{\mathbf{Z}}^{\times}$ denote the preimage of $\mathrm{Gal}(\mathbf{Q}(\zeta_m)/F) \subset (\mathbf{Z}/m\mathbf{Z})^{\times}$ under the natural map $\widehat{\mathbf{Z}}^{\times} \to (\mathbf{Z}/m\mathbf{Z})^{\times}$ (note that U_F does not depend on m). For any compact open subgroup $K \subset \mathrm{GL}_2(\mathbf{A}_f)$ with $\det(K) = \widehat{\mathbf{Z}}^\times$, let

$$K_F := \{k \in K ; \det(k) \in U_F\}. \tag{34}$$

Let $\mathrm{pr} : \mathbf{A}_f^\times \to \widehat{\mathbf{Z}}^\times$ be the projection associated to the decomposition $\mathbf{A}_f^\times \cong \mathbf{Q}_{>0} \times \widehat{\mathbf{Z}}^\times$.

Definition 11. Let $\gamma : \mathrm{GL}_2(\mathbf{A}_f) \to G$ be the composite morphism

$$\mathrm{GL}_2(\mathbf{A}_f) \xrightarrow{\det} \mathbf{A}_f^\times \xrightarrow{\mathrm{pr}} \widehat{\mathbf{Z}}^\times \to (\frac{\mathbf{Z}}{m\mathbf{Z}})^\times \to G. \tag{35}$$

Note that there is an exact sequence

$$1 \to K_F \to K \xrightarrow{\gamma|_K} G \to 1. \tag{36}$$

The sequence (36) induces a right action of G on $\overline{M}_{K_F}$, and thus a left action of G on $\Omega^1(\overline{M}_{K_F})$. Moreover, the curve $\overline{M}_{K_F}$ can be identified with $\overline{M}_K \otimes F$ as a curve over $\mathbf{Q}$, and we have a bijection

$$\begin{aligned} \overline{M}_{K_F}(\mathbf{C}) &\xrightarrow{\cong} G \times \overline{M}_K(\mathbf{C}) \\ [\tau, g] &\mapsto (\gamma(g), [\tau, g]). \end{aligned} \tag{37}$$

The action of G on $\overline{M}_{K_F}(\mathbf{C})$ corresponds via (37) to the action by translation on the first factor of $G \times \overline{M}_K(\mathbf{C})$.
Now let us consider the case $K = K_1(N)$, so that $\overline{M}_{K_F} \cong X_1(N)_F$. By the previous discussion, the image of G in $\mathrm{End}\,\Omega^1(X_1(N)_F) \otimes \overline{\mathbf{Q}}$ is contained in $\mathbf{T}_{X_1(N)\otimes F}$. In order to ease notations, let $\mathcal{T} = \mathbf{T}^{(Nm)}_{X_1(N)\otimes F} \subset \mathrm{End}\,\Omega^1(X_1(N)_F) \otimes \overline{\mathbf{Q}}$. Let $\mathcal{T}G$ be the subalgebra of $\mathbf{T}_{X_1(N)\otimes F}$ generated by $\mathcal{T}$ and G.

Lemma 12. *The algebra $\mathcal{T}G$ is commutative.*

Proof. Note that $K(Nm) \subset K_1(N)_F$, so $\mathcal{T}$ is commutative and commutes with G by Lemma 10. Since G is abelian, the result follows. ☐

Since $\Omega^1(X_1(N)_F) \cong \Omega^1(X_1(N)) \otimes F$, we can define the base change morphism $\nu_F : \mathrm{End}\,\Omega^1(X_1(N)) \to \mathrm{End}\,\Omega^1(X_1(N)_F)$ by $\nu_F(T) = T \otimes \mathrm{id}_F$. For any $\alpha \in (\mathbf{Z}/m\mathbf{Z})^\times$, let σ_α be its image in G.

Lemma 13. *For any prime $p \nmid Nm$, we have*

$$\nu_F\big(T(p)_{X_1(N)}\big) = T(p)_{X_1(N)\otimes F} \cdot \sigma_p \in \mathcal{T}G \tag{38}$$
$$\nu_F\big(T(p,p)_{X_1(N)}\big) = T(p,p)_{X_1(N)\otimes F} \cdot \sigma_p^2 \in \mathcal{T}G. \tag{39}$$

Proof. Let $g := \begin{pmatrix} \varpi_p & 0 \\ 0 & 1 \end{pmatrix}$ and $K := K_1(N) \cap g^{-1} K_1(N) g = K_1(N) \cap K_0(p)$. Note that $\det K = \widehat{\mathbf{Z}}^\times$. Consider the following correspondence

$$\begin{tikzcd} & \overline{M}_K \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X_1(N) \arrow[rr, dashed, "\widetilde{T}(p)_{X_1(N)}"] & & X_1(N) \end{tikzcd} \tag{40}$$

where $\alpha = \pi_{K,K_1(N)}$ and $\beta = g^{-1} \circ \pi_{K, g^{-1}K_1(N)g} = \pi_{gKg^{-1},K_1(N)} \circ g^{-1}$. Then $T(p)_{X_1(N)} = \beta_* \circ \alpha^*$ on $\Omega^1(X_1(N))$. Similarly $T(p)_{X_1(N)\otimes F}$ is defined by

$$\begin{tikzcd} & \overline{M}_{K_F} \arrow[dl, "\alpha_F"'] \arrow[dr, "\beta_F"] & \\ X_1(N)_F \arrow[rr, dashed, "\widetilde{T}(p)_{X_1(N)\otimes F}"] & & X_1(N)_F \end{tikzcd} \tag{41}$$

where α_F is the natural projection and β_F is induced by g^{-1}. Using the identification $\overline{M}_{K_F} \cong \overline{M}_K \otimes F$ and the description (37) of the complex points, we obtain $\alpha_F = \alpha \otimes \mathrm{id}_F$ and $\beta_F = \beta \otimes \gamma(g^{-1})$. Since $\gamma(g) = \sigma_{p^{-1}}$, we get $T(p)_{X_1(N)\otimes F} = \nu_F(T(p)_{X_1(N)}) \circ (\sigma_p)_*$ and thus (38). The proof of (39) is similar. □

5 A divisibility in the Hecke algebra

In this section we define and study a projection associated to E_F using the Hecke algebra of $X_1(N)_F$.

Let $\varphi : X_1(N) \to E$ be a modular parametrization of the elliptic curve E, and let $\varphi_F : X_1(N)_F \to E_F$ be the base change of φ to F. Consider the map $e_F = \frac{1}{\deg \varphi_F} (\varphi_F)^* (\varphi_F)_*$ on $\Omega^1(X_1(N)_F)$.

Lemma 14. *We have $e_F^2 = e_F$ and $e_F \in \mathcal{T}G$.*

Proof. The first equality follows from $(\varphi_F)_*(\varphi_F)^* = \deg \varphi_F$.

We have $e_F = \nu_F(e)$ where $e = \frac{1}{\deg \varphi} \varphi^* \varphi_* \in \mathrm{End}_{\mathbf{Q}} \Omega^1(X_1(N))$. The image of e is the $\mathbf{Q}$-vector space generated by $\omega_f = 2i\pi f(z) dz$. Since f is a newform of level N, the Atkin-Lehner-Li theory implies that $e \in \mathbf{T}^{(Nm)}_{X_1(N)}$. The result now follows from Lemma 13. □

The space $\Omega = \varinjlim_K \Omega^1(\overline{M}_K) \otimes \overline{\mathbf{Q}}$ has a natural $\mathrm{GL}_2(\mathbf{A}_f)$-action and decomposes as a direct sum of irreducible admissible representations Ω_π of $\mathrm{GL}_2(\mathbf{A}_f)$. For any K we have $\Omega^K = \Omega^1(\overline{M}_K) \otimes \overline{\mathbf{Q}}$. Let $\Pi(K)$ be the set of such π satisfying $\Omega_\pi^K \neq \{0\}$. By [12, p. 393], we have

$$\Omega^1(\overline{M}_K) \otimes \overline{\mathbf{Q}} = \bigoplus_{\pi \in \Pi(K)} \Omega_\pi^K \tag{42}$$

where each Ω_π^K is a simple $\mathbf{T}_K$-module. In particular $\mathbf{T}_K$ is a semisimple algebra. By Lemma 10, the algebra $\mathcal{T}$ is contained in the center of $\mathbf{T}_{K_1(N)_F}$. Using [12, Prop 2.11], we deduce that $\mathcal{T}$ acts by scalar multiplication on each $\Omega_\pi^{K_1(N)_F}$, so there exists a morphism $\theta_\pi : \mathcal{T} \to \overline{\mathbf{Q}}$ such that T acts as $\theta_\pi(T)$ on $\Omega_\pi^{K_1(N)_F}$. The multiplicity one and strong multiplicity one theorems [14] ensure that the characters $(\theta_\pi)_{\pi \in \Pi(K_1(N)_F)}$ are pairwise distinct.

For any $\chi \in \widehat{G}$, let $\pi(f \otimes \chi)$ be the automorphic representation of $\mathrm{GL}_2(\mathbf{A}_f)$ corresponding to the modular form $f \otimes \chi$. We have $\pi(f \otimes \chi) \cong \pi(f) \otimes (\chi \circ \det)$, where $\chi : \mathbf{A}_f^\times / \mathbf{Q}_{>0} \to \mathbf{C}^\times$ denotes the adèlization of χ, sending ϖ_p to $\chi(p)$ for every $p \nmid m$. Since $\pi(f) \in \Pi(K_1(N))$, it follows that $\pi(f \otimes \chi) \in \Pi(K_1(N)_F)$.

LEMMA 15. *For any prime $p \nmid Nm$, we have*

$$\theta_{\pi(f\otimes\chi)}(T(p)) = a_p \chi(p) \tag{43}$$

$$\theta_{\pi(f\otimes\chi)}(T(p,p)) = \chi(p)^2. \tag{44}$$

Proof. We know that $\theta_{\pi(f)}(T(p)) = a_p$ and $\theta_{\pi(f)}(T(p,p)) = 1$. The equalities (43) and (44) follow formally from the fact that $\chi \circ \det$ is equal to $\chi(p)$ on the double coset $K_1(N)_F \begin{pmatrix} \varpi_p & 0 \\ 0 & 1 \end{pmatrix} K_1(N)_F$. □

Let $e_{f\otimes\chi} : \Omega^1(X_1(N)_F) \otimes \overline{\mathbf{Q}} \to \Omega^{K_1(N)_F}_{\pi(f\otimes\chi)}$ be the projection induced by (42). The multiplicity one theorems imply that $e_{f\otimes\chi} \in \mathcal{T}$.

PROPOSITION 16. *The element $e_\chi e_F$ is divisible by $e_{f\otimes\chi}$ in $\mathcal{T}G$.*

Proof. Since e_χ, e_F and $e_{f\otimes\chi}$ are commuting projections, it suffices to prove that the image of $e_\chi e_F$ is contained in the image of $e_{f\otimes\chi}$. We know that the image of $\varphi^* : \Omega^1(E) \to \Omega^1(X_1(N))$ lies in the kernel of $T(p) - a_p \in \mathbf{T}_{X_1(N)}$. Therefore the image of φ_F^* lies in the kernel of $\nu_F(T(p)) - a_p$. Using Lemma 13, it follows that in $\mathcal{T}G$ we have

$$T(p)\sigma_p e_F = a_p e_F. \tag{45}$$

Applying e_χ to both sides and using the identity $e_\chi \sigma_p = \overline{\chi}(p) e_\chi$ yields

$$T(p) e_\chi e_F = a_p \chi(p) e_\chi e_F. \tag{46}$$

The same argument shows that $T(p,p) e_\chi e_F = \chi(p)^2 e_\chi e_F$. The proposition now follows from Lemma 15 and the multiplicity one theorems. □

6 Proof of the main results

Recall that $\varphi : X_1(N) \to E$ is a modular parametrization, and that φ_F is the base change of φ to F. We have a commutative diagram

$$\begin{tikzcd} K_2(X_1(N)_F) \otimes \mathbf{Q} \arrow[r] \arrow[d, "(\varphi_F)_*"'] & H^1(X_1(N)_F(\mathbf{C}), \mathbf{R})^- \arrow[d, "(\varphi_F)_*"] \\ K_2(E_F) \otimes \mathbf{Q} \arrow[r] & H^1(E_F(\mathbf{C}), \mathbf{R})^- \end{tikzcd} \tag{47}$$

where the horizontal maps are the regulator maps on $X_1(N)_F$ and E_F.
The strategy of the proof is to use Beilinson's theorem on $X_1(N)_F$ and then to get back to E_F using the Hecke algebra.
Let $\mathcal{P}_{E/F} = (\varphi_F)_* \mathcal{P}_{X_1(N)/F} \subset K_2(E_F) \otimes \mathbf{Q}$. By [15, 1.1.2(iii)], we have $\mathcal{P}_{E/F} \subset H^2_{\mathcal{M}/\mathbf{Z}}(E_F, \mathbf{Q}(2))$. We want to prove that $R_{E/F} \coloneqq \mathrm{reg}_{E/F}(\mathcal{P}_{E/F})$ is a $\mathbf{Q}$-structure satisfying (4). Since $\mathcal{P}_{X_1(N)/F}$ is stable by the Hecke algebra, the spaces $\mathcal{P}_{E/F}$ and $R_{E/F}$ are stable by G.
For any $\chi \in \widehat{G}$, let $R_\chi = e_\chi(R_{E/F} \otimes \overline{\mathbf{Q}})$ and $H_\chi = e_\chi(H_{E/F} \otimes \overline{\mathbf{Q}})$. We want to compare R_χ and H_χ. We have

$$\begin{aligned} \varphi_F^* R_\chi &= e_\chi \varphi_F^* (R_{E/F} \otimes \overline{\mathbf{Q}}) \\ &= e_\chi e_F \,\mathrm{reg}_{X_1(N)/F}(\mathcal{P}_{X_1(N)/F} \otimes \overline{\mathbf{Q}}). \end{aligned} \tag{48}$$

Similarly, we have

$$\varphi_F^* H_\chi = e_\chi e_F (H_{X_1(N)/F} \otimes \overline{\mathbf{Q}}). \tag{49}$$

We will build on the following theorem of Schappacher and Scholl. Let λ_χ be the unique element of $(\overline{\mathbf{Q}} \otimes \mathbf{R})^\times$ such that for every $\psi : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, we have $\psi(\lambda_\chi) = L'(f \otimes \chi^\psi, 0) \in \mathbf{C}^\times$. By [15, 1.2.4 and 1.2.6], we have

$$e_{f\otimes\chi}\big(\mathrm{reg}_{X_1(N)/F}(\mathcal{P}_{X_1(N)/F} \otimes \overline{\mathbf{Q}})\big) = \lambda_\chi \cdot e_{f\otimes\chi}(H_{X_1(N)/F} \otimes \overline{\mathbf{Q}}). \tag{50}$$

By Prop. 16, the equality (50) remains true when $e_{f\otimes\chi}$ is replaced by $e_\chi e_F$, so that $\varphi_F^* R_\chi = \lambda_\chi \cdot \varphi_F^* H_\chi$ by (48) and (49). Since φ_F^* is injective, we get $R_\chi = \lambda_\chi \cdot H_\chi$. Put $V = H^1(E_F(\mathbf{C}), \mathbf{R})^-$ and $V_\chi = e_\chi(V \otimes \overline{\mathbf{Q}})$ for any $\chi \in \widehat{G}$.

Lemma 17. *The $\mathbf{R}[G]$-module V is free of rank 1.*

Proof. By Poincaré duality $V \cong \mathrm{Hom}_{\mathbf{Q}}(\Omega^1(E_F), \mathbf{R})$, and $\Omega^1(E_F) \cong \Omega^1(E) \otimes F$ is free of rank 1 over $\mathbf{Q}[G]$ by the normal basis theorem. ☐

We will use the following lemma from linear algebra. Recall that if B is an A-algebra and N is a B-module, an *A-structure of N* is an A-submodule $M \subset N$ such that $M \otimes_A B \xrightarrow{\cong} N$.

Lemma 18. *Let M be a $\mathbf{Q}[G]$-submodule of V. The following conditions are equivalent :*

(i) *M is a $\mathbf{Q}$-structure of the real vector space V.*

(ii) *For any $\chi \in \widehat{G}$, the space $M_\chi := e_\chi(M \otimes \overline{\mathbf{Q}})$ is a $\overline{\mathbf{Q}}$-structure of the $\overline{\mathbf{Q}} \otimes \mathbf{R}$-module V_χ.*

Moreover, if these conditions hold, then M is free of rank 1 over $\mathbf{Q}[G]$.

Proof. The implication (i) ⇒ (ii) follows from the isomorphisms $M_\chi \otimes_{\overline{\mathbf{Q}}} (\overline{\mathbf{Q}} \otimes \mathbf{R}) \cong e_\chi(M \otimes \overline{\mathbf{Q}} \otimes \mathbf{R}) \cong V_\chi$. Let us assume (ii). By Lemma 17, the $\overline{\mathbf{Q}} \otimes \mathbf{R}$-module V_χ is free of rank 1, so that $\dim_{\overline{\mathbf{Q}}} M_\chi = 1$. Since $M \otimes \overline{\mathbf{Q}} \cong \bigoplus_{\chi \in \widehat{G}} M_\chi$, we get $\dim_{\mathbf{Q}} M = d$. Moreover $M \otimes \overline{\mathbf{Q}} \otimes \mathbf{R}$ generates $V \otimes \overline{\mathbf{Q}}$ over $\overline{\mathbf{Q}} \otimes \mathbf{R}$, so that any $\mathbf{Q}$-basis of M is actually free over $\mathbf{R}$.
Finally, if (i) holds, then M is isomorphic to the regular representation of G by Lemma 17, so that M is free of rank 1 over $\mathbf{Q}[G]$. □

Using Lemma 18 with the $\mathbf{Q}$-structure $H_{E/F}$, we see that H_χ is a $\overline{\mathbf{Q}}$-structure of V_χ. By Lemma 8, the 1-form η_χ is a $\overline{\mathbf{Q}}$-basis of H_χ.

Proof of Theorem 2. Since $R_\chi = \lambda_\chi \cdot H_\chi$ is a $\overline{\mathbf{Q}}$-structure of V_χ, Lemma 18 implies that $R_{E/F}$ is a $\mathbf{Q}$-structure of V. Moreover, the determinant of $R_{E/F} \otimes \overline{\mathbf{Q}}$ with respect to $H_{E/F} \otimes \overline{\mathbf{Q}}$ is represented by $\delta := \prod_{\chi \in \widehat{G}} \lambda_\chi \in (\overline{\mathbf{Q}} \otimes \mathbf{R})^\times$. Note that $\sigma(\lambda_\chi) = \lambda_{\chi^\sigma}$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, so that δ lies in the image of $\mathbf{R}^\times$ in $(\overline{\mathbf{Q}} \otimes \mathbf{R})^\times$. Using the natural evaluation map $(\overline{\mathbf{Q}} \otimes \mathbf{R})^\times \xrightarrow{\iota} \mathbf{C}^\times$, we get in fact $\delta = \prod_{\chi \in \widehat{G}} L'(f \otimes \chi, 0)$. Since the natural map $\mathbf{R}^\times/\mathbf{Q}^\times \to (\overline{\mathbf{Q}} \otimes \mathbf{R})^\times/\overline{\mathbf{Q}}^\times$ is injective, we conclude that $\det(R_{E/F}) = L^{(d)}(E_F, 0) \cdot \det(H_{E/F})$ by (8). □

Proof of Theorem 1. We know from Theorem 2 that $R_{E/F}$ is a $\mathbf{Q}$-structure of V. Since $R_{E/F}$ is stable by G, it is free of rank 1 over $\mathbf{Q}[G]$ by Lemma 18. Let $\gamma \in \mathcal{P}_{E/F}$ such that $R_{E/F} = \mathbf{Q}[G] \cdot \mathrm{reg}_{E/F}(\gamma)$. Replacing γ by a suitable integer multiple, we may assume that γ has a representative $\widetilde{\gamma} \in F(E)^\times \otimes F(E)^\times$. Let $\ell = \beta(\widetilde{\gamma})$. For any $\chi \in \widehat{G}$, we have $R_\chi = \mu_\chi(\ell) H_\chi$ by Prop. 9, where $\mu_\chi(\ell)$ is given by (26). It follows that $\mu_\chi(\ell)/\lambda_\chi \in \overline{\mathbf{Q}}^\times$. Since λ_χ and $\mu_\chi(\ell)$ belong to $\mathbf{Q}(\chi) \otimes \mathbf{R}$, we have in fact $\mu_\chi(\ell)/\lambda_\chi \in \mathbf{Q}(\chi)^\times$. Moreover, the definitions of λ_χ and $\mu_\chi(\ell)$ show that

$$\tau(\lambda_\chi) = \lambda_{\chi^\tau} \quad \text{and} \quad \tau(\mu_\chi(\ell)) = \mu_{\chi^\tau}(\ell) \qquad (\tau \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})). \tag{51}$$

Lemma 19. *Let $(a_\chi)_{\chi \in \widehat{G}}$ be a family of algebraic numbers, with $a_\chi \in \mathbf{Q}(\chi)^\times$, such that $\tau(a_\chi) = a_{\chi^\tau}$ for any χ and any $\tau \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})$. Then there exists a unique $a \in \mathbf{Q}[G]^\times$ such that for every $\chi \in \widehat{G}$, we have $\chi(a) = a_\chi$.*

Proof. The canonical morphism of $\mathbf{Q}$-algebras $\Psi : \mathbf{Q}[G] \to \prod_{\chi \in \widehat{G}} \mathbf{Q}(\chi)$ is injective and its image is contained in the subalgebra W of families $(b_\chi)_\chi$ satisfying $\tau(b_\chi) = b_{\chi^\tau}$ for any χ and τ. Writing $\widehat{G}$ as a disjoint union of Galois orbits, we have $\dim_{\mathbf{Q}} W = \#\widehat{G} = d$, so that Ψ is an isomorphism. □

Using Lemma 19 with $a_\chi := \mu_\chi(\ell)/\lambda_\chi$, we get $a \in \mathbf{Q}[G]^\times$ such that $\mu_\chi(\ell) = \chi(a)\lambda_\chi$ for any χ. Since $\mu_\chi(a\ell) = \overline{\chi}(a)\mu_\chi(\ell)$, replacing ℓ with a suitable integer multiple of $a\ell$ results in $\mu_\chi(\ell) \sim_{\mathbf{Q}^\times} \lambda_\chi$ for any χ. Evaluating everything in $\mathbf{C}$ yields (1). □

Proof of Corollary. Let us first recall the Dedekind-Frobenius formula for group determinants. If $a : G \to \mathbf{C}$ is an arbitrary function, let A be the matrix $(a(gh^{-1}))_{g,h\in G}$. Then

$$\det(A) = \prod_{\chi\in\widehat{G}} \sum_{g\in G} \chi(g)a(g). \tag{52}$$

Let $\ell \in \mathbf{Z}[E(\overline{\mathbf{Q}})]^{G_F}$ be a divisor satisfying the identities (1) of Theorem 1. Assume first F is real. Put $\ell_i := \ell^{\sigma_i^{-1}}$ for $1 \leq i \leq d$. Using (52) with $a(\sigma) = D_E(\ell^\sigma)$ yields

$$\det\bigl(D_E(\ell_i^{\sigma_j})\bigr)_{1\leq i,j\leq d} \sim_{\mathbf{Q}^\times} \prod_{\chi\in\widehat{G}} \pi L'(E\otimes\chi,0) \sim_{\mathbf{Q}^\times} \pi^{-d}L(E_F,2) \tag{53}$$

where the last relation follows from (8) and Prop. 4.
Assume now F is complex. Put $\ell_i := \ell^{\sigma_i^{-1}}$ for $1 \leq i \leq d/2$. We use (52) with the function $a(\sigma) = D_E(\ell^\sigma) + J_E(\ell^\sigma)$. Indexing the lines and columns of A by $\sigma_1, \overline{\sigma_1}, \ldots, \sigma_{d/2}, \overline{\sigma_{d/2}}$, we see that A consists of blocks of the form $\begin{pmatrix} x+y & x-y \\ x-y & x+y \end{pmatrix}$, where $x = D_E(\ell^{\sigma_j\sigma_i^{-1}})$ and $y = J_E(\ell^{\sigma_j\sigma_i^{-1}})$. Elementary operations on the lines and columns of A thus gives

$$\det A = 2^d \det\bigl(D_E(\ell_i^{\sigma_j})\bigr)_{1\leq i,j\leq d/2} \cdot \det\bigl(J_E(\ell_i^{\sigma_j})\bigr)_{1\leq i,j\leq d/2}. \tag{54}$$

On the other hand, we have

$$\sum_{\sigma\in G} \chi(\sigma)a(\sigma) = \begin{cases} \sum_{\sigma\in G} \chi(\sigma)D_E(\ell^\sigma) & \text{if } \chi \text{ is even,} \\ \sum_{\sigma\in G} \chi(\sigma)J_E(\ell^\sigma) & \text{if } \chi \text{ is odd,} \end{cases} \tag{55}$$

so that we conclude as in the first case. □

Further remarks and a conjecture

The proof of Theorem 2 relies crucially on the hypothesis that $F/\mathbf{Q}$ is abelian. Since the field of constants of a modular curve is always an abelian extension of $\mathbf{Q}$, it is not possible to cover a non-abelian base change of E by a usual modular curve. In fact, in the case $F/\mathbf{Q}$ is not abelian, we have no example of a (non CM) elliptic curve E over $\mathbf{Q}$ for which we can prove Zagier's conjecture for $L(E_F,2)$. However, Theorem 1 suggests the following conjecture for Artin-twisted L-values. For simplicity, we restrict to the case F is totally real.

Conjecture 20. *Let E be an elliptic curve defined over $\mathbf{Q}$, and let F be a finite Galois totally real extension of $\mathbf{Q}$. There exists a divisor $\ell \in \mathbf{Z}[E(\overline{\mathbf{Q}})]^{\mathrm{Gal}(\overline{\mathbf{Q}}/F)}$ satisfying Goncharov and Levin's conditions such that for every Artin representation $\rho : \mathrm{Gal}(F/\mathbf{Q}) \to \mathrm{GL}_d(\mathbf{C})$, we have*

$$L^{(d)}(E \otimes \rho, 0) \sim_{\mathbf{Q}^\times} \pi^{-d} \det\Big(\sum_{\sigma \in G} \rho(\sigma) D_E(\ell^\sigma)\Big). \tag{56}$$

Conversely, for every $\ell \in \mathbf{Z}[E(\overline{\mathbf{Q}})]^{\mathrm{Gal}(\overline{\mathbf{Q}}/F)}$ satisfying Goncharov and Levin's conditions and for every $\rho : \mathrm{Gal}(F/\mathbf{Q}) \to \mathrm{GL}_d(\mathbf{C})$, we have

$$\pi^{-d} \det\Big(\sum_{\sigma \in G} \rho(\sigma) D_E(\ell^\sigma)\Big) \in L^{(d)}(E \otimes \rho, 0) \cdot \mathbf{Q}(\operatorname{tr} \rho), \tag{57}$$

where $\mathbf{Q}(\operatorname{tr}\rho)$ is the field generated by the traces of ρ.

Note that the identities (56) and (57) are compatible with taking direct sums of Artin representations. In fact, Conjecture 20 is a refinement of Zagier's conjecture for $L(E_F, 2)$, in the sense that taking the product over irreducibles ρ with multiplicities $\dim(\rho)$ gives the conjecture for E_F. Note that the analytic continuation and the functional equation of $L(E \otimes \rho, s)$ are only conjectural in general.

It would be interesting to investigate the rational factors arising in Theorem 1. As a matter of fact, even for $F = \mathbf{Q}$, we don't know how to predict the rational factor appearing in Zagier's conjecture. The Bloch-Kato conjecture predicts the exact value of $L(E_F, 2)$ (at least up to a unit in the ring of integers of F), but the link between both conjectures remains to be worked out. In fact, in this setting it may be more natural to investigate the equivariant Tamagawa number conjecture of Burns and Flach [9, Part 2, Conjecture 3], which predicts the equivariant L-value $L({}_F E, 2) \in \mathbf{R}[G]^\times$ up to a unit in an order of $\mathbf{Q}[G]$. Taking norms down to $\mathbf{Q}$, this predicts $L(E_F, 2)$ up to sign. The deep work of Gealy [10] on the Bloch-Kato conjecture for modular forms, which uses Kato's Euler system, could be used to tackle this equivariant conjecture. Note also that if F is abelian and real, then Theorem 1 gives a link between $L({}_F E, 2)$ and the vector-valued elliptic dilogarithm $\vec{D}_E(\ell) := \sum_{\sigma \in G} D_E(\ell^\sigma)[\sigma]$.

Finally, although the divisor ℓ produced by Theorem 1 is inexplicit in general, it would be interesting to try to bound the number field generated by the support of ℓ, as well as the heights of the points involved.

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