Realizability and Admissibility under Extension of p-Adic and Number Fields

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Abstract. A finite group G is K-admissible if there is a G-crossed product K-division algebra. In this manuscript we study the behavior of admissibility under extensions of number fields M/K. We show that in many cases, including Sylow metacyclic and nilpotent groups whose order is prime to the number of roots of unity in M, a K-admissible group G is M-admissible if and only if G satisfies the easily verifiable Liedahl condition over M.

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1. Introduction

Let K be a field. A field $L \supseteq K$ is called K-adequate if it is contained as a maximal subfield in a finite dimensional central K-division algebra. A group G is K-admissible if there is a G-extension L/K, i.e. L/K is a Galois extension with $\mathrm{Gal}(L/K) \cong G$, so that L is K-adequate. Equivalently, G is K-admissible if there is a G-crossed product K-division algebra. Ever since adequacy and admissibility were introduced in [19], they were studied extensively over various types of fields, especially over number fields.
As oppose to realizability of groups as Galois groups, there are known restrictions on the number fields K over which a given group is K-admissible. Liedahl's condition (which was shown by Schacher [19] over $\mathbb{Q}$, and generalized by Liedahl [9, Theorem 28]) describes such a restriction. We say that G satisfies Liedahl's condition over K, if for every prime p dividing $|G|$, one of the following holds:

(i) p decomposes in K (has at least two prime divisors),
(ii) p does not decompose in K, and a p-Sylow subgroup $G(p)$ of G is metacyclic and admits a Liedahl presentation over K (for details see Definition 2.4 which is based on [9]).

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In [19, Theorem 9.1], Schacher showed that any finite group G is admissible over some number field K. However, for many groups G it is an open problem to determine the number fields over which they are admissible. In fact, searching for an explicit description for all groups seems hopeless.
In this paper we fix a field K over which G is admissible and ask over which finite extensions of K, G is still admissible. We assume our group G is realizable over M and furthermore can be realized over M with prescribed local conditions, i.e. it admits the Grunwald-Neukirch (GN) property. This question then leads (see Section 5.1) via Schacher's criterion (Theorem 2.1) to the following local realization problem:

Problem 1.1. *Let m/k be an extension of p-adic fields and G a group that is realizable over k. Is there a subgroup H of G which is realizable over m and contains a p-Sylow subgroup of G?*

At first we consider the case of p-groups, where the problem is whether a p-group that is realizable over k, is realizable over an extension m of k. For p odd, we notice that the maximal pro-p quotient $\overline{G_k(p)}$ of the absolute Galois group G_k is covered by $\overline{G_m(p)}$, providing a positive answer:

Proposition 1.2. *Let m/k be a finite extension of p-adic fields where p is an odd prime. Then any p-group that is realizable over k is also realizable over m.*

The simplest behavior one can hope for in terms of admissibility, is that a K-admissible group G would be M-admissible if and only if it satisfies Liedahl's condition over M. This is indeed the case for various classes of groups:

(1) When all Sylow subgroups of G are cyclic [19, Theorem 2.8];
(2) When G is abelian and does not fall into a special case over M [2];
(3) For metacyclic groups [9],[10];
(4) For $G = \mathrm{SL}_2(5)$ [5];
(5) For $G = A_6$ or $G = A_7$ [20];
(6) For $G = \mathrm{PGL}_2(7)$ [1], and
(7) For the Symmetric groups $G = S_n$, $1 \leq n \leq 17$, $n \neq 12, 13$ (by [4], [9] and [19]).

Using Proposition 1.2, we are able to add all the odd-order p-groups having the GN-property over M to this list:

Proposition 1.3. *Let M/K be an extension of number fields and p an odd prime. Let G be a p-group that is K-admissible and has the GN-property over M. Then G is M-admissible if and only if G satisfies Liedahl's condition over M.*

Propositions 1.2 and 1.3 are proved in Subsection 3.1. We note (in Section 3) that Proposition 1.3 extends to nilpotent groups of odd order and (by Remark 5.1) to Sylow metacyclic groups (having metacyclic Sylow subgroups). However, the following example shows that Problem 1.1 can have a negative answer for some 2-groups:

Example 1.4. *There is a group G, of order 2^6, which is realizable over $\mathbb{Q}_2$ but not over $\mathbb{Q}_2(\sqrt{-1})$.*

For a proof, see Corollary 3.6. In Proposition 3.8 we interpret this example globally:

Example 1.5. *There is a rational prime q for which the group G of Example 1.4 is $\mathbb{Q}(\sqrt{q})$-admissible, satisfies Liedahl's condition over $\mathbb{Q}(\sqrt{-1}, \sqrt{q})$ but is not $\mathbb{Q}(\sqrt{-1}, \sqrt{q})$-admissible.*

Liedahl showed that a similar phenomena happens for the groups S_n, $n = 12, 13$ and the local extension $\mathbb{Q}_2(\sqrt{-3})/\mathbb{Q}_2$ (see [4]). We shall restrict our discussion to groups which are either of odd order, or with metacyclic 2-Sylow subgroups. For some p-adic extensions m/k, for p odd, including extensions in which the inertia degree $f(m/k)$ is a p-power and $[m:k] > 5$ we show that G_m covers the maximal quotient of G_k with a normal p-Sylow subgroup.
We use this method to answer Problem 1.1 positively for odd primes, under the following assumptions. The list of 'sensitive' extensions of p-adic fields, (16 with $p = 3$ and one for $p = 5$) is described in Subsection 4.2.

Theorem 1.6. *Let p be an odd prime. Let m/k be a non-sensitive extension of p-adic fields and G a group with a normal p-Sylow subgroup P. Assume G is realizable over k. Then there is a subgroup $H \leq G$ that contains P and is realizable over m.*

The question as to whether the non-sensitivity assumption can be removed remains open. However, the assumption that the Sylow subgroup is normal is essential:

Example 1.7. *Let $G = C_7 \wr D$ where*

$$D = \langle a, b \mid a^7 = b^{29} = 1, a^{-1}ba = b^7 \rangle;$$

thus the 7-Sylow subgroups of G are neither normal nor metacyclic.
In Example 4.11 we show there exists an extension m/k of 7-adic fields such that G is realizable over k, although no subgroup of G that contains a 7-Sylow subgroup is realizable over m.

We say that an extension of number fields M/K is sensitive if it has a sensitive completion. The main theorem follows from Theorem 1.6 by combining the local data (see Subsection 5.1):

Theorem 1.8. *Let M/K be a non-sensitive extension of number fields. Let G be a group for which every Sylow subgroup is either normal or metacyclic, and the 2-Sylow subgroups are metacyclic.*
Assume G is K-admissible and has the GN-property over M. Then G is M-admissible if and only if G satisfies Liedahl's condition over M.

Let μ_n denote the set of n-th roots of unity. As a consequence of Theorem 1.8 and [13, Corollary 2] we have:

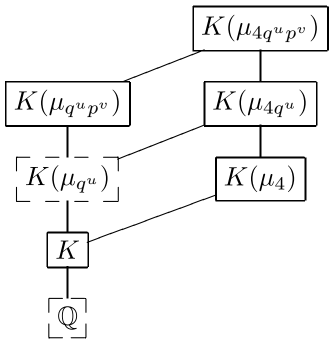

Figure 1. Admissibility of $C_{13} \wr M_{3^3}$: the group is admissible over the solid-boxed fields, but not over the dash-boxed ones (see Example 1.11; here $u \geq 2, v \geq 1, p = 13, q = 3$ and μ_n are the n-th roots of unity.)

Corollary 1.9. *Let G be an odd order group for which every Sylow subgroup is either normal or metacyclic. Let M/K be a non-sensitive extension of number fields so that G is K-admissible and $\mu_{|G|} \cap M = \{1\}$. Then G is M-admissible if and only if G satisfies Liedahl's condition over M.*
In particular, if every prime dividing $|G|$ decomposes in M or if $M \cap \mathbb{Q}(\mu_{|G|}) = K \cap \mathbb{Q}(\mu_{|G|})$, then G is M-admissible.

We also show that the assumption that every Sylow subgroup is either normal or metacyclic is essential in Theorem 1.8:

Example 1.10. *Let G be the group defined in Example 1.7. In Example 5.7, we show furthermore that there is an extension of number fields M/K so that G is K-admissible, satisfies Liedahl's condition over M, has the GN-property over M, but is not M-admissible.*

As an example we use Theorem 1.8 to understand the behavior of admissibility for a specific group (see Example 5.4):

Example 1.11. *Let $K = \mathbb{Q}(\sqrt{14})$ and $G = C_{13} \wr M_{3^3}$, where M_{3^3} is the modular group*

$$\langle x, y | x^{-1}yx = y^4, x^3 = y^9 = 1 \rangle,$$

and $\wr$ is the standard wreath product. In Example 5.4, we show G is K-admissible and deduce from Theorem 1.8 that for a number field $M \supseteq K$, G is M-admissible if and only if G satisfies Liedahl's condition. We therefore deduce the admissibility behavior in Figure 1 by checking Liedahl's condition.

Similar examples (also given in Example 5.4) show that the rank (the minimal number of generators) of the p-Sylow subgroups of K-admissible groups is not bounded (as apposed to the case of admissible p-groups discussed in [19, Section 10]).

The basic facts about admissibility of groups over number fields are reviewed in Section 2. We also discuss the behavior of wild and tame admissibility under extension of number fields and the connection between these types of admissibility to parts (i) and (ii) (respectively) in Liedahl's condition.

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2. Preliminaries

2.1. Admissibility and Preadmissibility. For a prime v of a field K, we denote by K_v the completion of K with respect to v. If L/K is a finite Galois extension, L_v denotes the completion of L with respect to some prime divisor of v in L.
The basic criterion for admissibility over global fields is due to Schacher:

Theorem 2.1 (Schacher, [19]). *Let L/K be a finite Galois extension of global fields. Then L is K-adequate if for every rational prime p dividing $|G|$, where $G = \mathrm{Gal}(L/K)$, there is a pair of primes v_1, v_2 of K such that each of $\mathrm{Gal}(L_{v_i}/K_{v_i})$ contains a p-Sylow subgroup of G.*

Extracting the necessary local conditions for K-admissibility from Theorem 2.1, we arrive at the following definition. For a group G, $G(p)$ denotes a p-Sylow subgroup.

Definition 2.2. Let K be a number field. The group G is K-preadmissible if G is realizable over K, and there exists a finite set $S = \{v_i(p) : p \mid |G|, i = 1, 2\}$ of primes of K, and, for each $v \in S$, a subgroup $G^v \leq G$, such that

(1) $v_1(p) \neq v_2(p)$ for every p dividing $|G|$,
(2) $G^{v_i(p)} \supseteq G(p)$ for every p and $i = 1, 2$, and
(3) G^v is realizable over K_v for every $v \in S$.

(Notice that a p-group G is K-preadmissible if and only if there is a pair of primes v_1 and v_2 of K, such that G is realizable over K_{v_1} and over K_{v_2}.)

Clearly, every K-admissible group is also K-preadmissible. However the opposite does not always hold (see [11, Example 2.14]).
For an extension of fields L/K, $\mathrm{Br}(L/K)$ denotes the kernel of the restriction map $\mathrm{res} : \mathrm{Br}(K) \to \mathrm{Br}(L)$. For number fields we have the following isomorphism of groups, where Π_K is the set of places of K:

$$\mathrm{Br}(L/K) \cong \left(\bigoplus_{\pi \in \Pi_K} \frac{1}{\gcd_{\pi' \mid \pi} [L_{\pi'} : K_\pi]} \mathbb{Z}/\mathbb{Z} \right)_0 ,$$

where $(\,\cdot\,)_0$ denotes that the sum of invariants is zero.
Over a number field K, the exponent of a division algebra is equal to its degree, and so L is K-adequate if and only if there is an element of order $[L:K]$ in $\mathrm{Br}(L/K)$ [19, Proposition 2.1].

2.2. Tame and wild admissibility. We denote by $k_{\rm un}$ the maximal unramified extension of a local field k, and by $k_{\rm tr}$ the maximal tamely ramified extension.
The tamely ramified subgroup $\mathrm{Br}(L/K)_{\rm tr}$ of $\mathrm{Br}(L/K)$ is the subgroup of algebras which are split by the tamely ramified part of every completion of L; namely the subgroup corresponding under the above isomorphism to

$$\left(\bigoplus_{\pi\in\Pi_K} \frac{1}{\gcd_{\pi'|\pi}\left[L_{\pi'}\cap (K_\pi)_{\rm tr} : K_\pi\right]}\mathbb{Z}/\mathbb{Z}\right)_0 . \tag{2.1}$$

Following the above local description of adequacy we define:

Definition 2.3. We say that a finite extension L of K is *tamely K-adequate* if there is an element of order $[L:K]$ in $\mathrm{Br}(L/K)_{\rm tr}$.
Likewise, a finite group G is *tamely K-admissible* if there is a tamely K-adequate Galois G-extension L/K.

The structure of tamely admissible groups is related to Liedahl presentations: For t prime to n, let $\sigma_{t,n}$ be the automorphism of $\mathbb{Q}(\mu_n)/\mathbb{Q}$ defined by $\sigma_{t,n}(\zeta) = \zeta^t$ for $\zeta \in \mu_n$.

Definition 2.4 ([9]). We say that a metacyclic p-group has a *Liedahl presentation* over K, if it has a presentation

$$\mathcal{M}(m,n,i,t) := \left\langle x, y \mid x^m = y^i,\ y^n = 1,\ x^{-1}yx = y^t\right\rangle \tag{2.2}$$

such that $\sigma_{t,n}$ fixes $K\cap\mathbb{Q}(\mu_n)$.

Example 2.5. *The dihedral group D_4 has a Liedahl presentation over $\mathbb{Q}$, but not over $\mathbb{Q}(\sqrt{-1})$. Thus D_4 satisfies the Liedahl condition over $\mathbb{Q}$, but not over $\mathbb{Q}(\sqrt{-1})$.*

The existence of Liedahl's presentation for a p-group G over K implies G is K-tame-preadmissible (namely, Definition 2.2 holds with realizability within $(K_v)_{\rm tr}$ in 2.2.(3)).

Remark 2.6 (Liedahl, follows directly from [9, Proofs of theorems 28 and 29]). Let G be a finite group. If G is realizable over infinitely many completions of K (at infinitely many primes), then G has a presentation as above. If G is a p-group then the converse also holds. In addition a p-group is realizable over infinitely many completions of K if and only if it is realizable over a completion K_v at one prime v that does not divide p.

This allows us to simplify the definition of preadmissibility by noticing that the primes $v_i(p)$, $i = 1,2$, $p \mid |G|$, in Definition 2.2 can be chosen to be distinct:

Lemma 2.7. *Let K be a number field. A group G is K-preadmissible if and only if it is realizable over K, and there are distinct primes $v_i(p)$, p runs over the primes dividing $|G|$ and $i = 1,2$, such that for every p and $i = 1,2$, there is a subgroup $H \leq G$ that contains a p-Sylow subgroup of G and is realizable over $K_{v_i(p)}$.*

Proof. The if part holds by definition. To prove the only if part let

$$T = \{v_i(p) \mid p \mid |G|, i = 1, 2\}$$

be a set of primes of K and for every prime $v \in T$ a corresponding subgroup G^v so that

(1) $v_1(p) \neq v_2(p)$,
(2) G^v is realizable over over K_v,
(3) $G^{v_i(p)}$ contains a p-Sylow subgroup of G,

for every p dividing $|G|$ and $i = 1, 2$. We shall define primes $w_i(p)$, $p \mid |G|$ and $i = 1, 2$, such that all primes are distinct and for every $w_i(p)$ there is a subgroup of G that contains a p-Sylow subgroup of G and is realizable over $K_{w_i(p)}$.
If $v_i(p)$ divides p define $w_i(p) = v_i(p)$ for any $p \mid |G|$ and $i = 1, 2$. If $v_i(p)$ does not divide p then $G(p)$ is metacyclic and has a Liedahl presentation over K (by Remark 2.6). Thus, there are infinitely many primes w of K for which $G(p)$ is realizable over K_w. For all primes $v_i(p)$ that do not divide p (running over both i and p) choose distinct primes $w_i(p)$ which are not in T and for which $G(p)$ is realizable over $K_{w_i(p)}$ (such a choice is possible since there are infinitely many such w's). We have chosen distinct primes $w_i(p)$, $p \mid |G|$ and $i = 1, 2$, as required. □

Remark 2.8. If a p-group G has a Liedahl presentation over M, then G also has a Liedahl presentation over any subfield K of M.

Theorem 2.9 (Liedahl [9], see also [11]). *If G is tamely K-admissible, then $G(p)$ has a Liedahl presentation over K for every prime p dividing $|G|$.*

There are no known counterexamples to the opposite implication. However, the following two results are proved for p-groups in [9, Theorem 30] and in general in [11]:

Theorem 2.10. *Let K be a number field and let G be a solvable group with metacyclic Sylow subgroups. Then G is tamely K-admissible if and only if its Sylow subgroups have Liedahl presentations.*

Theorem 2.11. *Let K be a number field. Let G be a solvable group such that the rational primes dividing $|G|$ do not decompose in K. Then G is K-admissible if and only if its Sylow subgroups are metacyclic and have Liedahl presentations.*

In particular if a solvable group is tamely M-admissible then it is also tamely K-admissible for every $K \subseteq M$, i.e. tame admissibility goes down for solvable groups. Also, if G is solvable and any prime $p \mid |G|$ satisfies Item (i) in Liedahl's condition over M, i.e. does not decompose in M, then G is M-admissible if and only if G is tamely M-admissible.
In particular for $M = \mathbb{Q}$ one has that any solvable group G that is $\mathbb{Q}$-admissible is tamely $\mathbb{Q}$-admissible. However over larger number fields this is no longer the case. Let us define wild K-admissibility:

DEFINITION 2.12. A G-extension L/K is wildly K-adequate if L/K is K-adequate and there is a prime p dividing $|G|$ such that every prime v of K for which

$$\mathrm{Gal}(L_v/K_v) \supseteq G(p),$$

divides p. A K-admissible group G is called wildly K-admissible if every K-adequate G-extension is wildly K-adequate.

Clearly a tamely K-admissible group is not wildly K-admissible. Theorems 2.10 and 2.11 guarantee that a solvable group which is K-admissible but not wildly, is tamely K-admissible. In particular:

Remark 2.13. Every K-admissible p-group which is not tamely K-admissible is wildly K-admissible. So, every non-metacyclic K-admissible p-group is wildly K-admissible.

2.3. THE GRUNWALD-NEUKIRCH (GN) PROPERTY. A group G HAS THE GN-PROPERTY (named after Grunwald and Neukirch) over a number field K if for every finite set S of primes of K and corresponding subgroups $G^v \leq G$ for $v \in S$, there is a Galois G-extension L/K for which $\mathrm{Gal}(L_v/K_v) \cong G^v$ for every $v \in S$.
The Grunwald-Wang Theorem shows that except for special cases (see [25]), abelian groups A have the GN-property over K. A large set of examples comes from a Theorem of Neukirch [13, Corollary 2]. Let $m(K)$ denote the number of roots of unity in a number field K.

THEOREM 2.14 (Neukirch, [13]). *Let K be a number field and G a group for which $|G|$ is prime to $m(K)$. Then G has the GN-property over K.*

Another important source of examples is having a generic extension ([18, Theorem 5.9]):

THEOREM 2.15 (Saltman). *If G has a generic extension over a number field K then G has the GN-property over K.*

By [17], if $\mu_p \subseteq K$ then any group of order p^3 which is not the cyclic group of order 8 has a generic extension over K. In [16], many groups are proved to have a generic extension over number fields, in particular, any abelian group that does not have an element of order 8. In [16] it is also proved that the class of groups with a generic extension is closed under wreath products. In particular we have:

COROLLARY 2.16 (Saltman). *Let q be an odd prime and let K be a number field that contains the q-th roots of unity. Then any iterated wreath product of odd order cyclic groups and groups of order q^3 has the GN-property over K.*

For more examples see [11]. Under the assumption of the GN-property one has the following characterization of wild admissibility:

LEMMA 2.17. *Let K be a number field and G a K-admissible group that has the GN-property over K. Then G is wildly K-admissible if and only if there is a prime $p_0 \,|\, |G|$ for which $G(p_0)$ does not have a Liedahl presentation over K.*

Proof. Assume p_0 is a prime for which $G(p_0)$ does not have a Liedahl presentation over K. Assume on the contrary there is a K-adequate G-extension L/K such that for every $p \mid |G|$ there is a prime v of K that does not divide p, with $\mathrm{Gal}(L_v/K_v) \supseteq G(p)$. Then $G(p_0)$ has a Liedahl presentation over $L^{G(p_0)}$ and by Remark 2.8 $G(p_0)$ has a Liedahl presentation over K, contradiction.
On the other hand if all Sylow subgroups have Liedahl presentations then by Remark 2.6 every Sylow subgroup is realizable over infinitely many completions. One can therefore choose distinct primes $\{v_i(p) \mid i = 1, 2, p \mid |G|\}$ of K such that $G(p)$ is realizable over $K_{v_i(p)}$ and $v_i(p) \nmid p$ for every $p \mid |G|, i = 1, 2$. Since G has the GN-property it follows that G is tamely K-admissible. □

2.4. Galois groups of local fields. Let k be a p-adic field of degree n over $\mathbb{Q}_p$. Let q be the size of the residue field $\overline{k}$, and let p^s be the size of the group of p-power roots of unity inside k_{tr}. Then

(1) $\mathrm{Gal}(k_{\mathrm{un}}/k)$ is (topologically) generated by an automorphism σ, and isomorphic to $\hat{\mathbb{Z}}$;
(2) $\mathrm{Gal}(k_{\mathrm{tr}}/k_{\mathrm{un}})$ is (topologically) generated by an automorphism τ, isomorphic to $\hat{\mathbb{Z}}^{(p')}$ (which is the complement of $\mathbb{Z}_p$ in $\hat{\mathbb{Z}}$);
(3) The group $\mathrm{Gal}(k_{\mathrm{tr}}/k)$ is a pro-finite group generated by σ (lifting the above mentioned automorphism) and τ, subject to the single relation $\sigma^{-1}\tau\sigma = \tau^q$.

Moreover, σ and τ act on μ_{p^s} by exponentiation by some $g \in \mathbb{Z}_p$ and $h \in \mathbb{Z}_p$, respectively (Note that g and h are well defined modulo p^s).
Let $\overline{G_k(p)}$ denote the Galois group $\mathrm{Gal}(m/k)$, where m is the maximal p-extension of k inside a separable closure $\widetilde{k}$. Let p^{s_0} be the number of roots of unity of p-power order in k. Note that if $p^{s_0} \neq 2$ then n must be even. The following Theorem summarizes results of Shafarevich [23], Demushkin [3], Serre [21] and Labute [8]:

Theorem 2.18 ([22, Section II.5.6]). *When $p^{s_0} \neq 2$, $\overline{G_k(p)}$ have the following presentation of pro-p groups:*

$$\overline{G_k(p)} \cong \begin{cases} \langle x_1, \dots, x_{n+2} \mid x_1^{p^{s_0}}[x_1, x_2] \cdots [x_{n+1}, x_{n+2}] = 1\rangle, & \text{if } s_0 > 0 \\ \langle x_1, \dots, x_{n+1}\rangle, & \text{if } s_0 = 0 \end{cases};$$

When $p^{s_0} = 2$ and n is odd,

$$\overline{G_k(p)} \cong \left\langle x_1, \dots, x_{n+2} \mid x_1^2 x_2^4 [x_2, x_3] \cdots [x_{n+1}, x_{n+2}] = 1 \right\rangle,$$

otherwise there is an $f \geq 2$ for which $\overline{G_k(p)}$ has one of the pro-p presentations:

$$\langle x_1, \dots, x_{n+2} \mid x_1^2 [x_1, x_2] x_3^{2^f} [x_3, x_4] \cdots [x_{n+1}, x_{n+2}] = 1\rangle, \text{ or}$$

$$\langle x_1, \dots, x_{n+2} \mid x_1^{2+2^f}[x_1, x_2] \cdots [x_{n+1}, x_{n+2}] = 1\rangle.$$

Theorem 2.19 (Jannsen, Wingberg, [7], see also [15, Theorem 7.5.10]). *The group G_k has the following presentation (as a profinite group):*

$$G_k = \langle \sigma, (\tau)_{p'}, (x_0, \dots, x_n)_p^{\mathrm{N}} \mid \tau^\sigma = \tau^q,\, x_0^\sigma = \langle x_0, \tau\rangle^g x_1^{p^s}[x_1, x_2] \cdots [x_{n-1}, x_n]\rangle$$

if n is even, and

$$G_k = \langle \sigma, (\tau)_{p'}, (x_0, \ldots, x_n)_p^{\mathrm{N}} \mid \tau^\sigma = \tau^q,$$
$$x_0^\sigma = \langle x_0, \tau\rangle^g x_1^{p^s}[x_1, y_1][x_2, x_3]\cdots[x_{n-1}, x_n]\rangle$$

if n is odd, where $(\tau)_{p'}$ denotes that τ is a pro-p' element (has order prime to p in every finite quotient), and $(x_0, \ldots, x_n)_p^{\mathrm{N}}$ denotes the condition that the closed normal subgroup generated by $x_0, \ldots, x_n$ is required to be a pro-p group. Here, the closed subgroup generated by σ and τ is isomorphic to $\mathrm{Gal}(k_{\mathrm{tr}}/k)$. The notation $\langle x_0, \tau\rangle$ stands for $(x_0^{h^{p-1}}\tau x_0^{h^{p-2}}\tau\cdots x_0^h\tau)^{\frac{\pi_p}{p-1}}$, where $\pi_p \in \hat{\mathbb{Z}}$ is an element such that $\pi_p\hat{\mathbb{Z}} = \mathbb{Z}_p$. Also, y_1 is a multiple of $x_1^{\tau^{\pi_2(p+1)}}$ by an element in the maximal pro-p quotient of the pro-finite group generated by x_1, σ^{π_2} and τ^{π_2}. In particular, in every pro-odd quotient of G_k, $[x_1, y_1]$ is trivial.

Remark 2.20. Notice that G_k is a semidirect product of a pro-p group P_k and a profinite metacyclic group D_k, where P_k is the closed normal subgroup generated by $x_0, \ldots, x_n$ and D_k is the closed subgroup generated by σ and τ. The p-Sylow subgroup of G_k is therefore the pro-p closure of $\langle \sigma^{\pi_p}\rangle \cdot P_k$.

Remark 2.21. If G is admissible over a number field K, then for every p there is a subgroup $H \supseteq G(p)$ which is realizable over a completion of K. In particular, H is a product of a metacyclic group and a normal p-subgroup.

The following result on realizability of metacyclic p-groups will be used in Section 5.

LEMMA 2.22. *Let k be a p-adic field. Then any metacyclic p-group G is realizable over k.*

Proof. Let $G = \mathcal{M}(m, n, i, t)$ (see *(2.2)*). The proof for $k \neq \mathbb{Q}_2$ is in [12]. For $k = \mathbb{Q}_2$ we cover 2-groups, so m and n are 2-powers and t is odd. In this case $\overline{G_k(2)}$ has the pro-2 presentation $\langle a, b, c \mid a^2b^4[b, c] = 1\rangle$ (by Theorem 2.18), i.e. $\overline{G_k(2)}$ is isomorphic to the free pro-2 group on three generator modulo the normal closure of the single relation. So the map $\phi : \overline{G_k(2)} \to G$ defined by:

$$a \mapsto x^{-2}y^s,\ b \mapsto x,\ c \mapsto y,$$

is well defined (and surjective) whenever $(x^{-2}y^s)^2x^4[x, y] = 1$.
As t is odd, $t^2+1 \equiv 2 \pmod 4$, $\frac{t^2+1}{2}$ is odd and we can choose an $s \equiv \frac{1}{\frac{t^2+1}{2}}\frac{1}{t^2}\frac{1-t}{2}$ (mod n) so that $s(t^2+1) \equiv \frac{1-t}{t^2}$ (mod n). For such s one has:

$$(x^{-2}y^s)^2x^4[x, y] = x^{-4}y^{st^{-2}+s}x^4y^{t-1} = y^{s(t^{-2}+1)t^4+t-1} = 1.$$

Thus ϕ is well defined. □

3. p-GROUPS

A nilpotent group G is K-admissible if and only if all Sylow subgroups of G are K-admissible. In particular studying the behavior of the admissibility of G under extension of number fields is reduced to understanding the behavior of its Sylow subgroups.

3.1. The case p odd. We begin by proving the observation on realizability over extensions of p-adic fields, p odd.

Proof of Proposition 1.2. Let n denote the rank $[k:\mathbb{Q}_p]$ and let $t = [m:k]$. If $t = 1$ there is nothing to prove. For $t = 2$, $([m:k], p) = 1$ and hence from a G-extension l/k we can form a G-extension lm/m. Now let $t > 2$. It suffice to show that $\overline{G_k(p)}$ is a quotient of $\overline{G_m(p)}$.
By Theorem 2.18, $\overline{G_k(p)}$ and $\overline{G_m(p)}$ have the following presentations of pro-p groups:

$$\overline{G_k(p)} \cong \begin{cases} \langle x_1, \ldots, x_{n+2} \mid x_1^{p^{s_0}}[x_1,x_2]\cdots[x_{n+1},x_{n+2}] = 1\rangle, & \text{if } s_0 > 0 \\ \langle x_1, \ldots, x_{n+1}\rangle, & \text{if } s_0 = 0 \end{cases},$$

and

$$\overline{G_m(p)} \cong \begin{cases} \langle x_1, \ldots, x_{nt+2} \mid x_1^{p^{s'_0}}[x_1,x_2]\cdots[x_{nt+1},x_{nt+2}] = 1\rangle, & \text{if } s'_0 > 0 \\ \langle x_1, \ldots, x_{nt+1}\rangle, & \text{if } s'_0 = 0 \end{cases},$$

where p^{s_0} and $p^{s'_0}$ are the numbers of p-power roots of unity in k and m, respectively. Clearly $s_0 \le s'_0$. Let $F_p(y_1, \ldots, y_k)$ denote the free pro-p group of rank k with generators $y_1, \ldots, y_k$. If $s'_0 = 0$ then we are done since $F_p(x_1, \ldots, x_{n+1})$ is a quotient of $F_p(x_1, \ldots, x_{nt+1})$.
Suppose $s'_0 > 0$. Let $\phi : \overline{G_m(p)} \to F_p(y_1, \ldots, y_{\frac{nt+2}{2}})$ be the epimorphism defined by $\phi(x_{2i-1}) = 1$ and $\phi(x_{2i}) = y_i$, $i = 1, \ldots, \frac{nt+2}{2}$. Now as $t > 2$ we have:

$$\frac{nt+2}{2} = n\frac{t}{2} + 1 \ge n + 2$$

and hence there is a projection π:

$$\pi : F_p(y_1, \ldots, y_{\frac{nt+2}{2}}) \to \overline{G_k(p)}.$$

Thus $\pi \circ \phi : \overline{G_m(p)} \to \overline{G_k(p)}$ is an epimorphism. We deduce that every epimorphic image of $\overline{G_k(p)}$ is also an epimorphic image of $\overline{G_m(p)}$. $\square$

We can now prove Proposition 1.3. It suffices to prove:

Proposition 3.1. *Let M/K be an extension of number fields. Let p be an odd prime and G a p-group that is K-admissible and has the GN-property over M. If G satisfies Liedahl's condition over M, then G is M-admissible.*

Proof. As G is K-admissible, G is realizable over K_{v_1}, K_{v_2} for two primes v_1, v_2 of K. We claim there are two primes w_1, w_2 of M for which G is realizable over M_{w_1}, M_{w_2}. Since G has the GN-property over M proving the claim shows G is M-admissible. There are two cases:
Case I: p decomposes in M. If one of the primes v_1, v_2 does not divide p, then G is metacyclic and hence by Lemma 2.22, G is realizable over any M_{w_1}, M_{w_2} for any two primes w_1, w_2 of M that divide p. If on the other hand both v_1, v_2 divide p then by Proposition 1.2, G is realizable over M_{w_i} for $w_i | v_i$, $i = 1, 2$,

Case II: p does not decompose in M. Since G satisfies Liedahl's condition over M, G has a Liedahl presentation over M. In particular by Theorem 2.11, G is M-admissible. □

As a corollary we deduce that wild admissibility goes up for p-groups:

Corollary 3.2. *Let p be an odd prime. Let M/K be an extension of number fields and G a wildly K-admissible p-group that has the GN-property over M. Then G is also wildly M-admissible.*

Proof. Since G is wildly K-admissible, p decomposes in K and hence in M. Thus G satisfies Liedahl's condition over M and by Proposition 3.1 G is M-admissible. By remarks 2.6 and 2.13, the wild K-admissibility of G implies that G is not realizable over K_v for any prime v which does not divide p. By Remark 2.8, G is also not realizable over M_w for any prime w of M which does not divide p. Therefore an M-adequate G-extension must also be wildly M-adequate and hence G is wildly M-admissible. □

Apply Theorem 2.14, we have:

Corollary 3.3. *Let p be an odd prime. Let M/K be an extension of number fields so that M does not contain the p-th roots of unity. Let G be a p-group that is wildly K-admissible. Then G is wildly M-admissible.*

3.2. The case $p = 2$. As in Proposition 1.2 we have:

Lemma 3.4. *Let m/k is a finite extension of 2-adic fields, which is either of degree greater than 2, or such that m and k contain $\sqrt{-1}$ and have the same 2-power roots of unity. Then any 2-group realizable over k is also realizable over m.*

Proof. If $[m:k] > 2$, the same proof as of Proposition 1.2 holds in all cases of Theorem 2.18. If $\sqrt{-1} \in k$, and k and m have the same number of 2-power roots of unity then $\overline{G_k(p)}$ and $\overline{G_m(p)}$ have the same type of presentations in Theorem 2.18 and one can obtain an epimorphism: $\overline{G_m(p)} \to \overline{G_k(p)}$ simply by dividing by the redundant generators of $\overline{G_m(p)}$. □

However, we show that Proposition 1.2 may fail for $p = 2$. We begin with some group-theoretic preparations.

Lemma 3.5. *The group*

$$G \quad = \quad \langle a_1, a_2, a_3 \mid G' \text{ is central of exponent } 2, \quad a_1^2 = [a_2, a_3], \;\; a_2^2 = a_3^2 = 1 \rangle$$

is not a quotient of the pro-2 group

$$\Gamma = \langle x_1, \dots, x_4 \mid x_1^4 [x_1, x_2][x_3, x_4] = 1 \rangle.$$

Proof. For $j, k = 1, 2, 3$, write $\alpha_{jk} = [a_j, a_k] \in G$. Suppose $x_i \mapsto a_1^{t_{i,1}} a_2^{t_{i,2}} a_3^{t_{i,3}} z_i$ $(i = 1, \dots, 4)$ is an epimorphism $\Gamma \to G$, where $z_i \in G'$. Then $[x_{2i-1}, x_{2i}] \mapsto$

$\prod_{j,k=1}^{3}[a_j^{t_{2i-1,j}}, a_k^{t_{2i,k}}] = \prod_{1\le k<j\le 3}\alpha_{jk}^{t_{2i-1,j}t_{2i,k}-t_{2i-1,k}t_{2i,j}}$. Since $\exp(G) = 4$, the defining relation of Γ translates to

$$\prod_{i=1}^{2}\prod_{1\le k<j\le 3}\alpha_{jk}^{t_{2i-1,j}t_{2i,k}-t_{2i-1,k}t_{2i,j}} = 1,$$

from which it follows that $t_{1,j}t_{2,k} - t_{1,k}t_{2,j} + t_{3,j}t_{4,k} - t_{3,k}t_{4,j} \equiv 0 \pmod 2$ for every $1 \le k < j \le 3$.
Let V denote the vector space $\mathbb{F}_2^4$, endowed with the bilinear form $b : V\times V \to \mathbb{F}_2$ defined by $b((v_i)_{i=1}^4, (v_i')_{i=1}^4) = v_1v_2' - v_2v_1' + v_3v_4' - v_4v_3'$. This is an alternating non-degenerate form (in fact, hyperbolic), and letting $t^j \in V$ be the vectors $t_i^j = t_{i,j}$, we have that $b(t^j, t^k) = 0$ for every $j, k = 1, 2, 3$. It follows that $T = \operatorname{span}\{t^1, t^2, t^3\} \subset V$ is orthogonal to itself. But then $\dim T \le \frac{1}{2}\dim V = 2$, contradicting the assumption that the induced map $\Gamma \to G/G' = C_2^3$ is surjective. □

Corollary 3.6. *There is a group of order 2^6 which is realizable over $k = \mathbb{Q}_2$ but not over $m = \mathbb{Q}_2(\sqrt{-1})$.*

Proof. As before, we construct a quotient of $\overline{G_k(2)}$ which is not a quotient of $\overline{G_m(2)}$. Let G and Γ be as in Lemma 3.5. By Theorem 2.18, $\overline{G_m(2)} \cong \Gamma$ and

$$\overline{G_k(2)} = \big\langle x_1, x_2, x_3 \mid x_1^2x_2^4[x_2, x_3] = 1\big\rangle.$$

Mapping $x_i \mapsto a_i$ projects $\overline{G_k(2)}$ onto G, which is not a quotient of Γ. □

It seems that 2^6 is the minimal possible order for such a 2-group.

Remark 3.7. Let m/k be an extension of local fields. If there is one 2-group which is realizable over k but not over m, then there are infinitely many such groups. Indeed, let G be such a group, and let k' be a G-Galois extension of k; the Galois group of any 2-extension of k' which is Galois over k has G as a quotient, and so is not realizable over m.

Let us apply this example to construct an extension of number fields M/K for which the group G of Lemma 3.5 is wildly K-admissible but not M-admissible and not even M-preadmissible.
Let p and q be two primes for which:
1) $p \equiv 5 \pmod 8$
2) $q \equiv 1 \pmod 8$
3) q is not a square mod p.

Proposition 3.8. *Let $K = \mathbb{Q}(\sqrt{q}), M = K(i)$ and G be the group from Lemma 3.5. Then G is wildly K-admissible but not M-preadmissible.*

Proof. Since G is a 2-group that is not metacyclic it is realizable only over completions at primes dividing 2. In particular if G is K-admissible then G is wildly K-admissible. As 2 splits in K, any (of the two) prime divisor v of 2 in M has a completion $M_v \cong \mathbb{Q}_2(i)$. By Corollary 3.6, G is not realizable over $\mathbb{Q}_2(i)$ and hence not M-preadmissible. It therefore remains to show G is K-admissible.

The rational prime p is inert in K. Let $\mathfrak{p}$ be the unique prime of K that divides p. We have $N(\mathfrak{p}) := \left|\overline{K_\mathfrak{p}}\right| = p^2 \equiv 1 \pmod 8$. Thus, $K_\mathfrak{p}$ has a totally ramified C_8-extension. Let $\mathfrak{q}_1, \mathfrak{q}_2$ be the two primes of K dividing 2.

Consider the field extension $L_0 = K(\mu_8, \sqrt{p})/K$. It has a Galois group

$$\mathrm{Gal}(L_0/K) \cong C_2^3 \cong G/Z(G).$$

This extension is ramified only at $\mathfrak{q}_1, \mathfrak{q}_2$ and $\mathfrak{p}$. As $K_{\mathfrak{q}_i} \cong \mathbb{Q}_2$ and $\mathrm{Gal}((L_0)_{\mathfrak{q}_i}/K_{\mathfrak{q}_i}) \cong C_2^3$, $(L_0)_{\mathfrak{q}_i}/K_{\mathfrak{q}_i}$ is the maximal abelian extension of $K_{\mathfrak{q}_i}$ of exponent 2, for $i = 1, 2$. Note that $N(\mathfrak{p}) \equiv 1 \pmod 8$ and hence $K_\mathfrak{p}$ contains μ_8.

Notice that $Z(G) = G' \cong C_2^3$ and $G/Z(G) \cong C_2^3$. Let us show that the central embedding problem

(3.1)
$$\begin{tikzcd} & & & G_K \arrow[d] \arrow[dl, dashed] & \\ 0 \arrow[r] & Z(G) \arrow[r] & G \arrow[r] & G/Z(G) \arrow[r] & 0, \end{tikzcd}$$

has a solution. Let π denote the epimorphism $G \to G/Z(G)$. By theorems 2.2 and 4.7 in [14], there is a global solution to Problem 3.1 if and only if there is a local solution at every prime of K. There is always a solution at primes of K which are unramified in L_0 so it suffices to find solutions at $\mathfrak{p}, \mathfrak{q}_1, \mathfrak{q}_2$. Any G-extension of $K_{\mathfrak{q}_i}$ contains $(L_0)_{\mathfrak{q}_i}$ (as it is the unique C_2^3 extensions of $\mathbb{Q}_2$), $i = 1, 2$. Since G is realizable over $\mathbb{Q}_2$ we deduce that the induced local embedding problem

(3.2)
$$\begin{tikzcd} & G_{K_{\mathfrak{q}_i}} \arrow[d] \arrow[dl, dashed] & \\ \pi^{-1}(\mathrm{Gal}((L_0)_{\mathfrak{q}_i}/K_{\mathfrak{q}_i})) = G \arrow[r] & \mathrm{Gal}((L_0)_{\mathfrak{q}_i}/K_{\mathfrak{q}_i}) \cong G/Z(G) \arrow[r] & 0, \end{tikzcd}$$

has a solution for $i = 1, 2$. Since $(L_0)_\mathfrak{p}$ is the ramified C_2-extension of $K_\mathfrak{p}$, it can be embedded into the totally ramified C_4-extension and hence the local embedding problem at $\mathfrak{p}$ has a solution.

Therefore, Embedding problem 3.1 has a solution. Let L be the corresponding solution field. As Problem 3.1 is a Frattini embedding problem such a solution must be surjective globally and at $\{\mathfrak{q}_1, \mathfrak{q}_2\}$. Thus, L_0/K can be embedded in a Galois G-extension L/K for which $\mathrm{Gal}(L_{\mathfrak{q}_i}/K_{\mathfrak{q}_i}) \cong G$, for $i = 1, 2$. The field L is clearly K-adequate and hence G is K-admissible. $\square$

4. Realizability under extension of local fields

Realizability of a group G as a Galois group over a field k is clearly a necessary condition for k-admissibility. When k is a local field, the conditions are equivalent since a division algebra of index n is split by every extension of degree n.
In this section we study realizability of groups under field extensions, assuming the fields are local.

4.1. Totally ramified extensions. We first note what happens under prime to p local extensions:

Lemma 4.1. *Let G_1 be a subgroup of G that contains a p-Sylow subgroup of G and is realizable over the p-adic field k. Let m/k be a finite extension for which $([m:k],p) = 1$. Then there is a subgroup $G_2 \leq G_1$ that contains a p-Sylow subgroup of G and is realizable over m.*

Proof. Let l/k be a G_1-extension. Then lm/m is a Galois extension with Galois group G_2 which is a subgroup of G_1 and for which $[l : l \cap m] = |G_2|$. Since $([l \cap m:k], p) = 1$, any $p^s \mid [l:k] = |G_1|$ also divides $p^s \mid [l:l \cap m] = |G_2|$. Thus G_2 must also contain a p-Sylow subgroup of G. □

The case where p divides the degree $[m:k]$ is more difficult. Let us consider next totally ramified extensions:

Lemma 4.2. *Let $p \neq 2$. Let G be a group, k a p-adic field with $n = [k : \mathbb{Q}_p]$ and m/k a totally ramified finite extension. Assume furthermore that m/k is not the extension $\mathbb{Q}_3(\zeta_9 + \overline{\zeta}_9)/\mathbb{Q}_3$. If G is realizable over k then G is also realizable over m.*

Remark 4.3. This shows that if G has a subgroup G_1 that contains a p-Sylow of G and is realizable over k then we can pick $G_2 := G_1$ as a subgroup which contains a p-Sylow subgroup of G and is realizable over m.

Proof. Let m/k be a totally ramified extension of degree $r = [m:k]$. We shall construct an epimorphism $G_m \to G_k$. For this we shall consider the presentations given in Theorem 2.19. Denote the parameters of k by n, q, s, g and h. Then the degree of m over $\mathbb{Q}_p$ is nr and its residue degree remains q. Denote the rest of the parameters over m by s', g' and h' (the parameters that correspond to s, g and h in Theorem 2.19). Then by Theorem 2.19, G_m has the following presentation (as a profinite group):

$$G_m = \langle \sigma, (\tau)_{p'}, (x_0, \ldots, x_{nr})_p^{\mathrm{N}} \mid \tau^\sigma = \tau^q, x_0^\sigma = \langle x_0, \tau \rangle^{g'} x_1^{p^{s'}} [x_1, x_2] \cdots [x_{nr-1}, x_{nr}] \rangle,$$

if nr is even and

$$G_m = \langle \sigma, (\tau)_{p'}, (x_0, \ldots, x_{nr})_p^{\mathrm{N}} \mid \tau^\sigma = \tau^q, x_0^\sigma = \langle x_0, \tau \rangle^{g'} x_1^{p^{s'}} [x_1, y_1][x_2, x_3] \cdots [x_{nr-1}, x_{nr}] \rangle,$$

if nr is odd. Let P_k be the closed normal subgroup of G_k generated by $x_0, \ldots, x_n$ and let D_k (resp. D_m) be the closed subgroup generated by σ and τ. By assumption, P_k is a pro-p group. Note that as k and m have the same residue degree (same q), $D_k \cong D_m$.
Let us construct the epimorphism from G_m to G_k. First send x_0 and every x_k with k odd in the presentation of G_m to 1. We get an epimorphism

$$G_m \twoheadrightarrow \langle \sigma, (\tau)_{p'}, (z_1, \ldots, z_d)_p^{\mathrm{N}} \mid \sigma\tau\sigma^{-1} = \tau^q \rangle$$

where $d = \lceil \frac{nr-1}{2} \rceil$ and $\lceil \gamma \rceil$ denotes the smallest integer $\geq \gamma$. Let us continue under the assumption $d \geq n+1$. Then there is an epimorphism $F_p(d) \twoheadrightarrow F_p(n+1)$. We therefore obtain epimorphisms:

$$G_m \twoheadrightarrow \langle \sigma, (\tau)_{p'}, (z_1, \ldots, z_{n+1})_p^{\mathrm{N}} \mid \sigma\tau\sigma^{-1} = \tau^q \rangle \twoheadrightarrow D_k \ltimes P_k \twoheadrightarrow G_k.$$

The numerical condition $\lceil \frac{nr-1}{2} \rceil \geq n+1$ fails if and only if:

(1) $r = 1$, or
(2) $r = 2$, or
(3) $r = 3$ and $n = 1$.

The case $r = 1$ is trivial. Since p is an odd prime, the cases $r = 2$ and $r = 3$ are done by Lemma 4.1, unless $p = 3$ and $[m:k] = r = 3$, in which case $n = 1$, so $k = \mathbb{Q}_3$. If $m \neq \mathbb{Q}_3(\zeta_9 + \overline{\zeta}_9)$ then $m \cap k_{\mathrm{tr}} = k$ and the parameters g, h, s in the presentation of G_k remain the same in the presentation of G_m. In such case there is an epimorphism from G_m onto G_k whose kernel is generated by $\langle x_2, x_3 \rangle$. □

So the case $k = \mathbb{Q}_3$ and $m = \mathbb{Q}_3(\zeta_9 + \overline{\zeta}_9)$ remains open. This will be one of several *sensitive* cases.

4.2. The sensitive cases.

DEFINITION 4.4. We call the extension m/k *sensitive* if it is one of the following:

(1) $k = \mathbb{Q}_3$ and m is the totally ramified 3-extension $\mathbb{Q}_3(\zeta_9 + \overline{\zeta}_9)$,
(2) $k = \mathbb{Q}_5$ and $m = \mathbb{Q}_5(\zeta_{11})$ is the unramified 5-extension,
(3) $[k:\mathbb{Q}_3] = 1, 2, 3$ and m/k is the unramified 3-extension,
(4) $k = \mathbb{Q}_3$ and $m = \mathbb{Q}_3(\zeta_7)$ is the unramified 6-extension.

Remark 4.5. There are 17 sensitive field extensions, up to isomorphism: one over $\mathbb{Q}_5$ and 16 over $\mathbb{Q}_3$. Fixing the algebraic closures of the respective p-adic fields, there is one sensitive 5-adic extension and 27 3-adic ones. This can be verified using the automated tools in [6]. We provide details in the Appendix.

Let us formulate the problem in case (1) for odd order groups:

Remark 4.6. Given a field F denote by G_F^{odd} the Galois group that corresponds to the maximal pro-odd Galois extension of F. In the p-adic case, for odd p, this is obtained from the presentation of G_k (see Theorem 2.19) simply by dividing by the 2-part of σ and τ. In such case we get a presentation of G_F^{odd} by identifying $\sigma_2 = \tau_2 = 1$. We get that y_1 is a power of x_1 and hence $[x_1, y_1] = 1$.

Question 4.7. *Let m/k be the sensitive extension (1). Then $q = 3$; also $p^s = 3$ so we can choose $h = -1$. For m we have $p^{s_m} = 9$ and $\tau(\zeta_9 + \zeta_9^{-1}) = \zeta_9 + \zeta_9^{-1}$, so $h_m = -1$ as well. Theorem 2.19 gives us the presentations:*

$$G_k^{\mathrm{odd}} = \langle \sigma, (\tau)_{3'}, (x_0, x_1)_3^{\mathrm{N}} \mid \tau^\sigma = \tau^3, x_0^\sigma = \langle x_0, \tau\rangle x_1^3 \rangle,$$

while:

$$G_m^{\mathrm{odd}} = \langle \sigma, (\tau)_{3'}, (x_0, x_1, x_2, x_3)_3^{\mathrm{N}} \mid \tau^\sigma = \tau^3, x_0^\sigma = \langle x_0, \tau\rangle x_1^9 [x_2, x_3] \rangle,$$

where σ, τ are of order prime to 2 and $\langle x_0, \tau\rangle = (x_0 \tau x_0^{-1} \tau)^{\frac{\pi}{2}}$, which has order a power of 3 in every finite quotient. Does the following hold: Let G be an epimorphic image of G_k^{odd}, is there necessarily a subgroup $G(p) \le G_0 \le G$ so that G_0 is an epimorphic image of G_m^{odd}?

Note that for a 3-group G the claim was proved in Proposition 1.2.

Remark 4.8. In fact quotients of G_k^{odd} with $\tau = 1$ can be covered: the group $\left\langle \sigma, (x_0, x_1)_3^{\mathrm{N}} \mid x_0^\sigma = x_1^3 \right\rangle = \left\langle \sigma, (x_1)_3^{\mathrm{N}} \right\rangle$ is covered by $\sigma \mapsto \sigma$, $\tau \mapsto 1$, $x_0 \mapsto 1$, $x_1 \mapsto 1$, $x_2 \mapsto x_1$ and $x_3 \mapsto 1$. This corresponds to realization of G over k whose ramification index is a 3-power.

4.3. Extensions of local fields. We can now approach the general case. Recall the presentation of G_k from Theorem 2.19. Let P_k denote the closed normal subgroup generated by $x_0, \ldots, x_n$ and D_k the closed subgroup generated by σ and τ, as in Remark 2.20.

Remark 4.9.
(1) Decompose $\langle\sigma\rangle$ into its p-primary part generated by σ_p and its complement generated by $\sigma_{p'}$ so that $\sigma = \sigma_p \sigma_{p'}$, where $[\sigma_p, \sigma_{p'}] = 1$. Then the pro-$p$ closure of $\langle\sigma_p\rangle \cdot P_k$ is a p-Sylow subgroup of G_k.
(2) In every finite quotient σ_p is a power of σ, and so normalizes τ. It follows that $[\tau, \sigma_p]$ is a power of τ, and so a pro-p' element.
(3) The image of the closure of $\langle\sigma_p\rangle P_k$ is normal in a quotient of G_k if and only if τ conjugates σ_p into the closure of $\langle\sigma_p\rangle P_k$; but then the image of $[\tau, \sigma_p]$ is a pro-p element, so by (2) this is the case if and only if the image of $[\tau, \sigma_p]$ is trivial.
(4) Let $\bar{G}_k$ denote the maximal quotient of G_k with a normal p-Sylow subgroup. It follows from (3) that $\bar{G}_k$ is defined by the relation $[\tau, \sigma_p] = 1$.

Lemma 4.10. *Let p be an odd prime. Let m/k be an extension of p-adic fields with $f = [\bar{m} : \bar{k}]$ a p-power, and $\lceil \frac{nr-1}{2} \rceil \ge n+2$ where $n = [k : \mathbb{Q}_p]$ and $r = [m : k]$. Then $\bar{G}_k$ is a quotient of G_m.*

Proof. We construct an epimorphism from G_m to $\bar{G}_k$. Let s_m, g_m, h_m be the invariants s, g, h in Theorem 2.19 that correspond to m, and let $n = [k : \mathbb{Q}_p]$. Theorem 2.19 gives the following presentation of G_m:

$$G_m = \langle \sigma, (\tau)_{p'}, (x_0, \ldots, x_{nr})_p^{\mathrm{N}} \mid \\ \tau^\sigma = \tau^{q^f}, x_0^\sigma = \langle x_0, \tau\rangle^{g_m} x_1^{p^{s_m}} [x_1, x_2] \cdots [x_{nr-1}, x_{nr}] \rangle,$$

if nr is even and

$$G_m = \langle \sigma, (\tau)_{p'}, (x_0, \dots, x_{nr})_p^{\mathrm{N}} \mid \tau^\sigma = \tau^{q^f},$$
$$x_0^\sigma = \langle x_0, \tau\rangle^{g_m} x_1^{p^{sm}}[x_1, y_1][x_2, x_3]\cdots[x_{nr-1}, x_{nr}]\rangle,$$

if nr is odd.
Let P_k (resp. P_m) be the closed normal subgroup generated by $x_0, \dots, x_n$ (resp. $x_0, \dots, x_{nr}$) in G_k (resp. G_m) and let $D_k \le G_k$ (resp. $D_m \le G_m$) be the closed subgroup generated by σ, τ in G_k (resp. in G_m).
Set $d = \lceil \frac{nr-1}{2} \rceil$, so by assumption $d \ge n + 2$. Similarly to Lemma 4.2 (noting that this time D_m can be viewed as a subgroup of index f in D_k), we have an epimorphism

$$\begin{aligned} G_m &\twoheadrightarrow \left\langle \sigma, (\tau)_{p'}, (z_1, \dots, z_m)_p^{\mathrm{N}} \mid \tau^\sigma = \tau^{q^f} \right\rangle \\ &\cong \left\langle (\sigma_p)_p, (\sigma_{p'})_{p'}, (\tau)_{p'}, (z_1, \dots, z_m)_p^{\mathrm{N}} \mid \tau^{\sigma_p \sigma_{p'}} = \tau^{q^f}, [\sigma_p, \sigma_{p'}] = 1 \right\rangle. \end{aligned}$$

Let us divide by the relations $z_m^f = \sigma_p$ and $\tau^{z_m} = \tau^{q\sigma_{p'}^{-1/f}}$, where $\sigma_{p'}^{-1/f}$ is well defined since f is a p-power. We then obtain an epimorphism to

$$\langle (\sigma_{p'})_{p'}, (\tau)_{p'}, (z_1, \dots, z_m)_p^{\mathrm{N}} \mid \tau^{z_m} = \tau^{q\sigma_p'^{-1/f}}, [z_m^f, \sigma_{p'}] = 1 \rangle.$$

Adding the relation $[z_m, \sigma_{p'}]$ and sending $z_m \mapsto \sigma_p$ maps this group onto

$$\langle (\sigma_{p'})_{p'}, (\tau)_{p'}, (z_1, \dots, z_{m-1}, \sigma_p)_p^{\mathrm{N}} \mid \tau^{\sigma_p \sigma_{p'}^{1/f}} = \tau^q, [\sigma_p, \sigma_{p'}] = 1 \rangle.$$

Mapping $\sigma_{p'} \mapsto \sigma_{p'}^f$, this groups maps onto

$$\left\langle (\sigma_p)_p, (\sigma_{p'})_{p'}, (\tau)_{p'}, (x_0, \cdots, x_n)_p^{\mathrm{N}} \mid \tau^{\sigma_p \sigma_{p'}} = \tau^q,\ [\sigma_p, \sigma_{p'}] = [\sigma_p, \tau] = 1 \right\rangle,$$

since by Remark 4.9 the assumption that the normal subgroup generated by σ_p is a p-group is equivalent to $[\sigma_p, \tau] = 1$.
But $\bar{G}_k$ is a quotient of this group by Theorem 2.19 and Remark 4.9.(4). $\square$

Using Lemma 4.10, we can prove:

Proof of Theorem 1.6. Let n, q be as defined above for k, and let $r = [m{:}k]$. Let $f = [\overline{m}{:}\overline{k}] = f_p f_{p'}$ where f_p is a p-power and $f_{p'}$ is prime to p. There is an unramified C_f-extension m'/k which lies in m, and then m/m' is totally ramified. Denote by m_p the subfield of m' which is fixed by C_{f_p}. Let $r' = \frac{r}{f_{p'}} = [m{:}m_p]$. By Lemma 4.1, there is a subgroup $G_0 \le G$ that contains a p-Sylow subgroup of G and an epimorphism $\phi : G_{m_p} \to G_0$. The list of sensitive cases satisfies that if m/k is non-sensitive and m_p/k is unramified and prime to p, then m/m_p is also non-sensitive and therefore we can assume without loss of generality that $m_p = k$, $G = G_0$, i.e $f_{p'} = 1$, $f = f_p$, and $r' = r$.
If $\lceil \frac{nr-1}{2} \rceil \ge n + 2$ then G is a quotient of G_m by Lemma 4.10. This numerical condition fails if and only if

(1) $r = 4, 5$ and $n = 1$;
(2) $r = 3$ and $n = 1, 2, 3$;
(3) $r = 1, 2$.

The cases $r = 1, 2, 4$ are covered by Lemma 4.1. We are left with cases $r = 3, 5$. For $r = 5$, $n = 1$ so $k = \mathbb{Q}_5$ and by Lemma 4.2 we may assume m/k is not totally ramified, so m/k is the unramified 5-extension which is sensitive.
Let $r = 3$. Lemma 4.1 covers the case $p \neq 3$, so we may assume $p = 3$. Note that $f \,|\, r$. If $f = 1$, Lemma 4.2 applies, except for $m = \mathbb{Q}_3(\zeta_9 + \overline{\zeta_9})$ and $k = \mathbb{Q}_3$, which is sensitive. If $f = 3$, then m/k is the unramified 3-extension and $n = [k : \mathbb{Q}_3] = 1, 2, 3$ which are all sensitive. $\square$

The following example shows that the assumption in Theorem 1.6 that the normal p-Sylow subgroup of G is normal, is essential.

Example 4.11. *Let $p < q$ be odd primes such that $p^p \equiv 1 \pmod q$ and $p \not\equiv 1 \pmod q$ (for example $p = 7$ and $q = 29$). Let $G = C_p \wr D$ where*

$$D = \langle a, b \mid a^p = b^q = 1, a^{-1}ba = b^p \rangle.$$

Let $k = \mathbb{Q}_p$ and m/k the unramified extension of degree p. Then G is realizable over k but there is no subgroup of G that contains a p-Sylow subgroup of G and is realizable over m.

Proof. Let $P = C_p^{pq}$, so that $G = P \rtimes D$. Then one has the projections

$$G_k \to G_k^{\mathrm{odd}} = \langle \sigma, (\tau)_{p'}, (x_0, x_1)_p^{\mathrm{N}} \mid \tau^\sigma = \tau^p, x_0^\sigma = \langle x_0, \tau \rangle x_1^p \rangle \to$$
$$\to \langle \sigma, (\tau)_{p'}, (x_1)_p^{\mathrm{N}} \mid \tau^\sigma = \tau^p, x_1^p = 1 \rangle,$$

where the latter two group are pro-odd and the second epimorphism is obtained by dividing by x_0. The latter group maps onto G by $\sigma \mapsto a$ and $\tau \mapsto b$. It is therefore left to prove that for any homomorphism $\phi : G_m \to G$, $\mathrm{Im}(\phi)$ does not contain a p-Sylow subgroup of G. Assume on the contrary that $H = \mathrm{Im}(\phi)$ does. Recall:

$$G_m = \langle \sigma, (\tau)_{p'}, (x_0, \ldots, x_p)_p^{\mathrm{N}} \mid$$
$$\tau^\sigma = \tau^{p^p}, x_0^\sigma = \langle x_0, \tau \rangle x_1^p [x_1, y_1][x_2, x_3] \cdots [x_{p-1}, x_p] \rangle.$$

Since q is the only prime dividing $|G|$ other than p, and τ is pro-p', any map into G must split through:

$$G_m \to \langle \sigma, (\tau)_{p'}, (x_0, \ldots, x_p)_p^{\mathrm{N}} \mid [\sigma, \tau] = 1, \tau^q = 1,$$
$$x_0^\sigma = \langle x_0, \tau \rangle x_1^p [x_1, y_1][x_2, x_3] \cdots [x_{p-1}, x_p] \rangle.$$

However the latter group has a normal p-Sylow subgroup which is the product of the closed normal subgroup generated by the x_i's and the pro-p group generated by σ^{π_p}. In particular, letting $\pi : G \to G/P = D$ be the projection, the image of $\pi\phi$ has a normal p-Sylow subgroup. This implies $\pi\phi$ is not surjective. But H contains a p-Sylow subgroup of G, so we must have $\mathrm{Im}(\pi\phi) = C_p$. Again since H contains a p-Sylow subgroup, and in particular P, we must have $H = P \rtimes C$ where $C = C_p$ is a subgroup of D and the action of C on P is induced from the action of D. Thus:

$$\mathrm{rank}(H) = \mathrm{rank}(H/[H, H]) = \mathrm{rank}((P/[P, C]) \times C) = q + 1.$$

Since H is a p-group any epimorphism to it must split through $\overline{G_m(p)}$. However $\operatorname{rank}(\overline{G_m(p)}) = [m:k]+1 = p+1$, leading to a contradiction. □

5. Extensions of number fields

We shall now apply Theorem 1.6 to study admissibility and wild admissibility.

5.1. Main Theorem.

Proof of Theorem 1.8. As mentioned in the introduction, Liedahl's condition is necessary. Let us show that if G satisfies this condition then G is M-admissible. We claim that one can choose distinct primes $w_i(p)$, $i = 1, 2, p \mid |G|$, of M and corresponding subgroups $H_i(p) \leq G$ so that $H_i(p)$ contains a p-Sylow subgroup of G and is realizable over $M_{w_i(p)}$, $i = 1, 2$.
As G is K-admissible, for every $p \mid |G|$ there are two options:

(1) there are two primes v_1, v_2 of K dividing p and two subgroups $G(p) \leq G_i \leq G$ so that G_i is realizable over K_{v_i}, $i = 1, 2$.
(2) $G(p)$ is realizable over K_v for v which does not divide p.

In case (1) with p odd and $G(p)$ normal in G, by Theorem 1.6, for any prime w dividing v_1 or v_2 there is a subgroup $G(p) \leq H_w \leq G$ that is realizable over M_w. Choose two such primes $w_1(p), w_2(p)$ and set $H_i(p) := H_{w_i(p)}$ (the subgroups Theorem 1.6 constructs). In case $G(p)$ is not normal or $p = 2$, we assumed $G(p)$ is metacyclic and by Lemma 2.22, $G(p)$ is realizable over any M_w for any prime w dividing v_1 or v_2. In such case similarly choose two such primes $w_1(p), w_2(p)$ and set $H_i(p) = G(p)$.
In case (2), $G(p)$ is metacyclic. If p has more than one prime divisor in M then there are two primes $w_1(p), w_2(p)$ so that $w_i(p)$ divides p and by Lemma 2.22 $H_i(p) := G(p)$ is realizable over $M_{w_i(p)}$, $i = 1, 2$.
If p has a unique prime divisor in M then $G(p)$ is assumed to have a Liedahl presentation. Liedahl's condition implies that there are infinitely many primes $w(p)$ for which $G(p)$ is realizable over $M_{w(p)}$ (see Theorem 28 and Theorem 30 in [9]). Thus, we can choose two primes $w_1(p), w_2(p)$ for every prime $p \mid |G|$ that has only one prime divisor in M, so that the primes $w_i(p), i = 1, 2$, are not divisors of any prime $q \mid |G|$ and are all distinct. For such p, we also choose $H_i(p) := G(p)$.
We have covered all cases of behavior of divisors of rational primes in M and hence proved the claim. It follows that G is M-preadmissible and as G has the GN-property over M, G is M-admissible. □

Remark 5.1. If G has metacyclic Sylow subgroups, the proof of Theorem 1.8 does not use Theorem 1.6 and holds for sensitive extensions as well.

5.2. Wild admissibility. As to wild admissibility Theorem 1.8 and Lemma 2.17 give:

Corollary 5.2. *Let M/K be a non-sensitive extension. Let G be a K-admissible group for which every Sylow subgroup is either normal or metacyclic and the 2-Sylow subgroups are metacyclic. Assume G has the GN-property over*

M, satisfies Liedahl's condition over M but there is a prime p for which $G(p)$ does not have a Liedahl presentation over M. Then G is wildly M-admissible.

We deduce that for groups as in Theorem 1.8, wild admissibility goes up in the following sense that generalizes Corollary 3.2:

Corollary 5.3. *Let M/K be a non-sensitive extension. Let G be a wildly K-admissible group for which every Sylow subgroup is either normal or metacyclic and the 2-Sylow subgroups are metacyclic. Assume G has the GN-property over K and M satisfies Liedahl's condition over M. Then G is wildly M-admissible.*

Proof. By Theorem 1.8, G is M-admissible. The assertion now follows from Lemma 2.17, applied to both K and M, and the fact that if $G(p)$ does not have a Liedahl presentation over K then $G(p)$ does not have a Liedahl presentation over M (see Remark 2.8). □

5.3. Examples. The following is an example in which Theorem 1.8 is used to understand how admissibility behaves under extensions of a given number field:

Example 5.4. *Let p, q be odd primes and m an integer so that m is not square mod q but is a square mod p and $p \equiv q+1$ (mod q^2). For example $p = 13$, $q = 3$, $m = 14$. Let $K = \mathbb{Q}(\sqrt{m})$ and $G = C_p \wr H$, where H is one of the following groups:*

(1) *$H = M_{q^3}$ is the modular group of order q^3, i.e.*
$$H = \langle x, y | x^{-1}yx = y^{q+1}, x^q = y^{q^2} = 1\rangle.$$
(2) *$H = C_{pq} \times C_q$.*
(3) *$H = C_t$ where $t \in \mathbb{N}_{\mathrm{odd}}$ is prime to p.*

We shall show in each of the cases G is K-admissible. Let M be any non-sensitive extension of K.
By Theorem 2.14, in case (1) G satisfies the GN-property over any number field that does not have any p-th and q-th roots of unity, in particular over K.
By Corollary 2.16, in cases (2),(3), G satisfies the GN-property over any M and in case (1) if M contains the q-th roots of unity.
In cases (2),(3), G satisfies Liedahl's condition over any M and in case (1), G satisfies Liedahl's condition over M if and only if q decomposes in M or $M \cap \mathbb{Q}(\mu_{q^2}) \subseteq \mathbb{Q}(\mu_q)$.
It follows from Theorem 1.8, that in cases (2) and (3) G is M-admissible. In case (1), if one assumes M does not contain any p-th and q-th roots of unity or that M contains the q-th roots of unity then G is M-admissible if and only if G satisfies Liedahl's condition.

Proof. The prime p splits (completely) in K. Denote it's prime divisors in K by v_1, v_2. Then $K_{v_i} \cong \mathbb{Q}_p$ for $i = 1, 2$. Using the presentation of $G_{\mathbb{Q}_p}^{\mathrm{odd}}$ given in Question 4.7 and dividing by $x_0 = 1$ one obtains an epimorphism:

$$G_{\mathbb{Q}_p} \to \langle \sigma, (\tau)_{p'}, (x_1)_p^{\mathrm{N}} \mid \tau^\sigma = \tau^p, x_1^3 = 1\rangle. \tag{5.1}$$

Since $p \equiv q+1 \pmod{q^2}$ there is an epimorphism

$$\langle \sigma, (\tau)_{p'} \mid \tau^\sigma = \tau^p \rangle \to M_{q^3}$$

which together with Epimorphism 5.1 shows that $C_p \wr M_{q^3}$ is an epimorphic image of $G_{\mathbb{Q}_p}$. The group G in case 3 can obtained as an epimorphic image of $G_{\mathbb{Q}_p}$ after dividing 5.1 by $\tau = 1$. In case 2, since $q|p-1$, there is an epimorphism

$$\langle \sigma, (\tau)_{p'} \mid \tau^\sigma = \tau^p \rangle \to C_{pq} \times C_q,$$

which together with 5.1 can be used to construct an epimorphism onto G. In particular $C_p \wr H$ is realizable over K_{v_1}, K_{v_2} in all cases. Since M_{q^3}, $C_q \times C_q$ and C_t have Liedahl presentations over K, they are realizable over completions at infinitely many primes of K. As G has the GN-property over K, it follows that G is K-admissible in all cases. $\square$

Remark 5.5. As Case 3 of Example 5.4 shows, the rank of p-Sylow subgroups of K-admissible groups is not bounded as apposed to the case of admissible p-group in which the rank of the group is bounded (see [19, Section 10]).

Remark 5.6. Case 2 in Example 5.4 is an example of a group for which proving M-admissibility requires the use of all steps in the proof of Theorem 1.6.

The following example shows that the assumption that every Sylow subgroup is either normal or metacyclic is essential for Theorem 1.8 even for odd order groups and non-sensitive extensions:

EXAMPLE 5.7. *As in Example 4.11, let $p < q$ be odd primes such that $p^p \equiv 1 \pmod{q}$ and $p \not\equiv 1 \pmod{q}$. Let $G = C_p \wr D$ where*

$$D = \langle a, b \mid a^p = b^q = 1, a^{-1}ba = b^p \rangle.$$

Let d be a non-square integer that is a square mod pq and $K = \mathbb{Q}(\sqrt{d})$. Let v_1, v_2 be the primes of K dividing p. Let M/K be a C_p-extension in which both v_1 and v_2 are inert and M does not contain any p-th and q-th roots of unity. Since both p and q have more than one prime divisor in M, G satisfies Liedahl's condition over M. As M does not have any p-th and q-th roots of unity, by Theorem 2.14, G has the GN-property over M and K. We shall now show G is K-admissible but not M-admissible. Note that the only condition of Theorem 1.8 that fails is that either G has a normal p-Sylow subgroup or a metacyclic one.

Proof. By Example 4.11, G is realizable over $\mathbb{Q}_p$ and hence over K_{v_1}, K_{v_2}. As G has the GN-property over K, G is K-admissible. On the other hand, since $G(p)$ is not metacyclic, a subgroup of G that contains $G(p)$ is realizable only over completions of M at prime divisors of p. Let w_i be the prime dividing v_i in M, $i = 1, 2$. Then w_1, w_2 are the only primes dividing p in M but by Example 4.11 G is not realizable over M_{w_i}, for $i = 1, 2$. In particular G is not M-preadmissible and not M-admissible. $\square$

Appendix

We use the Jannsen-Wingberg presentation of $G_{\mathbb{Q}_3}$ to count the sensitive extensions, as defined in Subsection 4.2, up to isomorphism. There is a single extension in each of cases (1), (2) and (4). In case (3) the extension is unramified, so it suffices to count the ground field k, which we do by degrees over $\mathbb{Q}_3$. In degree 1 there is one case. In degree 2 there are $\left|\mathbb{Q}_3^\times/\mathbb{Q}_3^{\times 2}\right|-1=3$ quadratic extensions.

Since the abelianization of $\overline{G_{\mathbb{Q}_3}(3)}$ is C_3^2, there are $\frac{3^2-1}{2}=4$ Galois cubic extensions of $\mathbb{Q}_3$. For every non-Galois cubic extension k there is a unique S_3-Galois extension of $\mathbb{Q}_3$ (generated over k by the square root of the discriminant). The S_3-Galois extensions of $\mathbb{Q}_3$ are in one-to-one correspondence to the normal subgroups of $G_{\mathbb{Q}_3}$ with quotient S_3. The number of such subgroups is the number of epimorphisms from $G_{\mathbb{Q}_3}$ to S_3, divided by $|\mathrm{Aut}(S_3)|=6$. When counting epimorphisms $\varphi: G_{\mathbb{Q}_3}\to S_3$, we may assume the generators are in S_3, which simplifies the presentation a great deal. Since $p^s=q=3$ and we may assume $g=1$ and $h=-1$, the presentation is

$$G_{\mathbb{Q}_3}=\langle\sigma,(\tau)_{3'},(x_0,x_1)_3^{\mathrm{N}}\mid \tau^\sigma=\tau^3,\, x_0^\sigma=(x_0\tau x_0^{-1}\tau)^{\frac{\pi}{2}}x_1^3[x_1,y_1]\rangle.$$

However, since x_1 is a pro-3 element, we may assume $x_1^3=1$. Since all elements of order 3 in S_3 commute with each other, we may assume $[x_1,y_1]=1$ (see Theorem 2.19 for more details on y_1). Since τ is a pro-$3'$ element, $\varphi(\tau)$ has order at most 2, so $\varphi(x_0\tau x_0^{-1}\tau)$ is a commutator, whose order must divide 3. Exponentiation by $\frac{\pi}{2}$ squares such elements. So every epimorphism to S_3 splits through

$$\langle\sigma,(\tau)_{3'},(x_0,x_1)_3^{\mathrm{N}}\mid \tau^2=x_0^3=x_1^3=1,\,\tau^\sigma=\tau^3,\,x_0^\sigma=(x_0\tau x_0^{-1}\tau)^2\rangle,$$

and we count epimorphisms from this group. If $\varphi(\tau)=1$ then $\varphi(x_0)=1$, so $\varphi(x_1)$ is a non-trivial element of order 3 and $\varphi(\sigma)\notin\langle\varphi(x_1)\rangle$. There are 6 such epimorphisms. So assume $\varphi(\tau)$ is a non-trivial involution. The relations give $[\varphi(\sigma),\varphi(\tau)]=1$, so $\varphi(\sigma)$ is either 1 or $\varphi(\tau)$. Moreover, it turns out that $\varphi(x_0\tau x_0^{-1}\tau)^2=\varphi(x_0)$ whenever $\varphi(x_0)$ has order dividing 3. For each possible value of $\varphi(\tau)$ we get 8 epimorphisms with $\varphi(\sigma)=1$ and 2 more with $\varphi(\sigma)=\varphi(\tau)$. There are 3 involutions, providing us with $6+3\cdot 10=36$ epimorphisms all together. Dividing by the number of automorphisms, we have 6 Galois extensions of $\mathbb{Q}_3$ with Galois group S_3.

Each Galois extension of this type contains 3 non-Galois cubic extensions of $\mathbb{Q}_3$. The S_3-extension is determined by the cubic extension, being its Galois closure. So we have $3\cdot 6=18$ non-Galois cubic subfields of a fixed algebraic closure $\bar{\mathbb{Q}}_3$, consisting of 6 isomorphism classes. Summing up, there are $1+1+(1+3+(4+6))+1=17$ sensitive field extensions, up to isomorphism.

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