Transportation-Cost Inequalities on Path Space over Manifolds with Boundary[1]

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Received: May 9, 2011
Revised: October 11, 2011

Communicated by Friedrich Götze

Abstract. Let $L = \Delta + Z$ for a C^1 vector field Z on a complete Riemannian manifold possibly with a boundary. A number of transportation-cost inequalities on the path space for the (reflecting) L-diffusion process are proved to be equivalent to the curvature condition $\mathrm{Ric} - \nabla Z \geq -K$ and the convexity of the boundary (if exists). These inequalities are new even for manifolds without boundary, and are partly extended to non-convex manifolds by using a conformal change of metric which makes the boundary from non-convex to convex.

2010 Mathematics Subject Classification: 60J60, 58G60.
Keywords and Phrases: Transportation-cost inequality, curvature, second fundamental form, path space.

1 Introduction

In 1996 Talagrand [13] found that the L^2-Wasserstein distance to the standard Guassian measure can be dominated by the square root of twice relative entropy. This inequality is called (Talagrand) transportation-cost inequality, and has been extended to distributions on finite- and infinite-dimensional spaces. In particular, this inequality was established on the path space of diffusion processes with respect to several different distances (i.e. cost functions): see e.g. [7] for the study on the Wiener space with the Cameron-Martin distance, [17, 5] on the path space of diffusions with the L^2-distance, [18] on the Riemannian path space with intrinsic distance induced by the Malliavin gradient

[1]Supported in Part By NNSFC(11131003), SRFDP and the Fundamental Research Funds for the Central Universities.

operator, and [6, 27] on the path space of diffusions with the uniform distance. The main purpose of this paper is to investigate the Talagrand inequality on the path space of reflecting diffusion process, for which both the curvature and the second fundamental form of the boundary will take important roles.

Let M be a connected complete Riemannian manifold possibly with a boundary ∂M. Let $L = \Delta + Z$ for a C^1 vector field Z on M. Let X_t be the (reflecting if $\partial M \neq \emptyset$) diffusion process generated by L with initial distribution $\mu \in \mathscr{P}(M)$, where $\mathscr{P}(M)$ is the set of all probability measures on M. Assume that X_t is non-explosive, which is the case if ∂M is convex and the curvature condition

$$\mathrm{Ric} - \nabla Z \geq -K \tag{1.1}$$

holds for some constant $K \in \mathbb{R}$. In this case, for any $T > 0$, the distribution Π_μ^T of $X_{[0,T]} := \{X_t : \ t \in [0,T]\}$ is a probability measure on the (free) path space

$$M^T := C([0,T]; M).$$

When $\mu = \delta_o$, the Dirac measure at point $o \in M$, we simply denote $\Pi_{\delta_o}^T = \Pi_o^T$. For any nonnegative measurable function F on M_T such that $\Pi_\mu^T(F) = 1$, one has

$$\mu_F^T(\mathrm{d}x) := \Pi_x^T(F)\mu(\mathrm{d}x) \in \mathscr{P}(M). \tag{1.2}$$

Let ρ be the Riemannian distance on M; i.e. for $x, y \in M, \rho(x,y)$ is the length of the shortest curve on M linking x and y. Then M^T is a Polish space under the uniform distance

$$\rho_\infty(\gamma, \eta) = \sup_{t\in[0,T]} \rho(\gamma_t, \eta_t), \quad \gamma, \eta \in M^T.$$

Let W_{2,ρ_∞} be the L^2-Wasserstein distance (or L^2-transportation cost) induced by ρ_∞. In general, for any $p \geq 1$ and for two probability measures Π_1, Π_2 on M^T,

$$W_{p,\rho_\infty}(\Pi_1, \Pi_2) := \inf_{\pi\in\mathscr{C}(\Pi_1,\Pi_2)} \bigg\{ \int_{M_T\times M_T} \rho_\infty(\gamma,\eta)^p \pi(\mathrm{d}\gamma, \mathrm{d}\eta) \bigg\}^{1/p}$$

is the L^p-Warsserstein distance (or L^p-transportation cost) of Π_1 and Π_2 induced by the uniform norm, where $\mathscr{C}(\Pi_1, \Pi_2)$ is the set of all couplings for Π_1 and Π_2.

Before moving on, let us recall the Talagrand transportation-cost inequality established in [6] on the path space over Riemannian manifolds without boundary. Let $\partial M = \emptyset$ and $\rho_o = \rho(o, \cdot)$. If

$$|Z| \leq \psi \circ \rho_o \tag{1.3}$$

holds for some positive function ψ such that $\int_0^\infty \frac{1}{\psi(s)}\mathrm{d}s = \infty$, then (see [6, Theorem 1.1])

$$W_{2,\rho_\infty}(F\Pi_o^T, \Pi_o^T)^2 \le \frac{2}{K}(\mathrm{e}^{2KT}-1)\Pi_o^T(F\log F), \quad F \ge 0, \Pi_o^T(F) = 1. \quad (1.4)$$

According to [12, 4, 18], the log-Sobolev inequality for a smooth elliptic diffusion implies the Talagrand transportation-cost inequality with the intrinsic distance. So, (1.4) was proved in [6] by using a known damped log-Sobolev inequality on the path space and finite-dimensional approximations. To ensure the smoothness of the approximating diffusions, one needs the boundedness of curvature. To get rid of this condition, a sequence of new metric approximating the original one were constructed in [6], which satisfy (1.1) and have bounded curvatures. In this way (1.4) was established without using curvature upper bounds. But to realize this approximation argument, the technical condition (1.3) with $\int_0^\infty \frac{1}{\psi(s)}\mathrm{d}s = \infty$ was adopted.

In this paper we adopt a different argument developed in [27] for diffusions on $\mathbb{R}^d$ by using the martingale representation theorem and Girsanov transformations, so that this technical condition was avoided. Furthermore, we present a number of cost inequalities which are equivalent to the convexity of ∂M (if exists) and the curvature condition (1.1).

When $\partial M \neq \emptyset$, let N be the inward unit normal vector field of ∂M. Then the second fundamental form of ∂M is defined by

$$\mathbb{I}(U,V) = -\langle \nabla_U N, V\rangle, \quad U,V \in T\partial M,$$

where $T\partial M$ is the tangent space of ∂M. If $\mathbb{I} \ge 0$, i.e. $\mathbb{I}(U,U) \ge 0$ for all $U \in T\partial M$, we call M (or ∂M) convex.

Theorem 1.1. *Let $P_T(o,\cdot)$ be the distribution of X_T with $X_0 = o$, and let P_T be the corresponding semigroup. The following statements are equivalent to each other:*

(1) *∂M is either convex or empty, and (1.1) holds.*

(2) *For any $T > 0, \mu \in \mathscr{P}(M)$ and nonnegative F with $\Pi_\mu^T(F) = 1$,*

$$W_{2,\rho_\infty}(F\Pi_\mu^T, \Pi_{\mu_F^T}^T)^2 \le \frac{2}{K}(\mathrm{e}^{2KT}-1)\Pi_\mu^T(F\log F)$$

holds, where $\mu_F^T \in \mathscr{P}(M)$ is fixed by (1.2).

(3) *(1.4) holds for any $o \in M$ and $T > 0$.*

(4) *For any $o \in M$ and $T > 0$,*

$$W_{2,\rho}\big(P_T(o,\cdot), fP_T(o,\cdot)\big)^2 \le \frac{2}{K}(\mathrm{e}^{2KT}-1)P_T(f\log f)(o),$$
$$f \ge 0, P_T f(o) = 1.$$

(5) *For any $T > 0$, $\mu, \nu \in \mathscr{P}(M)$, and $p \geq 1$,*

$$W_{p,\rho_\infty}(\Pi_\mu^T, \Pi_\nu^T) \leq \mathrm{e}^{KT} W_{p,\rho}(\mu, \nu),$$

where $W_{p,\rho}$ is the L^p-Wasserstein distance for probability measures on M induced by ρ.

(6) *For any $x, y \in M$ and $T > 0$,*

$$W_{2,\rho}\big(P_T(x, \cdot), P_T(y, \cdot)\big) \leq \mathrm{e}^{KT} \rho(x, y).$$

(7) *For any $T > 0$, $\mu \in \mathscr{P}(M)$, and $F \geq 0$ with $\Pi_\mu^T(F) = 1$,*

$$W_{2,\rho_\infty}(F\Pi_\mu^T, \Pi_\mu^T) \leq \Big\{\frac{2}{K}(\mathrm{e}^{2KT} - 1)\Pi_\mu^T(F \log F)\Big\}^{1/2} + \mathrm{e}^{KT} W_{2,\rho}(\mu_F^T, \mu).$$

(8) *For any $\mu \in \mathscr{P}(M)$ and $C \geq 0$ such that*

$$W_{2,\rho}(f\mu, \mu)^2 \leq C\mu(f \log f), \quad f \geq 0, \mu(f) = 1,$$

there holds

$$W_{2,\rho_\infty}(F\Pi_\mu^T, \Pi_\mu^T)^2 \leq \bigg(\sqrt{\frac{2}{K}(\mathrm{e}^{2KT} - 1)} + \mathrm{e}^{KT}\sqrt{C}\bigg)^2 \Pi_\mu^T(F \log F),$$
$$F \geq 0, \Pi_\mu^T(F) = 1.$$

When ∂M is empty or convex, there exist many equivalent semigroup inequalities for the curvature condition (1.1): see e.g. [3, 10] for equivalent statements on gradient estimates, log-Sobolev/Poincaré inequalities, and isoperimetric inequality; [19, 22, 23] for equivalent Harnack type inequalities; and [11] for equivalent inequalities on Wasserstein distances. The corresponding results have been partly extended in [23, 24] to the non-convex case. Theorem 1.1 provides seven equivalent inequalities for the convexity of ∂M (if exists) and the curvature condition (1.1), which are new even for manifolds without boundary. We would like to mention that the log-Sobolev inequality has been established in [26] on the path space over manifolds with convex boundary, and in a forthcoming paper we will extend the results on intrinsic ultracontractivity derived in [25] to non-compact manifolds with boundary.
To prove Theorem 1.1, we shall use a formula of the second fundamental form established in [22] for compact manifolds with boundary. Since in this paper the manifold is allowed to be non-compact, we shall reprove this formula in Section 2 by using the reflecting diffusion process up to the exit time of a compact domain. This formula implies the equivalence of Theorem 1.1(1) and the semigroup log-Sobolev/Poincaré inequalities (see Theorem 2.4 below). In Section

3 we prove Theorem 1.1 by using results in Section 2, the martingale representation and Girsanov transformation for (reflecting) diffusions on (convex) manifolds.
To establish transportation-cost inequalities on the path space for non-convex manifolds, we shall adopt a conformal change of metric $\langle\cdot,\cdot\rangle' = f^{-2}\langle\cdot,\cdot\rangle$ such that ∂M is convex under the new metric (see [21, Lemma 2.1]). Let Δ' be the Laplacian induced by the new metric, we have (see [21, Lemma 2.2])

$$L = f^{-2}\Big\{\Delta' + f^2 Z + \frac{d-2}{2}\nabla f^2\Big\}. \tag{1.5}$$

According to this fact, we will modify our arguments in Section 4 to study the reflecting diffusion process generated by $L := \psi^2(\Delta + Z)$ for a smooth function ψ on a convex manifold, then extend Theorem 1.1 in Section 5 to the non-convex setting.

2 Formulae for the second fundamental form and applications

When M is compact, the following formula on ∂M has been found in [22]:

$$\lim_{t\to 0}\frac{|\nabla f|^2}{\sqrt{t}}\log\frac{|\nabla P_t f|}{(P_t|\nabla f|^p)^{1/p}} = -\frac{2}{\sqrt{\pi}}\mathbb{I}(\nabla f,\nabla f),\quad p\geq 1, \tag{2.1}$$

where f is a smooth function satisfying the Neumann boundary condition. When M is non-compact, some technical problems appear in the original proof when e.g. a dominated convergence is used. To fix these problems, we shall stop the process in a compact domain, so that we shall first study the behavior of hitting times.
Recall that the reflecting L-diffusion process can be constructed by solving the SDE

$$\mathrm{d}X_t = \sqrt{2}\,\Phi_t\circ \mathrm{d}B_t + Z(X_t)\mathrm{d}t + N(X_t)\mathrm{d}l_t, \tag{2.2}$$

where Φ_t is the horizontal lift of X_t onto the frame bundle $O(M)$, B_t is the d-dimensional Brownian motion.
By the Itô formula, for any $f\in C^2(M)$ we have

$$\mathrm{d}f(X_t) = \sqrt{2}\langle\nabla f(X_t),\Phi_t\circ \mathrm{d}B_t\rangle + Lf(X_t)\mathrm{d}t + Nf(X_t)\mathrm{d}l_t, \tag{2.3}$$

where $Nf = \langle N,\nabla f\rangle$. For any $R>0$, let

$$\tau_R = \inf\{t\geq 0:\ \rho(X_0,X_t)\geq R\}.$$

Proposition 2.1. *Let $R>0$ and $X_0 = o\in M$ be fixed. Then there exist two constants $c_1, c_2>0$ such that*

$$\mathbb{P}(\tau_R\leq t)\leq c_1\mathrm{e}^{-c_2/t},\quad t>0.$$

Proof. This result is well known on manifolds without boundary (cf. [2, Lemma 2.3]), and the proof works also when ∂M is convex. As in the present case the boundary is not necessarily convex, we shall follow [21] to make the boundary convex under a conformal change of metric. Since

$$\mathbf{B}_R := \{x \in M : \ \rho(o,x) \le R\}$$

is compact, there exists a constant $\sigma > 0$ such that $\mathbb{I} \ge -\sigma$ holds on $\partial M \cap \mathbf{B}_R$. Let $f \ge 1$ be smooth such that

$$N \log f \ge \sigma \quad \text{on } \partial M \cap \mathbf{B}_R. \tag{2.4}$$

Such a function can be constructed by using the distance function ρ_∂ to the boundary ∂M. Since ρ_∂ is smooth in a neighborhood of the boundary, there exists a constant $r_0 > 0$ such that ρ_∂ is smooth on $\{x \in \mathbf{B}_{2R} : \ \rho_\partial(x) \le r_0\}$. Let $h \in C^\infty([0,\infty))$ such that $h' \ge 0, h(0) = 1, h'(0) = \sigma$ and $h'(r) = 0$ for $r \ge r_0$. Then $h \circ \rho_\partial$ is smooth on $\mathbf{B}_{2R}$ and $N \log h \circ \rho_\partial|_{\partial M \cap \mathbf{B}_{2R}} = \sigma$. Thus, it suffices to take smooth $f \ge 1$ such that $f = h \circ \rho_\partial$ on $\mathbf{B}_R$.
By [21, Lemma 2.1] and (2.4), ∂M is convex in $\mathbf{B}_R$ under the new metric

$$\langle \cdot, \cdot \rangle' := f^{-2} \langle \cdot, \cdot \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the original metric. Let Δ' be the Laplacian induced by the new metric. We have (see [21, Lemma 2.2])

$$L = f^{-2}(\Delta' + Z')$$

for some C^1-vector field Z'. Let $\tilde\rho_o$ be the Riemannian distance to o induced by the new metric. By the Laplacian comparison theorem,

$$L\tilde\rho_o^2 \le c \quad \text{on } \mathbf{B}_R \tag{2.5}$$

holds for some constant $c > 0$ outside the cut-locus induced by $\langle \cdot, \cdot \rangle'$. Since ∂M is convex on $\mathbf{B}_R$ and N is still the inward normal vector under the new metric, we have

$$N\tilde\rho_o \le 0 \quad \text{on } \partial M \cap \mathbf{B}_R.$$

Therefore, by using Kendall's Itô formula for the distance (cf. [9] for $f = 1$), (2.5) implies

$$\mathrm{d}\tilde\rho_o^2(X_t) \le 2\sqrt{2}\, f^{-2}(X_t)\tilde\rho_o(X_t)\mathrm{d}b_t + c\mathrm{d}t, \quad t \le \tau_R,$$

or equivalently,

$$t \mapsto 2\sqrt{2} \int_0^{t\wedge\tau_R} f^{-2}(X_s)\tilde\rho_o(X_s)\mathrm{d}b_s + c(t \wedge \tau_R) - \tilde\rho_0^2(X_{t\wedge\tau_R})$$

is an increasing process, where b_t is some one-dimensional Brownian motion. Since $f^{-2} \le 1$, this implies that for any $\delta > 0$, the process

$$Z_s := \exp\left[\frac{\delta}{t}\tilde{\rho}_o^2(X_s) - \frac{\delta cs}{t} - \frac{4\delta^2}{t^2}\int_0^s \tilde{\rho}_o^2(X_u)\mathrm{d}u\right], \quad s \le \tau_R$$

is a super martingale. Therefore, letting $C > 1$ be a constant such that $f \le C$ on $\mathbf{B}_R$ and thus, $\rho_o \ge \tilde{\rho}_o \ge C^{-1}\rho_o$ holds on $\mathbf{B}_R$, we obtain

$$\begin{aligned}\mathbb{P}(\tau_R \le t) &= \mathbb{P}\Big(\max_{s\in[0,t]} \rho_o(X_{s\wedge\tau_R}) \ge R\Big) \le \mathbb{P}\Big(R \ge \max_{s\in[0,t]} \tilde{\rho}_o(X_{s\wedge\tau_R}) \ge \frac{R}{C}\Big)\\ &\le \mathbb{P}\Big(\max_{s\in[0,t]} Z_{s\wedge\tau_R} \ge \exp\Big[\frac{\delta R^2}{tC^2} - \delta c - \frac{4\delta^2 R^2}{t}\Big]\Big)\\ &\le \exp\Big[c\delta - \frac{R^2}{tC^2}(\delta - 4C^2\delta^2)\Big], \quad \delta > 0.\end{aligned}$$

The proof is then completed by taking e.g. $\delta = 1/(8C^2)$. □

Proposition 2.2. *Let $X_0 = o \in \partial M$. Then for any $R > 0$,*

$$\limsup_{t\to 0} \frac{1}{t}\big|\mathbb{E}l_{t\wedge\tau_R} - 2\sqrt{t/\pi}\big| < \infty.$$

Proof. Repeating the proof of [22, Lemma 2.2] by using $t \wedge \tau_R$ in place of t, we obtain

$$\mathbb{E}l_{t\wedge\tau_R}^2 \le ct, \quad t \in [0,1] \tag{2.6}$$

for some constant $c > 0$. Let $r_0 > 0$ be such that ρ_∂ is smooth on $\{\rho_\partial \le r_0\} \cap \mathbf{B}_R$. Let

$$\tau = \inf\{t \ge 0 : \ \rho_\partial(X_t) \ge r_0\}.$$

By the Itô formula we have

$$\mathrm{d}\rho_\partial(X_t) = \sqrt{2}\,\mathrm{d}b_t + L\rho_\partial(X_t)\mathrm{d}t + \mathrm{d}l_t, \quad t \le \tau \wedge \tau_R, \tag{2.7}$$

where, as before, b_t is some one-dimensional Brownian motion. By the proof of [22, Theorem 2.1] using $\tau \wedge \tau_R$ in place of τ, we have, instead of (2.4) in [22],

$$\mathbb{E}\big(\rho_\partial(X_{t\wedge\tau\wedge\tau_R}) - \sqrt{2}\,|\tilde{b}_{t\wedge\tau\wedge\tau_R}|\big)^2 \le c_1 t^2, \quad t \in [0,1] \tag{2.8}$$

for some constant $c_1 > 0$, where $\tilde{b}_t$ is some one-dimensional Brownian motion. Due to (2.7),

$$\big|\mathbb{E}l_{t\wedge\tau\wedge\tau_R} - \mathbb{E}\rho_\partial(X_{t\wedge\tau\wedge\tau_R})\big| \le c_2 t$$

holds for some constant $c_2 > 0$. Combining this with (2.8) we arrive at

$$\big|\mathbb{E}l_{t\wedge\tau\wedge\tau_R} - \sqrt{2}\,\mathbb{E}|\tilde{b}_{t\wedge\tau\wedge\tau_R}|\big| \le c_3 t, \quad t \in [0,1]$$

for some constant $c_3 > 0$. Since $\mathbb{E}|\tilde{b}_t| = \sqrt{2t/\pi}$ and $\mathbb{E}|\tilde{b}_t|^2 = t$, this and (2.6) imply

$$\begin{aligned}
&\Big|\mathbb{E}l_{t\wedge\tau_R} - \frac{2\sqrt{t}}{\sqrt{\pi}}\Big| = \big|\mathbb{E}l_{t\wedge\tau_R} - \sqrt{2}\,\mathbb{E}|\tilde{b}_t|\big| \\
&\le c_3 t + \mathbb{E}1_{\{t\ge\tau\wedge\tau_R\}}(l_{t\wedge\tau_R} + \sqrt{2}\,|\tilde{b}_t|) \\
&\le c_3 t + c_4\sqrt{t\mathbb{P}(t\ge\tau\wedge\tau_R)}, \quad t\in[0,1].
\end{aligned} \tag{2.9}$$

Moreover, noting that

$$\mathbb{P}(\tau\wedge\tau_R \le t, \tau_R > \tau) \le \mathbb{P}\Big(\max_{s\in[0,t]}\rho_\partial(X_{s\wedge\tau\wedge\tau_R}) \ge r_0\Big),$$

by using $\tau\wedge\tau_R$ to replace τ in the proof of [22, Proposition A.2], we conclude that

$$\mathbb{P}(\tau\wedge\tau_R \le t,\ \tau_R > \tau) \le c_5\exp[-r_0^2/(16t)], \quad t>0$$

holds for some constant $c_5 > 0$. Combining this with Proposition 2.1, we obtain

$$\mathbb{P}(t\ge\tau\wedge\tau_R) \le c_6 \mathrm{e}^{-c_7/t}, \quad t>0$$

for some constants $c_6, c_7 > 0$. Therefore, the proof is completed by (2.9). □

THEOREM 2.3. *Let $f\in C^\infty(M)$ with $Nf|_{\partial M} = 0$.*
(1) *For any $p\ge 1$ and $R>0$,*

$$\lim_{t\to 0}\frac{|\nabla f|^2}{\sqrt{t}}\log\frac{(\mathbb{E}|\nabla f|^p(X_{t\wedge\tau_R})|)^{1/p}}{|\nabla f|} = \frac{2}{\sqrt{\pi}}\mathbb{I}(\nabla f,\nabla f) \tag{2.10}$$

holds at points on ∂M such that $|\nabla f|>0$.
(2) *Assume that for any $g\in C_0^1(M)$ the function $|\nabla P.g|$ is bounded on $[0,1]\times M$. If moreover f has a compact support, then (2.1) holds points on ∂M such that $|\nabla f|>0$.*

Proof. (2.10) follows immediately from the proof of [22, Theorem 1.2] by using Proposition 2.2 in place of [22, Theorem 2.1], and using $t\wedge\tau_R$ in place of t. Next, let $f\in C_0^\infty(M)$. By the assumption of (2) and that $Lf\in C_0^1(M)$, $|\nabla P.Lf|$ is bounded on $[0,1]\times M$. So, the proof of [22, (3.1)] implies that

$$\lim_{t\to 0}\frac{|\nabla f|^2}{\sqrt{t}}\log\frac{|\nabla P_t f|}{(P_t|\nabla f|^p)^{1/p}} = -\lim_{t\to 0}\frac{|\nabla f|^2}{\sqrt{t}}\log\frac{(P_t|\nabla f|^p)^{1/p}}{|\nabla f|}. \tag{2.11}$$

Since by Proposition 2.1, there exist two constant $c_1, c_2 > 0$ such that

$$\big|P_t|\nabla f|^p - \mathbb{E}|\nabla f|^p(X_{t\wedge\tau_R})\big| \le \|\nabla f\|_\infty^p\mathbb{P}(t>\tau_R) \le c_1\mathrm{e}^{-c_2/t}, \quad t>0,$$

we conclude that (2.1) follows from (2.11) and (2.10). □

As an application of (2.10), the following result provides equivalent semigroup log-Sobolev/Poincaré inequalities for Theorem 1.1(1).

Theorem 2.4. *Each of the following statements is equivalent to Theorem 1.1*(1)*:*

(9) *For any $T>0$ and $f \in C_b(M)$,*

$$P_T f^2 \log f^2 \le (P_T f^2) \log P_T f^2 + \frac{\mathrm{e}^{2KT}-1}{2K} P_T |\nabla f|^2.$$

(10) *For any $T>0$ and $f \in C_b(M)$,*

$$P_T f^2 \le (P_T f)^2 + \frac{\mathrm{e}^{2KT}-1}{K} P_T |\nabla f|^2.$$

Proof. According to e.g. [16, Lemma 3.1], which holds also for the non-symmetric case, Theorem 1.1(1) implies the semigroup log-Sobolev inequality (9). It is well known that the log-Sobolev inequality implies the Poincaré inequality. So, (10) follows from (9). Hence, it remains to show that (10) implies Theorem 1.1(1). Below we shall prove the convexity of ∂M and the curvature condition (1.1) respectively.

(a) Let $\partial M \neq \emptyset$. For any $o \in \partial M$ and non-trivial $U \in T_o \partial M$, we aim to show that $\mathbb{I}(U,U) \ge 0$. Let $f \in C_b^\infty(M)$ such that $Nf|_{\partial M} = 0$ and $\nabla f(o) = U$. Let $X_0 = o$ and

$$\tau_1 = \inf\{t \ge 0 : \rho(o, X_t) \ge 1\}.$$

Since f and f^2 satisfies the Neumann boundary condition, we have

$$\begin{aligned} &\mathbb{E} f(X_{t\wedge\tau_1}) = f(o) + \mathbb{E}\int_0^{t\wedge\tau_1} Lf(X_s)\mathrm{d}s,\\ &\mathbb{E} f^2(X_{t\wedge\tau_1}) = f^2(o) + 2\mathbb{E}\int_0^{t\wedge\tau_1} (fLf)(X_s)\mathrm{d}s + 2\mathbb{E}\int_0^{t\wedge\tau_1} |\nabla f|^2(X_s)\mathrm{d}s. \end{aligned}$$

So,

$$\begin{aligned} \mathbb{E} f^2(X_{t\wedge\tau_1}) - \{\mathbb{E} f(X_{t\wedge\tau_1})\}^2 &= 2\mathbb{E}\int_0^{t\wedge\tau_1} \{f(X_s) - f(X_0)\} Lf(X_s)\mathrm{d}s \\ &- \left(\mathbb{E}\int_0^{t\wedge\tau_1} Lf(X_s)\mathrm{d}s\right)^2 + 2\mathbb{E}\int_0^{t\wedge\tau_1} |\nabla f|^2(X_s)\mathrm{d}s. \end{aligned} \tag{2.12}$$

Since Lf is bounded on $\mathbf{B}_1 := \{x : \rho(o,x) \le 1\}$, we have

$$\left(\mathbb{E}\int_0^{t\wedge\tau_1} Lf(X_s)\mathrm{d}s\right)^2 \le ct^2 \tag{2.13}$$

for some $c>0$. Moreover, due to Proposition 2.1,

$$\mathbb{P}(\tau_1\le t)\le c_1\mathrm{e}^{-c_2/t},\quad t>0 \tag{2.14}$$

holds for some constants $c_1,c_2>0$. Thus,

$$\begin{aligned}&\big|P_tf^2(o)-(P_tf)^2(o)-\big(\mathbb{E}f^2(X_{t\wedge\tau_1})-\{\mathbb{E}f(X_{t\wedge\tau_1})\}^2\big)\big|=\mathrm{o}(t^2),\\ &\mathbb{E}\int_0^{t\wedge\tau_1}|\nabla f|^2(X_s)\mathrm{d}s=t|\nabla f(o)|^2+\int_0^t\mathbb{E}\big\{|\nabla f|^2(X_{s\wedge\tau_1})-|\nabla f(o)|^2\big\}\mathrm{d}s+\mathrm{o}(t^2),\end{aligned} \tag{2.15}$$

where and in what follows, $\mathrm{o}(s)$ stands for a function of $s>0$ such that $\lim_{s\to0}\mathrm{o}(s)/s=0$.
Similarly, applying the Itô formula to $\{f(X_s)-f(o)\}Lf(X_s)$, we obtain (note that $Nf|_{\partial M}=0$)

$$\begin{aligned}&\mathbb{E}\int_0^{t\wedge\tau_1}\{f(X_s)-f(o)\}Lf(X_s)\mathrm{d}s\\ &=\mathrm{o}(t^2)+\int_0^t\mathbb{E}\big[(f(X_{s\wedge\tau_1})-f(o))Lf(X_{s\wedge\tau_1})\big]\mathrm{d}s\\ &=\mathrm{o}(t^2)+\mathbb{E}\int_0^t\mathrm{d}s\int_0^{s\wedge\tau_1}L\{(f-f(o))Lf\}(X_r)\mathrm{d}r\\ &\quad+\mathbb{E}\int_0^t\mathrm{d}s\int_0^{s\wedge\tau_1}\{(f-f(o))NLf\}(X_r)\mathrm{d}l_r.\end{aligned} \tag{2.16}$$

Noting that

$$f(X_r)-f(o)=\sqrt{2}\int_0^r\langle\nabla f(X_u),\Phi_u\circ\mathrm{d}B_u\rangle+\int_0^rLf(X_u)\mathrm{d}u,\quad u\le\tau_1,$$

and that

$$\mathbb{E}\sup_{r\in[0,t]}\bigg(\int_0^r\langle\nabla f(X_u),\Phi_u\circ\mathrm{d}B_u\rangle\bigg)^2\le c_2t,\quad t\in[0,1]$$

holds for some constant $c_2>0$, we obtain from (2.16) and (2.6) that

$$\bigg|\mathbb{E}\int_0^{t\wedge\tau_1}\{f(X_s)-f(o)\}Lf(X_s)\mathrm{d}s\bigg|\le c_3t^2,\quad t\in[0,1] \tag{2.17}$$

holds for some constant $c_3>0$. Finally, by Theorem 2.3(1), we have

$$\mathbb{E}|\nabla f|^2(X_{t\wedge\tau_1})=|\nabla f|^2(o)+\frac{4\sqrt{t}}{\sqrt{\pi}}\mathbb{I}(\nabla f,\nabla f)(o)+\mathrm{o}(t^{1/2}) \tag{2.18}$$

for small $t > 0$. Combining this with (2.12), (2.13), (2.15) and (2.17), and noting that $U = \nabla f(o)$, we conclude that

$$P_t f^2(o) - (P_t f)^2(o) = 2t|\nabla f(o)|^2 + \frac{16t^{3/2}}{3\sqrt{\pi}} \mathbb{I}(U,U) + \mathrm{o}(t^{3/2}). \tag{2.19}$$

Finally, (2.18) and (2.14) imply that

$$\frac{\mathrm{e}^{2Kt}-1}{K} P_t|\nabla f|^2(o) = 2t|\nabla f(o)|^2 + \frac{8t^{3/2}}{\sqrt{\pi}} \mathbb{I}(U,U) + \mathrm{o}(t^{3/2}).$$

Since $\frac{16}{3} < 8$, combining this with (10) and (2.19) we conclude that $\mathbb{I}(U,U) \geq 0$.
(b) Let $X_0 = o \in M \setminus \partial M$, we aim to show that $\mathrm{Ric} - \nabla Z \geq -K$ holds on T_oM. Let $R > 0$ such that $\mathbf{B}_R \cap \partial M = \emptyset$. Since l_t increases only when $X_t \in \partial M$, $l_t = 0$ for $t \leq \tau_R$. Hence, due to Proposition 2.1, for any $f \in C_b^\infty(M)$,

$$\begin{aligned} P_t f^2(o) - (P_t f)^2(o) &= \mathrm{o}(t^2) + \mathbb{E} f^2(X_{t\wedge\tau_R}) - \big(\mathbb{E} f(X_{t\wedge\tau_R})\big)^2 \\ &= \mathrm{o}(t^2) + \int_0^t \big\{\mathbb{E} L f^2(X_{s\wedge\tau_R}) - 2f(o)\mathbb{E} Lf(X_{s\wedge\tau_R})\big\}\mathrm{d}s - \bigg(\int_0^t \mathbb{E} Lf(X_{s\wedge\tau_R})\mathrm{d}s\bigg)^2. \end{aligned} \tag{2.20}$$

By the continuity of $s \mapsto Lf(X_{s\wedge\tau_R})$, we have

$$\bigg(\int_0^t \mathbb{E} Lf(X_{s\wedge\tau_R})\mathrm{d}s\bigg)^2 = (Lf)^2(o)t^2 + \mathrm{o}(t^2). \tag{2.21}$$

Similarly, it is easy to see that

$$\begin{aligned} &\mathbb{E} L f^2(X_{s\wedge\tau_R}) - 2f(o)\mathbb{E} Lf(X_{s\wedge\tau_R}) \\ &= Lf^2(o) - 2f(o)Lf(o) + s\big\{LLf^2 - 2fLLf\big\}(o) + \mathrm{o}(s) \\ &= 2|\nabla f|^2(o) + 2s\{L|\nabla f|^2(o) + (Lf)^2(o) + 2\langle \nabla f, \nabla Lf\rangle(o)\} + \mathrm{o}(s). \end{aligned}$$

Combining this with (2.20) and (2.21) we obtain

$$P_t f^2(o) - (P_t f)^2(o) = 2t|\nabla f|^2(o) + t^2(L|\nabla f|^2 + 2\langle\nabla f, \nabla Lf\rangle\}(o) + \mathrm{o}(t^2). \tag{2.22}$$

Finally, by Proposition 2.1 and noting that $l_s = 0$ for $s \leq \tau_R$, we have

$$P_t|\nabla f|^2(o) = \mathrm{o}(t^2) + \mathbb{E}|\nabla f|^2(X_{t\wedge\tau_R}) = |\nabla f|^2(o) + tL|\nabla f|^2(o) + \mathrm{o}(t).$$

Combining this with (10) and (2.22), we conclude that

$$\frac{1}{2}L|\nabla f|^2(o) - \langle\nabla f, \nabla Lf\rangle(o) \geq -K|\nabla f|(o), \quad f \in C_b^\infty(M).$$

This completes the proof by the Bochner-Weitzenböck formula. □

3 PROOF OF THEOREM 1.1

By taking $\mu = \delta_o$, we have $\mu_F^T = \Pi_o^T(F)\delta_o = \delta_o$. So, (3) follows from each of (2), (7) and (8). Next, (4) follows from (3) by taking $F(X_{[0,T]}) = f(X_T)$, and (5) implies (6) by taking $p = 2$ and $\mu = \delta_x, \nu = \delta_y$. Moreover, it is clear that (8) follows from (7) while (7) is implied by (2) and (5). So, it suffices to prove that (1) $\Rightarrow$ (3) $\Rightarrow$ (2), (4) $\Rightarrow$ (1) $\Rightarrow$ (6) $\Rightarrow$ (5) and (6) $\Rightarrow$ (1), where " $\Rightarrow$" stands for "implies".

(a) (1) $\Rightarrow$ (3). We shall only consider the case where ∂M is non-empty and convex. For the case without boundary, the following argument works well by taking $l_t = 0$ and $N = 0$. The idea of the proof comes from [27], where elliptic diffusions on $\mathbb{R}^d$ were concerned. Let B_t be the d-dimensional Brownian motion on the naturally filtered probability space $(\Omega, \mathscr{F}_t, \mathbb{P})$. Let $\{X_t : \ t \geq 0\}$ solve (2.2) with $X_0 = o$.

Next, let F be a positive bounded measurable function on M^T such that $\inf F > 0$ and $\Pi_o^T(F) = 1$. Then

$$m_t := \mathbb{E}_{\mathbb{P}}(F(X_{[0,T]})|\mathscr{F}_t) \ \text{ and } \ L_t := \int_0^t \frac{\mathrm{d}m_s}{m_s}, \quad t \in [0,T]$$

are square-integrable $\mathscr{F}_t$-martingales under $\mathbb{P}$, where $\mathbb{E}_{\mathbb{P}}$ is the expectation taken for the probability measure $\mathbb{P}$. Obviously, we have

$$m_t = \mathrm{e}^{L_t - \frac{1}{2}\langle L\rangle_t}, \quad t \in [0,T]. \tag{3.1}$$

Since $\mathscr{F}_t$ is the natural filtration of B_t, by the martingale representation theorem (cf. [8, Theorem 6.6]), there exists a unique $\mathscr{F}_t$-predictable process β_t on $\mathbb{R}^d$ such that

$$L_t = \int_0^t \langle \beta_s, \mathrm{d}B_s\rangle, \quad t \in [0,T]. \tag{3.2}$$

Let $\mathrm{d}\mathbb{Q} = F(X_{[0,T]})\mathrm{d}\mathbb{P}$. Since $\mathbb{E}_{\mathbb{P}} F(X_{[0,T]}) = \Pi_\mu^T(F) = 1$, $\mathbb{Q}$ is a probability measure on Ω. Due to (3.1) and (3.2) we have

$$F(X_{[0,T]}) = m_T = \mathrm{e}^{\int_0^T \langle \beta_s, \mathrm{d}B_s\rangle - \frac{1}{2}\int_0^T \|\beta_s\|^2 \mathrm{d}s}.$$

Moreover, by the Girsanov theorem,

$$\tilde{B}_t := B_t - \int_0^t \beta_s \mathrm{d}s, \quad t \in [0,T] \tag{3.3}$$

is a d-dimensional Brownian motion under the probability measure $\mathbb{Q}$.

Let Y_t solve the SDE

$$\mathrm{d}Y_t = \sqrt{2}\, P_{X_t,Y_t}\Phi_t \circ \mathrm{d}\tilde{B}_t + Z(Y_t)\mathrm{d}t + N(Y_t)\mathrm{d}\tilde{l}_t, \quad Y_0 = o, \tag{3.4}$$

where P_{X_t,Y_t} is the parallel displacement along the minimal geodesic from X_t to Y_t and $\tilde{l}_t$ is the local time of Y_t on ∂M. As explained in e.g. [1, Section

3], we may assume that the minimal geodesic is unique so that $P_{x,y}$ is smooth in $x, y \in M$. Since, under $\mathbb{Q}$, $\tilde{B}_t$ is a d-dimensional Brownian motion, the distribution of $Y_{[0,T]}$ is Π_o^T.
On the other hand, by (2.2) and (3.3), we have

$$\mathrm{d}X_t = \sqrt{2}\,\Phi_t \circ \mathrm{d}\tilde{B}_t + Z(X_t) + \sqrt{2}\,\Phi_t\beta_t\mathrm{d}t + N(X_t)\mathrm{d}l_t. \tag{3.5}$$

Since for any bounded measurable function G on M^T

$$\mathbb{E}_{\mathbb{Q}}G(X_{[0,T]}) = \mathbb{E}_{\mathbb{P}}(FG)(X_{[0,T]}) = \Pi_o^T(FG),$$

we conclude that under $\mathbb{Q}$ the distribution of $X_{[0,T]}$ coincides with $F\Pi_\mu^T$. Therefore,

$$W_{2,\rho_\infty}(F\Pi_o^T, \Pi_o^T)^2 \le \mathbb{E}_{\mathbb{Q}}\rho_\infty(X_{[0,T]}, Y_{[0,T]})^2 = \mathbb{E}_{\mathbb{Q}} \max_{t\in[0,T]} \rho(X_t, Y_t)^2. \tag{3.6}$$

By the convexity of ∂M we have

$$\langle N(x), \nabla\rho(y,\cdot)(x)\rangle = \langle N(x), \nabla\rho(\cdot,y)(x)\rangle \le 0, \quad x \in \partial M.$$

Combining this with the Itô formula for (X_t, Y_t) given by (3.4) and (3.5), we obtain from (1.1) that

$$\begin{aligned} \mathrm{d}\rho(X_t, Y_t) &\le K\rho(X_t, Y_t)\mathrm{d}t + \sqrt{2}\,\langle \Phi_t\beta_t, \nabla\rho(\cdot, Y_t)(X_t)\rangle \mathrm{d}t \\ &\le \Big(K\rho(X_t, Y_t) + \sqrt{2}\,\|\beta_t\|\Big)\mathrm{d}t, \end{aligned}$$

see e.g. [15, Lemmas 2.1 and 2.2]. Since we are using the coupling by parallel displacement instead of the mirror reflection, the martingale part here disappears (cf. Theorem 2 and (2.5) in [9]). Since $X_0 = Y_0$, this implies

$$\rho(X_t, Y_t)^2 \le \mathrm{e}^{2Kt}\left(\sqrt{2}\int_0^t \mathrm{e}^{-Ks}\|\beta_s\|\,\mathrm{d}s\right)^2 \le \frac{\mathrm{e}^{2Kt}-1}{K}\int_0^t \|\beta_s\|^2\mathrm{d}s, \quad t \in [0,T].$$

Therefore,

$$\mathbb{E}_{\mathbb{Q}} \max_{t\in[0,T]} \rho(X_t, Y_t)^2 \le \frac{\mathrm{e}^{2KT}-1}{K}\int_0^T \mathbb{E}_{\mathbb{Q}}\|\beta_s\|^2\mathrm{d}s. \tag{3.7}$$

It is clear that

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}\|\beta_s\|^2 = \mathbb{E}_{\mathbb{P}}\big(m_T\|\beta_s\|^2\big) \\ &= \mathbb{E}_{\mathbb{P}}\big(\|\beta_s\|^2\mathbb{E}_{\mathbb{P}}(m_T|\mathscr{F}_s)\big) = \mathbb{E}_{\mathbb{P}}\big(m_s\|\beta_s\|^2\big), \quad s \in [0,T]. \end{aligned} \tag{3.8}$$

Finally, since (3.1) and (3.2) yield

$$\mathrm{d}\langle m\rangle_t = m_t^2 \mathrm{d}\langle L\rangle_t = m_t^2 \|\beta_t\|^2 \mathrm{d}t,$$

we have

$$\begin{aligned}\mathrm{d}m_t \log m_t &= (1+\log m_t)\mathrm{d}m_t + \frac{\mathrm{d}\langle m\rangle_t}{2m_t}\\ &= (1+\log m_t)\mathrm{d}m_t + \frac{m_t}{2}\|\beta_t\|^2 \mathrm{d}t.\end{aligned}$$

As m_t is a $\mathbb{P}$-martingale, combining this with (3.8) we obtain

$$\int_0^T \mathbb{E}_{\mathbb{Q}} \|\beta_s\|^2 \mathrm{d}s = 2\mathbb{E}_{\mathbb{P}} F(X_{[0,T]}) \log F(X_{[0,T]}). \tag{3.9}$$

Therefore, (1.4) follows from (3.6), (3.7) and (3.9).
(b) (3) $\Rightarrow$ (2). By (3), for each $x \in M$, there exists

$$\pi_x \in \mathscr{C}\Big(\frac{F}{\Pi_x^T(F)}\Pi_x^T, \Pi_x^T\Big)$$

such that

$$\int_{M^T\times M^T} \rho_\infty(\gamma,\eta)^2 \pi_x(\mathrm{d}\gamma,\mathrm{d}\eta) \le \frac{2}{K}(\mathrm{e}^{2KT}-1)\Pi_x^T\Big(\frac{F}{\Pi_x^T(F)}\log\frac{F}{\Pi_x^T(F)}\Big). \tag{3.10}$$

If $x \mapsto \pi_x(G)$ is measurable for bounded continuous functions G on $M^T\times M^T$, then

$$\pi := \int_M \pi_x \mu_F^T(\mathrm{d}x) \in \mathscr{C}(F\Pi_\mu^T, \Pi_{\mu_F^T}^T)$$

is well defined and by (3.10)

$$\begin{aligned}\int_{M^T\times M^T} \rho_\infty^2 \mathrm{d}\pi &\le \frac{2}{K}(\mathrm{e}^{2KT}-1)\int_M \Pi_x^T\Big(F\log\frac{F}{\Pi_x^T(F)}\Big)\mu(\mathrm{d}x)\\ &\le \frac{2}{K}(\mathrm{e}^{2KT}-1)\Pi_\mu^T(F\log F).\end{aligned}$$

This implies the inequality in (2).
To confirm the measurability of $x\mapsto \pi_x$, we first consider discrete μ, i.e. $\mu = \sum_{n=1}^\infty \varepsilon_n\delta_{x_n}$ for some $\{x_n\}\subset M$ and $\varepsilon_n \ge 0$ with $\sum_{n=1}^\infty \varepsilon_n = 1$. In this case

$$\pi_x = \sum_{n=1}^\infty 1_{\{x=x_n\}}\pi_{x_n}, \quad \mu\text{-a.e.}$$

which is measurable in x and $\pi = \sum_{n=1}^\infty \mu_F^T(\{x_n\})\pi_{x_n}$. Hence, the inequality in (2) holds. Then, for general μ, the desired inequality can be derived by approximating μ with discrete distributions in a standard way, see (b) in the proof of [6, Theorem 4.1].

(c) (4) $\Rightarrow$ (1). According to [12, Section 7] (see also [4, Section 4.1]), by first applying the transportation-cost inequality in (3) to $1-\varepsilon+\varepsilon f$ in place of f, then letting $\varepsilon \to 0$, we obtain the Poincaré inequality

$$P_T f^2 \le \frac{e^{2KT}-1}{K} P_T |\nabla f|^2 + (P_T f)^2, \quad f \in C_b^1(M), T > 0. \tag{3.11}$$

Thus, the proof is finished by Theorem 2.4.

(d) (1) $\Rightarrow$ (6). Let X_t solve (2.2) with $X_0 = x$ and Y_t solve

$$dY_t = \sqrt{2}\, P_{X_t,Y_t}\Phi_t \circ dB_t + Z(Y_t)dt - N(Y_t)d\tilde{l}_t,\ Y_0 = y, \tag{3.12}$$

where $\tilde{l}_t$ is the local time of Y_t on ∂M. Since ∂M is convex and (1.1) holds, as explained in (a), we have

$$d\rho(X_t, Y_t) \le K\rho(X_t, Y_t)dt.$$

Thus, $\rho_\infty(X_\cdot, Y_\cdot) \le e^{KT}\rho(x, y)$. This implies (6).

(e) (6) $\Rightarrow$ (5). By (6), for any $x, y \in M$, there exists $\pi_{x,y} \in \mathscr{C}(\Pi_x^T, \Pi_y^T)$ such that

$$\int_{M^T\times M^T} \rho_\infty^p d\pi_{x,y} \le e^{KT}\rho(x, y)^p.$$

As explained in (b), we assume that μ and ν are discrete, so that for any $\pi^0 \in (\mu, \nu)$, $\pi_{x,y}$ has a π^0-version measurable in (x, y). Thus,

$$\pi := \int_{M\times M} \pi_{x,y}\pi^0(dx, dy) \in \mathscr{C}(\Pi_\mu^T, \Pi_\nu^T)$$

satisfies

$$\int_{M^T\times M^T} \rho_\infty^p d\pi \le e^{KT} \int_{M\times M} \rho(x, y)^p \pi^0(dx, dy).$$

This implies the desired inequality in (5).

(f) (6) $\Rightarrow$ (1). Let $T > 0$ be fixed. For any $x, y \in M$, let $\pi_{x,y} \in \mathscr{C}(P_T(x, \cdot), P_T(y, \cdot))$ be the optimal coupling for $W_{2,\rho}$, i.e.

$$W_{2,\rho}(P_T(x, \cdot), P_T(y, \cdot))^2 = \int_{M\times M} \rho^2 d\pi_{x,y}. \tag{3.13}$$

Then for any $f \in C_b^2(M)$, (6) implies

$$\begin{aligned}\frac{|P_Tf(x)-P_Tf(y)|}{\rho(x,y)} &\le \int_{M\times M}\frac{|f(z_1)-f(z_2)|}{\rho(z_1,z_2)}\cdot\frac{\rho(z_1,z_2)}{\rho(x,y)}\pi_{x,y}(\mathrm{d}z_1,\mathrm{d}z_2)\\ &\le \frac{W_{2,\rho}(P_T(x,\cdot),P_T(y,\cdot))}{\rho(x,y)}\bigg\{\int_{M\times M}\frac{(f(z_1)-f(z_2))^2}{\rho(z_1,z_2)^2}\pi_{x,y}(\mathrm{d}z_1,\mathrm{d}z_2)\bigg\}^{1/2}\\ &\le \mathrm{e}^{KT}\bigg\{\int_{M\times M}\frac{(f(z_1)-f(z_2))^2}{\rho(z_1,z_2)^2}\pi_{x,y}(\mathrm{d}z_1,\mathrm{d}z_2)\bigg\}^{1/2}.\end{aligned} \tag{3.14}$$

Noting that $f \in C_b^2(M)$ implies

$$|f(z_1)-f(z_2)|^2 \le \rho(z_1,z_2)^2|\nabla f|^2(z_1)+c\rho(z_1,z_2)^3$$

for some constant $c > 0$, by (6) and (3.13) we obtain

$$\int_{M\times M}\frac{(f(z_1)-f(z_2))^2}{\rho(z_1,z_2)^2}\pi_{x,y}(\mathrm{d}z_1,\mathrm{d}z_2) \le P_T|\nabla f|^2(x)+c\mathrm{e}^{KT}\rho(x,y).$$

Therefore, letting $y \to x$ in (3.14) we arrive at

$$|\nabla P_Tf(x)| \le \mathrm{e}^{KT}(P_T|\nabla f|^2(x))^{1/2}.$$

By a standard argument of Bakry and Emery, this implies the Poincaré inequality (3.11). Thus, (1) holds according to Theorem 2.4.

4 The case with a diffusion coefficient

Let $\psi > 0$ be a smooth function on M, and let $\Pi_{\mu,\psi}^T$ be the distribution of the (reflecting if $\partial M \neq \emptyset$) diffusion process generated by $L_\psi := \psi^2(\Delta + Z)$ on time interval $[0,T]$ with initial distribution μ, and let $\Pi_{x,\psi}^T = \Pi_{\delta_x,\psi}^T$ for $x \in M$. Moreover, for $F \ge 0$ with $\Pi_{\mu,\psi}^T(F) = 0$, let

$$\mu_{F,\psi}^T(\mathrm{d}x) = \Pi_{x,\psi}^T(F)\mu(\mathrm{d}x).$$

Theorem 4.1. *Assume that ∂M is either empty or convex and let (1.1) hold. Let $\psi \in C_b^\infty(M)$ be strictly positive. Let*

$$K_\psi = K^+\|\psi\|_\infty^2 + 2\|Z\|_\infty\|\nabla\psi\|_\infty\|\psi\|_\infty + (d-1)\|\nabla\psi\|_\infty^2.$$

Then

$$W_{2,\rho_\infty}(F\Pi_{\mu,\psi}^T, \Pi_{\mu_{F,\psi}^T,\psi}^T)^2 \le 2C(T,\psi)\Pi_{\mu,\psi}^T(F\log F),$$
$$\mu \in \mathscr{P}(M),\ F \ge 0,\ \Pi_{\mu,\psi}^T(F) = 1$$

holds for

$$C(T,\psi) := \inf_{R>0}\Big\{(1+R^{-1})\|\psi\|_\infty^2\frac{e^{2K_\psi T}-1}{K_\psi}\exp\Big[2(1+R)\|\nabla\psi\|_\infty^2\frac{e^{2K_\psi T}-1}{K_\psi}\Big]\Big\}.$$

Proof. As explained in (a) of the proof of Theorem 1.1, we shall only consider the case that ∂M is non-empty and convex. According to the proof of "(3) $\Rightarrow$ (2)", it suffices to prove for $\mu=\delta_o, o\in M$. In this case the desired inequality reduces to

$$W_{2,\rho_\infty}(F\Pi^T_{o,\psi},\Pi^T_{o,\psi}) \le 2C(T,\psi)\Pi^T_{o,\psi}(F\log F),\quad F\ge 0, \Pi^T_{o,\psi}(F)=1. \tag{4.1}$$

Since the diffusion coefficient is non-constant, it is convenient to adopt the Itô differential d_I for the Girsanov transformation. So, the reflecting diffusion process generated by $L_\psi := \psi^2(\Delta+Z)$ can be constructed by solving the Itô SDE

$$d_I X_t = \sqrt{2}\,\psi(X_t)\Phi_t dB_t + \psi^2(X_t)Z(X_t)dt + N(X_t)dl_t, \tag{4.2}$$

where $X_0=o$ and B_t is the d-dimensional Brownian motion with natural filtration $\mathscr{F}_t$. Let $\beta_t,\mathbb{Q}$ and $\tilde B_t$ be fixed in the proof of Theorem 1.1. Then

$$d_I X_t = \sqrt{2}\,\psi(X_t)\Phi_t d\tilde B_t + \big\{\psi^2(X_t)Z(X_t)+\sqrt{2}\,\psi(X_t)\Phi_t\beta_t\big\}dt + N(X_t)dl_t. \tag{4.3}$$

Let Y_t solve

$$d_I Y_t = \sqrt{2}\,\psi(Y_t)P_{X_t,Y_t}\Phi_t d\tilde B_t + \psi^2(Y_t)Z(Y_t)dt + N(Y_t)d\tilde l_t,\quad Y_0=o, \tag{4.4}$$

where $\tilde l_t$ is the local time of Y_t on ∂M. As in (a) of the proof of Theorem 1.1, under $\mathbb{Q}$, the distributions of $Y_{[0,T]}$ and $X_{[0,T]}$ are $\Pi^T_{o,\psi}$ and $F\Pi^T_{o,\psi}$ respectively. So,

$$W_{2,\rho_\infty}(F\Pi^T_{o,\psi},\Pi^T_{o,\psi})^2 \le \mathbb{E}_{\mathbb{Q}}\max_{t\in[0,T]}\rho(X_t,Y_t)^2. \tag{4.5}$$

Noting that due to the convexity of ∂M

$$\langle N(x),\nabla\rho(y,\cdot)(x)\rangle = \langle N(x),\nabla\rho(\cdot,y)(x)\rangle \le 0,\quad x\in\partial M,$$

by (4.3), (4.4) and the Itô formula, we obtain

$$\begin{aligned} \mathrm{d}\rho(X_t,Y_t) \le \sqrt{2}\,\big\{&\psi(X_t)\langle\nabla\rho(\cdot,Y_t)(X_t),\Phi_t\mathrm{d}\tilde B_t\rangle \\ &+\psi(Y_t)\langle\nabla\rho(X_t,\cdot)(Y_t),P_{X_t,Y_t}\Phi_t\mathrm{d}\tilde B_t\rangle\big\} \\ +\Big\{&\sum_{i=1}^{d-1}U_i^2\rho(X_t,Y_t)+\langle\psi(X_t)^2Z(X_t)+\sqrt{2}\,\psi(X_t)\Phi_t\beta_t,\nabla\rho(\cdot,Y_t)(X_t)\rangle \\ &+\psi(Y_t)^2\langle Z(Y_t),\nabla\rho(X_t,\cdot)(Y_t)\rangle\Big\}\mathrm{d}t, \end{aligned} \tag{4.6}$$

where $\{U_i\}_{i=1}^{d-1}$ are vector fields on $M\times M$ such that $\nabla U_i(X_t,Y_t)=0$ and

$$U_i(X_t,Y_t)=\psi(X_t)V_i+\psi(Y_t)P_{X_t,Y_t}V_i,\quad 1\le i\le d-1$$

for $\{V_i\}_{i=1}^d$ an OBN of $T_{X_t}M$ with $V_d=\nabla\rho(\cdot,Y_t)(X_t)$.
In order to calculate $U_i^2\rho(X_t,Y_t)$, we adopt the second variational formula for the distance. Let $\rho_t=\rho(X_t,Y_t)$ and let $\{J_i\}_{i=1}^{d-1}$ be Jacobi fields along the minimal geodesic $\gamma:[0,\rho_t]\to M$ from X_t to Y_t such that $J_i(0)=\psi(X_t)V_i$ and $J_i(\rho_t)=\psi(Y_t)P_{X_t,Y_t}V_i, 1\le i\le d-1$. Note that the existence of γ is ensured by the convexity of ∂M. Then, by the second variational formula and noting that $\nabla U_i(X_t,Y_t)=0$, we have

$$I:=\sum_{i=1}^{d-1}U_i^2\rho(X_t,Y_t)=\sum_{i=1}^{d-1}\int_0^{\rho_t}\big\{|\nabla_{\dot\gamma}J_i|^2-\langle\mathscr{R}(\dot\gamma,J_i)J_i,\dot\gamma\rangle\big\}(s)\mathrm{d}s, \tag{4.7}$$

where $\mathscr{R}$ is the curvature tensor. Let

$$\tilde J_i(s)=\Big(\frac{s}{\rho_t}\psi(Y_t)+\frac{\rho_t-s}{\rho_t}\psi(X_t)\Big)P_{\gamma(0),\gamma(s)}V_i,\ \ 1\le i\le d-1.$$

We have $\tilde J_i(0)=J_i(0)$ and $\tilde J_i(\rho_t)=J_i(\rho_t), 1\le i\le i-1$. By the index lemma,

$$\begin{aligned} I&\le\sum_{i=1}^{d-1}\int_0^{\rho_t}\big\{|\nabla_{\dot\gamma}\tilde J_i|^2-\langle\mathscr{R}(\dot\gamma,\tilde J_i)\tilde J_i,\dot\gamma\rangle\big\}(s)\mathrm{d}s \\ &\le(d-1)\|\nabla\psi\|_\infty^2\rho_t-\frac{1}{\rho_t^2}\int_0^{\rho_t}\big\{s\psi(Y_t)+(\rho_t-s)\psi(X_t)\big\}^2\mathrm{Ric}\big(\dot\gamma(s),\dot\gamma(s)\big)\mathrm{d}s. \end{aligned} \tag{4.8}$$

Moreover,

$$
\begin{aligned}
&\psi(X_t)^2\langle Z(X_t),\nabla\rho(\cdot,Y_t)(X_t)\rangle+\psi(Y_t)^2\langle Z(Y_t),\nabla\rho(X_t,\cdot)(Y_t)\rangle\\
&=\frac{1}{\rho_t^2}\int_0^{\rho_t}\frac{\mathrm{d}}{\mathrm{d}s}\Big\{\big(s\psi(Y_t)+(\rho_t-s)\psi(X_t)\big)^2\langle Z(\gamma(s)),\dot\gamma(s)\rangle\Big\}\mathrm{d}s\\
&=\frac{1}{\rho_t^2}\int_0^{\rho_t}\big(s\psi(Y_t)+(\rho_t-s)\psi(X_t)\big)^2\langle(\nabla_{\dot\gamma}Z)\circ\gamma,\dot\gamma\rangle(s)\mathrm{d}s\\
&\qquad+\frac{2}{\rho_t^2}\int_0^{\rho_t}\langle Z\circ\gamma,\dot\gamma\rangle(s)(\psi(Y_t)-\psi(X_t))\big(s\psi(Y_t)+(\rho_t-s)\psi(X_t)\big)\mathrm{d}s.\\
&\le\frac{1}{\rho_t^2}\int_0^{\rho_t}\big(s\psi(Y_t)+(\rho_t-s)\psi(X_t)\big)^2\langle(\nabla_{\dot\gamma}Z)\circ\gamma,\dot\gamma\rangle(s)\mathrm{d}s\\
&\qquad+2\|Z\|_\infty\|\psi\|_\infty\|\nabla\psi\|_\infty\rho_t.
\end{aligned}
\tag{4.9}
$$

Finally, we have

$$
\begin{aligned}
\langle\nabla\rho(X_t,\cdot)(Y_t),P_{X_t,Y_t}\Phi_t\mathrm{d}\tilde B_t\rangle=\langle P_{Y_t,X_t}\nabla\rho(X_t,\cdot)(Y_t),\Phi_t\mathrm{d}\tilde B_t\rangle=\\
=-\langle\nabla\rho(\cdot,Y_t)(X_t),\Phi_t\mathrm{d}\tilde B_t\rangle.
\end{aligned}
$$

Combining this with (4.6), (4.7), (4.8) and (4.9), we arrive at

$$
\begin{aligned}
\mathrm{d}\rho(X_t,Y_t)\le&\sqrt2\,(\psi(X_t)-\psi(Y_t))\langle\nabla\rho(\cdot,Y_t)(X_t),\Phi_t\mathrm{d}\tilde B_t\rangle\\
&+K_\psi\rho(X_t,Y_t)\mathrm{d}t+\sqrt2\,\|\psi\|_\infty\|\beta_t\|\mathrm{d}t.
\end{aligned}
$$

Then

$$
M_t:=\sqrt2\int_0^t\mathrm{e}^{-K_\psi s}(\psi(X_s)-\psi(Y_s))\langle\nabla\rho(\cdot,Y_s)(X_s),\Phi_s\mathrm{d}\tilde B_s\rangle
$$

is a $\mathbb{Q}$-martingale such that

$$
\rho(X_t,Y_t)\le\mathrm{e}^{K_\psi t}M_t+\sqrt2\,\mathrm{e}^{K_\psi t}\int_0^t\mathrm{e}^{-K_\psi s}\|\psi\|_\infty\|\beta_s\|\mathrm{d}s,\quad t\in[0,T].\tag{4.10}
$$

So, by the Doob inequality we obtain

$$\begin{aligned}
h_t &:= \mathbb{E}_{\mathbb{Q}} \max_{s\in[0,t]} \rho(X_s,Y_s)^2\\
&\le (1+R)\mathrm{e}^{2K_\psi t}\mathbb{E}_{\mathbb{Q}} \max_{s\in[0,t]} M_s^2\mathrm{d}s\\
&\qquad + 2\|\psi\|_\infty^2(1+R^{-1})\mathrm{e}^{2K_\psi t}\mathbb{E}_{\mathbb{Q}}\bigg(\int_0^t \mathrm{e}^{-K_\psi s}\|\beta_s\|\mathrm{d}s\bigg)^2\\
&\le 4(1+R)\mathrm{e}^{2K_\psi t}\,\mathbb{E}_{\mathbb{Q}}M_t^2 + (1+R^{-1})\|\psi\|_\infty^2\frac{\mathrm{e}^{2K_\psi t}-1}{K_\psi}\int_0^t \mathbb{E}_{\mathbb{Q}}\|\beta_s\|^2\mathrm{d}s\\
&\le 4(1+R)\|\nabla\psi\|_\infty^2\mathrm{e}^{2K_\psi t}\int_0^t \mathrm{e}^{-2K_\psi s}h_s\mathrm{d}s\\
&\qquad + (1+R^{-1})\|\psi\|_\infty^2\frac{\mathrm{e}^{2K_\psi T}-1}{K_\psi}\int_0^t \mathbb{E}_{\mathbb{Q}}\|\beta_s\|^2\mathrm{d}s
\end{aligned}$$

for any $R>0$. Since $\mathrm{e}^{-2K_\psi s}$ is decreasing in s while h_s is increasing in s, by the FKG inequality we have

$$\int_0^t \mathrm{e}^{-2K_\psi s}h_s\mathrm{d}s \le \bigg(\frac{1}{t}\int_0^t \mathrm{e}^{-2Ks}\mathrm{d}s\bigg)\int_0^t h_s\mathrm{d}s = \frac{1-\mathrm{e}^{-2K_\psi t}}{2K_\psi t}\int_0^t h_s\mathrm{d}s.$$

Therefore,

$$h_t \le 2(1+R)\|\nabla\psi\|_\infty^2\frac{\mathrm{e}^{2K_\psi T}-1}{K_\psi T}\int_0^t h_s\mathrm{d}s + (1+R^{-1})\|\psi\|_\infty^2\frac{\mathrm{e}^{2K_\psi T}-1}{K_\psi}\int_0^t \mathbb{E}_{\mathbb{Q}}\|\beta_s\|^2\mathrm{d}s$$

holds for $t\in[0,T]$. Since $h_0=0$, this implies that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\max_{t\in[0,T]}\rho(X_t,Y_t)^2 = h_T\\
&\le (1+R^{-1})\|\psi\|_\infty^2\frac{\mathrm{e}^{2K_\psi T}-1}{K_\psi}\exp\Big[2(1+R)\|\nabla\psi\|_\infty^2\frac{\mathrm{e}^{2K_\psi T}-1}{K_\psi}\Big]\int_0^T \mathbb{E}_{\mathbb{Q}}\|\beta_s\|^2\mathrm{d}s.
\end{aligned}$$

Combining this with the (4.5) and (3.9), we complete the proof. □

Theorem 4.2. *In the situation of Theorem 4.1,*

$$W_{2,\rho_\infty}(\Pi_{\mu,\psi}^T,\Pi_{\nu,\psi}^T) \le 2\mathrm{e}^{(K_\psi+\|\nabla\psi\|_\infty^2)T}W_{2,\rho}(\mu,\nu),\quad \mu,\nu\in\mathscr{P}(M), T>0.$$

Proof. As explained in the proof of "(6) ⇒ (5)", we only consider $\mu=\delta_x$ and $\nu=\delta_y$. Let X_t solve (4.2) with $X_0=x$, and let Y_t solve, instead of (4.4),

$$\mathrm{d}_I Y_t = \sqrt{2}\,\psi(Y_t)P_{X_t,Y_t}\Phi_t\mathrm{d}\tilde{B}_t + \psi^2(Y_t)Z(Y_t)\mathrm{d}t + N(Y_t)\mathrm{d}\tilde{l}_t,\quad Y_0=y.$$

Then, repeating the proof of Theorem 4.1, we have, instead of (4.10),

$$\rho(X_t,Y_t) \le \mathrm{e}^{K_\psi t}(M_t+\rho(x,y)),\quad t\ge 0 \tag{4.11}$$

for

$$M_t := \sqrt{2}\int_0^t \mathrm{e}^{-K_\psi s}(\psi(X_s)-\psi(Y_s))\langle \nabla\rho(\cdot,Y_s)(X_s), \Phi_s \mathrm{d}B_s\rangle.$$

So,

$$\mathbb{E}\rho(X_t,Y_t)^2 \le \mathrm{e}^{2K_\psi t}\bigg\{\rho(x,y)^2 + 2\|\nabla\psi\|_\infty^2 \int_0^t \mathrm{e}^{-2K_\psi s}\mathbb{E}\rho(X_s,Y_s)^2\mathrm{d}s\bigg\},$$

which implies

$$\mathbb{E}\rho(X_t,Y_t)^2 \le \mathrm{e}^{2(K_\psi+\|\nabla\psi\|_\infty^2)t}\rho(x,y)^2.$$

Combining this with (4.11) and the Doob inequality, we arrive at

$$\begin{aligned}
&W_{2,\rho_\infty}(\Pi_{x,\psi}^T,\Pi_{y,\psi}^T)^2 \le \mathbb{E}\max_{t\in[0,T]}\rho(X_t,Y_t)^2 \le \mathrm{e}^{2K_\psi T}\mathbb{E}\max_{t\in[0,T]}(M_t+\rho(x,y))^2\\
&\le 4\mathrm{e}^{2K_\psi T}\mathbb{E}(M_T+\rho(x,y))^2 = 4\mathrm{e}^{2K_\psi T}\big(\mathbb{E}M_T^2+\rho(x,y)^2\big)\\
&= 4\mathrm{e}^{2K_\psi T}\bigg(\rho(x,y)^2 + 2\|\nabla\psi\|_\infty^2\int_0^T \mathrm{e}^{-2K_\psi t}\mathbb{E}\rho(X_t,Y_t)^2\mathrm{d}t\bigg)\\
&\le 4\mathrm{e}^{2(K_\psi+\|\nabla\psi\|_\infty^2)T}\rho(x,y)^2.
\end{aligned}$$

This implies the desired inequality for $\mu=\delta_x$ and $\nu=\delta_y$. □

5 Extensions to non-convex manifolds

As explained in the end of Section 1, combining Theorem 4.1 with a proper conformal change of metric, we are able to establish the following transportation-cost inequality on a class of manifolds with non-convex boundary.

Theorem 5.1. *Let $\partial M \ne \emptyset$ with $\mathbb{I} \ge -\sigma$ for some constant $\sigma > 0$, and let (1.1) hold for some $K \in \mathbb{R}$. Then for any $f \in C_b^\infty(M)$ with $f \ge 1$ and $N\log f|_{\partial M} \ge \sigma$, and for any $\mu \in \mathscr{P}(M)$,*

$$W_{2,\rho_\infty}(F\Pi_\mu^T,\Pi_{\mu_F^T}^T)^2 \le 2\|f\|_\infty^2 c(T,f)\Pi_\mu^T(F\log F),\quad F\ge 0, \Pi_\mu^T(F)=1$$

holds for

$$c(T,f) = \inf_{R>0}\Big\{(1+R^{-1})\frac{\mathrm{e}^{2\kappa_f T}-1}{\kappa_f}\exp\Big[2(1+R)\|\nabla f\|_\infty^2\frac{\mathrm{e}^{2\kappa_f T}-1}{\kappa_f}\Big]\Big\},$$

where

$$\kappa_f = 5\|f\|_\infty\|\nabla f\|_\infty\|Z\|_\infty + \big\{3d-5+(d-3)^+\big\}\|\nabla f\|_\infty^2 + \|(Kf^2-f\Delta f)^+\|_\infty.$$

In particular,

$$W_{2,\rho_\infty}(F\Pi_o^T,\Pi_o^T)^2 \le 2\|f\|_\infty^2 c(T,f)\Pi_o^T(F\log F),\quad o\in M, F\ge 0, \Pi_o^T(F)=1.$$

Proof. Let $f \in C_b^\infty(M)$ such that $f \geq 1$. Since $\mathbb{I} \geq -\sigma$ and $N \log f|_{\partial M} \geq \sigma$, by [21, Lemma 2.1] the boundary ∂M is convex under the new metric

$$\langle \cdot, \cdot \rangle' = f^{-2} \langle \cdot, \cdot \rangle.$$

Let Δ' and ∇' be induced by the new metric. Then (see formula (2.2) in [14])

$$L = f^{-2}(\Delta' + Z'), \quad Z' := f^2 Z + \frac{d-2}{2} \nabla f^2.$$

Let Ric' be the Ricci curvature induced by the new metric, we have (cf. formula (3.2) in [6])

$$\mathrm{Ric}' = \mathrm{Ric} + (d-2) f^{-1} \mathrm{Hess}_f + (f^{-1} \Delta f - (d-3)|\nabla \log f|^2) \langle \cdot, \cdot \rangle. \tag{5.1}$$

Since the Levi-Civita connection induced by $\langle \cdot, \cdot \rangle'$ satisfies (cf. [3, Theorem 1.59(a)])

$$\nabla'_U V = \nabla_U V - \langle U, \nabla \log f \rangle V - \langle V, \nabla \log f \rangle U + \langle U, V \rangle \nabla \log f, \quad U, V \in TM,$$

we have

$$\begin{aligned}
\langle \nabla'_U Z', U \rangle' &= f^{-2} \big\{ \langle \nabla_U Z', U \rangle - \langle Z', \nabla \log f \rangle |U|^2 \big\} \\
&= 2 \langle U, \nabla \log f \rangle \langle Z, U \rangle + \langle \nabla_U Z, U \rangle + \frac{d-2}{2f^2} \mathrm{Hess}_{f^2}(U, U) \\
&\quad - \langle Z, \nabla \log f \rangle |U|^2 - \frac{d-2}{2} \langle \nabla \log f^2, \nabla \log f \rangle |U|^2 \\
&\leq \langle \nabla_U Z, U \rangle + 3 |\nabla \log f| \cdot |Z| \cdot |U|^2 + (d-2) f^{-1} \mathrm{Hess}_f(U, U).
\end{aligned}$$

Combining this with (5.1), we obtain

$$\begin{aligned}
&\mathrm{Ric}'(U, U) - \langle \nabla_U Z', U \rangle' \\
&\geq \mathrm{Ric}(U, U) - \langle \nabla_U Z, U \rangle + \big\{ f^{-1} \Delta f - (d-3) |\nabla \log f| - 3|Z| \cdot |\nabla \log f| \big\} |U|^2 \\
&\geq -K' \langle U, U \rangle', \quad U \in TM,
\end{aligned}$$

where

$$K' = \sup_M \{ K f^2 - f \Delta f + (d-3) |\nabla f|^2 + 3 |Z| f |\nabla f| \}. \tag{5.2}$$

Noting that $f \geq 1$, we have

$$\begin{aligned}
&\sqrt{\langle Z', Z' \rangle'} = f^{-1} |f^2 Z + (d-2) f \nabla f| \leq \|f\|_\infty \|Z\|_\infty + (d-2) \|\nabla f\|_\infty, \\
&\sqrt{\langle \nabla' f^{-1}, \nabla' f^{-1} \rangle'} = f |\nabla f^{-1}| \leq \|\nabla f\|_\infty.
\end{aligned} \tag{5.3}$$

Letting K_ψ be defined in Theorem 4.1 for the manifold $(M, \langle\cdot,\cdot\rangle')$ and $L = \psi^2(\Delta' + Z')$ with $\psi = f^{-1}$, we deduce from $f \geq 1$, (5.2) and (5.3) that

$$K_\psi \leq \kappa_f.$$

Therefore, $C(T,\psi) \leq c(T,f)$ and thus, Theorem 4.1 implies

$$W_{2,\rho'_\infty}(F\Pi^T_\mu, \Pi^T_{\mu^T_F})^2 \leq 2c(T,f)\Pi^T_\mu(F\log F), \quad F \geq 0, \Pi^T_\mu(F) = 1,$$

where ρ'_∞ is the uniform distance on M^T induced by the metric $\langle\cdot,\cdot\rangle'$. The proof is completed by noting that $\rho_\infty \leq \|f\|_\infty \rho'_\infty$. □

Similarly, since $K_\psi \leq \kappa_f$ and

$$\rho' \leq \rho \leq \|f\|_\infty \rho,$$

the following result from Theorem 4.2 by taking $\psi = f^{-1}$.

Theorem 5.2. *In the situation of Theorem 5.1,*

$$W_{2,\rho_\infty}(\Pi^T_\mu, \Pi^T_\nu) \leq 2\|f\|_\infty \mathrm{e}^{(\kappa_f + \|\nabla f^{-1}\|^2_\infty)T} W_{2,\rho}(\mu,\nu), \quad \mu,\nu \in \mathscr{P}(M), T > 0.$$

As a consequence of Theorems 5.1 and 5.2, we present below an explicit transportation-cost inequalities for a class of non-convex manifolds.

Corollary 5.3. *Assume that (1.1) holds for some $K \geq 0$ and the injectivity radius $\mathrm{i}_{\partial M}$ of ∂M is strictly positive. Let $\sigma \geq 0$ and $\gamma, k, > 0$ be such that $-\sigma \leq \mathbb{I} \leq \gamma$ and $\mathrm{Sect}_M \leq k$. Let*

$$0 < r \leq \min\left\{\mathrm{i}_{\partial M}, \ \frac{1}{\sqrt{k}}\arcsin\left(\frac{\sqrt{k}}{\sqrt{k+\gamma^2}}\right)\right\}.$$

(i) *The transportation-cost inequality*

$$W_{2,\rho_\infty}(F\Pi^T_\mu, \Pi^T_{\mu^T_F})^2 \leq (2 + rd\sigma)^2 \frac{\mathrm{e}^{2\theta T} - 1}{\theta} \exp\left[\frac{4(\mathrm{e}^{2\theta T} - 1)}{\theta}\right]\Pi^T_\mu(F\log F)$$

holds for all $\mu \in \mathscr{P}(M)$ and $F \geq 0$ with $\Pi^T_\mu(F) = 1$, where

$$\theta = K\Big(1 + rd\sigma + \frac{r^2d^2\sigma^2}{4}\Big) + \frac{d\sigma}{r}\Big(3d - 5 + (d-3)^+ + \frac{d^2}{2}\Big)\sigma^2 + 5\|Z\|_\infty \sigma\Big(1 + \frac{rd\sigma}{2}\Big).$$

In particular,

$$W_{2,\rho_\infty}(F\Pi^T_o, \Pi^T_o)^2 \leq (2 + rd\sigma)^2 \frac{\mathrm{e}^{2\theta T} - 1}{\theta} \exp\left[\frac{4(\mathrm{e}^{2\theta T} - 1)}{\theta}\right]\Pi^T_o(F\log F)$$

holds for all $F \geq 0$ *with* $\Pi_\mu^T(F) = 1$.
(ii) *For any* $T > 0$ *and* $\mu, \nu \in \mathscr{P}(M)$,

$$W_{2,\rho_\infty}(\Pi_\mu^T, \Pi_\nu^T) \leq (2 + \sigma r d)\mathrm{e}^{(\theta+\sigma^2)T} W_{2,\rho}(\mu, \nu).$$

Proof. Let

$$h(s) = \cos\big(\sqrt{k}\, s\big) - \frac{\gamma}{\sqrt{k}} \sin\big(\sqrt{k}\, s\big), \quad s \geq 0.$$

Then h is the unique solution to the equation

$$h'' + kh = 0, \quad h(0) = 1, h'(0) = -\gamma.$$

Up to an approximation argument presented in the proof of [20, Theorem 1.1], we may apply Theorem 5.1 to

$$f = 1 + \sigma\varphi \circ \rho_{\partial M},$$

where ρ_∂ is the Riemannian distance to ∂M, which is smooth on $\{\rho_{\partial M} < \mathrm{i}_{\partial M}\}$, and

$$\alpha = (1 - h(r))^{1-d} \int_0^r (h(s) - h(r))^{d-1} \mathrm{d}s,$$
$$\varphi(s) = \frac{1}{\alpha} \int_0^s (h(t) - h(r))^{1-d} \mathrm{d}t \int_{t \wedge r}^r (h(u) - h(r))^{d-1} \mathrm{d}u, \quad s \geq 0.$$

We have $\varphi(0) = 1, 0 \leq \varphi' \leq \varphi'(0) = 1$. Moreover, as observed in [20, Proof of Theorem 1.1],

$$\alpha \geq \frac{r}{d}, \quad \varphi(r) \leq \frac{r^2}{2\alpha} \leq \frac{dr}{2}, \quad \Delta\varphi \circ \rho_{\partial M} \geq -\frac{1}{\alpha} \geq -\frac{d}{r}.$$

So,

$$\|f\|_\infty \leq 1 + \sigma\varphi(r) \leq 1 + \frac{rd\sigma}{2}, \quad \|\nabla f\|_\infty \leq \varphi'(0) = \sigma, \quad \Delta f \geq -\frac{\sigma d}{r}. \tag{5.4}$$

Noting that (recall that $K \geq 0$)

$$\sup(Kf^2) \leq K\Big(1 + rd\sigma + \frac{r^2 d^2 \sigma^2}{4}\Big),$$

from (5.4) we conclude that $\kappa_f \leq \theta$. So, (i) follows from (1.5) and 5.1 for $R = 1$, and (ii) follows from Theorem 4.2 and (5.4). □

ACKNOWLEDGEMENT. The author would like to thank the referee for very helpful comments and corrections.

References

[1] M. Arnaudon, A. Thalmaier, F.-Y. Wang, *Harnack inequality and heat kernel estimates on manifolds with curvature unbounded below,* Bull. Sci. Math. 130(2006), 223–233.

[2] M. Arnaudon, A. Thalmaier, F.-Y. Wang, *Gradient estimates and Harnack inequalities on non-compact Riemannian manifolds,* Stoch. Proc. Appl. 2009.

[3] A. L. Besse, *Einstein Manifolds,* Springer, Berlin, 1987.

[4] S. Bobkov, I. Gentil, M. Ledoux, *Hypercontractivity of Hamilton-Jacobi equations*, J. Math. Pure Appl. 80 (2001), 669–696.

[5] H. Djellout, A. Guillin, L.-M. Wu, *Transportation cost information inequalities and applications to random dynamical systems and diffusions*, Ann. Probab. 32 (2004), 2702–2732.

[6] S. Fang, F.-Y. Wang, B. Wu, *Transportation-cost inequality on path spaces with uniform distance,* Stoch. Proc. Appl. 118(2008), 2181–197.

[7] D. Feyel, A. Üstünel, *Measure transport on Wiener space and the Girsanov theorem,* C.R. Acad. Paris 334(2002), 1025–1028.

[8] N. Ikeda, S. Watanabe, *Stochastic Differential Equations,* North-Holland, New York, 1989.

[9] W.S. Kendall, *Nonnegative Ricci curvature and the Brownian coupling property,* Stochastics 19 (1986), 111C 129.

[10] M. Ledoux, *The geometry of Markov diffusion generators,* Ann. Facul. Sci. Toulouse 9(2000), 305–366.

[11] M.-K. von Reness, K.-T. Sturm, *Transport inequalities, gradient estimates, entropy, and Ricci curvature,* Comm. Pure Math. 58(2005), 923–940.

[12] F. Otto, C. Villani, *Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality,* J. Funct. Anal. 173(2000), 361–400.

[13] M. Talagrand, *Transportation cost for Gaussian and other product measures,* Geom. Funct. Anal. 6(1996), 587–600.

[14] A. Thalmaier, F.-Y. Wang, *Gradient estimates for harmonic functions on regular domains in Riemannian manifolds,* J. Funct. Anal. 155:1(1998),109–124.

[15] F.-Y. Wang, *Application of coupling methods to the Neumann eigenvalue problem,* Probab. Theory Related Fields 98 (1994), 299–306.

[16] F.-Y. Wang,*On estimation of the logarithmic Sobolev constant and gradient estimates of heat semigroups,* Probability Theory Related Fields 108(1997), 87–101.

[17] F.-Y. Wang, *Transportation cost inequalities on path spaces over Riemannian manifolds,* Illinois J. Math. 46 (2002), 1197–1206.

[18] F.-Y. Wang, *Probability distance inequalities on Riemannian manifolds and path spaces*, J. Funct. Anal. 206 (2004), 167–190.

[19] F.-Y. Wang, *Equivalence of dimension-free Harnack inequality and curvature condition,* Integral Equation and Operator Theory 48(2004), 547–552.

[20] F.-Y. Wang, *Gradient estimates and the first Neumann eigenvalue on manifolds with boundary,* Stoch. Proc. Appl. 115(2005), 1475–1486.

[21] F.-Y. Wang, *Estimates of the first Neumann eigenvalue and the log-Sobolev constant on nonconvex manifolds,* Math. Nachr. 280(2007), 1431–1439.

[22] F.-Y. Wang, *Second fundamental form and gradient of Neumann semigroups,* J. Funct. Anal. 256(2009), 3461–3469.

[23] F.-Y. Wang, *Harnack inequalities on manifolds with boundary and applications,* J. Math. Pures Appl. 94(2010), 304–321.

[24] F.-Y. Wang, *Semigroup properties for the second fundamental form,* Docum. Math. 15(2010), 543–559.

[25] F.-Y. Wang, *Intrinsic ultracontractivity on Riemannian manifolds with infinite volume measures,* Science in China (Mathematics) 53(2010), 895–904.

[26] F.-Y. Wang, *Analysis on path space over Riemannian manifolds with boundary,* Comm. Math. Sci. 9(2011), 1203–1213.

[27] L.-M. Wu, Z.-L. Zhang, *Talagrand's T_2-transportation inequality w.r.t. a uniform metric for diffusions*, Acta Math. Appl. Sin. Engl. Ser. 20 (2004), 357–364.

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