Factorial Cluster Algebras

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Abstract. We show that cluster algebras do not contain non-trivial units and that all cluster variables are irreducible elements. Both statements follow from Fomin and Zelevinsky's Laurent phenomenon. As an application we give a criterion for a cluster algebra to be a factorial algebra. This can be used to construct cluster algebras, which are isomorphic to polynomial rings. We also study various kinds of upper bounds for cluster algebras, and we prove that factorial cluster algebras coincide with their upper bounds.

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Contents

1. Introduction and main results

1.1. Introduction. The introduction of cluster algebras by Fomin and Zelevinsky [FZ1] triggered an extensive theory. Most results deal with the combinatorics of seed and quiver mutations, with various categorifications of cluster algebras, and with cluster phenomena occuring in various areas of mathematics, like representation theory of finite-dimensional algebras, quantum groups and Lie theory, Calabi-Yau categories, non-commutative Donaldson-Thomas invariants, Poisson geometry, discrete dynamical systems and algebraic combinatorics.

On the other hand, there are not many results on cluster algebras themselves. As a subalgebra of a field, any cluster algebra $\mathcal{A}$ is obviously an integral domain. It is also easy to show that its field of fractions $\mathrm{Frac}(\mathcal{A})$ is isomorphic to a field $K(x_1, \dots, x_m)$ of rational functions. Several classes of cluster algebras are known to be finitely generated, e.g. acyclic cluster algebras [BFZ, Corollary 1.21] and also a class of cluster algebras arising from Lie theory [GLS2, Theorem 3.2]. Berenstein, Fomin and Zelevinsky gave an example of a cluster algebra which is not finitely generated. (One applies [BFZ, Theorem 1.24] to the example mentioned in [BFZ, Proposition 1.26].) Only very little is known on further ring theoretic properties of an arbitrary cluster algebra $\mathcal{A}$. Here are some basic questions we would like to address:

- Which elements in $\mathcal{A}$ are invertible, irreducible or prime?
- When is $\mathcal{A}$ a factorial ring?
- When is $\mathcal{A}$ a polynomial ring?

In this paper, we work with cluster algebras of geometric type.

1.2. Definition of a cluster algebra. In this section we repeat Fomin and Zelevinsky's definition of a cluster algebra.

A matrix $A = (a_{ij}) \in M_{n,n}(\mathbb{Z})$ is *skew-symmetrizable* (resp. *symmetrizable*) if there exists a diagonal matrix $D = \mathrm{Diag}(d_1, \dots, d_n) \in M_{n,n}(\mathbb{Z})$ with positive diagonal entries $d_1, \dots, d_n$ such that DA is skew-symmetric (resp. symmetric), i.e. $d_i a_{ij} = -d_j a_{ji}$ (resp. $d_i a_{ij} = d_j a_{ji}$) for all i, j.

Let m,n and p be integers with

$$m \geq p \geq n \geq 1 \quad \text{and} \quad m > 1.$$

Let $B = (b_{ij}) \in M_{m,n}(\mathbb{Z})$ be an $(m \times n)$-matrix with integer entries. By $B^\circ \in M_{n,n}(\mathbb{Z})$ we denote the *principal part* of B, which is obtained from B by deleting the last $m - n$ rows.

Let $\Delta(B)$ be the graph with vertices $1, \dots, m$ and an edge between i and j provided b_{ij} or b_{ji} is non-zero. We call B *connected* if the graph $\Delta(B)$ is connected.

Throughout, we assume that K is a field of characteristic 0 or $K = \mathbb{Z}$. Let $\mathcal{F} := K(X_1, \dots, X_m)$ be the field of rational functions in m variables.

A *seed* of $\mathcal{F}$ is a pair $(\mathbf{x}, B)$ such that the following hold:

(i) $B \in M_{m,n}(\mathbb{Z})$,
(ii) B is connected,
(iii) B° is skew-symmetrizable,
(iv) $\mathbf{x} = (x_1, \dots, x_m)$ is an m-tuple of elements in $\mathcal{F}$ such that $x_1, \dots, x_m$ are algebraically independent over K.

For a seed $(\mathbf{x}, B)$, the matrix B is the *exchange matrix* of $(\mathbf{x}, B)$. We say that B has *maximal rank* if $\operatorname{rank}(B) = n$.

Given a seed $(\mathbf{x}, B)$ and some $1 \leq k \leq n$ we define the *mutation* of $(\mathbf{x}, B)$ at k as

$$\mu_k(\mathbf{x}, B) := (\mathbf{x}', B'),$$

where $B' = (b'_{ij})$ is defined as

$$b'_{ij} := \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise,} \end{cases}$$

and $\mathbf{x}' = (x'_1, \dots, x'_m)$ is defined as

$$x'_s := \begin{cases} x_k^{-1} \prod_{b_{ik}>0} x_i^{b_{ik}} + x_k^{-1} \prod_{b_{ik}<0} x_i^{-b_{ik}} & \text{if } s = k, \\ x_s & \text{otherwise.} \end{cases}$$

The equality

$$x_k x'_k = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}} \tag{1}$$

is called an *exchange relation.* We write

$$\mu_{(\mathbf{x},B)}(x_k) := x'_k$$

and

$$\mu_k(B) := B'.$$

It is easy to check that $(\mathbf{x}', B')$ is again a seed. Furthermore, we have $\mu_k\mu_k(\mathbf{x}, B) = (\mathbf{x}, B)$.

Two seeds $(\mathbf{x}, B)$ and $(\mathbf{y}, C)$ are *mutation equivalent* if there exists a sequence $(i_1, \dots, i_t)$ with $1 \leq i_j \leq n$ for all j such that

$$\mu_{i_t} \cdots \mu_{i_2}\mu_{i_1}(\mathbf{x}, B) = (\mathbf{y}, C).$$

In this case, we write $(\mathbf{y}, C) \sim (\mathbf{x}, B)$. This yields an equivalence relation on all seeds of $\mathcal{F}$. (By definition $(\mathbf{x}, B)$ is also mutation equivalent to itself.)

For a seed $(\mathbf{x}, B)$ of $\mathcal{F}$ let

$$\mathcal{X}_{(\mathbf{x},B)} := \bigcup_{(\mathbf{y},C)\sim(\mathbf{x},B)} \{y_1, \dots, y_n\},$$

where the union is over all seeds $(\mathbf{y}, C)$ with $(\mathbf{y}, C) \sim (\mathbf{x}, B)$. By definition, the *cluster algebra* $\mathcal{A}(\mathbf{x}, B)$ associated to $(\mathbf{x}, B)$ is the L-subalgebra of $\mathcal{F}$ generated by $\mathcal{X}_{(\mathbf{x},B)}$, where

$$L := K[x_{n+1}^{\pm 1}, \dots, x_p^{\pm 1}, x_{p+1}, \dots, x_m]$$

is the localization of the polynomial ring $K[x_{n+1}, \dots, x_m]$ at $x_{n+1} \cdots x_p$. (For $p = n$ we set $x_{n+1} \cdots x_p := 1$.) Thus $\mathcal{A}(\mathbf{x}, B)$ is the K-subalgebra of $\mathcal{F}$ generated by

$$\{x_{n+1}^{\pm 1}, \dots, x_p^{\pm 1}, x_{p+1}, \dots, x_m\} \cup \mathcal{X}_{(\mathbf{x},B)}.$$

The elements of $\mathcal{X}_{(\mathbf{x},B)}$ are the *cluster variables* of $\mathcal{A}(\mathbf{x}, B)$.

We call $(\mathbf{y}, C)$ a *seed* of $\mathcal{A}(\mathbf{x}, B)$ if $(\mathbf{y}, C) \sim (\mathbf{x}, B)$. In this case, for any $1 \leq k \leq n$ we call $(y_k, \mu_{(\mathbf{y},C)}(y_k))$ an *exchange pair* of $\mathcal{A}(\mathbf{x}, B)$. Furthermore, the m-tuple $\mathbf{y}$ is a *cluster* of $\mathcal{A}(\mathbf{x}, B)$, and monomials of the form $y_1^{a_1} y_2^{a_2} \cdots y_m^{a_m}$ with $a_i \geq 0$ for all i are called *cluster monomials* of $\mathcal{A}(\mathbf{x}, B)$.

Note that for any cluster $\mathbf{y}$ of $\mathcal{A}(\mathbf{x}, B)$ we have $y_i = x_i$ for all $n + 1 \leq i \leq m$. These $m - n$ elements are the *coefficients* of $\mathcal{A}(\mathbf{x}, B)$. There are no invertible coefficients if $p = n$.

Clearly, for any two seeds of the form $(\mathbf{x}, B)$ and $(\mathbf{y}, B)$ there is an algebra isomorphism $\eta\colon \mathcal{A}(\mathbf{x}, B) \to \mathcal{A}(\mathbf{y}, B)$ with $\eta(x_i) = y_i$ for all $1 \leq i \leq m$, which respects the exchange relations. Furthermore, if $(\mathbf{x}, B)$ and $(\mathbf{y}, C)$ are mutation equivalent seeds, then $\mathcal{A}(\mathbf{x}, B) = \mathcal{A}(\mathbf{y}, C)$ and we have $K(x_1, \dots, x_m) = K(y_1, \dots, y_m)$.

1.3. Trivial cluster algebras and connectedness of exchange matrices. Note that we always assume $m > 1$. For $m = 1$ we would get the trivial cluster algebra $\mathcal{A}(\mathbf{x}, B)$ with exactly two cluster variables, namely x_1 and $x_1' := \mu_{(\mathbf{x},B)}(x_1) = x_1^{-1}(1 + 1)$. In particular, for $K \neq \mathbb{Z}$, both cluster variables are invertible in $\mathcal{A}(\mathbf{x}, B)$, and $\mathcal{A}(\mathbf{x}, B)$ is just the Laurent polynomial ring $K[x_1^{\pm 1}]$.

Furthermore, for any seed $(\mathbf{x}, B)$ of $\mathcal{F}$ the exchange matrix B is by definition connected. For non-connected B one could write $\mathcal{A}(\mathbf{x}, B)$ as a product $\mathcal{A}(\mathbf{x}_1, B_1) \times \mathcal{A}(\mathbf{x}_2, B_2)$ of smaller cluster algebras and study the factors $\mathcal{A}(\mathbf{x}_i, B_i)$ separately. The connectedness assumption also ensures that there are no exchange relations of the form $x_k x_k' = 1 + 1$.

1.4. The Laurent phenomenon. It follows by induction from the exchange relations that for any cluster $\mathbf{y}$ of $\mathcal{A}(\mathbf{x}, B)$, any cluster variable z of $\mathcal{A}(\mathbf{x}, B)$ is of the form

$$z = \frac{f}{g},$$

where $f, g \in \mathbb{N}[y_1, \dots, y_m]$ are integer polynomials in the cluster variables $y_1, \dots, y_m$ with non-negative coefficients. For any seed $(\mathbf{x}, B)$ of $\mathcal{F}$ let

$$\mathscr{L}_{\mathbf{x}} := K[x_1^{\pm 1}, \dots, x_n^{\pm 1}, x_{n+1}^{\pm 1}, \dots, x_p^{\pm 1}, x_{p+1}, \dots, x_m]$$

be the localization of $K[x_1, \dots, x_m]$ at $x_1 x_2 \cdots x_p$, and let

$$\mathscr{L}_{\mathbf{x},\mathbb{Z}} := \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, x_{n+1}, \dots, x_m]$$

be the localization of $\mathbb{Z}[x_1, \dots, x_m]$ at $x_1 x_2 \cdots x_n$. We consider $\mathscr{L}_{\mathbf{x}}$ and $\mathscr{L}_{\mathbf{x},\mathbb{Z}}$ as subrings of the field $\mathcal{F}$. The following remarkable result, known as the *Laurent phenomenon*, is due to Fomin and Zelevinsky and is our key tool to derive some ring theoretic properties of cluster algebras.

Theorem 1.1 ([FZ1, Theorem 3.1], [FZ2, Proposition 11.2]). *For each seed $(\mathbf{x}, B)$ of $\mathcal{F}$ we have*

$$\mathcal{A}(\mathbf{x}, B) \subseteq \overline{\mathcal{A}}(\mathbf{x}, B) := \bigcap_{(\mathbf{y},C)\sim(\mathbf{x},B)} \mathscr{L}_{\mathbf{y}}$$

and

$$\mathcal{X}_{(\mathbf{x},B)} \subset \bigcap_{(\mathbf{y},C)\sim(\mathbf{x},B)} \mathscr{L}_{\mathbf{y},\mathbb{Z}}.$$

The algebra $\overline{\mathcal{A}}(\mathbf{x}, B)$ is called the *upper cluster algebra* associated to $(\mathbf{x}, B)$, compare [BFZ, Section 1].

1.5. Upper bounds. For a seed $(\mathbf{x}, B)$ and $1 \leq k \leq n$ let $(\mathbf{x}_k, B_k) := \mu_k(\mathbf{x}, B)$. Berenstein, Fomin and Zelevinsky [BFZ] called

$$U(\mathbf{x}, B) := \mathscr{L}_{\mathbf{x}} \cap \bigcap_{k=1}^{n} \mathscr{L}_{\mathbf{x}_k}$$

the *upper bound* of $\mathcal{A}(\mathbf{x}, B)$. They prove the following:

Theorem 1.2 ([BFZ, Corollary 1.9]). *Let $(\mathbf{x}, B)$ and $(\mathbf{y}, C)$ be mutation equivalent seeds of $\mathcal{F}$. If B has maximal rank and $p = m$, then $U(\mathbf{x}, B) = U(\mathbf{y}, C)$. In particular, we have $\overline{\mathcal{A}}(\mathbf{x}, B) = U(\mathbf{x}, B)$.*

For clusters $\mathbf{y}$ and $\mathbf{z}$ of $\mathcal{A}(\mathbf{x}, B)$ define

$$U(\mathbf{y}, \mathbf{z}) := \mathscr{L}_{\mathbf{y}} \cap \mathscr{L}_{\mathbf{z}}.$$

1.6. Acyclic cluster algebras. Let $(\mathbf{x}, B)$ be a seed of $\mathcal{F}$ with $B = (b_{ij})$. Let $\Sigma(B)$ be the quiver with vertices $1, \dots, n$, and arrows $i \to j$ for all $1 \leq i, j \leq n$ with $b_{ij} > 0$, compare [BFZ, Section 1.4]. So $\Sigma(B)$ encodes the sign-pattern of the principal part B° of B.

The seed $(\mathbf{x}, B)$ and B are called *acyclic* if $\Sigma(B)$ does not contain any oriented cycle. The cluster algebra $\mathcal{A}(\mathbf{x}, B)$ is *acyclic* if there exists an acyclic seed $(\mathbf{y}, C)$ with $(\mathbf{y}, C) \sim (\mathbf{x}, B)$.

1.7. Skew-symmetric exchange matrices and quivers. Let $B = (b_{ij})$ be a matrix in $M_{m,n}(\mathbb{Z})$ such that B° is skew-symmetric. Let $\Gamma(B)$ be the quiver with vertices $1, \dots, m$ and b_{ij} arrows $i \to j$ if $b_{ij} > 0$, and $-b_{ij}$ arrows $j \to i$ if $b_{ij} < 0$. Thus given $\Gamma(B)$, we can recover B. In the skew-symmetric case one often works with quivers and their mutations instead of exchange matrices.

1.8. Main results. For a ring R with 1, let $R^\times$ be the set of invertible elements in R. Non-zero rings without zero divisors are called *integral domains*. A non-invertible element a in an integral domain R is *irreducible* if it cannot be written as a product $a = bc$ with $b, c \in R$ both non-invertible. Cluster algebras are integral domains, since they are by definition subrings of fields.

Theorem 1.3. *For any seed* $(\mathbf{x}, B)$ *of* $\mathcal{F}$ *the following hold:*

(i) *We have* $\mathcal{A}(\mathbf{x}, B)^\times = \left\{ \lambda x_{n+1}^{a_{n+1}} \cdots x_p^{a_p} \mid \lambda \in K^\times, a_i \in \mathbb{Z} \right\}$.
(ii) *Any cluster variable in* $\mathcal{A}(\mathbf{x}, B)$ *is irreducible.*

For elements a, b in an integral domain R we write $a|b$ if there exists some $c \in R$ with $b = ac$. A non-invertible element a in a commutative ring R is *prime* if whenever $a|bc$ for some $b, c \in R$, then $a|b$ or $a|c$. Every prime element is irreducible, but the converse is not true in general. Non-zero elements $a, b \in R$ are *associate* if there is some unit $c \in R^\times$ with $a = bc$. An integral domain R is *factorial* if the following hold:

(i) Every non-zero non-invertible element $r \in R$ can be written as a product $r = a_1 \cdots a_s$ of irreducible elements $a_i \in R$.
(ii) If $a_1 \cdots a_s = b_1 \cdots b_t$ with $a_i, b_j \in R$ irreducible for all i and j, then $s = t$ and there is a bijection $\pi\colon \{1, \dots, s\} \to \{1, \dots, s\}$ such that a_i and $b_{\pi(i)}$ are associate for all $1 \le i \le s$.

For example, any polynomial ring is factorial. In a factorial ring, all irreducible elements are prime.

Two clusters $\mathbf{y}$ and $\mathbf{z}$ of a cluster algebra $\mathcal{A}(\mathbf{x}, B)$ are *disjoint* if $\{y_1, \dots, y_n\} \cap \{z_1, \dots, z_n\} = \varnothing$.

The next result gives a useful criterion when a cluster algebra is a factorial ring.

Theorem 1.4. *Let* $\mathbf{y}$ *and* $\mathbf{z}$ *be disjoint clusters of* $\mathcal{A}(\mathbf{x}, B)$. *If there is a subalgebra* U *of* $\mathcal{A}(\mathbf{x}, B)$, *such that* U *is factorial and*

$$\{y_1, \dots, y_n, z_1, \dots, z_n, x_{n+1}^{\pm 1}, \dots, x_p^{\pm 1}, x_{p+1}, \dots, x_m\} \subset U,$$

then

$$U = \mathcal{A}(\mathbf{x}, B) = U(\mathbf{y}, \mathbf{z}).$$

In particular, $\mathcal{A}(\mathbf{x}, B)$ *is factorial and all cluster variables are prime.*

We obtain the following corollary on upper bounds of factorial cluster algebras.

Corollary 1.5. *Assume that $\mathcal{A}(\mathbf{x}, B)$ is factorial.*

(i) *If $\mathbf{y}$ and $\mathbf{z}$ are disjoint clusters of $\mathcal{A}(\mathbf{x}, B)$, then $\mathcal{A}(\mathbf{x}, B) = U(\mathbf{y}, \mathbf{z})$.*
(ii) *For any $(\mathbf{y}, C) \sim (\mathbf{x}, B)$ we have $\mathcal{A}(\mathbf{x}, B) = U(\mathbf{y}, C)$.*

In Section 7 we apply the above results to show that many cluster algebras are polynomial rings. In Section 8 we discuss some further applications concerning the dual of Lusztig's semicanonical basis and monoidal categorifications of cluster algebras.

1.9. Factoriality and maximal rank. In Section 6.1 we give examples of cluster algebras $\mathcal{A}(\mathbf{x}, B)$, which are not factorial. In these examples, B does not have maximal rank.

After we presented our results at the Abel Symposium in Balestrand in June 2011, Zelevinsky asked the following question:

Problem 1.6. *Suppose $(\mathbf{x}, B)$ is a seed of $\mathcal{F}$ such that B has maximal rank. Does it follow that $\mathcal{A}(\mathbf{x}, B)$ is factorial?*

After we circulated a first version of this article, Philipp Lampe [La] discovered an example of a non-factorial cluster algebra $\mathcal{A}(\mathbf{x}, B)$ with B having maximal rank. With his permission, we explain a generalization of his example in Section 6.2.

2. Invertible elements in cluster algebras

In this section we prove Theorem 1.3(i), classifying the invertible elements of cluster algebras.

The following lemma is straightforward and well-known.

Lemma 2.1. *For any seed $(\mathbf{x}, B)$ of $\mathcal{F}$ we have*

$$\mathscr{L}_{\mathbf{x}}^{\times} = \{\lambda x_1^{a_1} \cdots x_p^{a_p} \mid \lambda \in K^{\times}, a_i \in \mathbb{Z}\}.$$

Theorem 2.2. *For any seed $(\mathbf{x}, B)$ of $\mathcal{F}$ we have*

$$\mathcal{A}(\mathbf{x}, B)^{\times} = \left\{\lambda x_{n+1}^{a_{n+1}} \cdots x_p^{a_p} \mid \lambda \in K^{\times}, a_i \in \mathbb{Z}\right\}.$$

Proof. Let u be an invertible element in $\mathcal{A}(\mathbf{x}, B)$, and let $(\mathbf{y}, C)$ be any seed of $\mathcal{A}(\mathbf{x}, B)$. By the Laurent phenomenon Theorem 1.1 we know that $\mathcal{A}(\mathbf{x}, B) \subseteq \mathscr{L}_{\mathbf{y}}$. It follows that u is also invertible in $\mathscr{L}_{\mathbf{y}}$. Thus by Lemma 2.1 there are $a_1, \ldots, a_p \in \mathbb{Z}$ and $\lambda \in K^{\times}$ such that $u = \lambda M$, where

$$M = y_1^{a_1} \cdots y_k^{a_k} \cdots y_p^{a_p}.$$

If all a_i with $1 \leq i \leq n$ are zero, we are done. To get a contradiction, assume that there is some $1 \leq k \leq n$ with $a_k \neq 0$.

Let $y_k^* := \mu_{(\mathbf{y},C)}(y_k)$. Again the Laurent phenomenon yields $b_1, \dots, b_p \in \mathbb{Z}$ and $\nu \in K^\times$ such that

$$u = \nu y_1^{b_1} \cdots y_{k-1}^{b_{k-1}} (y_k^*)^{b_k} y_{k+1}^{b_{k+1}} \cdots y_p^{b_p}.$$

Without loss of generality let $b_k \geq 0$. (Otherwise we can work with u^{-1} instead of u.)

If $b_k = 0$, we get

$$\lambda y_1^{a_1} \cdots y_k^{a_k} \cdots y_p^{a_p} = \nu y_1^{b_1} \cdots y_{k-1}^{b_{k-1}} y_{k+1}^{b_{k+1}} \cdots y_p^{b_p},$$

where $\lambda, \nu \in K^\times$. This is a contradiction, because $a_k \neq 0$ and $y_1, \dots, y_m$ are algebraically independent, and therefore Laurent monomials in $y_1, \dots, y_m$ are linearly independent in $\mathcal{F}$.

Next, assume that $b_k > 0$. By definition we have

$$y_k^* = M_1 + M_2$$

with

$$M_1 = y_k^{-1} \prod_{c_{ik}>0} y_i^{c_{ik}} \quad \text{and} \quad M_2 = y_k^{-1} \prod_{c_{ik}<0} y_i^{-c_{ik}},$$

where the products run over the positive, respectively negative, entries in the kth column of the matrix C.

Thus we get an equality of the form

$$u = \lambda M = \nu (y_1^{b_1} \cdots y_{k-1}^{b_{k-1}}) (M_1 + M_2)^{b_k} (y_{k+1}^{b_{k+1}} \cdots y_p^{b_p}). \tag{2}$$

We know that $M_1 \neq M_2$. (Here we use that $m > 1$ and that exchange matrices are by definition connected. Otherwise, one could get exchange relations of the form $x_k x_k' = 1 + 1$.) Thus the right-hand side of Equation (2) is a non-trivial linear combination of $b_k + 1 \geq 2$ pairwise different Laurent monomials in $y_1, \dots, y_m$. This is again a contradiction, since $y_1, \dots, y_m$ are algebraically independent. □

Corollary 2.3. *For any seed $(\mathbf{x}, B)$ of $\mathcal{F}$ the following hold:*

(i) *Let y and z be non-zero elements in $\mathcal{A}(\mathbf{x}, B)$. Then y and z are associate if and only if there exist $a_{n+1}, \dots, a_p \in \mathbb{Z}$ and $\lambda \in K^\times$ with*

$$y = \lambda x_{n+1}^{a_{n+1}} \cdots x_p^{a_p} z.$$

(ii) *Let y and z be cluster variables of $\mathcal{A}(\mathbf{x}, B)$. Then y and z are associate if and only if $y = z$.*

Proof. Part (i) follows directly from Theorem 2.2. To prove (ii), let $\mathbf{y}$ and $\mathbf{z}$ be clusters of $\mathcal{A}(\mathbf{x}, B)$. Assume y_i and z_j are associate for some $1 \leq i, j \leq n$. By (i) there are $a_{n+1}, \dots, a_p \in \mathbb{Z}$ and $\lambda \in K^\times$ with $y_i = \lambda x_{n+1}^{a_{n+1}} \cdots x_p^{a_p} z_j$. By

Theorem 1.1 we know that there exist $b_1, \dots, b_n \in \mathbb{Z}$ and a polynomial f in $\mathbb{Z}[z_1, \dots, z_m]$ with

$$y_i = \frac{f}{z_1^{b_1} \cdots z_n^{b_n}}$$

and f is not divisible by any $z_1, \dots, z_n$. The polynomial f and $b_1, \dots, b_n$ are uniquely determined by y_i. It follows that $\lambda \in \mathbb{Z}$ and $a_{n+1}, \dots, a_p \geq 0$. But we also have $z_j = \lambda^{-1} x_{n+1}^{-a_{n+1}} \cdots x_p^{-a_p} y_i$. Reversing the role of y_i and z_j we get $-a_{n+1}, \dots, -a_p \geq 0$ and $\lambda^{-1} \in \mathbb{Z}$. This implies $y_i = z_j$ or $-y_i = z_j$. By the remark at the beginning of Section 1.4 we know that $z_j = f/g$ for some $f, g \in \mathbb{N}[y_1, \dots, y_m]$. Assume that $-y_i = z_j$. We get $z_j = -y_i = f/g$ and therefore $f + y_i g = 0$. This is a contradiction to the algebraic independence of $y_1, \dots, y_m$. Thus we proved (ii). □

We thank Giovanni Cerulli Irelli for helping us with the final step of the proof of Corollary 2.3(ii).

Two clusters $\mathbf{y}$ and $\mathbf{z}$ of a cluster algebra $\mathcal{A}(\mathbf{x}, B)$ are *non-associate* if there are no $1 \leq i, j \leq n$ such that y_i and z_j are associate.

Corollary 2.4. *For clusters $\mathbf{y}$ and $\mathbf{z}$ of $\mathcal{A}(\mathbf{x}, B)$ the following are equivalent:*

(i) *The clusters $\mathbf{y}$ and $\mathbf{z}$ are non-associate.*
(ii) *The clusters $\mathbf{y}$ and $\mathbf{z}$ are disjoint.*

Proof. Non-associate clusters are obviously disjoint. The converse follows directly from Corollary 2.3(ii). □

3. Irreducibility of cluster variables

In this section we prove Theorem 1.3(ii). The proof is very similar to the proof of Theorem 2.2.

Theorem 3.1. *Let $(\mathbf{x}, B)$ be a seed of $\mathcal{F}$. Then any cluster variable in $\mathcal{A}(\mathbf{x}, B)$ is irreducible.*

Proof. Let $(\mathbf{y}, C)$ be any seed of $\mathcal{A}(\mathbf{x}, B)$. We know from Theorem 2.2 that the cluster variables of $\mathcal{A}(\mathbf{x}, B)$ are non-invertible in $\mathcal{A}(\mathbf{x}, B)$.

Assume that y_k is not irreducible for some $1 \leq k \leq n$. Thus $y_k = y_k' y_k''$ for some non-invertible elements y_k' and y_k'' in $\mathcal{A}(\mathbf{x}, B)$. Since y_k is invertible in $\mathscr{L}_{\mathbf{y}}$, we know that y_k' and y_k'' are both invertible in $\mathscr{L}_{\mathbf{y}}$. Thus by Lemma 2.1 there are $a_i, b_i \in \mathbb{Z}$ and $\lambda', \lambda'' \in K^\times$ with

$$y_k' = \lambda' y_1^{a_1} \cdots y_s^{a_s} \cdots y_p^{a_p} \quad \text{and} \quad y_k'' = \lambda'' y_1^{b_1} \cdots y_s^{b_s} \cdots y_p^{b_p}.$$

Since $y_k = y_k' y_k''$, we get $a_s + b_s = 0$ for all $s \neq k$ and $a_k + b_k = 1$.

Assume that $a_s = 0$ for all $1 \leq s \leq n$ with $s \neq k$. Then $y_k' = \lambda' y_k^{a_k} y_{n+1}^{a_{n+1}} \cdots y_p^{a_p}$ and $y_k'' = \lambda'' y_k^{b_k} y_{n+1}^{b_{n+1}} \cdots y_p^{b_p}$. If $a_k \leq 0$, then y_k' is invertible in $\mathcal{A}(\mathbf{x}, B)$, and if $a_k > 0$, then y_k'' is invertible in $\mathcal{A}(\mathbf{x}, B)$. In both cases we get a contradiction.

Next assume $a_s \neq 0$ for some $1 \leq s \leq n$ with $s \neq k$. Let $y_s^* := \mu_{(\mathbf{y},C)}(y_s)$. Thus we have

$$y_s^* = M_1 + M_2$$

with

$$M_1 = y_s^{-1} \prod_{c_{is} > 0} y_i^{c_{is}} \quad \text{and} \quad M_2 = y_s^{-1} \prod_{c_{is} < 0} y_i^{-c_{is}},$$

where the products run over the positive, respectively negative, entries in the sth column of the matrix C.

Since $s \neq k$, we see that y_k and therefore also y_k' and y_k'' are invertible in $\mathscr{L}_{\mu_s(\mathbf{y},C)}$. Thus by Lemma 2.1 there are $c_i, d_i \in \mathbb{Z}$ and $\nu', \nu'' \in K^\times$ with

$$y_k' = \nu' y_1^{c_1} \cdots y_{s-1}^{c_{s-1}} (y_s^*)^{c_s} y_{s+1}^{c_{s+1}} \cdots y_p^{c_p}$$

and

$$y_k'' = \nu'' y_1^{d_1} \cdots y_{s-1}^{d_{s-1}} (y_s^*)^{d_s} y_{s+1}^{d_{s+1}} \cdots y_p^{d_p}.$$

Note that $c_s + d_s = 0$. Without loss of generality we assume that $c_s \geq 0$. (If $c_s < 0$, we continue to work with y_k'' instead of y_k'.) If $c_s = 0$, we get

$$y_k' = \lambda' y_1^{a_1} \cdots y_s^{a_s} \cdots y_p^{a_p} = \nu' y_1^{c_1} \cdots y_s^0 \cdots y_p^{c_p}.$$

This is a contradiction, since $a_s \neq 0$ and $y_1, \ldots, y_m$ are algebraically independent. If $c_s > 0$, then

$$\begin{aligned} y_k' &= \lambda' y_1^{a_1} \cdots y_{s-1}^{a_{s-1}} y_s^{a_s} y_{s+1}^{a_{s+1}} \cdots y_p^{a_p} \\ &= \nu' y_1^{c_1} \cdots y_{s-1}^{c_{s-1}} (y_s^*)^{c_s} y_{s+1}^{c_{s+1}} \cdots y_p^{c_p} \\ &= \nu' y_1^{c_1} \cdots y_{s-1}^{c_{s-1}} (M_1 + M_2)^{c_s} y_{s+1}^{c_{s+1}} \cdots y_p^{c_p}. \end{aligned}$$

We know that $M_1 \neq M_2$. Thus the Laurent monomial y_k' is a non-trivial linear combination of $c_s + 1 \geq 2$ pairwise different Laurent monomials in $y_1, \ldots, y_m$, a contradiction. $\square$

Note that the coefficients $x_{p+1}, \ldots, x_m$ of $\mathcal{A}(\mathbf{x}, B)$ are obviously irreducible in $\mathscr{L}_{\mathbf{x}}$. Since $\mathcal{A}(\mathbf{x}, B) \subseteq \mathscr{L}_{\mathbf{x}}$, they are also irreducible in $\mathcal{A}(\mathbf{x}, B)$.

4. Factorial cluster algebras

4.1. A factoriality criterion.

This section contains the proofs of Theorem 1.4 and Corollary 1.5.

Theorem 4.1. *Let $\mathbf{y}$ and $\mathbf{z}$ be disjoint clusters of $\mathcal{A}(\mathbf{x}, B)$, and let U be a factorial subalgebra of $\mathcal{A}(\mathbf{x}, B)$ such that*

$$\{y_1, \ldots, y_n, z_1, \ldots, z_n, x_{n+1}^{\pm 1}, \ldots, x_p^{\pm 1}, x_{p+1}, \ldots, x_m\} \subset U.$$

Then we have

$$U = \mathcal{A}(\mathbf{x}, B) = U(\mathbf{y}, \mathbf{z}).$$

Proof. Let $u \in U(\mathbf{y}, \mathbf{z}) = \mathscr{L}_{\mathbf{y}} \cap \mathscr{L}_{\mathbf{z}}$. Thus we have

$$u = \frac{f}{y_1^{a_1} y_2^{a_2} \cdots y_p^{a_p}} = \frac{g}{z_1^{b_1} z_2^{b_2} \cdots z_p^{b_p}},$$

where f is a polynomial in $y_1, \dots, y_m$, and g is a polynomial in $z_1, \dots, z_m$, and $a_i, b_i \geq 0$ for all $1 \leq i \leq p$. By the Laurent phenomenon it is enough to show that $u \in U$.

Since $y_i, z_i \in U$ for all $1 \leq i \leq m$, we get the identity

$$f z_1^{b_1} z_2^{b_2} \cdots z_n^{b_n} z_{n+1}^{b_{n+1}} \cdots z_p^{b_p} = g y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n} y_{n+1}^{a_{n+1}} \cdots y_p^{a_p}$$

in U.

By Theorem 3.1 the cluster variables y_i and z_i with $1 \leq i \leq n$ are irreducible in $\mathcal{A}(\mathbf{x}, B)$. In particular, they are irreducible in the subalgebra U of $\mathcal{A}(\mathbf{x}, B)$. The elements $y_{n+1}^{a_{n+1}} \cdots y_p^{a_p}$ and $z_{n+1}^{b_{n+1}} \cdots z_p^{b_p}$ are units in U. (Recall that $x_i = y_i = z_i$ for all $n+1 \leq i \leq m$.)

The clusters $\mathbf{y}$ and $\mathbf{z}$ are disjoint. Now Corollary 2.4 implies that the elements y_i and z_j are non-associate for all $1 \leq i, j \leq n$. Thus, by the factoriality of U, the monomial $y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n}$ divides f in U. In other words there is some $h \in U$ with $f = h y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n}$. It follows that

$$u = \frac{f}{y_1^{a_1} y_2^{a_2} \cdots y_p^{a_p}} = \frac{h y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n}}{y_1^{a_1} y_2^{a_2} \cdots y_p^{a_p}} = \frac{h}{y_{n+1}^{a_{n+1}} \cdots y_p^{a_p}} = h y_{n+1}^{-a_{n+1}} \cdots y_p^{-a_p}.$$

Since $h \in U$ and $y_{n+1}^{\pm 1}, \dots, y_p^{\pm 1} \in U$, we get $u \in U$. This finishes the proof. $\square$

Corollary 4.2. *Assume that $\mathcal{A}(\mathbf{x}, B)$ is factorial.*

(i) *If $\mathbf{y}$ and $\mathbf{z}$ are disjoint clusters of $\mathcal{A}(\mathbf{x}, B)$, then $\mathcal{A}(\mathbf{x}, B) = U(\mathbf{y}, \mathbf{z})$.*
(ii) *For any $(\mathbf{y}, C) \sim (\mathbf{x}, B)$ we have $\mathcal{A}(\mathbf{x}, B) = U(\mathbf{y}, C)$.*

Proof. Part (i) follows directly from Theorem 1.4. To prove part (ii), assume $(\mathbf{y}, C) \sim (\mathbf{x}, B)$ and let $u \in U(\mathbf{y}, C)$. For $1 \leq k \leq n$ let $(\mathbf{y}_k, C_k) := \mu_k(\mathbf{y}, C)$ and $y_k^* := \mu_{(\mathbf{y}, C)}(y_k)$. We get

$$u = \frac{f}{y_1^{a_1} \cdots y_k^{a_k} \cdots y_p^{a_p}} = \frac{f_k}{y_1^{b_1} \cdots (y_k^*)^{b_k} \cdots y_p^{b_p}}$$

for a polynomial f in $y_1, \dots, y_k, \dots, y_m$, a polynomial f_k in $y_1, \dots, y_k^*, \dots, y_m$, and $a_i, b_i \geq 0$. This yields an equality

$$f y_1^{b_1} \cdots (y_k^*)^{b_k} \cdots y_p^{b_p} = f_k y_1^{a_1} \cdots y_k^{a_k} \cdots y_p^{a_p} \tag{3}$$

in $\mathcal{A}(\mathbf{x}, B)$. Now we argue similarly as in the proof of Theorem 4.1. The cluster variables $y_1, \dots, y_n, y_1^*, \dots, y_n^*$ are obviously pairwise different. Now

Corollary 2.3(ii) implies that they are pairwise non-associate, and by Theorem 3.1 they are irreducible in $\mathcal{A}(\mathbf{x}, B)$. Thus by the factoriality of $\mathcal{A}(\mathbf{x}, B)$, Equation (3) implies that $y_k^{a_k}$ divides f in $\mathcal{A}(\mathbf{x}, B)$. Since this holds for all $1 \leq k \leq n$, we get that $y_1^{a_1} \cdots y_k^{a_k} \cdots y_n^{a_n}$ divides f in $\mathcal{A}(\mathbf{x}, B)$. It follows that $u \in \mathcal{A}(\mathbf{x}, B)$. □

4.2. Existence of disjoint clusters. One assumption of Theorem 4.1 is the existence of disjoint clusters in $\mathcal{A}(\mathbf{x}, B)$. We can prove this under a mild assumption. But it should be true in general.

Proposition 4.3. *Assume that the cluster monomials of $\mathcal{A}(\mathbf{x}, B)$ are linearly independent. Let $(\mathbf{y}, C)$ be a seed of $\mathcal{A}(\mathbf{x}, B)$, and let*

$$(\mathbf{z}, D) := \mu_n \cdots \mu_2 \mu_1(\mathbf{y}, C).$$

Then the clusters $\mathbf{y}$ and $\mathbf{z}$ are disjoint.

Proof. Set $(\mathbf{y}[0], C[0]) := (\mathbf{y}, C)$, and for $1 \leq k \leq n$ let $(\mathbf{y}[k], C[k]) := \mu_k(\mathbf{y}[k-1], C[k-1])$ and $(y_1[k], \ldots, y_m[k]) := \mathbf{y}[k]$. We claim that

$$\{y_1[k], \ldots, y_k[k]\} \cap \{y_1, \ldots, y_n\} = \varnothing.$$

For $k = 1$ this is straightforward. Thus let $k \geq 2$, and assume that our claim is true for $k-1$. To get a contradiction, assume that $y_k[k] = y_j$ for some $1 \leq j \leq n$. (By the induction assumption we know that $\{y_1[k], \ldots, y_{k-1}[k]\} \cap \{y_1, \ldots, y_n\} = \varnothing$, since $y_i[k] = y_i[k-1]$ for all $1 \leq i \leq k-1$.)

We have $\mathbf{y}[k] = (y_1[k], \ldots, y_k[k], y_{k+1}, \ldots, y_m)$.

Since $y_1[k], \ldots, y_k[k], y_{k+1}, \ldots, y_m$ are algebraically independent and $y_k[k] \neq y_k$, we get $1 \leq j \leq k-1$. Since $(\mathbf{y}[j], C[j]) = \mu_j(\mathbf{y}[j-1], C[j-1])$, it follows that $(y_j[j-1], y_j[j])$ is an exchange pair of $\mathcal{A}(\mathbf{x}, B)$. Next, observe that $y_k[k] = y_j = y_j[j-1]$ and $y_j[k] = y_j[j]$. Thus $y_j[j-1]$ and $y_j[j]$ are both contained in $\{y_1[k], \ldots, y_m[k]\}$, and therefore $y_j[j-1]y_j[j]$ is a cluster monomial. The corresponding exchange relation gives a contradiction to the linear independence of cluster monomials. □

Fomin and Zelevinsky [FZ3, Conjecture 4.16] conjecture that the cluster monomials of $\mathcal{A}(\mathbf{x}, B)$ are always linearly independent. Under the assumptions that B has maximal rank and that B° is skew-symmetric, the conjecture follows from [DWZ, Theorem 1.7].

5. The divisibility group of a cluster algebra

Let R be an integral domain, and let $\mathrm{Frac}(R)$ be the field of fractions of R. Set $\mathrm{Frac}(R)^* := \mathrm{Frac}(R) \setminus \{0\}$. The abelian group

$$G(R) := (\mathrm{Frac}(R)^*/R^\times, \cdot)$$

is the *divisibility group* of R.

For $g, h \in \mathrm{Frac}(R)^*$ let $g \leq h$ provided $hg^{-1} \in R$. This relation is reflexive and transitive and it induces a partial ordering on $G(R)$.

Let I be a set. The abelian group $(\mathbb{Z}^{(I)}, +)$ is equipped with the following partial ordering: We set $(x_i)_{i\in I} \leq (y_i)_{i\in I}$ if $x_i \leq y_i$ for all i. (By definition, the elements in $\mathbb{Z}^{(I)}$ are tuples $(x_i)_{i\in I}$ of integers x_i such that only finitely many x_i are non-zero.)

There is the following well-known criterion for the factoriality of R, see for example [C, Section 2].

Proposition 5.1. *For an integral domain R the following are equivalent:*

(i) *R is factorial.*
(ii) *There is a set I and a group isomorphism*
$$\phi\colon G(R) \to \mathbb{Z}^{(I)}$$
such that for all $g, h \in G(R)$ we have $g \leq h$ if and only if $\phi(g) \leq \phi(h)$.

Not all cluster algebras $\mathcal{A}(\mathbf{x}, B)$ are factorial, but at least one part of the above factoriality criterion is satisfied:

Proposition 5.2. *For any seed $(\mathbf{x}, B)$ of $\mathcal{F}$ the divisibility group $G(\mathcal{A}(\mathbf{x}, B))$ is isomorphic to $\mathbb{Z}^{(I)}$, where*
$$I := \{f \in K[x_1, \ldots, x_m] \mid f \text{ is irreducible and } f \neq x_i \text{ for } n+1 \leq i \leq p\}/K^\times$$
is the set of irreducible polynomials unequal to any $x_{n+1}, \ldots, x_p$ in $K[x_1, \ldots, x_m]$ up to non-zero scalar multiples.

Proof. By the Laurent phenomenon and the definition of a seed we get
$$\mathrm{Frac}(\mathcal{A}(\mathbf{x}, B)) = \mathrm{Frac}(\mathscr{L}_{\mathbf{x}}) = K(x_1, \ldots, x_m).$$
Furthermore, by Theorem 2.2 we have
$$\mathcal{A}(\mathbf{x}, B)^\times = \{\lambda x_{n+1}^{a_{n+1}} \cdots x_p^{a_p} \mid \lambda \in K^\times, a_i \in \mathbb{Z}\}.$$
Any element in $K(x_1, \ldots, x_m)$ is of the form $f_1 \cdots f_s g_1^{-1} \cdots g_t^{-1}$ with f_i, g_j irreducible in $K[x_1, \ldots, x_m]$. Using that the polynomial ring $K[x_1, \ldots, x_m]$ is factorial, and working modulo $\mathcal{A}(\mathbf{x}, B)^\times$ yields the result. $\square$

6. Examples of non-factorial cluster algebras

6.1. For a matrix $A \in M_{m,n}(\mathbb{Z})$ and $1 \leq i \leq n$ let $c_i(A)$ be the ith column of A.

Proposition 6.1. *Let $(\mathbf{x}, B)$ be a seed of $\mathcal{F}$. Assume that $c_k(B) = c_s(B)$ or $c_k(B) = -c_s(B)$ for some $k \neq s$ with $b_{ks} = 0$. Then $\mathcal{A}(\mathbf{x}, B)$ is not factorial.*

Proof. Define $(\mathbf{y}, C) := \mu_k(\mathbf{x}, B)$ and $(\mathbf{z}, D) := \mu_s(\mathbf{y}, C)$. We get

$$y_k = z_k = x_k^{-1}(M_1 + M_2),$$

where

$$M_1 := \prod_{b_{ik}>0} x_i^{b_{ik}} \quad \text{and} \quad M_2 := \prod_{b_{ik}<0} x_i^{-b_{ik}}.$$

By the mutation rule, we have $c_k(C) = -c_k(B)$, and since $b_{ks} = 0$, we get $c_s(C) = c_s(B)$. Since $c_s(B) = c_k(B)$ or $c_s(B) = -c_k(B)$, this implies that

$$z_s = x_s^{-1}(M_1 + M_2).$$

The cluster variables x_k, x_s, z_k, z_s are pairwise different. Thus they are pairwise non-associate by Corollary 2.3(ii), and by Theorem 3.1 they are irreducible in $\mathcal{A}(\mathbf{x}, B)$. Obviously, we have

$$x_k z_k = x_s z_s.$$

Thus $\mathcal{A}(\mathbf{x}, B)$ is not factorial. □

To give a concrete example of a cluster algebra, which is not factorial, assume $m = n = p = 3$, and let $B \in M_{m,n}(\mathbb{Z})$ be the matrix

$$B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The matrix B obviously satisfies the assumptions of Proposition 6.1. Note that $B = B^\circ$ is skew-symmetric, and that $\Gamma(B)$ is the quiver

$$3 \longrightarrow 2 \longrightarrow 1.$$

Thus $\mathcal{A}(\mathbf{x}, B)$ is a cluster algebra of Dynkin type $\mathbb{A}_3$. (Cluster algebras with finitely many cluster variables are classified via Dynkin types, for details see [FZ2].)

Define $(\mathbf{z}, D) := \mu_3\mu_1(\mathbf{x}, B)$. We get $z_1 = x_1^{-1}(1 + x_2)$, $z_3 = x_3^{-1}(1 + x_2)$ and therefore $x_1 z_1 = x_3 z_3$.

Clearly, the cluster variables x_1, x_3, z_1, z_3 are pairwise different. Using Corollary 2.3(ii) we get that x_1, x_3, z_1, z_3 are pairwise non-associate, and by Theorem 3.1 they are irreducible. Thus $\mathcal{A}(\mathbf{x}, B)$ is not factorial.

6.2. The next example is due to Philipp Lampe. It gives a negative answer to Zelevinsky's Question 1.6.

Proposition 6.2 ([La]). *Let $K = \mathbb{C}$, $m = n = 2$ and*

$$B = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}.$$

Then $\mathcal{A}(\mathbf{x}, B)$ is not factorial.

The proof of the following result is a straightforward generalization of Lampe's proof of Proposition 6.2.

PROPOSITION 6.3. *Let* $(\mathbf{x}, B)$ *be a seed of* $\mathcal{F}$. *Assume that there exists some* $1 \leq k \leq n$ *such that the polynomial* $X^d + Y^d$ *is not irreducible in* $K[X,Y]$, *where* $d := \gcd(b_{1k}, \dots, b_{mk})$ *is the greatest common divisor of* $b_{1k}, \dots, b_{mk}$. *Then* $\mathcal{A}(\mathbf{x}, B)$ *is not factorial.*

Proof. Let $X^d + Y^d = f_1 \cdots f_t$, where the f_j are irreducible polynomials in $K[X,Y]$. Since $X^d + Y^d$ is not irreducible in $K[X,Y]$, we have $t \geq 2$. Let $y_k := \mu_{(\mathbf{x},B)}(x_k)$. The corresponding exchange relation is

$$x_k y_k = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}} = M^d + N^d = \prod_{j=1}^{t} f_j(M,N),$$

where

$$M := \prod_{b_{ik}>0} x_i^{b_{ik}/d} \quad \text{and} \quad N := \prod_{b_{ik}<0} x_i^{-b_{ik}/d}.$$

Clearly, each $f_j(M,N)$ is contained in $\mathcal{A}(\mathbf{x}, B)$. To get a contradiction, assume that $\mathcal{A}(\mathbf{x}, B)$ is factorial. By Theorem 2.2 none of the elements $f_j(M,N)$ is invertible in $\mathcal{A}(\mathbf{x}, B)$. Since $\mathcal{A}(\mathbf{x}, B)$ is factorial, each $f_j(M,N)$ is equal to a product $f_{1j} \cdots f_{a_j j}$, where the f_{ij} are irreducible in $\mathcal{A}(\mathbf{x}, B)$ and $a_j \geq 1$. By Theorem 3.1 the cluster variables x_k and y_k are irreducible in $\mathcal{A}(\mathbf{x}, B)$. It follows that $a_1 + \cdots + a_t = 2$, since $\mathcal{A}(\mathbf{x}, B)$ is factorial. This implies $t = 2$ and $a_1 = a_2 = 1$. In particular, $f_1(M,N)$ and $f_2(M,N)$ are irreducible in $\mathcal{A}(\mathbf{x}, B)$, and we have $x_k y_k = f_1(M,N) f_2(M,N)$. For $j = 1, 2$ the elements x_k and $f_j(M,N)$ cannot be associate, since $f_j(M,N)$ is just a K-linear combination of monomials in $\{x_1, \dots, x_m\} \setminus \{x_k\}$. (Here we use Corollary 2.3(i) and the fact that $b_{kk} = 0$.) This is a contradiction to the factoriality of $\mathcal{A}(\mathbf{x}, B)$. □

Note that a polynomial of the form $X^d + Y^d$ is irreducible if and only if $X^d + 1$ is irreducible.

COROLLARY 6.4. *Let* $K = \mathbb{C}$, $m = n = 2$ *and*

$$B = \begin{pmatrix} 0 & -c \\ d & 0 \end{pmatrix}$$

with $c \geq 1$ *and* $d \geq 2$. *Then* $\mathcal{A}(\mathbf{x}, B)$ *is not factorial.*

Proof. For $k = 1$ the assumptions of Proposition 6.3 hold. (We have $\gcd(0, d) = d$, and the polynomial $X^d + 1$ is not irreducible in $\mathbb{C}[X]$.) □

COROLLARY 6.5. *Let* $m = n = 2$ *and*

$$B = \begin{pmatrix} 0 & -c \\ d & 0 \end{pmatrix}$$

with $c \geq 1$ *and* $d \geq 3$ *an odd number. Then* $\mathcal{A}(\mathbf{x}, B)$ *is not factorial.*

Proof. For $k = 1$ the assumptions of Proposition 6.3 hold. (We have $\gcd(0, d) = d$, and for odd d we have

$$X^d + 1 = (X + 1) \left(\sum_{j=0}^{d-1} (-1)^j X^j \right).$$

Thus $X^d + 1$ is not irreducible in $K[X]$.) $\square$

7. Examples of factorial cluster algebras

7.1. Cluster algebras of Dynkin type $\mathbb{A}$ as polynomial rings. Assume $m = n + 1 = p + 1$, and let $B \in M_{m,n}(\mathbb{Z})$ be the matrix

$$B = \left(\begin{array}{cccccc} 0 & -1 & & & & \\ 1 & 0 & -1 & & & \\ & 1 & 0 & \ddots & & \\ & & 1 & \ddots & -1 & \\ & & & \ddots & 0 & -1 \\ & & & & 1 & 0 \\ \hline & & & & & 1 \end{array} \right).$$

Obviously, B° is skew-symmetric, $\Gamma(B)$ is the quiver

$$m \longrightarrow \cdots \longrightarrow 2 \longrightarrow 1,$$

and $\mathcal{A}(\mathbf{x}, B)$ is a cluster algebra of Dynkin type $\mathbb{A}_n$. Note that $\mathcal{A}(\mathbf{x}, B)$ has exactly one coefficient, and that this coefficient is non-invertible.

Let $(\mathbf{x}[0], B[0]) := (\mathbf{x}, B)$. For each $1 \leq i \leq m - 1$ we define inductively a seed by

$$(\mathbf{x}[i], B[i]) := \mu_{m-i} \cdots \mu_2 \mu_1 (\mathbf{x}[i-1], B[i-1]).$$

For $0 \leq i \leq m - 1$ set $(x_1[i], \dots, x_m[i]) := \mathbf{x}[i]$.

For simplicity we define $x_0[i] := 1$ and $x_{-1}[i] := 0$ for all i.

Lemma 7.1. *For $0 \leq i \leq m - 2$, $1 \leq k \leq m - 1 - i$ and $0 \leq j \leq i$ we have*

$$\mu_{(\mathbf{x}[i], B[i])}(x_k[i]) = \frac{x_{k-1}[i] + x_{k+1}[i]}{x_k[i]} = \frac{x_{k-1+j}[i-j] + x_{k+1+j}[i-j]}{x_{k+j}[i-j]}. \tag{4}$$

Proof. The first equality follows from the definition of $(\mathbf{x}[i], B[i])$ and the mutation rule. The second equality is proved by induction on i. $\square$

Corollary 7.2. *For $0 \leq i \leq m - 2$ we have*

$$x_{i+2} = x_1[i+1] x_{i+1} - x_i, \tag{5}$$

$$x_{i+1}[1] = x_1[i+1] x_i[1] - x_{i-1}[1]. \tag{6}$$

Proof. Equation (5) follows from (4) for $k = 1$ and $j = i$. The case $k = 1$ and $j = i - 1$ yields Equation (6). □

PROPOSITION 7.3. *The elements $x_1[0], x_1[1], \dots, x_1[m-1]$ are algebraically independent and*

$$K[x_1[0], x_1[1], \dots, x_1[m-1]] = \mathcal{A}(\mathbf{x}, B).$$

In particular, the cluster algebra $\mathcal{A}(\mathbf{x}, B)$ is a polynomial ring in m variables.

Proof. It follows from Equation (5) that

$$x_1[i] \in K(x_1, \dots, x_{i+1}) \setminus K(x_1, \dots, x_i)$$

for all $1 \leq i \leq m-1$. Since $x_1, \dots, x_m$ are algebraically independent, this implies that $x_1[0], x_1[1], \dots, x_1[m-1]$ are algebraically independent as well. Thus

$$U := K[x_1[0], x_1[1], \dots, x_1[m-1]]$$

is a polynomial ring in m variables. In particular, U is factorial. Equation (5) implies that $x_1, \dots, x_m \in U$, and Equation (6) yields that $x_1[1], \dots, x_m[1] \in U$. Clearly, the clusters $\mathbf{x}$ and $\mathbf{x}[1]$ are disjoint. Thus the assumptions of Theorem 4.1 are satisfied, and we get $U = \mathcal{A}(\mathbf{x}, B)$. □

The cluster algebra $\mathcal{A}(\mathbf{x}, B)$ as defined above has been studied by several people. It is related to a T-system of Dynkin type $\mathbb{A}_1$ with a certain boundary condition, see [DK]. Furthermore, for $K = \mathbb{C}$ the cluster algebra $\mathcal{A}(\mathbf{x}, B)$ is naturally isomorphic to the complexified Grothendieck ring of the category $\mathcal{C}_n$ of finite-dimensional modules of level n over the quantum loop algebra of Dynkin type $\mathbb{A}_1$, see [HL, N2]. It is well known, that $\mathcal{A}(\mathbf{x}, B)$ is a polynomial ring. We just wanted to demonstrate how to use Theorem 4.1 in practise.

7.2. ACYCLIC CLUSTER ALGEBRAS AS POLYNOMIAL RINGS. Let $C = (c_{ij}) \in M_{n,n}(\mathbb{Z})$ be a *generalized Cartan matrix*, i.e. C is symmetrizable, $c_{ii} = 2$ for all i and $c_{ij} \leq 0$ for all $i \neq j$.

Assume that $m = 2n = 2p$, and let $(\mathbf{x}, B)$ be a seed of $\mathcal{F}$, where $B = (b_{ij}) \in M_{2n,n}(\mathbb{Z})$ is defined as follows: For $1 \leq i \leq 2n$ and $1 \leq j \leq n$ let

$$b_{ij} := \begin{cases} 0 & \text{if } i = j, \\ -c_{ij} & \text{if } 1 \leq i < j \leq n, \\ c_{ij} & \text{if } 1 \leq j < i \leq n, \\ 1 & \text{if } i = n + j, \\ c_{i-n,j} & \text{if } n+1 \leq i \leq 2n \text{ and } i - n < j, \\ 0 & \text{if } n+1 \leq i \leq 2n \text{ and } i - n > j. \end{cases}$$

Thus we have

$$B = \left(\begin{array}{ccccc} 0 & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & 0 & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & b_{n-1,n} \\ b_{n1} & b_{n2} & \cdots & b_{n,n-1} & 0 \\ \hline 1 & -b_{12} & -b_{13} & \cdots & -b_{1n} \\ 0 & 1 & -b_{23} & \cdots & -b_{2n} \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -b_{n-1,n} \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Clearly, $(\mathbf{x}, B)$ is an acyclic seed. Namely, if $i \to j$ is an arrow in $\Sigma(B)$, then $i < j$. Up to simultaneous reordering of columns and rows, each acyclic skew-symmetrizable matrix in $M_{n,n}(\mathbb{Z})$ is of the form B° with B defined as above. Note that $\mathcal{A}(\mathbf{x}, B)$ has exactly n coefficients, and that all these coefficients are non-invertible.

For $1 \le i \le n$ let

$$(\mathbf{x}[1], B[1]) := \mu_n \cdots \mu_2\mu_1(\mathbf{x}, B)$$

and $(x_1[1], \dots, x_{2n}[1]) := \mathbf{x}[1]$. Let $B_0 := B$, and for $1 \le i \le n$ let $B_i := \mu_i(B_{i-1})$. Thus we have $B_n = B[1]$. It is easy to work out the matrices B_i explicitly: The matrix B_i is obtained from B_{i-1} by changing the sign in the ith row and the ith column of the principal part B_{i-1}°. Furthermore, the $(n+i)$th row

$$(0, \dots, 0, 1, -b_{i,i+1}, -b_{i,i+2}, \dots, -b_{in})$$

of B_{i-1} gets replaced by

$$(-b_{i1}, -b_{i2}, \dots, -b_{i,i-1}, -1, 0, \dots, 0).$$

If we write N_+ (resp. N_-) for the upper (resp. lower) triangular part of B°, we get

$$B = \left(\begin{array}{ccc} 1 & & N_+ \\ & \ddots & \\ N_- & & 1 \\ \hline 1 & & -N_+ \\ & \ddots & \\ 0 & & 1 \end{array}\right) \quad \text{and} \quad B[1] = \left(\begin{array}{ccc} 1 & & N_+ \\ & \ddots & \\ N_- & & 1 \\ \hline -1 & & 0 \\ & \ddots & \\ -N_- & & -1 \end{array}\right).$$

In particular, the principal part B° of B is equal to the principal part $B[1]^\circ$ of $B[1]$.

Now the definition of seed mutation yields

$$x_k[1] = x_k^{-1}\left(x_{n+k} + \prod_{i=1}^{k-1} x_i[1]^{b_{ik}} \prod_{i=k+1}^{n} x_i^{-b_{ik}}\right) \tag{7}$$

for $1 \leq k \leq n$.

Proposition 7.4. *The elements $x_1, \ldots, x_n, x_1[1], \ldots, x_n[1]$ are algebraically independent and*

$$K[x_1, \ldots, x_n, x_1[1], \ldots, x_n[1]] = \mathcal{A}(\mathbf{x}, B).$$

In particular, the cluster algebra $\mathcal{A}(\mathbf{x}, B)$ is a polynomial ring in $2n$ variables.

Proof. By Equation (7) and induction we have

$$x_k[1] \in K(x_1, \ldots, x_{n+k}) \setminus K(x_1, \ldots, x_{n+k-1})$$

for all $1 \leq k \leq n$. It follows that $x_1, \ldots, x_n, x_1[1], \ldots, x_n[1]$ are algebraically independent, and that the clusters $\mathbf{x}$ and $\mathbf{x}[1]$ are disjoint. Let

$$U := K[x_1, \ldots, x_n, x_1[1], \ldots, x_n[1]].$$

Thus U is a polynomial ring in $2n$ variables. In particular, U is factorial. It follows from Equation (7) that

$$x_{n+k} = x_k[1]x_k - \prod_{i=1}^{k-1} x_i[1]^{b_{ik}} \prod_{i=k+1}^{n} x_i^{-b_{ik}}. \tag{8}$$

This implies $x_{n+k} \in U$ for all $1 \leq k \leq n$. Thus the assumptions of Theorem 4.1 are satisfied, and we can conclude that $U = \mathcal{A}(\mathbf{x}, B)$. □

Proposition 7.4 is a special case of a much more general result proved in [GLS2]. But the proof presented here is new and more elementary.

Next, we compare the basis

$$\mathcal{P}_{\mathrm{GLS}} := \left\{x[\mathbf{a}] := x_1^{a_1} \cdots x_n^{a_n} x_1[1]^{a_{n+1}} \cdots x_n[1]^{a_{2n}} \mid \mathbf{a} = (a_1, \ldots, a_{2n}) \in \mathbb{N}^{2n}\right\}$$

of $\mathcal{A}(\mathbf{x}, B)$ resulting from Proposition 7.4 with a basis constructed by Berenstein, Fomin and Zelevinsky [BFZ]. For $1 \leq k \leq n$ let

$$x'_k := \mu_{(\mathbf{x},B)}(x_k) = x_k^{-1}\left(x_{n+k}\prod_{i=1}^{k-1} x_i^{b_{ik}} + \prod_{i=k+1}^{n} x_i^{-b_{ik}} \prod_{i=1}^{k-1} x_{n+i}^{b_{ik}}\right) \tag{9}$$

and set

$$\mathcal{P}_{\mathrm{BFZ}} := \{x'[\mathbf{a}] := x_1^{a_1} \cdots x_n^{a_n} x_{n+1}^{a_{n+1}} \cdots x_{2n}^{a_{2n}} (x'_1)^{a_{2n+1}} \cdots (x'_n)^{a_{3n}} \mid \\ \mathbf{a} = (a_1, \ldots, a_{3n}) \in \mathbb{N}^{3n}, a_k a_{2n+k} = 0 \text{ for } 1 \leq k \leq n\}.$$

Proposition 7.5 ([BFZ, Corollary 1.21]). *The set $\mathcal{P}_{\mathrm{BFZ}}$ is a basis of $\mathcal{A}(\mathbf{x}, B)$.*

Note that the basis $\mathcal{P}_{\mathrm{GLS}}$ is constructed by using cluster variables from two seeds, namely $(\mathbf{x}, B)$ and $\mu_n \cdots \mu_1(\mathbf{x}, B)$, whereas $\mathcal{P}_{\mathrm{BFZ}}$ uses cluster variables from $n+1$ seeds, namely $(\mathbf{x}, B)$ and $\mu_k(\mathbf{x}, B)$, where $1 \leq k \leq n$.

Now we insert Equation (8) into Equation (9) and obtain

$$x_k x_k' = \left(x_k[1]x_k - \prod_{i=1}^{k-1} x_i[1]^{b_{ik}} \prod_{i=k+1}^{n} x_i^{-b_{ik}} \right) \prod_{i=1}^{k-1} x_i^{b_{ik}} + \prod_{i=k+1}^{n} x_i^{-b_{ik}} \prod_{i=1}^{k-1} \left(x_i[1]x_i - \prod_{j=1}^{i-1} x_j[1]^{b_{ji}} \prod_{j=i+1}^{n} x_j^{-b_{ji}} \right)^{b_{ik}}. \tag{10}$$

Then we observe that the right-hand side of Equation (10) is divisible by x_k and that x_k' is a polynomial in $x_1, \ldots, x_n, x_1[1], \ldots, x_n[1]$. Thus we can express every element of the basis $\mathcal{P}_{\mathrm{BFZ}}$ explicitely as a linear combination of vectors from the basis $\mathcal{P}_{\mathrm{GLS}}$.

One could use Equation (10) to get an alternative proof of Proposition 7.4 as pointed out by Zelevinsky [Z]. Vice versa, using Propostion 7.4 yields another proof that $\mathcal{P}_{\mathrm{BFZ}}$ is a basis.

As an illustration, for $n = 3$ the matrices B_i look as follows:

$$B_0 = \left(\begin{array}{ccc} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \\ \hline 1 & -b_{12} & -b_{13} \\ 0 & 1 & -b_{23} \\ 0 & 0 & 1 \end{array}\right), \qquad B_1 = \left(\begin{array}{ccc} 0 & -b_{12} & -b_{13} \\ -b_{21} & 0 & b_{23} \\ -b_{31} & b_{32} & 0 \\ \hline -1 & 0 & 0 \\ 0 & 1 & -b_{23} \\ 0 & 0 & 1 \end{array}\right),$$

$$B_2 = \left(\begin{array}{ccc} 0 & b_{12} & -b_{13} \\ b_{21} & 0 & -b_{23} \\ -b_{31} & -b_{32} & 0 \\ \hline -1 & 0 & 0 \\ -b_{21} & -1 & 0 \\ 0 & 0 & 1 \end{array}\right), \qquad B_3 = \left(\begin{array}{ccc} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \\ \hline -1 & 0 & 0 \\ -b_{21} & -1 & 0 \\ -b_{31} & -b_{32} & -1 \end{array}\right).$$

For example, for

$$B = B_0 = \left(\begin{array}{ccc} 0 & 2 & 0 \\ -2 & 0 & 1 \\ 0 & -1 & 0 \\ \hline 1 & -2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right)$$

the quivers $\Gamma(B_0)$ and $\Gamma(B_3)$ look as follows:

$$\Gamma(B_0)\colon\quad \begin{tikzcd}[column sep=small]
 & & 4 \arrow[rr] & & 1 \arrow[dl, shift left] \arrow[dl, shift right] \\
 & 5 \arrow[rr] & & 2 \arrow[ul, shift left] \arrow[ul, shift right] \arrow[dl] & \\
6 \arrow[rr] & & 3 \arrow[ul] & &
\end{tikzcd}
\qquad
\Gamma(B_3)\colon\quad \begin{tikzcd}[column sep=small]
 & & 4 & & 1 \arrow[ll] \arrow[dl, shift left] \arrow[dl, shift right] \\
 & 5 \arrow[urrr, shift left] \arrow[urrr, shift right] & & 2 \arrow[ll] \arrow[dl] & \\
6 \arrow[urrr] & & 3 \arrow[ll] & &
\end{tikzcd}$$

The cluster algebra $\mathcal{A}(\mathbf{x}, B)$ is a polynomial ring in the 6 variables x_1, x_2, x_3 and

$$\begin{aligned}
x_1[1] &= \frac{x_2^2 + x_4}{x_1},\\
x_2[1] &= \frac{x_2^4 x_3 + 2x_2^2 x_3 x_4 + x_3 x_4^2 + x_1^2 x_5}{x_1^2 x_2},\\
x_3[1] &= \frac{x_2^4 x_3 + 2x_2^2 x_3 x_4 + x_3 x_4^2 + x_1^2 x_5 + x_1^2 x_2 x_6}{x_1^2 x_2 x_3}.
\end{aligned}$$

7.3. Cluster algebras arising in Lie theory as polynomial rings. The next class of examples can be seen as a fusion of the examples discussed in Sections 7.1 and 7.2. In the following we use the same notation as in [GLS2].

Let $C \in M_{n,n}(\mathbb{Z})$ be a symmetric generalized Cartan matrix, and let $\mathfrak{g}$ be the associated Kac-Moody Lie algebra over $K = \mathbb{C}$ with triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$, see [K].

Let $U(\mathfrak{n})^*_{\rm gr}$ be the graded dual of the enveloping algebra $U(\mathfrak{n})$ of $\mathfrak{n}$. To each element w in the Weyl group W of $\mathfrak{g}$ one can associate a subalgebra $\mathcal{R}(\mathcal{C}_w)$ of $U(\mathfrak{n})^*_{\rm gr}$ and a cluster algebra $\mathcal{A}(\mathcal{C}_w)$, see [GLS2]. Here $\mathcal{C}_w$ denotes a Frobenius category associated to w, see [BIRS, GLS2].

In [GLS2] we constructed a natural algebra isomorphism

$$\mathcal{A}(\mathcal{C}_w) \to \mathcal{R}(\mathcal{C}_w).$$

This yields a cluster algebra structure on $\mathcal{R}(\mathcal{C}_w)$.

Let $\mathbf{i} = (i_r, \ldots, i_1)$ be a reduced expression of w. In [GLS2] we studied two cluster-tilting modules $V_{\mathbf{i}} = V_1 \oplus \cdots \oplus V_r$ and $T_{\mathbf{i}} = T_1 \oplus \cdots \oplus T_r$ in $\mathcal{C}_w$, which are associated to $\mathbf{i}$. These modules yield two disjoint clusters $(\delta_{V_1}, \ldots, \delta_{V_r})$ and $(\delta_{T_1}, \ldots, \delta_{T_r})$ of $\mathcal{R}(\mathcal{C}_w)$. The exchanges matrices are of size $r \times (r - n)$. In contrast to our conventions in this article, the n coefficients are $\delta_{V_k} = \delta_{T_k}$ with $k^+ = r + 1$, where k^+ is defined as in [GLS2], and none of these coefficients is invertible. Furthermore, we studied a module $M_{\mathbf{i}} = M_1 \oplus \cdots \oplus M_r$ in $\mathcal{C}_w$, which yields cluster variables $\delta_{M_1}, \ldots, \delta_{M_r}$ of $\mathcal{R}(\mathcal{C}_w)$. (These do not form a cluster.) Using methods from Lie theory we obtained the following result.

Theorem 7.6 ([GLS2, Theorem 3.2]). *The cluster algebra $\mathcal{R}(\mathcal{C}_w)$ is a polynomial ring in the variables $\delta_{M_1}, \ldots, \delta_{M_r}$.*

To obtain an alternative proof of Theorem 7.6, one can proceed as follows:

(i) Show that the cluster variables $\delta_{M_1}, \dots, \delta_{M_r}$ are algebraically independent.
(ii) Show that for $1 \le k \le r$ the cluster variables δ_{V_k} and δ_{T_k} are polynomials in $\delta_{M_1}, \dots, \delta_{M_r}$.
(iii) Apply Theorem 4.1.

Part (i) can be done easily using induction and the mutation sequence in [GLS2, Section 13]. Part (ii) is not at all straightforward.

Let us give a concrete example illustrating Theorem 7.6. Let $\mathfrak{g}$ be the Kac-Moody Lie algebra associated to the generalized Cartan matrix

$$C = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},$$

and let $\mathbf{i} = (2,1,2,1,2,1,2,1)$. Then $\mathcal{A}(\mathcal{C}_w) = \mathcal{A}(\mathbf{x}, B_{\mathbf{i}})$, where $r = n+2 = 8$, x_7 and x_8 are the (non-invertible) coefficients, and

$$B_{\mathbf{i}} = \left(\begin{array}{cccccc} 0 & 2 & -1 & & & \\ -2 & 0 & 2 & -1 & & \\ 1 & -2 & 0 & 2 & -1 & \\ & 1 & -2 & 0 & 2 & -1 \\ & & 1 & -2 & 0 & 2 \\ & & & 1 & -2 & 0 \\ \hline & & & & 1 & -2 \\ & & & & & 1 \end{array}\right).$$

The principal part $B_{\mathbf{i}}^{\circ}$ of $B_{\mathbf{i}}$ is skew-symmetric, and the quiver $\Gamma(B_{\mathbf{i}})$ looks as follows:

$$\begin{tikzcd} 8 \arrow[rr] & & 6 \arrow[rr] \arrow[dl, shift left] \arrow[dl, shift right] & & 4 \arrow[rr] \arrow[dl, shift left] \arrow[dl, shift right] & & 2 \arrow[dl, shift left] \arrow[dl, shift right] & \\ & 7 \arrow[rr] & & 5 \arrow[rr] \arrow[ul, shift left] \arrow[ul, shift right] & & 3 \arrow[rr] \arrow[ul, shift left] \arrow[ul, shift right] & & 1 \arrow[ul, shift left] \arrow[ul, shift right] \end{tikzcd}$$

Define

$$\begin{aligned} &(\mathbf{x}[0], B[0]) := (\mathbf{x}, B_{\mathbf{i}}), \\ &(\mathbf{x}[1], B[1]) := \mu_5\mu_3\mu_1(\mathbf{x}[0], B[0]), && (\mathbf{x}[2], B[2]) := \mu_6\mu_4\mu_2(\mathbf{x}[1], B[1]), \\ &(\mathbf{x}[3], B[3]) := \mu_3\mu_1(\mathbf{x}[2], B[2]), && (\mathbf{x}[4], B[4]) := \mu_4\mu_2(\mathbf{x}[3], B[3]), \\ &(\mathbf{x}[5], B[5]) := \mu_1(\mathbf{x}[4], B[4]), && (\mathbf{x}[6], B[6]) := \mu_2(\mathbf{x}[5], B[5]), \end{aligned}$$

and for $0 \le k \le 6$ let $(x_1[k], \dots, x_8[k]) := \mathbf{x}[k]$.

Under the isomorphism $\mathcal{A}(\mathcal{C}_w) \to \mathcal{R}(\mathcal{C}_w)$ the cluster $\mathbf{x}[0]$ of $\mathcal{A}(\mathcal{C}_w) = \mathcal{A}(\mathbf{x}, B_{\mathbf{i}})$ corresponds to the cluster $(\delta_{V_1}, \dots, \delta_{V_8})$ of $\mathcal{R}(\mathcal{C}_w)$, the cluster $\mathbf{x}[6]$ corresponds to $(\delta_{T_1}, \dots, \delta_{T_8})$, and we have

$$
\begin{array}{llll}
x_1[0] \mapsto \delta_{M_1}, & x_2[0] \mapsto \delta_{M_2}, & x_1[2] \mapsto \delta_{M_3}, & x_2[2] \mapsto \delta_{M_4}, \\
x_1[4] \mapsto \delta_{M_5}, & x_2[4] \mapsto \delta_{M_6}, & x_1[6] \mapsto \delta_{M_7}, & x_2[6] \mapsto \delta_{M_8}.
\end{array}
$$

By Theorem 7.6 we know that the cluster algebra $\mathcal{A}(\mathbf{x}, B_{\mathbf{i}})$ is a polynomial ring in the variables $x_1[0], x_2[0], x_1[2], x_2[2], x_1[4], x_2[4], x_1[6], x_2[6]$.

8. Applications

8.1. Prime elements in the dual semicanonical basis. As in Section 7.3 let $C \in M_{n,n}(\mathbb{Z})$ be a symmetric generalized Cartan matrix, and let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the associated Lie algebra.

As before let W be the Weyl group of $\mathfrak{g}$. To C one can also associate a preprojective algebra Λ over $\mathbb{C}$, see for example [GLS2, R]

Lusztig [Lu] realized the universal enveloping algebra $U(\mathfrak{n})$ of $\mathfrak{n}$ as an algebra of constructible functions on the varieties Λ_d of nilpotent Λ-modules with dimension vector $d \in \mathbb{N}^n$. He also constructed the semicanonical basis $\mathcal{S}$ of $U(\mathfrak{n})$. The elements of $\mathcal{S}$ are naturally parametrized by the irreducible components of the varieties Λ_d.

An irreducible component Z of Λ_d is called *indecomposable* if it contains a Zariski dense subset of indecomposable Λ-modules, and Z is *rigid* if it contains a rigid Λ-module M, i.e. M is a module with $\mathrm{Ext}^1_\Lambda(M, M) = 0$.

Let $\mathcal{S}^*$ be the dual semicanonical basis of the graded dual $U(\mathfrak{n})^*_{\mathrm{gr}}$ of $U(\mathfrak{n})$. The elements ρ_Z in $\mathcal{S}^*$ are also parametrized by irreducible components Z of the varieties Λ_d. We call ρ_Z *indecomposable* (resp. *rigid*) if Z is indecomposable (resp. rigid). An element $b \in \mathcal{S}^*$ is called *primitive* if it cannot be written as a product $b = b_1 b_2$ with $b_1, b_2 \in \mathcal{S}^* \setminus \{1\}$.

Theorem 8.1 ([GLS1, Theorem 1.1]). *If ρ_Z is primitive, then Z is indecomposable.*

Theorem 8.2 ([GLS2, Theorem 3.1]). *For $w \in W$ all cluster monomials of the cluster algebra $\mathcal{R}(\mathcal{C}_w)$ belong to the dual semicanonical basis $\mathcal{S}^*$ of $U(\mathfrak{n})^*_{\mathrm{gr}}$. More precisely, we have*

$$
\begin{aligned}
\{\textit{cluster variables of } \mathcal{R}(\mathcal{C}_w)\} &\subseteq \{\rho_Z \in \mathcal{S}^* \mid Z \textit{ is indecomposable and rigid}\}, \\
\{\textit{cluster monomials of } \mathcal{R}(\mathcal{C}_w)\} &\subseteq \{\rho_Z \in \mathcal{S}^* \mid Z \textit{ is rigid}\}.
\end{aligned}
$$

Combining Theorems 3.1, 7.6 and 8.2 we obtain a partial converse of Theorem 8.1.

Theorem 8.3. *The cluster variables in $\mathcal{R}(\mathcal{C}_w)$ are prime, and they are primitive elements of $\mathcal{S}^*$.*

Conjecture 8.4. *If $\rho_Z \in \mathcal{S}^*$ is indecomposable and rigid, then ρ_Z is prime in $U(\mathfrak{n})^*_{\mathrm{gr}}$.*

8.2. Monoidal categorifications of cluster algebras. Let $\mathcal{C}$ be an abelian tensor category with unit object $I_{\mathcal{C}}$. We assume that $\mathcal{C}$ is a Krull-Schmidt category, and that all objects in $\mathcal{C}$ are of finite length. Let $\mathcal{M}(\mathcal{C}) := \mathcal{K}_0(\mathcal{C})$ be the Grothendieck ring of $\mathcal{C}$. The class of an object $M \in \mathcal{C}$ is denoted by $[M]$. The addition in $\mathcal{M}(\mathcal{C})$ is given by $[M] + [N] := [M \oplus N]$ and the multiplication is defined by $[M][N] := [M \otimes N]$. We assume that $[M \otimes N] = [N \otimes M]$. (In general this does not imply $M \otimes N \cong N \otimes M$.) Thus $\mathcal{M}(\mathcal{C})$ is a commutative ring.

Tensoring with K over $\mathbb{Z}$ yields a K-algebra $\mathcal{M}_K(\mathcal{C}) := K \otimes_{\mathbb{Z}} \mathcal{K}_0(\mathcal{C})$ with K-basis the classes of simple objects in $\mathcal{C}$. Note that the unit object $I_{\mathcal{C}}$ is simple.

A *monoidal categorification* of a cluster algebra $\mathcal{A}(\mathbf{x}, B)$ is an algebra isomorphism

$$\Phi\colon \mathcal{A}(\mathbf{x}, B) \to \mathcal{M}_K(\mathcal{C}),$$

where $\mathcal{C}$ is a tensor category as above, such that each cluster monomial $y = y_1^{a_1} \cdots y_m^{a_m}$ of $\mathcal{A}(\mathbf{x}, B)$ is mapped to a class $[S_y]$ of some simple object $S_y \in \mathcal{C}$. In particular, we have

$$[S_y] = [S_{y_1}]^{a_1} \cdots [S_{y_m}]^{a_m} = [S_{y_1}^{\otimes a_1} \otimes \cdots \otimes S_{y_m}^{\otimes a_m}].$$

For an object $M \in \mathcal{C}$ let x_M be the element in $\mathcal{A}(\mathbf{x}, B)$ with $\Phi(x_M) = [M]$.

The concept of a monoidal categorification of a cluster algebra was introduced in [HL, Definition 2.1]. But note that our definition uses weaker conditions than in [HL].

An object $M \in \mathcal{C}$ is called *invertible* if $[M]$ is invertible in $\mathcal{M}_K(\mathcal{C})$. An object $M \in \mathcal{C}$ is *primitive* if there are no non-invertible objects M_1 and M_2 in $\mathcal{C}$ with $M \cong M_1 \otimes M_2$.

Proposition 8.5. *Let $\Phi\colon \mathcal{A}(\mathbf{x}, B) \to \mathcal{M}_K(\mathcal{C})$ be a monoidal categorification of a cluster algebra $\mathcal{A}(\mathbf{x}, B)$. Then the following hold:*

(i) *The invertible elements in $\mathcal{M}_K(\mathcal{C})$ are*

$$\mathcal{M}_K(\mathcal{C})^\times = \left\{\lambda[I_{\mathcal{C}}][S_{x_{n+1}}]^{a_{n+1}} \cdots [S_{x_p}]^{a_p} \mid \lambda \in K^\times, a_i \in \mathbb{Z}\right\}.$$

(ii) *Let M be an object in $\mathcal{C}$ such that the element x_M is irreducible in $\mathcal{A}(\mathbf{x}, B)$. Then M is primitive.*

Proof. Part (i) follows directly from Theorem 2.2. To prove (ii), assume that M is not primitive. Thus there are non-invertible objects M_1 and M_2 in $\mathcal{C}$ with $M \cong M_1 \otimes M_2$. Thus in $\mathcal{M}_K(\mathcal{C})$ we have $[M] = [M_1][M_2]$. Since Φ is an algebra isomorphism, we get $x_M = x_{M_1} x_{M_2}$ with x_{M_1} and x_{M_2} non-invertible in $\mathcal{A}(\mathbf{x}, B)$. Since x_M is irreducible, we have a contradiction. □

Combining Proposition 8.5 with Theorem 3.1 we get the following result.

Corollary 8.6. *Let $\Phi\colon \mathcal{A}(\mathbf{x}, B) \to \mathcal{M}_K(\mathcal{C})$ be a monoidal categorification of a cluster algebra $\mathcal{A}(\mathbf{x}, B)$. For each cluster variable y of $\mathcal{A}(\mathbf{x}, B)$, the simple object S_y is primitive.*

Examples of monoidal categorifications of cluster algebras can be found in [HL, N1], see also [Le].

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